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# Implication des théories effectives des champs dans la phénoménologie de la physique des particules

# Implication of Effective Field Theories in Particle Physics Phenomenology

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# Résumé

Cette thèse de doctorat porte sur les applications du formalisme de l'intégrale du chemin aux théories effectives des champs (TEC), qui constituent une méthode puissante pour étudier la physique des particules au-delà du modèle standard. Nous présentons d'abord le Modèle Standard (MS) en tant que Théorie Effective des Champs et nous présentons brièvement quelques déformations possibles du MS par la TEC, qui jouent un rôle central dans la recherche d'une nouvelle physique. Nous présentons un paradigme dans lequel les théories effectives des champs sont utilisées comme un pont (pouvant être traversé dans les deux sens) pour relier la nouvelle physique à l'échelle des hautes énergies aux mesures expérimentales à basse énergie. Pour dériver le Lagrangien effectif à basse énergie d'une théorie ultraviolette (UV), il faut intégrer les modes lourds et ne garder que les degrés de liberté légers qui sont pertinents pour les mesures à basse énergie. Dans le corps de cette thèse, nous nous concentrons sur les méthodes fonctionnelles pour intégrer les particules lourdes jusqu'à l'ordre d'une boucle. Contrairement à l'approche traditionnelle du diagramme de Feynman, le formalisme de l'intégrale de chemin fournit un cadre universel et efficace pour mener à bien les tâches de correspondance à une boucle, ce qui est un exercice difficile en pratique. Enfin, nous appliquons nos techniques à l'étude des anomalies dans la théorie quantique des champs (TQC), et nous dérivons le lagrangien TEC de l'axion à basse énergie à partir d'un modèle complet UV de l'axion. En outre, nous utilisons également notre technique TEC pour explorer les récentes divergences de saveurs dans le MS.

**Mots clés:** Théories de champ effectives, intégrale de chemin, anomalies TQC, théories de jauge, physique de l'axion, physique des saveurs, physique au-delà du modèle standard.

# Abstract

This PhD thesis covers the applications of the path integral formalism to Effective Field Theories (EFTs), which serve as a powerful method for studying particle physics beyond the Standard Model. We first introduce the Standard Model (SM) as an Effective Field Theory and briefly present some possible EFT deformations of the SM, which play a central role in the search of new physics. We provide a paradigm in which Effective Field Theories are used as a bridge (can be crossed both ways) to connect new physics at the high-energy scale with the low-energy experimental measurements. To derive the low-energy effective Lagrangian of an ultraviolet (UV) theory, one needs to integrate out heavy modes and keep only the light degrees of freedom which are relevant to the low-energy measurements. In the body of this thesis, we concentrate on the functional methods for integrating out heavy particles up to one-loop order. In contrast with the traditional Feynman diagram approach, the path integral formalism provides a universal and efficient framework to carry out the one-loop matching tasks, which is a challenging exercise in practice. Finally, we apply our techniques to study anomalies in Quantum Field Theory (QFT), and derive the low-energy axion EFT Lagrangian from a given axion UV complete model. Besides, we also use our EFT technique to explore the recent flavour discrepancies in the SM.

**Key words:** Effective field theories, Path integral, QFT anomalies, Gauge theories, Axion physics, Flavour physics, Physics beyond the Standard Model.

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# List of Publications

**Published articles:**

[1] **Anomalies from an effective field theory perspective**, B. Filoche, R. Larue, J. Quevillon, P. N.H. Vuong, *accepted for publication in Phys. Rev. D*, [2205.02248].

[2] **Axion Effective Action**, J. Quevillon, C. Smith, P. N.H. Vuong, *JHEP* **08** (2022) 137 [2112.00553].

[3] **The Fermionic Universal One-Loop Effective Action**, S.A. Ellis, J. Quevillon, P. N.H. Vuong, T. You and Z. Zhang, *JHEP* **11** (2020) 078 [2006.16260].

[4] **Quark Flavor Phenomenology of the QCD Axion**, J. Martin Camalich, M. Pospelov, P. N.H. Vuong, R. Ziegler and J. Zupan, *Phys. Rev. D* 102 (1) (2020) 015023 [2002.04623].

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# Introduction

Our present knowledge of elementary particles and their interactions is described in the so-called Standard Model (SM) of particle physics. The discovery of a scalar boson with a mass of 125 GeV, corresponding to the Higgs boson in the SM, is the most remarkable confirmation of the predictions of the SM. In recent years, the direct discovery of new physics states has become less frequent, and there is no sign of new physics beyond the SM so far. Therefore, new physics states might be on a very high energy scale that current experiments cannot achieve. However, it is possible that new BSM states can manifest themselves via tree-level exchange or loop-level effects and then indirectly affect SM precision measurements at the low-energy scale. Recently, many indirect searches for the hints of new physics states have been performed and received much attention. Following this aspect, Effective Field Theories (EFT) rise as a robust theoretical framework that allows for hunting for new physics effects in a model-independent way (bottom-up approach) or connecting UV complete models with theories at low energies and their precision measurements (top-down approach). In this thesis, we have developed functional methods for EFT top-down approach and applied our techniques to study axion physics, QFT anomalies and some recent flavour discrepancies in the SM. The outline of this thesis is the following:

The first part of this thesis is devoted to presenting the basic knowledge when constructing effective field theories.

In chapter 1, we briefly review the Standard Model of particle physics and the motivation to consider EFT as a general framework for the extension of the SM. We present two ways to apply the EFT idea, the top-down and the bottom-up EFT approach, to study physics beyond the SM. We close this chapter by discussing two possible EFT deformations to extend the SM: the Standard Model Effective Field Theory (SMEFT) and the Higgs Effective Field Theory (HEFT).

In chapter 2, we present how to integrate out a heavy particle using the Feynman diagram approach and the functional approach. We focus on studying the path integral formulation of one-loop matching calculations. Such matching computations are the most crucial point in the EFT top-down paradigm. We provide the master formulas that allow us to match a UV model onto an EFT directly and elegantly. We briefly present our ongoing project where we integrate out leptoquarks and study flavour anomalies in the SM. Eventually, we show that the functional approach can easily derive the renormalization group equations (RGEs), thus improving the technique to derive the anomalous matrix dimension.

In chapter 3, which is based on [3], written in collaboration with S. A. Ellis, J. Quevillon, T. You and Z. Zhang, we show that one-loop matching by the functional method will generate universal structures, which makes repeated evaluation of the loop integrals redundant. Ultimately, this set of universal structures can be pre-computed once and for all, forming the so-called Universal One-Loop Effective Action (UOLEA). Starting from the UOLEA, one-loop matching calculations are reduced to an algebraic manipulation of matrix traces. In this part, we studied all universal structures arising from integrating out heavy chiral fermions. The final results provides an analytic expression for all coefficients of the universal structures and serves as a reference to cross-check if one would like to decouple heavy chiral fermions.

The second part of this thesis is devoted to studying effective field theories in the presence of chiral anomalies.

In chapter 4, we briefly review the anomalies in quantum field theory which is the foundation for many upcoming discussions.

In chapter 5, we briefly review the theoretical background of the axion physics. We focus on how QCD instantons solve the $U(1)_A$ problem of the low-energy QCD theory but simultaneously induce the so-called strong CP problem. We concentrate on the Peccei-Quinn mechanism which solves the strong CP problem elegantly. We end this chapter by presenting the axion UV complete models.

In chapter 6, which is based on [2], written in collaboration with J. Quevillon and C. Smith, we present the construction of EFTs in which a chiral fermion, charged under both gauge and global symmetries, is integrated out. These symmetries can be spontaneously broken, and the global ones might also be anomalous. This setting is typically used to study the structure of low-energy axion EFTs, where the anomalous global symmetry can be the well-known $U(1)_{PQ}$ and the local symmetries can be the SM electroweak chiral gauge symmetries. Spontaneous symmetry breaking will generate Goldstone bosons, and in the meantime, chiral fermions also become massive. In this setup, we emphasize that the derivative couplings of the Goldstone bosons to fermion will lead to severe divergences and ambiguities when evaluating one-loop computations. Within the path integral formalism, we show how to solve the ambiguity problem by adapting the anomalous Ward identities to the EFT context, and thus enforcing the gauge invariance of the result. Our methodology provides a neat, generic and consistent, result when evaluating the Wilson coefficients of EFT operators involving axion and gauge bosons.

In chapter 7, which is based on [1], written in collaboration with B. Filoche, R. Larue, and J. Quevillon, we show that the technical developments of the functional one-loop matching can be used to evaluate the path-integral measure. Such non-invariant fermionic measures will lead to anomalous Ward-Takahashi identities. We present several ways to customize the crucial regularization such that the anomaly is located in the desired current, which is unprecedented within the path integral approach. With our techniques, we can derive the covariant, consistent, gravitational and scale anomalies in a transparent and unified way.

Eventually, we summarize what we found and developed in the section conclusion and outlook.

# Part I

# Effective Field Theories

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# Chapter 1

# The Standard Model as an Effective Field Theory

In this chapter, we briefly review the making of the Standard Model of particle physics and present the motivation to consider effective field theory as a general framework to extend the Standard Model. This chapter is dedicated to the basic concept of effective field theories.

## 1.1 The Standard Model of particle physics

The Standard Model (SM) of particle physics is a renormalizable quantum field theory (QFT) that describes the interactions of our current understanding of elementary particles (quarks and leptons), which are the main building blocks of all the matter we know at present. The SM has been extensively tested for many years and is still providing excellent agreement with nearly all experimental results, Refs. [7–10].

According to the SM, the matter is described by the spin-$\frac{1}{2}$ fermion fields. There are three generations of fermion. Each generation contains two quark flavours ($u$ and $d$ like) and two leptons (neutrino and electron like). Coexisting with all these particles is their corresponding antiparticle with the same mass but opposite charges. The fermionic content of the SM is listed in Table 1.1.

| | 1st generation | 2nd generation | 3rd generation | $Q$ |
|---|---|---|---|---|
| Quarks | $u$ | $c$ | $t$ | $2/3$ |
| | $d$ | $s$ | $b$ | $-1/3$ |
| Leptons | $e^-$ | $\mu^-$ | $\tau^-$ | $-1$ |
| | $\nu_e$ | $\nu_\mu$ | $\nu_\tau$ | $0$ |

**Table 1.1:** *Fermion fields in the SM. $Q$ is the electromagnetic charge. Each quark has three color degrees of freedom, in this table, we omitted the color indices to simplify the notation. Notice that neutrinos are massless and only left-handed polarized.*

Quarks and leptons interact by exchanging the strong and electroweak (EW) forces. In QFT, these features are formulated by applying the principles of gauge theory. The key points of gauge theory are first choosing the gauge group. Corresponding to each generator of the gauge group is a spin-1 gauge boson, which plays a role as the mediator carrying the interaction force. Second, by defining the gauge transformations of the quantum fields in this theory, one can build the gauge invariant Lagrangian density. When this work is achieved, we will be able to study all physical phenomena of this theory. A priori, the choice of the gauge groups and the properties of spinor field representations in the SM are suggested by many observations in particle physics experiments. Before delving into details, we briefly present the making of essential ingredients in the SM.

**The theory of electromagnetic interactions.** From the fact that fermions can have an electric charge, by requiring the Lagrangian gauge invariance under the phase rotations of the fermion fields with the electric charge $Q$, the electromagnetic interaction is embedded in the SM. The symmetry that describes this interaction is then $U(1)_Q$ Abelian gauge symmetry. Since $U(1)_Q$ has one generator, there is only one corresponding gauge boson, the photon field. The quantum field theory that describes the electromagnetic interactions is often called Quantum Electrodynamics (QED).

**The theory of strong interactions.** Historically, quarks were introduced to explain the hadronic spectrum. In the model of Gell-Mann [11], denoting the members of quarks as $q$ and the members of antiquarks as $\bar{q}$, in terms of quark bound states, meson are $|q\bar{q}\rangle$ states and baryon are $|qqq\rangle$ states. In 1951, a new baryon $\Delta^{++}(J = 3/2)$ was discovered and brought the first hint that quark has *color* charges, Ref. [12]. Surprisingly, the new particle $\Delta^{++}$ is in the ground state $L = 0$ with all the spins of up quarks aligned along the same direction, $|u^\uparrow u^\uparrow u^\uparrow\rangle$. Thus this state will lead to a symmetric wavefunction. Since the quark is a spin-$\frac{1}{2}$ particle, this would have forbidden the $\Delta^{++}$ due to the Dirac statistics and Pauli exclusion principle. To solve this problem, color charges were introduced as the new quantum numbers of quarks. Within this assumption, the wavefunction of $\Delta^{++}$ is,

$$\Delta^{++}(J = 3/2) = \frac{1}{\sqrt{6}}\epsilon^{\alpha\beta\gamma}|u^\uparrow_\alpha u^\uparrow_\beta u^\uparrow_\gamma\rangle \,, \tag{1.1}$$

with $\alpha$ being a color index. Eq. (1.1) teaches us that each quark needs at least three colors in order to make the wavefunction of $\Delta^{++}$ antisymmetric. The value $N_c$ is also measured from the ratio between the hadronic and leptonic decays, $R^{\pi^0}_{e^+e^-} = \Gamma(\pi^0 \to \gamma\gamma)/\Gamma(e^+e^- \to \gamma\gamma) \sim N_c \sum_f Q_f^2$ . The result of these measurements indicate that $N_c = 3$.

In 1954, Yang and Mills [13] enlarged the concept of Abelian gauge theory (QED) to a non-Abelian gauge theory and thus opened the door to study the interactions between quarks. Considering quarks require color charges, one can require the Lagrangian invariance under the $SU(3)_C$ gauge transformation. Since $SU(3)_C$ has eight generators, there are eight corresponding gauge bosons, often called gluons. These gluons are responsible for the strong interactions between quarks. Notice that $SU(3)_C$ is a non-Abelian gauge theory, hence gluons can have self-interactions. The theory that describes the strong interactions of quarks and gluons is called Quantum Chromodynamics (QCD).

**The theory of weak interactions.** Unlike QED, the theory of weak interactions contain more subtle details, and it took a long time to formulate completely. In the 1930s, with the discovery of the neutron, it was quickly realized that the $\beta$-decay process of the neutron ($n \to p^+e^-\bar{\nu}_e$). At the level of quarks, this decay process is $d \to u + e^- + \bar{\nu}_e$ . Following the historical timeline, 1930s – 1970s, there were many developments in the theoretical backgrounds to precisely explain the $\beta$-decay of the neutron.

- **Fermi's theory.** In the 1930s, Enrico Fermi introduced the four-fermions effective theory, Ref. [14]. In his theory, the neutron $\beta$-decay is governed by the non-renormalizable effective operator,

$$\mathcal{L}_{\text{Fermi}} = -\frac{G_F}{\sqrt{2}}\big(\bar{p}\gamma^{\mu}n\big)\big(\bar{e}\gamma_{\mu}\nu_e\big) + \text{h.c.}\,, \tag{1.2}$$

where $G_F = 1.166\times10^{-5}$ GeV$^{-2}$ is the Fermi constant. The value of this constant is obtained from the precision measurement of the muon lifetime, $\frac{1}{\tau_\mu} = \frac{G_F^2\, m_\mu^5}{192\pi^3}$, Ref. [15].

- **The V − A theory.** In the middle of the 1950s, there was an experimental breakthrough in understanding the weak interactions. Lee and Yang [16] first pointed out no experimental evidence of parity conservation in weak interactions. Motivated by the work of Lee & Yang, Wu and collaborators [17] led an experiment to test the parity properties of neutron and nuclear decay directly. From the $\beta$-decay of cobalt-60, $^{60}\text{Co} \to ^{60}\text{Ni} + \text{e}^- + \bar{\nu}_\text{e} + 2\gamma$, they observed an extremely shocking phenomenon: the neutrino is only left-handed polarized. This is the first experimental hint suggesting that weak interactions violate parity maximally. To explain the experimental result of Wu et al., Feynman and Gell-Mann [18] modified Fermi's theory by using the chiral projector operators,

$$\mathcal{L}_{\text{Fermi}} = -\frac{G_F}{\sqrt{2}}\big(\bar{d}_L\gamma^{\mu}u_L\big)\big(\bar{\nu}_{e,L}\gamma_{\mu}e_L\big)\,, \tag{1.3}$$

where $\nu_{e,L} = P_L\nu_e$ and $P_L = (1-\gamma^5)/2$. Because the fermionic chiral currents in Eq. (1.3) can be decomposed in terms of vector and axial currents, for instance,

$$J^{\mu}_{e,L} = \bar{\nu}_{e,L}\gamma^{\mu}e_L = \frac{1}{2}\big(\bar{\nu}_e\gamma^{\mu}e\big) - \frac{1}{2}\big(\bar{\nu}_e\gamma^{\mu}\gamma^5 e\big) = J^{\mu}_{e,V} - J^{\mu}_{e,A}\,, \tag{1.4}$$

the model of Feynman & Gell-mann in Eq. (1.3) is also called the $V-A$ theory. In the past, this theory could successfully explain many charged processes including $\beta$-decay, muon decay and charged pion decay.

However, the $V-A$ theory should not be understood as a fundamental theory because it is non-renormalizable and hence we only consider it as a low-energy effective field theory (EFT). The problem of $V-A$ theory can be seen easily via unitarity bound. Considering the scattering process, $e^- + \nu_e \to e^- + \nu_e$, the conservation of probability gives the unitarity bound on the differential cross-section of this scattering process,

$$\frac{d\sigma}{d\Omega} \sim G_F^2 s^2 \leqslant 1\,, \tag{1.5}$$

with $s$ being the energy of the neutrino in the center of mass frame, hence $s \leqslant G_F^{-1} \sim 300\,\text{GeV}$. From this rough estimation, one knows that above the energy scale $\Lambda \geqslant 300\,\text{GeV}$ the $V-A$ theory becomes inconsistent with unitarity condition. Thus it must be replaced by a UV completion.

- **The intermediate vector boson (IVB) theory.** In the early 1960s, to overcome the contradiction with unitarity in $V-A$ theory, Glashow [7] proposed a theory with new massive vector bosons which mediates weak interactions in nuclear decay process. The new vector bosons, $W^\pm$ with electric charge $\pm1$, are supposed to be massive, $m_W \sim 100\,\text{GeV}$, so that at a low-energy scale the IVB theory will coincide with Fermi's effective theory. Since $W^\pm$ can couple to the charged currents, the Lagrangian for weak interactions is then,

$$\mathcal{L}_{\text{IVB}} \supset g\big(\bar{\psi}^i_L\gamma^{\mu}\psi^j_L\big)W^-_\mu + \text{h.c.}\,, \tag{1.6}$$

with $i$ stands for the flavour indices, e.g. $i = (u, d, \nu_e, e, \dots)$, and $g$ is the dimensionless coupling constant. The Fermi constant can be expressed in terms of the weak coupling constant $g$ through $G_F \sim g^2/m_W^2$ . However, the IVB theory with $W^\pm$ bosons is still incomplete. The scattering amplitude of the process, $e^+e^- \to W^+W^-$ with the photon exchange, becomes divergent. To cancel the divergences, the Z boson is then introduced. This boson has the mass around $m_W$ and can couple to the neutral currents. The full Lagrangian of IVB theory now has a form,

$$\mathcal{L}_{\text{IVB}} = g\big(\bar{\psi}^i_L \gamma^\mu \psi^j_L\big) W^-_\mu + \big(g^Z_L\, \bar{\psi}^i_L \gamma^\mu \psi^i_L + g^Z_R\, \bar{\psi}^i_R \gamma^\mu \psi^j_R\big) Z_\mu + \text{h.c.}\,, \tag{1.7}$$

where $g^Z_L$, $g^Z_R$ are the coupling constants of the Z boson with the left-handed and right-handed fermion, respectively. Although it has many phenomenological successes, the IVB theory cannot explain the origin of massive vector bosons and hence demands a new sophisticated theory. The IVB theory also requires an additional scalar field (the Higgs field) to cure further divergences.

**The unification of electromagnetic and weak interactions.** In the original work of IVB theory, Glashow [7] also raised an idea of unifying electromagnetic and weak interactions by using the $SU(2)_L \otimes U(1)_Y$ gauge group. However, to successfully complete this idea, one needed two key points: spontaneous symmetry breaking and Higgs mechanism.

- **Spontaneous symmetry breaking.** In the early 1960s, inspired by the superconductor solution in condensed matter physics, Nambu and Goldstone [19, 20] developed the concept of spontaneous symmetry breaking (SSB) and the Goldstone theorem. For a given theory, if its ground state does not respect the internal symmetry of this theory, new massless scalar particles will emerge in the spectrum. Corresponding to each generator of the broken symmetry is the so-called Nambu-Goldstone boson. This idea is then explicitly proved and systematically generalized in the context of QFT by Goldstone, Salam, and Weinberg [21].

  In 1967–1968, Weinberg and Salam [8, 9] synthesized Glashow's unification framework [7] with the symmetry breaking mechanism to build a gauge theory for electroweak interactions. At the energy above the electroweak scale (unbroken phase), the theory is charged under the $SU(2) \otimes U(1)_Y$ gauge group, and the weak gauge bosons are all massless. Below the electroweak scale (broken phase), the theory is spontaneously broken into the $U(1)_Q$ gauge group. Since $SU(2)_L$ has three generators, three corresponding Goldstone bosons appeared in the symmetry breaking process, and all of them are absorbed to generate the mass of $W^\pm$ and $Z$ bosons in the Higgs mechanism.

- **The Higgs boson.** Because the theory in the unbroken phase must be invariant under the $SU(2)_L \otimes U(1)_Y$ gauge group, all fermions are also massless. To generate the gauge boson mass and naturally break the $SU(2)_L \otimes U(1)_Y$ gauge group, Higgs, Brout and Englert [22, 23] introduced a new scalar field, the Brout-Englert-Higgs boson, commonly called the Higgs boson, and its potential terms in the electroweak Lagrangian. In their mechanism, the electroweak symmetry is broken when the Higgs field obtains its vacuum expectation value (VEV) via minimizing the Higgs potential. In this way, the theory in both unbroken and broken phases can be renormalizable and consistently explain the origin of massive gauge bosons. Weinberg then coupled the Higgs to fermions to give them mass in his model of leptons [8].

Eventually, combining the Glashow-Weinberg-Salam model and the Higgs mechanism allows us to construct the standard model of electroweak interactions comprehensively.

### 1.1.1 Structure of the Standard Model

From the above discussions, we are now able to investigate in detail the formalism of Standard Model. The gauge group of SM is,

$$G_{\rm SM} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \,. \tag{1.8}$$

The particle content of SM includes the fermion fields as listed in Table 1.1, one scalar field (Higgs), and the vector gauge bosons such as the photon, gluon, $W^\pm$, and $Z$. The matter fields in the SM are chiral, and their left-handed components are doublets of the $SU(2)_L$ gauge symmetry while the right-handed components are $SU(2)_L$ singlets. Notice that the right-handed neutrinos do not exist in the SM. All properties of the SM particles are summarized in Table 1.2, where we study their quantum numbers.

| | Fields | Representation $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ | Isospin $I_3$ | Electric charges $Q = Y + I_3$ |
|---|---|---|---|---|
| Spin $\frac{1}{2}$ | $q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$ | $(\mathbf{3}, \mathbf{2}, 1/6)$ | $\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$ | $\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$ |
| | $u_R$ | $(\mathbf{3}, \mathbf{1}, 2/3)$ | 0 | 2/3 |
| | $d_R$ | $(\mathbf{3}, \mathbf{1}, -1/3)$ | 0 | −1/3 |
| | $l_L = \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}$ | $(\mathbf{1}, \mathbf{2}, -1/2)$ | $\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ |
| | $e_R$ | $(\mathbf{1}, \mathbf{1}, -1)$ | 0 | −1 |
| Spin 0 | $H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$ | $(\mathbf{1}, \mathbf{2}, 1/2)$ | $\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
| Spin 1 | $G^A_\mu$ | $(\mathbf{8}, \mathbf{1}, 0)$ | 0 | 0 |
| | $W^I_\mu$ | $(\mathbf{1}, \mathbf{3}, 0)$ | $(1, -1, 0)$ | $(1, -1, 0)$ |
| | $B_\mu$ | $(\mathbf{1}, \mathbf{1}, 0)$ | 0 | 0 |

**Table 1.2:** *Fields and their quantum numbers under the SM gauge group. For simplicity, we omitted the color and generation indices. The notations $(\mathbf{8}, \mathbf{3}, \mathbf{2}, \mathbf{1})$ denote the octet, triplet, doublet and singlet representations of the $SU(N)$ group. $I_3$ stands for the third generator of $SU(2)_L$. The breaking of electroweak symmetry, $SU(2)_L \otimes U(1)_Y \to U(1)_Q$ implies the relation $Q = I_3 + Y$.*

**The SM Lagrangian.** Before spontaneous symmetry breaking, the SM Lagrangian reads,

$$\mathcal{L}_{\rm SM} = -\frac{1}{4} G^A_{\mu\nu} G^{A,\mu\nu} - \frac{1}{4} W^I_{\mu\nu} W^{I,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \tag{1.9}$$

$$+ \left( \bar{q}_L i \not{D} q_L + \bar{u}_R i \not{D} u_R + \bar{d}_R i \not{D} d_R + \bar{l}_L i \not{D} l_L + \bar{e}_R i \not{D} e_R \right) \tag{1.10}$$

$$+ (D_\mu H)^\dagger (D^\mu H) + \mu^2 H^\dagger H - \frac{\lambda}{2} (H^\dagger H)^2 \tag{1.11}$$

$$- \left( \bar{l}_L \Gamma_e \, e_R H + \bar{q}_L \Gamma_u \, u_R \tilde{H} + \bar{q}_L \Gamma_d \, d_R H + \text{h.c.} \right) \tag{1.12}$$

$$+ \mathcal{L}_{GF} + \mathcal{L}_{FP} \,, \tag{1.13}$$

where the complex conjugate of the Higgs fields is occurred as $\tilde{H}^j = \epsilon_{jk}(H^k)^*$ , with the convention $\epsilon_{12} = +1$. In what follows, we present the detail of this Lagrangian:

- **Yang-Mill terms.** The kinetic terms of gauge bosons are described by Eq. (1.9). Chronologically, $G^A_\mu$ and $W^I_\mu$ with the gauge indices $A = (1, \dots, 8)$, $I = (1, 2, 3)$, are the non-Abelian gauge bosons of $SU(3)_C$, $SU(2)_L$, while $B_\mu$ is the Abelian gauge boson of $U(1)_Y$. The field strength tensors are defined by,

$$\begin{aligned}G^A_{\mu\nu} &= \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g_s f^{ABC} G^B_\mu G^C_\nu \,,\\ W^I_{\mu\nu} &= \partial_\mu W^I_\nu - \partial_\nu W^I_\mu - g\epsilon^{IJK} W^J_\mu W^K_\nu \,,\\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \,,\end{aligned} \tag{1.14}$$

  where $f^{ABC}$, $\epsilon^{IJK}$ are the structure constants of $SU(3)_C$ and $SU(2)_L$, respectively. The convention for dual tensors are $\tilde{X}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} X_{\rho\sigma}$ with the choice $\epsilon^{0123} = +1$, and $X_\mu$ stands for $G^A_\mu, W^I_\mu, B_\mu$.

- **Gauge interactions.** The gauge interactions between fermions and gauge bosons are described by Eq (1.10). The covariant derivative $D_\mu$ is defined such that its action to a field $\psi_f$ is also a gauge covariant object. Suppose $\psi_f$ is charged under the SM gauge group with the field representation as $(\mathbf{3}, \mathbf{2}, Y_f)$, the covariant derivative then reads,

$$(D_\mu \psi_f) = \big(\partial_\mu + ig_s T^A G^A_\mu + igT^I W^I_\mu + ig' Y_f B_\mu\big)\psi_f \,, \tag{1.15}$$

  where $g_s, g, g'$ are dimensionless gauge coupling constants. $T^A$, $T^I$ are $SU(3)_C$ and $SU(2)_L$ generators, respectively. All of them satisfy the Lie algebra,

$$\big[T^A, T^B\big] = if^{ABC} T^A \,, \quad \big[T^I, T^J\big] = i\epsilon^{IJK} T^K \,. \tag{1.16}$$

  In the fundamental representation, the $SU(3)_C$ generators are expressed as $T^A = \lambda^A/2$ with $\lambda^A$ are the Gell-Mann matrices. Analogously, the $SU(2)_L$ generators read, $T^I = \sigma^I/2$, with $\sigma^I$ are the Pauli matrices. For the SM fermion fields, where their field representations and quantum numbers are explicitly listed in Table 1.2, the covariant derivative acting on these fields will be represented similarly to Eq. (1.15).

- **Higgs self interactions.** The symmetry breaking part is stated in Eq. (1.11) which contains the Higgs kinetic term $(D_\mu H)^\dagger (D^\mu H)$, and the Higgs potential shaped like a Mexican hat,

$$-\mathrm{V}(H) = \mu^2 H^\dagger H - \frac{\lambda}{2}(H^\dagger H)^2 \,. \tag{1.17}$$

  This potential is minimized when the Higgs field obtains the vacuum expectation value. For $\mu^2$, $\lambda > 0$, the scalar potential does not occur at the minimum for $\langle 0|H|0\rangle = 0$, however, all non-vanishing field configurations, with $H^\dagger H = \mu^2/\lambda$, will satisfy the minimum condition. Selecting the configuration which is real and electrically neutral, the *vacuum expectation value* (VEV) of the Higgs field reads,

$$\langle 0|H|0\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} v \end{pmatrix} \,, \text{ with } v = \sqrt{\frac{2\mu^2}{\lambda}} \,, \tag{1.18}$$

  and $v \simeq 246$ GeV at tree-level. Notice that this ground state is only invariant under the $U(1)_Q$ gauge transformation; hence the symmetry of SM is spontaneously broken with the breaking pattern $SU(2)_L \otimes U(1)_Y \to U(1)_Q$. The Higgs and Goldstone bosons appear as excitation states when expanding around the vacuum of Higgs field,

$$H = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + h + iG^0) \end{pmatrix}, \tag{1.19}$$

where $h$ is the Higgs boson, and $G^\pm, G^0$ stand for the Goldstone bosons. Plugging Eq. (1.19) into the Higgs potential, one obtains the squared-mass of Higgs boson at tree-level,

$$M_h^2 = \lambda v^2 = 2\mu^2 . \tag{1.20}$$

**The mass of gauge bosons.** The origin of massive gauge bosons is then explained by expanding the Higgs kinetic term with the field configuration in Eq. (1.19). We begin with,

$$(D_\mu H)^\dagger (D^\mu H) \supset \frac{g^2 v^2}{8}\left(W_\mu^1 W^{1,\mu} + W_\mu^2 W^{2,\mu}\right) + \frac{v^2}{8}\begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix}\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W^{3,\mu} \\ B^\mu \end{pmatrix}, \tag{1.21}$$

where the gauge fields are still in the weak eigenstates basis. The physical basis of $W^\pm$ bosons with the well defined electric charges is constructed from $[Q, T^1 \mp T^2] = \pm(T^1 \mp T^2)$. The physical combinations of $W_\mu^3$ and $B_\mu$ are obtained by diagonalizing the mass matrix in Eq. (1.21). Explicitly, the physical states of electroweak gauge fields are given by,

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i W_\mu^2\right), \quad \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix}\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}, \tag{1.22}$$

where $Z_\mu, A_\mu$ are the $Z$ boson and photon field, respectively. The parameter $\theta_w$ is a weak mixing angle which is determined from,

$$\cos\theta_w = \tan\theta_w = \frac{g'}{g} . \tag{1.23}$$

Within the physical mass basis, from the Higgs kinetic terms we obtain,

$$(D_\mu H)^\dagger (D^\mu H) \supset M_W^2 W_\mu^+ W^{-,\mu} + \begin{pmatrix} Z_\mu & A_\mu \end{pmatrix}\begin{pmatrix} M_Z^2 & 0 \\ 0 & M_A^2 \end{pmatrix}\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}, \tag{1.24}$$

whereas for the mass of gauge bosons read,

$$M_W = \frac{gv}{2}, \quad M_Z = \sqrt{g'^2 + g^2}\,\frac{v}{2}, \quad M_A = 0 . \tag{1.25}$$

Also from the Higgs kinetic term, one observes the mixing of gauge and Goldstone bosons,

$$(D_\mu H)^\dagger (D^\mu H) \supset i M_W \left(W_\mu^+ \partial^\mu G^- - W_\mu^- \partial^\mu G^+\right) - M_Z Z_\mu \partial^\mu G^0 . \tag{1.26}$$

From the above equation, we know that the three Goldstone bosons are "eaten" by $W^\pm$ and $Z$. They generate the longitudinal mode of the gauge field and hence making them become massive. Last but not least, the above gauge-Goldstone mixing terms in Eq. (1.26) can be discarded in practical computations thank to the gauge fixing procedure.

- **Yukawa interactions and flavour changing charged currents.** The interactions between Higgs and fermion fields are expressed in Eq. (1.12) with $\Gamma_e, \Gamma_u, \Gamma_d$ are the general Yukawa couplings matrices in the generation space. After SSB, with the vacuum configuration in Eq. (1.18) the Yukawa interactions give rise to a mass matrix for quarks and electron-like leptons.However, these matrices are not diagonal in the generation space. The physical states are obtained by performing the unitary transformations,

$$f_{L,R} = U^f_{L,R}\, f'_{L,R}\,, \tag{1.27}$$

where $f = u, d, e$ and $f'$ denotes the mass eigenstates fields. Hence, the mass matrices in the Yukawa interactions are diagonolized,

$$\begin{aligned} M_u &= U_L^{u\dagger}\frac{v}{\sqrt{2}}\Gamma_u U_R^u = \mathrm{diag}(m_u, m_c, m_t)\,, \\ M_d &= U_L^{d\dagger}\frac{v}{\sqrt{2}}\Gamma_d U_R^d = \mathrm{diag}(m_d, m_s, m_b)\,, \\ M_e &= U_L^{e\dagger}\frac{v}{\sqrt{2}}\Gamma_e U_R^e = \mathrm{diag}(m_e, m_\mu, m_\tau)\,. \end{aligned} \tag{1.28}$$

It is worth noting that the redefinition of the fermion fields will modify the flavour changing charged currents,

$$\bar{q}_L i\not{D} q_L \supset \frac{g}{\sqrt{2}}\bar{u}_L\gamma^\mu W_\mu^- d_L + \text{h.c.} = \frac{g}{\sqrt{2}}\bar{u}'_L\gamma^\mu W_\mu^-\left(U_L^{u\dagger}U_L^d\right)d'_L + \text{h.c.} \tag{1.29}$$

Obviously, the physical states now interact effectively with the charged currents through a non-trivial unitary matrix,

$$V_{CKM} = U_L^{u\dagger}U_L^d\,, \tag{1.30}$$

where $V_{CKM}$ is the well-known Cabibbo-Kowayashi-Maskawa (CKM) matrix, Refs. [24, 25]. The CKM matrix is parameterized in terms of four independent parameters, with three angles $\theta_{1,2,3}$ and one phase $\delta$. Explicitly, the CKM matrix reads,

$$V_{CKM} = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2c_1 & c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ s_1s_2c_1 & s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix}\,, \tag{1.31}$$

where $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$. Since $\delta$ is non-vanishing, the CKM matrix can provide a natural explanation for the CP-violating interactions which have been observed a long time ago in the neutral meson mixing (e.g. $K^0 - \bar{K}^0$ [26]) process.

- **Gauge fixing and Faddeev-Popov ghost terms.** In practice, to derive a propagator function of a massless vector field one needs to add the gauge fixing term (see Ref. [27] for a pedagogical example). Since the SM Lagrangian is defined in unbroken phase, the gauge fixing procedure is mandatory. Precisely, the SM gauge fixing terms are,

$$\begin{aligned} \mathcal{L}_{GF} = &-\frac{1}{2\xi_G}(\partial_\mu G^{A,\mu})^2 - \frac{1}{2\xi_A}(\partial_\mu A^\mu)^2 - \frac{1}{2\xi_Z}(\partial_\mu Z^\mu + \xi_Z M_Z G^0)^2 \\ &- \frac{1}{2\xi_W}(\partial_\mu W^{+,\mu} + i\xi_W M_W G^+)(\partial_\nu W^{-,\nu} - i\xi_W M_W G^-)\,, \end{aligned} \tag{1.32}$$

where $\xi_i$ are gauge parameters. Of course, all physical observations are independent with $\xi_i$ parameters, hence one can choose a gauge that allows us to simplify the computation of $S$-matrix elements. For instance, $\xi_i = 1$ corresponds to the Feynman - 't Hooft gauge [28], where the gauge-Goldstone mixing terms in Eq. (1.26) is entirely cancelled out. The caveat of this procedure is that the gauge fixing terms are not gauge invariant, and hence the Faddeev-Popov ghost fields are introduced to systematically quantize the vector gauge fields. In terms of the ghost fields, the SM Lagrangian is invariant under the Becchi, Rouet, Stora, and Tyutin (BRST) transformations [28, 29] instead of gauge transformations. Explicitly, by defining a BRST operator, called $\hat{\mathbf{s}}$ operator, under the BRST transformations the gauge fixing and Faddeev-Popov ghost terms behave as follows,

$$\hat{\mathbf{s}}(\mathcal{L}_{GF}) = -\hat{\mathbf{s}}(\mathcal{L}_{FP}) \,, \tag{1.33}$$

thus they ensure the SM Lagrangian is invariant under the BRST transformations.

From the above discussions, we have presented the formalism of the SM. To wrap up this section, we highlight two main properties of the SM from a formal theory point of view: First, the SM is a renormalizable theory. This key feature offers the power of predicting physical phenomena at various energy scales. Second, the SM is a consistent theory in the sense that all gauge anomalies[1] are cancelled. Such anomalous gauge current will lead to inconsistencies in the quantization of gauge theory. We will see later that the gauge symmetries of the SM are potentially anomalous, and the quantum numbers of SM fermions in Table (1.2) are carefully chosen such that all gauge anomalies cancel [28, 30]. We leave some concrete discussions of QFT anomalies to Chapter 4 and Chapter 7 of this PhD thesis.

## 1.2 Motivation for physics beyond the Standard Model

As we have seen, the SM was tremendously successful in describing the strong and electroweak interactions of Nature. Here we list some of the remarkable achievements of SM:

- The prediction of SM for the anomalous magnetic moment of the electron was in excellent agreement (at a precision level $10^{-12}$) with experimental measurements, Ref. [31].

- At CERN in 1983, $W^\pm$ and $Z$ boson were directly discovered from the UA1 experiment led by Carlo Rubbia and Simon van der Meer and confirmed from the UA2 experiment led by Pierre Darriulat. These experiments were a significant breakthrough in experimental physics and accelerator technology. Furthermore, as we know $\cos\theta_w = M_W/M_Z$, the observations of $W^\pm, Z$ bosons also allow us to test the weak mixing angle $\theta_w$ at very high accuracy (including radiative corrections).

- The discovery of a scalar boson with a mass of 125 GeV in 2012 at LHC [32] was a milestone in the history of particle physics. This greatest achievement finally confirms the prediction of the Higgs boson in the SM.

**Why beyond the Standard Model?** Despite many successes, the SM still contains several unsolved puzzles. We state here a few of them:

- **Hierarchy problem.** In the SM, we did not include the effect of gravity. Thus the theory should have an ultraviolet cut-off, $\Lambda_{SM} \sim M_{pl} = 10^{19}$ GeV, above this energy scale, we cannot

[1] A symmetry is anomalous when its associated Noether's current is no longer conserved at the quantum level.

neglect the contribution of gravity and the predictions of SM are longer valid. The radiative corrections for the mass of Higgs boson [33] suggest,

$$\Delta M_h^2 \simeq \frac{3y_t^2}{8\pi^2}\left[-\Lambda_{SM}^2 + 6m_t^2 \ln\frac{\Lambda_{SM}}{m_t} - 2m_t^2\right] + \mathcal{O}\left(\frac{1}{\Lambda_{SM}^2}\right), \tag{1.34}$$

where we only consider the dominant contribution from the top quark, and $y_t \simeq 1$ is the top quark Yukawa coupling. The Eq. (1.34) tells us that the quadratic divergence term, $\frac{-3y_t^2}{8\pi^2}\Lambda_{SM}^2$, will dominantly contribute to the mass of the Higgs boson. We then naively expect that $M_h$ is around the Plank scale, $M_{pl} = 10^{19}$ GeV. This result contrasts with the experimental results, whereas we only observe the Higgs boson with the mass of 125 GeV [32, 34] and the SM could not explain why the mass of the Higgs boson is so light. In the past, this hierarchy problem was one of the best reasons to motivate us to look forward to a new theory. *Supersymmetry* is a typical BSM extension that beautifully solves this problem.

- **Neutrino mass.** We recall that neutrinos in the SM are massless. However, the observations of neutrino oscillation directly proved that the neutrinos are massive [35], and the origin of the mass of neutrinos has been unsolved until now.

- **Strong CP problem.** The structure of the QCD vacuum brings a new term into the SM Lagrangian, the *θ-term*,

$$\mathcal{L}_{SM} \supset \frac{\theta}{32\pi^2} G_{\mu\nu}^A \tilde{G}^{A,\mu\nu}\,, \tag{1.35}$$

  with $\theta$ being a dimensionless parameter, and the operator $G_{\mu\nu}^A \tilde{G}^{A,\mu\nu}$ violates the CP transformation. We then naively expect some CP-violating phenomena in the QCD sector. However, the measurements of neutron electric dipole moment (nEDM) [36–38] tell us: $\theta < 10^{-10}$. This situation is analogous to the hierarchy problem, where again, the SM does not have any mechanism to explain why the $\theta$ parameter is so tiny. We will rigorously come back to the strong CP problem and its solution in Chapter 5 and Chapter 6 of this thesis.

- **Some discrepancies in flavour physics.** In recent years, we have witnessed the rise of several observations in the flavour physics that tension with the SM. For instance, we observed the so-called $(g-2)_\mu$ anomaly, whereas the measurement of muon's anomalous magnetic moment had a $3.3\sigma$ discrepancy compared to the SM prediction, Ref. [39]. From the so-called B anomalies, we observed some hints of the breaking of lepton universality[2] in the b-quark decays, with a significance of $3.1\sigma$, Refs. [40, 41]. All these deviations have currently drawn great attention to the particle physics community.

In the literature, we have seen that many solutions for the problems of SM are often leading to new BSM states, at the energy scale $\Lambda_{NP} >$ TeV. Unfortunately, except for several deviations like $(g-2)_\mu$ or B anomalies, there is no new excitation states have been discovered so far. Besides, the current experimental data have not suggested any clear orientations to extend the SM. However, the optimistic point in this situation is that the precision of many ongoing experiments is promptly increasing, and we might be able to observe the effect of new BSM states from the tiny error bar of these experiments. To prepare for this near future situation, we can consider Effective Field

[2] In the SM, the charged leptons (e.g. $e, \mu, \tau$) couples identically with the electroweak gauge bosons. The violation of lepton universality will then imply a new fundamental interaction between quarks and leptons.

Theories (EFTs) as a robust paradigm that allows for hunting for new physics effects in a model-independent way (bottom-up approach) or connecting the UV complete model with the theories at low energies and their precision measurements (top-down approach). In the next section and the following chapters, we will then present the concepts and core techniques of EFT.

## 1.3 The Standard Model from an EFT point of view

If the scale separation between the SM and new physics is sufficiently large, it is then possible to consider the SM as a leading order approximation in the EFT expansion of a new fundamental theory. Ultimately, in the EFT framework, it is legitimate to parametrize new physics effects in terms of effective operators without referring to any BSM models. The effective Lagrangian can be written as,

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{\text{SM}} + \frac{1}{\Lambda}\sum_i c_i^{(5)}\mathcal{O}_i^{(5)} + \frac{1}{\Lambda^2}\sum_i c_i^{(6)}\mathcal{O}_i^{(6)} + \cdots , \tag{1.36}$$

with $\Lambda$ being an energy scale until where the EFT is still valid. $\mathcal{O}_i^{(n)}$ is a set of dimension-$n$ non-renormalizable operators that respect some postulated symmetries (e.g Lorentz and gauge invariance), and $c_i^{(n)}$ are their corresponding Wilson coefficients that run as a function $c_i^{(n)}(\mu)$ of the renormalization group equation (RGE) scale $\mu$. For a recent review of SM as an EFTs see Ref. [42]. This EFT Lagrangian (1.36) is then making used as a bridge to connect BSM models at the high-energy scale with experimental measurements at the low-energy scale. The UV-EFT connection could be crossly constructed on both sides, commonly known as bottom-up and top-down approaches.

- **The bottom-up EFT approach.** The main idea of the bottom-up approach is to parametrize the EFT Lagrangian without any assumption of specific UV theories. All new physics effects are encoded inside the Wilson coefficients of higher-dimensional operators. From the power counting arguments, one can keep only relevant terms in $\mathcal{L}_{\text{EFT}}$ that are expected to give significant deviations from the SM. By this construction, the truncated EFT will coincide with the low-energy limit of several classes of BSM models. Experimental data are used to put some constraints on the parameter space spanned the finite set of Wilson coefficients. The knowledge we gain from the bottom-up approach can be used to interpret the low-energy behaviour of some well-motivated BSM models when it is necessary.

- **The top-down EFT approach.** The spirit of the top-down approach is to study the implication at the low-energy scale of some specific UV theories. Due to a large scale separation, new BSM states will indirectly affect our current experiments[3]. To connect the UV theories to the low-energy observations, one needs the so-called three-step procedure, namely *matching-running-mapping*. First, matching the UV theory onto an EFT Lagrangian by integrating out the BSM states. Second, using the renormalization group equation to run the Wilson coefficients from the UV scale down to lower energy scales where experimental measurements are performed. Finally, using the EFT Lagrangian at these low-energy scales to compute physical quantities of interest (mapping). For convenience, we leave the technical discussion of the top-down approach to the next chapters.

[3] New physics can manifest itself via tree or loop effects. Since we have not observed any significant deviations from the current experimental data, new physics are expected to hide inside the loop level.

We would like to emphasize that both bottom-up and top-down approaches are complementary. Hence, combining both approaches will provide a powerful tool to search for BSM physics in the era of precision measurements. The main point of the bottom-up approach is to model-independently constrain the value of $c_i^{(n)}$ from precision measurements. This approach will suggest what types of BSM extensions are still valid with the experimental data. The central point of the top-down approach is to compute the value of $c_i^{(n)}$ from a given UV theory. The top-down studies will help us understand which EFT operators have significant contributions and also guide us on which effective field theories we should use in the bottom-up approach.

### 1.3.1 Building EFT from IR perspective

The Wilson coefficients $c_i^{(n)}$ in the EFT Lagrangian are defined due to the choice of operator basis $\mathcal{O}_i^{(n)}$. There exist different operator bases motivated by the broad class of BSM models. The operator basis must be declared for both bottom-up and top-down approaches. To construct the EFT operators, one needs to identify the field content and the symmetries that the EFT must respect. There are two well-known approaches to constructing the EFT, called SMEFT and HEFT, which are now presented as follows:

#### 1.3.1.1 The Standard Model Effective Field Theory (SMEFT)

Assuming that the UV theories at the high energy scale $\Lambda$ involve new massive particles, the idea of SMEFT is to construct the effective Lagrangian above the measured electroweak scale. This effective Lagrangian is a consistent generalization of the SM Lagrangian. Explicitly, we still keep both the SM field content as defined in Table 1.2 and the SM gauge symmetries. The crucial points of SMEFT are presented as follows:

- The Higgs field is kept as a doublet like the one in SM. This important choice is what defines the SMEFT.
- The SMEFT Lagrangian is defined above the electroweak scale (unbroken phase). We assumed that the Higgs field is the unique source that spontaneously breaks electroweak symmetries.
- The higher-dimensional operators are constructed with the SM fields as listed in Table 1.2 and invariant under the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ gauge transformations. Their corresponding Wilson coefficients are dimensionless. The SMEFT Lagrangian is organized as an expansion in powers of new physics scale $(1/\Lambda)$, and thus the order of effective operators can be classified via the canonical power counting rules.

Following these assumptions, the SMEFT Lagrangian is written analogously to the EFT Lagrangian in Eq. (1.36),

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \frac{1}{\Lambda} \sum_i c_i^{(5)} \mathcal{O}_i^{(5)} + \frac{1}{\Lambda^2} \sum_i c_i^{(6)} \mathcal{O}_i^{(6)} + \cdots , \tag{1.37}$$

where $\mathcal{O}_i^{(n)}$ is now a set of non-redundant operators. It is worth noting that gauge invariance criteria only allow us to build a complete but still redundant operator basis. The source of redundancies comes in the equation of motions (EOMs)[4], total derivatives, integration by parts, Bianchi

[4]Using EOMs, one can eliminate the linear combinations of several effective operators because those linear combinations give no contributions to the on-shell $\mathcal{S}$-matrix elements at all orders in the perturbation theory, Ref. [43].

identities, Fierz transformations and field redefinitions. Such redundant operators are physically equivalent, hence, taking one of them is enough to build the EFT Lagrangian. Although the principle to identify redundant operators looks pretty simple, the number of redundancies increases rapidly with the dimensions, starting from dimension six. For instance, we state here the main points of the operator bases at a certain dimensionality:

- **Dimension-5 operator.** There is only one operator that violates the lepton-number conservation, called the Weinberg's operator [44]. After the spontaneous symmetry breaking, this operator will generate the Majorana mass for neutrinos.
- **Dimension-6 operators.** In 1986, Buchmuller and Wyler [45] firstly introduced a complete set of dimension-6 operators, including 80 operators. After 24 years, a complete and non-redundant basis was derived in Ref. [46], in the literature, it is often called the "Warsaw basis". In this minimal basis, assuming flavour-blind structure, together with lepton and baryon-number conservation, there are in total 59 dimension-6 operators. If we consider a fully generic flavour structure, for three generations, the dimension-six SMEFT Lagrangian contains 2499 hermitian operators, Ref. [47].
- **Beyond dimension-six.** The complete and non-redundant bases are known for dimension-7 [48, 49], dimension-8 [49, 50], and dimension-9 [51, 52] operators[5]. Specifically, Refs. [49, 50, 53, 54] making use of the Hilbert series method as a universal algorithm to identify the non-redundant operator bases at a certain dimensionality. This achievement makes the SMEFT is now defined at all orders in the local operator expansion.

#### 1.3.1.2 The Higgs Effective Field Theory (HEFT)

The HEFT, alternatively called the Electroweak Chiral Lagrangian (EWCh$\mathcal{L}$), is the most general approach (i.e. using minimal assumptions) to build an EFT Lagrangian involving only the SM fields. Without delving into technical details, the different main features of HEFT compare to SMEFT are

- HEFT relaxes the assumption of writing the Higgs field as a doublet. The field content in the bosonic sector of HEFT now includes a singlet Higgs-like scalar[6] $h$, and an independent set of Nambu-Goldstone bosons $\pi^I$ which play a role as the longitudinal modes of the EW gauge bosons.
- The HEFT Lagrangian is defined below the electroweak scale (broken phase). The HEFT Lagrangian only manifest $SU(3)_C \otimes U(1)_Q$ gauge invariance. Conversely with the SMEFT, the $SU(2)_L \otimes U(1)_Y$ electroweak symmetry is non-linearly realized by making use of the EFT formalism for Goldstone bosons. Since the electroweak symmetry has no longer existed, there are no relations between the Higgs-like scalar boson and the NGBs[7]. The HEFT approach also implies that the Higgs field is not compulsorily responsible for the SSB.
- The HEFT Lagrangian is organized as an expansion in powers of chiral derivatives (momenta in Fourier space) over the electroweak (and/or new physics) scale [55, 57], and the power counting rules are not the same as those applied in SMEFT. The power counting rules in

[5] We note that all dimension-5, 7, 9 operators violate the lepton or baryon-number conservation.

[6] In the SM, $h$ is a physical Higgs boson which is a scalar component of the Higgs doublet $H$.

[7] SMEFT now becomes a special case of HEFT because the Higgs and Goldstone fields are enforced to transform linearly as a complex scalar doublet [55, 56].

HEFT have been intensely debated a long ago in literature. Several general power counting rules were only proposed recently, see Ref. [58] as an example.

Unlike SMEFT, building the HEFT Lagrangian is not trivial due to the lack of gauge symmetry restrictions. The key point of constructing the HEFT Lagrangian is to identify the symmetry breaking pattern, then construct an EFT of the NGBs by making use of the Callan-Coleman-Wess-Zumino (CCWZ) formalism, Refs. [59, 60]. Since the Higgs sector in the SM Lagrangian has an accidental $SO(4)$ global symmetry [61], a natural choice for the symmetry breaking pattern in HEFT is,

$$SO(4) \cong SU(2)_L \otimes SU(2)_R \to SU(2)_{V=L+R}\,, \tag{1.38}$$

where the global symmetry $SU(2)_L \otimes SU(2)_R$ is spontaneously broken by the Higgs vev to its vectorial subgroup $SU(2)_V$. By this choice, the the candidate NGBs will have suitable properties to become the longitudinal components of the EW gauge bosons. The HEFT Lagrangian is constructed in following.

**The HEFT Lagrangian.** By working at a low energy scale and only assuming the symmetry breaking pattern in Eq. (1.38), one can write down a generic EFT Lagrangian without referring to the UV dynamics. Let us begin with the main building blocks of HEFT Lagrangian. Following the conventions in Refs. [42, 62], we have

- **The Goldstone bosons.** The Goldstone bosons are encapsulated into a $2\times 2$ unitary matrix which is transformed under the $SU(2)_L \otimes SU(2)_R$ global transformations as follows,

$$\mathbf{U}(x) = \exp\left[i\sigma^I \frac{\pi^I(x)}{v}\right]\,, \quad \mathbf{U}(x) \to L\mathbf{U}(x)R^\dagger\,, \quad v = 246 \text{ GeV}\,, \tag{1.39}$$

where $\sigma^I$ are the Pauli matrices and $L, R$ are the group elements of $SU(2)_{L/R}$ global transformations. The covariant derivative acting on $\mathbf{U}(x)$ reads,

$$\left[D_\mu \mathbf{U}(x)\right] = \left[\partial_\mu \mathbf{U}(x)\right] + i\frac{g}{2}W^I_\mu \sigma^I \mathbf{U}(x) + i\frac{g'}{2}B_\mu \mathbf{U}(x)\sigma^3\,. \tag{1.40}$$

**$\mathbf{SU(2)_L}$ covariant objects.** To keep the EFT Lagrangian still manifest the $SU(2)_L \otimes SU(2)_R$ global symmetries, it is necessary to introduce the scalar and vector objects made up of the Goldstone unitary matrix,

$$\begin{aligned} \mathbf{T}(x) &= \mathbf{U}(x)\sigma^3\mathbf{U}(x)^\dagger\,, & \mathbf{T}(x) &\to L\mathbf{T}(x)L^\dagger\,, \\ \mathbf{V}_\mu(x) &= \left[D_\mu \mathbf{U}(x)\right]\mathbf{U}^\dagger(x)\,, & \mathbf{V}_\mu(x) &\to L\mathbf{V}_\mu(x)L^\dagger\,, \end{aligned} \tag{1.41}$$

where $\mathbf{T}, \mathbf{V}_\mu$ transform in the adjoint representation of the $SU(2)_L$, these notations will allow us easily identify the operators that break the $SU(2)_V$ custodial symmetry.

- **Higgs interaction terms.** Since we relax the Higgs doublet assumption, the couplings of Higgs-like scalar boson are arbitrarily parametrized. In the literature, the interactions of Higgs boson are embedded in the generic polynomial functions [55, 56, 63],

$$\mathcal{F}_i(h) = 1 + 2a_i\frac{h}{v} + b_i\frac{h^2}{v^2} + \sum_{n\geqslant 3} a_i^{(n)}\left[\frac{h}{v}\right]^n\,, \quad \mathcal{V}(h) = \frac{m_h^2}{2}h^2 + d_3\frac{m_h^2}{2v}h + d_4\frac{m_h^2}{8v^2}h^4 + \cdots\,, \tag{1.42}$$

where $a_i, b_i, \cdots$ are arbitrary coefficients, and $\mathcal{V}(h)$ stands for the Higgs-like scalar potential. Essentially, the SM Higgs doublet can be reconstructed by composing $h$ and $\mathbf{U}(x)$,

$$(\tilde{H} \quad H) = \frac{v}{\sqrt{2}}\left(1+\frac{h}{v}\right)\mathbf{U}\,. \tag{1.43}$$

- **Fermion fields.** The SM fermion fields are conveniently grouped into the doublets of $SU(2)_{L/R}$ global symmetries,

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}\,,\; Q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}\,,\; L_L = \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}\,,\; L_R = \begin{pmatrix} 0 \\ e_R \end{pmatrix}\,. \tag{1.44}$$

Since the Yukawa terms depend on the Higgs field, in HEFT we also parametrize these interactions analogously to the Higgs interactions,

$$\mathcal{Y}_Q(h) = \text{diag}\left(\sum_{n\geqslant 0} Y_u^{(n)}\left[\frac{h}{v}\right]^n,\, \sum_{n\geqslant 0} Y_d^{(n)}\left[\frac{h}{v}\right]^n\right)\,,\; \mathcal{Y}_L(h) = \text{diag}\left(0\,,\, \sum_{n\geqslant 0} Y_L^{(n)}\left[\frac{h}{v}\right]^n\right)\,. \tag{1.45}$$

The HEFT Lagrangian consists of the Goldstone unitary matrix, Higgs-like scalar field, SM fermion fields, and the field strength tensors of gauge bosons. The HEFT Lagrangian is then defined as follows,

$$\mathcal{L}_{\text{HEFT}} = \mathcal{L}_0 + \Delta\mathcal{L}\,. \tag{1.46}$$

The first term in Eq. (1.46), $\mathcal{L}_0$, contains the leading order (LO) operators which reproduces the SM Lagrangian if we properly fix the value of the arbitrary coefficients. Explicitly, the leading order HEFT Lagrangian reads,

$$\begin{aligned}\mathcal{L}_0 = &-\frac{1}{4}G^A_{\mu\nu}G^{A,\mu\nu} - \frac{1}{4}W^I_{\mu\nu}W^{I,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \sum_{f=Q_L,Q_R,L_L,L_R} \bar{f}i\not{D}f \\ &+\frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \mathcal{V}(h) - \frac{v^2}{4}\text{tr}\big(\mathbf{V}_\mu\mathbf{V}^\mu\big)\mathcal{F}_C(h) \\ &- \frac{v}{\sqrt{2}}\Big[\bar{Q}_L\mathbf{U}\mathcal{Y}_Q(h)Q_R + \text{h.c.}\Big] - \frac{v}{\sqrt{2}}\Big[\bar{L}_L\mathbf{U}\mathcal{Y}_L(h)L_R + \text{h.c.}\Big]\,,\end{aligned} \tag{1.47}$$

where the generic polynomial functions $\mathcal{F}_C(h)$, $\mathcal{V}(h)$, $\mathcal{Y}_{Q/L}(h)$ are defined in Eqs. (1.42), (1.45). By choosing $a_C = b_C = d_3 = d_4 = 1$ and neglecting the remaining terms in the Higgs interaction functions, one obtains the SM Higgs sector. From the Yukawa terms, the $n = 0$ contributions will give mass to the SM fermions. Eventually, the kinetic terms of Goldstone bosons and the mass terms of gauge bosons are encapsulated inside the term $\text{tr}(\mathbf{V}_\mu\mathbf{V}^\mu)$. For further details about the HEFT Lagrangian, see Refs. [42, 62].

The second term in Eq. (1.46), $\Delta\mathcal{L}$, contains both new operators beyond the SM Lagrangian and the next-to-leading order (NLO) contributions from $\mathcal{L}_0$. HEFT is a synthesis of linear description (for gauge and fermion sectors) and non-linear description (for Higgs and Goldstone fields), and each description has different power counting schemes. Therefore, we cannot straightforwardly present $\Delta\mathcal{L}$ in a generic form like the SMEFT Lagrangian. Explicitly, see Ref. [62] for a complete NLO set of operator basis in HEFT. Besides, a complete chiral Lagrangian for axion-like-particles (ALPs) has been constructed very recently by the authors in Ref. [64].

#### 1.3.1.3 Which effective field theories?

In the previous subsection, we have presented the formalism of SMEFT and HEFT. These EFTs now become prime frameworks for interpreting experimental data, especially if we observe new significant deviations compared to SM. It is indeed an excellent moment to ask a deeper question regarding these EFT deformations of the SM. From the symmetry point of view, clearly, the parameter space of HEFT encompasses SMEFT. Therefore, an essential question is how do we distinguish HEFT and SMEFT? In order words, what are the concrete criteria to formalistic classify these EFTs? The answer to these questions will teach us which EFT descriptions we should use according to a given UV model, and guide us to interpret the data in terms of EFTs.

Classifying HEFT and SMEFT is not an easy task. It has been intensely studied for a long time in the literature, but satisfactory answers were only obtained recently, Refs. [65–69]. There are three different approaches and their combinations to tackle this problem, namely *unitarity, analyticity, geometry.*

*Unitarity.* From the Naive Dimensional Analysis [70], since HEFT non-linearly realizes the electroweak symmetry, this EFT will violate the unitarity at the energy scale $E \sim 4\pi v$. On the other hand, SMEFT linearly realizes the electroweak symmetry, therefore, one can obtain a separate scale of unitarity violation. Unfortunately, this approach alone did not bring many meaningful distinctions between HEFT and SMEFT. The combination of unitarity and the remaining two approaches will offer the key points to distinguish these EFTs [69].

*Analyticity.* Falkowski and Rattazzi first proposed this approach in Ref. [67]. This approach uses the properties of the Lagrangian in a given parametrization to distinguish HEFT and SMEFT via analyticity and non-analyticity. Starting from the most general scalar potential, it pointed out that the HEFT-like Lagrangian arises if the scalar potential (written in terms of the Higgs doublet, $H$) is non-analytic at $|H| = 0$, such non-analyticities cannot be removed by the field redefinition[8].

*Geometry.* From the seminal work of Alonso, Jenkins, and Manohar [65], the Higgs and Goldstone fields are treated as the coordinates on a Riemannian manifold, then HEFT and SMEFT are distinguished by the invariant properties of the EFT geometry. More importantly, by combining analyticity criteria and geometric formalism of HEFT and SMEFT, the work of Cohen, Craig, Lu, and Sutherland [68], derive concrete criteria that can be used to distinguish these EFTs. Their approach also shows which UV models prefer HEFT rather than SMEFT in the low energy limit.

From the Ref. [68], the UV models must be matched onto HEFT are the following: First, the UV models have additional sources that break electroweak symmetry. Second, the UV models with new BSM states acquire their whole mass from electroweak symmetry breaking. For instance, matching to HEFT is required when integrating out chiral fermions in SM, or "Loryons" BSM candidates[9] [71]. Furthermore, when integrating out new BSM states lie near weak scale, one should match onto HEFT.

In the following chapters, we will present the core techniques to integrate out heavy particles, which is the heart of this PhD manuscript. Afterwards, we will revisit the above second scenario (integrating out chiral fermions), which violates the decoupling theorem and contains many subtleties detail.

[8]Conversely, the SMEFT-like Lagrangian will arise if at least one field redefinition exists, making the scalar potential analytic at $|H| = 0$.

[9]Loryons candidates are characterised by non-decoupling effects after being integrated out of the UV action. The physical mass of these new states is very heavy and dominated by the contribution of Higgs VEV.

# Chapter 2

# Building Effective Field Theories from UV perspective

In this chapter, we present in detail the top-down EFT approach as was mentioned in chapter 1. A key object in this approach is the one-particle irreducible (1PI) effective action, which can be generally evaluated via the path integral. This chapter is dedicated to the functional methods for computing Wilson coefficients of the effective operators and the renormalization group equations (RGEs).

## 2.1 From UV to IR procedure: *matching-running-mapping*

Within the EFTs framework, we now present in details the three-step procedure used to connect UV models with low-energy observables. This process is schematically described in Fig. 2.1.

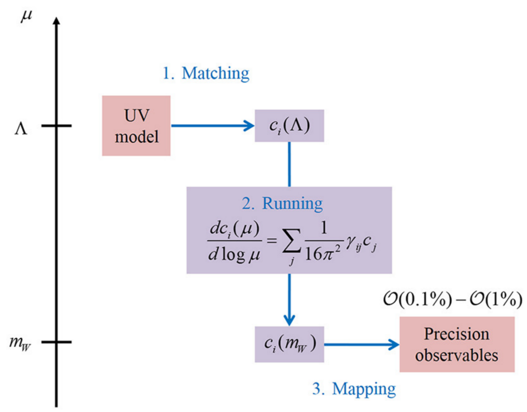



**Figure 2.1:** *EFTs as a bridge to connect a UV model to low-energy scale precision observables.*

- **Matching.** As a heart of this procedure, in the first step, the UV model is matched onto the EFT Lagrangian at the new physics scale $\Lambda$. To construct a low-energy EFT from a given UV model, one needs to identify the relevant degrees of freedom for the measurements of interest and integrates out all new BSM states[1]. Formally, the massive BSM states are integrated out

[1]Suppose the UV model is matched onto SMEFT, then the relevant light fields are the SM particles while the remaining massive BSM states are integrated out.

of the action $S_{UV}[\phi_L, \Phi_H]$ by evaluating the path integral over the heavy states only,

$$e^{iS_{\text{eff}}[\phi_L](\mu=\Lambda)} = \int \mathcal{D}\Phi_H \, e^{S_{UV}[\phi_L,\Phi_H](\mu=\Lambda)} \,, \tag{2.1}$$

where $\phi_L$, $\Phi_H$ are light and heavy fields, respectively. There are two ways to evaluate the effective action in the matching process. First, a traditional approach is commonly known as the Feynman diagram method. The key point of this approach is to compute the scattering amplitudes of some relevant processes in both the UV and EFT theories and then use matching conditions to extract the value of Wilson coefficients. Second, a more elegant method is based on the direct evaluation of the functional path integral, which is the core technique of this manuscript. The central point of the functional method is the so-called technique of Covariant Derivative Expansion (CDE). The details and comparisons between these two methods will be presented in the following section.

The matching coefficients $c_i^{(n)}(\Lambda)$ are determined such that the $\mathcal{S}$-matrix elements of the EFT and the UV model are equivalent at the RGE scale $\mu = \Lambda$. In practice, one can choose $\Lambda = M_H$, the typical mass scale of the new heavy BSM states. The resulting effective Lagrangian can be organized into a finite set of dimension-four or less operators (which will be matched onto $\mathcal{L}_{\text{SM}}$) and a tower of higher dimensional effective operators (which can be matched onto $\mathcal{L}_{\text{SMEFT}}$ or $\mathcal{L}_{\text{HEFT}}$).

The matching process is performed order-by-order in perturbation theory. As recent and near future experimental measurements of Higgs boson and electroweak observables are typically sensitive to quantum loop effects, the desire for one-loop order calculations in the matching process is highly needed. In this case, the contribution of the new BSM states to the low-energy effective Lagrangian consists of tree-level and one-loop level terms. Besides, we highlight that sometimes new physics effects only appear at the one-loop level, and there are no tree-level effects. Thus calculating at one-loop order and beyond is key to discovering BSM physics.

- **Running.** In order to connect the EFT Lagrangian with experimental measurements, the Wilson coefficients $c_i^{(n)}(\mu)$ determined at the matching scale $\Lambda$ are run down to the experimental energy scale (e.g. weak scale $\sim m_W$) according to the RGEs of the given EFT Lagrangian. The essential point of the running step is: when does the RG running become important? In practice, depending on the case understudying and the sensitivity of precision measurements, one can estimate whether the effects of RG running are actually relevant or not. If the scale separation of the UV model and experimental measurements is too large, the higher-order corrections from RGEs improvement must be included.

- **Mapping.** The main idea of the mapping step is to use the EFT Lagrangian at the low-energy scale to compute physical quantities of interests in terms of the Wilson coefficients (e.g. $c_i^{(n)}(\mu = m_W)$). The central question in the mapping step is how effective operators impact the low-energy physical observables (as a function of $c_i^{(n)}$). For example, one can compute the Higgs decay width for a given channel, then estimate its deviation from the SM result. Let us consider the $h \to \gamma\gamma$ decay process, within the SMEFT Lagrangian, the deviation compare to the prediction of SM is [72],

$$\epsilon = \frac{\Gamma^{SMEFT}_{h\to\gamma\gamma}}{\Gamma^{SM}_{h\to\gamma\gamma}} - 1 = \frac{\Gamma_{h\to\gamma\gamma}[c_i^{(n)}]}{\Gamma_{h\to\gamma\gamma}[c_i^{(n)} = 0]} - 1 = \frac{16\pi^2}{Re\big(\mathcal{A}^{SM}_{h\gamma\gamma}\big)} \frac{8v^2}{\Lambda^2}\big(c_{WW} + c_{BB} - c_{WB}\big) \,, \tag{2.2}$$

where $\mathcal{A}^{SM}_{h\gamma\gamma}$ is the decay amplitude of $h \to \gamma\gamma$ decay process in the SM. The coefficients $\{c_{WW}, c_{BB}, c_{WB}\}$ are the Wilson coefficients associated to the Higgs-gauge effective couplings, chronologically, $|H|^2 W^I_{\mu\nu} W^{I,\mu\nu}$, $|H|^2 B_{\mu\nu} B^{\mu\nu}$, and $(H^\dagger \sigma^I H) W^I_{\mu\nu} B^{\mu\nu}$.

Eventually, we emphasize that the running and mapping steps can be done once and for all since they only require the knowledge of EFT operator bases, which has been built from the EFT bottom-up approach. However, the matching step must be performed for any UV model one would like to confront with experiments. In the following sections, we will present the recent development of matching calculations up to one-loop order. We will see that the path integral approach, especially the covariant derivative expansion (CDE) method, significantly simplifies the one-loop matching computations.

### 2.1.1 Toy example: matching by Feynman diagram method

We present a toy example to illustrate the spirit of matching via the Feynman diagram approach. Let us consider the simplest Yukawa theory consisting of a massless fermion $\psi$ interacting with a real heavy scalar field $\Phi_H$. The UV Lagrangian reads [5],

$$\mathcal{L}_{\rm UV} = i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\left(\partial_\mu \Phi_H\right)\left(\partial^\mu \Phi_H\right) - \frac{1}{2}M^2\Phi_H^2 - \lambda\bar{\psi}\psi\Phi_H\,, \tag{2.3}$$

where $M$ is a mass of heavy scalar field $\Phi_H$, and $\lambda$ is the Yukawa coupling between the light field $\psi$ and heavy field $\Phi_H$. Our main goal is to find an EFT that only consists of the light field $\psi$. In the EFT framework, the interactions generated by the exchange heavy field $\Phi_H$ will be replaced by new contact interactions involving the light field $\psi$. To construct this EFT, one needs to integrate out the heavy field $\Phi_H$. This task is performed by comparing the $\mathcal{S}$-matrix elements of the UV and EFT theories. In this toy example, we have to compute the scattering amplitude of $\psi\psi \to \psi\psi$.

**Tree-level matching.** Since the matching process is performed order-by-order in perturbation expansion, we begin with matching the UV theory onto EFT Lagrangian at leading order. The UV scattering amplitude to order $\lambda^2$ is given by the Feynman diagrams in Fig. 2.2.

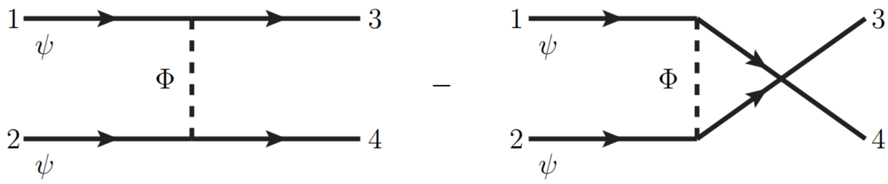



**Figure 2.2:** *Tree-level diagrams proportional to $\lambda^2$ that contribute to $\psi\psi \to \psi\psi$ scattering process, Ref. [5].*

The amplitude of these diagrams read,

$$\mathcal{A}_{\rm UV} = (-i\lambda)^2 \frac{i}{(p_3-p_1)^2 - M^2}\bar{u}(p_3)u(p_1)\bar{u}(p_4)u(p_2) - \{p_3 \leftrightarrow p_4\}\,, \tag{2.4}$$

where $\{p_3 \leftrightarrow p_4\}$ indicates the interchange of external momenta $p_3^\mu$, $p_4^\mu$. A relative sign of these two diagrams is minus due to the requirement of Fermi statistics. Since the spinor structures are identical in both UV and effective theory, we only need to concentrate on the coefficients that come from the massive propagator $\Phi_H$,

$$(-i\lambda)^2 \frac{i}{(p_3-p_1)^2 - M^2} = i\frac{\lambda^2}{M^2}\frac{1}{1-\dfrac{(p_3-p_1)^2}{M^2}} \simeq i\frac{\lambda^2}{M^2}\left[1+\frac{(p_3-p_1)^2}{M^2}+\mathcal{O}\left(\frac{p^4}{M^4}\right)\right]. \tag{2.5}$$

In the limit $p^2 \ll M^2$, one can consider the ratio $p^2/M^2$ as an expansion parameter and construct the effective theory at a certain order in this expansion. The EFT is designed such that the $\mathcal{S}$-matrix elements of the EFT and the full theory are equivalent at the matching scale. At zeroth order in external momenta of Eq. (2.5), the amplitude of $\psi\psi \to \psi\psi$ process can be reproduced by considering the EFT Lagrangian with the four-fermion operator,

$$\mathcal{L}_{\rm EFT} = i\bar{\psi}\partial\!\!\!/\psi + c_1\frac{1}{2}\bar{\psi}\psi\bar{\psi}\psi, \tag{2.6}$$

where $c_1$ is the Wilson coefficient of the four-fermion operator. The amplitude of $\psi\psi \to \psi\psi$ process given by this EFT Lagrangian reads,

$$\mathcal{A}_{\rm EFT} = [ic_1]\,\bar{u}(p_3)u(p_1)\bar{u}(p_4)u(p_2) - \{p_3 \leftrightarrow p_4\}\,. \tag{2.7}$$

Comparing Eq. (2.4) with Eq. (2.7) at zeroth order in $(p^2/M^2)$ EFT expansion, we are able to extract the value of Wilson coefficient $c_1$. Eventually, the EFT Lagrangian reads,

$$\mathcal{L}_{\rm EFT} = i\bar{\psi}\partial\!\!\!/\psi + \frac{\lambda^2}{M^2}\frac{1}{2}\bar{\psi}\psi\bar{\psi}\psi \tag{2.8}$$

As we have seen, the spirit of the Feynman diagram approach is to "match" the amplitudes of a given physical process. From the full theory, one needs to determine the processes involving the heavy fields that we want to integrate out, then evaluate the $\mathcal{S}$-matrix elements and organize the result order-by-order according to the expansion parameter. From the EFT theory, one needs to choose a suitable power counting scheme to select relevant effective operators and also evaluate the $\mathcal{S}$-matrix elements. By solving the matching conditions (comparing the matrix elements of these two theories) at the UV scale,

$$\langle \mathbf{p}|\mathcal{S}|\mathbf{k}\rangle^{\rm EFT} = \langle \mathbf{p}|\mathcal{S}|\mathbf{k}\rangle^{\rm UV}\Big|_{p^2 \ll M^2}\,, \tag{2.9}$$

we are able to extract the value of Wilson coefficients. This comparison steps precisely why the Feynman diagram approach is cumbersome. For instance, in practical calculations, one has to evaluate many Feynman diagrams in both UV and EFT theories, fix the gauge parameters, and recombine the results to obtain gauge invariant operators. We will see in the next section that the spirit of the path integral approach is to "match" the generating functionals of these theories once and for all. Hence one can directly evaluate the value of Wilson coefficients.

## 2.2 One-loop matching by functional method

In recent years, the path integral formalism for one-loop matching has been intensely studied in the literature. Unlike the Feynman diagram approach, this formalism took a while to develop comprehensively, Refs. [73–79]. In what follows, we present the state-of-the-art techniques in this formalism, which will conveniently adapt for different cases in the one-loop matching process. For instance, one might have only heavy fields [3, 75, 80], mixed heavy and light fields [81, 82], or mixed boson-fermion fields [6], inside the loops. Each scenario was studied in the literature. We aim to present a unified framework recently derived in Ref. [79].

**Set up the one-loop effective action.** We consider a general UV theory $\mathcal{L}_{\rm UV}[\varphi]$, whose the mass scale of the field content are largely separated. Explicitly, the multiplet $\varphi$ consists of heavy and light fields[2],

$$\varphi = \begin{pmatrix} \Phi_H \\ \phi_L \end{pmatrix}\,, \text{ assuming } M_{\Phi_H} \gg m_{\phi_L}\,. \tag{2.10}$$

Using the background field method, one can write $\varphi$ in terms of its classical configuration $\varphi_c$ which satisfy the equation of motions (EOMs) and its quantum fluctuations $\varphi$. From the path integral formalism, the effective action of the theory reads,

$$e^{i\Gamma_{\rm UV}[\varphi_c]} = \int \mathcal{D}\varphi\, e^{i\int d^4x\, \mathcal{L}_{\rm UV}[\varphi_c+\varphi]} \tag{2.11}$$

The effective action at the one-loop order can be computed via saddle point approximation. We then expand the Lagrangian around the classical configurations and collect the contributions involving heavy fields,

$$\begin{aligned} \mathcal{L}_{\rm UV}[\varphi_c+\varphi] \supset\, & \mathcal{L}_{\rm UV}[\varphi]\Big|_{\Phi_H=\Phi_{H,c}} + \frac{1}{2}\bar{\varphi}_i\left(\frac{\delta^2\mathcal{L}_{\rm UV}}{\delta\bar{\varphi}_i\delta\varphi_j}\Big|_{\Phi_H=\Phi_{H,c}}\right)\varphi_j + \mathcal{O}(\varphi^3) \\ =\, & \mathcal{L}_{\rm UV}[\varphi]\Big|_{\Phi_H=\Phi_{H,c}} + \frac{1}{2}\bar{\varphi}\,\mathcal{Q}\,\varphi + \mathcal{O}(\varphi^3)\,, \end{aligned} \tag{2.12}$$

where the first order functional derivative is vanished due to EOMs. The matrix $\mathcal{Q}$ is obtained by taking the second order functional derivative of the UV Lagrangian. From the kinetic and interactions terms of the UV Lagrangian, the matrix $\mathcal{Q}$ can be decomposed generally in terms of inverse propagator matrix $\mathbf{K}$ and an interaction matrix $\mathbf{X}$,

$$\mathcal{Q} = \frac{\delta^2\mathcal{L}_{\rm UV}}{\delta\varphi\delta\bar{\varphi}}\Big|_{\Phi_H=\Phi_{H,c}} = \mathbf{K} - \mathbf{X} = \mathbf{K}\big(1-\mathbf{K}^{-1}\mathbf{X}\big)\,, \tag{2.13}$$

where we have used similar notations as Ref. [79]. The matrix $\mathbf{K}$ is block-diagonal and its matrix elements are given by

$$K_i = \begin{cases} P^2 - M_i^2 & \text{(scalar)} \\ \gamma^\mu P_\mu - M_i & \text{(fermion)} \\ -g^{\mu\nu}(P^2-M_i^2) + \big(1-\dfrac{1}{\xi}\big)P^\mu P^\nu & \text{(vector)} \end{cases}\,, \tag{2.14}$$

where we have used the Hermitian covariant derivative operator $P_\mu = iD_\mu$ instead of $D_\mu$. We note that the propagators of vector gauge fields are gauge dependence and one need to choose a convenient gauge by fixing the value of parameter $\xi$[3]. The interaction matrix $\mathbf{X}$ encodes the effects of new BSM states. Generally, one can parameterize this matrix as the form, Ref. [79]:

$$\mathbf{X}[\phi_L, P_\mu] = \mathbf{U}\Big[\phi_L,\, [P_\mu,\phi_L],\, [P_\mu,[P_\nu,\phi_L]],\cdots\Big] + \big(P_\mu \mathbf{Z}^\mu[\phi_L] + \bar{\mathbf{Z}}^\mu[\phi_L]P_\mu\big) + \cdots\,. \tag{2.15}$$

[2]The multiplet of scalar, fermion or gauge boson fields will be presented explicitly in section 2.2.2.

[3]For the UV models with new scalar or fermion states, a convenient choice is Feynman - 't Hooft gauge with $\xi = 1$. For the case integrating out massive vector gauge fields, see Ref. [72] for elaboration examples.

Technically, $\mathbf{X}$ is parametrized as a series expansion in terms of "open" covariant derivative $P_\mu$. The matrices $\mathbf{U}, \mathbf{Z}^\mu, \bar{\mathbf{Z}}^\mu$ are functions of lights fields $\phi_L$ and "closed" covariant derivatives[4]. In practice, it is quite rare for a UV model to contain derivative interactions with the "open" covariant derivatives. Most of the time, we only use the first term of Eq. (2.15). It is worth emphasising that we only use the EOMs of heavy fields. Substituting the EOMs of the light fields can be done in the final step when it is necessary.

Substituting Eq. (2.12) into Eq. (2.11), the effective action is obtained as follows,

$$\begin{aligned} e^{i\Gamma_{\mathrm{UV}}[\phi_L]} &= e^{i\int d^dx \mathcal{L}_{\mathrm{UV}}\left[\Phi_{H,c}[\phi_L],\phi_L\right]} \int \mathcal{D}\varphi \exp\left(i\int d^dx\, \frac{1}{2}\bar{\varphi}\,\mathcal{Q}\,\varphi\right) \\ &= e^{i\int d^dx \mathcal{L}_{\mathrm{UV}}\left[\Phi_{H,c}[\phi_L],\phi_L\right]} \mathrm{SDet}\big[\mathcal{Q}\big]^{-1/2} \\ \Leftrightarrow \Gamma_{\mathrm{UV}}[\phi_L] &= \int d^dx \mathcal{L}_{\mathrm{UV}}\big[\Phi_{H,c}[\phi_L],\, \phi_L\big] + \frac{i}{2}\mathrm{STr}\ln\mathcal{Q}\,, \end{aligned} \tag{2.16}$$

where in the last equation we have used the fact that $\ln\mathrm{SDet}\,\mathcal{Q} = \mathrm{STr}\ln\mathcal{Q}$. The notation "S" abbreviate by the word "Super" so that SDet and STr are the generalizations of the regular functional determinant and functional trace. The main idea is that the supertrace acting on a functional operator $\mathcal{O}$ yields a $\pm$ sign,

$$\mathrm{STr}\,\mathcal{O} = \pm\mathrm{Tr}\,\mathcal{O}\,, \tag{2.17}$$

depending on the bosonic or fermionic nature of the particle inside the propagator. We will come back this point in the next section for further technical discussions. The notation "Tr" denotes the regular functional trace, i.e. the trace over functional space (momentum or position space) and the trace over internal space (i.e. trace over color indices, gauge indices, $\cdots$). Evaluating this functional trace is the central point over the scope of this manuscript. The first term in Eq. (2.16), only depending on the classical field configurations, hence it will yield the tree-level effective action when integrate out heavy fields. The second term in Eq. (2.16) will contribute to the effective action at the one-loop order. Using the explicit form of the matrix $\mathcal{Q}$ given by Eq. (2.13), the one-loop effective action reads,

$$\Gamma^{(1-\mathrm{loop})}_{\mathrm{UV}}[\phi_L] = \frac{i}{2}\mathrm{STr}\ln\mathcal{Q} = \frac{i}{2}\,\mathrm{STr}\ln\mathbf{K} + \frac{i}{2}\,\mathrm{STr}\ln\left(1-\mathbf{K}^{-1}\mathbf{X}\right), \tag{2.18}$$

where we have used the identity, $\mathrm{STr}\ln(AB) = \mathrm{STr}\ln A + \mathrm{STr}\ln B$, which holds for any non-commuting operators. It is worth emphasizing that Eq. (2.16) and (2.18) only tell us the UV theory's effective action up to one-loop order. To determine the value of Wilson coefficients, one needs to define matching conditions properly.

**Method of regions.** The one-loop effective action in Eq. (2.18) contains all possible quantum fluctuations including heavy and light fields. More precisely, Eq. (2.18) includes the contributions at one-loop order resulting from the tree-level EFT Lagrangian. Our aim is directly derive the EFT Lagrangian from the action of the full theory at one-loop order and beyond. This goal can be achieved by using the so-called method of "expansion by regions", Refs. [76, 83, 84]. The key idea of this method is before evaluating the loop integrals by using dimensional regularization, one can perform Taylor expansion to split $\Gamma^{(1-\mathrm{loop})}_{\mathrm{UV}}$ into *hard* and *soft* region contributions,

$$\Gamma^{(1-\mathrm{loop})}_{\mathrm{UV}}[\phi_L] = \Gamma^{(1-\mathrm{loop})}_{\mathrm{UV}}[\phi_L]\Big|_{hard} + \Gamma^{(1-\mathrm{loop})}_{\mathrm{UV}}[\phi_L]\Big|_{soft}\,, \tag{2.19}$$

[4] $P_\mu$ is a functional operator which act on everything to their right. Besides, the "closed" covariant derivative is obtained by writing as a commutator, $[P_\mu, \phi_L] = (P_\mu\phi_L)$.

where the *hard* and *soft* regions correspond to the momentum regimes $q \sim M_{\Phi_H} \gg m_{\phi_L}$ and $q \sim m_{\phi_L} \ll M_{\Phi_H}$, respectively. Afterwards, the contribution of each region is obtained by integrating over all possible loop momenta. The method of regions states that the contributions from the *hard* region involve heavy fields (or mixed heavy-light fields) in the loops while the *soft* region involves light fields only.

**Matching conditions and EFT Lagrangian.** Ultimately, in the path integral formalism, one can systematically derive the matching conditions once and for all. The key point is to match the generating functions of the two theories,

$$\Gamma_{\text{EFT}}[\phi_L] = \Gamma_{\text{UV}}[\phi_L]\,. \tag{2.20}$$

Solving Eq. (2.20) order-by-order, one can determine the value of Wilson coefficients and write down the EFT Lagrangian. Using the first term in Eq. (2.16), we obtain the solution of Eq. (2.20) at tree-level as follows,

$$\mathcal{L}_{\text{EFT}}^{(\text{tree})}[\phi_L] = \mathcal{L}_{\text{UV}}\big[\Phi_{H,c}[\phi_L], \phi_L\big]\,. \tag{2.21}$$

As we have seen, the matching at tree-level is simply achieved by plugging the classical configuration of heavy fields (which satisfy the EOMs) into the UV Lagrangian. The matching at one-loop order is obtained by taking the contribution from the *hard* region of the effective action, the solution of Eq. (2.20) at one-loop order is,

$$\int d^dx \mathcal{L}_{\text{EFT}}^{(1-\text{loop})} = \Gamma_{\text{UV}}^{(1-\text{loop})}[\phi_L]\Big|_{hard}\,, \tag{2.22}$$

where the short distance behaviour of the UV theory is encoded inside the Wilson coefficients of the local EFT operators. The solution (2.22) directly yields the EFT Lagrangian resulting from one-loop matching computations [79],

$$\int d^dx\, \mathcal{L}_{\text{EFT}}^{(1-\text{loop})}[\phi_L] = \frac{i}{2}\,\text{STr}\ln\mathbf{K}\Big|_{hard} - \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}\text{STr}\Big[\big(\mathbf{K}^{-1}\mathbf{X}\big)^n\Big]\Big|_{hard}\,. \tag{2.23}$$

At this stage, we highlight that Eq. (2.21) and Eq. (2.23) make the matching tasks ultimately transparent. The Wilson coefficients are now directly computed without any prior knowledge of the EFT Lagrangian. Unlike the Feynman diagram approach, we do not have to compute the amplitudes of the EFT theory, and thus the comparison steps are entirely ignored.

The EFT Lagrangian given by Eq. (2.23) is decomposed in terms of two types of contributions: The first term in (2.23) is often called *log-type* contributions while the second term corresponds to *power-type* contributions. The log-type contributions only depend on the heavy field propagators which arise from the kinetic terms of the UV Lagrangian. The computations of log-type terms will generate pure gauge interactions[5]. The power-type contributions depend on the interaction terms of the UV Lagrangian. Importantly, all effects of the heavy field only, mixed heavy-light fields, and mixed boson-fermion fields inside the loops, are encapsulated in the second terms of Eq (2.23).

[5] For example: $F_{\mu\nu}F^{\mu\nu}, F^{\mu}_{\nu}F^{\nu}_{\rho}F^{\rho}_{\mu}, \cdots$

### 2.2.1 Evaluating the functional trace: Covariant Derivative Expansion (CDE)

We now turn to evaluate those functional supertraces given by Eq. (2.23). Without loss of generality, in position space, the functional operators appearing in Eq. (2.23) can be written in a form, $\mathcal{O}\big[P_\mu, U\big]$, where $P_\mu = iD_\mu$ is a hermitian covariant derivative[6] and $U$ contains a set of momentum-independent terms. As a first step, we act the supertrace on functional operator $\mathcal{O}[\hat{P}_\mu, U]$,

$$\mathrm{STr}\,\mathcal{O}\big[\hat{P}_\mu, U\big] = \pm \mathrm{Tr}\,\mathcal{O}\big[\hat{P}_\mu, U\big]\,. \tag{2.24}$$

The $\pm$ sign is determined by the bosonic or fermionic nature of the **first** propagator appearing in the functional trace[7]. We note that the Faddeev-Popov ghost propagator is an exceptional case. Since it has the Grassmannian numbers in the path integral measure, one has to treat the sign of ghost propagator similar to the fermion case.

#### 2.2.1.1 The simplified CDE method.

We now evaluate the functional trace "Tr" over all $d$-dimensional momentum space, then the unity identity $\int d^dx |x\rangle\langle x| = \mathbb{1}$ and project the functional operator $\mathcal{O}[\hat{P}_\mu, U]$ into position space[8]. Eventually, we obtain,

$$\begin{aligned}
\mathrm{STr}\,\mathcal{O}\big[\hat{P}_\mu, U_k\big] &= \pm \mathrm{Tr}\,\mathcal{O}\big[\hat{P}_\mu, U_k\big] = \pm \int \frac{d^dq}{(2\pi)^d} \langle q|\mathrm{tr}\,\mathcal{O}\big[\hat{P}_\mu, U_k\big]|q\rangle \\
&= \pm \int d^dx \int \frac{d^dq}{(2\pi)^d}\, \langle q|\, x\rangle\, \langle x|\mathrm{tr}\,\mathcal{O}\big[\hat{P}_\mu, U_k\big]|q\rangle \\
&= \pm \int d^dx \int \frac{d^dq}{(2\pi)^d} e^{iq\cdot x}\Big(\mathrm{tr}\,\mathcal{O}\big[P_\mu, U_k\big]\Big) e^{-iq\cdot x}\,,
\end{aligned} \tag{2.25}$$

where we have used the plane wave basis, $\langle q|\, x\rangle = \langle x|\, q\rangle^\dagger = e^{iq\cdot x}$, and the remaining trace "tr" will act to the internal space of operator $\mathcal{O}[P_\mu, U]$ . Under the sandwich $e^{iq\cdot x}\mathrm{tr}\,\mathcal{O}[P_\mu, U]e^{-iq\cdot x}$, we have

$$e^{iq\cdot x} P_\mu e^{-iq\cdot x} = P_\mu + q_\mu \to P_\mu - q_\mu\ , \quad e^{iq\cdot x} U(x) e^{-iq\cdot x} = U(x)\,, \tag{2.26}$$

where we used the fact the momentum integral is invariant under the change of sign of integral variables, $q_\mu \to -q_\mu$. We choose this convention for the convenience of the following computations. Substituting Eq. (2.26) into Eq. (2.25) and using the matching condition at one-loop order (2.22), we obtain the master formula of the simplified CDE method,

$$\mathrm{STr}\,\mathcal{O}\big[\hat{P}_\mu, U\big]\Big|_{hard} = \pm \int d^dx \int \frac{d^dq}{(2\pi)^d}\, \mathrm{tr}\,\mathcal{O}\big[P_\mu - q_\mu, U\big]\Big|_{hard}\,. \tag{2.27}$$

The crucial points of the simplified CDE method are presented as follows:

[6] $P_\mu$ is hermitian conjugate up to integration by part, i.e. $(AP_\mu B)^\dagger = B^\dagger(-i)\overleftarrow{D}_\mu A^\dagger = B^\dagger i\overrightarrow{D}_\mu A^\dagger = B^\dagger P_\mu A^\dagger$.

[7] This convenient trick helps us not to get confused in the case mixed boson-fermion fields. The fact that, $\mathrm{tr}(K_bU_{bf}K_fU_{fb}) = \mathrm{tr}(K_fU_{fb}K_bU_{bf})$ and $\mathrm{tr}(U_{bf}U_{fb}) = -\mathrm{tr}(U_{fb}U_{bf})$, so that the $+(-)$ sign from the inverse propagator function $K_b(K_f)$ that appears first in those traces will not change the final result.

[8] Generally, in functional space $\hat{P}_\mu = i\hat{q}_\mu + A_\mu(\hat{x})$, and its expression in position space is $P_\mu = i(\partial_\mu - iA_\mu(x)) = iD_\mu$.

- **Evaluating loop integrals.** From the master formula (2.27), the effective Lagrangian is obtained by performing an inverse mass expansion. Explicitly, using the Taylor expansion,

$$\frac{1}{\Delta^{-1}(1-\Delta A)} = \Delta + \Delta A \Delta + \Delta A \Delta A \Delta + \cdots = \sum_{n=0}^{\infty} (\Delta A)^n \Delta \,, \tag{2.28}$$

  where in practical calculations $\Delta$ play a role as propagator functions. and then integrating over all possible loop momenta. The next step is to expand the propagator functions in the hard momentum regimes. If the loop integrals contain only heavy propagators, one can skip this step.

  **Dealing with light propagators.** For the case of mixed heavy-light fields in the loops, one needs to Taylor expand light propagators in the hard momentum regions, $q^2 \sim m_H^2 \gg m_L^2$, before performing the loop integrals. Explicitly,

$$\left.\frac{1}{q^2 - m_L^2}\right|_{hard} = \frac{1}{q^2} + \frac{1}{q^4} m_L^2 + \cdots \,, \tag{2.29}$$

  where the truncation of this expansion is optional, usually, we only take the contribution from the zeroth order of the mass of light fields (i.e. $1/q^2$).

  **Loop integrals.** After expanding light propagators in hard momentum regions, one can factorize out loop integrals and evaluate them once and for all. The loop integrals that often appear in the computations have the following generic form,

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^{\mu_1} \cdots q^{\mu_{2n_c}}}{(q^2 - M_i^2)^{n_i} (q^2 - M_j^2)^{n_j} \cdots (q^2)^{n_L}} \,. \tag{2.30}$$

  We note that these tensorial integrals can be decomposed in terms of scalar integrals given by Appendix A. If the loop integral is divergence, we use dimensional regularization and the $\overline{MS}$-scheme for calculation and renormalization.

- **Forming gauge invariant operators.** The most crucial point in the simplified CDE method is that the inverse mass expansion (2.28) will generate a series in the power of the "open" covariant derivative $P_\mu$. To obtain the effective Lagrangian with local operators, one must combine the output of simplified CDE to form commutator structures. Such operators with the "closed" covariant derivative will manifest the gauge invariant results. More detail about forming commutator structures will be presented in the next chapter.

In summary, the main advantage of the simplified CDE method is that the one-loop matching calculations can be performed very straightforwardly and efficiently[9]. However, it still contains some drawbacks where we have to close the "open" covariant derivative to obtain the gauge invariant results[10].

#### 2.2.1.2 The Gaillard-Cheyette CDE method.

As mentioned in the above discussion, in the simplified CDE method, one has to close the covariant derivative to avoid the operators which are not manifest gauge invariant. To systematically avoid those operators, one can use the technique developed by Gaillard and Cheyette [73, 74] in the 1980s.

[9] In the next chapter, we will use the so-called Covariant Diagram technique, which allows us to select the terms that we are interested in directly. Hence drastically simplify the calculations.

[10] In the next chapter, we will present an algorithm to form the commutator structures efficiently.

The key idea is to sandwich the naive CDE expansion (2.27) by the pair of operator insertions $e^{P\cdot\partial_q}$ and $e^{-P\cdot\partial_q}$, we have

$$\mathrm{STr}\,\mathcal{O}[\hat{P}_\mu, U]\Big|_{hard} = \pm\int d^dx \int \frac{d^dq}{(2\pi)^d}\, e^{P\cdot\partial_q}\Big(\mathrm{tr}\,\mathcal{O}[P_\mu - q_\mu, U]\Big)e^{-P\cdot\partial_q}\Big|_{hard}\,, \tag{2.31}$$

where $\partial_q^\mu = \dfrac{\partial}{\partial q_\mu}$ is the partial derivative with respect to the loop momentum variables $q_\mu$. We note that the operator $e^{-P\cdot\partial_q}$ will act to the right and yields a trivial unity function $\mathbb{1}$[11]. The remaining operator $e^{P\cdot\partial_q}$ also becomes a trivial unity function if we integrate by part and act it to the left. Eventually, the sandwich of Gaillard and Cheyette will not change our final results. However, these operator insertions will put all the "open" covariant derivatives into commutators. Hence, the CDE expansion will yield the desired manifestly gauge covariant operators. Explicitly,

$$e^{P\cdot\partial_q}(P_\mu - q_\mu)e^{-P\cdot\partial_q} = -q_\mu + F^{CDE}_{\mu\nu}\frac{\partial}{\partial q_\nu}\,, \quad e^{P\cdot\partial_q}\,U\,e^{-P\cdot\partial_q} = U^{CDE}\,, \tag{2.32}$$

where the quantities $F^{CDE}_{\mu\nu}$ and $U^{CDE}$ are defined as follows,

$$\begin{aligned} F^{CDE}_{\mu\nu} &= -\sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}\Big(P_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}[P_\mu, P_\nu]\Big)\frac{\partial^n}{\partial q_{\mu_1}\partial q_{\mu_2}\cdots\partial q_{\mu_n}}\,, \\ U^{CDE} &= \sum_{n=0}^{\infty}\frac{1}{n!}\Big(P_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}U\Big)\frac{\partial^n}{\partial q_{\mu_1}\partial q_{\mu_2}\cdots\partial q_{\mu_n}}\,, \end{aligned} \tag{2.33}$$

where the parenthesis denotes that the covariant derivative is closed, i.e. they only locally act on $U$ and $[P_\mu, P_\nu] = iF_{\mu\nu}$ with $F_{\mu\nu}$ is the usual field strength tensor. Besides, the closed covariant derivatives can always be written in terms of commutators,

$$(P_\mu X) = [P_\mu, X]\,,\ (P_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}X) = \Big[P_{\mu_1}, \Big[P_{\mu_2}, [\cdots[P_{\mu_n}, X]]\Big]\Big]\,. \tag{2.34}$$

In summary, the master formula of the Gaillard-Cheyette CDE method is presented as follows,

$$\mathrm{STr}\,\mathcal{O}[\hat{P}_\mu, U]\Big|_{hard} = \pm\int d^dx\int\frac{d^dq}{(2\pi)^d}\,\mathrm{tr}\,\mathcal{O}\Big[-q_\mu + F^{CDE}_{\mu\nu}\frac{\partial}{\partial q_\nu}, U^{CDE}\Big]\Big|_{hard}\,, \tag{2.35}$$

where the explicit expression of $F^{CDE}_{\mu\nu}$ and $U^{CDE}$ are given by Eqs. (2.33). Technically, the different main features of Gaillard-Cheyette CDE compare to simplified CDE are

- With the Gaillard-Cheyette CDE method, one can skip the step of closing the "open" covariant derivative. Since the covariant derivative now always appears in terms of commutators, the one-loop matching procedure will manifest gauge invariant in any steps of the computations.
- To obtain the effective Lagrangian, we still need to follow the standard procedure discussed in the above subsection. More precisely, the inverse mass expansion and the expansion in hard regions are analogous to the simplified CDE. However, an expenditure of this method

[11] Since derivative of one is zero, we have $e^{-P\cdot\partial_q}\mathbb{1} = \mathbb{1}$.

is that one must compute many momentum derivatives which act on propagator functions before evaluating the loop integrals. In practice, such computations are often lengthy and tedious. Fortunately, last year, the Gaillard-Cheyette CDE method was fully automatized in the Mathematical packages, which two different groups developed. See, Cohen et al. [79, 85], and Fuentes-Martin et al. [86].

#### 2.2.1.3 Toy example: integrating out heavy complex scalar field.

To illustrate several technical points that we previously discussed, we present here a mini example where we integrate out a heavy complex scalar field $\Phi$. Consider an UV Lagrangian,

$$\mathcal{L}_{\rm UV} \supset (D_\mu\Phi)^\dagger(D^\mu\Phi) - M^2|\Phi|^2 - \Phi^\dagger U\Phi \ . \tag{2.36}$$

In the first step, we treat the scalar field and its conjugate as independent variables and put them in the scalar multiplet $\varphi_\Phi = (\Phi\ \Phi^\star)^\intercal$. The UV Lagrangian can be written as follows,

$$\mathcal{L}_{\rm UV} \supset \frac{1}{2}\,\bar{\varphi}_\Phi\big(P^2 - M^2 - U\big)\varphi_\Phi \ , \tag{2.37}$$

where $\bar{\varphi}_\Phi = \varphi_\Phi^\dagger$, and we have used integration by part and hermitian covariant derivative when rewriting this UV Lagrangian. Taking the second-order functional derivative with respect to $\varphi_\Phi$, one can identify the matrix $K = P^2 - M^2$ and the matrix $X = U$. The effective Lagrangian resulting from integrating out the heavy field is obtained using the Eq. (2.23). Suppose we are interested in a term,

$$\int d^dx\mathcal{L}_{\rm eft} \supset -\frac{i}{2}{\rm STr}\big[K^{-1}X\big] = -\frac{i}{2}{\rm STr}\left[\frac{1}{P^2-M^2}U\right] , \tag{2.38}$$

in this case $\mathcal{O}[\hat{P}, U] = [1/(P^2 - M^2)]U$. This functional supertrace can be directly evaluated by the simplified CDE or Gaillard-Cheyette CDE techniques.

**Evaluating with simplified CDE.** Using the master formula (2.27), we have

$$\begin{aligned}
-\frac{i}{2}{\rm STr}\left[\frac{1}{P^2-M^2}U\right] &= -\frac{i}{2}\int d^dx\int\frac{d^dq}{(2\pi)^d}{\rm tr}\frac{1}{(q^2-M^2)\left[1+\frac{1}{q^2-M^2}(P^2-2q\cdot P)\right]}U \\
&= -\frac{i}{2}\int d^dx\int\frac{d^dq}{(2\pi)^d}{\rm tr}\big[\Delta_b + \Delta_b(P^2-2q\cdot P)\Delta_b + \cdots\big]U \qquad (2.39)\\
&= -\frac{i}{2}\int d^dx\,{\rm tr}\left[M^2\left(1-\log\frac{M^2}{\mu}\right)U - \frac{1}{6M^2}(P^\mu P^\nu P_\mu P_\nu + P^\mu P^\nu P_\nu P_\mu)U + \cdots\right] \qquad (2.40)\\
&= -\frac{i}{2}\int d^dx\,{\rm tr}\left[M^2\left(1-\log\frac{M^2}{\mu}\right)U - \frac{1}{12M^2}\big[P_\mu,P_\nu\big]\big[P^\mu,P^\nu\big]U + \cdots\right] , \qquad (2.41)
\end{aligned}$$

where $\Delta_b = \dfrac{1}{q^2-M^2}$ stands for bosonic propagators. In the last line of Eq. (2.41), we rewrite

$$P^\mu P^\nu P_\mu P_\nu = \frac{1}{2}\big[P_\mu,P_\nu\big]\big[P^\mu,P^\nu\big] - P^\mu P^\nu P_\nu P_\mu \ , \tag{2.42}$$

to eliminate the operator contains open covariant derivative. Eventually, we get rid of the usual loop factor $\dfrac{i}{(4\pi)^2}$.

**Evaluating with Gaillard-Cheyette CDE.** We will also obtain the same result by evaluating Eq. (2.38) with the help of the master formula (2.35), see Ref [79] for further technical detail,

$$\begin{aligned}
-\frac{i}{2}\mathrm{STr}\left[\frac{1}{P^2-M^2}U\right] &= -\frac{i}{2}\int d^dx \int \frac{d^dq}{(2\pi)^d}\mathrm{tr}\big[\Delta_b \\
&\quad + F_{\mu\nu}^{CDE}F_{\rho\sigma}^{CDE}\big(-g^{\mu\rho}\Delta_b\frac{\partial^2}{\partial q_\nu \partial q_\sigma}\Delta_b + 4\Delta_b\, q^\mu\frac{\partial}{\partial q_\nu}\Delta_b\, q^\rho\frac{\partial}{\partial q_\sigma}\Delta_b\big) + \cdots\big]U^{CDE} \\
&= -\frac{i}{2}\int d^dx\,\mathrm{tr}\left[M^2\left(1-\log\frac{M^2}{\mu}\right)U - \frac{1}{12M^2}\left[P_\mu,P_\nu\right]\left[P^\mu,P^\nu\right]U + \cdots\right] ,
\end{aligned} \tag{2.43}$$

where we performed the inverse mass expansion for the propagator function,

$$\Delta_b^{CDE} = \frac{1}{\left[-q_\mu + F_{\mu\nu}^{CDE}\dfrac{\partial}{\partial q_\nu}\right]^2 - M^2} , \tag{2.44}$$

to obtain a series expansion in power of $F_{\mu\nu}^{CDE}$. Additionally, we have used $U^{CDE} = U$ since $U^{CDE}$ stayed in the right most and thus, the momentum derivatives act on unity function $\mathbb{1}$ have trivially vanished. To obtain the final results, one needs to act momentum derivatives to the right before evaluating loop integrals.

#### 2.2.1.4 Some universal results

As seen in the toy example, one can write down a generic UV Lagrangian and perform the one-loop matching without reference to any BSM models. The result of this exercise contains universal operator structures where the loop integrals are evaluated once and for all (for instance, see Eq. (2.41)). The one-loop matching calculations for a given UV model are reduced to the matrix algebra manipulations (e.g. trace over internal indices). This idea is a prototype of the Universal One-Loop Effective Action (UOLEA), Refs. [3, 80–82]. We leave the detailed discussions about this idea in the next chapter.

### 2.2.2 Summary: one-loop matching using functional method

We have discussed in detail the main points of the CDE technique. It is undoubtedly a good time to summarise the main steps and derive the prescription for one-loop matching via the functional approach. More precisely, a step-by-step procedure is presented as follows:

1. **Solving EOMs and matching at tree-level.** From the first order functional derivative of the UV Lagrangian with respect to the heavy fields, one can derive the EOMs by requiring,

$$\left.\frac{\delta\mathcal{L}_{\mathrm{UV}}}{\delta\Phi_H}\right|_{\Phi_H=\Phi_{H,c}} = 0 . \tag{2.45}$$

The solution of these EOMs yields the classical configuration of the heavy fields. Substituting $\Phi_{H,c}$ into the UV Lagrangian, one obtains the tree-level effective Lagrangian involving only light fields,

$$\mathcal{L}_{\mathrm{EFT}}^{(\mathrm{tree})}[\phi_L] = \mathcal{L}_{\mathrm{UV}}\left[\Phi_{H,c}[\phi_L],\phi_L\right] . \tag{2.46}$$

In practical calculations, one needs to perform the inverse mass expansion for $\mathcal{L}_{\mathrm{UV}}\left[\Phi_{H,c}[\phi_L],\phi_L\right]$ to obtain a tower of effective operators.

2. **Deriving the fluctuation matrices.** First, identify all independent fields in the UV theory (including the conjugation of the complex fields). Afterwards, arrange those fields into suitable field multiplets. A convenient choice is that the multiplets are formed from the fields and their conjugates. For instance, the field multiplets of real scalar; complex scalar; Dirac fermion; real gauge boson, are respectively presented as follows:

$$\varphi_\phi = \phi \ ; \quad \varphi_\Phi = \begin{pmatrix} \Phi \\ \Phi^\star \end{pmatrix} \ ; \quad \varphi_\Psi = \begin{pmatrix} \Psi \\ \Psi^c \end{pmatrix} \ ; \quad \varphi_A = A_\mu \ , \tag{2.47}$$

and the hermitian conjugate of these field multiplets are

$$\bar{\varphi}_\phi = \phi \ ; \quad \bar{\varphi}_\Phi = \begin{pmatrix} \Phi^\dagger & \Phi^T \end{pmatrix} \ ; \quad \bar{\varphi}_\Psi = \begin{pmatrix} \bar{\Psi} & \bar{\Psi}^c \end{pmatrix} \ ; \quad \bar{\varphi}_A = A_\mu \ . \tag{2.48}$$

We emphasize that the step of identifying field multiplets must be done for fields of the UV theory. In case the fermion fields are chiral, one must write down the chiral projectors (i.e. $P_{L/R}$) explicitly. Eventually, the UV Lagrangian can be written in terms of these field multiplets (see the above toy example). The fluctuation matrices are obtained by taking second order functional derivative of the UV Lagrangian with respect to the field multiplets $\varphi_i$ ,

$$\mathcal{Q}_{ij} = \left.\frac{\delta^2 \mathcal{L}_{\rm UV}}{\delta\bar{\varphi}_i \delta\varphi_j}\right|_{\Phi_H=\Phi_{H,c}} = \delta_{ij} K_i - X_{ij}\Big|_{\Phi_H=\Phi_{H,c}} \ , \tag{2.49}$$

hence we can derive the inverse propagator matrix $\mathbf{K}$ and the interaction matrix $\mathbf{X}$. Notice that if $\mathbf{X}$ involves the heavy fields, one must replace them with their classical configurations.

3. **Deriving the effective Lagrangian at one-loop order.** The EFT Lagrangian resulting from integrating out heavy fields at one-loop order is obtained from Eq. (2.23). In practice, a priori, one should know the mass dimension of the interaction matrices. Hence we can truncate the expansion (2.23) and enumerate relevant terms that need to compute. Next step, we evaluate the functional supertraces using the master formula of the simplified CDE given by Eq. (2.27) or the Gaillard-Cheyette CDE given by Eq. (2.35). The effective Lagrangian, $\mathcal{L}_{\rm EFT}^{(1-loop)}[\phi_L]$, is obtained after finishing the standard calculations, namely inverse mass expansion, expansion in hard regions, evaluating loop integrals, trace over the internal indices. We emphasize that the computations with CDE techniques has been implemented in the Mathematica packages, Refs. [85, 86].

4. **Simplifying the final results.** It is worth noting that the effective operators obtained from the functional matching approach only respect the Lorentz invariant and gauge invariant. To match the final results onto a given EFT operator basis (e.g. SMEFT basis), one needs to apply the EOMs of light fields, integration by part, and Fierz transformations[12]. From the pragmatic point of view, this task can be done delicately by first mapping the results of the functional approach to Green's operator basis[13] and then converting from the Green's basis to the SMEFT basis. Such converting tasks have already been done in the literature, Ref [87].

[12]Since we used dimensional regularization to evaluate divergence integrals, evanescent operators might appear when applying Fierz identities.

[13]SMEFT off-shell operator basis, where the effective operators are independent up to integration by part, but still contain redundancies from EOMs.

#### 2.2.2.1 Application: integrating out Leptoquarks (ongoing project)

In chapter 1 of this manuscript, we have introduced some discrepancies in the flavour sector of the SM, namely $(g-2)_\mu$ anomaly and B-anomalies. Tree-level explanations of $B$ anomalies have been extensively studied already. At the one-loop order, there are many possibilities. In our project, we consider systematically one- or two-particle explanations of B-anomalies assuming a common origin with $(g-2)_\mu$ anomaly. More precisely, the new particles must all enter the mechanisms that explain the $(g-2)_\mu$ and B-anomalies, not partially. This restriction is beneficial because it maximises the overlap of flavour anomaly explanations while minimising the extensions for the SM. We then apply the EFT top-down paradigm to study the implications of these UV models to the low-energy experimental measurements. We aim to show that if the muon anomalies have a common origin and do not arise at the tree level, they will be excluded or discovered soon. Since this project is still going, we state here our main tasks in this project as follows:

- **Systematically classifying UV models.** We base our study of the muon anomalies on a catalogue of simplified models, which we derive through a diagram-topology analysis. This catalogue represents all possible models — subject to the minimality assumptions discussed below — that explain $(g-2)_\mu$ or the $b \to s$ anomalies through heavy $(\Lambda > v)$ new physics. The models are derived by enumerating the exotic field content that furnishes topologies generating the leading-order effective operators appropriate to each process such that:

  *i)* The operators are generated through one-loop processes.

  *ii)* The only Lorentz representations allowed are scalars and vector-like fermions.

  *iii)* The allowed SM irreducible representations are restricted to be finite.

  These assumptions serve to define our use of the term *minimal* when applied to the simplified models we study. Scalars and vector-like fermions live at an energy scale $\Lambda \gg v$, and working exclusively with vector-like fermions will avoid issues of gauge anomaly cancellation. The allowed colour representations are $\mathbf{1}$, $\mathbf{3}$, $\bar{\mathbf{3}}$, $\mathbf{6}$, $\bar{\mathbf{6}}$ and $\mathbf{8}$. We accept isospin representations of dimension-4 and lower, and only allow hypercharges in the range $[-2, 2]$. This range is chosen to coincide with the range of quantum numbers characterising exotic multiplets that generate dimension-6 operators at tree level [88]. After enumerating possible Feynman diagrams that contribute to the mechanisms to explain $(g-2)_\mu$ and B-anomalies, we write the implied UV Lagrangians. Our approach reproduced many UV models that were proposed in the literature before, Refs. [89–95]. Since we aim to find a common origin of the muon anomalies, there are only three UV models that pass our criteria.

  1. There are only two possibilities for single scalar particle extensions of the SM, they are $S_1\big(\bar{\mathbf{3}}, \mathbf{1}, 1/3\big)$ model and $R_2\big(\mathbf{3}, \mathbf{2}, 7/6\big)$ model.
  2. The last scenario is a two-particle extension involving only scalar and fermion. We found uniquely one UV model, a model with the $R_2$ scalar leptoquark and a vector-like quark $\Psi\big(\mathbf{3}, \mathbf{3}, 2/3\big)$.

- **Applying EFT top-down approach.** Our analysis's main point is to integrate these new heavy particles and match the UV models onto SMEFT Lagrangian. This task is carried by the path integral formulation for one-loop matching. We also cross-check our results by the calculations with the Feynman diagram approach, which has been fully automatized recently by the so-called MatchMakerEFT program, Ref. [96]. We note that the one-loop matching for the $S_1$ model has been entirely done by the diagrammatic and functional approaches, Refs. [87, 97]. However, the model with $R_2$ or $(R_2 + \Psi)$ particles has not been intensively studied in the EFT context yet. Eventually, several effective operators interested in this analysis are listed in Table 2.1.

| SMEFT operators | physical observables | tree-level matching | one-loop matching |
|---|---|---|---|
| $\mathcal{O}_{eu}$ | Semi-leptonic decays | ✓ | ✓ |
| $\mathcal{O}_{lq}^{(1)}, \mathcal{O}_{lq}^{(3)}$ | Semi-leptonic decays | ✓ | ✓ |
| $\mathcal{O}_{eB}, \mathcal{O}_{eW}$ | $(g-2)_\mu$ | | ✓ |
| $\mathcal{O}_{Hl}^{(1)}, \mathcal{O}_{Hl}^{(3)}, \mathcal{O}_{He}$ | $Z$-pole | | ✓ |
| $\mathcal{O}_{qq}^{(1)}, \mathcal{O}_{qq}^{(3)}$ | Meson-mixing | | ✓ |

**Table 2.1:** *Some relevant SMEFT operators imply the presence of new heavy particles, e.g.* $S_1(\bar{\mathbf{3}}, \mathbf{1}, 1/3)$, $R_2(\mathbf{3}, \mathbf{2}, 7/6)$, *and* $\Psi(\mathbf{3}, \mathbf{3}, 2/3)$.

## 2.3 RGEs running by functional method

As mentioned in the EFT top-down paradigm, one needs to run the Wilson coefficients from the matching scale, $\mu = \Lambda$, down to the experimental energy scale, e.g. $\mu = v$. In this section, we show how to derive the RGEs by using functional method. For further examples of RG running by the CDE method, see Ref. [75, 78]. The key idea is to extract the RGEs from the one-loop effective action $\Gamma^{(1-\text{loop})}[\varphi]$. Suppose we are interested in the RG evolution of the Wilson coefficient of operator $\mathcal{O}_Y$ given by the EFT Lagrangian,

$$\mathcal{L}_{\text{EFT}} = \mathcal{O}_K[\varphi] + c_Y \mathcal{O}_Y[\varphi] \ , \tag{2.50}$$

where $\mathcal{O}_K$ denotes kinetic operators, who have been canonically normalized. To extract the RGEs, the first step is to compute at one-loop order the effective action resulting from the Lagrangian given by Eq. (2.50),

$$\Gamma^{\text{full}}[\varphi] = \Gamma^{(\text{tree})} + \Gamma^{(1-\text{loop})} \supset \int d^4x \left[ \left(1 + c_K^{(1-\text{loop})}\right) \mathcal{O}_K[\varphi] + \left(c_Y + c_Y^{(1-\text{loop})}\right) \mathcal{O}_Y[\varphi] \right] \ , \tag{2.51}$$

where $c_K^{(1-\text{loop})}, c_Y^{(1-\text{loop})}$ encoded the one-loop corrections for the operator $\mathcal{O}_K$ and $\mathcal{O}_Y$, respectively. The crucial point is that $c_i^{(1-\text{loop})}$ can be calculated very efficiently by the CDE method or directly recycle some universal results already pre-computed from Refs. [3, 6, 80–82, 98]. Next step, we rescale the field $\varphi$ to normalize the kinetic terms canonically,

$$\varphi \to \frac{1}{\sqrt{1 + c_K^{(1-\text{loop})}}} \varphi = \left[ 1 - \frac{1}{2} c_K^{(1-\text{loop})} + \cdots \right] \varphi \ , \tag{2.52}$$

where we performed Taylor expansion around the small coupling $c_K^{(1-\text{loop})}$. The effective action after normalize the kinetic terms reads,

$$\Gamma^{\text{full}}[\varphi] \supset \int d^4x \left[ \mathcal{O}_K[\varphi] + \left(c_Y + f_Y^{(1-\text{loop})}\right) \mathcal{O}_Y[\varphi] \right] \ , \tag{2.53}$$

where $f_Y^{(1-\text{loop})}$ is obtained after rescaling the field $\varphi$. Eventually, we extract the RG evolution of the coupling $c_Y$ by solving,

$$\mu \frac{d}{d\mu} \left[ c_Y + f_Y^{(1-\text{loop})} \right] = 0 \; . \tag{2.54}$$

# Chapter 3

# The Fermionic Universal One-Loop Effective Action

Recent development of path integral matching techniques based on the covariant derivative expansion has made manifest a universal structure of one-loop effective Lagrangians. The universal terms can be computed once and for all to serve as a reference for one-loop matching calculations and to ease their automation. Here we present the fermionic universal one-loop effective action (UOLEA), resulting from integrating out heavy fermions (Dirac or Majorana) with scalar, pseudo-scalar, vector and axial-vector couplings. We also clarify the relation of the new terms computed here to terms previously computed in the literature and those that remain to complete the UOLEA. Our results can be readily used to efficiently obtain analytical expressions for effective operators arising from heavy fermion loops.

## 3.1 Introduction and summary of results

The methods of effective field theory (EFT) have seen a resurgence lately in particle physics, due in part to the lack of new physics discovery at the weak scale. If new physics is indeed decoupled to heavier scales, as observations seem to be indicating, then the Standard Model (SM) should be properly considered as an EFT supplemented by higher-dimensional operators. The coefficients of these higher-dimensional operators encapsulate the new physics integrated out at some higher energy scale. Calculating these coefficients from ultraviolet (UV) theories has traditionally been performed using Feynman diagrams, where amplitudes involving the heavy degrees of freedom are explicitly "matched" to the EFT amplitudes. However, a more elegant approach is to "integrate out" the heavy particles by evaluating the path integral directly [6, 72–78, 80–82, 99]. While the adoption of this approach for practical phenomenological calculations has been limited in the past by cumbersome expansion techniques and the misconception that it could not account for matching with both heavy and light particles in the loop, these technical issues have been addressed in the last few years [75–77, 81]. New methods were developed to evaluate the path integral at one loop more efficiently using improved expansion techniques (as for example the covariant diagram method [77]), that could also include mixed heavy-light matching.

Compared to the traditional approach of matching Feynman diagrams, these path integral methods have several advantages: they can be calculated more generally, directly and systematically when computing a set of operator coefficients. Ultimately, it was pointed out in Refs. [72, 80] that the one-loop effective action has a universal structure which makes repeated evaluation of the path integral redundant. It is this set of universal terms and coefficients, evaluated once and for all, that forms the so-called Universal One-Loop Effective Action (UOLEA). Starting from the UOLEA, a one-loop matching calculation is reduced to an algebraic manipulation of matrix traces.

The piece of the UOLEA that was first worked out, for the simplified case of degenerate masses in Ref. [72] and generalised to the non-degenerate case in Ref. [80], contains terms arising from integrating out heavy bosonic fields $\Phi$ which couple to light fields $\phi$ via a Lagrangian of the form

$$\mathcal{L}_{\rm UV}[\phi,\Phi] = \mathcal{L}_0[\phi] + \Phi^\dagger\left(P^2 - M^2 - U[\phi]\right)\Phi + \mathcal{O}(\Phi^3)\,, \tag{3.1}$$

where $P_\mu \equiv iD_\mu$ is the Hermitian covariant derivative, $M$ is a diagonal mass matrix for the heavy fields $\Phi$, and the model-dependent couplings of $\Phi$ to $\phi$ are encapsulated in the matrix $U[\phi]$. By virtue of keeping the covariant derivatives intact, the UOLEA can thus be written as an expansion in covariant derivatives, i.e. a covariant derivative expansion (CDE) [73, 74, 99]. In the end, to obtain the low-energy EFT Lagrangian up to dimension-six operators, one simply needs to insert the matrix $U[\phi]$ into the UOLEA:

$$\begin{aligned}
\mathcal{L}^{\rm bosonic,heavy}_{\rm UOLEA} &= -ic_s\,\mathrm{tr}\Big\{ f_2^i\,U_{ii} + f_3^i\,G_i'^{\mu\nu}G'_{\mu\nu,i} + f_4^{ij}\,U_{ij}U_{ji} \\
&\quad + f_5^i\,[P^\mu, G'_{\mu\nu,i}][P_\rho, G_i'^{\rho\nu}] + f_6^i\,G'^{\mu}{}_{\nu,i}G'^{\nu}{}_{\rho,i}G'^{\rho}{}_{\mu,i} \\
&\quad + f_7^{ij}\,[P^\mu,U_{ij}][P_\mu,U_{ji}] + f_8^{ijk}\,U_{ij}U_{jk}U_{ki} + f_9^i\,U_{ii}G_i'^{\mu\nu}G'_{\mu\nu,i} \\
&\quad + f_{10}^{ijkl}\,U_{ij}U_{jk}U_{kl}U_{li} + f_{11}^{ijk}\,U_{ij}[P^\mu,U_{jk}][P_\mu,U_{ki}] \\
&\quad + f_{12}^{ij}\left[P^\mu,[P_\mu,U_{ij}]\right]\left[P^\nu,[P_\nu,U_{ji}]\right] + f_{13}^{ij}\,U_{ij}U_{ji}G_i'^{\mu\nu}G'_{\mu\nu,i} \\
&\quad + f_{14}^{ij}\,[P^\mu,U_{ij}][P^\nu,U_{ji}]G'_{\nu\mu,i} \\
&\quad + f_{15}^{ij}\left(U_{ij}[P^\mu,U_{ji}] - [P^\mu,U_{ij}]U_{ji}\right)[P^\nu,G'_{\nu\mu,i}] \\
&\quad + f_{16}^{ijklm}\,U_{ij}U_{jk}U_{kl}U_{lm}U_{mi} \\
&\quad + f_{17}^{ijkl}\,U_{ij}U_{jk}[P^\mu,U_{kl}][P_\mu,U_{li}] + f_{18}^{ijkl}\,U_{ij}[P^\mu,U_{jk}]U_{kl}[P_\mu,U_{li}] \\
&\quad + f_{19}^{ijklmn}\,U_{ij}U_{jk}U_{kl}U_{lm}U_{mn}U_{ni}\Big\} \qquad (3.2)\\
&= \sum_N f_N^{(P)}\mathbb{O}_N^{(P)} + f_N^{(U)}\mathbb{O}_N^{(U)} + f_N^{(PU)}\mathbb{O}_N^{(PU)}\,. \qquad (3.3)
\end{aligned}$$

The prefactor $c_s = \frac{1}{2}$ for each real degree of freedom (e.g. real scalar, vector) and can be taken as $c_s = \pm 1$ in some other cases [72]. The *universal coefficients* $f_N^{ij\cdots}$ are functions of heavy particle masses $m_i, m_j, \ldots$, and are expressed in terms of a set of master integrals. The field strength matrix is defined as $G'_{\mu\nu} = -[P_\mu,\, P_\nu] = -igG_{\mu\nu}$, and the subscripts $i,j,\ldots$ on $G$ and $U$ instruct us to take the corresponding block for particles $i,j,\ldots$. In Eq. (3.3), we have schematically summarised the entire expression by three UOLEA operator classes: those involving only covariant derivatives ($\mathbb{O}_N^{(P)}$), only interaction matrices ($\mathbb{O}_N^{(U)}$), and both ($\mathbb{O}_N^{(PU)}$). We refer the reader to Refs. [72, 77, 80] for the derivation of this bosonic UOLEA, though we stress that it is no longer necessary to re-do the path integral calculation for each specific model given the availability of these universal results. The UOLEA operator structures, written in terms of the matrices $P$ and $U$, become EFT operators when substituting in the specific forms of these matrices, in terms of the light fields and for a given UV model, which can then be rearranged into the desired non-redundant EFT basis.[1]

There are, however, additional structures that arise in some UV Lagrangians which lead to new terms in the UOLEA beyond those in Eq. (3.2). In particular, for UV Lagrangians containing

[1]Note that the UOLEA can be expanded indefinitely in the CDE; in Eq. (3.2) and, later, the fermionic UOLEA we terminate the CDE to keep only all UOLEA operator structures necessary for obtaining EFT operators up to dimension six.

heavy fermion fields, further terms in the UOLEA arise from fermionic loops. While some of them can be obtained from the bosonic UOLEA (3.2) by "squaring" the functional determinant (see, e.g., Appendix A1 of Ref. [72], Appendix E of Ref. [80] and Eq. (3.19) or Ref.[77]) to put the UV Lagrangian into the form of Eq. (3.1), this only yields partial results when the interactions involve $\gamma$ matrices[2]. It is therefore necessary to extend the UOLEA to properly include fermionic loops.

In this work, we present this fermionic UOLEA. It can be applied straightforwardly to the case of fermions in an analogous manner to the bosonic case described above. Specifically, we consider a UV Lagrangian capable of parametrising all possible renormalisable UV theories for a heavy multiplet of fermions $\Psi$ interacting with a light multiplet of bosons or fermions $\phi$ of the following form[3]:

$$\mathcal{L}_{\text{UV}}[\phi,\Psi] = \mathcal{L}_0[\phi] + \bar{\Psi}\left(\not{P} - M - X[\phi]\right)\Psi\,, \tag{3.4}$$

where we decompose the interaction matrix $X[\phi]$ into scalar, pseudo-scalar, vector and axial-vector couplings matrices as[4]

$$X[\phi] = W_0[\phi] + W_1[\phi]\, i\gamma^5 + V_\mu[\phi]\gamma^\mu + A_\mu[\phi]\gamma^\mu\gamma^5\,. \tag{3.5}$$

The low-energy EFT at one loop is then obtained by substituting these matrices into our fermionic UOLEA, which reads schematically

$$\begin{aligned}
\mathcal{L}_{\text{UOLEA}}^{\text{fermionic,heavy}} = \sum_N & f_N^{(P)}\mathbb{O}_N^{(P)} + f_N^{(W_0)}\mathbb{O}_N^{(W_0)} + f_N^{(W_1)}\mathbb{O}_N^{(W_1)} + f_N^{(W_0W_1)}\mathbb{O}_N^{(W_0W_1)} \\
& + f_N^{(PW_0)}\mathbb{O}_N^{(PW_0)} + f_N^{(PW_1)}\mathbb{O}_N^{(PW_1)} + f_N^{(PW_0W_1)}\mathbb{O}_N^{(PW_0W_1)} \\
& + f_N^{(V)}\mathbb{O}_N^{(V)} + f_N^{(A)}\mathbb{O}_N^{(A)} + f_N^{(VA)}\mathbb{O}_N^{(VA)} \\
& + f_N^{(PV)}\mathbb{O}_N^{(PV)} + f_N^{(PA)}\mathbb{O}_N^{(PA)} + f_N^{(PVA)}\mathbb{O}_N^{(PVA)} \\
& + f_N^{(W_0V)}\mathbb{O}_N^{(W_0V)} + f_N^{(W_1V)}\mathbb{O}_N^{(W_1V)} + f_N^{(W_0W_1V)}\mathbb{O}_N^{(W_0W_1V)} \\
& + f_N^{(PW_0V)}\mathbb{O}_N^{(PW_0V)} + f_N^{(PW_1V)}\mathbb{O}_N^{(PW_1V)} + f_N^{(PW_0W_1V)}\mathbb{O}_N^{(PW_0W_1V)} \\
& + f_N^{(W_0A)}\mathbb{O}_N^{(W_0A)} + f_N^{(W_1A)}\mathbb{O}_N^{(W_1A)} + f_N^{(W_0W_1A)}\mathbb{O}_N^{(W_0W_1A)} \\
& + f_N^{(PW_0A)}\mathbb{O}_N^{(PW_0A)} + f_N^{(PW_1A)}\mathbb{O}_N^{(PW_1A)} + f_N^{(PW_0W_1A)}\mathbb{O}_N^{(PW_0W_1A)} \\
& + f_N^{(W_0VA)}\mathbb{O}_N^{(W_0VA)} + f_N^{(W_1VA)}\mathbb{O}_N^{(W_1VA)} + f_N^{(W_0W_1VA)}\mathbb{O}_N^{(W_0W_1VA)} \\
& + f_N^{(PW_0VA)}\mathbb{O}_N^{(PW_0VA)} + f_N^{(PW_1VA)}\mathbb{O}_N^{(PW_1VA)} + f_N^{(PW_0W_1VA)}\mathbb{O}_N^{(PW_0W_1VA)}\,.
\end{aligned} \tag{3.6}$$

There are a large combinatorial number of possibilities for the fermionic UOLEA operator structures when including all coupling matrices in Eq. (3.5). We will see, however, that when calculating specific cases one can employ power counting to pick out the UOLEA operator structures that are relevant for matching to a set of desired EFT operators. Moreover, if the low-energy EFT does not contain massive vector bosons (e.g. arising from a broken gauge symmetry), then only the first

[2] Squaring the functional determinant with the coupling matrix in Eq. (3.5) would yield "open" derivative couplings. Besides, an "open" covariant derivative will get shifted, $P_\mu \to P_\mu - q_\mu$, and lead to non-trivial structures that were not encapsulated with the existing bosonic UOLEA.

[3] The parametrization in Eq. (3.4) works for the case with pure heavy Dirac (Majorana) fermions in the loop. See Appendix B for more details about integrating out Majorana fermions.

[4] Some UV models might contain derivative couplings, for example, $V_\mu = (\partial_\mu\phi)$.

**Universal terms available in the UOLEA**

| | Heavy-only | Mixed heavy-light | + derivative couplings |
|---|---|---|---|
| Bosonic | ✓ [80] | ✓ [82] | – |
| Fermionic | ✓ [this work] | (✓) | –$^{(*)}$ |
| Mixed statistics | (✓) | (✓) | –$^{(*)}$ |

$^{(*)}$ do not arise in renormalizable UV theories.

**Table 3.1:** *Status of the UOLEA. Entries marked by "✓" are available in the form of operator structures built from the various types of couplings that appear in the quadratic Lagrangian. Entries marked by "(✓)" are not available in the same form, but can be computed by plugging fermion couplings into the results of Ref. [6] and evaluating Dirac matrix traces. Entries marked by "–" have not been computed in the literature, though the techniques for computing them are available. See text for details.*

two lines of Eq. (3.6) are needed, comprising a relatively compact set of UOLEA operators. These UOLEA operator structures, along with their universal coefficients for the degenerate mass case, are tabulated below in Tables 3.4, 3.5, 3.6 and 3.7. Explicit results for the non-degenerate case and for the rest of Eq. (3.6) are available in a Mathematica notebook on `GitHub`, [100], as explained in Sec. 3.2.4.

The bosonic UOLEA presented in Ref. [80] (summarised above in Eq. (3.3)) and fermionic UOLEA presented in this chapter (summarised above in Eq. (3.6)) complete the one-loop matching master formula that includes loops involving heavy bosonic fields and heavy fermionic fields, respectively, for UV theories whose Lagrangians take the form of Eq. (3.1) or Eq. (3.4), and for up to dimension-6 operators in the EFT. Other UV theories exhibit additional coupling structures which are not captured by these UOLEAs, such as tensor current coupling (involving $\sigma_{\mu\nu}$), derivative couplings (which give rise to "open covariant derivatives" in the quadratic Lagrangian) and mixed bosonic-fermionic loops. If the UV Lagrangian includes terms coupling heavy fields linearly to the light fields, $\mathcal{L}_{\text{UV}} \supset \Phi^\dagger F[\phi] + \text{h.c.}$, then mixed heavy-light loops also contribute.[5] For the bosonic case, the mixed heavy-light terms, $\mathcal{L}_{\text{UOLEA}}^{\text{bosonic,mixed}}$, were computed in Ref. [82], where it was found that the operator structures in $\mathcal{L}_{\text{UOLEA}}^{\text{bosonic,mixed}}$ mirror those in $\mathcal{L}_{\text{UOLEA}}^{\text{bosonic,heavy}}$ with a much larger number of terms due to the heavy-light combinatorics. We expect the same for the fermionic UOLEA, though given the proliferation of terms already in the heavy-only case we find it less compelling to also tabulate the mixed heavy-light terms explicitly.

In Table 3.1 we summarise the progress of the UOLEA program. Fermionic results are also available in Ref. [6], though not in the same form as our expressions since the various matrix substructures are not expanded as in Eq. (3.5) — they can be computed, together with additional structures such as mixed fermion-boson and heavy-light loops, after plugging in these substructures and further evaluating the resulting Dirac matrix traces. Finally, UV theories involving derivative couplings generate additional terms in the UOLEA which have not yet been computed, though

[5] Linear couplings also generate tree-level contributions, but loop-level mixed heavy-light contributions can in certain cases be the leading terms for certain operators [101].

for the fermionic case they only arise when matching to non-renormalisable UV Lagrangians. The UOLEA has, so far, also been limited to those terms necessary for obtaining EFT operators up to dimension six only. Nevertheless, it is worth emphasising that, following the technical development of evaluating one-loop functional determinants with general structures [75–78], *one-loop functional matching is a fully solved problem*, independently of the availability of the UOLEA that captures those additional structures. The usefulness of the UOLEA lies in its packaging of certain universal steps of the calculation into the form of a master formula.

The chapter is organised as follows. In Sec. 3.2, we describe our calculation of the fermionic UOLEA and present the final results for the universal coefficients of the UOLEA operators. We then present examples illustrating the use of the fermionic UOLEA for efficient one-loop matching calculations in Sec. 3.3, before concluding in Sec. 3.4.

## 3.2 The Fermionic UOLEA

Fermions, by virtue of their symmetry properties, necessitate additional care as compared with spin-0 bosons, which have been the primary focus of CDE developments thus far. Some previous work on using the CDE to integrate out heavy fermions had employed the approach of squaring the argument of the functional trace in the effective action so as to bring it into the same form as bosonic loops, for subsequent insertion into the bosonic UOLEA as written in Eq. (3.2) [72, 102]. However, this approach cannot be straightforwardly applied to the case where fermion coupling structures contain gamma matrices beyond that accompanying the covariant derivative $\not{P}$.

As was pointed out for example in Refs. [75, 77], the argument of the functional trace need not be squared, in which case a CDE and universal one-loop action may be still be formulated, with a somewhat different structure from the bosonic UOLEA of the previous section but one that simplifies the UOLEA as applied to fermions. This procedure was employed in Ref. [6] to obtain contributions to the UOLEA from integrating out heavy fermions, though they do not decompose the general coupling matrix $X$ into its Hermitian matrix substructure constituents so that their final result still requires taking the trace over $\gamma$ matrices.

Here we provide a master formula in terms of these matrix substructures. In this case, as will be expanded upon in detail in the rest of this section, it is straightforward to account for all possible Lorentz structures of fermionic coupling matrices to light fields, thereby allowing for the completion of a *fermionic UOLEA*.

### 3.2.1 One-loop matching from the path integral

Let us begin by reviewing the basic idea of one-loop functional matching, focusing on the case of integrating out heavy fermions. Consider a UV Lagrangian containing a multiplet of heavy Dirac fermion fields $\Psi$ and light fields $\phi$. Assuming the heavy fermions $\Psi$ couple to the light fields only via bilinears, the UV Lagrangian can be written in the form

$$\mathcal{L}_{\text{UV}}[\phi,\Psi] = \mathcal{L}_0[\phi] + \bar{\Psi}\left(\not{P} - M - X[\phi]\right)\Psi\,, \tag{3.7}$$

where $P_\mu \equiv iD_\mu$ and $M$ is the diagonal mass matrix for the multiplet $\Psi$. In order to maximise the analytical and physical utility of the universal structures obtained by using the CDE method to obtain the fermionic UOLEA, it is useful to decompose the interaction matrix $X[\phi]$ into scalar, pseudoscalar, vector, axial-vector and tensor parts. As we restrict our scope to renormalizable UV theories here, we exclude the tensor coupling, and write

$$X[\phi] = W_0[\phi] + iW_1[\phi]\gamma^5 + V_\mu[\phi]\gamma^\mu + A_\mu[\phi]\gamma^\mu\gamma^5\,, \tag{3.8}$$

where the $W_0$, $W_1$, $V_\mu$, $A_\mu$ coupling matrices are Hermitian. Obtaining the effective action for the UV lagrangian above is performed in the standard way, by integrating out the heavy fermion $\Psi$:

$$\begin{aligned} e^{iS_{\text{eff}}[\phi]} &= \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi\, e^{iS_{\text{UV}}[\phi,\Psi]} \\ &\simeq e^{iS_{\text{UV}}[\phi,\Psi_c]} \int \mathcal{D}\bar{\eta}\,\mathcal{D}\eta\, e^{i\int d^dx\,\bar{\eta}\left(\not{P}-M-X[\phi]\right)\eta} \\ &= e^{iS_{\text{UV}}[\phi,\Psi_c]} \det\left(\not{P}-M-X[\phi]\right) = e^{iS_{\text{UV}}[\phi,\Psi_c]+\text{Tr}\ln\left(\not{P}-M-X[\phi]\right)} . \end{aligned} \tag{3.9}$$

In going from the first to the second line, we have expanded the heavy fields around their classical background values, $\Psi = \Psi_c + \eta$, so that the integration is performed over the quantum fluctuations $\eta$, around the UV action evaluated at this classical solution. We therefore arrive at the one-loop effective action arising from integrating out heavy fermions:

$$S_{\text{eff}}^{\text{1-loop}} = -i\,\text{Tr}\ln\left(\not{P}-M-X[\phi]\right), \tag{3.10}$$

where "Tr" denotes a trace over both internal indices and over the functional space of the operator $\left(\not{P}-M-X[\phi]\right)$. We then evaluate the functional trace by making use of the momentum eigenstate basis, and employing the standard trick of inserting the identity,

$$\begin{aligned} S_{\text{eff}}^{\text{1-loop}} &= -i\int \frac{d^dq}{(2\pi)^d}\langle q|\text{tr}\ln\left(\not{P}-M-X[\phi]\right)|q\rangle \\ &= -i\int d^dx\int \frac{d^dq}{(2\pi)^d}\,\langle q|\,x\rangle\,\langle x|\text{tr}\ln\left(\not{P}-M-X[\phi]\right)|q\rangle \\ &= -i\int d^dx\int \frac{d^dq}{(2\pi)^d}\text{tr}\ln\left(\not{P}-\not{q}-M-X[\phi]\right), \end{aligned} \tag{3.11}$$

where now "tr" denotes a trace over internal indices only. In the last line of (3.11), we have used $\langle x|\,q\rangle = e^{-iq\cdot x}$ and made a conventional change in the integration variable $q \to -q$. Further details of these functional manipulations are reviewed in Refs.[72, 77].

The one-loop effective action of Eq. (3.11) must then be expanded in the hard region, where the loop momenta $q^2 \sim M^2$, to obtain the low-energy effective Lagrangian consisting of local operators, as explained, for example, in Refs. [76, 77]. This method of regions ensures that both heavy-only and mixed heavy-light loops are correctly accounted for in the matching calculation. In the present case of heavy-only terms, the hard region contribution coincides with the full integral, so we obtain

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{\text{1-loop}} &= -i\int \frac{d^dq}{(2\pi)^d}\text{tr}\ln\left(\not{P}-\not{q}-M-X[\phi]\right) \\ &= i\,\text{tr}\sum_{n=1}^{\infty}\frac{1}{n}\int\frac{d^dq}{(2\pi)^d}\left[\frac{-1}{\not{q}+M}\left(-\not{P}+W_0[\phi]+iW_1[\phi]\gamma^5+V_\mu[\phi]\gamma^\mu+A_\mu[\phi]\gamma^\mu\gamma^5\right)\right]^n . \end{aligned} \tag{3.12}$$

The second equality makes explicit the universal operator structures that appear in the one-loop effective action, and hints at the universality of the corresponding operator coefficients. After expansion and computation of the integrals over the loop momenta, this expression is clearly the fermionic analog of the familiar expression for the bosonic UOLEA of Eq. (3.2). We can also see that by virtue of separating $X$ into the sum over the $W_0$, $W_1$, $V_\mu$ and $A_\mu$ components, we can apply our physical intuition for what types of combinations of structures can appear both in the UOLEA itself, and when considering specific models. This will become more clear in the rest of this section, where we discuss the universal structures in more depth, and in Sec. 3.3 when we apply the fermionic UOLEA to several examples.

### 3.2.2 Universal operator structures in the fermionic UOLEA

In the previous subsection we have described how to obtain a general expression for the fermionic UOLEA. However, as written in Eq. (3.12), the utility of the UOLEA is not yet apparent.

It is important to recall that an attractive feature of the bosonic UOLEA is that once the analog of Eq. (3.12) is expanded out to obtain e.g. Eq. (3.2) (for heavy-only loop contributions to EFT operators up to dimension 6), all necessary structures in the one-loop effective Lagrangian are known and enumerated, and their universal coefficients are calculated once-and-for-all. Having all the possible structures enumerated makes for intuitive application to integrating out particles in specific UV models. Knowing the specific form of the interaction matrix $U$ of Eq. (3.2) for the UV model being studied allows for dramatic simplification of computation of the one-loop effective action, since not all the bosonic UOLEA operators would contribute to the specific EFT operators of interest. As a trivial example, let us consider a quartic $|\Phi|^2|\phi|^2$ interaction in the UV, such that $U \sim |\phi|^2$. If one is interested in the bosonic UOLEA at dimension 6, it is evident that the term $f_{19}\, U^6$ in Eq. 3.2 is of higher dimension so it will not contribute, and therefore $U^6$ can be discarded without being computed.

Turning to the Fermionic UOLEA, from the form of Eq. (3.12), we can see that ultimately there will be a proliferation of universal structures in the final one-loop effective Lagrangian, which can be written compactly as

$$\mathcal{L}^{\text{fermionic}}_{\text{UOLEA}} = \sum_N f_N\, \mathbb{O}_N^{\{P,W_0,W_1,A,V\}} \;. \tag{3.13}$$

Due to the variety of matrix coupling structures denoted in the superscript set, the fermionic heavy-only UOLEA has a large number of operators in the sum arising from all the (non-vanishing) combinatorial possibilities, in contrast to the bosonic heavy-only UOLEA's 19 operator structures. An expanded sum of the UOLEA operator classes is presented in Eq. (3.6) and Tables 3.2-3.3, where we enumerated all the different classes of possible UOLEA operator structures.

The advantage of separating $X$ into $W_0, W_1, V, A$ is now apparent: all possible universal fermionic UOLEA operators are obtained and their coefficients computed and tabulated once and for all, analogously to the bosonic UOLEA. When inserting a UV model into the fermionic UOLEA, computation of the $W_0, W_1, V, A$ structures then allows for transparent power counting, as well as enabling simple symmetry cross-checks. We now describe this in more detail for each of these (non-vanishing) structures and their combinations listed in Tables 3.2 and 3.3.

#### Scalar and pseudo-scalar structures ($W_0$, $W_1$)

From the Lagrangian as written in Eq. (3.7) and the expansion of $X$ in Eq. (3.8), it is clear that if the heavy fermion that is integrated out has couplings to scalar operators, these will be captured by the $W_0$ matrix structure. Likewise, in the case of couplings to pseudoscalar operators, these will be captured by the $W_1$ matrix. The $W_0$ ($W_1$) matrix is therefore even (odd) under parity, which will allow us to easily intuit what UOLEA operators might be formed and therefore contribute to the final result of Eq. (3.6). All such structures are listed in Table 3.2. Indeed, the parity properties of the matrices and their impact on the operator structures is clear. As a scalar structure, $W_0$ can appear in both even and odd powers. In contrast, $W_1$, as a pseudo-scalar structure, must always appear in even powers, or accompanied by $P^4$. That the latter is permitted follows from $\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) \neq 0$, so that one can already see that the only EFT operators arising from such a structure will involve pseudo-scalars coupling to $F\tilde{F}$.

| Operator class | Non-vanishing structures |
|---|---|
| $\mathbb{O}^{(P)}$ | $P^4,\ P^6$ |
| $\mathbb{O}^{(W_0)}$ | $W_0, W_0^2, W_0^3, W_0^4, W_0^5, W_0^6$ |
| $\mathbb{O}^{(W_1)}$ | $W_1^2, W_1^4, W_1^6$ |
| $\mathbb{O}^{(W_0W_1)}$ | $W_0W_1^2, W_0^2W_1^2, W_0^3W_1^2, W_0W_1^4, W_0^4W_1^2, W_0^2W_1^4$ |
| $\mathbb{O}^{(PW_0)}$ | $P^2W_0^2, P^2W_0^3, P^4W_0, P^2W_0^4, P^4W_0^2$ |
| $\mathbb{O}^{(PW_1)}$ | $P^2W_1^2, P^4W_1, P^2W_1^4, P^4W_1^2$ |
| $\mathbb{O}^{(PW_0W_1)}$ | $P^2W_0W_1^2, P^4W_0W_1, P^2W_0^2W_1^2$ |
| $\mathbb{O}^{(V)}$ | $V^2, V^4, V^6$ |
| $\mathbb{O}^{(A)}$ | $A^2, A^4, A^6$ |
| $\mathbb{O}^{(VA)}$ | $VA^3, V^2A^2, V^3A, VA^5, V^2A^4, V^3A^3, V^4A^2, V^5A$ |
| $\mathbb{O}^{(PV)}$ | $PV^3, P^2V^2, P^3V, PV^5, P^2V^4, P^3V^3, P^4V^2, P^5V$ |
| $\mathbb{O}^{(PA)}$ | $PA^3, P^2A^2, P^3A, PA^5, P^2A^4, P^3A^3, P^4A^2, P^5A$ |
| $\mathbb{O}^{(PAV)}$ | $PAV^2, PA^2V, P^2AV, PAV^4, PA^2V^3, PA^3V^2, PA^4V,$ $P^2AV^3, P^2A^2V^2, P^2A^3V, P^3AV^2, P^3A^2V, P^4AV$ |

**Table 3.2:** *Non-vanishing operator structures in the fermionic UOLEA that involve covariant derivatives ($P$) plus either scalar and pseudo-scalar structures ($W_0, W_1$), or vector and axial-vector structures ($V, A$).*

### Vector and axial-vector structures ($V$, $A$)

These structures will appear if the UV Lagrangian contains fermionic couplings to vector bosons that do *not* appear in the covariant derivative operator $P$. This would occur, for example, if the heavy fermion current is coupled to a light gauge boson such as the $Z_\mu$ of the SM (in this case the low-energy effective theory with $Z_\mu$ not in a covariant derivative would not be the SMEFT), or an $A'_\mu$ associated with a broken $U(1)'$ whose mass was sufficiently small compared with that of the fermion being integrated out. These results are particularly useful if one is interested in matching to low-energy EFTs containing massive vector bosons. In this case, it should be noted that the covariant derivative operator $P$ only contains the gauge fields associated with the remaining unbroken symmetries.

Even if the gauge boson content of the low-energy theory is purely that of the SM, these structures must be included in a complete fermionic UOLEA if one wishes to apply it to matching with general EFTs. We will see examples in Sec. 3.3 where the $V$ and $A$ structures appear.

As in the case above of $W_0$, $W_1$ operator structures, the power counting and enumeration of non-vanishing $V$ and $A$ combinations is straightforwardly obtained from symmetry arguments. All structures that contribute to EFT operators up to dimension 6 are listed in Table 3.2. We can see that by virtue of the symmetry properties of both $V$ and $A$, they must always appear in the

| Operator class | Non-vanishing structures |
|---|---|
| $\mathbb{O}^{(VW_0)}$ | $V^2W_0^2, V^2W_0^3, V^4W_0, V^2W_0^4, V^4W_0^2$ |
| $\mathbb{O}^{(VW_1)}$ | $V^2W_1^2, V^4W_1, V^2W_1^4, V^4W_1^2$ |
| $\mathbb{O}^{(VW_0W_1)}$ | $V^2W_0W_1^2, V^4W_1W_0, V^2W_0^2W_1^2$ |
| $\mathbb{O}^{(PVW_0)}$ | $PVW_0, PVW_0^2, PVW_0^3, PVW_0^4, PV^3W_0, P^3VW_0, P^2V^2W_0,$ <br> $P^3VW_0^2, PV^3W_0^2, P^2V^2W_0^2$ |
| $\mathbb{O}^{(PVW_1)}$ | $PVW_1^2, PVW_1^4, PV^3W_1, P^3VW_1, P^2V^2W_1, PV^3W_1^2, P^3VW_1^2, P^2V^2W_1^2$ |
| $\mathbb{O}^{(PVW_0W_1)}$ | $PVW_0W_1^2, PVW_0^2W_1^2, P^3VW_0W_1, P^2V^2W_0W_1, PV^3W_0W_1$ |
| $\mathbb{O}^{(AW_0)}$ | $A^2W_0^2, A^2W_0^3, A^4W_0, A^2W_0^4, A^4W_0^2$ |
| $\mathbb{O}^{(AW_1)}$ | $A^2W_1^2, A^4W_1, A^2W_1^4, A^4W_1^2$ |
| $\mathbb{O}^{(AW_0W_1)}$ | $A^2W_0W_1^2, A^4W_1W_0, A^2W_0^2W_1^2$ |
| $\mathbb{O}^{(PAW_0)}$ | $PA^3W_0, P^3AW_0, P^2A^2W_0, P^3AW_0^2, PA^3W_0^2, P^2A^2W_0^2$ |
| $\mathbb{O}^{(PAW_1)}$ | $PAW_1, PAW_1^3, PA^3W_1, P^3AW_1, P^2A^2W_1, PA^3W_1^2, P^3AW_1^2, P^2A^2W_1^2$ |
| $\mathbb{O}^{(PAW_0W_1)}$ | $PAW_0W_1, PAW_0^2W_1, PAW_0W_1^3, PAW_0^3W_1, P^3AW_0W_1, P^2A^2W_0W_1, PA^3W_0W_1$ |
| $\mathbb{O}^{(AVW_0)}$ | $VA^3W_0, V^3AW_0, V^2A^2W_0, V^3AW_0^2, VA^3W_0^2, V^2A^2W_0^2$ |
| $\mathbb{O}^{(AVW_1)}$ | $VAW_1, VAW_1^3, VA^3W_1, V^3AW_1, V^2A^2W_1, VA^3W_1^2, V^3AW_1^2, V^2A^2W_1^2$ |
| $\mathbb{O}^{(AVW_0W_1)}$ | $VAW_0W_1, VAW_0^2W_1, VAW_0W_1^3, VAW_0^3W_1, V^3AW_0W_1, VA^3W_0W_1, V^2A^2W_0W_1$ |
| $\mathbb{O}^{(PAVW_0)}$ | $PAV^2W_0, PAV^2W_0^2, PA^2VW_0, PA^2VW_0^2, P^2AVW_0, P^2AVW_0^2$ |
| $\mathbb{O}^{(PAVW_1)}$ | $PAV^2W_1, PAV^2W_1^2, PA^2VW_1, PA^2VW_1^2, P^2AVW_1, P^2AVW_1^2$ |
| $\mathbb{O}^{(PAVW_0W_1)}$ | $PAV^2W_0W_1, PA^2VW_0W_1, P^2AVW_0W_1$ |

**Table 3.3:** *Non-vanishing operator structures in the fermionic UOLEA that involve both (pseudo-)scalar and (axial-)vector couplings.*

combinations $P^kV^lA^m$ with $k+l+m$ even.

### General case ($W_0$, $W_1$, $V$, $A$ all present)

The above discussion can be extended further to the situation when all possible structures in Eq. (3.7) are present. In this most general case, one gets a proliferation of possible combinations and operator classes, all of which are listed in Table 3.3.

Once again, the power counting is straightforward, and follows from trace identities of gamma matrices. As before, scalar structures $W_0$ can appear without restriction, while pseudo-scalar structures $W_1$ can only appear in combination with operators such that the overall number of $\gamma^5$ matrices is even, or in combination with four $\gamma^\mu$ matrices.

### 3.2.3 Computing UOLEA operators with covariant diagrams

To evaluate the expansion in (3.12), we use the covariant diagrams technique of Ref.[77] to keep track of the expansion and directly compute the Wilson coefficient for each EFT operator. Each term in the CDE expansion (3.12) can be represented by a covariant diagram, which can be written down directly by a systematic set of rules. We then straightforwardly obtain the prefactor coefficient and the one-loop master integral associated with the diagram we are considering. The details of the covariant diagram technique are described in Ref.[77]. Here we summarise the essential ingredients relevant for the present case of heavy fermion loops.

- **Fermion propagator:**
  Each fermion propagator can be decomposed into two terms,
  $$\frac{-1}{\not{q}+M} = \frac{M}{q^2-M^2} + \frac{-q_\mu\gamma^\mu}{q^2-M^2}, \tag{3.14}$$
  where the first term is the heavy bosonic propagator multiplied by the mass. The second term involves the loop momentum $q_\mu$ in the numerator, which will contribute to the loop integral. The loop integrals have the general form
  $$\int \frac{d^dq}{(2\pi)^d} \frac{q^{\mu_1}\cdots q^{\mu_{2n_c}}}{(q^2-M_i^2)^{n_i}(q^2-M_j^2)^{n_j}\cdots(q^2)^{n_L}} = g^{\mu_1\cdots\mu_{2n_c}}\mathcal{I}[q^{2n_c}]^{n_in_j\cdots n_L}_{ij\cdots 0}, \tag{3.15}$$
  where $g^{\mu_1\cdots\mu_{2n_c}}$ is the completely symmetric tensor, e.g. $g^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}$, and $\mathcal{I}$ are master integrals, a useful set of which can be found in Appendix A or Ref. [77]. The symmetric tensor in (3.15) will contract the Lorentz indices of Dirac matrices in the fermionic propagator, then we must sum over all possibilities of the contractions. In the covariant diagram, we shall use dotted lines to indicate the contractions among the fermionic part of the propagator in Eq. (3.14), following the conventions of Ref. [77].

- **Vertex insertions:** From Eq. (3.12), all vertex insertions, $\gamma^\mu P_\mu, W_0, i\gamma^5 W_1, \gamma^\mu V_\mu$ and $\gamma^\mu\gamma^5 A_\mu$ are independent of the loop momentum $q_\mu$ and thus do not change the loop integrals. We note that the vertex insertions will not be contracted with each other or with the propagators.

- **Dirac trace evaluations:** By construction, $P_\mu, W_0, W_1, V_\mu$ and $A_\mu$ do not involve additional Dirac matrices. Therefore, after reading off the value of each covariant diagram, the *trace* over Dirac matrices is factorized out and evaluated once-and-for-all. The trace in the final results, still denoted by "tr" is over the remaining internal indices, e.g. $SU(2)$ and color indices.

- **Renormalisation and divergences:** For the one-loop divergent integrals, we use dimensional regularisation and the $\overline{\text{MS}}$-scheme for renormalisation. The important point in the case with divergent integrals is that the *trace* over all Dirac matrices have to be evaluated in $D = 4-\epsilon$ dimensions, and the $\epsilon$-term resulting from the contractions of the metric tensor, $g_{\mu\nu}g^{\mu\nu} = D$, must be kept in the computations. This term can hit the $1/\epsilon$ pole resulting from a divergent integral and yield a finite contribution to the Wilson coefficient. It is well-known that in $D = 4-\epsilon$ dimensions, the relations $\{\gamma^\mu,\gamma^5\} = 0$ and $\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) \neq 0$ cannot be satisfied simultaneously [103, 104]. In our calculations, we use the Breitenlohner-Maison-'t Hooft-Veltman (BMHV) scheme [105, 106].

- **Covariant derivatives in commutators:** By expanding the one-loop effective action in Eq.(3.12), we will obtain operator structures that carry "open" covariant derivatives, $P_\mu$. We emphasise that the $P_\mu$ in the CDE expansion is a functional operator, i.e. $P_\mu$ will act on everything to the right. To construct an effective operator we need a "closed" covariant derivative where $P_\mu$ will only act on its immediate nearest neighbour in the operator; we thus need to organise the final results such that $P_\mu$'s only appear in commutators (see e.g. Refs. [72, 77, 78]). To be concrete, let us consider for example a functional operator $P_\mu W_0$ acting on a generic functional $\phi$:

$$P_\mu W_0 \; \phi = (P_\mu W_0)_{\text{local}} \, \phi + W_0 \, (P_\mu \phi) \,, \tag{3.16}$$

we then combine all operator structures with $P_\mu$ into commutators,

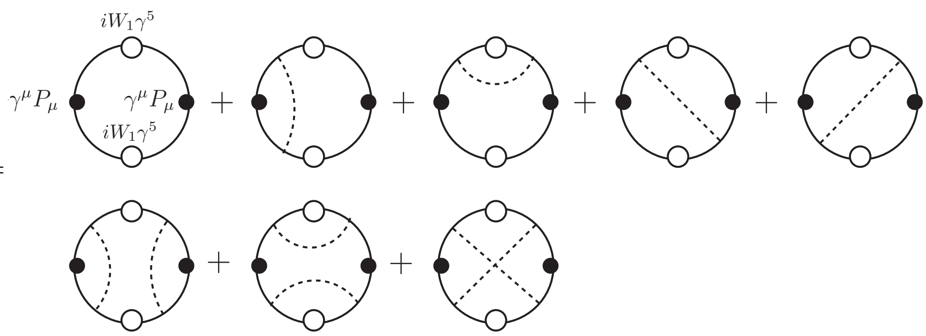
$$(P_\mu W_0)_{\text{local}} \, \phi = (P_\mu W_0 - W_0 P_\mu) \, \phi = [P_\mu, W_0] \, \phi \,. \tag{3.17}$$

In practice, we first write down a basis set of independent operators where $P_\mu$'s only appear in the commutators, and then expand the commutators and match the results from the functional trace expansion to solve the system of equations and determine the coefficient of the elements in the "commutator" basis that we chose. We note that the operator structures with adjacent covariant derivatives, $\text{tr}\left(\cdots P^2 \cdots\right)$, can be dropped to simplify this computation, since the non-$P^2$ terms are sufficient for reconstructing the universal operators written in the commutator basis when matching to the expanded form (see Ref. [77] for details).

- **Hermiticity of the operator structures:** Since covariant diagrams that are mirror images of each other are related by hermitian conjugation, only one in each such pair needs to be computed. We will also use the hermiticity of the Lagrangian to identify the number of irreducible operator structures. In particular, when the vector and axial-vector structures are included in the matrix $X[\phi]$, the hermitian conjugate relations can drastically reduce the number of operator structures we need to evaluate.

Let us consider a simple example to illustrate concretely some of these points, taking a coupling matrix $X[\phi]$ that only contains pseudo-scalar structures. We would then compute the universal coefficient of the operator structure $P^2 W_1^2$ as follows:

$\mathbb{O}^{P^2 W_1^2} =$




(3.18)

Making use of the covariant diagram rules given by Ref. [77], we readout the value of each diagram in Eq. (3.18),

$$
\begin{aligned}
\mathbb{O}^{P^2W_1^2} = &\ \frac{i}{2}m^4\mathcal{I}_i^4\,\mathrm{tr}\left(\not{P}iW_1\gamma^5\not{P}iW_1\gamma^5\right)\\
&+ im^2\mathcal{I}[q^2]_i^4\Big[\mathrm{tr}\left(\gamma^\mu\not{P}\gamma_\mu iW_1\gamma^5\not{P}iW_1\gamma^5\right)+\mathrm{tr}\left(\not{P}\gamma^\mu iW_1\gamma^5\gamma_\mu\not{P}iW_1\gamma^5\right)\\
&\qquad+\frac{1}{2}\mathrm{tr}\left(\not{P}\gamma^\mu iW_1\gamma^5\not{P}\gamma_\mu iW_1\gamma^5\right)+\frac{1}{2}\mathrm{tr}\left(\not{P}iW_1\gamma^5\gamma^\mu\not{P}iW_1\gamma^5\gamma_\mu\right)\Big]\\
&+ i\mathcal{I}[q^4]_i^4\Big[\frac{1}{2}\mathrm{tr}\left(\gamma^\mu\not{P}\gamma_\mu iW_1\gamma^5\gamma^\nu\not{P}\gamma_\nu iW_1\gamma^5\right)+\frac{1}{2}\mathrm{tr}\left(\not{P}\gamma^\mu iW_1\gamma^5\gamma_\mu\not{P}\gamma^\nu iW_1\gamma^5\gamma_\nu\right)\\
&\qquad+\frac{1}{2}\mathrm{tr}\left(\not{P}\gamma^\mu iW_1\gamma^5\gamma^\nu\not{P}\gamma_\mu iW_1\gamma^5\gamma_\nu\right)\Big]\\
= &\ i\left(2m^4\mathcal{I}_i^4-16m^2\mathcal{I}[q^2]_i^4+[48-4\epsilon]\,\mathcal{I}[q^4]_i^4\right)\mathrm{tr}\left(P_\mu W_1 P_\mu W_1\right),
\end{aligned}
\tag{3.19}
$$

where the loop integral $\mathcal{I}[q^4]_i^4$ is divergent and thus we evaluated the Dirac trace in $D = 4 - \epsilon$ dimensions and kept the $\mathcal{O}(\epsilon)$ terms. Note that we have omitted diagrams where the two $\not{P}$ insertions are adjacent, because they lead to terms proportional to $\mathrm{tr}(\dots P^2 \dots)$, which provide redundant information for constructing independent operators as discussed above. Finally, we re-write the operator structures in Eq. (3.19) in terms of commutators, using

$$
2f_N^{(P^2W_1^2)}\,\mathrm{tr}\left(P_\mu W_1 P_\mu W_1\right) \supset f_N^{(P^2W_1^2)}\,\mathrm{tr}\left([P_\mu, W_1]\,[P_\mu, W_1]\right), \tag{3.20}
$$

and therefore obtain the final result

$$
\begin{aligned}
\mathcal{L}_{\mathrm{EFT}}^{1-\mathrm{loop}}[\phi] &\supset i\left(m^4\mathcal{I}_i^4-8m^2\mathcal{I}[q^2]_i^4+[24-2\epsilon]\,\mathcal{I}[q^4]_i^4\right)\mathrm{tr}\left([P_\mu, W_1]\,[P_\mu, W_1]\right)\\
&\supset \frac{i^2}{(4\pi)^2}\left(-\log\frac{m^2}{\mu^2}+\frac{2}{3}\right)\mathrm{tr}\left([P_\mu, W_1]\,[P_\mu, W_1]\right),
\end{aligned}
\tag{3.21}
$$

making use of the master integrals listed in Ref. [77].

### 3.2.4 Results for the universal coefficients

We now present the results of the calculation outlined above, listing here only the UOLEA operators with $P$, $W_0$ and $W_1$ terms where all fermions in the loop are degenerate in mass. In this case, there are 52 distinct operator structures in the UOLEA, and we tabulate their coefficients in Tables 3.4, 3.5, 3.6 and 3.7. The coefficients and operators containing only $P$'s can be found in Table 3.4. The operators contain the coupling with scalar structures $\mathbb{O}^{(W_0)}$, $\mathbb{O}^{(PW_0)}$ are tabulated in Table 3.5, while the coupling with pseudo-scalar structures $\mathbb{O}^{(W_1)}$, $\mathbb{O}^{(PW_1)}$ are in Table 3.6. Finally, the coefficients of the operators containing a mixture of scalar and pseudo-scalar structures $\mathbb{O}^{(PW_0W_1)}$ are listed in Table 3.7. Note that each universal coefficient in the Tables 3.4, 3.5, 3.6 and 3.7 has to be multiplied by the factor $i$, and that repeated Lorentz indices are implied to be contracted (though they are all written as subscripts for typographical convenience).

Results for the more general non-degenerate mass spectrum and including the vector ($V$) and axial-vector ($A$) structures in the degenerate case are lengthy, so we include them in a Mathematica notebook made available on `GitHub` [100]. Some of the UOLEA operators involving $V$ and $A$ that will be used in the examples in Sec. 3.3 are shown in Table 3.8.

For the user's convenience, we organised the Mathematica notebook as follows:

- We remind the user that the effective Lagrangian will be a summation of all universal operators we have tabulated in the Mathematica notebook. The coefficient of each operator has to be multiplied by a factor of $i$. Afterward, we have to read off the value of the master integrals, as tabulated in Ref. [77]. We note that the coefficients include the $\mathcal{O}(\epsilon)$ terms that can cancel the $\frac{1}{\epsilon}$ pole from the loop integrals and yield finite contributions.

| $\mathbb{O}^{(P)}$ terms | |
|---|---|
| $-\frac{1}{2}\mathcal{I}_i^4 m_i^4 + 4m_i^2\mathcal{I}[q^2]_i^4 + (5\epsilon - 8)\mathcal{I}[q^4]_i^4$ | $[P_\mu, P_\nu][P_\mu, P_\nu]$ |
| $24m_i^2\mathcal{I}[q^4]_i^6 - 2m_i^4\mathcal{I}[q^2]_i^6 - 64\mathcal{I}[q^6]_i^6$ | $[P_\mu, [P_\mu, P_\nu]][P_\rho, [P_\rho, P_\nu]]$ |
| $-\frac{2}{3}\mathcal{I}_i^6 m_i^6 + 4m_i^4\mathcal{I}[q^2]_i^6 - \frac{128}{3}\mathcal{I}[q^6]_i^6$ | $[P_\mu, P_\nu][P_\nu, P_\rho][P_\rho, P_\mu]$ |

**Table 3.4:** *Pure gauge operator structures in the fermionic UOLEA.*

| $\mathbb{O}^{(W_0)}$ terms | |
|---|---|
| $4m_i\mathcal{I}_i$ | $W_0$ |
| $2\mathcal{I}_i^2 m_i^2 + (8 - 2\epsilon)\mathcal{I}[q^2]_i^2$ | $W_0^2$ |
| $\frac{4}{3}\mathcal{I}_i^3 m_i^3 + (16m_i - 4\epsilon m_i)\mathcal{I}[q^2]_i^3$ | $W_0^3$ |
| $\mathcal{I}_i^4 m_i^4 + 24m_i^2\mathcal{I}[q^2]_i^4 + (24 - 10\epsilon)\mathcal{I}[q^4]_i^4$ | $W_0^4$ |
| $\frac{4}{5}\mathcal{I}_i^5 m_i^5 + 96m_i\mathcal{I}[q^4]_i^5 + 32m_i^3\mathcal{I}[q^2]_i^5$ | $W_0^5$ |
| $\frac{2}{3}\mathcal{I}_i^6 m_i^6 + 240m_i^2\mathcal{I}[q^4]_i^6 + 40m_i^4\mathcal{I}[q^2]_i^6 + 128\mathcal{I}[q^6]_i^6$ | $W_0^6$ |

| $\mathbb{O}^{(PW_0)}$ terms | |
|---|---|
| $\mathcal{I}_i^4 m_i^4 + (24 - 10\epsilon)\mathcal{I}[q^4]_i^4$ | $[P_\mu, W_0][P_\mu, W_0]$ |
| $4\mathcal{I}_i^5 m_i^5 + 192m_i\mathcal{I}[q^4]_i^5 + 16m_i^3\mathcal{I}[q^2]_i^5$ | $W_0[P_\mu, W_0][P_\mu, W_0]$ |
| $-2\mathcal{I}_i^5 m_i^5 - 16m_i\mathcal{I}[q^4]_i^5 + 16m_i^3\mathcal{I}[q^2]_i^5$ | $W_0[P_\mu, P_\nu][P_\mu, P_\nu]$ |
| $4\mathcal{I}_i^6 m_i^6 + 432m_i^2\mathcal{I}[q^4]_i^6 + 36m_i^4\mathcal{I}[q^2]_i^6 + 192\mathcal{I}[q^6]_i^6$ | $W_0[P_\mu, W_0]W_0[P_\mu, W_0]$ |
| $6\mathcal{I}_i^6 m_i^6 + 576m_i^2\mathcal{I}[q^4]_i^6 + 60m_i^4\mathcal{I}[q^2]_i^6 + 576\mathcal{I}[q^6]_i^6$ | $W_0^2[P_\mu, W_0][P_\mu, W_0]$ |
| $2\mathcal{I}_i^6 m_i^6 - 16m_i^2\mathcal{I}[q^4]_i^6 - 16m_i^4\mathcal{I}[q^2]_i^6$ | $[P_\mu, W_0][P_\nu, W_0][P_\mu, P_\nu]$ |
| $-5\mathcal{I}_i^6 m_i^6 + 72m_i^2\mathcal{I}[q^4]_i^6 + 36m_i^4\mathcal{I}[q^2]_i^6 - 64\mathcal{I}[q^6]_i^6$ | $W_0^2[P_\mu, P_\nu][P_\mu, P_\nu]$ |
| $-2\mathcal{I}_i^6 m_i^6 - 8m_i^2\mathcal{I}[q^4]_i^6 + 18m_i^4\mathcal{I}[q^2]_i^6 + 96\mathcal{I}[q^6]_i^6$ | $(W_0[P_\mu, W_0] - [P_\mu, W_0]W_0)\,[P_\nu, [P_\mu, P_\nu]]$ |
| $8m_i^2\mathcal{I}[q^4]_i^6 + 2m_i^4\mathcal{I}[q^2]_i^6 + 96\mathcal{I}[q^6]_i^6$ | $[P_\mu, [P_\mu, W_0]][P_\nu, [P_\nu, W_0]]$ |

**Table 3.5:** *Operator structures in the degenerate fermionic UOLEA involving the scalar coupling $W_0$.*

- In the first section of the Mathematica notebook, we summarise all universal structures as presented in the Tables 3.2 and 3.3 where each entry is hyperlinked such that a click takes the user directly to the table of operator structures and their corresponding coefficients.

| $\mathbb{O}^{(W_1)}$ terms | |
|---|---|
| $2(\epsilon+4)\mathcal{I}[q^2]_i^2 - 2\mathcal{I}_i^2 m_i^2$ | $W_1^2$ |
| $\mathcal{I}_i^4 m_i^4 - 8m_i^2\mathcal{I}[q^2]_i^4 + 2(11\epsilon+12)\mathcal{I}[q^4]_i^4$ | $W_1^4$ |
| $-\frac{2}{3}\mathcal{I}_i^6 m_i^6 - 48m_i^2\mathcal{I}[q^4]_i^6 + 8m_i^4\mathcal{I}[q^2]_i^6 + 128\mathcal{I}[q^6]_i^6$ | $W_1^6$ |

| $\mathbb{O}^{(PW_1)}$ terms | |
|---|---|
| $\mathcal{I}_i^4 m_i^4 - 8m_i^2\mathcal{I}[q^2]_i^4 - 2(\epsilon-12)\mathcal{I}[q^4]_i^4$ | $[P_\mu, W_1][P_\mu, W_1]$ |
| $24m_i\mathcal{I}[q^4]_i^5 - 8m_i^3\mathcal{I}[q^2]_i^5 + \mathcal{I}_i^5 m_i^5$ | $\epsilon_{\mu\nu\rho\sigma}\, W_1[P_\mu, P_\nu][P_\rho, P_\sigma]$ |
| $-48m_i^2\mathcal{I}[q^4]_i^6 + 4m_i^4\mathcal{I}[q^2]_i^6 + 192\mathcal{I}[q^6]_i^6$ | $W_1[P_\mu, W_1]W_1[P_\mu, W_1]$ |
| $-2\mathcal{I}_i^6 m_i^6 - 192m_i^2\mathcal{I}[q^4]_i^6 + 28m_i^4\mathcal{I}[q^2]_i^6 + 576\mathcal{I}[q^6]_i^6$ | $W_1^2[P_\mu, W_1][P_\mu, W_1]$ |
| $-2\mathcal{I}_i^6 m_i^6 - 48m_i^2\mathcal{I}[q^4]_i^6 + 16m_i^4\mathcal{I}[q^2]_i^6$ | $[P_\mu, W_1][P_\nu, W_1][P_\mu, P_\nu]$ |
| $\mathcal{I}_i^6 m_i^6 + 56m_i^2\mathcal{I}[q^4]_i^6 - 12m_i^4\mathcal{I}[q^2]_i^6 - 64\mathcal{I}[q^6]_i^6$ | $W_1^2[P_\mu, P_\nu][P_\mu, P_\nu]$ |
| $-24m_i^2\mathcal{I}[q^4]_i^6 + 2m_i^4\mathcal{I}[q^2]_i^6 + 96\mathcal{I}[q^6]_i^6$ | $[P_\mu, [P_\mu, W_1]][P_\nu, [P_\nu, W_1]]$ |
| $-24m_i^2\mathcal{I}[q^4]_i^6 + 2m_i^4\mathcal{I}[q^2]_i^6 + 96\mathcal{I}[q^6]_i^6$ | $(W_1[P_\mu, W_1] - [P_\mu, W_1]W_1)\,[P_\nu, [P_\mu, P_\nu]]$ |

**Table 3.6:** *Operator structures in the degenerate fermionic UOLEA involving the pseudoscalar coupling $W_1$.*

| $\mathbb{O}^{(W_0W_1)}$ terms | |
|---|---|
| $4(3\epsilon+4)m_i\mathcal{I}[q^2]_i^3 - 4\mathcal{I}_i^3 m_i^3$ | $W_0W_1^2$ |
| $8(\epsilon+12)\mathcal{I}[q^4]_i^4 - 4\mathcal{I}_i^4 m_i^4$ | $W_0^2W_1^2$ |
| $-2\mathcal{I}_i^4 m_i^4 + 16m_i^2\mathcal{I}[q^2]_i^4 + 4(5\epsilon-12)\mathcal{I}[q^4]_i^4$ | $W_0W_1W_0W_1$ |
| $-4\mathcal{I}_i^5 m_i^5 + 288m_i\mathcal{I}[q^4]_i^5 - 32m_i^3\mathcal{I}[q^2]_i^5$ | $W_0^3W_1^2$ |
| $-4\mathcal{I}_i^5 m_i^5 - 96m_i\mathcal{I}[q^4]_i^5 + 32m_i^3\mathcal{I}[q^2]_i^5$ | $W_0^2W_1W_0W_1$ |
| $4\mathcal{I}_i^5 m_i^5 + 96m_i\mathcal{I}[q^4]_i^5 - 32m_i^3\mathcal{I}[q^2]_i^5$ | $W_0W_1^4$ |
| $-4\mathcal{I}_i^6 m_i^6 + 96m_i^2\mathcal{I}[q^4]_i^6 + 16m_i^4\mathcal{I}[q^2]_i^6 - 768\mathcal{I}[q^6]_i^6$ | $W_0^3W_1W_0W_1$ |
| $-2\mathcal{I}_i^6 m_i^6 - 144m_i^2\mathcal{I}[q^4]_i^6 + 24m_i^4\mathcal{I}[q^2]_i^6 + 384\mathcal{I}[q^6]_i^6$ | $W_0^2W_1W_0^2W_1$ |
| $-4\mathcal{I}_i^6 m_i^6 + 480m_i^2\mathcal{I}[q^4]_i^6 - 80m_i^4\mathcal{I}[q^2]_i^6 + 768\mathcal{I}[q^6]_i^6$ | $W_0^4W_1^2$ |
| $4\mathcal{I}_i^6 m_i^6 + 288m_i^2\mathcal{I}[q^4]_i^6 - 48m_i^4\mathcal{I}[q^2]_i^6 - 768\mathcal{I}[q^6]_i^6$ | $W_0W_1W_0W_1^3$ |
| $2\mathcal{I}_i^6 m_i^6 - 48m_i^2\mathcal{I}[q^4]_i^6 - 8m_i^4\mathcal{I}[q^2]_i^6 + 384\mathcal{I}[q^6]_i^6$ | $W_0W_1^2W_0W_1^2$ |
| $4\mathcal{I}_i^6 m_i^6 - 96m_i^2\mathcal{I}[q^4]_i^6 - 16m_i^4\mathcal{I}[q^2]_i^6 + 768\mathcal{I}[q^6]_i^6$ | $W_0^2W_1^4$ |

| $\mathbb{O}^{(PW_0W_1)}$ terms | |
|---|---|
| $48m_i\mathcal{I}[q^4]_i^5 - 8m_i^3\mathcal{I}[q^2]_i^5$ | $W_1[P_\mu, W_0][P_\mu, W_1] + \text{h.c.}$ |
| $4\mathcal{I}_i^5 m_i^5 + 96m_i\mathcal{I}[q^4]_i^5 - 32m_i^3\mathcal{I}[q^2]_i^5$ | $W_0[P_\mu, W_1][P_\mu, W_1]$ |
| $24m_i^2\mathcal{I}[q^4]_i^6 - 8m_i^4\mathcal{I}[q^2]_i^6 + \mathcal{I}_i^6 m_i^6$ | $\epsilon_{\mu\nu\rho\sigma}\, W_0W_1[P_\mu, P_\nu][P_\rho, P_\sigma] + \text{h.c.}$ |
| $24m_i^2\mathcal{I}[q^4]_i^6 - 8m_i^4\mathcal{I}[q^2]_i^6 + \mathcal{I}_i^6 m_i^6$ | $\epsilon_{\mu\nu\rho\sigma}\, W_0[P_\mu, P_\nu]W_1[P_\rho, P_\sigma]$ |
| $2\mathcal{I}_i^6 m_i^6 + 192m_i^2\mathcal{I}[q^4]_i^6 - 28m_i^4\mathcal{I}[q^2]_i^6 - 576\mathcal{I}[q^6]_i^6$ | $W_1W_0[P_\mu, W_1][P_\mu, W_0] + \text{h.c.}$ |
| $48m_i^2\mathcal{I}[q^4]_i^6 - 4m_i^4\mathcal{I}[q^2]_i^6 - 192\mathcal{I}[q^6]_i^6$ | $W_1[P_\mu, W_0]W_1[P_\mu, W_0]$ |
| $4\mathcal{I}_i^6 m_i^6 + 144m_i^2\mathcal{I}[q^4]_i^6 - 36m_i^4\mathcal{I}[q^2]_i^6 - 192\mathcal{I}[q^6]_i^6$ | $W_0[P_\mu, W_1]W_0[P_\mu, W_1]$ |
| $96m_i^2\mathcal{I}[q^4]_i^6 - 24m_i^4\mathcal{I}[q^2]_i^6 + 384\mathcal{I}[q^6]_i^6$ | $W_1[P_\mu, W_1]W_0[P_\mu, W_0] + \text{h.c.}$ |
| $6\mathcal{I}_i^6 m_i^6 - 36m_i^4\mathcal{I}[q^2]_i^6 + 576\mathcal{I}[q^6]_i^6$ | $W_0^2[P_\mu, W_1][P_\mu, W_1]$ |
| $-2\mathcal{I}_i^6 m_i^6 - 4m_i^4\mathcal{I}[q^2]_i^6 + 576\mathcal{I}[q^6]_i^6$ | $W_0W_1[P_\mu, W_1][P_\mu, W_0] + \text{h.c.}$ |
| $-2\mathcal{I}_i^6 m_i^6 - 4m_i^4\mathcal{I}[q^2]_i^6 + 576\mathcal{I}[q^6]_i^6$ | $W_1^2[P_\mu, W_0][P_\mu, W_0]$ |

**Table 3.7:** *Operator structures in the degenerate fermionic UOLEA involving both the scalar coupling $W_0$ and the pseudoscalar coupling $W_1$.*

- In the following sections, we present the full results in both degenerate and non-degenerate cases where the coupling matrix $X[\phi]$ contains only scalar and pseudo-scalar structures.
- Finally, we present the full results in the degenerate case including the $V$ and $A$ structures.

| $\mathbb{O}^{(P^2A^2W_1)}$ terms | |
|---|---|
| $4m_i^5\mathcal{I}_i^5 - 16m_i^3\mathcal{I}[q^2]_i^5$ | $\epsilon^{\mu\nu\rho\sigma}P_\mu A_\nu P_\rho A_\sigma W_1 + \text{h.c.}$ |
| $-4m_i^5\mathcal{I}_i^5 + 16m_i^3\mathcal{I}[q^2]_i^5$ | $\epsilon^{\mu\nu\rho\sigma}P_\mu P_\nu A_\rho A_\sigma W_1 + \text{h.c.}$ |
| $4m_i^5\mathcal{I}_i^5 - 96m_i\mathcal{I}[q^4]_i^5$ | $\epsilon^{\mu\nu\rho\sigma}P_\mu W_1 P_\nu A_\rho A_\sigma$ |
| $4m_i^5\mathcal{I}_i^5 - 32m_i^3\mathcal{I}[q^2]_i^5 + 96m_i\mathcal{I}[q^4]_i^5$ | $\epsilon^{\mu\nu\rho\sigma}P_\mu P_\nu A_\rho W_1 A_\sigma$ |

| $\mathbb{O}^{(P^2V^2W_1)}$ terms | |
|---|---|
| $-4m_i^5\mathcal{I}_i^5 + 32m_i^3\mathcal{I}[q^2]_i^5 - 96m_i\mathcal{I}[q^4]_i^5$ | $\epsilon^{\mu\nu\rho\sigma}P_\mu W_1 P_\nu V_\rho V_\sigma$ |
| $-4m_i^5\mathcal{I}_i^5 + 32m_i^3\mathcal{I}[q^2]_i^5 - 96m_i\mathcal{I}[q^4]_i^5$ | $\epsilon^{\mu\nu\rho\sigma}P_\mu P_\nu V_\rho W_1 V_\sigma$ |
| $4m_i^5\mathcal{I}_i^5 - 32m_i^3\mathcal{I}[q^2]_i^5 + 96m_i\mathcal{I}[q^4]_i^5$ | $\epsilon^{\mu\nu\rho\sigma}P_\mu P_\nu V_\rho V_\sigma W_1 + \text{h.c.}$ |
| $4m_i^5\mathcal{I}_i^5 - 32m_i^3\mathcal{I}[q^2]_i^5 + 96m_i\mathcal{I}[q^4]_i^5$ | $\epsilon^{\mu\nu\rho\sigma}P_\mu V_\nu P_\rho V_\sigma W_1 + \text{h.c.}$ |

| $\mathbb{O}^{(P^3VW_1)}$ terms | |
|---|---|
| $-4m_i^5\mathcal{I}_i^5 + 32m_i^3\mathcal{I}[q^2]_i^5 - 96m_i\mathcal{I}[q^4]_i^5$ | $\epsilon^{\mu\nu\rho\sigma}P_\mu P_\nu P_\rho V_\sigma W_1 + \text{h.c.}$ |
| $-4m_i^5\mathcal{I}_i^5 + 32m_i^3\mathcal{I}[q^2]_i^5 - 96m_i\mathcal{I}[q^4]_i^5$ | $\epsilon^{\mu\nu\rho\sigma}P_\mu P_\nu V_\rho P_\sigma W_1 + \text{h.c.}$ |

**Table 3.8:** *Subset operator structures in the degenerate fermionic UOLEA involving the pseudoscalar, vector and axial-vector structures. This subset will be used in the various examples we present in Sec. 3.3.*

Due to a large number of combinations, we divide this section into subcategories: vector only, axial-vector only, and mixed vector/axial-vector. We also note that the results for mixed structures are written in functional space with open covariant derivatives. Depending on the effective operators one needs to construct, a subset of operators in the UOLEA will need to be selected and reorganized into the form of commutators. The non-degenerate results are available upon request.

- We use the same notation in the Mathematica notebook as in the Eq. (3.12) where `P`,`W0`, `W1` stand for the covariant derivative, scalar, and pseudo-scalar structures, respectively. To avoid conflict with other Mathematica packages, we denote $\bar{v}^b$, $\bar{a}^b$ for vector and axial-vector structures. We follow the conventions of Ref. [28] for $\gamma^5$ and the total anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$, $\epsilon^{0123} = +1$. The trace of Dirac matrices is evaluated using the `FeynCalc` package [107–109] and thus the output operator structures are also written in the language of this package.
- Regarding the hermiticity of the operator structures, the operators which are not self-hermitian need to be accompanied with their hermitian conjugates. The non-self-hermitian operators appear with "+ h.c." in the table of operators. We also checked that the operator and its hermitian conjugate have the same coefficients that result from the process of functional matching computations.

## 3.3 Examples

In this Section we present a few examples involving the top quark, as a cross-check of our results and to illustrate concretely how to use the fermionic UOLEA for practical calculations.

### 3.3.1 Integrating out the top quark in the Standard Model

In the broken phase of the electroweak symmetry, the terms quadratic in the top quark field interacting with the SM Higgs via a Yukawa interaction are

$$\mathcal{L}_{\text{SM}} \supset \bar{t}\left(i\partial_\mu - g_s G_\mu^a T^a - eQ_t F_\mu\right)\gamma^\mu t - m_t \bar{t}t - \frac{y_t}{\sqrt{2}} h\bar{t}t\,, \tag{3.22}$$

where $G^a_\mu$ is the gluon field, $T^a$ is the $SU(3)_c$ generator, and $F_\mu$ is the notation chosen for the photon field so as to avoid confusion with the axial-vector matrix $A_\mu$.

The above Lagrangian can be written in the canonical form that provides the starting point for a UOLEA analysis as

$$\mathcal{L}_{\text{SM}}^{(\text{UOLEA form})} \supset \bar{t}\,(\gamma^\mu P_\mu - m_t - W_0)\,t\ , \tag{3.23}$$

where, for this example, the covariant derivative $P_\mu$ and the coupling matrix $W_0$ are

$$P_\mu = iD_\mu = i\partial_\mu - g_s G^a_\mu T^a - eQ_t F_\mu\,, \quad W_0 = \frac{y_t}{\sqrt{2}} h. \tag{3.24}$$

We focus on the following operators in the EFT Lagrangian: $h$, $h^2$, $(\partial_\mu h)^2$, $h\,F_{\mu\nu}F^{\mu\nu}$ and $h\,G^a_{\mu\nu}G^{a,\mu\nu}$. This selects the following relevant terms in the UOLEA:

$$\begin{aligned}\mathcal{L}_{\text{EFT}} \supset -\frac{1}{(4\pi)^2}\Bigg[&4m_t^3\left(1-\log\frac{m_t^2}{\mu^2}\right)\text{tr}W_0 + 2m_t^2\left(1-3\log\frac{m_t^2}{\mu^2}\right)\text{tr}W_0^2 \\ &-\left(\frac{2}{3}+\log\frac{m_t^2}{\mu^2}\right)\text{tr}[P_\mu,W_0][P_\mu,W_0] + \left(\frac{2}{3m_t}\right)\text{tr}\big([P_\mu,P_\nu][P_\mu,P_\nu]W_0\big)\Bigg]\,, \end{aligned}\tag{3.25}$$

where the coefficient of each operator in Eq. (3.25) can be found in Table 3.5, and note that we must multiply those coefficients by $i$. To obtain the pre-computed coefficients in Eq. (3.25), we must retain the $1/\epsilon$ poles in the master integrals. These poles can be multiplied by the $\epsilon$ terms appearing in the prefactor multiplying the master integral coming from the trace over gamma matrices in the operator. For example,

$$\begin{aligned}(8-2\epsilon)\mathcal{I}[q^2]^2_i &= (8-2\epsilon)\frac{m_t^2}{2}\left(1-\log\frac{m_t^2}{\mu^2}+\frac{2}{\epsilon}-\gamma_E+\log 4\pi\right) \\ &= 4m_t^2\left(1-\log\frac{m_t^2}{\mu^2}\right) - 2m_t^2\ ,\end{aligned}\tag{3.26}$$

where in going from the first to the second line, we take the limit $\epsilon \to 0$ and drop the terms $2/\epsilon - \gamma_E + \log 4\pi$, since we use the $\overline{\text{MS}}$-scheme for renormalisation.

Next, we evaluate the trace over all internal indices, which in this case corresponds to the colour and $SU(3)_c$ indices carried by the top quark and gluon fields respectively, obtaining

$$\begin{aligned}&\text{tr}W_0 = \text{tr}\,\frac{y_t}{\sqrt{2}}h\,\delta_{ab} = N_c\frac{y_t}{\sqrt{2}}h\,, \quad \text{tr}W_0^2 = \text{tr}\,\frac{y_t^2}{2}h^2\,\delta_{ab}\delta_{ba} = N_c\frac{y_t^2}{2}h^2\ , \\ &\text{tr}[P_\mu,W_0][P_\mu,W_0] = \text{tr}\left[i\partial_\mu - g_sG^a_\mu T^a - eQ_tF_\mu, \frac{y_t}{\sqrt{2}}h\,\delta_{ab}\right]\left[i\partial_\mu - g_sG^a_\mu T^a - eQ_tF_\mu, \frac{y_t}{\sqrt{2}}h\,\delta_{ba}\right] \\ &\qquad = -N_c\frac{y_t^2}{2}(\partial_\mu h)^2\ .\end{aligned}\tag{3.27}$$

The field strength tensors can be obtained by using $\big[P_\mu,P_\nu\big] = i\left(-g_sG^a_{\mu\nu}T^a\right) + i\left(-eQ_tF_{\mu\nu}\right)$,

$$\begin{aligned}\text{tr}\left([P_\mu,P_\nu][P_\mu,P_\nu]W_0\right) &= \text{tr}\Bigg[\Big((-ig_s)^2G^a_{\mu\nu}G^b_{\mu\nu}T^aT^b + (-ieQ_t)^2F_{\mu\nu}F_{\mu\nu} \\ &\qquad - 2(g_s\,eQ_t)G^a_{\mu\nu}T^aF_{\mu\nu}\Big)\left(\frac{y_t}{\sqrt{2}}h\,\delta^{cd}\right)\Bigg] \\ &= -\left(N_c\,g_s^2\frac{y_t}{2\sqrt{2}}\right)hG^a_{\mu\nu}G^a_{\mu\nu} - \left(N_c\,(eQ_t)^2\frac{y_t}{\sqrt{2}}\right)hF_{\mu\nu}F_{\mu\nu}\ ,\end{aligned}\tag{3.28}$$

where $\mathrm{tr}\left(T^aT^b\right) = \delta^{ab}/2$ for generators of the fundamental representation of an $SU(N)$ gauge group. Inserting Eqs. (3.27), (3.28) into Eq. (3.25), we obtain

$$\begin{aligned}\mathcal{L}_{\mathrm{EFT}} \supset \frac{-1}{(4\pi)^2}\Bigg[\, & y_t^2\, N_c \left(\frac{1}{3}+\frac{1}{2}\log\frac{m_t^2}{\mu^2}\right)(\partial_\mu h)^2 \\ & + 4\frac{y_t}{\sqrt{2}} N_c\, m_t^3 \left(1-\log\frac{m_t^2}{\mu^2}\right) h + y_t^2\, N_c\, m_t^2\left(1-3\log\frac{m_t^2}{\mu^2}\right)h^2\Bigg] \\ + \left(\frac{y_t}{\sqrt{2}}\right)\Bigg[ & \frac{g_s^2}{48\pi^2\, m_t} N_c\, h G^a_{\mu\nu}G^a_{\mu\nu} + \frac{e^2 Q_t^2}{24\pi^2\, m_t} N_c\, h F_{\mu\nu}F_{\mu\nu}\Bigg] \;, \end{aligned} \tag{3.29}$$

where $N_c = 3$, $Q_t = 2/3$. The kinetic term for the Higgs may then be canonically normalised by a suitable field redefinition. The results of the first two lines of Eq. (3.29) agree with those of Ref. [6]. The third line agrees with the results of Ref. [110, 111].

### 3.3.2 Integrating out the top quark coupling to a light pseudo-scalar Higgs $A^0$

#### 3.3.2.1 The effective coupling $A^0\gamma\gamma$

In this example, we consider the top quark with a coupling to a light pseudo-scalar, denoted $A^0$. We assume this field is lighter than the top quark, so that we may integrate the latter out in order to obtain the Wilson coefficient for the dimension−5 operator coupling between $A^0$ and two photons. We assume a coupling structure of the pseudo-scalar to the top quark taking the same form as in the type II Two Higgs Doublet Model (2HDM) or the MSSM. Note, however, that our result may be generalised to any model involving a pseudo-scalar coupling to the top quark, by a simple rescaling.

The terms in the UV Lagrangian relevant for computing the effective $A^0\gamma\gamma$ coupling can be written in the form

$$\mathcal{L}_{\mathrm{UV}} \supset \bar{t}\left[\left(i\partial_\mu - eQ_tF_\mu\right)\gamma^\mu - m_t + i\frac{m_t}{v}\cot\beta A^0\gamma^5\right]t, \tag{3.30}$$

where $g/2M_W = 1/v$ and we use the notation of the 2HDM of type II, $\tan\beta = v_1/v_2$ with $v = \sqrt{v_1^2+v_2^2}$.[6]

Upon integrating out the top-quark, we know that the effective interaction $A^0\gamma\gamma$ should be of the form

$$\mathcal{L}_{\mathrm{EFT}} \supset C_{A^0\gamma\gamma}\, A^0 F_{\mu\nu}\tilde{F}^{\mu\nu} \;, \tag{3.31}$$

where our convention for the dual field strength tensor is $\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, with $\epsilon^{0123} = +1$. The aim of this example is therefore to compute $C_{A^0\gamma\gamma}$ arising from the heavy top quark loop.

**UV Lagrangian in the UOLEA form:**

Before using the pre-computed coefficients from the tables above, we need to write the UV Lagrangian in Eq. (3.30) in the UOLEA form in terms of the relevant structures, which in this case comprises of only $W_1$ in addition to the covariant derivative,

$$\mathcal{L}_{\mathrm{UV}}^{(\mathrm{UOLEA\ form})} \supset \bar{t}\left[P_\mu\gamma^\mu - m_t - i\gamma^5 W_1\right]t, \tag{3.32}$$

[6]Note also the use of $F_\mu$ for the photon field, to avoid confusion with the axial-vector coupling matrix $A_\mu$.

where the covariant derivative $P_\mu$ (omitting the gluon piece which does not contribute) and the coupling $W_1$ are

$$P_\mu \supset i\partial_\mu - eQ_t F_\mu\,, \quad W_1 = -\frac{m_t}{v}\cot\beta A^0\ . \tag{3.33}$$

Clearly, the existence of only these two structures means we will only need operators and coefficients from Table 3.6 above in order to compute $C_{A^0\gamma\gamma}$. Furthermore, we know that since both $P_\mu$ and $W_1$ are dimension 1, we will need only operators from the table of dimension 5 to form the EFT Lagrangian operator. While in this example the power counting may seem superfluous since we are only interested in one operator with a transparent structure, in more complicated examples this counting can be extremely helpful.

**Relevant structures in the UOLEA:**

Now, referring to Table 3.6, we can immediately identify the necessary combinations of $P_\mu$ and $W_1$ that will form the effective operator $A^0F_{\mu\nu}\tilde{F}^{\mu\nu}$, along with their universal coefficients (recalling that we must multiply the coefficients from the table by $i$). This therefore yields the effective Lagrangian as obtained from the UOLEA

$$\begin{aligned}\mathcal{L}_{\rm EFT} &\supset i\left(m_t^5\mathcal{I}_i^5 - 8m_t^3\mathcal{I}[q^2]_i^5 + 24m_t\mathcal{I}[q^4]_i^5\right)\mathrm{tr}\epsilon^{\mu\nu\rho\sigma}W_1[P_\mu,P_\nu][P_\rho,P_\sigma]\\ &= \frac{1}{32\pi^2 m_t}\mathrm{tr}\left(\epsilon^{\mu\nu\rho\sigma}\,W_1[P_\mu,P_\nu][P_\rho,P_\sigma]\right)\ .\end{aligned} \tag{3.34}$$

The trace over the internal indices is then evaluated, to obtain

$$\begin{aligned}\mathrm{tr}\left(\epsilon^{\mu\nu\rho\sigma}W_1[P_\mu,P_\nu][P_\rho,P_\sigma]\right) &= \mathrm{tr}\left(\left[-\frac{m_t}{v}\cot\beta A^0\right](-ieQ_t)^2\delta_{ab}\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)\\ &= 2\,\frac{m_t}{v}\cot\beta(eQ_t)^2N_c\,A^0F_{\mu\nu}\tilde{F}_{\mu\nu}\ ,\end{aligned} \tag{3.35}$$

where we have used that the commutator $[P_\mu,P_\nu]\phi = i(-eQ_t)F_{\mu\nu}\phi$. Putting the two pieces together, we thus obtain the EFT operator corresponding to the effective interaction $A^0\gamma\gamma$,

$$\mathcal{L}_{\rm EFT} \supset \frac{e^2}{16\pi^2 v}Q_t^2N_c\cot\beta A^0F_{\mu\nu}\tilde{F}_{\mu\nu}\ . \tag{3.36}$$

Comparing Eqs. (3.31) and (3.36), we conclude that

$$C_{A^0\gamma\gamma} = \frac{e^2}{16\pi^2 v}Q_t^2N_c\cot\beta\,. \tag{3.37}$$

We have checked that this agrees with the result obtained by the usual Feynman diagram derivation. Eq. (3.36) also matches the one in Refs. [112–114] once the different convention for the dual field strength used in those references, $\tilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$, is taken into account. In contrast to the Feynman diagram computation, here the effective operator and its Wilson coefficient were trivially obtained using the pre-calculated universal results of the UOLEA and the simple evaluation of a trace over internal indices.

#### 3.3.2.2 The effective coupling $A^0ZZ$

We next consider a more complicated matching procedure than in the previous examples. Indeed, since we wish to obtain the coefficient of the dimension-5 operator coupling the pseudo-scalar $A^0$

to $Z$ bosons, it is immediately apparent that we will now need to make use of the vector and axial-vector coupling matrices $V_\mu$ and $A_\mu$.

The relevant terms in the Lagrangian are

$$\mathcal{L}_{\rm UV} \supset \bar{t}\left[(i\partial_\mu)\,\gamma^\mu - m_t + \left(i\frac{m_t}{v}\cot\beta A^0\right)\gamma^5 \right. \\ \left. -\frac{g}{\cos\theta_w}\left(\frac{T_3}{2} - Q_t\sin^2\theta\right)Z_\mu\gamma^\mu + \left(\frac{g}{\cos\theta_w}\frac{T_3}{2}\right)Z_\mu\gamma^\mu\gamma^5\right]t\,, \tag{3.38}$$

where we used the same conventions as in Ref.[114, 115]. Meanwhile, the effective Lagrangian for the $A^0ZZ$ effective coupling is

$$\mathcal{L}_{\rm EFT} \supset C_{A^0ZZ}A^0 Z_{\mu\nu}\tilde{Z}^{\mu\nu}, \tag{3.39}$$

where $\tilde{Z}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_{[\rho}Z_{\sigma]}$, with $\epsilon^{0123} = +1$.

**UV Lagrangian in the UOLEA form:**

As in the previous examples, we first re-write the UV Lagrangian in terms of the UOLEA structures,

$$\mathcal{L}_{\rm UV}^{\rm (UOLEA\ form)} \supset \bar{t}\left[P_\mu\gamma^\mu - m_t - i\gamma^5 W_1 - V_\mu\gamma^\mu - A_\mu\gamma^\mu\gamma^5\right]t, \tag{3.40}$$

where the $P_\mu$, $W_1$, $V_\mu$, $A_\mu$ are defined as

$$\begin{cases} P_\mu & \supset i\partial_\mu\,, \\ W_1 & = -\dfrac{m_t}{v}\cot\beta A^0\,, \\ V_\mu & = g_V Z_\mu\,; \quad g_V = \dfrac{g}{\cos\theta_w}\left(\dfrac{T_3}{2} - Q_t\sin^2\theta_w\right)\,, \\ A_\mu & = g_A Z_\mu\,; \quad g_A = -\left(\dfrac{g}{\cos\theta_w}\dfrac{T_3}{2}\right)\,. \end{cases} \tag{3.41}$$

We have dropped the gluon and photon pieces in the covariant derivative, since they do not contribute to the matching calculation here.

**Relevant structures in the UOLEA:**

Having identified the UOLEA structures that will appear in the construction of the EFT operator, we must now decompose the latter to determine what UOLEA structures will contribute. The EFT Lagrangian is

$$\mathcal{L}_{\rm EFT} \supset C_{A^0ZZ}A^0 Z_{\mu\nu}\tilde{Z}^{\mu\nu} = \frac{1}{2}C_{AZZ}A^0\epsilon^{\mu\nu\rho\sigma}\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right)\left(\partial_\rho Z_\sigma - \partial_\sigma Z_\rho\right) \tag{3.42}$$

Thus, to reconstruct the EFT operator in terms of UOLEA structures, we need

- One insertion of $W_1$ to account for the pseudo-scalar $A^0$.
- Two $P_\mu$ insertions to account for the partial derivatives.
- To account for the two $Z$ bosons, one might expect some combination of the structures $AA$, $VV$ or $AV$ to be required. However, due to the structure of the effective operator, it is clear that the product of the various UOLEA coupling matrices should have an odd number of $\gamma^5$ insertions. Since $W_1$ carries a $\gamma^5$, $AV$, which also has one $\gamma^5$, will not contribute.

The EFT Lagrangian will therefore be given by the following classes of UOLEA operators,

$$\mathcal{L}_{\text{EFT}} \supset \sum_N f_N \mathbb{O}_N^{(P^2V^2W_1)} + f_N \mathbb{O}_N^{(P^2A^2W_1)} . \tag{3.43}$$

Note that since we are integrating out the top-quark coupling to a single type of vector boson, the $W_1$, $V_\mu$, $A_\mu$ terms are proportional to the identity matrix and so commute with each other, which will simplify the calculation. Owing to the commutativity of $W_1$ with $A_\mu$ and $V_\mu$, the combination of all the $\mathbb{O}^{(P^2A^2W_1)}$ and $\mathbb{O}^{(P^2V^2W_1)}$ UOLEA operators of Table 3.8 can be written as

$$\begin{aligned}\mathcal{L}_{\text{EFT}} \supset\, & f_1 \text{tr}\left(\epsilon^{\mu\nu\rho\sigma} P_\mu P_\nu V_\rho V_\sigma W_1\right) + f_2 \text{tr}\left(\epsilon^{\mu\nu\rho\sigma} P_\mu W_1 P_\nu V_\rho V_\sigma\right) \\ & + f_3 \text{tr}\left(\epsilon^{\mu\nu\rho\sigma} P_\mu P_\nu A_\rho A_\sigma W_1\right) + f_4 \text{tr}\left(\epsilon^{\mu\nu\rho\sigma} P_\mu W_1 P_\nu A_\rho A_\sigma\right) ,\end{aligned} \tag{3.44}$$

where the values of the universal coefficients are

$$\begin{aligned}f_1 &= i\left(4m_i^5 \mathcal{I}_i^5 - 32 m_i^3 \mathcal{I}[q^2]_i^5 + 96 m_i \mathcal{I}[q^4]_i^5\right) = \frac{1}{8\pi^2 m_t} , \\ f_2 &= i\left(-4m_i^5 \mathcal{I}_i^5 + 32 m_i^3 \mathcal{I}[q^2]_i^5 - 96 m_i \mathcal{I}[q^4]_i^5\right) = \frac{-1}{8\pi^2 m_t} , \\ f_3 &= i\left(-4m_i^5 \mathcal{I}_i^5 + 96 m_i \mathcal{I}[q^4]_i^5\right) = \frac{1}{24\pi^2 m_t} , \\ f_4 &= i\left(4m_i^5 \mathcal{I}_i^5 - 96 m_i \mathcal{I}[q^4]_i^5\right) = \frac{-1}{24\pi^2 m_t} .\end{aligned} \tag{3.45}$$

These UOLEA operators and their coefficients can be read off from Table 3.8; a complete tabulation of UOLEA results for the degenerate vector and axial-vector case can be found in the accompanying Mathematica notebook, as described in Sec. 3.2.4. Due to the proliferation of UOLEA operators involving $V$ and $A$, these are not listed in a commutator basis. Instead, it is preferable to perform the rearrangement into the commutator basis for the small subset of operators contributing to a specific application. We will now demonstrate this for the effective $A^0ZZ$ coupling.

**Constructing the EFT operators:**

We begin with the vector structure case,

$$\mathcal{L}_{\text{EFT}}^{(\text{vector})} = f_1 \text{tr}\, \epsilon^{\mu\nu\rho\sigma} P_\mu P_\nu V_\rho V_\sigma W_1 + f_2 \text{tr}\, \epsilon^{\mu\nu\rho\sigma} P_\mu W_1\, P_\nu V_\rho V_\sigma , \tag{3.46}$$

and rearrange it into the following commutator basis (note that e.g. $[V, W_1] = 0$),

$$\begin{aligned}\mathcal{L}_{\text{EFT}}^{(\text{vector})} \supset\, & c_1 \text{tr}\, \epsilon^{\mu\nu\rho\sigma} W_1 \left[P_\mu, V_\nu\right]\left[P_\rho, V_\sigma\right] + c_2 \text{tr}\, \epsilon^{\mu\nu\rho\sigma} W_1 \left[P_\mu, P_\nu\right]\left[V_\rho, V_\sigma\right] \\ =\, & -c_1 \text{tr}\, \epsilon^{\mu\nu\rho\sigma} P_\mu W_1 P_\nu V_\rho V_\sigma + (4c_2 - c_1) \text{tr}\, \epsilon^{\mu\nu\rho\sigma} P_\mu P_\nu V_\rho V_\sigma W_1 .\end{aligned} \tag{3.47}$$

In the second line we have expanded the commutators, so that comparing with the non-commutator basis of Eqs.(3.46) then allows us to solve a system of linear equations that take us from the non-commutator to the commutator basis of operators,

$$\begin{cases} -c_1 = f_2 \\ 4c_2 - c_1 = f_1 \end{cases} \Leftrightarrow \begin{cases} c_1 = -f_2 = \dfrac{1}{8\pi^2 m_t} \\ c_2 = \dfrac{f_1 - f_2}{4} = \dfrac{1}{16\pi^2 m_t} \end{cases} \tag{3.48}$$

Using $[P_\mu, V_\nu] = i(\partial_\mu V_\nu)$, we may rewrite the operator $\mathrm{tr}\,\epsilon^{\mu\nu\rho\sigma}\,W_1[P_\mu, V_\nu][P_\rho, V_\sigma]$ as

$$\begin{aligned} c_1\,\mathrm{tr}\,\epsilon^{\mu\nu\rho\sigma}\,W_1[P_\mu, V_\nu][P_\rho, V_\sigma] &= (i^2)\,c_1\,\mathrm{tr}\,\epsilon^{\mu\nu\rho\sigma} W_1(\partial_\mu V_\nu)(\partial_\rho V_\sigma) \\ &= c_1\frac{i^2}{4}\mathrm{tr}\,W_1\,[\epsilon^{\mu\nu\rho\sigma}(\partial_\mu V_\nu)(\partial_\rho V_\sigma) + \epsilon^{\mu\nu\sigma\rho}(\partial_\mu V_\nu)(\partial_\sigma V_\rho) \\ &\qquad + \epsilon^{\nu\mu\rho\sigma}(\partial_\nu V_\mu)(\partial_\rho V_\sigma) + \epsilon^{\nu\mu\sigma\rho}(\partial_\nu V_\mu)(\partial_\sigma V_\rho)\,] \\ &= c_1\frac{i^2}{4}\mathrm{tr}\,\epsilon^{\mu\nu\rho\sigma} W_1[\partial_\mu V_\nu - \partial_\nu V_\mu][\partial_\rho V_\sigma - \partial_\sigma V_\rho]. \end{aligned} \tag{3.49}$$

Putting it all together, we obtain the contributions from the vector terms $\mathbb{O}^{(P^2V^2W_1)}$,

$$\begin{aligned} \mathcal{L}_{\mathrm{EFT}}^{(\mathrm{vector})} &\supset \frac{1}{8\pi^2 m_t}\frac{i^2}{4}\mathrm{tr}\,\epsilon^{\mu\nu\rho\sigma}\left(-\frac{m_t}{v}\cot\beta A^0\right)\delta_{ab}\,g_V^2\,Z_{\mu\nu}Z_{\rho\sigma} \\ &= \frac{1}{16\pi^2 v}N_c\cot\beta\frac{g^2}{\cos^2\theta_w}\left(\frac{T_3}{2} - Q_t\sin^2\theta_w\right)^2 A^0 Z_{\mu\nu}\tilde{Z}_{\mu\nu}\,. \end{aligned} \tag{3.50}$$

The computation for the UOLEA operators involving the axial-vector coupling matrix $A$ proceeds similarly. We find

$$\begin{aligned} \mathcal{L}_{\mathrm{EFT}}^{(\mathrm{axial\text{-}vector})} &\supset \frac{1}{24\pi^2 m_t}\frac{i^2}{4}\mathrm{tr}\,\epsilon^{\mu\nu\rho\sigma}\left(-\frac{m_t}{v}\cot\beta A^0\right)\delta_{ab}\,g_A^2 Z_{\mu\nu}Z_{\rho\sigma} \\ &= \frac{1}{48\pi^2 v}N_c\cot\beta\left(\frac{g}{\cos\theta_w}\frac{T_3}{2}\right)^2 A^0 Z_{\mu\nu}\tilde{Z}_{\mu\nu}. \end{aligned} \tag{3.51}$$

Adding (3.50) and (3.51) gives the final result,

$$\mathcal{L}_{\mathrm{EFT}} \supset \frac{1}{48\pi^2 v}N_c\cot\beta\frac{g^2}{\cos^2\theta_w}\left(T_3^2 + 3Q_t\sin^2\theta_w[Q_t\sin^2\theta_w - T_3]\right) A^0 Z_{\mu\nu}\tilde{Z}_{\mu\nu}. \tag{3.52}$$

This result agrees with the one in Ref.[112]. However, the calculation here is carried out in a more streamlined manner using the UOLEA.

#### 3.3.2.3 The effective coupling $A^0 Z\gamma$

To construct the effective coupling $A^0 Z\gamma$ resulting from integrating out the top quark coupling to a light pseudo-scalar $A^0$, we split the interaction with the $Z$ boson into vector and axial-vector currents. The relevant terms in the UV Lagrangian are then

$$\begin{aligned} \mathcal{L}_{\mathrm{UV}} \supset \bar{t}\Big[&(i\partial_\mu - eQ_tF_\mu)\gamma^\mu - m_t + \left(i\frac{m_t}{v}\cot\beta A^0\right)\gamma^5 \\ &-\frac{g}{\cos\theta_w}\left(\frac{T_3}{2} - Q_t\sin^2\theta_w\right)Z_\mu\gamma^\mu + \left(\frac{g}{\cos\theta_w}\frac{T_3}{2}\right)Z_\mu\gamma^\mu\gamma^5\Big]\,t\,, \end{aligned} \tag{3.53}$$

where $F_\mu$ denotes the photon field. We now integrate out the top quark to obtain the following $\mathcal{CP}$-even effective operator,

$$\mathcal{L}_{\mathrm{EFT}} \supset C_{A^0Z\gamma}A^0 Z_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{3.54}$$

**UV Lagrangian in the UOLEA form:**

We write the UV Lagrangian (3.53) in the canonical form,

$$\mathcal{L}_{\text{UV}}(\text{UOLEA form}) = \bar{t}\left[P_\mu\gamma^\mu - m_t - i\gamma^5 W_1 - V_\mu\gamma^\mu - A_\mu\gamma^\mu\gamma^5\right] t, \tag{3.55}$$

where the structures $P_\mu, W_1, V_\mu, A_\mu$ correspond to

$$\begin{cases} P_\mu & \supset i\partial_\mu - eQ_t F_\mu\,, \\ W_1 & = -\dfrac{m_t}{v}\cot\beta A^0\,, \\ V_\mu & = g_V Z_\mu\,; \quad g_V = \dfrac{g}{\cos\theta_w}\left(\dfrac{T_3}{2} - Q_t\sin^2\theta_w\right)\,, \\ A_\mu & = g_A Z_\mu\,; \quad g_A = -\left(\dfrac{g}{\cos\theta_w}\dfrac{T_3}{2}\right)\,. \end{cases} \tag{3.56}$$

Note that after the broken phase, our theory still respects $U(1)_{\text{QED}}$, thus the photon field still lives in the covariant derivative (together with the gluon field, which does not contribute in the present case and has been omitted), while the $Z$ boson should be put into the $V$ and $A$ structures.

**Relevant structures in the UOLEA:**

To obtain the EFT operator (3.54), we need:

- One insertion of $W_1$ to account for the appearance of $A^0$.
- Three insertions of $P_\mu$. Two of them form the photon field strength. The last one will act on the $Z_\mu$. Then combining with the anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ we can construct the dual field-strength tensor of the $Z$ boson.
- One insertion of $V$ to account for $Z_\mu$. As in the previous example, we can count $\gamma^5$ insertions to see that no operator involving $A$ can contribute to the EFT operator (3.54).

Putting it all together, the relevant class of UOLEA operators which contribute to the EFT operator (3.54) is then

$$\mathcal{L}_{\text{EFT}}^{(\text{UOLEA})} \supset \sum_N f_N \mathbb{O}_N^{(P^3VW_1)}\,. \tag{3.57}$$

Since in this case $[W_1, V_\mu] = 0$, we have only one UOLEA operator to consider,

$$\mathcal{L}_{\text{EFT}}^{(\text{vector})} = i\left(-4m_i^5\mathcal{I}_i^5 + 32m_i^3\mathcal{I}[q^2]_i^5 - 96m_i\mathcal{I}[q^4]_i^5\right)\left[\text{tr}\left(\epsilon^{\mu\nu\rho\sigma}P_\mu P_\nu V_\rho P_\sigma W_1\right) + \text{h.c.}\right], \tag{3.58}$$

we note that the operator structure $\left[\text{tr}\left(\epsilon^{\mu\nu\rho\sigma}P_\mu P_\nu P_\rho V_\sigma W_1\right) + \text{h.c.}\right]$ vanishes due to the antisymmetry of the $\epsilon^{\mu\nu\rho\sigma}$ tensor and $[W_1, V_\mu] = 0$. We then rearrange the operator structures in (3.58) into the basis where $P$'s only appear in the commutators. The operator structures we expect in the commutator basis are

$$\begin{aligned}\mathcal{L}_{\text{EFT}}^{(\text{vector})} &\supset f_1\big(\text{tr}\,\epsilon^{\mu\nu\rho\sigma}[P_\mu, P_\nu][P_\rho, V_\sigma]W_1 + \text{tr}\,\epsilon^{\mu\nu\rho\sigma}W_1[P_\mu, V_\nu][P_\rho, P_\sigma]\big) \\ &= 2f_1\big(\text{tr}\,\epsilon^{\mu\nu\rho\sigma}P_\mu P_\nu V_\rho P_\sigma W_1 + \text{tr}\,\epsilon^{\mu\nu\rho\sigma}P_\mu P_\nu W_1 P_\rho V_\sigma\big)\end{aligned} \tag{3.59}$$

As in the previous examples, we expand the commutators and, using the fact that $[W_1, V_\mu] = 0$, match with the non-commutator basis of Eq. (3.58) and fix the value of the coefficient $f_1$:

$$f_1 = i\,\frac{1}{2}\left(-4m_i^5\mathcal{I}_i^5 + 32m_i^3\mathcal{I}[q^2]_i^5 - 96m_i\mathcal{I}[q^4]_i^5\right) = \frac{-1}{16\pi^2 m_t}\,. \tag{3.60}$$

Plugging $P_\mu$, $V_\mu$ and $W_1$ from Eq. (3.56) into

$$\mathcal{L}_{\text{EFT}}^{(\text{vector})} \supset f_1\big(\,\text{tr}\,\epsilon^{\mu\nu\rho\sigma}\big[P_\mu, P_\nu\big]\big[P_\rho, V_\sigma\big]W_1 + \text{tr}\,\epsilon^{\mu\nu\rho\sigma}W_1\big[P_\mu, V_\nu\big]\big[P_\rho, P_\sigma\big]\,\big)\,, \tag{3.61}$$

and using $[P_\mu, P_\nu] = i\,(-eQ_t)\,F_{\mu\nu}$ and $[P_\mu, V_\nu] = ig_V\,(\partial_\mu Z_\nu)$, we obtain

$$\begin{aligned}\mathcal{L}_{\text{EFT}} &\supset f_1\,(g_V\,eQ_t)\,\text{tr}\left[2\epsilon^{\mu\nu\rho\sigma}\,(\partial_\mu Z_\nu)\,F_{\rho\sigma}W_1\right]\\ &= f_1\,(g_V\,eQ_t)\,\text{tr}\left[\epsilon^{\mu\nu\rho\sigma}\,(\partial_\mu Z_\nu)\,F_{\rho\sigma}W_1 + \epsilon^{\nu\mu\rho\sigma}\,(\partial_\nu Z_\mu)\,F_{\rho\sigma}W_1\right]\\ &= \frac{-1}{16\pi^2 m_t}\,(g_V\,eQ_t)\,\text{tr}\left[\left(-\frac{m_t}{v}\cot\beta A^0\,\delta^{ab}\right)\epsilon^{\mu\nu\rho\sigma}Z_{\mu\nu}F_{\rho\sigma}\right]\,.\end{aligned} \tag{3.62}$$

Taking the trace over colour degrees of freedom and using $g_V = \dfrac{g}{\cos\theta_w}\left(\dfrac{T_3}{2} - Q_t\sin^2\theta_w\right)$ from Eq. (3.56), we obtain the final result,

$$\mathcal{L}_{\text{EFT}} \supset \frac{1}{16\pi^2 v}N_c\cot\beta(eQ_t)\frac{g}{\cos\theta_w}\left(T_3 - 2Q_t\sin^2\theta_w\right)A^0 Z_{\mu\nu}\tilde{F}_{\mu\nu}\,. \tag{3.63}$$

This result agrees with the ones in Refs.[110, 112]. Once again we note the relative ease and efficiency with which the same result can be derived in the UOLEA.

## 3.4 Conclusion

The universality of the one-loop effective action obtained by integrating out heavy degrees of freedom has emerged as a byproduct of improved path integral methods for performing these calculations. This so-called UOLEA makes the repeated evaluation of functional determinants redundant and provides a more efficient way of matching at one loop compared to Feynman diagrams, especially when systematically obtaining an ensemble of operator coefficients at once. It also has the advantage of being easier to automate.

Previous work developed the bosonic UOLEA for integrating out heavy bosons, including mixed heavy-light loops. While these results could be used for integrating out fermions as well in some cases, they did not account for $\gamma$ matrices in the fermion couplings, and were also not as straightforward to use as in the bosonic case. It was therefore necessary to extend the UOLEA to the fermionic case, and desirable to do so in a way that maintained the simplicity of the UOLEA approach.

In this work we presented the fermionic UOLEA, which can be used for one-loop matching with heavy fermions (Dirac or Majorana) in the loop, coupling with structures involving $\gamma$ matrices. The starting point is the UV Lagrangian of Eq. (3.4), for which the UOLEA is given by Eq. (3.6). A subset of our results for the new UOLEA operators and the corresponding universal coefficients are tabulated in Tables 3.4, 3.5, 3.6, 3.7 and 3.8 for the degenerate mass case, while the full results, in the non-degenerate case for $P, W_0, W_1$ structures and in the degenerate case for $V, A$ structures, are available in the accompanying Mathematica notebook, [100].[7] These expressions can be readily incorporated into codes that automate the tracing over the internal indices and the rearranging of the resulting EFT operators into a non-redundant basis.[8]

The status of the UOLEA terms available and those that remain to be computed is summarised in Table 3.1. This is listed for completeness though we note that the majority of UV Lagrangian structures of interest are now included in the UOLEA for obtaining EFT operators up to dimension 6. Nevertheless, further efforts to complete the UOLEA, including all possible structures and extending to higher dimensional operators, would then enable and be a part of a fully general automated one-loop matching tool. This ambitious goal is left for future work.

[7]The non-degenerate results for $V, A$ structures can also be made available on request.

[8]See, for example, Ref. [116] for automated tree-level matching and Ref. [117] that implements the degenerate bosonic UOLEA results.

# Part II

# Chiral anomalies and Effective Field Theories

# Table of Contents

# Chapter 4

# Introduction to anomalies in QFT

This chapter is dedicated to introducing the anomalies in quantum field theory. For a complete review of this topic; see, for instance, Refs. [118–120]. An extension of this chapter will be presented in the forthcoming chapters, where we will discuss in-depth the regularization techniques to compute anomalies and how anomalies affect the construction of effective field theories.

## 4.1 Anomalies in QFT

The foundation of modern quantum field theory is based on the principle of gauge symmetry. There is an anomaly when the symmetry current is conserved at a classical level but no longer conserved at the quantum level. This anomaly is a signature for the breakdown of the quantized gauge symmetry, in other words, the ruin of the consistency of the gauge theory.

To understand how anomalies arise in the perturbative theory, let us present the simplest example, the axial anomaly (the ABJ anomaly [121, 122]). We aim to show how the axial current becomes anomalous at the one-loop level. We begin with the bilinear fermion terms in the massless QED Lagrangian,

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\gamma^{\mu}(i\partial_{\mu} + A_{\mu})\psi \ . \tag{4.1}$$

**Symmetries and conservation laws.** The massless QED Lagrangian given by Eq. (4.1) is invariant under the local vector gauge transformation $U(1)_V$,

$$\psi \to e^{i\alpha(x)}\psi \ , \quad \bar{\psi} \to \bar{\psi}e^{-i\alpha(x)} \ , \quad A_{\mu} \to A_{\mu} + \partial_{\mu}\alpha(x) \ . \tag{4.2}$$

In addition, this Lagrangian also remains invariant under the global axial transformation $U(1)_A$,

$$\psi \to e^{i\beta\gamma^5}\psi \ , \quad \bar{\psi} \to \bar{\psi}e^{i\beta\gamma^5} \ . \tag{4.3}$$

From Noether's theorem, we know conservation laws are connected with symmetries. The symmetry currents respectively associated with the $U(1)_V$ and $U(1)_A$ transformations are

$$J_V^{\mu} = \bar{\psi}\gamma^{\mu}\psi \ , \quad J_5^{\mu} = \bar{\psi}\gamma^{\mu}\gamma^5\psi \ . \tag{4.4}$$

With the help of Dirac equations, at the classical level (at the massless fermion limit), one can derive the conservation laws for the vector and axial currents as follows,

$$\partial_{\mu}J_V^{\mu} = 0 \ , \quad \partial_{\mu}J_5^{\mu} = 0 \ . \tag{4.5}$$

We will show that the axial current is no longer conserved at the quantum level. We then define the axial anomaly as follows,

$$\partial_\mu J_5^\mu = \mathcal{A} \ . \tag{4.6}$$

In our toy example, $\mathcal{A}$ is precisely the value ABJ anomaly. Explicitly, we will prove that

$$\mathcal{A} = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \ . \tag{4.7}$$

**Anomaly from the triangle Feynman diagrams.** In the momentum space, the derivative of the symmetry current is replaced by the Ward-Takahashi identity. At one-loop order, the derivative of axial current received the contributions from the well-known AVV triangle diagrams given by Fig. (4.1).

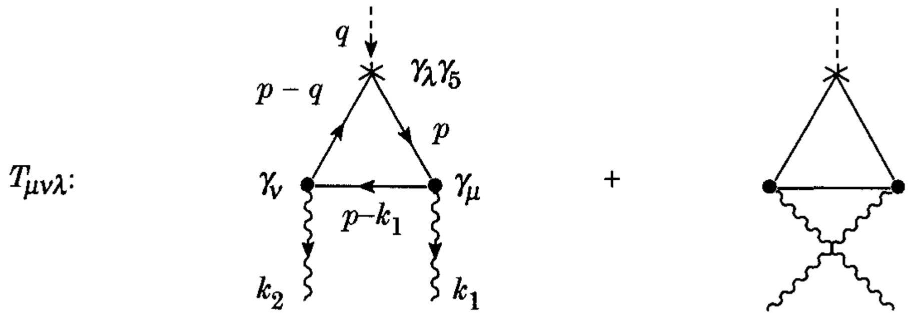



**Figure 4.1:** *Triangle diagrams for the ABJ anomaly.*

Explicitly, the axial Ward identity is defined as follows,

$$q^\lambda\Gamma_{\mu\nu\lambda} = \int d^4x d^4y \, e^{ik_1x+ik_2y-iqz}\langle 0|T\,\frac{\partial}{\partial z^\lambda}J_5^\lambda(z)J_V^\nu(y)J_V^\mu(x)|0\rangle \ . \tag{4.8}$$

Readout the amplitude of the two Feynman diagrams in Fig. (4.1), we have

$$q^\lambda\Gamma_{\mu\nu\lambda} = \int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\frac{1}{\not{p}-\not{k}_2}\gamma_5\gamma_\nu\frac{1}{\not{p}-\not{q}}\gamma_\mu - \frac{1}{\not{p}}\gamma_5\gamma_\nu\frac{1}{\not{p}-\not{k}_1}\gamma_\mu\right] + \begin{pmatrix}\mu\leftrightarrow\nu \\ k_1\leftrightarrow k_2\leftrightarrow\end{pmatrix} \tag{4.9}$$

The key point is that this momentum integral is linearly divergent, and we are not allowed to perform the shift of the internal momentum integration by

$$p^\mu \to p^\mu + k_2^\mu \ \text{, and by } p^\mu \to p^\mu + k_1^\mu \ , \tag{4.10}$$

which seems to make this integral vanish. Following Ref. [119], the correct way to evaluate this integral is to shift the internal momentum integration by

$$p^\mu \to p^\mu + a^\mu \ , \quad a^\mu = \alpha k_1^\mu + (\alpha-\beta)k_2^\mu \ , \tag{4.11}$$

where $\alpha, \beta$ are free parameters that allow us to keep track of the behaviour of different momentum routes. This leads to the anomalous axial Ward identity,

$$q^\lambda\Gamma_{\mu\nu\lambda} = -\frac{1-\beta}{4\pi^2}\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \tag{4.12}$$

Similarly, one can also evaluate the vector Ward identity,

$$k_1^\mu\Gamma_{\mu\nu\lambda} = \frac{1+\beta}{8\pi^2}\epsilon_{\nu\lambda\alpha\beta}k_1^\alpha k_2^\beta \tag{4.13}$$

**Anomalous axial Ward identity.** In order to derive the physical result, one must fix the value of the free parameter $\beta$. Since the vector current is associated to the gauge symmetry, this current should be conserved. Hence, the value of $\beta$ is fixed by $\beta = -1$. This choice makes the axial current becomes anomalous,

$$q^{\lambda}\Gamma_{\mu\nu\lambda} = \frac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}k_1^{\alpha}k_2^{\beta} \ . \tag{4.14}$$

Back to the position space, we observe that the derivative of axial current is divergence

$$\partial_{\mu}J_5^{\mu} = \frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \tag{4.15}$$

**Important remark.** There is no way to fix the value of the $\beta$ parameter such that both vector and axial Ward identities are satisfied. Additionally, Adler and Bardeen [122] showed that higher order loop corrections only renormalize the fields and charged. Hence anomaly is one-loop exact.

## 4.2 On the good use of anomalies.

The importance of anomalies for physics is twofold.

- **Anomalies and experimental observations.** If the current is associated with an external symmetry (i.e. global symmetry), breaking the conservation law of this current due to an anomaly will lead to interesting experimental observations. In fact, the decay amplitude of $\pi^0 \to \gamma\gamma$ is totally determined by the computation of anomaly. Additionally, we will see in the chapter 5 how chiral anomaly solves the $U(1)_A$ problem of the low-energy QCD theory.

- **Anomalies and the quantization of gauge theory.** Conversely, anomalies are catastrophic in a quantized perturbative gauge theory. If the Ward-Takahashi identities become anomalous, the gauge theory will lose the renormalisability and hence the consistency. If one constructs a gauge theory, all gauge anomalies must be cancelled. The dark side of anomaly implies non-trivial restrictions to the physical content of a gauge theory.

## 4.3 Cancellation of gauge anomalies in the Standard Model

When evaluating the anomalous triangle diagram, one must trace over the gauge group generators. For each triangle diagram, it is convenient to define an anomaly coefficient as follows,

$$\mathcal{D}_{\mathcal{R}}^{ABD} = \mathrm{tr}_{\mathcal{R}}T^{A}T^{B}T^{C} \ , \tag{4.16}$$

where $\mathcal{R}$ denotes the representation of the group generators. To guarantee that the gauge theory is not anomalous, one must satisfy the condition,

$$\sum_{\psi}\mathcal{D}_{\mathcal{R}}^{ABD} = 0 \ , \tag{4.17}$$

where we have summed over all fermion fields in our theory. It is easy to see that only two options satisfy the condition (4.17). First, the combinations of the gauge groups are safe, so the trace over gauge group generators automatically vanish. The second option is to carefully choose the fermion gauge charges so that the summation over fermionic content satisfies the condition (4.17).

From the above discussion, a priori, the combination of the SM gauge group might be anomalous. Hence, for each generation, the SM fermions must satisfy the following conditions,

$$
\begin{aligned}
0 &= \sum_{f=u,d} (y_L^f - y_R^f) \\
0 &= N_c \sum_{f=u,d} y_L^f + \sum_{f=\nu,e} y_L^f \\
0 &= N_c \sum_{f=u,d} \left[ (y_L^f)^3 - (y_R^f)^3 \right] + \sum_{f=\nu,e} \left[ (y_L^f)^3 - (y_R^f)^3 \right] .
\end{aligned}
\tag{4.18}
$$

# Chapter 5

# Introduction to the strong CP problem and Axions

This chapter is dedicated to presenting the theoretical background of the Strong CP problem in the SM and its chiral solution, which leads to a new BSM state, called the axion. For a complete review of this topic; see, for instance, Refs. [123–127]. We begin with the $U(1)_A$ problem [128] of the low-energy QCD theory (alternatively called the missing $\eta$-meson problem), and the necessity of the QCD vacuum structure to resolve this problem. The QCD instantons solution predicts the existence of a new term – proportional to the vacuum angle $\bar{\theta}$ – in the SM Lagrangian, which violates the charge conjugation (C) or time-reversal (T) symmetries but is still invariant under the parity (P) transformation. Hence, this new term will induce the CP violation observables in the strong sector of the SM. However, the experimental bound of the $\bar{\theta}$ parameter is $\bar{\theta} \leq 10^{-10}$, which is too small for what we expected. This contradiction and the fact that the SM cannot provide any mechanism to explain the tiny value of $\bar{\theta}$ is the well-known strong CP problem. We will concentrate on the Peccei-Quinn mechanism [129, 130] that solves the strong CP problem and discuss the properties and dynamics of axions through well-known UV models [131–135].

## 5.1 Low-energy QCD and the $U(1)_A$ problem

In the 1970s, the low-energy QCD theory had a puzzling problem which later motivated much development in the theoretical background of this theory. To present this problem, let us recall the classical QCD Lagrangian for $N_f$ quark flavours as stated in chapter 1,

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^A_{\mu\nu} \tilde{G}^{A,\mu\nu} + \sum_f \bar{q}_f \big(i\not{D} - m_f\big) q_f \ , \tag{5.1}$$

where we summed over all quark flavour indices $N_f = \{u, d, c, s, b, t\}$. This Lagrangian is invariant under the $SU(N_c = 3)$ gauge transformation. When all quarks are massless, this Lagrangian has an additional global symmetry,

$$G_{\text{QCD}} = U(N_f)_V \otimes U(N_f)_A \ , \tag{5.2}$$

which is broken explicitly by the mass terms of the QCD Lagrangian. Before delving into technical details, we emphasize that the limit $m_f \to 0$ is sensible since given the fact that $m_u, m_d \ll \Lambda_{QCD}$, with $\Lambda_{QCD}$ being a typical energy scale of QCD where the non-perturbative effects must be taken into account. Hence, at least for two quark flavours, $u$ and $d$ quarks, the low-energy QCD Lagrangian is approximately invariant under the $U(N_f)_V \otimes U(N_f)_A$ global transformation.

**The vector part of QCD global symmetry.** The vector part of $G_{\rm QCD}$ can be decomposed as follows,

$$U(N_f)_V = SU(N_f)_I \otimes U(1)_B \ , \tag{5.3}$$

the vector symmetries $SU(N_f)_I$ and $U(1)_B$ correspond to the Isospin and Baryon number symmetry, respectively. These symmetries have been realized as a good approximation due to the occurrence of nucleon and pion multiplets in the hadronic spectrum.

**The axial part of QCD global symmetry.** Analogously, the axial part of $G_{\rm QCD}$ is decomposed as follows,

$$U(N_f)_A = SU(N_f)_A \otimes U(1)_A \ . \tag{5.4}$$

Unlike the vector part, these axial symmetries will be spontaneously broken by the quark condensates $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \neq 0$ [136]. Considering $N_f = 2$ for the case $m_u = m_d \simeq 0$, we expect to observe four Nambu-Goldstone bosons. On the other hand, with the help of Goldstone bosons, it is possible to write down a massive QCD Lagrangian, which is invariant under the axial transformation. This goal is achieved by encapsulating the NGBs into a unitary matrix,

$$U = \exp\left( i \frac{\Sigma^i}{f_\pi} \sigma^i \right) \ , \tag{5.5}$$

where $i = \{0, 1, 2, 3\}$, $\sigma^0 = \mathbb{1}$ and $\sigma^{1,2,3}$ are Pauli matrices. The Goldstone boson $\Sigma^0$ is associated to the breaking of $U(1)_A$ symmetry, namely $\eta$ meson. The remaining Goldstone bosons $\Sigma^{1,2,3}$ are associated to the three broken generators of the $SU(2)_A$ symmetry, they are referred to the pions $\Sigma^3 = \pi^0$ and $\Sigma^{1,2} = \pi^{1,2}$. The charged pions are obtained through a change of basis,

$$\pi^\pm = \frac{1}{\sqrt{2}} (\pi^1 \pm i\pi^2) \tag{5.6}$$

At this stage, one can non-linearly realize the global axial symmetries and write down the effective Lagrangian of NGBs. The spirit of this exercise is very similar to what was presented in chapter 1 of this manuscript. The leading order of the low-energy QCD Lagrangian reads,

$$\mathcal{L}_{\rm QCD}^{\rm low-energy} = f_\pi^2 \, {\rm tr} (\partial_\mu U)(\partial^\mu U)^\dagger + f_\pi^3 \, {\rm tr} \left( M^\dagger U + M U^\dagger \right) \ , \tag{5.7}$$

where the mass matrix $M$ has some typical properties under the global transformations. The potential terms of the meson particles are obtained by Taylor expanding the Lagrangian given by Eq. (5.7). For a detailed computation, see, for example, the TASI lecture notes [126]. We show here the mass term of the QCD Lagrangian written in terms of pion fields,

$$\mathcal{L}_{\rm QCD}^{\rm low-energy} \supset \alpha f_\pi (m_u + m_d) \pi^+ \pi^- + \alpha \frac{f_\pi}{2} \begin{pmatrix} \pi^0 & \eta \end{pmatrix} \begin{pmatrix} m_u + m_d & m_u - m_d \\ m_u - m_d & m_u + m_d \end{pmatrix} \begin{pmatrix} \pi^0 \\ \eta \end{pmatrix} \ , \tag{5.8}$$

where $\alpha$ is an arbitrary constant determined from the experimental data. Theoretically, we observe that the four NGBs will have a light mass which should obey the sum rule $2m_{\pi^+} = m_{\pi^0} + m_\eta$. This result is a problem of the old QCD theory since it contradicts the experimental observations. Explicitly, $m_{\pi^0} \simeq m_{\pi^+} \simeq 140$ MeV while $m_\eta \simeq 960$ MeV $\gg m_{\pi^0}$. In other words, we only observe three light states instead of four as predicted by the QCD theory. In 1975, Weinberg [128] figured out this problem and suggested that $U(1)_A$ is not a good symmetry of strong interactions.

### 5.1.1 QCD instanton resolve the $U(1)_A$ problem

This section aims to show why $U(1)_A$ is not a good symmetry for the QCD theory. We then show how the QCD instantons resolve the $U(1)_A$ problem and justify why we can add a new CP violation term, $\int d^4x \frac{\theta}{32\pi^2} g_s^2 G\tilde{G}$, to the QCD action. Eventually, we unravel the mystery of the missing $\eta$-meson.

The solution to this problem originated from an incredible remark in the seminal work of Weinberg [128], where he doubted that $U(1)_A$ is not a genuine symmetry of QCD theory even though it seemed to be true in the quark massless limit. In 1976, 't Hooft explicitly proved Weinberg's idea by realizing that QCD vacuum had a sophisticated structure, making the $U(1)_A$, despite being an apparent symmetry of the QCD in the limit of zero quark masses, not a true symmetry of the strong interactions. This crucial point has solved the $U(1)_A$ problem; hence, the missing $\eta$ Goldstone boson is no longer a mystery.

We now discuss more quantitatively the QCD instanton solution proposed by 't Hooft, see Ref. [137–139]. The key points of this solution are presented as follows:

**Chiral anomaly.** A priori, from the previous chapter, we know that $U(1)_A$ is an anomalous symmetry. In the limit of vanishing fermion masses, the Noether current, which is associated to the $U(1)_A$ symmetry $J_5^\mu$ is no longer conserved due to the non-vanishing quantum corrections [121, 140]. Explicitly, under the $U(1)_A$ transformation,

$$q_f \to \exp\left(i\frac{\alpha}{2}\gamma^5\right) q_f \ , \tag{5.9}$$

where $\alpha$ is a $U(1)_A$ charge. The current $J_5^\mu$ is broken at the one-loop level,

$$\partial_\mu J_5^\mu = \alpha \frac{g_s^2 N_f}{32\pi^2} (G^A_{\mu\nu} \tilde{G}^{A,\mu\nu}) \ . \tag{5.10}$$

The chiral anomaly might contribute to the QCD action,

$$\begin{aligned} \delta S_{\rm QCD} &= \alpha \frac{g_s^2 N_f}{32\pi^2} \int d^4x \, G^A_{\mu\nu} \tilde{G}^{A,\mu\nu} = \alpha \frac{g_s^2 N_f}{32\pi^2} \int d^4x \, \partial_\mu \left[ \epsilon^{\mu\nu\rho\sigma} \left( G^A_\nu G^A_{\rho\sigma} - \frac{g_s}{3} f_{ABC} G^A_\nu G^B_\rho G^C_\sigma \right) \right] \\ &= \alpha \frac{g_s^2 N_f}{32\pi^2} \int d^4x \, \partial_\mu K^\mu = \alpha \frac{g_s^2 N_f}{32\pi^2} \int_{S_3} d\sigma \cdot K \ , \end{aligned} \tag{5.11}$$

where we have re-written the operator $G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}$ as a total derivative $\partial_\mu K^\mu$, and hence $\delta S_{\rm QCD}$ is a pure surface integral. $S_3$ is the three-dimensional sphere at infinity, and $d\sigma_\mu$ is an element of its hypersurface. The crucial point is that if the surface integral does not vanish, $\int d\sigma \cdot K \neq 0$, the chiral anomaly will contribute to the QCD action and hence explicitly prove that $U(1)_A$ is not a symmetry of the strong interactions.

**Surface integral in the presence of instantons background.** The surface integral given by Eq. (5.14) is a special object in the sense that we cannot use the naive boundary condition $G^A_\mu(x \to \infty) = 0$, which makes the integral vanish. In the original work, 't Hooft [137, 138] showed that the correct boundary condition at spatial infinity $G^A_\mu(x \to \infty)$ should be a *pure* gauge field. Explicitly, $G^A_\mu(x \to \infty)$ is either zero or a gauge transformation of zero. To understand this point properly, let us perform a gauge transformation for the gluon field $G_\mu = G^A_\mu T^A$,

$$G_\mu \to G'_\mu = g^{-1}\Big(G_\mu(x) + \frac{i}{g_s}\partial_\mu\Big)g\,\Big|_{x\to\infty}$$
$$= \frac{i}{g_s}g^{-1}\partial_\mu g \ , \tag{5.12}$$

where $g$ is an element of the $SU(N)$ group and we have used $G_\mu(\infty) = 0$. The field configuration $G'_\mu = (i/g_s)g^{-1}\partial_\mu g$ is often called *pure gauges*. With these boundary conditions, the non-trivial topological configurations exist such that the surface integral is non-vanishing. To clarify this point, we need to know whether the pure gauge configurations of $G_\mu(x\to\infty)$ are equivalent. From the mathematical point of view, this is a question of homotopy, where we look for all equivalent mappings between the $SU(3)$ gauge group and the sphere $S_3$ at infinity. The answer to this question is

$$M_{S_3}\big[SU(3)\big] = \mathbb{Z} \ , \tag{5.13}$$

where $M_{S_3}$ indicates the mapping from $SU(3)$ to $S_3$. Therefore, we now understand that the behaviour of the gauge fields at spatial infinity is characterized by an integer $n \in \mathbb{Z}$, typically called the winding number. Importantly, we are now able to evaluate the surface integral (5.11) correctly, see Refs. [141, 142] about the details of this calculation,

$$\int d^4x\,\frac{1}{32\pi^2}G^A_{\mu\nu}\tilde{G}^{A,\mu\nu} = m - n = \nu \ , \tag{5.14}$$

where $m$ is the gluon field's winding number at infinity (e.g. at $x = +\infty$) while $n$ is the winding number at the origin point (e.g. at $x = -\infty$), and $\nu$ is often called the Pontryagin index.

**QCD $\theta$-vacuum.** First, we denote $n$ states, $|n\rangle$, as the pure gauge configurations characterized by a $n$ winding number. Keep in mind that all $n$ states have the same energy, so one can refer to them as an $n$-vacuum. The result given by Eq. (5.14) implies the non-zero transition amplitude for a tunneling from the vacuum state $|n\rangle$ to its gauge rotated state $|m\rangle$. Such tunneling events are described by the so-called instanton solutions [137, 138]. Since the vacuum states are degenerate and able to switch to each other by instantons, the physical vacuum state must be the superposition of the $n$-vacua. Hence, the genuine vacuum is referred to as the so-called $\theta$-vacuum and written as,

$$|\theta\rangle = \sum_{n=-\infty}^{n=+\infty} e^{i\theta\,n}|n\rangle \ , \tag{5.15}$$

where $\theta$ is an angular parameter which has the value $\theta \in [0, 2\pi)$. The crucial point is that $\theta$ plays a the role of a super-selection rule. More precisely, different QCD vacuums will have different values of the *theta* parameter, and it is impossible to switch from one vacuum state to another. Therefore, $\theta$ is a fundamental parameter, each value of $\theta$ labels a different theory.

**QCD Lagrangian done correctly.** We are now in the last step to solve the famous $U(1)_A$ problem. All we need to do now is to show explicitly how the *theta* parameter appears in the action of QCD. Considering the vacuum to vacuum transition amplitude,

$$\begin{aligned}\langle\theta_+|\,\theta_-\rangle &= \sum_{m,n} e^{i\theta(m-n)}\,\langle m_+|\,n_-\rangle = \sum_{\nu,n} e^{i\theta\nu}\,\langle(n+\nu)_+|\,n_-\rangle = \sum_{\nu,n} e^{i\theta\int d^4x\frac{1}{32\pi^2}G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}}\,\langle(n+\nu)_+|\,n_-\rangle \\ &= \sum_{\nu}\int \mathcal{D}A\, e^{i\theta\int d^4x\frac{1}{32\pi^2}G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}+i\int d^4x\mathcal{L}}\;\delta\left[\nu-\int d^4x\,\frac{1}{32\pi^2}G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}\right] \\ &= \int \mathcal{D}A\,\exp\left(i\int d^4x\,\mathcal{L}+\theta\frac{1}{32\pi^2}G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}\right)\ ,\end{aligned} \tag{5.16}$$

notice that in the last equation, we have used the path integral formalism to express the transition amplitude $\sum_n \langle(n+\nu)_+|\,n_-\rangle$ where the value of $\nu$ is fixed. Eventually, by realizing the non-trivial topologies of the QCD vacua, from Eq. (5.16), we conclude that the QCD Lagrangian must include a new term,

$$\mathcal{L}_{\rm QCD} \supset \theta\frac{g_s^2}{32\pi^2}G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}\ . \tag{5.17}$$

This new term will help us solve the $U(1)_A$ problem. It is worth noting that the operator $G\tilde{G}$ violates $P, T$, and hence $CP$ symmetries.

**Chiral anomaly in the presence of $\theta$-vacuum.** After a lengthy discussion about the development of the QCD, we now come back to the $U(1)_A$ problem, precisely, equation (5.11). Since the surface integral does not vanish in the presence of an instanton background, the $U(1)_A$ anomalous symmetry can affect the $\theta$ parameter by shifting it to another value,

$$\theta \to \theta + \alpha\ , \tag{5.18}$$

which explicitly shows that $U(1)_A$ was never a true symmetry of QCD. In fact, the mass of the $\eta$-meson is generated by the instanton effects dominantly, hence naturally explaining the significant mass scale separation between the $\eta$-meson and the other pseudo-Goldstone bosons $(\pi^0, \pi^\pm)$, Ref. [139]. The $U(1)_A$ problem now has been solved.

### 5.1.2 QCD instanton induces the strong CP problem

The fact that quarks are massive and the SM includes both strong and weak interactions, without loss of generality, the mass term in the QCD Lagrangian reads,

$$\mathcal{L}_{\rm QCD} \supset \bar{q}^i_L M_{ij} q^j_R + \text{h.c.}\ , \tag{5.19}$$

where the quark mass matrix $M$ is complex. The physical mass basis is obtained by diagonalizing the mass matrix $M$. This step can be done by re-defining the fermion fields, in other words, performing a chiral transformation. The vacuum angle is then shifted by $\arg\det M$. Therefore, in the full theory, the physical vacuum angle reads[1],

$$\bar{\theta} = \theta + \arg\det M\ , \tag{5.20}$$

where $\bar{\theta}$ receives the contributions both from the strong and electroweak sectors. For completeness, we state here the CP violation term in the QCD Lagrangian,

$$\mathcal{L}_{\rm QCD} = \bar{\theta}\frac{g_s^2}{32\pi^2}G^A_{\mu\nu}\tilde{G}^{A,\mu\nu}\ . \tag{5.21}$$

[1] In terms of Yukawa matrices, the vacuum angle is written as $\bar{\theta} = \theta + \arg\det Y_u Y_d$.

Since $\bar{\theta}$ is a physical parameter, it can be measured. The most sensitive CP-violating observable is the neutron Electric Dipole Moment (nEDM) which is directly proportional to the $\bar{\theta}$ vacuum angle. In the Hamiltonian formalism, we define the nEDM, denoted $d_n$, as follows,

$$H = -d_n \overrightarrow{E} \cdot \widehat{S} \ . \tag{5.22}$$

Translating this definition to the effective Lagrangian, we have

$$\mathcal{L}_{\text{eft}} = -d_n \frac{i}{2} F_{\mu\nu} \left( \bar{n}\, \sigma^{\mu\nu} \gamma^5 n \right) \ , \tag{5.23}$$

where $d_n$ plays the role of a Wilson coefficient of the effective operator $F_{\mu\nu}\left(\bar{n}\, \sigma^{\mu\nu}\gamma^5 n\right)$. This coefficient is obtained by matching the low-energy QCD theory onto the effective Lagrangian given by Eq. (5.23) at one-loop order. The result from the computation in the chiral perturbation theory ($\chi$PT) reads [143, 144],

$$|d_n| \approx \frac{e}{8\pi^2} \frac{m_q}{m_n^2} \bar{\theta} \approx (2.4 \pm 1.0) \times 10^{-16}\, \bar{\theta}\, e \text{ cm} \ . \tag{5.24}$$

Besides, the most recent upper limit of $d_n$ is given by the nEDM measurement at PSI published in 2020, see Ref. [145][2],

$$|d_n| < 1.8 \times 10^{-26}\, e \text{ cm} \ . \tag{5.25}$$

The upper bound of the $\bar{\theta}$ vacuum angle resulting from this measurement is,

$$|\bar{\theta}| \lesssim 10^{-10} \ . \tag{5.26}$$

As we have seen, $\bar{\theta}$ is almost zero. The result of $\bar{\theta}$ surprises us, since the parameter receives contributions from two disconnected sectors in the SM which, somehow, appear to cancel unnaturally. Understanding why $\bar{\theta}$ is so tiny is the central question of the strong CP problem.

### 5.1.3 Possible solutions of the strong CP problem

As we showed in the previous section, the strong CP problem is definitely a major problem. There are three possible solutions to this problem. We will go over these options briefly in this section. In the following section, we will concentrate on the most viable solution, the axion, which was proposed by Peccei and Quinn [129, 130].

- **Massless quark solution.** From Eq. (5.24), if we set up the quark fields (e.g. up-quark) are massless, the value of $|d_n|$ will trivially vanish without any constrains. Nowadays, this solution has been excluded by the simulations of lattice QCD, which show that the up-quark has non-zero mass, $m_u = 2.32$ MeV, Ref. [146]. Additionally, the experimentally observed properties of mesons also suggest that quark fields require some sort of bare mass.

- **Spontaneous CP (P) breaking solution.** If CP is a true symmetry of nature which has been spontaneously broken at a high-energy scale, one can set $\bar{\theta} = 0$ in the tree-level UV Lagrangian before the symmetry breaking. The problematic point is that at the low-energy scale, when CP is spontaneously broken, one must account for the presence of CP violation in the weak sector of SM while keeping $\bar{\theta} = 0$ up to the one-loop level. There exist some UV models that bypassed this difficulty [147, 148]. However, it costs some tuning, e.g. complex Higgs VEVs, which leads to difficulties with flavour-changing neutral currents (FCNC). Importantly, the current experimental data is still in good agreement with the Cabibbo-Kobayashi-Maskawa model, where the CP symmetry is not spontaneously broken.

[2] The old upper bound of $d_n$ is given by Ref. [37], explicitly $|d_n| < 2.9 \times 10^{-26}\, e$ cm.

- **Additional chiral symmetry: axion solution.** The key point of this idea is to add a new chiral symmetry in the SM Lagrangian. Due to chiral anomaly, one can effectively rotate the $\bar{\theta}$-vacua away. This solution will provide a natural explanation for the tiny value of the $\bar{\theta}$ parameter.

## 5.2 Introduction to axion models

### 5.2.1 The Peccei-Quinn mechanism

In 1977, R. Peccei and H. Quinn [129, 130] proposed a chiral solution which can deal with the strong CP problem. The crucial point is to require the SM Lagrangian to have an additional $U(1)$ chiral symmetry, later called the Peccei-Quinn (PQ) symmetry, $U(1)_{PQ}$. At the typical energy scale $f_a$, the $U(1)_{PQ}$ symmetry will be spontaneously broken and give rise to the so-called axion as a Goldstone boson associated with its broken generator. In this section, we aim to show how the axion field dynamically cleans up the $\bar{\theta}$-dependent term out of the SM Lagrangian.

We begin by non-linearly realising the $U(1)_{PQ}$ symmetry, this way, one can write down the effective Lagrangian without referring to the axion UV complete models. The possible effective Lagrangian reads,

$$\mathcal{L}_{\text{eft}} \supset \mathcal{L}_{\text{SM}} + \left(\bar{\theta} + \frac{a}{f_a}\right) \frac{g_s^2}{32\pi^2} G^A_{\mu\nu} G^{A,\mu\nu} + \frac{1}{2}(\partial_\mu a)(\partial^\mu a) + \mathcal{L}_{int}\left[\frac{\partial_\mu a}{f_a}, \psi\right] , \tag{5.27}$$

where the axion field $a$ can be shifted under the $U(1)_{PQ}$ transformation,

$$a(x) \to a(x) + \alpha_{PQ} f_a , \tag{5.28}$$

where $\alpha_{PQ}$ is a dimensionless parameter. To understand the dynamic behind the PQ solution, we need to take a look at the axion potential when the low-energy QCD effects are taken into account,

$$V(a)_{\text{eff}} \approx m_a^2 f_a^2 \left[1 - \cos\left(\bar{\theta} + \frac{a}{f_a}\right)\right] . \tag{5.29}$$

The VEV of the axion field is obtained by minimizing this effective potential,

$$\bar{\theta} + \frac{\langle a \rangle}{f_a} = 0 \Leftrightarrow \langle a \rangle = -\bar{\theta} f_a . \tag{5.30}$$

Expanding the axion field around its VEV, $a = a + \langle a \rangle$, and plugging it into Eq. (5.27), we can easily see that $\bar{\theta}$ is cancelled by the VEV of axion field. In other words, the effective vacuum angle $\theta_{\text{eff}} = \bar{\theta} + a/f_a$ is now dynamically relaxed to zero. The strong CP problem is solved. As a cost of this solution, a new excitation state – the axion field – appears in the SM Lagrangian.

In the following sections, we will briefly present the UV complete models which can provide the axion Lagrangian (5.27).

### 5.2.2 The Peccei-Quinn-Weinberg-Wilczek (PQWW) model

Weinberg and Wilczek [131, 132] proposed the original axion model based on the PQ mechanism. Sometimes this model is called the visible axion model since the breaking scale of the PQ symmetry is around the electroweak scale, so the axion couplings are not highly suppressed.

Since we need to embed the $U(1)_{PQ}$ symmetry into the UV theory, it is clear that the SM Lagrangian needs to be extended. In the next chapter, we will discuss this model in detail. To avoid overlap of information, we state here the crucial points of the Weinberg-Wilczek model:

- The axion will appear in the phase of the Higgs doublet. However, one Higgs field is insufficient to produce the axion since the three NGBs generated by this doublet have been absorbed to generate the longitudinal mode of the SM gauge bosons. The minimal extension still invariant under the $U(1)_{PQ}$ transformation is to add a new scalar doublet. In summary, the original axion model is nothing but the two Higgs doublet model (2HDM) enlarged by the $U(1)_{PQ}$ symmetry. The 2HDM Lagrangian reads,

$$\mathcal{L}^{\text{2HDM}}_{\text{Yuk.}} = -\Big[Y_u \bar{u}_R\, \Phi_1\, q_L + Y_d \bar{d}_R\, \Phi_2^\dagger\, q_L\Big] - Y_e \bar{e}_R\, \Phi_2^\dagger\, l_L + \text{h.c.}\ , \tag{5.31}$$

where both scalar and fermion fields charged under the PQ symmetry. After identifying the would be Goldstone $G^0$ and the axion $a$, the Higgs fields can be written as follows,

$$\Phi_1 = \frac{1}{\sqrt{2}} e^{i\, x \frac{a}{v}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}\ ,\ \Phi_2 = \frac{1}{\sqrt{2}} e^{i\left(-\frac{1}{x}\right)\frac{a}{v}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix}\ , \tag{5.32}$$

with $x = v_2/v_1$ and $v_1, v_2$ are the VEV of $\Phi_1$ and $\Phi_2$, respectively.

- Explicitly, the PQ charges of the scalar fields are

$$PQ(\Phi_1; \Phi_2) = \left(x; -\frac{1}{x}\right)\ . \tag{5.33}$$

Since the Yukawa term is invariant under the PQ transformation, one can identify the PQ charge of the fermion fields, see Refs. [149, 150],

$$PQ\big(q_L, u_R, d_R\ ;\ l_L, e_R\big) = \left(\alpha, \alpha + x, \alpha + \frac{1}{x}\ ;\ \beta, \beta + \frac{1}{x}\right)\ , \tag{5.34}$$

with $\alpha, \beta$ are the free parameters, corresponding to the conservation of baryon and lepton numbers. In the original axion model, the parameters $\alpha, \beta$ were set to zero.

- To obtain the axion Lagrangian, we perform the fermion field reparametrization,

$$\psi \to \exp\left[iPQ(\psi)\frac{a}{v}\right]\psi\ ,\quad (\psi = q_L, u_R, d_R\ ;\ l_L, e_R). \tag{5.35}$$

This chiral rotation brings three effects: First, it eliminates the dependence of the axion field in the Yukawa terms. Second, it generates anomalous interactions, e.g. $aG\tilde{G}$, $aW\tilde{W}$, $aB\tilde{B}$. The coupling of these interactions are read from the value of their anomalous coefficients. Without loss of generality, the anomalous interactions can be written as

$$\mathcal{L}_{anomalous} = \frac{1}{16\pi^2\, v}\ \text{tr} \sum_{\psi} PQ(\psi) G(\psi) G(\psi) \Big[aG\tilde{G} + aW\tilde{W} + aB\tilde{B}\Big] \tag{5.36}$$

where $G(\psi)$ are fermion gauge charges. Third, the axion-fermion derivative couplings will appear,

$$\mathcal{L}_{int}\left[\frac{\partial_\mu a}{v}, \psi\right] = \frac{\partial_\mu a}{v}\ \bar{\psi}\gamma^\mu \big(g^V_{PQ} - g^A_{PQ}\gamma^5\big)\psi\ , \tag{5.37}$$

with the vector and axial-vector charges are obtained from

$$g^V_{PQ}(\psi) = \frac{1}{2}\big[PQ(\psi_L) + PQ(\psi_R)\big]\ ,\quad g^A_{PQ}(\psi) = \frac{1}{2}\big[PQ(\psi_L) - PQ(\psi_R)\big]\ . \tag{5.38}$$

Hence, we obtain the axion Lagrangian (5.27) with the couplings coming from PQWW model[3].

[3] See next chapter for the full Lagrangian of the PQWW model.

It is worth noting that experiments have already ruled out the PQWW model since its PQ breaking scale is too small $\simeq 246$ GeV which means that the axion should have been observed.

This problem led to the invisible axion models where the $U(1)_{PQ}$ symmetry breaking occurs at an energy scale far higher than the electroweak scale, $f_a \gg v_{ew}$. The axion couplings in this type of model are suppressed by the factor $1/f_a \ll 1/v$. Hence the axion will be very light, weakly coupled and long-lived.

Depending on how $U(1)_{PQ}$ is realized, there are two types of invisible axion models. The Dine-Fischler-Srednicki-Zhitnitsky (DFSZ)-type [134, 135] and the Kim-Shifman-Vainshtein-Zkharov (KSVZ)-type [133, 151] are the two axion models that are still viable under many experimental constrains. In the following subsections, we will briefly present these models.

### 5.2.3 The DFSZ axion model

The DFSZ model is an extension of the Weinberg-Wilczek model. Analogously, the QCD anomaly is driven by the SM chiral fermions. The crucial point is to decouple the PQ breaking scale $f_a$ far from the electroweak scale by introducing a new singlet complex scalar field, acquiring its vacuum expectation value at an energy scale $f_a \gg v$. Explicitly, the scalar content in the DFSZ model is $\Phi_1 = \big(\mathbf{1}, \mathbf{2}, -1/2\big)$, $\Phi_2 = \big(\mathbf{1}, \mathbf{2}, +1/2\big)$, and $\phi = \big(\mathbf{1}, \mathbf{1}, 0\big)$. Notice that these scalar fields are also charged under the PQ global symmetry. The new scalar field $\phi$ will only directly couple with the Higgs doublet $\Phi_1$ and $\Phi_2$, so it will indirectly modify the axion couplings. More precisely, the scalar potential in the DFSZ model is

$$V(\Phi_1, \Phi_2, \phi) = \frac{\lambda_1}{4}(|\Phi_1|^2 - v_1^2)^2 + \frac{\lambda_1}{4}(|\Phi_2|^2 - v_2^2)^2 + \frac{\lambda_1}{4}(|\phi|^2 - f_a^2)^2 + (a|\Phi_1|^2 + b|\Phi_2|^2)|\phi|^2 \\ + c\big([\Phi_1 \cdot \Phi_2]\phi^2 + \text{h.c.}\big) + d|\Phi_1 \cdot \Phi_2|^2 + e|\Phi_1^\dagger \Phi_2|^2 \ , \tag{5.39}$$

where $\{a, b, c, d, e\}$ are arbitrary coefficients. Because we will work with the DFSZ model in the next chapter, we only state here the key points of this model:

- **The Yukawa terms.** In the 2HDM model, we have two ways to write the Yukawa Lagrangian. The DFSZ type-I is determined by the following Lagrangian,

$$\mathcal{L}_{\text{DFSZ}-\text{I}} \supset -\Big[Y_u \bar{q}_L \Phi_1 u_R + Y_d \bar{q}_L \Phi_2 d_R\Big] - Y_e l_L \Phi_2 e_R + \text{h.c.} \ . \tag{5.40}$$

  The DFSZ type-II is obtained if we couple $\tilde{\Phi}_1 = i\sigma_2 \Phi_1^*$ with the leptons,

$$\mathcal{L}_{\text{DFSZ}-\text{II}} \supset -\Big[Y_u \bar{q}_L \Phi_1 u_R + Y_d \bar{q}_L \Phi_2 d_R\Big] - Y_e l_L \tilde{\Phi}_1 e_R + \text{h.c.} \ . \tag{5.41}$$

- **Axion couplings.** Ignoring the radial degree of freedoms, the neutral components of scalar fields in the DFSZ potential are,

$$\Phi_1 = \frac{1}{\sqrt{2}} e^{i\frac{\eta_1}{v_1}} \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \ , \quad \Phi_2 = \frac{1}{\sqrt{2}} e^{i\frac{\eta_2}{v_2}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \ , \quad \phi = \frac{1}{\sqrt{2}} v_\phi e^{i\frac{\eta_\phi}{v_\phi}} \ , \tag{5.42}$$

  where $v_\phi \gg v_1, v_2$. The physical axion field is the linear combination of $\eta_1, \eta_2$ and $\eta_\phi$,

$$a = \frac{1}{f_a} \sum_i c_i \big(v_i \, \eta_i\big) \ , \quad f_a^2 = \sum_i c_i^2 v_i^2 \ . \tag{5.43}$$

More precisely, the value of the coefficients $c_i$ are

$$c_{\Phi_1} = 2\cos^2\beta \ , \quad c_{\Phi_2} = 2\sin^2\beta \ , \quad c_\phi = 1 \ , \tag{5.44}$$

where the angle $\beta$ is defined by $\cot\beta = v_2/v_1 = x$. The PQ charges of fermions are determined by inverting Eq. (5.43) and expressing $\eta_i$ in terms of the axion field $a$. From the Yukawa Lagrangian, one can read out the PQ charges of fermion fields, then performing the chiral rotations, the DFSZ axion Lagrangian will emerge. This procedure is exactly the same as what we mentioned in the PQWW model. Comparing to the Weinberg-Wilcezk model, one just needs to do some simple replacements,

$$v \to f_a \ , \quad x \to \frac{2x^2}{x^2+1} \ , \quad \left(\frac{1}{x}\right) \to \frac{2}{x^2+1} \ , \tag{5.45}$$

and the axion couplings will be obtained very easily.

### 5.2.4 The KSVZ axion model

The KSVZ model also has the complex scalar field $\phi = (\mathbf{1}, \mathbf{1}, 0)$ where the axion can be embedded in its phase, and the PQ breaking scale $f_a$ is also very far from the electroweak scale. The main difference is how QCD anomalous interaction is realized. The KSVZ model introduced a new vector-like fermion, $Q = (\mathbf{3}, \mathbf{1}, 0)$ that carries the PQ charge while the SM fermions are uncharged under the PQ symmetry. The KSVZ Lagrangian reads,

$$\mathcal{L}_{\text{KFSZ}} \supset \bar{Q} i \not{D} Q - \left(y_q \bar{Q}_L \phi Q_R + \text{h.c.}\right) + (\partial_\mu \phi)^2 - \lambda_\phi \left(|\phi|^2 - \frac{f_a^2}{2}\right)^2 \ . \tag{5.46}$$

When the $U(1)_{PQ}$ is spontaneously broken, the axion field appears as a NGB of the scalar field $\phi$,

$$\phi = \frac{1}{\sqrt{2}} f_a e^{i\frac{a}{f_a}} \ . \tag{5.47}$$

The PQ charges of the field content are the following,

$$PQ(\phi) = \alpha \ , \quad PQ(Q_L, Q_R) = \left(\frac{\alpha}{2}, -\frac{\alpha}{2}\right) \ , \quad PQ(f_{SM}) = 0 \ . \tag{5.48}$$

Eventually, the axion Lagrangian is obtained via performing the chiral rotations. Since we choose that $Q$ only has colours and electromagnetic gauge charges, the model provides the anomalous interactions with gluons and photons at tree-level.

# Chapter 6

# Axion Effective Action

In this chapter, we discuss the construction of Effective Field Theories (EFTs) in which a chiral fermion, charged under both gauge and global symmetries, is integrated out. Inspired by typical axion models, these symmetries can be spontaneously broken, and the global ones might also be anomalous. In this context, particular emphasis is laid on the derivative couplings of the Goldstone bosons to the fermions, as these lead to severe divergences and ambiguities when building the EFT. We show how to precisely solve these difficulties within the path integral formalism, by adapting the anomalous Ward identities to the EFT context. Our results are very generic, and when applied to axion models, they reproduce the non-intuitive couplings between the massive SM gauge fields and the axion. Altogether, this provides an efficient formalism, paving the way for a systematic and consistent methodology to build entire EFTs involving anomalous symmetries, as required for axion or ALP searches.

## 6.1 Introduction

The Peccei-Quinn (PQ) mechanism [129, 130] is probably the best solution to the strong CP problem of the Standard Model (SM) [145]. This solution predicts a new Goldstone boson, the so-called axion [131, 132], which is hunted by many experiments. Recent constrains from astrophysics [152, 153] and particle physics [154, 155] imply that the breaking scale of PQ symmetry, $f_a$, is much larger than the electroweak scale. Due to this large-scale separation, Effective Field Theory (EFT) is a well-suited framework to describe the interactions between the axion and other light particles (usually from the SM).

In previous works, the authors of Refs. [149, 150, 156] have shown that axion models exhibit intrinsic ambiguities in their formulation, and this has a dramatic impact on the coupling of axions to massive gauge fields. One of the main conclusion of Ref. [149] states as follows: when axion models are specified in a representation in which the axion has only derivative couplings to SM chiral fermions, such as in DFSZ-like axion models [134, 135] [1], some chiral reparametrisation of the fermionic fields are implicit and lie at the root of the so-called anomalous axion couplings to gauge field strengths. For vector gauge interactions, it is well known that derivative couplings to fermions decouple faster than local anomalous operators, which thus capture the whole axion to gauge boson couplings. By contrast, derivative interactions do not systematically decouple for chiral gauge interactions, ultimately because the gauge symmetry is necessarily spontaneously broken when the chiral fermions get their masses. Importantly, non-decoupling contributions from deriva-

[1] KSVZ-like models [133, 151] involve vector-like fermions, whose masses are decoupled from the spontaneous electroweak symmetry breaking, and the discussion is much simpler.

tive interactions can arise from the usual axial coupling to fermions, but also from the vector one. Both can be anomalous in the presence of chiral gauge interactions. In practice, keeping track of these non-decoupling effects is crucial to get consistent, parametrisation-independent couplings of the axion to gauge bosons. Only with them, one can match the results obtained using a linear representation of the complex Peccei-Quinn scalar field, in which the axion has pseudoscalar couplings to chiral fermions, and no anomaly-related ambiguities ever arise.

On a more technical side, these results have been derived by appropriately computing anomalous triangle diagrams regularised using Weinberg's method [118, 157] which allow to parametrise the initial ambiguity inherent to the momentum rooting in the amplitude integrals. This rigorous treatment allows to obtain generalised Ward identities, in which one can tune which current is anomalous, or not, which is physically mandatory and not guaranteed in a more naive computation. The Ref. [158] reaches similar conclusions from a more anomaly matching EFT-oriented point of view, which brings interesting insights to axion couplings to chiral gauge fields.

In this work, our goal is not only to add up on the understanding and construction of low energy axion EFTs, but also more generally on the possible interplays or entanglements between spontaneous and anomalous symmetry breaking that can arise when chiral fermions are integrated out. Further, our goal is to perform this analysis exclusively in a functional context, by building the low-energy EFT following a step-by-step integration of the chiral fermion fields, without recourse to triangle Feynman diagrams or Ward identities, and take advantage of the elegant and convenient techniques developed recently to integrate out heavy fermionic fields [3, 6, 72, 75–77, 79–82, 98]. The only ingredients will thus be dimensionally-regulated functional traces, and the order-by-order invariance of the EFT operators under gauge transformations, when the appropriate would-be-Goldstone bosons are accounted for. Ultimately, the same non-decoupling of derivative interactions will be observed, in the sense that the EFT built from them will start with dimension-five operators.

The main novelties of our approach are the following: within the path integral formalism for one-loop matching, we show how to consistently integrate out heavy chiral fermions, which are charged under both the local and global symmetries. Focusing on the Goldstone-gauge bosons couplings, we show how to deal with $\gamma^5$ within dimensional regularisation to properly keep track of the ambiguities arising in the one-loop effective action. In the functional matching, we show that the gauge invariant combinations of the EFT operators can be used to fix these ambiguities. Hence, for the first time, with the functional method for one-loop matching, we can fully control which symmetry currents are anomalous and which ones are anomaly free. Therefore, the Wilson coefficients can be obtained correctly. We then derive the universal formula that captures all EFT couplings of Goldstone bosons with gauge bosons (both massive and massless) consistently and generically. Our results can be easily applied to various axion UV models.

The plan of the manuscript is the following: In section 6.2, we integrate out a chiral fermion from a toy model to obtain a gauge and Goldstone boson EFT. This section will mainly concentrate on the physical interpretations so the reader can understand the logic behind the construction without entering into the details of the calculation. The crucial point of this section is to show how ambiguous coefficients can be fixed by enforcing the Ward identities, which are now written in terms of gauge invariant combinations of the EFT operators. For the reader who is familiar to this topic, one can skip this section and go directly to the following section.

In section 6.3, we will detail how to evaluate the one-loop effective action from the path integral functional approach and how we deal with the ambiguities originating from the QFT anomalies. The main outcome of this section is Eq. (6.53), a master formula which can be easily used to obtain effective couplings between gauge fields and Goldstone bosons, and which encapsulates the subtleties occurring when dealing with anomalous global symmetries in a chiral gauge theory.

In section 6.4, we apply this master formula to various models starting with a simple chiral toy model with an additional global $U(1)$ symmetry. We then explicitly apply our results to axion models and recover, for instance, the non-intuitive axion couplings involving massive gauge fields in DFSZ-like models. We conclude in section 6.5 while additional computational details regarding master integrals can be found in Appendix A.

## 6.2 EFTs with spontaneously and anomalously broken symmetries

Readers familiar to the topic of anomalous symmetries in the EFT may directly jump to the following sections. Our goal here is to introduce the formalism using a simplified setting. More precisely, our goal is to integrate out fermions that can be charged under both global and local symmetries. Further, those fermions will not be assumed vector-like: their left- and right-handed components need not have the same charges under these symmetries. This generates two complications. First, obviously, such fermions can only acquire a mass, and thus be integrated out, when the chiral components of the symmetries are spontaneously broken. Second, the classical symmetries cannot all survive quantisation, and there must be some anomalies. These two effects are entangled, and further, they induce some freedom in how the fermionic part of the UV Lagrangian is to be parametrised. So, before any attempt at integrating out the fermions, it is necessary to fix this freedom. As we will discuss in this section, from a functional point of view, one parametrisation emerges as the most natural, but requires a specific treatment of anomalous effects and derivative interactions.

### 6.2.1 EFTs and gauge invariance

We start from a generic UV Lagrangian exhibiting some set of local symmetries and involving fermionic degrees of freedom. Typically, the fermionic part of the Lagrangian is of the form, including for simplicity only one axial and one vector gauge field,

$$\mathcal{L}_{\text{UV}}^{\text{fermion}} = \bar{\Psi}\big(i\partial_\mu\gamma^\mu + g_{_V} V_\mu\gamma^\mu - g_{_A} A_\mu\gamma^\mu\gamma^5\big)\Psi\,. \tag{6.1}$$

Let us first consider abelian gauge symmetries for simplicity [2]. At the classical level, this theory is invariant under $U(1)_V$ and $U(1)_A$ gauge transformations, which we define as:

$$U(1)_V: \; V_\mu \to V_\mu + \frac{1}{g_{_V}}\partial_\mu\theta_{_V}\;,\; \Psi \to \exp(i\theta_{_V})\Psi\,, \tag{6.2}$$

$$U(1)_A: \; A_\mu \to A_\mu + \frac{1}{g_{_A}}\partial_\mu\theta_{_A}\;,\; \Psi \to \exp(-i\theta_{_A}\gamma^5)\Psi\,. \tag{6.3}$$

Our goal is to integrate out the fermion to get the tower of effective interactions by performing an inverse mass expansion [3]. This obviously means that the fermion to be integrated out should be massive, which forces the axial gauge symmetry to be spontaneously broken. To be able to consistently account for this, let us include the complex scalar field $\phi_A$ which, by acquiring a vacuum expectation value $v$, will spontaneously break the axial gauge symmetry,

$$\mathcal{L}_{\text{UV}}^{\text{fermion}} = \bar{\Psi}\big(i\partial_\mu\gamma^\mu + g_{_V} V_\mu\gamma^\mu - g_{_A} A_\mu\gamma^\mu\gamma^5\big)\Psi - y_\Psi\big(\bar{\Psi}_{_L}\phi_A\Psi_{_R} + \text{h.c.}\big)\,, \tag{6.4}$$

with $y_\Psi$ the Yukawa coupling, and two Weyl components $\Psi_{R,L} = P_{R,L}\,\Psi$ with $P_{R,L} = (1 \pm \gamma^5)/2$.

[2] We will discuss later the peculiarities arising in the non abelian case.

[3] More precisely we will use convenient Covariant Derivative Expansion (CDE) techniques.

If one wants to focus on manifest gauge invariance, it is convenient to include the Goldstone boson, $\pi_A$, and adopt an exponential or polar representation for the complex scalar field,

$$\phi_A = \frac{1}{\sqrt{2}}(v+\sigma_A)\exp\left[i\,\frac{\pi_A(x)}{v}\right]. \tag{6.5}$$

Indeed, thanks to the exponential parametrisation of the Goldstone boson, this theory is still manifestly gauge invariant provided, together with the transformation of Eq. (6.3),

$$\pi_A \to \pi_A + 2v\theta_{_A}\,, \tag{6.6}$$

while $\sigma_A$ is gauge invariant and plays no rôle in that regard. Said differently, with this representation, it is sufficient to keep only the gauge bosons and the Goldstone fields explicitly to construct the EFT, which will involve only these fields in a gauge invariant way.

By contrast, if one insists on manifest renormalisability, the Goldstone boson has to enter linearly, that is, by writing the scalar field acquiring a vacuum expectation value $v$ as linear in all its components,

$$\phi_A = \frac{1}{\sqrt{2}}(v+\sigma_A+i\pi_A)\,. \tag{6.7}$$

The $\sigma_A$ is no longer gauge invariant since a $U(1)_A$ gauge transformation is nothing but a $SO(2)$ rotation for the $(v+\sigma_A,\pi_A)$ vector. If one insists on manifest gauge invariance, the difficulty then is that $\sigma_A$ should explicitly appear in $\mathcal{L}_{\rm UV}^{\rm fermion}$. Even if in the abelian case, this is quite simple, this would introduce unnecessary model-dependence in the non-abelian case.

So, at the end of the day, it is legitimate in order to be able to consistently account for spontaneous breaking of the axial gauge symmetry, to consider the exponential representation of the Goldstone boson,

$$\mathcal{L}_{\rm UV}^{\rm fermion} = \bar\Psi\left(i\partial_\mu\gamma^\mu + g_{_V}V_\mu\gamma^\mu - g_{_A}A_\mu\gamma^\mu\gamma^5 - M\exp\left[i\,\frac{\pi_A}{v}\gamma^5\right]\right)\Psi\,, \tag{6.8}$$

where $M \equiv y_\Psi v/\sqrt{2}$ stands for the mass of the fermion. Yet, at this stage, a Taylor expansion[4] produces the pseudoscalar $\pi_A\bar\Psi\gamma^5\Psi$ coupling, and is the same as it would be in the linear representation of $\phi_A$. So, the distinction between linear and polar representation may appear quite academic. Yet, the exponential parametrisation offers an alternative route. Instead of a Taylor expansion, there is a well-known exact procedure to recover a linearised Lagrangian that allows to transfer the Goldstone dependence from the Yukawa sector to the gauge sector. Based on the chiral rotation that is given by Eq. (6.3), it suffices to perform a field-dependent reparametrisation of the fermion fields

$$\Psi \to \Psi = \exp\left[-\,i\frac{\pi_{_A}(x)}{2v}\gamma^5\right]\Psi\,, \tag{6.9}$$

and the Lagrangian in Eq. (6.8) becomes

$$\mathcal{L}_{\rm UV}^{\rm fermion} = \bar\Psi\left(i\partial_\mu\gamma^\mu - M + g_{_V}V_\mu\gamma^\mu - \left[g_{_A}A_\mu - \frac{\partial_\mu\pi_A(x)}{2v}\right]\gamma^\mu\gamma^5\right)\Psi\,. \tag{6.10}$$

[4]For the purpose of evaluating the one-loop effective action using the Covariant Derivative Expansion (CDE), truncating this expansion is perfectly consistent since operators at most linear in a given Goldstone boson will be considered. Issues related to the apparent non-renormalisability of the exponential parametrisation will not affect our developments.

Under this form, $\Psi$ is invariant under the axial gauge transformation $U(1)_A$, so the mass term does not cause any trouble even for a chiral gauge symmetry and could easily be factored out for an EFT mass expansion. The quadratic operator defined in Eq. (6.10) has the virtue of being manifestly gauge invariant. The Goldstone boson itself ensures the theory stays invariant when $A_\mu \to A_\mu + \frac{1}{g_A}\partial_\mu \theta_A$ thanks to $\pi_A \to \pi_A + 2v\theta_A$. Evidently, for that to work, one should not get rid of them by moving to the unitary gauge.

However, as a side effect, the theory is still not manifestly renormalisable since the $\partial_\mu \pi_A(x)$ operator is of dimension five. Yet, this form looks particularly well suited for an inverse mass expansion since $M \sim v$. Let us stress, though, that one should not be tempted to conclude that the $\partial_\mu \pi_A(x)$ operator is subleading and can be neglected. Such considerations can only be consistently done after the fermion field has been integrated out, and as we will see in details in the following, this operator does contribute in general to the leading terms in the EFT.

### 6.2.2 EFTs and anomalies

The Lagrangian in Eq. (6.10) looks promising, but to reach it, we had to reparametrise the fermion field, Eq. (6.9), and there is one crucial caveat for that. The fermion being chiral, this reparametrisation does not leave the path integral fermionic measure invariant. In general, given that $\Psi$ is coupled to gauge fields, the Jacobian, obtained using the singlet anomaly result for chiral fermions [118, 119], sums up to additional terms in the Lagrangian of the form

$$\mathcal{L}_{\text{UV}} \supset \mathcal{L}_{\text{UV}}^{\text{Jac}} = \frac{1}{8\pi^2}\frac{\pi_A}{2v}\left[g_V^2 F_{V,\mu\nu}\tilde{F}_V^{\mu\nu} + \frac{1}{3}g_A^2 F_{A,\mu\nu}\tilde{F}_A^{\mu\nu}\right], \tag{6.11}$$

with $F_X^{\mu\nu} = \partial^\mu X^\nu - \partial^\nu X^\mu$ the usual field strength tensor applied to the generic gauge field X and $\tilde{F}_X^{\mu\nu} = (1/2)\epsilon_{\mu\nu\rho\sigma}F_X^{\rho\sigma}$ its dual field strength tensor with the suffix indicating if this apply to the vector gauge field (V) or the axial one (A). These terms explicitly break gauge invariance, since they get shifted under $\pi_A \to \pi_A + 2v\theta_A$.

There are two main ways to deal with the anomalous contributions shown in Eq. (6.11). If one wants to hold the interactions to be gauged, a first possibility consists in tuning the chiral fermionic content such that the total contribution to the anomaly vanishes (as it happens in the SM). The second possibility is to give up gauge invariance and reconsider the local symmetry as a global symmetry. We clarify in the following these two cases to consider:

- For gauge interactions that are meant to exist at the quantum level (then not being anomalous), the fermionic content is supposed to be just right so that the sum of all Jacobian terms sum up to zero. As is well known, this is the prototype of the gauge interactions in the SM, where gauge anomalies cancel out only when all matter fields are summed over. The important point is that the corresponding Goldstone fields are allowed to be moved to and from the mass terms without generating a Jacobian contribution. Indeed, the reparametrisation in Eq. (6.9) must not generate Jacobian terms since a gauge transformation acts like that on fermions, see Eq. (6.3). In this context, the strict equivalence between the $\bar{\Psi}\left(\partial_\mu \pi_A \gamma^\mu \gamma^5\right)\Psi$ and $\bar{\Psi}\left(M\gamma_5 \pi_A/v\right)\Psi$ couplings can be viewed as the transcription of the non-anomalous Ward identity $\partial_\mu A^\mu = 2iMP$ with $A^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi$ and $P = \bar{\Psi}\gamma^5\Psi$. Indeed, to the divergence of any correlation function of the axial gauge current, $\langle 0|A^\mu ...|0\rangle$, we can associate that with $\partial_\mu \pi_A$ from Eq. (6.10), which can then be equivalently calculated from Eq. (6.8) (after Taylor expanding the exponential term). Regarding the vector gauge interactions, the situation is simpler since the mass term is gauge invariant. Imposing $V_\mu \to V_\mu + \frac{1}{g_V}\partial_\mu \theta_V$ requires the $\partial_\mu \theta_V$ piece to cancel out, i.e. any correlation function of the vector gauge currents $V^\mu = \bar{\Psi}\gamma^\mu\Psi$ satisfies the non-anomalous Ward identity $\partial_\mu V^\mu = 0$.

- Some of the gauge interactions may simply be absent if their symmetry is kept global. In that case, one can simply remove the corresponding $\mathrm{A}_\mu$ from the Lagrangian, but keep the Goldstone bosons since they become independent physical degrees of freedom. These global symmetries may or may not have anomalies, but whenever they do, one should keep track of the Jacobian when passing from pseudoscalar to derivative Goldstone boson couplings to fermions. As explained in Refs.[149, 150], one must obtain the same results using either the Lagrangian with pseudoscalar couplings (after Taylor expanding the exponential term in Eq. (6.8)), or that using derivative couplings, Eq. (6.10), provided the local anomalous terms, Eq. (6.11), are then also included. Indeed, the point is that derivative couplings do induce anomalous effects that precisely cancel those in the local terms of Eq. (6.11). In the inverse mass expansion context, this shows that one must be careful not to perform the limit $M \to \infty$ too soon, that is, discard the derivative interaction in Eq. (6.10) on the basis of its relative $\mathcal{O}(M^2)$ suppression with respect to the fermion mass term, because it does provide terms of the same order in $M$ as those in Eq. (6.11).

### 6.2.3 EFTs with local and global symmetries

The goal of the present chapter is to consider scenarios combining both situations we have discussed so far, that is, with spontaneously broken gauge symmetries and anomalous global symmetries. Generically, our theory of interest corresponds to

$$\mathcal{L}_{\text{UV}} \supset \mathcal{L}_{\text{UV}}^{\text{fermion}} + \mathcal{L}_{\text{UV}}^{\text{Jac}} \,, \tag{6.12}$$

with

$$\begin{aligned}\mathcal{L}_{\text{UV}}^{\text{fermion}} = \bar{\Psi}\bigg[ i\partial_\mu\gamma^\mu - M + \left(V_\mu - \frac{\partial_\mu \pi_V}{2v_V}\right)\gamma^\mu - \left(A_\mu - \frac{\partial_\mu \pi_A}{2v_A}\right)\gamma^\mu\gamma^5 \\ - \left(0 - \frac{\partial_\mu \pi_S}{2v_S}\right)\gamma^\mu - \left(0 - \frac{\partial_\mu \pi_U}{2v_U}\right)\gamma^\mu\gamma^5\bigg]\Psi \,.\end{aligned} \tag{6.13}$$

and for the Jacobian, using the singlet anomaly result for chiral fermions [118, 119], and noting that $\Psi_{L/R}$ couples to $V^\mu \pm A^\mu$ and $\partial_\mu(\pi_S \pm \pi_U)$,

$$\begin{aligned}\mathcal{L}_{\text{UV}}^{\text{Jac}} &= \frac{1}{16\pi^2}\frac{\pi_U}{2v_U}\Big[ \big(F_{V,\mu\nu} + F_{A,\mu\nu}\big)\big(\tilde{F}_V^{\mu\nu} + \tilde{F}_A^{\mu\nu}\big) + \big(F_{V,\mu\nu} - F_{A,\mu\nu}\big)\big(\tilde{F}_V^{\mu\nu} - \tilde{F}_A^{\mu\nu}\big) \Big] \\ &+ \frac{1}{16\pi^2}\frac{\pi_S}{2v_S}\Big[ \big(F_{V,\mu\nu} + F_{A,\mu\nu}\big)\big(\tilde{F}_V^{\mu\nu} + \tilde{F}_A^{\mu\nu}\big) - \big(F_{V,\mu\nu} - F_{A,\mu\nu}\big)\big(\tilde{F}_V^{\mu\nu} - \tilde{F}_A^{\mu\nu}\big) \Big] \\ &= \frac{1}{8\pi^2}\frac{\pi_U}{2v_U}\left(F_{V,\mu\nu}\tilde{F}_V^{\mu\nu} + F_{A,\mu\nu}\tilde{F}_A^{\mu\nu}\right) + \frac{1}{4\pi^2}\frac{\pi_S}{2v_S}F_{A,\mu\nu}\tilde{F}_V^{\mu\nu} \,,\end{aligned} \tag{6.14}$$

where $\pi_A$ and $\pi_V$ have no contact interactions with field strength tensors since these gauge interactions are assumed anomaly-free. To insist on the fact that $\pi_S$ and $\pi_U$ are Goldstone bosons associated to global symmetries, we explicitly assign their respective would-be-gauge fields to 0 in Eq. (6.12). In this expression and throughout the rest of this section, we have set all the couplings to one to unclutter the derivation, but they can be straightforwardly reintroduced, as we will do in the following sections. This parametrisation of the fermion sector of the UV theory deserves several important comments:

- The UV theory necessarily involves several complex scalar fields, several species of fermions to cancel the gauge anomalies, along with some set of scalar and fermion couplings ensuring the existence of the gauge and global symmetries at the Lagrangian level. Further, as will be detailed in Sec. 4, the pseudoscalar components of these scalar fields in general mix, with some combinations eaten by the gauge fields, and some left over as true physical degrees of freedom. With the above parametrisation, we single out one of these fermions, and all the other UV features are encoded into the parameters $v_{S,U}$, $v_{A,V}$, which in general involves vacuum expectation values and some mixing angles, and in the fermion mass term $M$, which in general arises from several Yukawa couplings.

- Adopting a non-linear representation for the scalar fields, with its associated loss of renormalisability, is inevitable if one wishes to leave the details of the whole scalar sector unspecified and start at the UV scale with an effective theory involving only the Goldstone bosons. Indeed, those have to be constrained to live on the specific coset space corresponding to the assumed symmetry breaking pattern. Note that for an abelian global symmetry, the dynamics of the Goldstone bosons is particularly simple, as there are no contact interactions among them, and all that remains is the shift symmetry.

- One of the main goal of this work is to build an EFT by integrating out chiral fermions. As we have discussed, it is then convenient to reparametrise the fermion fields, so that the Goldstone boson couplings to fermions involve local partial derivative. This first ensures the gauge and shift symmetries are manifest, but it also makes the fermion mass term invariant under all symmetries. Though not compulsory, it then allows to construct the EFT by factoring the mass term out in a symmetry preserving way.

- For the abelian toy model described here, the Goldstone bosons involved in vector currents, $\pi_S$ and $\pi_V$, actually play no role. Indeed, for the vector gauge interaction, the $\partial_\mu \pi_V \bar{\Psi}\gamma^\mu\Psi$ interaction can always be eliminated by a non-anomalous reparametrisation $\Psi \to \exp\left(i\frac{\pi_V}{2v_V}\right)\Psi$, which leaves the fermion mass term invariant. Whether it is spontaneously broken or not is thus irrelevant. For the scalar $\pi_S$ Goldstone boson associated to a global symmetry, the reparametrisation $\Psi \to \exp\left(i\frac{\pi_S}{2v_S}\right)\Psi$ not only removes the $\partial_\mu \pi_S \bar{\Psi}\gamma^\mu\Psi$ interaction, but being anomalous, it induces a Jacobian that precisely kills the $\pi_S$ terms in Eq. (6.14). The field $\pi_S$ thus disappears entirely from the theory. These two facts are truly peculiar to the abelian gauge symmetry case, with the fermion in a one-dimensional representation. So, to set up the formalism to deal with more general theories, like the SM, we keep these fields explicitly in the UV parametrisation of the fermion couplings.[5]

So, let us proceed and integrate out the fermion field involving local partial derivatives in its quadratic operator. Details of the calculation will be presented in the following section, but let us already discuss some interesting generic features. If one decides to use Feynman diagrams to integrate out fermions, one will have to deal with divergent triangle amplitudes that one will have to carefully regularise. Even if this is a standard manipulation in QFT, the potential spread of the anomaly has to be considered with great care as discussed in Refs. [149, 150]. In the functional approach, that we will follow all along this work, the fact that the axial vector or vector couplings are anomalous manifests itself by the presence of ambiguities in the functional trace [6].

[5]Further, integrating out the fermion starting from Eq. (6.13), to verify that the $\pi_S$ derivative interaction indeed induces EFT operators that precisely cancel the Jacobian term in Eq. (6.14) provides a non-trivial check for our calculation, see Sec. 4.1.

[6]More precisely the ambiguity is localised in the Dirac matrices trace if one chooses to use dimensional regularisation, as we will do.

This means that starting from Eq. (6.13), the fermion-less EFT expansion will start with six dimension-five operators involving the Goldstone bosons $\pi_U$ and $\pi_S$ [7],

$$\mathcal{L}_{\text{UV}}^{\text{fermion}} \to \mathcal{L}_{\text{EFT}}^{\text{1loop}} = \omega_{_{AVV}} \frac{\partial_\mu \pi_U}{2v_U} V_\nu \tilde{F}_{_V}^{\mu\nu} + \omega_{_{AAA}} \frac{\partial_\mu \pi_U}{2v_U} \left( A_\nu - \frac{\partial_\nu \pi_A}{2v_A} \right) \tilde{F}_{_A}^{\mu\nu} + \omega_{_{VVA}} \frac{\partial_\mu \pi_S}{2v_S} V_\nu \tilde{F}_{_A}^{\mu\nu} + \omega_{_{VAV}} \frac{\partial_\mu \pi_S}{2v_S} \left( A_\nu - \frac{\partial_\nu \pi_A}{2v_A} \right) \tilde{F}_{_V}^{\mu\nu} \,. \tag{6.15}$$

The evaluation of the $\omega_i$ coefficients [8] involves divergent integrals and after their regularisation, those parameters end up fully ambiguous. We thus need to find a strategy to fix them.

Actually, these ambiguities are the exact analog of those arising for the triangle diagrams, whose expressions are ambiguous since they depend on the routing of the momenta when working at the Feynman diagram level. In that case, the ambiguities are removed by imposing the appropriate Ward identities, that is, gauge invariance. So, we would like to do the same here, and impose the vector and axial gauge invariance. However, all the operators in Eq. (6.15) are already gauge invariant! Actually, the would-be-Goldstone bosons $\pi_A$ are not even needed to ensure the gauge invariance, and they never contribute to S-matrix elements. The reason is that their contributions, or the $\theta_{_{V,A}}$ terms arising when $V_\mu \to V_\mu + \partial_\mu \theta_{_V}$ or $A_\mu \to A_\mu + \partial_\mu \theta_{_A}$, drop out by integration by parts [9] thanks to the antisymmetry of $\tilde{F}_{_{V,A}}^{\mu\nu}$ and the Bianchi identity.

In the initial Lagrangian of Eq. (6.12) we decided to treat both the would-be Goldstone bosons ($\pi_V$ and $\pi_A$) and the Goldstone bosons ($\pi_S$ and $\pi_U$) on equal footing by writing them with local derivative acting on them. Since this increases the degree of divergence of integrals one would then be tempted, in order to minimise the number of integrals to regularise, to preferentially consider the situation where the would-be Goldstone bosons enter the mass term (let us remind that this can be trivially done since the gauge symmetries are assumed not to be anomalous). Then, after Taylor expanding the mass term one obtains,

$$\mathcal{L}_{\text{UV}}^{\text{fermion}} = \bar{\Psi} \left[ i\partial_\mu \gamma^\mu - M \left( 1 + \frac{\pi_A}{v_A} i\gamma^5 \right) + V_\mu \gamma^\mu - A_\mu \gamma^\mu \gamma^5 + \frac{\partial_\mu \pi_S}{2v_S} \gamma^\mu + \frac{\partial_\mu \pi_U}{2v_U} \gamma^\mu \gamma^5 \right] \Psi \,. \tag{6.16}$$

Since by construction the $U(1)_V$ and $U(1)_A$ symmetries are gauged, the would-be-Goldstone bosons, $\pi_V$ and $\pi_A$, are not involved in bosonic operators up to dimension five, starting from Eq. (6.16). This means that the fermion-less EFT expansion will start again with four dimension-five operators involving only the Goldstone bosons $\pi_S$ and $\pi_U$ ,

$$\mathcal{L}_{\text{UV}}^{\text{fermion}} \to \mathcal{L}_{\text{EFT}} = \omega_{_{AVV}} \frac{\partial_\mu \pi_U}{2v_U} V_\nu \tilde{F}_{_V}^{\mu\nu} + \omega_{_{AAA}} \frac{\partial_\mu \pi_U}{2v_U} A_\nu \tilde{F}_{_A}^{\mu\nu} + \omega_{_{VVA}} \frac{\partial_\mu \pi_S}{2v_S} V_\nu \tilde{F}_{_A}^{\mu\nu} + \omega_{_{VAV}} \frac{\partial_\mu \pi_S}{2v_S} A_\nu \tilde{F}_{_V}^{\mu\nu} \,. \tag{6.17}$$

Since $\pi_A$ drops out of Eq. (6.15) under integration by part, we recover exactly the same effective interactions. Moving the would-be-Goldstone to the fermion mass term, that is, making it gauge-dependent, does not help to fix the $\omega_i$ coefficients because gauge invariance is still automatic for the leading dimension-five operators. The only way forward is to perturb the theory to break this automatic gauge invariance, so that non-trivial constraints on the $\omega_i$ coefficients can emerge. One possibility is to associate to the Goldstone bosons, $\pi_S$ and $\pi_U$, auxiliary gauge fields $S_\mu$ and $U_\mu$, respectively, as we will now discuss.

[7] A priori the only non vanishing dimension five operators have to involve Dirac traces with only one $\gamma^5$ matrix or with three $\gamma^5$ matrices.

[8] The $\omega_i$ coefficients carry the CP properties of their associated three Lorentz structures. As an example, $\partial_\mu \pi_U$ is CP-odd, $V_\nu$ and $\tilde{F}_{_V}^{\mu\nu}$ are CP-even so the associated coefficient reads $\omega_{AVV}$.

[9] One should note that integration by parts can be performed without any hesitation since the fermion has been formally integrated out.

### 6.2.4 Remove ambiguities with artificial gauging

One way to fix $\omega_{AVV}$, $\omega_{VVA}$, $\omega_{VAV}$ and $\omega_{AAA}$ using the constraint of gauge invariance is to introduce fictitious [10] vector and axial vector gauge fields associated to the $\pi_S$ and $\pi_U$ Goldstone bosons. These fictitious gauge fields then enter in the effective operators of Eq. (6.15), and prevent gauge invariance from being automatic under partial integration. They also prevent the contributions involving the would-be-Goldstone bosons of the true symmetries to vanish. This trick, introduced in Ref. [158], is the key to derive non-trivial constraints and fix the ambiguous coefficients.

One may be a bit uneasy about this gauging of the global symmetries since these are precisely the symmetries that are anomalous. Actually, in the following, we will never need to use the fictitious gauge invariance in any form. All that matters is that these fictitious gauge fields act as background fields for $\partial_\mu \pi_S$ and $\partial_\mu \pi_U$, so as to upset the automatic (true) gauge invariances. This is sufficient to derive non-trivial constraints from the true, non-anomalous gauge symmetries.

Yet, as advocated in Ref. [158], it can also be technically interesting to view these background fields as fictitious gauge fields, because then all the symmetries are treated on the same footing. As we will detail in section 6.3, the calculation of the EFT becomes fully generic. The nice feature is that under this form, one can decide only at the very end which of the gauge symmetries is to be anomalous, hence fictitious, by imposing the exact invariance of the EFT under the *other* gauge symmetries, those that are kept active.

To illustrate all that, let us thus rewrite our initial Lagrangian as

$$\mathcal{L}_{\text{UV,I}}^{\text{fermion}} = \bar{\Psi}\Big[i\partial_\mu\gamma^\mu - M + \Big(V_\mu - \frac{\partial_\mu \pi_V}{2v_V}\Big)\gamma^\mu - \Big(A_\mu - \frac{\partial_\mu \pi_A}{2v_A}\Big)\gamma^\mu\gamma^5 + \Big(S_\mu - \frac{\partial_\mu \pi_S}{2v_S}\Big)\gamma^\mu + \Big(U_\mu - \frac{\partial_\mu \pi_U}{2v_U}\Big)\gamma^\mu\gamma^5\Big]\Psi\,. \tag{6.18}$$

In this Lagrangian, the $\partial_\mu \pi_V$ piece is irrelevant, since it can be eliminated by an innocuous reparametrisation, but let us keep it anyway for now. Integrating out the fermion leads to the EFT :

$$\begin{aligned}\mathcal{L}_{\text{EFT,I}}^{\text{1loop}} = {}& \omega_{VVA}\Big(S_\mu - \frac{\partial_\mu \pi_S}{2v_S}\Big)\Big(V_\nu - \frac{\partial_\nu \pi_V}{2v_V}\Big)\tilde{F}_A^{\mu\nu} + \omega_{AVV}\Big(U_\mu - \frac{\partial_\mu \pi_U}{2v_U}\Big)\Big(V_\nu - \frac{\partial_\nu \pi_V}{2v_V}\Big)\tilde{F}_V^{\mu\nu} \\ &+ \omega_{VAV}\Big(S_\mu - \frac{\partial_\mu \pi_S}{2v_S}\Big)\Big(A_\nu - \frac{\partial_\nu \pi_A}{2v_A}\Big)\tilde{F}_V^{\mu\nu} + \omega_{AAA}\Big(U_\mu - \frac{\partial_\mu \pi_U}{2v_U}\Big)\Big(A_\nu - \frac{\partial_\nu \pi_A}{2v_A}\Big)\tilde{F}_A^{\mu\nu}\,,\end{aligned} \tag{6.19}$$

with again the ambiguous coefficients $\omega_{AVV}$, $\omega_{VVA}$, $\omega_{VAV}$ and $\omega_{AAA}$ (the details of the calculation will be presented in the next section). All these interactions are still automatically gauge invariant thanks to the presence of the would-be-Goldstone bosons. Now, the key is to remember that the true gauge interactions are anomaly-free by assumption. This means the $\pi_A$ can be freely moved to the mass term by a reparametrisation of the fermion field, without Jacobian, and as said above, the $\partial_\mu \pi_V$ term can be discarded, again without Jacobian. Thus, the UV Lagrangian can equivalently be written as

$$\mathcal{L}_{\text{UV,II}}^{\text{fermion}} = \bar{\Psi}\Big[i\partial_\mu\gamma^\mu - M\Big(1 + \frac{\pi_A}{v_A}i\gamma^5\Big) + V_\mu\gamma^\mu - A_\mu\gamma^\mu\gamma^5 + \Big(S_\mu - \frac{\partial_\mu \pi_S}{2v_S}\Big)\gamma^\mu + \Big(U_\mu - \frac{\partial_\mu \pi_U}{2v_U}\Big)\gamma^\mu\gamma^5\Big]\Psi\,. \tag{6.20}$$

[10] At the end of the day, we will still want the global symmetry to stay global and to set to zero these fictitious vector fields.

This time, there is no ambiguity in calculating the Wilson coefficients of the operators involving the would-be-Goldstone bosons. The five-dimensional effective interactions become

$$\begin{aligned}\mathcal{L}^{\rm 1loop}_{\rm EFT,II} \supset\, &\omega_{VVA}\left(S_\mu - \frac{\partial_\mu \pi_S}{2v_S}\right)V_\nu \tilde{F}^{\mu\nu}_A + \omega_{AVV}\left(U_\mu - \frac{\partial_\mu \pi_U}{2v_U}\right)V_\nu \tilde{F}^{\mu\nu}_V \\ &+ \omega_{VAV}\left(S_\mu - \frac{\partial_\mu \pi_S}{2v_S}\right)A_\nu \tilde{F}^{\mu\nu}_V + \eta_{ASV}\frac{\pi_A}{v_A}F_{S,\mu\nu}\tilde{F}^{\mu\nu}_V \\ &+ \omega_{AAA}\left(U_\mu - \frac{\partial_\mu \pi_U}{2v_U}\right)A_\nu \tilde{F}^{\mu\nu}_A + \eta_{AUA}\frac{\pi_A}{v_A}F_{U,\mu\nu}\tilde{F}^{\mu\nu}_A\,, \end{aligned} \tag{6.21}$$

where $\omega_{AVV}$, $\omega_{VVA}$, $\omega_{VAV}$ and $\omega_{AAA}$ are ambiguous, but not $\eta_{ASV}$ and $\eta_{AUA}$ since they arise from convergent integrals. Importantly, under this form, the true $U(1)_V$ and $U(1)_A$ gauge invariances are no longer automatic.

Now, we end up with two equivalent ways to fix the ambiguities. Either we enforce the matching of Eq. (6.21) with Eq. (6.19), or we impose gauge invariance on Eq. (6.21). In both cases, the constraints take the same form, but the latter is obviously more economical from a calculation point of view and will be adopted in the next sections.

For instance, for the vector gauge fields, since $\pi_V$ is absent from Eq. (6.21), matching with Eq. (6.19) requires $\omega_{VVA}$ and $\omega_{AVV}$ to vanish. Equivalently, invariance of Eq. (6.21) under $V_\mu \to V_\mu + \partial_\mu \theta_V$ immediately imposes $\omega_{VVA} = \omega_{AVV} = 0$. This corresponds to the usual result that for vector gauge interactions, the derivative interactions of a Goldstone boson with the fermions contributes only at the subleading order in the mass expansion, otherwise known as the Sutherland-Veltman theorem. The local Jacobian terms in Eq. (6.14) immediately catch the whole $\pi_U VV$ coupling.

For the axial gauge field, matching Eq. (6.21) with Eq. (6.19) obviously permits to fix the ambiguous $\omega_{VAV}$ and $\omega_{AAA}$ in terms of $\eta_{AVS}$ and $\eta_{AUA}$, which are fully calculable. Alternatively, performing a $U(1)_A$ gauge transformation $A_\mu \to A_\mu + \partial_\mu \theta_A$ together with $\pi_A \to \pi_A + 2v_A\theta_A$ in Eq. (6.21) generates the gauge variation, after integrating by part and using the Bianchi identity,

$$\delta_A\big(\mathcal{L}^{\rm 1loop}_{\rm EFT,II}\big) = \left(\frac{1}{2}\omega_{VAV} + 2\eta_{ASV}\right)\theta_A F^{\mu\nu}_S \tilde{F}_{V,\mu\nu} + \left(\frac{1}{2}\omega_{AAA} + 2\eta_{AUA}\right)\theta_A F^{\mu\nu}_U \tilde{F}_{A,\mu\nu}\,. \tag{6.22}$$

Hence, the requirement of gauge invariance asks for

$$\delta_A\big(\mathcal{L}^{\rm 1loop}_{\rm EFT,II}\big) = 0 \Leftrightarrow \omega_{VAV} = -4\eta_{ASV} \text{ and } \omega_{AAA} = -4\eta_{AUA}\,. \tag{6.23}$$

The effective axion-bosonic Lagrangian is obtained by adding $\mathcal{L}^{\rm Jac}_{\rm UV}$ and $\mathcal{L}^{\rm 1loop}_{\rm EFT,II}$ and finally setting the fictitious vector fields to zero, this gives the result,

$$\mathcal{L}_{\rm EFT} = \frac{1}{16\pi^2}\frac{\pi_U}{v_U}F_{V,\mu\nu}\tilde{F}^{\mu\nu}_V + \left[\frac{1}{16\pi^2} - \eta_{AUA}\right]\frac{\pi_U}{v_U}F_{A,\mu\nu}\tilde{F}^{\mu\nu}_A + \left[\frac{1}{8\pi^2} - \eta_{ASV}\right]\frac{\pi_S}{v_S}F_{A,\mu\nu}\tilde{F}^{\mu\nu}_V\,. \tag{6.24}$$

Let us stress again that the $\eta_{AUA}$ and $\eta_{ASV}$ are fully calculable, unambiguous coefficients originating from convergent integrals. The determination of $\omega_{VAV}$ and $\omega_{AAA}$ from the requirement of gauge invariance is now transparent, and precisely matches that using Ward identities in a Feynman diagram context [149]. This is the general procedure we will adopt in the following to derive our bosonic EFTs. Of course, in the physical case, none of the interactions parametrised by $\eta_{ASV}$ and $\eta_{AUA}$ exist since they require the presence of the fictitious $U_\mu$ and $S_\mu$ gauge fields as background values[11]. Yet, this derivation sheds a new light on the violation of the Sutherland-Veltman theorem

[11] Looking back, it is clear that gauge invariance under these fictitious symmetries is never imposed in any form. All that matters is to prevent the would-be-Goldstone bosons from being automatically absent from both Eq. (6.19) and Eq. (6.21), and true gauge invariance from being automatic in both EFT Lagrangians.

in the presence of spontaneously broken axial gauge interactions. Ultimately, it is due to the contribution of the associated would-be-Goldstone boson. The net effect is that the $\pi_S VA$ and $\pi_U AA$ couplings are not fully determined by the corresponding terms in the Jacobian, Eq. (6.14), since derivative interactions do contribute at leading order in the inverse mass expansion.

## 6.3 Integrating out chiral fermions

In the previous section we discussed, qualitatively, peculiarities arising when building an EFT, while integrating out fermionic fields, from a UV theory with exact or spontaneous gauge symmetries and anomalous global symmetries. In this section we will, quantitatively, construct these EFTs involving gauge fields and their associated would-be-Goldstone bosons and simple Goldstone bosons associated to global symmetries. While the would-be Goldstone bosons can display derivative or pseudo-scalar couplings to fermions, since ultimately this depends on the fermion parametrisation (as we have discussed before), the Goldstone bosons will have to be taken firmly with local derivative couplings to fermions. Strictly speaking, from a path integral point of view, those details of the model are not mandatory to perform the main computation part, meaning forming the operator basis, evaluating the loop integrals after regularizing them. The symmetry aspects of the model will only matter at the very last stage when matching a UV theory onto its EFT.

In this section, we will briefly review the core techniques for calculating Wilson coefficients of EFT higher dimensional operators at the one-loop level by utilizing the functional approach. Since our interest is about the anomaly structure of specific QFTs, we will concentrate, in a general way, on the task of integrating out chiral fermions or fields which chiraly interact with gauge fields. We will also remind the reader how anomalies arise depending on how the one-loop effective action is regularised.

### 6.3.1 Evaluation of the fermionic effective action

We consider a generic UV theory containing a heavy Dirac fermion $\Psi$ of mass $M$ interacting bilinearly with a light field $\phi$, which is encapsulated inside the background function $\mathrm{X}[\phi]$ [12]. The matter Lagrangian of this generic UV theory can be written as follows,

$$\mathcal{L}_{\text{UV}}^{\text{fermion}}[\Psi,\phi] \supset \bar{\Psi}\Big[P_\mu\gamma^\mu - M + X[\phi]\Big]\Psi = \bar{\Psi}\,\mathcal{Q}_{\text{UV}}[\phi]\,\Psi\,, \tag{6.25}$$

where $P_\mu = i\partial_\mu$ and introducing $\mathcal{Q}_{UV}[\phi]$ the fermionic quadratic operator. The background function $X[\phi]$ that we will consider throughout this chapter is

$$X[\phi] = V_\mu[\phi]\gamma^\mu - A_\mu[\phi]\gamma^\mu\gamma^5 - W_1[\phi]i\gamma^5\,, \tag{6.26}$$

where we decompose $X[\phi]$ in terms of vector $V_\mu[\phi]$, axial-vector $A_\mu[\phi]$ and pseudo-scalar $W_1[\phi]$ structures[13], which are all the different types of interactions we will need to match our "axion motivated" UV theory to an EFT. In order to obtain the fermionic one-loop effective action, the

[12] For simplicity, we will consider $\Psi$ and $\phi$ as singlets but the following procedure is more general and it is still possible to treat them as multiplets.

[13] We note that $V_\mu[\phi]$, $A_\mu[\phi]$ and $W_1[\phi]$ do not contain any Dirac matrices or momentum variables $q_\mu$. The structures $V_\mu[\phi]$ and $A_\mu[\phi]$ can include gauge fields or local derivative of scalar fields.

light field $\phi$ is treated classically, integrating out the fermion field $\Psi$ yields[14]

$$\begin{aligned} e^{iS_{\rm EFT}^{\rm 1loop}[\phi]} &= \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi\, e^{iS_{\rm UV}[\Psi,\,\phi]} \\ &\simeq e^{iS_{\rm UV}[\Psi_c,\,\phi]} \int \mathcal{D}\bar{\eta}\,\mathcal{D}\eta\, e^{i\int d^4x\,\bar{\eta}\mathcal{Q}_{\rm UV}[\phi]\eta} = e^{iS_{\rm UV}[\Psi_c,\,\phi]}\,\det \mathcal{Q}_{\rm UV}[\phi] \\ &= e^{iS_{\rm UV}[\Psi_c,\,\phi]} e^{\mathrm{Tr}\ln \mathcal{Q}_{\rm UV}[\phi]}\,, \end{aligned} \tag{6.27}$$

where in the second line of Eq. (6.27) we have expanded the fermion fields around their classical background values, $\Psi = \Psi_c + \eta$ and performed the integration over the quantum fluctuations $\eta$. Eventually, we have traded the functional determinant for the functional trace, "Tr", running over the functional space and internal indices of the quadratic operator, $\mathcal{Q}_{\rm UV}[\phi]$. We therefore arrive at the one-loop effective action arising from integrating out a fermion:

$$S_{\rm EFT}^{\rm 1loop} = -i\,\mathrm{Tr}\ln(\not{P} - M + V_\mu[\phi]\gamma^\mu - A_\mu[\phi]\gamma^\mu\gamma^5 - W_1[\phi]i\gamma^5)\,. \tag{6.28}$$

Generally, in the functional space, one can write the quadratic operator as a function of position, $\hat{x}$, and momentum, $\hat{p}$, operators. Projecting onto position space, these operators become $\hat{x} = x$ and $\hat{p}_\mu = i\partial_\mu$. The standard initial step is to evaluate the trace over functional space by inserting the momentum eigenstate basis together with employing the canonical quantum mechanical trick of inserting the identity matrix, $\int d^4x |x\rangle\langle x| = \mathbb{1}$ [15],

$$\begin{aligned} S_{\rm EFT}^{\rm 1loop} &= -i \int \frac{d^4q}{(2\pi)^4} \langle q|\mathrm{tr}\ln \mathcal{Q}_{UV}(\hat{x},\hat{p}_\mu)|q\rangle \\ &= -i \int d^4x \int \frac{d^4q}{(2\pi)^4}\, \langle q|\, x\rangle\, \langle x|\mathrm{tr}\ln \mathcal{Q}_{UV}(\hat{x},\hat{p}_\mu)|q\rangle \\ &= -i \int d^4x \int \frac{d^4q}{(2\pi)^4}\, e^{iq\cdot x}\,\mathrm{tr}\ln \mathcal{Q}_{UV}(x, i\partial_\mu)e^{-iq\cdot x} \\ &= \int d^4x \int \frac{d^4q}{(2\pi)^4}(-i)\,\mathrm{tr}\ln \mathcal{Q}_{UV}(x, i\partial_\mu - q_\mu)\,, \end{aligned} \tag{6.29}$$

where "tr" now denotes the trace over spinor and internal symmetry indices only. Here the $\langle x|$ denotes the eigenstate of local operator in position space, e.g. $\langle x|\mathcal{Q}_{UV}(\hat{x},\hat{p}) = \mathcal{Q}_{UV}(x, i\partial_\mu)\langle x|$, and the convention for inner product is $\langle x|\, q\rangle = e^{-iq\cdot x}$. An "open" derivative from the kinematic operator will get shifted due to $e^{iq\cdot x} i\partial_\mu e^{-iq\cdot x} = i\partial_\mu + q_\mu$. We perform also a conventional change of integration variable $q \to -q$. As we will study later, we emphasise that in the case where one has to deal with a local derivative of a bosonic field, e.g. $\left[\partial_\mu \pi(x)\right]$, this term will not be shifted under the sandwich of $e^{iq\cdot x}\left[\partial_\mu \pi(x)\right]e^{-iq\cdot x}$ since the partial derivative of this coupling is "closed". Therefore, on the computational side, depending on the vector or axial-vector nature of the local derivative couplings, one can absorb these terms into the vector ($V_\mu[\phi]$) and axial-vector ($A_\mu[\phi]$) structures of the UV quadratic operator [16].

Ultimately, the expansion of the logarithm in terms of a series of local operators suppressed by

[14] The quantity $S_{\rm EFT}^{\rm 1loop}$ corresponds to the fermion 1PI action and it is formally divergent. We will discuss its gauge variation and its regularisation in the following.

[15] For the reader who would like to investigate in details the whole computation steps, we recommend Refs.[72, 75, 76, 78].

[16] This underlines the practical usefulness of our initial choice of parametrisation made in Eq. (6.28)

the fermion mass scale can be performed by a variety of techniques,

$$\begin{aligned}\mathcal{L}_{\rm EFT}^{\rm 1loop} &= -i\,{\rm Tr}\ln(\not{P} - \not{q} - M - X[\phi]) \\ &= i\,{\rm tr}\sum_{n=1}^{\infty}\frac{1}{n}\int\frac{d^4q}{(2\pi)^4}\left[\frac{-1}{\not{q}+M}\Big(-\not{P} - V_\mu[\phi]\gamma^\mu + A_\mu[\phi]\gamma^\mu\gamma^5 + W_1[\phi]i\gamma^5\Big)\right]^n . \end{aligned} \tag{6.30}$$

The remarkable point at this stage is that the q-momentum integration can be factorised out from the generic operator structures. Indeed, regardless of the method used to evaluate the logarithm expansion, it can be done once-and-for-all, and the result is the same and universal in the sense that the final expression is independent of the details of the UV Lagrangian, which remain encapsulated in the X matrix of light fields, covariant derivative $P_\mu$, and mass matrix $M$. This leads to the so-called concept of the Universal One-Loop Effective Action (UOLEA) (see Refs [3, 77, 80–82]).

Note that in our calculations we will deal with multiple vector, axial vector and pseudo-scalar interactions so we will consider in all generalities

$$V_\mu[\phi] \equiv g_V^i V_\mu^i[\phi^i]\,,\ A_\mu[\phi] \equiv g_A^i A_\mu^i[\phi^i]\,,\ W_1[\phi] \equiv g_{W_1}^i W_1^i[\phi^i]\,, \tag{6.31}$$

with an implicit summation over the i index.

### 6.3.2 Ambiguities and regularisation of the functional trace

The evaluation of the one-loop effective Lagrangian Eq. (6.30) usually encounters divergent integrals and we use dimensional regularisation [105] to evaluate them along with the $\overline{MS}$ scheme for renormalisation. The traces over Dirac matrices have to be performed in $d = 4 - \epsilon$ dimensions, and the $\epsilon$-terms resulting from the contractions with the metric tensor (satisfying then $g^{\mu\nu}g_{\mu\nu} = d$) must be kept in the computations. These $\epsilon$-terms will then multiply with the $(1/\epsilon)$ pole of the divergent integrals and yield finite contributions. We emphasise that depending on the regularisation scheme for $\gamma^5$ in $d$-dimensions, different results for $\epsilon$-terms in Dirac traces will emerge (see for examples Refs. [103, 159, 160]). We will come back shortly to describe in details the prescription we used to evaluate ill-defined Dirac traces involving $\gamma^5$ matrices, in dimensional regularisation.

We now turn back on the ambiguities arising in some of our integrals in the 4-dimensional space. Usually, when computing one-loop divergent triangle Feynman diagrams (corresponding to the Adler–Bell–Jackiw anomaly [121, 140]), it is well-known that, in $d = 4$ dimensions, an ambiguity of the loop integral arises. It corresponds to an arbitrariness in the chosen integration variables (see Ref. [157]), and actually there can be surface terms that do depend on the chosen momentum routing. Those surface terms then contribute to the divergence of vector-currents and axial-vector-currents, and all the naive Ward identities cannot be satisfied simultaneously. At least some of them will be anomalous. The important point is that the arbitrariness of integral variable can be parametrised in terms of free parameters (see the standard Refs.[118, 157] and the more recent Refs.[149, 158]). By tuning the value of those free parameters, one can decide which symmetry is broken at the quantum level, and which are kept active. Evidently, to obtain the correct physical results, all the gauge symmetries must be preserved.

When switching to the d-dimensional space, the ambiguity on the loop integrals does not arise anymore from dependencies on the chosen momentum routing, but it is now inherent from the Dirac algebra sector. Indeed, not all the usual properties of the Dirac matrices can be maintained once in $d > 4$ dimensions, essentially because $\gamma^5$ and the anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ are intrinsically four-dimensional objects. Whatever the chosen definition, there is no way to consistently preserve both the anticommutativity properties of $\gamma^5$ matrices, i.e. $\{\gamma^\mu, \gamma^5\} = 0$, and the trace cyclicity property in $d > 4$ dimensions. In the original work of ’t Hooft and Veltman [105], they noted that the momentum routing ambiguity is replaced by an ambiguity in the location of $\gamma^5$ in the Dirac traces. Using their prescriptions for the Dirac algebra in $d > 4$ dimensions (see Refs.[105, 106]), it is then possible to introduce free parameters keeping track of all the possible $\gamma^5$ locations in a given Dirac matrices string [161]. As before, one can then tune these parameters to choose which symmetry is broken anomalously, and which one have to be preserved. This is the strategy we will employ to calculate the ambiguous Dirac traces in Eq. (6.30).

### 6.3.3 Evaluation of the anomaly related operators

We now concentrate on the derivation of the operators which ultimately involve a mixture of three gauge fields and Goldstone bosons with a derivative acting on them. With our parametrisation, they arise from combinations of the generic vector $V_\mu[\phi]$ and axial-vector $A_\mu[\phi]$ fields. Due to the presence of $\gamma_5$ Dirac matrices in their Wilson coefficients, they are truly ambiguous in dimensional regularisation. Then we will proceed with the evaluation of operators involving one Goldstone boson (without any derivative acting on it), namely the $W_1[\phi]$ field in our generic parametrisation. These operators have been evaluated using the usual Feynman diagrams technique (see Refs.[149, 158, 162]). Since those computations are subtle and lead to confusions, this is legitimate to wonder how one would perform them from a different point of view, such as within the path integral formalism. Which is what we present now.

#### 6.3.3.1 Evaluation of the ambiguous terms

We start with the exercise of computing the divergent terms that naturally arise when evaluating Eq. (6.30). The generic form of these operators is

$$G^i_\mu G^j_\nu \tilde{F}^k_{\mu\nu} = G^i_\mu G^j_\nu \left(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_{[\rho}G^k_{\sigma]}\right) , \tag{6.32}$$

where we use the notation $G^i_\mu$ to denote a generic gauge field and to avoid confusions with the vector and axial-vector structures in Eq. (6.30). We also introduce the upper indices $i$,$j$,$k$ to keep the computation as general as possible and offer us the possibility to apply this computations to multiple gauge field configurations later on. Since starting with Eq. (6.30), we chose to deal with vector and axial-vector structures, in order to reconstruct the ambiguous operators in the EFT, we need

- One insertion of $P_\mu$ to account for the partial derivative and then allow to form a field strength tensor.
- Several combinations of vector and axial-vector structures. It is clear that to generate the anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ the product of Dirac matrices must involve an odd number of $\gamma^5$ matrix. It exists only two possibilities, either an "$AVV$" contribution with one $\gamma^5$ or an "$AAA$" contribution with three $\gamma^5$.

While evaluating the one-loop effective Lagrangian of Eq. (6.30), several contributions to the ambiguous effective interaction would arise from the $n = 4$ polynomial terms

$$\begin{aligned}\mathcal{L}^{\text{1loop}}_{\text{EFT}} &\supset i\,\text{tr}\,\frac{1}{4}\int\frac{d^dq}{(2\pi)^d}\left[\frac{-1}{\not{q}+M}\Big(-P_\mu\gamma^\mu - V_\mu[\phi]\gamma^\mu + A_\mu[\phi]\gamma^\mu\gamma^5 + W_1[\phi]\gamma^5\Big)\right]^4 \\ &\supset \sum_N f_N^{AVV}\mathbb{O}^{(PAVV)} + f_N^{AAA}\mathbb{O}^{(PAAA)} ,\end{aligned} \tag{6.33}$$

where $\mathbb{O}^{(PAVV)}$ denotes the class of operator containing one $\gamma^5$ matrix and $\mathbb{O}^{(PAAA)}$ the one containing three $\gamma^5$ matrices.

**Evaluation of the $\mathbb{O}^{(PAVV)}$ structures.** There are three different types of combination that contribute to the $\mathbb{O}^{(PAVV)}$ structure, namely $\mathcal{O}(VVPA)$, $\mathcal{O}(VAPV)$, $\mathcal{O}(AVPV)$. Each type of combination contains four universal structures, and it is related together via trace cyclicity. At this stage, due to the ambiguity of $\gamma^5$-positions, one should not use trace cyclicity to minimise the number of universal structures that need to be evaluated. To present in detail the evaluation procedure of the Dirac trace and its regularisation, let us focus on one explicit example out of 12 universal structures included in Eq. (6.33)

$$\begin{aligned}\mathcal{O}(VVPA) \supset & \frac{1}{4}\int \frac{d^dq}{(2\pi)^d}\,\mathrm{tr}\left[\frac{-1}{\not{q}+M}V_\mu\gamma^\mu\frac{-1}{\not{q}+M}V_\nu\gamma^\nu\frac{-1}{\not{q}+M}P_\rho\gamma^\rho\frac{-1}{\not{q}+M}A_\sigma\gamma^\sigma\gamma^5\right]\\ =& \frac{i}{4}\left[-4M^4\mathcal{I}_i^4+16M^2\mathcal{I}[q^2]_i^4\right]\mathrm{tr}\Big(\epsilon^{\mu\nu\rho\sigma}V_\mu V_\nu P_\rho A_\sigma\Big)\\ &+\frac{1}{4}\mathcal{I}[q^4]_i^4\left[g^{ab}g^{cd}+g^{ac}g^{bd}+g^{ad}g^{bc}\right]\mathrm{tr}\Big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\gamma^5\Big)\Big(V_\mu V_\nu P_\rho A_\sigma\Big)\,,\end{aligned} \tag{6.34}$$

where the fermion propagators are decomposed into $\frac{-1}{\not{q}+M}=\frac{M}{q^2-M^2}+\frac{-\not{q}}{q^2-M^2}$. For the tensorial integrals, we use

$$\int\frac{d^dq}{(2\pi)^d}\frac{q^{\mu_1}\cdots q^{\mu_{2n_c}}}{(q^2-m_i^2)^{n_i}(q^2-m_j^2)^{n_j}\cdots}=g^{\mu_1\cdots\mu_{2n_c}}\mathcal{I}[q^{2n_c}]_{ij\cdots}^{n_in_j\cdots}\,, \tag{6.35}$$

where $g^{\mu_1\cdots\mu_{2n_c}}$ is the completely symmetric tensor, e.g. $g^{\mu\nu\rho\sigma}=g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}$, and we denote the master integrals as $\mathcal{I}[q^{2n_c}]_{ij\cdots}^{n_in_j\cdots}$. The explicit expression and the value of some useful master integrals are derived in the Appendix A. In the second line of Eq. (6.34), all the loop integrals are finite, one can then evaluate the various Dirac traces in the usual naive scheme. The last line of Eq. (6.34) contains divergent integrals, $\mathcal{I}[q^4]_i^4$, which have to be regularised. Let us show how to evaluate such an ambiguous quantity as $\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\gamma^5\big)$ of Eq. (6.34). We follow the procedure described earlier in the section 6.3.2. Before evaluating the Dirac trace in $d$-dimension, we first write down all possible structures that are equivalent to the original Dirac string by naively anti-commuting $\gamma^5$,

$$\begin{aligned}\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\gamma^5\big)\to\,&\bar{a}_1\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma^5\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\big)+\bar{a}_2\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma^5\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\big)\\ &+\bar{a}_3\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma^5\gamma_d\gamma^\sigma\big)+\bar{a}_4\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\gamma^5\big)\,,\end{aligned} \tag{6.36}$$

where we introduce the four free parameters, $\bar{a}_i$, to keep track of the position of the $\gamma_5$ matrix in Eq. (6.36). Let us briefly comment on the fact that

- In $d=4$ dimensions, all Dirac structures on the R.H.S of Eq. (6.36) are equivalent.
- In $d=4-\epsilon$ dimensions, by using the Breitenlohner-Maison-'t Hooft-Veltman (BMHV) scheme (see Refs.[105, 106]), the $\gamma^5$ matrix does not anti-commute anymore with Dirac $\gamma^\mu$ matrices. Therefore, each Dirac trace will give a different result due to the different position of $\gamma^5$ matrix. The free parameters $\bar{a}_i$ is a device to keep track of the $\gamma^5$-positions.
- Enforcing a consistent result in $d=4$ and $d=4-\epsilon$ dimensions requires that $\sum_{i=1}^4\bar{a}_i=1$.

After plugging Eq. (6.36) into Eq. (6.34), one obtains

$$\begin{aligned}\mathcal{O}(VVPA)\supset&\frac{1}{4}\left[4M^4\mathcal{I}_i^4-16M^2\mathcal{I}[q^2]_i^4-24\,\epsilon\,(-\bar{a}_1+\bar{a}_2-\bar{a}_3+\bar{a}_4)\,\mathcal{I}[q^4]_i^4\right]\mathrm{tr}\Big(\epsilon^{\mu\nu\rho\sigma}V_\mu V_\nu P_\rho A_\sigma\Big)\\ =&\frac{1}{32\pi^2}\left[-1-\bar{a}_1+\bar{a}_2-\bar{a}_3+\bar{a}_4\right]\mathrm{tr}\left[V_\mu^iV_\nu^j\tilde{F}_{\mu\nu}^{A^k}\right]\,,\end{aligned} \tag{6.37}$$

where in the last line of Eq. (6.37), we replace the vector and axial-vector structures by $V_\mu \equiv g^i_{V} V^i_\mu$, $A_\mu \equiv g^i_{A} A^i_\mu$, we also omit the gauge couplings to simplify the expression of (6.37) and highlight the final value of loop integrals and Dirac traces. We remind the reader that $g^i_V$ and $g^i_A$ will only appear when it is necessary. Also, keep in mind that in Eq. (6.37) the $\epsilon$-terms will hit the pole $\frac{1}{\epsilon}$ of the divergence integral, $\mathcal{I}[q^4]^4_i$, and generate finite contributions. We then apply the same method for the other contributions in $\mathbb{O}^{PAVV}$. One should note that since in Eq. (6.30), $P_\mu = i\partial_\mu$, is the "open" derivative one can therefore omit the operator structures which start with a $P_\mu$ since they lead to inert boundary terms. We underline one more time that at this stage, one cannot use the cyclicity property of the trace to reduce the number of terms that need to compute. Adding all the different contributions together gives

$$\begin{aligned}\mathcal{L}^{\text{1loop}}_{\text{EFT}} \supset\, & i\big(\, 24\epsilon\, \bar{a}_{V^iV^jA^k}\, \mathcal{I}[q^4]^4_i\,\big)\text{tr}\Big[V^i_\mu V^j_\nu \tilde{F}^{A^k}_{\mu\nu}\Big] \\ & + i\big(-4M^4\mathcal{I}^4_i + 16M^2\mathcal{I}[q^2]^4_i + 24\epsilon\, \bar{a}_{V^jA^kV^i}\, \mathcal{I}[q^4]^4_i\,\big)\text{tr}\Big[V^j_\mu A^k_\nu \tilde{F}^{V^i}_{\mu\nu}\Big] \\ & + i\big(4M^4\mathcal{I}^4_i - 16M^2\mathcal{I}[q^2]^4_i + i\,24\epsilon\, \bar{a}_{A^kV^iV^j}\, \mathcal{I}[q^4]^4_i\,\big)\text{tr}\Big[A^k_\mu V^i_\nu \tilde{F}^{V^j}_{\mu\nu}\Big]\,. \end{aligned} \tag{6.38}$$

Since the $\bar{a}_i$ coefficients are basically free, there are no reasons to give any physical meaning to the different contributions. For each operator structure, we redefine the total values of $\bar{a}_i$ by the new free parameters, e.g. $\bar{a}_{V^iV^jA^k}$, $\bar{a}_{V^jA^kV^i}$, $\bar{a}_{A^kV^iV^j}$. Readout the value of loop integrals, the above equation reduces to

$$\mathcal{L}^{\text{1loop}}_{\text{EFT}} \supset \frac{1}{8\pi^2}\bar{a}_{V^iV^jA^k}\text{tr}\Big[V^i_\mu V^j_\nu \tilde{F}^{A^k}_{\mu\nu}\Big] + \frac{1}{8\pi^2}\bar{a}_{V^jA^kV^i}\text{tr}\Big[V^j_\mu A^k_\nu \tilde{F}^{V^i}_{\mu\nu}\Big] + \frac{1}{8\pi^2}\bar{a}_{A^kV^iV^j}\text{tr}\Big[A^k_\mu V^i_\nu \tilde{F}^{V^j}_{\mu\nu}\Big]\,. \tag{6.39}$$

The three operators of Eq. (6.39) are not independent and by using integration by parts one should always end up with two independent operators and then two free parameters. As we will see later, in practice one decides to remove such or such operator by use of integration by parts based on the symmetries that are preserved or not since all operators are not invariant under the same vector or axial symmetries. As an example, if one supposes that, within our notation, the $V^i$ current might be anomalous, one may integrate by parts the first operator of Eq. (6.38), $\text{tr}\big(V^i_\mu V^j_\nu \tilde{F}^{A^k}_{\mu\nu}\big)$, and after discarding the total derivative operator, and redefining the free parameters, one obtains

$$\mathcal{L}^{\text{1loop}}_{\text{EFT}} \supset \frac{1}{8\pi^2}\bar{a}_{V^jA^kV^i}\text{tr}\Big[V^j_\mu A^k_\nu \tilde{F}^{V^i}_{\mu\nu}\Big] + \frac{1}{8\pi^2}\bar{a}_{A^kV^iV^j}\text{tr}\Big[A^k_\mu V^i_\nu \tilde{F}^{V^j}_{\mu\nu}\Big]\,. \tag{6.40}$$

At this point, one should comment on the fact that if one would have used the BMHV scheme without performing the decomposition of Eq. (6.36), one would have found each Wilson coefficients of the operators in Eq. (6.38) to vanish. This is ultimately due to the fact that, by default, vector currents cannot be anomalous while only following the BMHV procedure. Even if one would have expected to be able to write effective operators as displayed in Eq. (6.39) from the first principle, we have rigorously shown how to obtain it in dimensional regularisation, i.e the "AVV" interaction can be described by two independent operators for which it exists two Wilson coefficients which are ambiguous i.e free.

**Evaluation of the $\mathbb{O}^{(PAAA)}$ structures.** We now turn to the second class of operator, $\mathbb{O}^{(PAAA)}$, that contains three $\gamma^5$ matrices. Similarly to the previous case with $\mathbb{O}^{(PAVV)}$, we start here by giving an explicit example for an operator that belongs to this class,

$$\begin{aligned}\mathbb{O}^{AAPA} &\supset \frac{1}{4}\int \frac{d^dq}{(2\pi)^d}\,\mathrm{tr}\left[\frac{-1}{\not{q}+M}A_\mu\gamma^\mu\gamma^5\frac{-1}{\not{q}+M}A_\nu\gamma^\nu\gamma^5\frac{-1}{\not{q}+M}P_\rho\gamma^\rho\frac{-1}{\not{q}+M}A_\sigma\gamma^\sigma\gamma^5\right]^4 \\ &= i\,\frac{1}{4}\left[4M^4\mathcal{I}_i^4+16M^2\mathcal{I}[q^2]_i^4\right]\mathrm{tr}\Big(\epsilon^{\mu\nu\rho\sigma}A_\mu A_\nu P_\rho A_\sigma\Big) \\ &+\frac{1}{4}\mathcal{I}[q^4]_i^4\left[g^{ab}g^{cd}+g^{ac}g^{bd}+g^{ad}g^{bc}\right]\mathrm{tr}\Big(\gamma_a\gamma^\mu\gamma^5\gamma_b\gamma^\nu\gamma^5\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\gamma^5\Big)\Big(A_\mu A_\nu P_\rho A_\sigma\Big)\,, \end{aligned} \tag{6.41}$$

we then parameterise the ambiguous Dirac trace, $\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma^5\gamma_b\gamma^\nu\gamma^5\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\gamma^5\big)$, by using

$$\begin{aligned}\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma^5\gamma_b\gamma^\nu\gamma^5\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\gamma^5\big) &\to \bar{b}_1\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma^5\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\big)+\bar{b}_2\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma^5\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\big) \\ &+\bar{b}_3\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma^5\gamma_d\gamma^\sigma\big)+\bar{b}_4\mathrm{tr}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\gamma^5\big)\,.\end{aligned} \tag{6.42}$$

Afterwards, evaluating in $d=4-\epsilon$ dimensions with BMHV's scheme, we obtain

$$\begin{aligned}\mathbb{O}^{AAPA} &\supset \frac{i}{4}\left[4M^4\mathcal{I}_i^4+16M^2\mathcal{I}[q^2]_i^4+24\,\epsilon\,\big(-\bar{b}_1+\bar{b}_2-\bar{b}_3+\bar{b}_4\big)\,\mathcal{I}[q^4]_i^4\right]\mathrm{tr}\Big(\epsilon^{\mu\nu\rho\sigma}A_\mu A_\nu P_\rho A_\sigma\Big) \\ &= \frac{1}{32\pi^2}\left[\frac{1}{3}+\bar{b}_1-\bar{b}_2+\bar{b}_3-\bar{b}_4\right]\mathrm{tr}\left[A^i_\mu A^j_\nu \tilde{F}^{A^k}_{\mu\nu}\right]\,,\end{aligned} \tag{6.43}$$

where in the last step of the computation we evaluate the value of loop integrals, express $A_\mu \equiv A^i_\mu$. We also note that $g^i_A$ will appear when it is necessary. The computation for the other operators belonging to $\mathbb{O}^{(PAAA)}$ are similar and the full result reads

$$\begin{aligned}\mathcal{L}^{\text{1loop}}_{\text{EFT}} &\supset \big(i\,24\epsilon\bar{b}_{A^iA^jA^k}\,\mathcal{I}[q^4]_i^4\big)\mathrm{tr}\left[A^i_\mu A^j_\nu\tilde{F}^{A^k}_{\mu\nu}\right]+\big(i\,24\epsilon\bar{b}_{A^jA^kA^i}\,\mathcal{I}[q^4]_i^4\big)\mathrm{tr}\left[A^j_\mu A^k_\nu\tilde{F}^{A^i}_{\mu\nu}\right] \\ &+\big(i\,24\epsilon\bar{b}_{A^kA^iA^j}\,\mathcal{I}[q^4]_i^4\big)\mathrm{tr}\left[A^k_\mu A^i_\nu\tilde{F}^{A^j}_{\mu\nu}\right]\,.\end{aligned} \tag{6.44}$$

which basically resumes to

$$\mathcal{L}^{\text{1loop}}_{\text{EFT}} \supset \frac{1}{8\pi^2}\bar{b}_{A^iA^jA^k}\mathrm{tr}\left[A^i_\mu A^j_\nu\tilde{F}^{A^k}_{\mu\nu}\right]+\frac{1}{8\pi^2}\bar{b}_{A^jA^kA^i}\mathrm{tr}\left[A^j_\mu A^k_\nu\tilde{F}^{A^i}_{\mu\nu}\right]+\frac{1}{8\pi^2}\bar{b}_{A^kA^iA^j}\mathrm{tr}\left[A^k_\mu A^i_\nu\tilde{F}^{A^j}_{\mu\nu}\right]\,. \tag{6.45}$$

These three operators in Eq. (6.45) are not independent and one is free to remove one by the use of integration by parts. Consequently, in dimensional regularisation, the "AAA" interaction can be described by two independent operators attached to two free Wilson coefficients reflecting ambiguities in the evaluations of such interactions.

#### 6.3.3.2 Evaluation of the pseudo-scalar unambiguous terms

We are now looking for to evaluate operators involving a pseudo-scalar $\phi$ (without local partial derivative acting on it) and two field strength tensors. The generic operator form is given by

$$\phi\, F^j_{\mu\nu}\tilde{F}^k_{\mu\nu} = \phi\,\frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\big(\partial_{[\mu}G^j_{\nu]}\big)\big(\partial_{[\rho}G^k_{\sigma]}\big)\,, \tag{6.46}$$

To reconstruct the pseudo-scalar terms from the expansion of Eq. (6.30), we need

- Two insertions of $P_\mu$ to account for the two partial derivatives, then forming field strength tensors.
- One insertion of $W_1[\phi]$ to account for the pseudo-scalar field $\phi$.
- To account for the two gauge fields, we need $VV$ and $AA$ structures. Since $W_1[\phi]$ contains a $\gamma^5$, the combination with $AV$ structure will not contribute to the final result.

We collect the relevant classes of operators that contribute to the Wilson coefficients of these pseudo-scalar terms,

$$\begin{aligned}\mathcal{L}_{\rm EFT}^{\rm 1loop} &\supset i\,{\rm tr}\,\frac{1}{5}\int\frac{d^dq}{(2\pi)^d}\left[\frac{-1}{\not{q}+M}\Big(-P_\mu\gamma^\mu - V_\mu[\phi]\gamma^\mu + A_\mu[\phi]\gamma^\mu\gamma^5 + W_1[\phi]i\gamma^5\Big)\right]^5\\ &\supset \sum_N f_N^{\pi VV}\,\mathbb{O}^{(P^2V^2W_1)} + f_N^{\pi AA}\,\mathbb{O}^{(P^2A^2W_1)}\,,\end{aligned} \tag{6.47}$$

The evaluation of the class of operator $\mathbb{O}^{P^2V^2W_1}$ and $\mathbb{O}^{P^2A^2W_1}$ can be done very efficiently by using the One-Loop Universal Effective Action (UOLEA) [17]. One obtains

$$\begin{aligned}\mathcal{L}_{\rm EFT}^{\rm 1loop} &\supset -\frac{1}{8\pi^2 M}{\rm tr}\,\epsilon^{\mu\nu\rho\sigma}\Big(W_1[P_\mu,V_\nu][P_\rho,V_\sigma] + \frac{1}{3}W_1[P_\mu,A_\nu][P_\rho,A_\sigma]\Big)\\ &= \frac{1}{16\pi^2 M}{\rm tr}\Big(W_1^i F_{\mu\nu}^{Vj}\tilde{F}_{\mu\nu}^{Vk} + \frac{1}{3}W_1^i F_{\mu\nu}^{Aj}\tilde{F}_{\mu\nu}^{Ak}\Big)\,,\end{aligned} \tag{6.48}$$

where we form the field strength tensors by using

$$\epsilon^{\mu\nu\rho\sigma}[P_\mu, V_\nu^j][P_\rho, V_\sigma^k] = \frac{i^2}{4}\epsilon^{\mu\nu\rho\sigma}\big(\partial_{[\mu}V_{\nu]}^j\big)\big(\partial_{[\rho}V_{\sigma]}^k\big) = -\frac{1}{2}F_{\mu\nu}^{Vj}\tilde{F}_{\mu\nu}^{Vk}\,, \tag{6.49}$$

and similarly for the axial currents. We note that if $j \neq k$, one needs to sum over the exchange of $j,k$ indices to avoid the factor 2 problem.

### 6.3.4 Summary and master formula

We summarise the computations and the main outcome of section 6.3. Starting with a massive fermion which bilinearly involves some, yet undetermined, vector $V_\mu[\phi]$, axial vector $A_\mu[\phi]$ and pseudo scalar $W_1[\phi]$ interactions,

$$\mathcal{L}_{\rm UV}^{\rm fermion}[\Psi,\phi] \supset \bar{\Psi}\Big[i\gamma^\mu\partial_\mu - M + V_\mu[\phi]\gamma^\mu - A_\mu[\phi]\gamma^\mu\gamma^5 - W_1[\phi]i\gamma^5\Big]\Psi\,, \tag{6.50}$$

one obtains after integrating out the fermion field i.e evaluate the one-loop effective action by expanding the functional trace with CDE techniques,

$$\mathcal{L}_{\rm EFT}^{\rm 1loop} = i\,{\rm tr}\sum_{n=1}^{\infty}\frac{1}{n}\int\frac{d^4q}{(2\pi)^4}\left[\frac{-1}{\not{q}+M}\Big(-i\partial_\mu\gamma^\mu - V_\mu\gamma^\mu + A_\mu\gamma^\mu\gamma^5 + W_1 i\gamma^5\Big)\right]^n\,, \tag{6.51}$$

where in practice, the vector, axial-vector and pseudo-scalar structures are expressed as

$$V_\mu[\phi] \equiv g_V^i V_\mu^i[\phi^i]\,,\; A_\mu[\phi] \equiv g_A^i A_\mu^i[\phi^i]\,,\; W_1[\phi] \equiv g_{W_1}^i W_1^i[\phi^i]\,, \tag{6.52}$$

[17] These operators have been explicitly evaluated and are then available in the fermionic UOLEA in Ref. [3].

with an implicit summation over the $i$ index. One can proceed and form the low energy effective operators and evaluate their associated Wilson coefficients which are regularised in dimensional regularisation. After regularisation, it is important to identify ambiguities of some Wilson coefficients resulting from the fact that the gauge or anomalous aspects of the symmetries have not been addressed yet. The generic one-loop effective Lagrangian, still involving redundant operators as well as the ambiguous $\bar{a}$'s and $\bar{b}$'s coefficients, reads

$$\begin{aligned}\mathcal{L}_{\text{EFT}}^{\text{1loop}} \supset\ & \frac{1}{8\pi^2}\bar{a}_{V^iV^jA^k}\text{tr}\left[V_\mu^i V_\nu^j \tilde{F}_{\mu\nu}^{A^k}\right] + \frac{1}{8\pi^2}\bar{a}_{V^jA^kV^i}\text{tr}\left[V_\mu^j A_\nu^k \tilde{F}_{\mu\nu}^{V^i}\right] + \frac{1}{8\pi^2}\bar{a}_{A^kV^iV^j}\text{tr}\left[A_\mu^k V_\nu^i \tilde{F}_{\mu\nu}^{V^j}\right] \\ & + \frac{1}{8\pi^2}\bar{b}_{A^iA^jA^k}\text{tr}\left[A_\mu^i A_\nu^j \tilde{F}_{\mu\nu}^{A^k}\right] + \frac{1}{8\pi^2}\bar{b}_{A^jA^kA^i}\text{tr}\left[A_\mu^j A_\nu^k \tilde{F}_{\mu\nu}^{A^i}\right] + \frac{1}{8\pi^2}\bar{b}_{A^kA^iA^j}\text{tr}\left[A_\mu^k A_\nu^i \tilde{F}_{\mu\nu}^{A^j}\right] \\ & + \frac{1}{16\pi^2 M}\text{tr}\left(W_1^i F_{\mu\nu}^{V^j}\tilde{F}_{\mu\nu}^{V^k} + \frac{1}{3}W_1^i F_{\mu\nu}^{A^j}\tilde{F}_{\mu\nu}^{A^k}\right). \end{aligned} \tag{6.53}$$

This master formula is generic and encapsulates all the needed computations. Indeed, at this stage, imposing the EFT to respect specific gauge invariance relations will link several of these operators together and allow to fix the ambiguities of any free Wilson coefficients in a very simple and elegant way. Since doing so, presuppose having a concrete model in mind or set of symmetries, we now turn back to more phenomenological investigations where this master formula is applied to various models.

## 6.4 Application to axions

In this section, we use the results obtained in section 6.3 to build EFT involving would-be-Goldstone bosons of spontaneously broken symmetries and Goldstone bosons of global symmetries. As a first application, we apply the master formula of Eq. (6.53) to concretely build the intuited EFT of Eq. (6.15) from the toy model presented in section 6.2. We will then concentrate on more realistic constructions, e.g. building EFT involving the SM gauge fields and an axion or ALP. This task might precisely imply to integrate out chiral fermions which obtain their mass while the electroweak gauge symmetry is spontaneously broken, the global PQ symmetry being spontaneously and anomalously broken. We provide a simple expression adapted to SM gauge groups, and provide explicit use of it to derive axion couplings to massive gauge fields in the original 2HDM setup as proposed by Peccei and Quinn and in a more phenomenologically relevant version of it, the invisible axion DFSZ model [134, 135].

### 6.4.1 A chiral toy model

So far, we have evaluated the operators involving three vector structures (which can also incorporate derivative couplings) and the operators involving a pseudo-scalar field which couples with two field strength tensors. We now give an example how to use the results of the previous section to derive the EFT resulting from integrating out the chiral fermion of the toy model of section 6.2. We remind the fermionic quadratic operator of this toy model,

$$\mathcal{L}_{\text{UV}}^{\text{toy-model}} = \bar{\Psi}\left[i\partial_\mu\gamma^\mu - M + V_\mu\gamma^\mu - A_\mu\gamma^\mu\gamma^5 - W_1 i\gamma^5\right]\Psi \tag{6.54}$$

where the vector, axial-vector, and pseudo-scalar structures decompose as

$$V_\mu = \left\{V_\mu,\ \left[S_\mu - \frac{\partial_\mu\pi_S}{2v_S}\right]\right\},\ A_\mu = \left\{A_\mu,\ \left[U_\mu - \frac{\partial_\mu\pi_U}{2v_U}\right]\right\},\ W_1 = M\frac{\pi_A}{v_A}, \tag{6.55}$$

and the gauge couplings are omitted for simplicity. Making use of our master formula given in Eq. (6.53), one can straightforwardly obtain

$$\mathcal{L}_{\rm EFT}^{\rm 1loop} = \omega_{VAV}\left[S_\mu - \frac{\partial_\mu \pi_S}{2v_S}\right] A_\nu \tilde{F}_V^{\mu\nu} + \omega_{AAA}\left[U_\mu - \frac{\partial_\mu \pi_U}{2v_U}\right] A_\nu \tilde{F}_A^{\mu\nu} + \eta_{ASV}\left[\frac{\pi_A}{v_A} F_{S,\mu\nu}\tilde{F}_V^{\mu\nu}\right] + \eta_{AUA}\left[\frac{\pi_A}{v_A} F_{U,\mu\nu}\tilde{F}_A^{\mu\nu}\right]. \tag{6.56}$$

At this stage, $\omega_{VAV}$, $\omega_{AAA}$ and $\eta_{ASV}$, $\eta_{AUA}$ read,

$$\omega_{VAV} = \frac{1}{8\pi^2}(1-\bar{b})\,,\ \omega_{AAA} = -\frac{1}{8\pi^2}\bar{a}\,;\quad \eta_{ASV} = \frac{1}{8\pi^2}\,,\ \eta_{AUA} = \frac{1}{24\pi^2}\,, \tag{6.57}$$

with $\bar{a}$ and $\bar{b}$ the two free parameters. As presented in section 2, we now implement the consistency between the UV model of Eq. (6.54) and the associated EFT of Eq. (6.56) by fixing the nature of each symmetries i.e gauge or anomalous. We identify the precise value of the parameters $\bar{a}$ and $\bar{b}$ by requiring axial gauge invariance (leaving then the possibility that the other transformations, only, could be anomalous)

$$\delta_A\left(\omega_{VAV}\left[S_\mu - \frac{\partial_\mu \pi_S}{2v_S}\right] A_\nu \tilde{F}_V^{\mu\nu} + \frac{1}{8\pi^2}\frac{\pi_A}{v_A} F_{S,\mu\nu}\tilde{F}_V^{\mu\nu}\right) = 0\,,\ \text{and}$$
$$\delta_A\left(\omega_{AAA}\left[U_\mu - \frac{\partial_\mu \pi_U}{2v_U}\right] A_\nu \tilde{F}_A^{\mu\nu} + \frac{1}{24\pi^2}\frac{\pi_A}{v_A} F_{U,\mu\nu}\tilde{F}_A^{\mu\nu}\right) = 0\,, \tag{6.58}$$

where we perform the gauge variation of the axial current, $\delta_A A_\mu = \partial_\mu \theta_A$, the would-be-Goldstone, $\delta_A \pi_A = 2 v_A \theta_A$, integrate by parts and combine the various contributions proportional to $(\partial_\mu\theta_A)U_\nu\tilde{F}_A^{\mu\nu}$ and $(\partial_\mu\theta_A)S_\nu\tilde{F}_V^{\mu\nu}$. This straightforwardly leads to,

$$\omega_{VAV} = -\frac{1}{2\pi^2}\ \Leftrightarrow\ \bar{b} = 5\ ;\quad \omega_{AAA} = -\frac{1}{6\pi^2}\ \Leftrightarrow\ \bar{a} = \frac{4}{3}\,. \tag{6.59}$$

Finally, one can set to zero the artificial vector fields $S_\mu$ and $U_\mu$ and write the non-ambiguous dimension-five bosonic operators, simply as

$$\mathcal{L}_{\rm EFT}^{\rm 1loop} = \frac{1}{2\pi^2}\frac{\partial^\mu \pi_S}{2v_S} A^\nu \tilde{F}_{V,\mu\nu} + \frac{1}{6\pi^2}\frac{\partial^\mu \pi_U}{2v_U} A^\nu \tilde{F}_{A,\mu\nu} = -\frac{1}{8\pi^2}\frac{\pi_S}{v_S} F_A^{\mu\nu}\tilde{F}_{V,\mu\nu} - \frac{1}{24\pi^2}\frac{\pi_U}{v_U} F_A^{\mu\nu}\tilde{F}_{A,\mu\nu}\,. \tag{6.60}$$

This is the one-loop contributions to the EFT Lagrangian obtained by integrating out a chiral massive fermion in our toy model. To obtain the full EFT Lagrangian, one must add the Jacobian terms given by Eq. (6.14) with the one-loop terms of Eq. (6.60),

$$\mathcal{L}_{\rm EFT} = \frac{1}{16\pi^2}\frac{\pi_U}{v_U}\left(F_{V,\mu\nu}\tilde{F}_V^{\mu\nu} + \frac{1}{3}F_{A,\mu\nu}\tilde{F}_A^{\mu\nu}\right)\,. \tag{6.61}$$

We note that integrating out the fermion in a one-dimensional representation starting from Eq. (6.54), we obtain that the $\pi_S$ derivative interaction induces EFT operators that precisely cancel the Jacobian term in Eq. (6.14) as expected starting from an abelian gauge theory, as discussed earlier, and this provides a non-trivial check for our calculation. We should also remark that the $\pi_U VV$ coupling entirely arises from the Jacobian term, as predicted by the Sutherland-Veltman theorem. However, the $\pi_U AA$ coupling does not and displays an additional factor of 1/3 due to the one-loop contribution.

We now move on to more concrete axion models for which we will compute one-loop induced effective couplings between axions and gauge bosons, with a particular interest for those involving massive gauge fields [18].

[18] These results should and will reproduce those derived, using different techniques, in Refs. [149, 158].

### 6.4.2 Axion couplings to gauge fields

The axion field is a relic of the spontaneous symmetry breaking of a global $U(1)_{PQ}$ symmetry. A realistic model involving the QCD axion or an ALP, being a pseudo-scalar field $a(x)$, basically couples to fermions (of the SM or not) which have to be charged under the Global $U(1)_{PQ}$ group but also other abelian or non-abelian groups such as the one of the SM. For a massive chiral fermion, its bilinear form, after gauge symmetry breaking, generically reads

$$\mathcal{L}_{\rm UV} = \mathcal{L}_{\rm UV}^{\rm Jac} + \bar{\Psi}\Big[i\partial_\mu\gamma^\mu - M + V_\mu\gamma^\mu - A_\mu\gamma^\mu\gamma^5 - W_1 i\gamma^5\Big]\Psi\,, \tag{6.62}$$

where vector, axial-vector and pseudo-scalar structures include[19]

$$V_\mu = \big\{g_{_V}^i V_\mu^i\,,\, g_{_V}^{PQ}\big(\partial_\mu a - V_\mu^{PQ}\big)\big\}\,,\; A_\mu = \big\{g_{_A}^i A_\mu^i\,,\, g_{_A}^{PQ}\big(\partial_\mu a - A_\mu^{PQ}\big)\big\}\,,\; W_1 = M\frac{\pi_{_A}^i}{v_A}\,, \tag{6.63}$$

with $V_\mu^i$, $A_\mu^i$ stand for vector and axial-vector components of a generic chiral gauge field $G_\mu^i$, the term $\pi_{_A}^i(x)$ stands for the would-be-Goldstone boson in the case where $G_\mu^i$ obtains its mass from gauge spontaneous symmetry breaking. $V_\mu^{PQ}$ and $A_\mu^{PQ}$ are the fictitious auxiliary gauge fields associated to the global PQ symmetry. Writing Eq. (6.62) presupposes a chiral fermion reparametrisation which induces a Jacobian term, $\mathcal{L}_{\rm UV}^{\rm Jac}$. This contribution, before gauge spontaneous symmetry breaking, reads

$$\mathcal{L}_{\rm UV}^{\rm Jac} = \frac{1}{16\pi^2 f_a}\mathcal{N}_{PQ}\, a(x) F_{\mu\nu}^i \tilde{F}^{i,\,\mu\nu}\,, \tag{6.64}$$

where the $i$-index only runs for the gauge field strength tensors. The anomaly coefficient can be generally expressed as,

$$\mathcal{N}_{PQ} = \sum_{\Psi=\Psi_R,\Psi_L^\dagger} {\rm tr}\Big[PQ(\Psi)\otimes G(\Psi)\otimes G(\Psi)\Big]\,, \tag{6.65}$$

with $PQ(\Psi)$ and $G(\Psi)$ the PQ and gauge charge of the chiral fermion $\Psi$. Integrating out the chiral fermion and making use of the master formula Eq. (6.53), one obtains

$$\begin{aligned}
\mathcal{L}_{\rm EFT}^{\rm 1loop} &= \omega_{_{VAV}}\Big[g_{_V}^{PQ} g_{_A}^i g_{_V}^j\,\big(\partial_\mu a - V_\mu^{PQ}\big)A_\nu^i \tilde{F}^{V^j,\mu\nu}\Big] + \omega_{_{AAA}}\Big[g_{_A}^{PQ} g_{_A}^i g_{_A}^j\,\big(\partial_\mu a - A_\mu^{PQ}\big)A_\nu^i \tilde{F}^{A^j,\mu\nu}\Big] \\
&+ \frac{1}{8\pi^2}\big(g_{_V}^{PQ} g_{_V}^j\big)\frac{\pi_{_A}^i}{v_A}F_{\mu\nu}^{V^{PQ}}\tilde{F}^{V^j,\mu\nu} + \frac{1}{24\pi^2}\big(g_{_A}^{PQ} g_{_A}^j\big)\frac{\pi_{_A}^i}{v_A}F_{\mu\nu}^{A^{PQ}}\tilde{F}^{A^j,\mu\nu} \\
&= -\frac{1}{4\pi^2}\big(g_{_V}^{PQ} g_{_A}^i g_{_V}^j\big)\, a\, F_{\mu\nu}^{A^i}\tilde{F}^{V^j,\mu\nu} - \frac{1}{12\pi^2}\big(g_{_A}^{PQ} g_{_A}^i g_{_A}^j\big)\, a\, F_{\mu\nu}^{A^i}\tilde{F}^{A^j,\mu\nu}\,.
\end{aligned} \tag{6.66}$$

In order to get the last line of the above equation, we imposed the crucial axial gauge invariance, used integration by parts and Bianchi identity, neglected the surface terms, and at the end of the computation, we removed the fictitious fields $V_\mu^{PQ}$ and $A_\mu^{PQ}$. Adding all together, we are now able to build the axion-bosonic effective Lagrangian described by $\mathcal{L}_{\rm EFT} = \mathcal{L}_{\rm UV}^{\rm Jac} + \mathcal{L}_{\rm EFT}^{\rm 1loop}$ where the generic formula of $\mathcal{L}_{\rm UV}^{\rm Jac}$ and $\mathcal{L}_{\rm EFT}^{\rm 1loop}$ are given by Eqs. (6.64),(6.66).

[19] Note that for convenience, we have used a different normalisation convention for the PQ charges than the one used for gauge charges.

#### 6.4.2.1 SM gauge and PQ symmetries

We now present two examples where the axion field couples with the SM gauge fields. Our first example will be the original Peccei and Quinn scenario in which the axion is the pseudo-scalar component of a Two Higgs Doublet Model (2HDM). Our second application will be to consider the so-called DFSZ axion model [134, 135]. To illustrate the results and properties discussed in the previous sections, we will integrate out only one generation of quarks, let us say $\begin{pmatrix} u & d \end{pmatrix}$. This computation was performed in Ref.[149] by Feynman diagram technique, accompanied by Pauli-Villars regularisation. We will recover some of its results by using the functional method for one-loop matching.

We begin with the Jacobian terms which induce tree-level axion couplings to the SM gauge fields,

$$\mathcal{L}_{\text{UV}}^{\text{Jac}} = \frac{1}{16\pi^2 f_a}\left(g_s^2\mathcal{N}_C\, aG_{\mu\nu}\tilde{G}^{\mu\nu} + g^2\mathcal{N}_L\, aW_{\mu\nu}^i\tilde{W}^{i,\mu\nu} + g'^2\mathcal{N}_Y\, aB_{\mu\nu}\tilde{B}^{\mu\nu}\right), \tag{6.67}$$

with the anomaly coefficients $\mathcal{N}_i$ computable as follows,

$$\begin{aligned}
\mathcal{N}_C &= \sum_{\Psi=q_L^\dagger,u_R,d_R} C_{SU(3)_c}(\Psi)\, d_{SU(2)_L}(\Psi)\, PQ(\Psi)\,, \\
\mathcal{N}_L &= \sum_{\Psi=q_L^\dagger;l_L^\dagger} d_{SU(3)_c}(\Psi)\, C_{SU(2)_L}(\Psi)\, PQ(\Psi)\,, \\
\mathcal{N}_Y &= \sum_{\Psi=q_L^\dagger,u_R,d_R;l_L^\dagger,e_R} d_{SU(3)_c}(\Psi)\, d_{SU(2)_L}(\Psi) C_{U(1)_Y}(\Psi)\, PQ(\Psi)\,,
\end{aligned} \tag{6.68}$$

where we closely followed the conventions and notations of Ref.[149] with $d_{SU(3)_c}(\Psi)$, $d_{SU(2)_L}(\Psi)$ and $C_{SU(3)_c}(\Psi)$, $C_{SU(2)_L}(\Psi)$ are respectively the $SU(3)_c$ and $SU(2)_L$ dimensions and quadratic Casimir invariant of the representation carried by the chiral fermion field $\Psi$. Besides, $PQ(\Psi)$ is the PQ charge of the fermion $\Psi$ which is model-dependent. We will come back to these PQ charges when discussing a peculiar axion model.

The one-loop effective Lagrangian resulting from integrating out the SM chiral fermion is

$$\begin{aligned}
\mathcal{L}_{\text{EFT}}^{\text{1loop}} \supset \sum_f \frac{-1}{4\pi^2}\Big[ &(g_V^{PQ} g_A^Z g_V^Z)^f \left(a\, F_{\mu\nu}^{A^Z}\tilde{F}^{V^Z,\mu\nu}\right) + \frac{1}{3}(g_A^{PQ} g_A^Z g_A^Z)^f \left(a\, F_{\mu\nu}^{A^Z}\tilde{F}^{A^Z,\mu\nu}\right) \\
&+ (g_V^{PQ} g_A^W g_V^W)^f \left(a\, F_{\mu\nu}^{A^W}\tilde{F}^{V^W,\mu\nu}\right) + \frac{1}{3}(g_A^{PQ} g_A^W g_A^W)^f \left(a\, F_{\mu\nu}^{A^W}\tilde{F}^{A^W,\mu\nu}\right) \\
&+ (g_V^{PQ} g_A^Z g_V^\gamma)^f \left(a\, F_{\mu\nu}^{A^Z}\tilde{F}^{V^\gamma,\mu\nu}\right)\Big],
\end{aligned} \tag{6.69}$$

where $g_V^{PQ}$, $g_A^{PQ}$ are axion-fermion-fermion couplings written in terms of Dirac bilinear form. A summary of the gauge charges of SM fermions can be found in Table 6.1.
The only thing that remains to be determined in Eqs. (6.67), (6.68), (6.69) are the fermions PQ charge, that we discuss now for several axion models.

#### 6.4.2.2 PQ axion model

We first consider the original PQ scenario where the QCD axion is identified as the orthogonal state of the would-be-Goldstone of the Z boson in a 2HDM model (see Refs. [129, 130]). The starting

| $i$ | g | W | $\gamma$ | Z |
|---|---|---|---|---|
| $(g^i_V)^f$ | $g_s T^f_a$ | $\frac{g}{\sqrt{2}} T^f_3$ | $eQ^f$ | $\frac{g}{2\cos\theta_w}(T^f_3 - 2\sin^2\theta_w Q^f)$ |
| $(g^i_A)^f$ | 0 | $\frac{g}{\sqrt{2}} T^f_3$ | 0 | $\frac{g}{2\cos\theta_w} T^f_3$ |

**Table 6.1:** *SM fermion couplings to the SM gauge fields, where $T^f_a$, $T^f_3$, $Q^f$, $\theta_w$ are respectively the $SU(3)_C$ generators, the eigenvalue of the isospin operator, the electromagnetic charge and the weak mixing angle.*

point is a fermion-Higgs Yukawa interaction, that we assume of type II, which can be written as

$$\mathcal{L}^{\text{2HDM}}_{\text{Yukawa}} = -\left[Y_u \bar{u}_R\,\Phi_1\, q_L + Y_d \bar{d}_R\,\Phi_2^\dagger\, q_L\right] - Y_e \bar{e}_R\,\Phi_2^\dagger\, l_L + \text{h.c.}\,. \tag{6.70}$$

The two complex scalar fields can be written as

$$\Phi_1 = \frac{1}{\sqrt{2}} e^{i\frac{\eta_1}{v_1}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}\,,\ \Phi_2 = \frac{1}{\sqrt{2}} e^{i\frac{\eta_2}{v_2}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix}\,, \tag{6.71}$$

where $\eta_1$, $\eta_2$ are Goldstone bosons of the scalar fields $\Phi_1$ and $\Phi_2$. The vacuum expectation value of the scalar fields, $v_1$ and $v_2$ are related by $v_1^2 + v_2^2 \equiv v^2 \simeq \left(246\,\text{GeV}\right)^2$, and one usually introduces the $\beta$ angle such that $v_1 = v\sin\beta$, $v_2 = v\cos\beta$ and $v_2/v_1 = \left(1/\tan\beta\right) \equiv x$. The next step is to identify the would-be-Goldstone boson (that generates the mass of the Z-boson) from its orthogonal state, defining then the axion. One has the following relations

$$\begin{pmatrix} G^0 \\ a \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \eta_2 \\ \eta_1 \end{pmatrix}\,. \tag{6.72}$$

The Higgs doublets can be re-written as

$$\Phi_1 = \frac{1}{\sqrt{2}} e^{i\frac{G^0}{v_1}} e^{i\,x\,\frac{a}{v}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}\,,\ \Phi_2 = \frac{1}{\sqrt{2}} e^{i\frac{G^0}{v_2}} e^{i\left(-\frac{1}{x}\right)\frac{a}{v}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix}\,, \tag{6.73}$$

where $G^0$ is PQ neutral and the Higgs doublets carry the following PQ charge, $PQ(\Phi_1) = x$ and $PQ(\Phi_2) = -1/x$. In order to identify the PQ axion model with Eq. (6.62), we first make the Yukawa Lagrangian becomes PQ-invariant by performing the chiral rotation,

$$\Psi \to e^{iPQ(\Psi)\frac{a}{v}}\Psi\,. \tag{6.74}$$

The PQ charges for one generation of quarks $\left(u\ d\right)$ are assigned, such as

$$PQ\left(q_L; u_R, d_R\right) = \left(\alpha; \alpha + x, \alpha + \frac{1}{x}\right)\,. \tag{6.75}$$

$\alpha$ is a free parameter that corresponds to the conservation of the baryon number[20]. The chiral rotation leads to the derivative coupling of axion with SM fermions as defined in Eq. (6.62) and the axion couplings to fermions read

$$(g_V^{PQ})^u = -\frac{1}{2v}(2\alpha + x)\,,\ (g_A^{PQ})^u = \frac{1}{2v}x\,;\quad (g_V^{PQ})^d = -\frac{1}{2v}\left(2\alpha + \frac{1}{x}\right)\,,\ (g_A^{PQ})^d = \frac{1}{2v}\left(\frac{1}{x}\right)\,. \tag{6.76}$$

[20] For a general setup including also the lepton sector see Refs.[149, 150].

Plugging Eq. (6.75) into Eq. (6.67) and rotating the electroweak gauge fields from their interaction basis to their physical mass basis using $W^3_\mu = c_w Z_\mu + s_w A_\mu$, $B_\mu = -s_w Z_\mu + c_w A_\mu$ along with $e = g s_w = g' c_w$, one obtains the following Lagrangian for the Jacobian contribution

$$\begin{aligned}\mathcal{L}^{\{u,d\}}_{\rm Jac} = \frac{1}{16\pi^2 v}\Bigg(&\frac{g_s^2}{2}\left[x+\frac{1}{x}\right] a\, G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} + e^2 N_c\left[\frac{4}{9}x+\frac{1}{9x}\right] a\, F_{\mu\nu}\tilde{F}^{\mu\nu}\\ &-\left[g^2 N_c\,\alpha\right] a\, W^+_{\mu\nu}\tilde{W}^{-,\mu\nu} - \frac{2e^2}{c_w s_w}N_c\left[\frac{1}{2}\alpha + s_w^2\left(\frac{4}{9}x+\frac{1}{9x}\right)\right] a\, Z_{\mu\nu}\tilde{F}^{\mu\nu}\\ &+\frac{e^2}{c_w^2 s_w^2}N_c\left[-(1-2s_w^2)\frac{\alpha}{2} + s_w^4\left(\frac{4}{9}x+\frac{1}{9x}\right)\right] a\, Z_{\mu\nu}\tilde{Z}^{\mu\nu}\Bigg)\,, \end{aligned} \tag{6.77}$$

where $N_c = 3$. Plugging Eq. (6.76) into Eq. (6.69) and performing the same electroweak rotation lead to the following one-loop effective Lagrangian,

$$\begin{aligned}\mathcal{L}^{\rm 1loop-\{u,d\}}_{\rm EFT} = \frac{1}{16\pi^2 v}\Bigg(&g^2 N_c\left[\alpha + \frac{1}{6}\left(x+\frac{1}{x}\right)\right] a\, W^+_{\mu\nu}\tilde{W}^{-,\mu\nu} + \frac{e^2}{c_w s_w}N_c\left[\alpha + \left(\frac{1}{3}x+\frac{1}{6x}\right)\right] a\, Z_{\mu\nu}\tilde{F}^{\mu\nu}\\ &+\frac{e^2}{c_w^2 s_w^2}N_c\left[(1-2s_w^2)\frac{\alpha}{2} + \frac{1}{12}\left(x+\frac{1}{x}\right) - s_w^2\left(\frac{1}{3}x+\frac{1}{6x}\right)\right] a\, Z_{\mu\nu}\tilde{Z}^{\mu\nu}\Bigg)\,. \end{aligned} \tag{6.78}$$

The effective axion-bosonic Lagrangian is obtained by adding $\mathcal{L}^{\{u,d\}}_{\rm Jac}$ and $\mathcal{L}^{\rm 1loop-\{u,d\}}_{\rm EFT}$ and gives the compact result,

$$\begin{aligned}\mathcal{L}^{\rm a-bosonic}_{\rm EFT} = \frac{1}{16\pi^2 v}\Bigg(&\frac{g_s^2}{2}\left[x+\frac{1}{x}\right] a\, G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} + e^2 N_c\left[\frac{4}{9}x+\frac{1}{9x}\right] a\, F_{\mu\nu}\tilde{F}^{\mu\nu}\\ &+ g^2 N_c\frac{1}{6}\left[x+\frac{1}{x}\right] a\, W^+_{\mu\nu}\tilde{W}^{-,\mu\nu} + \frac{e^2}{c_w s_w}N_c\left[\left(\frac{1}{3}x+\frac{1}{6x}\right) - 2s_w^2\left(\frac{4}{9}x+\frac{1}{9x}\right)\right] a\, Z_{\mu\nu}\tilde{F}^{\mu\nu}\\ &+\frac{e^2}{c_w^2 s_w^2}N_c\left[\frac{1}{12}\left(x+\frac{1}{x}\right) - s_w^2\left(\frac{1}{3}x+\frac{1}{6x}\right) + s_w^4\left(\frac{4}{9}x+\frac{1}{9x}\right)\right] a\, Z_{\mu\nu}\tilde{Z}^{\mu\nu}\Bigg)\,. \end{aligned} \tag{6.79}$$

#### 6.4.2.3 DFSZ axion model

Concerning the case of the more realistic axion DFSZ model [134, 135], the Yukawa couplings are the same as in the 2HDM model, but now the scalar potential is modified. Typically, the 2HDM model is extended by a gauge-singlet complex scalar field $\phi$, with the scalar potential

$$V_{\rm DFSZ} = V_{\rm 2HDM} + V_{\phi 2HDM} + V_{\phi PQ} + V_\phi\,, \tag{6.80}$$

where we have

$$\begin{cases} V_{\phi 2HDM} = a_1\left(\phi^\dagger\phi\right)\left(\Phi_1^\dagger\Phi_1\right) + a_2\left(\phi^\dagger\phi\right)\left(\Phi_2^\dagger\Phi_2\right)\,,\\ V_{\phi PQ} = \lambda_{12}\left(\phi^\dagger\phi\right)\Phi_1^\dagger\Phi_2 + \text{h.c.}\,,\\ V_\phi = \mu^2\left(\phi^\dagger\phi\right) + \lambda\left(\phi^\dagger\phi\right)^2\,. \end{cases} \tag{6.81}$$

Similarly to $\Phi_i$ of Eq. (6.73), one can also write the new complex scalar field $\phi$ as

$$\phi = \frac{1}{\sqrt{2}}e^{i\frac{\eta_a}{f_a}}\begin{pmatrix}0\\ f_a\end{pmatrix}\,. \tag{6.82}$$

In summary, for the DFSZ axion model, one obtains the PQ-charges and the breaking-scale of the PQ-symmetry by rescaling their values in the axion PQ model, simply as follows,

$$x \to \frac{2x^2}{x^2+1}\,,\ \frac{1}{x} \to \frac{2}{x^2+1}\,,\ v \to f_a\,. \tag{6.83}$$

The effective DFSZ axion-bosonic Lagrangian, obtained by adding $\mathcal{L}_{\text{Jac}}^{\{u,d\}}$ and $\mathcal{L}_{\text{EFT}}^{\text{1loop}-\{\text{u,d}\}}$, is given by

$$\begin{aligned}\mathcal{L}_{\text{EFT}}^{\text{a}-\text{bosonic}} = \frac{1}{16\pi^2 f_a}\bigg(& g_s^2\, a\, G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} + e^2 N_c \frac{8x^2+2}{9(x^2+1)} a\, F_{\mu\nu}\tilde{F}^{\mu\nu} \\ &+ \frac{g^2 N_c}{3} a\, W^+_{\mu\nu}\tilde{W}^{-,\mu\nu} + \frac{e^2}{c_w s_w} N_c \frac{3+6x^2-4s_w^2(4x^2+1)}{9(x^2+1)} a\, Z_{\mu\nu}\tilde{F}^{\mu\nu} \\ &+ \frac{e^2}{c_w^2 s_w^2} N_c \left[\frac{1}{6} - s_w^2 \frac{2x^2+1}{3(x^2+1)} + s_w^4 \frac{8x^2+2}{9(x^2+1)}\right] a\, Z_{\mu\nu}\tilde{Z}^{\mu\nu}\bigg)\,. \end{aligned} \tag{6.84}$$

These results do agree with those derived in Refs. [149], using the more traditional approach of Feynman diagram computations.

It is certainly a good moment to pause and appreciate the difference in strategy with this last reference. The main and obvious distinction is that in this work, we favored the path integral method to evaluate one-loop processes. However, we believe that another elegant and insightful feature of this axionic EFT derivation is due to the direct and consistent way of dealing with gauge and anomalous symmetries. Indeed, one needs not to use the anomalous Ward-identities to alleviate ambiguities inherent to anomalies in QFTs. Equivalently, one can use the interplay between higher-dimensional operators involving the axion and the would-be-Goldstone bosons in order to consistently and easily derive axion EFTs. This offers a neat method to also explore other sectors of axion EFTs.

## 6.5 Conclusion

In this work, we have considered the task of building EFTs by integrating out fermions charged under both local and global symmetries. These symmetries can be spontaneously broken, and the global ones might also be anomalously broken. This setting is typically that encountered in axion models, where a new global but anomalous symmetry, $U(1)_{PQ}$, is spontaneously broken, so as to generate a Goldstone boson, the axion, coupled to gluons.

The main novelties of our approach are twofold. First, the heavy fermion to be integrated out is allowed to have chiral charges for both the local and global symmetries. The analysis is then much more intricate because of the presence of anomalies in various currents, and because the fermion can only have a mass when all the chiral symmetries are spontaneously broken. Second, we perform our analysis in a functional approach, by systematically building EFTs using an inverse mass expansion, that is, identifying leading operators and calculating their Wilson coefficients with the help of Covariant Derivative Expansion. Our calculations are adaptable to various UV models and allow us to correctly treat QFT anomalies.

In more details, our main results are the following:

- It exists many motivations for introducing Goldstone bosons of global symmetries using a polar representation. Once this choice has been made, we have identified an appropriate parametrisation of the fermionic part of the UV Lagrangian. Essentially, with the purpose of an inverse mass expansion, if one wants to perform an exact computation without truncating the initial UV theory, it is desirable to write the fermion mass term as an invariant quantity under the various symmetries, even for a chiral fermion. This requires some fermion field redefinitions. Only then one can clearly identify the fermion bilinear operator to be inverted.

- Usually, Ward identities are used to enforce the desired gauge symmetries. When dealing with anomalous quantities, these constraints are crucial to remove the ambiguities that creep in through the regularisation process. But in our functional approach, this cannot be immediately implemented because the leading operators in the EFT end up being automatically gauge invariant. The only way forward is to perturb the theory to upset this automatic gauge invariance. This is done with the help of background fields, in a way very similar as in Ref. [158]. Then, the necessary Ward identity constraints can be recovered thanks to EFT operators involving the would-be-Goldstone bosons of the exact gauge symmetries.

- The parametrisation of the fermion bilinear operator involves derivative interactions with scalar and pseudoscalar fields. To our knowledge, a precise description of how to perform the calculation of the determinant of such operators has never been presented. It should be noted that in that calculation, regularisation is necessary. For that, we adopt dimensional regularisation and follow the 't Hooft-Veltman prescription. We show that the two-parameter ambiguities, well known in the context of triangle Feynman diagrams, can be recovered. Those are crucial to allow one to enforce all the gauge constraints in a consistent way.

- We recover in the functional context the results of Refs. [149, 150, 158], that is, that the derivative coupling of the Goldstone boson $\pi$ to the fermions, $\bar{\Psi}(\partial_\mu \pi\, \gamma^\mu \gamma^5)\Psi$ and $\bar{\Psi}(\partial_\mu \pi\, \gamma^\mu)\Psi$, do not necessarily vanish in the infinite mass limit. They do contribute to the leading EFT operator $\pi VA$, $\pi AA$, but not $\pi VV$. In other words, this last coupling satisfies the Sutherland-Veltman theorem, and is fully driven by the anomaly, but not the other two.

In this chapter we have presented how to deal with scenarios combining both spontaneous and anomalous symmetry breaking. When building an EFT by integrating out chiral fermions charged under those various symmetries it is legitimate to keep local partial derivative interactions instead of traditional pseudo-scalar ones, but this has a cost. Now the anomaly is spread into several contributions which have to be recombined with high care when evaluating the S-matrix (see also Refs. [149, 150, 158]). We have integrated these peculiar fermions in the elegant and minimal functional approach and showed how to remove the ambiguities one has to face to evaluate the functional trace in dimensional regularisation. Inevitably, this corresponds to implement the anomalous Ward identities in a consistent way within the path integral formalism. We did so by introducing fictitious vector fields associated to the global symmetries so one can cure potential ambiguities undermining the theory while enforcing gauge invariance. More generally, this work shows a possible, neat and systematic path to follow to consistently build an entire EFT involving anomalous symmetries. It should also be very useful to derive other EFT higher dimensional operators. All in all, this procedure allowed us to compute in a transparent and in a very generic way the Wilson coefficients of higher dimensional operators involving Goldstone bosons, this is encapsulated in the master formula Eq. (6.53). Furthermore, we showed how to apply this master formula to the case of SM gauge interactions. Ultimately, we applied these results to the axion Goldstone boson (in the general sense i.e being the QCD axion or simply an ALP). We obtained in a closed form the higher dimensional operators involving the axion and SM gauge fields and collected them so that one can recover the non-intuitive physical coupling between axions and massive SM gauge fields which have been recently derived by some of us in Ref. [149]. The phenomenological relevance of these couplings are of particular interest for collider ALPs searches but also their imprints in the early universe.

# Chapter 7

# Anomalies from an effective field theory perspective

The path-integral measure of a gauge-invariant fermion theory is transformed under the chiral transformation and leads to an elegant derivation of the anomalous chiral Ward-Takahashi identities, as we know from the seminal work of Fujikawa. We present in this work an alternative and illuminating way to calculate the Jacobian in the path-integral measure from the Covariant Derivative Expansion technique used in Effective Field Theory. We present several ways to customise the crucial regularisation such that the anomaly is located in the desired current, which is unprecedented within the path integral approach. We are then able to derive, in a transparent and unified way the covariant, consistent, gravitational and scale anomalies.

## 7.1 Introduction

Symmetries play an important role in explaining the fundamental forces of nature. A symmetry valid in the classical theory might be violated in its quantised version. This defines what an anomaly is in Quantum Field Theory (QFT). The axial or chiral anomaly which has a long history is certainly the most well-known and had a huge impact in the building and understanding of QFT.

In 1967, Sutherland and Veltman [163, 164] proved that the neutral pion, $\pi_0$, cannot decay into two photons in obvious disagreement with the experimental results. The $\pi_0 \to \gamma\gamma$ puzzle has been solved in 1969 by Bell and Jackiw [140] who showed that the, unexpected, axial symmetry breaking perfectly explains this decay, later confirmed by Alder [121]. This is the so called ABJ anomaly now commonly computed through triangle Feynman diagrams involving one axial and two vector currents and involving a UV divergence which leads to $\partial^\mu j^5_\mu = 1/(8\pi^2)F\tilde{F}$, meaning that while the vector conservation law can be maintained, the axial current has to be broken.

As stated by the Adler-Bardeen theorem [165], this is actually quite astonishing that the anomaly does not receive radiative corrections and is totally given at the one-loop level. It has been realised later [166] that the anomaly was not just a perturbation effect arising from divergent diagrams requiring to be regularised. Indeed, anomalies, as opposed to divergences, essentially do not diverge even if they both emerge from the presence of an infinite number of degrees of freedom in the theory [1]. It seems more accurate to appreciate anomalies as a side effect of the quantisation which might break some symmetries.

This is really in the seventies [167–170] that the anomaly was interpreted in term of a topological invariant. Anomaly has been indeed determined by an index theorem by counting the zero-modes

[1] In that regard, the scale anomaly is singular.

of a chiral Dirac operator. This counting was made transparent by Fujikawa [120] as the anomaly arises in the path integral as the functional trace of $\gamma_5$.

In QFT the fundamental quantity is the generating functional which is a path integral for the classical action. How can an anomaly emerge when the classical action is invariant under a symmetry? This questions has been solved by Fujikawa in Ref. [171–173] by realising that the only quantity which contains the quantum aspects, the path integral measure, does not remain invariant under chiral transformations. The anomaly is precisely arising from the associated non-trivial Jacobian, which is ill-defined. To regularise it in a gauge-invariant manner, one can use, as Fujikawa did, an eigenbasis expansion associated to a Gaussian cutoff or alternatively a heat-kernel regularisation or a $\zeta$–function regularisation[2]. In any case, the anomaly technically arises as a finite term from the regularisation. Within this formalism the anomaly is truly independent of perturbation theory and indeed provides a conceptually and satisfactory derivation of the anomaly terms present from the beginning instead of discovering it after the evaluation of the divergence of a current.

In particle physics, the methods of Effective Field Theory (EFT) have recently seen a resurgence, mostly due to the lack of new physics discovery at the weak scale. Observations seem to indicate that new physics should indeed be decoupled to heavier scales, urging us to reconsider the Standard Model (SM) as a more humble EFT supplemented by higher-dimensional operators.

The new physics integrated out at some higher energy scale is technically encapsulated in the coefficients of these higher-dimensional operators. The task to evaluate these Wilson coefficients from ultraviolet (UV) theories has traditionally been done using Feynman diagrams, where amplitudes involving the heavy degrees of freedom are explicitly "matched" to the EFT amplitudes. However, a more elegant approach is to "integrate out" the heavy particles by evaluating the path integral directly [72–74, 80] even if in the past, this approach has been limited because, in practice, the expansion techniques could be cumbersome. However, recently a significant effort has been done for developing new methods to evaluate the path integral at one loop more efficiently using improved expansion techniques [3, 76, 77, 79, 82].

In this work, we propose to compute anomalies in QFT, identified as a Jacobian in the path integral formalism as Fujikawa did, but in view of recent developments made in EFTs and more especially the usefulness of a mass expansion technique such as the Covariant Derivative Expansion (CDE) [72–74, 80]. This offers a novel technical approach to evaluate anomalies in QFT within the path integral formalism. The novelty of our formalism is the following. First, it does not truly rely on the computation of the transformation of the measure through the existence and definition of the Dirac operator spectrum and more especially trying to properly deal with the zero modes of the chiral Dirac operators as Fujikawa did. Second, the anomalies emerge from a ratio of two ill-defined determinants which can be evaluated systematically and efficiently by the CDE technique.

In practice, in Fujikawa's method, the various symmetries have been in-forced to the model beforehand in order to define the eigenbasis of the Dirac operator and cure the illness of the Jacobian of the considered transformation. The choice of regulator (to count the zero modes) to evaluate the anomaly is crucial and depends on the active symmetries. It will lead to several type of anomalies (consistent, covariant, etc.). We will see that in many situations, it is possible to end-up to this situation when "bosonising" the fermionic functional determinants, then straightforwardly extracting the anomalous interactions with the CDE. Within our proposed alternative method, the regularisation procedure is fixed and always carried with the usual dimensional-regularisation scheme [105]. The illness of the Jacobian is then embodied in the ambiguity of Dirac traces involving $\gamma_5$ (see Ref. [2, 161]). These ambiguities are cured by imposing manually the invariance of the EFT under specific symmetries. Thus, our method is available to evaluate in a general way the covariant

[2]See Ref. [119, 120] and the references therein.

and consistent anomaly from the path integral having then the possibility to tune which current bears the anomaly.

These two approaches allow to treat gauge and mixed global-gauge anomalies i.e consistent and covariant anomalies, gravitational anomaly, as well as the scale anomaly, in a transparent, simple and unified way which certainly deserves to be presented due to the importance and phenomenological implications of anomalies in physics.

The plan of the paper is the following, in the next section we detail the outline of the proposed new method to compute QFT anomalies within the path integral formalism and use the example of the axial anomaly to concretely show how to connect the Jacobian of the transformation, to the functional determinants and how to conveniently expand them. In a third section, we apply our formalism to other anomalous transformations in chiral gauge field theory, namely fermionic vector and axial transformations, leading to so-called covariant and consistent anomalies. In the fourth section, we evaluate the axial-gravitational anomaly and technically show how to deal with this approach in curved space-time. In the fifth section, we evaluate the so-called scale anomaly, without having to introduce the curvature of space-time [174–178]. In a subsequent section, we discuss the approach of the new method presented in this work compared to the original approach of Fujikawa, before bringing our conclusion in the last section.

## 7.2 Outline of the new method

In this section we will introduce and present a method to compute QFT anomalies within the path integral formalism while dealing with EFTs. In order to make our points as clear as possible we will deal with the concrete case of the axial anomaly in a vector gauge theory. More general situations will be discussed in the following section.

### 7.2.1 Functional determinant and Jacobian

Let us start with a Dirac fermion field involved in a vector gauge theory with the following path integral,

$$Z \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left( i \int \mathrm{d}^4x\bar{\psi}(i\partial\!\!\!/ - V\!\!\!/ - m)\psi \right) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS}, \tag{7.1}$$

with $V$ a gauge field, element of $SU(N) \equiv G$ and "slashed" quantities are Lorentz-contracted with $\gamma$ matrices. Performing the integration on Grassmann variables $Z$ can be written as (in the eigenbasis with eigenvalues $\lambda_n$ of the Dirac operator),

$$Z = \prod_n (\lambda_n - m) = \det\left(i\partial\!\!\!/ - V\!\!\!/ - m\right), \tag{7.2}$$

where *det* is a functional determinant. Let us consider an infinitesimal chiral reparametrisation of the fermionic field, of parameter $\theta(x) = \theta^a T^a \in SU(N)$,

$$\psi \to e^{i\theta(x)\gamma_5}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\theta(x)\gamma_5}. \tag{7.3}$$

Under such a transformation, the path integral measure transforms with a Jacobian $J[\theta]$,

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to J[\theta]\mathcal{D}\psi\mathcal{D}\bar{\psi}, \tag{7.4}$$

on the other hand the action transforms like,

$$S \to S - \int \mathrm{d}^4x\,\bar{\psi}\left[2im\theta\gamma_5 + (D\!\!\!\!/\,\theta)\gamma_5\right]\psi, \tag{7.5}$$

with $(\not{D}\theta) = (\not{\partial}\theta) + i[\not{V}, \theta]$ and the parenthesis indicates the local derivative. The path integral after the chiral reparametrisation reads,

$$Z' = \int J[\theta]\mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(iS - i\int \mathrm{d}^4x\,\bar{\psi}\left[2im\theta\gamma_5 + (\not{D}\theta)\gamma_5\right]\psi\right). \tag{7.6}$$

Since the anomaly is fully determined by the structure of the gauge groups of the theory, the Jacobian $J[\theta]$ does not depend on the fermionic field, then one can perform the integration on the Grassmann variables and write,

$$Z' = J[\theta]\det(i\not{\partial} - \not{V} - m - 2im\theta\gamma_5 - (\not{D}\theta)\gamma_5). \tag{7.7}$$

As a result of the invariance under the labeling of the path integral variables ($Z = Z'$), the Jacobian reads,

$$J[\theta] = \frac{\det(i\not{D} - m)}{\det(i\not{D} - m - 2im\theta\gamma_5 - (\not{D}\theta)\gamma_5)} = \frac{\det(i\not{D} - m)}{\det(i\not{D} - m + i\{\theta\gamma_5, i\not{D} - m\})}. \tag{7.8}$$

The Jacobian can therefore be expressed as the exponential of the difference of two functional determinants,

$$J[\theta] = \exp\left(\log\det(i\not{D} - m) - \log\det(i\not{D} - m + i\{\theta\gamma_5, i\not{D} - m\})\right) \equiv \exp\left[\int \mathrm{d}^4x\,\mathcal{A}(x)\right]. \tag{7.9}$$

In the peculiar case of the chiral reparametrisation of Eq. (7.3) being disjoint from gauge transformations, injecting this solution for $J[\theta]$ in Eq. (7.6) leads to the relation,

$$D_\mu \left\langle\bar{\psi}\gamma^\mu\gamma_5\psi\right| = \Big\rangle\, 2im \left\langle\bar{\psi}\gamma_5\psi\right| + \Big\rangle \frac{\delta\mathcal{A}(x)}{\delta\theta(x)}, \tag{7.10}$$

which is the anomalous Ward identity of the axial current reflecting the anomalous behaviour of that chiral reparametrisation.

The main goal of this paper is to compute the anomaly operator of a theory, $\mathcal{A}$, directly from its path integral formulation. Yet, we will not revert to the procedure of Fujikawa to compute the determinants, which corresponds to a precise procedure to regularise the computation (the core of the problem). Instead we will call in the mass expansion method known as Covariant Derivative Expansion (CDE) [73, 74] that we will combine with different regularisation procedures. All in all, being very efficient to obtain anomalies in QFT.

One should also note that the CDE method has recently proved its usefulness while dealing with precisely this kind of EFTs and more especially the *matching* step which consists in expressing the Wilson coefficient of the low energy EFT as a function of the parameters of the high energy theory (see for example Refs. [3, 72, 75, 79–82]).

### 7.2.2 The ABJ anomaly from the Covariant Derivative Expansion

The principle of the CDE approach will be detailed below. Let $\mathcal{A}$ be,

$$\int \mathrm{d}^4x\,\mathcal{A}(x) = -\mathrm{Tr}\log\left(i\not{D} - m - 2im\theta\gamma_5 - (\not{\partial}\theta)\gamma_5\right) + \mathrm{Tr}\log\left(i\not{D} - m\right). \tag{7.11}$$

In this section, we restrain ourselves to a vector gauge theory, $D_\mu = \partial_\mu + iV_\mu$, with $V \in G = SU(N)$, and the chiral reparametrisation of the fermionic field is a simple axial $U(1)$ transformation. Thus we expect to obtain the so-called chiral or Adler–Bell–Jackiw (ABJ) anomaly [140, 165].

For clarity, we will first present the evaluation of the first functional trace in Eq. (7.11) that we label $\mathcal{A}_\theta$, before combining both needed to compute the axial current anomaly $\mathcal{A}$. We evaluate the trace over space-time using a plane wave basis, leaving the trace $tr$ over the internal space,

$$\mathcal{A}_\theta = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} e^{iq\cdot x} \mathrm{tr}\log\left(i\not{D} - m - 2im\theta\gamma_5 - (\not{\partial}\theta)\gamma_5\right) e^{-iq\cdot x} \ , \tag{7.12}$$

use the Baker-Campbell-Hausdorff formula to perform the spatial translation,

$$\mathcal{A}_\theta = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\log\left(i\not{D} + \not{q} - m - 2im\theta\gamma_5 - (\not{\partial}\theta)\gamma_5\right) \ , \tag{7.13}$$

and perform the change of variable $q \to -q$ to factorise an inverse propagator-like term,

$$\mathcal{A}_\theta = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\log\left[-(\not{q}+m)\left(1 + \frac{-1}{\not{q}+m}(i\not{D} - 2im\theta\gamma_5 - (\not{\partial}\theta)\gamma_5)\right)\right] \ . \tag{7.14}$$

The factorised term exhibits UV divergences, and involve the usual scale of renormalisation, which would be introduced through dimensional regularisation. It can be absorbed in redefinitions of the parameters of the model [3]. One could also notice that it would be anyway canceled by the other trace to evaluate in Eq. (7.11). Using Taylor expansion on the remaining logarithm,

$$\mathcal{A}_\theta = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}\left[\frac{-1}{\not{q}+m}\left(-i\not{D} + 2im\theta\gamma_5 + (\not{\partial}\theta)\gamma_5\right)\right]^n \ . \tag{7.15}$$

If we now apply the very same treatment to the other contribution, $\mathrm{Tr}\log(i\not{D} - m)$ of Eq. (7.11), in order to evaluate the anomaly, we find that the terms which do not involve the $\theta$ parameter do cancel with each other,

$$\mathcal{A} = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}\left[\frac{-1}{\not{q}+m}\left(-i\not{D} + 2im\theta\gamma_5 + (\not{\partial}\theta)\gamma_5\right)\right]^n \Bigg|_{\text{carrying } \theta \text{ dependence}} \ . \tag{7.16}$$

As shown in appendix C, it can alternatively be written as,

$$\mathcal{A} = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\ \left(2im\theta\gamma_5 + (\not{\partial}\theta)\gamma_5\right) \sum_{n=0}^{\infty} \left[\frac{-1}{\not{q}+m}\left(-i\not{D}\right)\right]^n \frac{-1}{\not{q}+m} \,. \tag{7.17}$$

So the reader should not be surprised if we switch between the two expressions. This expression might still look quite cumbersome to deal with, however, as we will see in the following section, it only calls for a basic power counting and use of master integrals [4].

### 7.2.3 Complete evaluation of the ABJ anomaly from the CDE

Since we are only interested in the terms linear [5] in $\theta$ in Eq. (7.16), the anomaly can be expressed as $\mathcal{A} = \mathcal{A}^{m\gamma_5} + \mathcal{A}^{\not{\partial}\gamma_5}$ [6] with,

[3] It corresponds to a renormalisation of the vacuum energy and it can be absorbed as a constant term in the Standard Model Higgs potential.

[4] Notice that our formalism does not support the $m = 0$ case. We suggest the Heat-kernel method if the reader would like to recover the ABJ anomaly in this case.

[5] This is the only possibility to obtain a $\theta$-dependent term times a gauge boundary term which is mass independent.

[6] This is nothing but the transcription of the Ward identity Eq. (7.10) in the present CDE context.

$$
\begin{aligned}
\mathcal{A}^{m\gamma_5} &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr} \left[ \frac{-1}{\not{q}+m} \left( -i\not{D} + 2im\theta\gamma_5 \right) \right]^n \Bigg|_{\mathcal{O}(\theta)} , \\
\mathcal{A}^{\not{\partial}\gamma_5} &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr} \left[ \frac{-1}{\not{q}+m} \left( -i\not{D} + (\not{\partial}\theta)\gamma_5 \right) \right]^n \Bigg|_{\mathcal{O}(\theta)} .
\end{aligned} \tag{7.18}
$$

The terms which contribute to $\mathcal{A}$ involve, here, exactly one $\gamma_5$ matrix and there can be no contribution from orders greater than $n = 5$, within the CDE approach, since they would carry a mass dependence.

Some of the integrals in Eq. (7.18) are divergent and we use dimensional regularisation [105] to evaluate them along with the $\overline{MS}$ scheme for renormalisation. The traces over Dirac matrices have to be performed in $d = 4 - \epsilon$ dimensions, and the $\epsilon$-terms resulting from the contractions with the metric tensor (satisfying then $g^{\mu\nu} g_{\mu\nu} = d$) must be kept in the calculations. These $\epsilon$-terms will then multiply with the $(1/\epsilon)$ pole of the divergent integrals and yield finite contributions. We also emphasise that depending on the regularisation scheme for $\gamma_5$ in $d$-dimensions, different results for $\epsilon$-terms in Dirac traces will emerge (see for examples Refs. [103, 159]). In the following sections, we will discuss in details several prescriptions that one can use to evaluate ill-defined Dirac traces involving $\gamma_5$ matrices, in dimensional regularisation. However, in this section, since we discuss the case of a vector gauge theory related to Eq. (7.1), the divergent contribution are regularised using Breitenlohner-Maison-'t Hooft-Veltman (BMHV) scheme of dimensional regularisation [105, 106] which is compatible with the conservation of the gauge vector current at the quantum level, as it is well-known, placing then the anomaly entirely in the classically conserved (only) axial current associated to Eq. (7.3). In the evaluation of Eq. (7.18), we will then maintain the trace cyclicity property which might not hold for another $\gamma_5$ regularisation scheme.

We are here by-passing, on purpose, an important difficulty regarding the crucial regularising step in order to focus on the standard but important CDE algebra. The reader willing to concentrate on a careful regularisation procedure should directly reach the next section.

Regarding the actual task of collecting operators from Eq. (7.18), we do not especially rely on it but the different contributions produced by these expansions could also be enumerated using the convenient formalism of covariant diagrams (see Ref. [77] for example).

To perform the computations straightforwardly from Eq. (7.18), we decompose the propagator $-1/(\not{q} + m)$ as follows,

$$
\frac{-1}{\not{q}+m} = \frac{m}{q^2 - m^2} + \frac{-\not{q}}{q^2 - m^2} . \tag{7.19}
$$

Let us consider first the expansion of $\mathcal{A}^{\not{\partial}\gamma_5}$. The first non zero contribution is to be found at $n = 4$, where there are finite and divergent contributions. The finite one leads to the following term,

$$
\mathcal{A}^{\not{\partial}\gamma_5}_{n=4,\mathrm{fin}} = i(m^4 \mathcal{I}_i^4 - 4m^2 \mathcal{I}[q^2]_i^4) \mathrm{tr}(\not{D}\not{D}\not{D}(\not{\partial}\theta)\gamma_5) , \tag{7.20}
$$

using standard and convenient master integrals $\mathcal{I}$, written explicitly in Appendix A. This contribution can be written as [7],

$$
\mathcal{A}^{\not{\partial}\gamma_5}_{n=4,\mathrm{fin}} = -\frac{1}{32\pi^2} \mathrm{tr}\big(\not{D}\not{D}\not{D}(\not{\partial}\theta)\gamma_5\big) = \frac{i}{8\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}(D_\mu D_\nu D_\rho (\partial_\sigma \theta)). \tag{7.21}
$$

[7] The trace over Dirac matrices is performed but the trace tr over the gauge group structure is left.

The divergent contribution is regularised using BMHV scheme of dimensional regularisation [105, 106]. With this choice, the divergent contribution is,

$$\mathcal{A}^{\partial\!\!\!/\gamma_5}_{n=4,\text{div}} = \frac{-i}{16\pi^2}\left(\varepsilon\frac{2}{\varepsilon}+\mathcal{O}(\varepsilon)\right)\epsilon^{\mu\nu\rho\sigma}\text{tr}(D_\mu D_\nu D_\rho(\partial_\sigma\theta)) \xrightarrow[\varepsilon\to 0]{} -\frac{i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{tr}(D_\mu D_\nu D_\rho(\partial_\sigma\theta)). \quad (7.22)$$

The full $n=4$ contribution therefore cancels as the divergent and finite contributions compensate exactly,

$$\mathcal{A}^{\partial\!\!\!/\gamma_5}_{n=4} = \mathcal{A}^{\partial\!\!\!/\gamma_5}_{n=4,\text{div}} + \mathcal{A}^{\partial\!\!\!/\gamma_5}_{n=4,\text{fin}} = 0\,. \quad (7.23)$$

Note that we talk about divergent contributions because the integrals are divergent, but in the end, the result is finite as the pole $2/\epsilon$ is compensated by an $\epsilon$ from the trace.

Turning now to the expansion of $\mathcal{A}^{m\gamma_5}$, its first contribution arises at $n=5$, and is fully finite. In this case there is no requirement to switch to $d$ dimensions [8]. This contribution reads,

$$\mathcal{A}^{m\gamma_5}_{n=5} = i\big(2m^6\mathcal{I}^5_i - 16m^4\mathcal{I}[q^2]^5_i + 48m^2\mathcal{I}[q^4]^5_i\big)\,\text{tr}(D\!\!\!/ D\!\!\!/ D\!\!\!/ D\!\!\!/\,\theta\gamma_5)\,. \quad (7.24)$$

Performing the Dirac matrix algebra and using the expression of the master integrals given in appendix A, the $n=5$ contributions reads,

$$\mathcal{A}^{m\gamma_5}_{n=5} = -\frac{i}{16\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\theta\,\text{tr}\,(F_{\mu\nu}F_{\rho\sigma})\,, \quad (7.25)$$

where the convention for the field strength is $F_{\mu\nu} = [D_\mu, D_\nu]$.

Within the CDE approach, this is the only surviving contribution, and it matches the well-known result for the axial current anomaly in a vector gauge field theory [120, 122, 171],

$$\mathcal{A} = \mathcal{A}^{m\gamma_5} + \mathcal{A}^{\partial\!\!\!/\gamma_5} = \mathcal{A}^{m\gamma_5}_{n=5} = -\frac{i}{8\pi^2}\theta\,\text{tr}\big(F_{\mu\nu}\tilde{F}^{\mu\nu}\big)\,, \quad (7.26)$$

where the convention for the dual tensor is $\tilde{F}^{\mu\nu} = 1/2\,\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, with the choice $\epsilon^{0123} = +1$.

One may be a bit surprised by the fact that the anomaly ends up extracted from a non-divergent integral, for which no regularisation is needed. Let us stress though that the crucial step was to show that the $\mathcal{A}^{\partial\!\!\!/\gamma_5}$ term gives no contribution in that particular case at order $m^0$.

Following a similar strategy, we will now discuss more generalities and details of the evaluation of the covariant and consistent anomalies in QFT based on a careful regularisation.

### 7.2.4 ABJ anomaly in a given $2n$ dimensions from the CDE

In this section, we provide a general approach to extend the computation of the ABJ anomaly in a given $2n$ dimensions. There is no obstruction to computing the anomaly for a given even dimension, but for arbitrary $2n$ dimensions it becomes more complicated to simplify Dirac traces.

Starting from Eq. (7.17) in the manuscript, we have in $d = 2n$ dimensions,

$$\mathcal{A} = \int\frac{\mathrm{d}^d q}{(2\pi)^d}\text{tr}\,(2im\theta\gamma_{2n+1} + (\partial\!\!\!/\theta)\gamma_{2n+1})\sum_{k=0}^{\infty}\text{tr}\left[\frac{-1}{q\!\!\!/ + m}\left(-iD\!\!\!/\right)\right]^k\frac{-1}{q\!\!\!/ + m}\,, \quad (7.27)$$

where we have generalised the definition of $\gamma_5$ in $2n$ dimensions as,

$$\gamma_{2n+1} = (i)^{n-1}\gamma^0\gamma^1\cdots\gamma^{2n-1}\,, \quad (7.28)$$

[8] for convenience with the notations, we are still using the master integrals which are technically defined in d-dimension.

and we have,

$$\mathrm{tr}\left[\gamma_{2n+1}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\right] \equiv (-i)2^n\,\epsilon^{\mu_1\mu_2\cdots\mu_{2n}}\,. \tag{7.29}$$

By power counting, we can isolate the terms of order $m^0$,

$$\mathcal{A}\Big|_{d=2n} = \int \frac{\mathrm{d}^{2n}q}{(2\pi)^{2n}}\mathrm{tr}\left[(2im\theta\gamma_{2n+1})\Big[\Delta(-i\not{D})\Big]^{2n}\Delta + \big[(\not{\partial}\theta)\gamma_{2n+1}\big]\Big[\Delta(-i\not{D})\Big]^{2n-1}\Delta\right]\,, \tag{7.30}$$

where $\Delta = -1/(\not{q}+m)$.

We rewrite the propagators $\Delta$ in terms of bosonic and fermionic propagators $\Delta = \Delta_b + \Delta_f$ with $\Delta_b = m/(q^2-m^2)$ and $\Delta_f = -\not{q}/(q^2-m^2)$. The integration over momentum is non-vanishing for even powers in $q$, which means that we have to account for all the terms that have an even number of fermionic propagators. Therefore, the number of terms to compute increases significantly with the dimension.

For $2k$ fermionic propagators among the $2n+1$ propagators, we have to compute traces of the form,

$$g_{\alpha_1\ldots\alpha_{2k}}\mathrm{tr}\,\gamma_{2n+1}\left(\gamma^{\alpha_1}\gamma^{\mu_1}\gamma^{\alpha_2}\gamma^{\mu_2}\ldots\gamma^{\mu_{2n+1}}+\ldots\right)\,, \tag{7.31}$$

where the dots encompass the remaining $\binom{2n+1}{2k}-1$ possible combinations of $2k$ fermionic propagators among $2n+1$ propagators, and $g_{\alpha_1\ldots\alpha_{2k}}$ is the fully symmetrised metric[9]. Then such traces have to computed for all $k \leq n$.

Such a trace is not trivial to compute for arbitrary $k$ and $n$, which is why the general formula for the anomaly in $2n$ dimensions is not straightforward to obtain, and is out of the scope of this paper. For the computation in an arbitrary $2n$ dimensions, we refer the reader to Refs. [179, 180].

Within our framework, we can compute the ABJ anomaly (and other anomalies) in 4, 6, 8, $\cdots$ dimensions, then extrapolate the result to $2n$ dimensions. This strategy is analogous to the computations of $l-$agon Feynman diagrams (with $l = n+1$) which have been performed by Frampton et al [181, 182].

One must also generalise the definition of the master integrals in $2n$ dimensions, but this presents no difficulty.

## 7.3 Anomalies in vector-axial gauge field theory

In the previous section, we have discussed the methodology to compute the Jacobian of a path integral measure by using EFT techniques, namely the CDE, and gave a concrete example by computing the well-known axial current anomaly in a vector gauge field theory. In this section, we apply this new formalism to recover the various and well-known anomalies in vector-axial gauge field theory. If $\theta$ is charged under the $SU(N)$ gauge group of the theory, then the anomaly can either be covariant (covariant anomaly), or respect the Wess-Zumino consistency conditions [183] (consistent anomaly). Our computations in the following sections are performed in Minkowski space-time [10], and our results agree with the traditional ones (see for example Refs. [118–120]).

In our computation, it is necessary to consider $\theta$ local. In practice, if $\theta \in SU(N)$ (and $V, A \in SU(N)$ as well) is associated to a global symmetry, we conduct the computation with $\theta$ local, but we should regularise in order to get the covariant anomaly. If $\theta \in SU(N)$ is associated to a local symmetry, i.e a gauge transformation, we should regularise in order to get the consistent anomaly (gauge anomaly).

[9]For example, $g_{\mu\nu\rho\sigma} = g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}$.

[10]The standard computations of Refs. [119, 120] are performed in Euclidian space.

### 7.3.1 Definiteness and regularisation

Consider the following Lagrangian,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - V\!\!\!/ - A\!\!\!/\gamma_5 - m)\psi\,, \tag{7.32}$$

with $V_\mu$ and $A_\mu$ a vector and axial gauge field, elements of $SU(N)$ [11]. It is anomalous under the fermion reparametrisation,

$$\psi \to e^{i\theta(x)\gamma_5}\psi\,, \quad \bar{\psi} \to \bar{\psi}e^{i\theta(x)\gamma_5}\,, \tag{7.33}$$

with $\theta$ infinitesimal, and can be charged under the $SU(N)$ gauge group. The Jacobian of this reparametrisation can be expressed as follows,

$$J[\theta] = \frac{\det(iD\!\!\!\!/ - m)}{\det(iD\!\!\!\!/ - m - (D\!\!\!\!/\theta)\gamma_5 - 2im\theta\gamma_5)}\,. \tag{7.34}$$

However, we know that the anomaly associated to the axial reparametrisation may as well appear in the vector current or the axial current (see for example Refs. [119, 157]). The Jacobian in Eq. (7.34) standing as it is can lead to any distribution of the anomaly in both currents.

Moreover, since the theory has an axial gauge field $A_\mu$, the reparametrisation in Eq. (7.33) can be interpreted as a gauge transformation (i.e local transformation) if $\theta$ is charged under the gauge group. For these reasons, the Jacobian in Eq. (7.34) is ill-defined.

To make sense of this ratio of formal determinants, we need to regularise it. In CDE, the most convenient regularisation scheme is dimensional regularisation. However, it is well-known that the definition of $\gamma_5$ in dimensional regularisation is ambiguous due to its intrinsic 4 dimensional nature [105]. We will propose two methods of regularising the Jacobian of Eq. (7.34). The first method consists in working with the formal determinant in dimensional regularisation and, throughout the computation, deal with the ambiguity related to $\gamma_5$ using free parameters [2, 161]. The second method consists in bosonising the determinant, making it finite, hence fixing the ambiguity before the calculation. The first method can be seen as more general (or maybe naïve and brutal) as one first regularises an ill-defined quantity inserting as much freedom as needed and secondly call for coherence (covariance, integrability/consistence) of the obtained theory to fix those ambiguities. We believe that a remarkable advantage of this approach is that its derivation is smooth and self-consistent within the path integral formalism. The second method works the opposite way, as one firstly calls for a well defined theory (free of any ambiguity) and secondly perform the regularisation. As we will see both have their own advantages and disadvantages and we find it illuminating to present them both. We should also notice that while we believe the first method is novel in its approach, the bosonisation method is well-known [72, 119, 180], however its combined used with the CDE to evaluate anomalies, is new and since this offers a powerful tool and interesting implications for EFTs related topics, it deserves to be duly studied here.

#### 7.3.1.1 Ambiguities and free parameters

In $d$ dimensions, $\gamma_5$ is ill-defined. One cannot maintain both the cyclicity of the trace and Clifford algebra. There exist many ways of defining $\gamma_5$ in $d$ dimensions consistently [103, 105, 106, 159, 161], although they may yield different results. The ambiguity in the Jacobian of Eq. (7.34) lies in the dependence on the choice of the $\gamma_5$ regularisation scheme.

[11] In order to clarify our manuscript, we postpone the important discussion about manifest gauge or global symmetry invariance, the mass term as an hard breaking source in the unitary basis and the introduction of Goldstone bosons to implement spontaneous symmetry breaking to section 7.3.3.1.

In a diagrammatic approach, the amplitude of a diagram is dictated by the Feynman rules. However, it does not specify by which vertex we should start writing the amplitude of the diagram, which results in different possible position for $\gamma_5$. Since in $d$ dimensions, the different positions of $\gamma_5$ are not equivalent, we have an ambiguity in the position of $\gamma_5$.

Nonetheless, it is possible to compute traces of $\gamma_5$ in $d$ dimensions while keeping track of the ambiguity by introducing free parameters [161]. We outline the method in the following.

Consider the trace,

$$\mathrm{tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right). \tag{7.35}$$

In 4 dimensions, one can use Clifford algebra to move the $\gamma_5$ at different positions,

$$\mathrm{tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = \mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma_5\gamma^\rho\gamma^\sigma\right) = \mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5\right). \tag{7.36}$$

However, this may not be true anymore in $d$ dimensions. Therefore, if there is an ambiguity in the position of $\gamma_5$

For example if we use BMHV scheme [106], we maintain the cyclicity of the trace but we have to abandon Clifford algebra. We then have an ambiguity on the position of $\gamma_5$ in the trace. The trick presented in Ref. [161] consists in implementing all the positions for $\gamma_5$ that are equivalent in 4 dimensions, with a free parameter for each,

$$\mathrm{tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \to \alpha\,\mathrm{tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) + \beta\,\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma_5\gamma^\rho\gamma^\sigma\right) + \delta\,\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5\right), \tag{7.37}$$

with the condition $\alpha + \beta + \delta = 1$, so that we recover $\mathrm{tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right)$ in 4 dimensions.

The introduction of those free parameters with all the equivalent positions (in 4 dimensions) of $\gamma_5$ makes the trace regularisation scheme independent. Therefore, we can choose a specific scheme to compute each separate trace. If the result depends on the free parameters in the end, it means that the initial trace itself is ambiguous.

For the example above, we compute each separate trace using BMHV scheme to get,

$$\begin{aligned}\mathrm{tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \to& \,\alpha\,\mathrm{tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) + \beta\,\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma_5\gamma^\rho\gamma^\sigma\right) + \delta\,\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5\right)\\ =&(\alpha+\beta+\delta)(-4i\epsilon^{\mu\nu\rho\sigma}) = -4i\epsilon^{\mu\nu\rho\sigma}\,,\end{aligned} \tag{7.38}$$

where we have used the condition $\alpha + \beta + \delta = 1$ to match with the result in 4 dimensions. It turns out that this trace is non-ambiguous.

However, consider the following trace with one contraction among the Dirac matrices,

$$\begin{aligned}&\mathrm{tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^a\gamma^\rho\gamma^\sigma\gamma_a\right)\\ \to& \,\alpha\mathrm{tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^a\gamma^\rho\gamma^\sigma\gamma_a\right) + \beta\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma_5\gamma^a\gamma^\rho\gamma^\sigma\gamma_a\right) + \gamma\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^a\gamma^\rho\gamma_5\gamma^\sigma\gamma_a\right) + \delta\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^a\gamma^\rho\gamma^\sigma\gamma_a\gamma_5\right)\\ =&(-1+2\gamma)\,4i(d-4)\epsilon^{\mu\nu\rho\sigma}\,.\end{aligned} \tag{7.39}$$

It is ambiguous because even after enforcing the condition $\alpha+\beta+\gamma+\delta = 1$, the result still depends on a free parameter. Actually, insofar as there is more than one contraction among the Dirac matrices, the trace will be ambiguous[12]. As a consequence, when computing the anomaly, the final result depends on free parameters. Those free parameters are then fixed under physical constraints, for example by enforcing gauge invariance and vector current conservation.

Although the positions of $\gamma_5$ in the computation of the path integral Jacobian are not arbitrary, as opposed to the diagrammatic approach, it may still bear traces that depends on the choice of $\gamma_5$ scheme. Despite the absence of arbitrariness in the position of $\gamma_5$ we will still rely on the free parameters trick to compute the ambiguous Jacobian, since it allows us to compute the traces in a $\gamma_5$ scheme independent way.

[12] See appendix D for the case with two contractions among the sequence of Dirac matrices.

#### 7.3.1.2 A well-known treatment : the bosonisation

Before delving into the expansion of the determinant, it is possible to regularise it. One way of achieving a regularised Jacobian is to bosonise it.

**Vector gauge theory:** Consider first a vector gauge theory, the Jacobian can be squared to bosonise it,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - V\!\!\!/ - m)\psi\,. \tag{7.40}$$

We will show in section 7.3.4.2 that the Jacobian in Eq. (7.34) can be written as,

$$J[\theta]^2 = \frac{\det(D\!\!\!\!/^{\,2} + m^2)}{\det(D\!\!\!\!/^{\,2} + m^2 + \{iD\!\!\!\!/, (D\!\!\!\!/\theta)\gamma_5\} + 4im^2\theta\gamma_5)}\,. \tag{7.41}$$

This Jacobian yields the same result as the fermionic Jacobian Eq. (7.8) insofar as the theory is not chiral.

**Vector-axial theory:** Now, consider a vector-axial gauge theory,

$$iD\!\!\!\!/ = i\partial\!\!\!/ - V\!\!\!/ - A\!\!\!/\gamma_5\,. \tag{7.42}$$

Now the operator $iD\!\!\!\!/ - m$ does not have a well-defined eigenvalue problem, the presence of the axial field spoils the hermitianity. It is however crucial to have a well-defined eigenvalue problem to make sense of the determinant, which is the product of the eigenvalues of the operator.

We will now present a solution for bosonising the Jacobian of Eq. (7.34) that let us deal with hermitian and gauge covariant operators.

One way to obtain a hermitian operator is to use the following Laplace operators,

$$D\!\!\!\!/^{\dagger}D\!\!\!\!/ \text{ and } D\!\!\!\!/D\!\!\!\!/^{\dagger}\,. \tag{7.43}$$

These operators are hermitian, hence have a well-defined eigenvalue problem. They preserve the spectrum of the theory (see for example Ref. [119]), hence do not change the value of the determinants (aside squaring them). Besides, they lead to a gauge covariant regularisation of the bosonised form of the Jacobian.

We will show in section 7.3.3, that the Jacobian of Eq. (7.34) can be written as,

$$J[\theta]^2 = \frac{\det\left(-(iD\!\!\!\!/)^{\dagger}iD\!\!\!\!/ + m^2\right)}{\det\left(-(iD\!\!\!\!/)^{\dagger}iD\!\!\!\!/ + m^2 + f(\theta)\right)}\,, \tag{7.44}$$

where,

$$f(\theta) = 4im^2\theta\gamma_5 - i[\theta, -D^2]\gamma_5 - \frac{1}{2}[\sigma.F^V, \theta]\gamma_5 - \frac{1}{2}[\sigma.F^A\gamma_5, \theta]\gamma_5\,. \tag{7.45}$$

$F^V$ and $F^A$ are the Bardeen curvatures defined a bit later in Eqs. (7.50) and (7.51).

This bosonised determinant is finite hence unambiguous. Besides, since the regularisation it provides is gauge covariant [119], the final result can only be gauge covariant, hence the so-called covariant anomaly.

On the other hand, if we want to compute the consistent anomaly, we can try to use the bosonisation as in the vector gauge theory. However, the operator $D\!\!\!\!/^{\,2} + m^2$ is still not hermitian. We palliate this problem using the analytic continuation $A_\mu \to iA_\mu$ that restores the hermitianity of $iD\!\!\!\!/$, hence of $D\!\!\!\!/^{\,2} + m^2$.

The Jacobian will then be written as,

$$J[\theta]^2 = \frac{\det(\not{D}^2 + m^2)}{\det(\not{D}^2 + m^2 + \{i\not{D}, (\not{D}\theta)\gamma_5\} + 4im^2\theta\gamma_5)} . \tag{7.46}$$

Unfortunately, as we will see, this does not suffice to fix the ambiguity. It does not necessarily yield the consistent anomaly.

### 7.3.2 A generic Lagrangian

To pave the way for our computations of covariant and consistent anomalies, we present briefly the generic Lagrangian we will consider and our notations. One can consider a gauge theory in which left and right-handed fermion components are charged under a non-Abelian gauge group, then described by the following Lagrangian,

$$\mathcal{L} = \bar{\psi}_L\gamma^\mu(i\partial_\mu - L_\mu)\psi_L + \bar{\psi}_R\gamma^\mu(i\partial_\mu - R_\mu)\psi_R - m\bar{\psi}\psi, \tag{7.47}$$

where $L_\mu = L^a_\mu T^a$ and $R_\mu = R^a_\mu T^a$ are gauge fields belonging to $SU(N)$. In term of the projector algebra, this Lagrangian can be written in terms of vector-axial gauge fields as follows,

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - \not{L}P_L - \not{R}P_R - m)\psi = \bar{\psi}(i\not{\partial} - \not{V} - \not{A}\gamma_5 - m)\psi \equiv \bar{\psi}(i\not{D} - m)\psi, \tag{7.48}$$

where we defined the fields $V_\mu$, $A_\mu$, and the covariant derivative as follows,

$$V_\mu \equiv \frac{L_\mu + R_\mu}{2}, \quad A_\mu \equiv \frac{R_\mu - L_\mu}{2}, \quad iD_\mu \equiv i\big(\partial_\mu + iV_\mu + iA_\mu\gamma_5\big) . \tag{7.49}$$

The computation of the commutator $[D_\mu, D_\nu]$ permits to define two Bardeen's curvatures (see Ref. [122]) by identifying the axial and vector part such that $[D_\mu, D_\nu] \equiv F^V_{\mu\nu} + F^A_{\mu\nu}\gamma_5$, which leads to the following expressions,

$$F^V_{\mu\nu} = i\left((\partial_\mu V_\nu) - (\partial_\nu V_\mu) + i[V_\mu, V_\nu] + i[A_\mu, A_\nu]\right), \tag{7.50}$$

$$F^A_{\mu\nu} = i\left((\partial_\mu A_\nu) - (\partial_\nu A_\mu) + i[A_\mu, V_\nu] + i[V_\mu, A_\nu]\right). \tag{7.51}$$

In the L/R basis the field strengths are,

$$F^L_{\mu\nu} = i\left((\partial_\mu L_\nu) - (\partial_\mu L_\mu) + i[L_\mu, L_\nu]\right) \tag{7.52}$$

$$F^R_{\mu\nu} = i\left((\partial_\mu R_\nu) - (\partial_\mu R_\mu) + i[R_\mu, R_\nu]\right) \ , \tag{7.53}$$

and the Bardeen curvatures are related to the L/R curvatures by,

$$F^V_{\mu\nu} = \frac{1}{2}(F^R_{\mu\nu} + F^L_{\mu\nu}) \tag{7.54}$$

$$F^A_{\mu\nu} = \frac{1}{2}(F^R_{\mu\nu} - F^L_{\mu\nu}) \ . \tag{7.55}$$

### 7.3.3 Covariant anomaly

#### 7.3.3.1 Mass term, manifest symmetry invariance and Goldstone bosons

All along our work, we constantly integrate out a massive chiral fermion. The mass term is a hard breaking source of axial symmetries (local or global). In order to make manifest those symmetries at tree-level one can evidently implement their spontaneous breaking introducing then their associated Goldstone bosons. We chose to work within the unitary basis and loose manifest tree-level axial invariance (when relevant) in order to deal with simpler functional determinants. The Goldstone bosons will be explicitly re-introduced only when it is necessary, see section 7.3.3.3. Consequently, one should not be surprised if we discuss an anomalous global symmetry which looks naively already broken at tree-level. [13]

#### 7.3.3.2 Case of an anomalous axial symmetry

We state here again, for convenience, the Lagrangian and the Jacobian associated to the axial transformation.

Starting from the vector-axial Lagrangian of Eq. (7.48), let us perform an axial fermion reparametrisation,

$$\psi \to e^{i\theta(x)\gamma_5}\psi, \quad \bar{\psi} \to \bar{\psi}\, e^{i\theta(x)\gamma_5} \ . \tag{7.56}$$

Under this fermion reparametrisation, the Lagrangian given by Eq. (7.48) becomes,

$$\mathcal{L} \to \bar{\psi}\big[i\not{D} - m - 2im\,\theta(x)\gamma_5 - \big(\not{D}\theta\gamma_5\big)\big]\psi \ , \tag{7.57}$$

where the quantity inside the parenthesis, $\big(\not{D}\theta\gamma_5\big) = \big((\not{\partial}\theta) + i[\not{V},\theta] + i[\not{A},\theta]\gamma_5\big)\,\gamma_5$, indicates that the covariant derivative locally acts on $\theta(x)$ (i.e not on everything on its right). The Jacobian produced by this transformation is therefore given by the following expression,

$$J[\theta] = \frac{\det\big(i\not{D} - m\big)}{\det e^{i\theta(x)\gamma_5}\big(i\not{D} - m\big)e^{i\theta(x)\gamma_5}} = \frac{\det\big(i\not{D} - m\big)}{\det\big(i\not{D} - m - 2im\theta\gamma_5 - (\not{D}\theta\gamma_5)\big)} \ . \tag{7.58}$$

As emphasised in the previous sections, this Jacobian is ill-defined. The next step is to explicitly compute it, according to the methods proposed in section 7.3.1.

**Fermionic expansion with free parameters** We are now in the situation where we are looking to evaluate an equivalent of Eq. (7.16) for a vector and axial gauge field theory,

$$\mathcal{A} = \int \frac{\mathrm{d}^d q}{(2\pi)^d}\mathrm{tr}\ \big((\not{D}\theta)\gamma_5 + 2im\theta\gamma_5\big)\sum_{n\geq 0}\left[\frac{-1}{\not{q} - m}(-i\not{D})\right]^n \frac{-1}{\not{q} - m}\,, \tag{7.59}$$

where $\theta$ belongs to $SU(N)$ and the covariant derivative is $D_\mu = \partial_\mu + iV_\mu + iA_\mu\gamma_5$.

Let's start by computing the mass term. The integrals are finite hence no ambiguity arises from this term.

The propagators that appear in the expansion need to be expanded as $-1/(\not{q} + m) = \Delta_b + \Delta_f$ where the bosonic propagator is $\Delta_b = m/(q^2 - m^2)$ and the fermionic propagator is $\Delta_f = -\not{q}/(q^2 - m^2)$. The integrals over momentum are non-vanishing only if the integrand has an even power in $q$ in the numerator (the denominator always has an even power in $q$). Therefore, the number of

[13] A detailed discussion on the parametrisation of local and global anomalous symmetries can be find in Ref. [2].

fermionic propagators must be even. This leaves us with three contributions. Note that each of those contributions is finite, thus the computation is performed in 4 dimensions.

- The contribution to the anomalous interaction involving only bosonic propagators is,

$$m^5\mathcal{I}[q^0]^5 2im\mathrm{tr}\left(\theta\gamma_5\not{D}^4\right) . \tag{7.60}$$

- The contribution to the anomalous interaction involving two fermionic propagators is,

$$m^3\mathcal{I}[q^2]^5 2im\mathrm{tr}\left(\theta\gamma_5[\gamma^a\not{D}\gamma_a\not{D}^3+\gamma^a\not{D}^2\gamma_a\not{D}^2+...]\right) , \tag{7.61}$$

where the dots bear all the remaining insertions of the two fermionic propagators ($\binom{5}{2}=10$ combinations).

- The contribution to the anomalous interaction involving four fermionic propagators is,

$$m\mathcal{I}[q^4]^5 2im\mathrm{tr}\left(\theta g_{abcd}\gamma_5[\gamma^a\not{D}\gamma^b\not{D}\gamma^c\not{D}\gamma^d\not{D}+\gamma^a\not{D}\gamma^b\not{D}\gamma^c\not{D}\not{D}\gamma^d+...]\right) , \tag{7.62}$$

where again the dots bear all the remaining insertions of the four fermionic propagators ($\binom{5}{4}=5$ combinations) and $g_{abcd}=g_{ab}g_{cd}+g_{ac}g_{bd}+g_{ad}g_{bc}$.

Then one needs to expand the covariant derivatives in order to extract the $\gamma_5$ from the axial fields and compute the Dirac traces. It is then simple algebra to form the field strengths as defined in Eqs. (7.50) and (7.51).

The mass term then yields a contribution that corresponds to the so-called Bardeen anomaly (with conserved vector current), that is to say the consistent anomaly,

$$\begin{aligned}\mathcal{A}^{m\gamma_5} &= \frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta^aT^a\Big(F^V_{\mu\nu}F^V_{\rho\sigma}+\frac{1}{3}F^A_{\mu\nu}F^A_{\rho\sigma}\\ &\qquad -\frac{8}{3}\left(iA_\mu iA_\nu F^V_{\rho\sigma}+iA_\mu F^V_{\nu\rho}iA_\sigma+F^V_{\mu\nu}iA_\rho iA_\sigma\right)+\frac{32}{3}iA_\mu iA_\nu iA_\rho iA_\sigma\Big)\\ &= \mathcal{A}^{\mathrm{Bardeen}} .\end{aligned} \tag{7.63}$$

Now let's focus on the derivative term,

$$\mathcal{A}_{\not{\partial}\gamma_5} = \int\frac{\mathrm{d}^dq}{(2\pi)^d}\mathrm{tr}\,(\not{D}\theta)\gamma_5\sum_{n\geq 0}\left[\frac{-1}{\not{q}-m}(-i\not{D})\right]^n\frac{-1}{\not{q}-m} . \tag{7.64}$$

We proceed similarly for the derivative term to obtain the following contributions:

- The contribution to the anomalous interaction involving only bosonic propagators is,

$$im^4\mathcal{I}[q^0]^4\mathrm{tr}\,\left((\not{D}\theta)\gamma_5(\not{D})^3\right) . \tag{7.65}$$

- The contribution to the anomalous interaction involving two fermionic propagators is,

$$im^2\mathcal{I}[q^2]^4\mathrm{tr}\,\left((\not{D}\theta)\gamma_5[\gamma^a\not{D}\gamma_a\not{D}\not{D}+\gamma^a\not{D}\not{D}\gamma_a\not{D}+\ldots]\right) , \tag{7.66}$$

  where the dots denote the other $\binom{4}{2}=6$ combinations for the insertions of the two fermionic propagators.

- The contribution to the anomalous interaction involving four fermionic propagators is,

$$i\mathcal{I}[q^4]^4 \mathrm{tr}\left((\not{D}\theta)\gamma_5[\gamma^a \not{D}\gamma^b \not{D}\gamma^c \not{D}\gamma^d g_{abcd}]\right) . \tag{7.67}$$

Now this last integral is divergent, thus the trace that appear in the term with four fermionic propagators is ambiguous. We use the trick described in section 7.3.1.1 to keep track of the ambiguity. Therefore, the three contributions above may be written, after integrating by parts, as a sum of operators with a free parameter for each. The result can thus be written fully in terms of free parameters associated to each possible operator (the finite contributions will just combine with a free parameter to give a different free parameter). We thus have,

$$\mathcal{A}_{\not{\partial}\gamma_5} = \frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta^a T^a \left(\sum_i a_i X_{i,\,\mu\nu\rho\sigma}\right) , \tag{7.68}$$

where $X_i$ are all the possible operators of the form $\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4$ with $\mathcal{O}_{1\leq i\leq 4} \in \{V, A, \partial\}$ that can be formed, provided it has an even number of $A$ fields (the number of $\gamma_5$ must be odd). Note that the operators with a partial derivative to the right vanish, and those with consecutive partial derivatives vanish due to the contraction with the $\epsilon$ tensor. This leaves us with 22 possible operators, with 22 free parameters $a_i$.

We then want to enforce the covariance of $\mathcal{A}_{m\gamma_5} + \mathcal{A}_{\not{\partial}\gamma_5}$ under the gauge transformation,

$$\begin{cases} V_\mu \to V_\mu + (D^V_\mu \varepsilon_V) + i[A_\mu, \varepsilon_A] \\ A_\mu \to A_\mu + i[A_\mu, \varepsilon_V] + (D^V_\mu \varepsilon_A) \end{cases} . \tag{7.69}$$

We can focus only on the gauge transformation associated to $\epsilon_A$, it will be sufficient to fix the free parameters.

A covariant operator $\mathcal{O}$ must transform as,

$$\delta\mathcal{O} = [\epsilon_A, \mathcal{O}] , \tag{7.70}$$

under the $\epsilon_A$ gauge transformation. Therefore, we enforce that the terms with derivatives of $\epsilon_A$ vanish, and also that $\epsilon_A$ must appear either at the beginning or at the end of each operator.

For example, after performing the gauge variation we have, among others, the following operator,

$$f(a_1, \dots, a_{22})\epsilon^{\mu\nu\rho\sigma}(\partial_\mu\, A_\nu)\epsilon_A V_\rho V_\sigma , \tag{7.71}$$

where $f$ is some linear function of the free parameters. This term must vanish for the result to be gauge covariant because $\epsilon_A$ is sandwished between operators, hence it cannot occur from a term of the form Eq. (7.70). We hence obtain a constraint on the free parameters.

It turns out that enforcing these conditions fixes 21 free parameters out of 22. We rely on the result from the ABJ anomaly to fix the last free parameter that we call $\beta$. Setting $A = 0$ we are left with,

$$\frac{-i}{16\pi^2}(1+\beta)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} , \tag{7.72}$$

where $F_{\mu\nu} = (\partial_\mu\, iV_\nu) - (\partial_\nu\, iV_\mu) + [iV_\mu, iV_\nu]$. Eq. (7.72) is covariant regardless of the normalisation, this is why it needs to be compared with the anomaly in a vector-like theory to fix $\beta$ (i.e the ABJ anomaly). This amounts to enforcing the conservation of the vector current. We thus deduce that $\beta = 0$. Now that all the free parameters are fixed, we obtain,

$$\mathcal{A}_{m\gamma_5} + \mathcal{A}_{\not{\partial}\gamma_5} = \frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta^a T^a \left(F^V_{\mu\nu}F^V_{\rho\sigma} + F^A_{\mu\nu}F^A_{\rho\sigma}\right) , \tag{7.73}$$

with $F^V$ and $F^A$ the Bardeen curvatures as defined in Eqs. (7.50),(7.51). This is the covariant non-Abelian anomaly in the axial current in a vector-axial theory. Note that the relative coefficient between $F^V\tilde{F}^V$ and $F^A\tilde{F}^A$ is fixed by requiring the covariance of the result, since $F^V\tilde{F}^V + bF^A\tilde{F}^A$ is not covariant unless $b = 1$.

It can also be written in the L-R basis as,

$$\mathcal{A}_{m\gamma_5} + \mathcal{A}_{\not{\partial}\gamma_5} = \frac{-i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta^a T^a\left(F^L_{\mu\nu}F^L_{\rho\sigma} + F^R_{\mu\nu}F^R_{\rho\sigma}\right) . \tag{7.74}$$

Finally, we can mention the Bardeen-Zumino polynomial (BZ polynomial) [184] that naturally appears in our computation. The BZ polynomial is the unique local function $\mathcal{P}^\mu$ such that the gauge variation of $D_\mu\mathcal{P}^\mu$ cancels exactly the gauge variation of the consistent anomaly.

The ambiguity in the derivative term $\mathcal{A}_{\not{\partial}\gamma_5}$ was fixed by requiring that the mass term and derivative term together are gauge covariant, that is to say, that the gauge variation of the derivative term cancels exactly the gauge variation of the unambiguous mass term. Since the mass term coincides with the consistent anomaly, then the derivative term cancelling its gauge variation is by definition the divergence of the BZ polynomial.

The derivative term thus reads,

$$\begin{aligned}\mathcal{A}_{\not{\partial}\gamma_5} &= \theta^a(D_\mu\mathcal{P}^\mu)^a \\ &= \frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta\left(\frac{2}{3}F^A_{\mu\nu}F^A_{\rho\sigma} + \frac{8}{3}\left(A_\mu A_\nu F^V_{\rho\sigma} + A_\mu F^V_{\nu\rho}A_\sigma + F^V_{\mu\nu}A_\rho A_\sigma\right) - \frac{32}{3}A_\mu A_\nu A_\rho A_\sigma\right) .\end{aligned} \tag{7.75}$$

Note that the divergence of the BZ polynomial was not obtained by substracting the covariant anomaly to the consistent anomaly, but truly by requiring the cancelation of the gauge variation of the consistent anomaly.

Eventually, we emphasise that the BZ polynomial itself can be obtained by summing Eqs. (7.65), (7.66) and (7.67), performing the Dirac trace using the values of the free parameters that cancel the gauge variation of the anomaly, as we did. We thus obtain a term of the form $(D_\mu\theta)\mathcal{P}^\mu$ where $\mathcal{P}^\mu$ is the BZ polynomial.

**Bosonisation method.** The previous method proceeds by carrying dimensional regularisation on the ill-defined functional determinants of the Jacobian of Eq. (7.58). A main difference with Fujikawa's approach is that one does not need to directly worry about whether the Dirac operator has a well defined eigenvalue problem, and then compute its spectrum. However, it exists a known trick which consists in transforming that Jacobian into a another well suited quantity, a Jacobian "squared".

As suggested in Refs. [179, 180, 185, 186], the operator $\not{D}^\dagger\not{D}$ and $\not{D}\not{D}^\dagger$ define a good eigenvalue problem in order to compute the spectrum of $i\not{D}$. In particular, since those two operators are Hermitian and covariant, they admit two orthogonal eigenbasis with real eigenvalues. For simplicity, we introduce $P_\mu = iD_\mu$, we then have,

$$\not{P}^\dagger\not{P}\phi_n = \lambda_n^2\phi_n, \quad \not{P}\not{P}^\dagger\varphi_n = \lambda_n^2\varphi_n, \quad n\in\mathbb{N}, \quad \lambda_n\in\mathbb{R}. \tag{7.76}$$

where

$$\not{P}\phi_n = \lambda_n\varphi_n\,,\ \not{P}^\dagger\varphi_n = \lambda_n\phi_n \ \text{ with } \lambda_n\in\mathbb{R}\,. \tag{7.77}$$

In order to form such operators from the orginal Jacobian Eq. (7.34), one can build the following quantity,

$$
\begin{aligned}
J^2[\theta] &= \frac{\det(\not{P}^\dagger - m)}{\det\left(e^{i\theta\gamma_5}(\not{P}^\dagger - m)e^{i\theta\gamma_5}\right)}\frac{\det(\not{P} - m)}{\det\left(e^{i\theta\gamma_5}(\not{P} - m)e^{i\theta\gamma_5}\right)} \\
&= \frac{\det(\not{P}^\dagger - m)}{\det\left(e^{i\theta\gamma_5}(\not{P}^\dagger - m)e^{i\theta\gamma_5}\right)}\frac{\det(-\not{P} - m)}{\det\left(e^{i\theta\gamma_5}(-\not{P} - m)e^{i\theta\gamma_5}\right)} \\
&= \frac{\det\left(-\not{P}^\dagger\not{P} + m^2\right)}{\det\left(-\not{P}^\dagger\not{P} + m^2 + m(\not{P} - \not{P}^\dagger) + f(\theta)\right)} \,,
\end{aligned}
\tag{7.78}
$$

where in the second line we have used the invariance of the determinant under the change $\gamma^\mu \to -\gamma^\mu$ [14], and we have defined,

$$
\begin{aligned}
f(\theta) =& 4im^2\theta\gamma_5 - i[\theta, P^2]\gamma_5 - \frac{1}{2}[\sigma.F^V, \theta]\gamma_5 - \frac{1}{2}[\sigma.F^A\gamma_5, \theta]\gamma_5 \\
&+ 2im\left(\theta\gamma_5\not{P} - \not{P}\theta\gamma_5\right) + im\left((\not{P}\theta) - (\not{P}^\dagger\theta)\right)\gamma_5 \,.
\end{aligned}
\tag{7.79}
$$

$F^V$ and $F^A$ are the Bardeen curvatures as defined in Eq. (7.50), (7.51), and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Details of the bosonisation are provided in appendix E.

$\theta$ is charged under the gauge group so $(\not{P}\theta) = (i\not{\partial}\theta) - [\not{V}, \theta] - [\not{A}\gamma_5, \theta]$. Therefore, we can a priori obtain the consistent or the covariant anomaly, but we will see that the bosonisation we have chosen selects the covariant result.

We shamelessly used the multiplicativity property, $\det(A)\det(B) = \det(AB)$, on non-regularised determinants. Indeed, this has to be admitted since CDE assumes that for a non-regularised determinant one can write $\log\det = \mathrm{Tr}\log$.

The computation of this Jacobian can be performed following the same principle given in section 7.2,

$$
2\mathcal{A} = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} e^{iqx} \mathrm{tr}\left(f(\theta) + m(\not{P} - \not{P}^\dagger)\right)\frac{1}{-\not{P}^\dagger\not{P} + m^2} e^{-iqx} \,.
\tag{7.80}
$$

At this point, one can recall that the terms that have an odd number of Dirac matrices vanish under the trace. Therefore, in the above we can drop the term $m(\not{P} - \not{P}^\dagger)$, and the terms $2im\left(\theta\gamma_5\not{P} - \not{P}\theta\gamma_5\right)$ and $im\left((\not{P}\theta) - (\not{P}^\dagger\theta)\right)\gamma_5$ from $f(\theta)$, since $1/(-\not{P}^\dagger\not{P} + m^2)$ has en even number of Dirac matrices. We are then left with,

$$
2\mathcal{A} = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} e^{-iqx} \mathrm{tr}\left(-i[\theta, P^2]\gamma_5 - \frac{1}{2}[\sigma.F^V, \theta]\gamma_5 - \frac{1}{2}[\sigma.F^A\gamma_5, \theta]\gamma_5 + 4im^2\theta\gamma_5\right)\frac{1}{-\not{P}^\dagger\not{P} + m^2} e^{iqx}.
\tag{7.81}
$$

This produces in the end,

$$
2\mathcal{A} = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\, h(\theta) \sum_{n \geq 0}\left[\Delta\left(-P^2 + \frac{i}{2}\sigma.F^V + \frac{i}{2}\sigma.F^A\gamma_5 + 2q \cdot P\right)\right]^n \Delta,
\tag{7.82}
$$

[14] It uses the fact that $\det = \mathrm{Tr}\exp$, and that the trace of an odd number of Dirac matrices always vanishes. Since they have to come in by pairs, the sign flip does not affect the result. Under this sign flip, $\gamma_5$ is unchanged since it has an even number of Dirac matrices.

where

$$h(\theta) = -i[\theta, P^2]\gamma_5 - \frac{1}{2}[\sigma.F^V, \theta]\gamma_5 - \frac{1}{2}[\sigma.F^A\gamma_5, \theta]\gamma_5 + 4im^2\theta\gamma_5 \ , \tag{7.83}$$

and $\Delta = 1/(q^2 - m^2)$.

After scrutinizing the various terms, they appear to be all finite (hence non-ambiguous), and in the end only one term contributes,

$$\begin{aligned} 2\mathcal{A} &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \Delta^3 \mathrm{tr}\, 4im^2\theta\gamma_5 (\frac{i}{2}\sigma.F^V + \frac{i}{2}\sigma.F^A\gamma_5)^2 \\ &= 2\frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta(F^V_{\mu\nu}F^V_{\rho\sigma} + F^A_{\mu\nu}F^A_{\rho\sigma})\,, \end{aligned} \tag{7.84}$$

where we have discarded terms with even number of $\gamma_5$ matrices (they cannot yield a boundary term so cannot contribute to the final result). We thus obtain the so-called covariant anomaly,

$$\mathcal{A} = \frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta(F^V_{\mu\nu}F^V_{\rho\sigma} + F^A_{\mu\nu}F^A_{\rho\sigma}) = \frac{-i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta(F^L_{\mu\nu}F^L_{\rho\sigma} + F^R_{\mu\nu}F^R_{\rho\sigma}) \ . \tag{7.85}$$

Additional details on the calculation are provided in the appendix E.

As an important remark, in the bosonised form of the Jacobian, it turns out that the derivative coupling contribution vanishes at order $m^0$, only the mass term $4im^2\theta\gamma_5$ (which stems from the term $2im\theta\gamma_5$ before bosonising) contributes. As a result, the computation is finite (in the sense that no divergent integral appears), therefore no ambiguity arises due to the definition of $\gamma_5$ since we can perform the calculation in 4 dimensions. Although the operators stemming from the derivative coupling $((\not{D}\theta)\gamma_5$ before bosonising) do not contribute to the anomaly, they are required to compensate the finite higher order (of order $1/m^k$ with $k > 0$) terms in the mass expansion, since the final result has to be exact at order $m^0$. Note also that we bosonised using $\not{D}^\dagger\not{D}$, but we could have equivalently used $\not{D}\not{D}^\dagger$ to get the same result.

#### 7.3.3.3 Case of an anomalous vector symmetry

Starting from the vector-axial Lagrangian of Eq. (7.48), let us perform now an $SU(N)$ vector fermion reparametrisation,

$$\psi \to e^{i\theta(x)}\psi, \quad \bar{\psi} \to \bar{\psi}\, e^{-i\theta(x)} \ . \tag{7.86}$$

Under this fermion reparametrisation, the Lagrangian given by Eq. (7.48) becomes,

$$\mathcal{L} \to \bar{\psi}\big[i\not{D} - m - \big(\not{D}\theta\big)\big]\psi \ , \tag{7.87}$$

with $D_\mu \equiv \big(\partial_\mu + iV_\mu + iA_\mu\gamma_5\big)$, and again the quantity inside the parenthesis, $\big(\not{D}\theta\big) \equiv \gamma^\mu(\partial_\mu\theta + i[V_\mu, \theta] + i[A_\mu, \theta]\gamma_5)$, indicates that the covariant derivative locally acts on $\theta(x)$, with $\theta$ charged under the gauge group. The Jacobian produced by this transformation is therefore given by the following expression,

$$J[\theta] = \frac{\det\big(i\not{D} - m\big)}{\det\big(i\not{D} - m - (\not{D}\theta)\big)} \ . \tag{7.88}$$

As in the axial rotation, the functional determinants of Eq. (7.88) are ill-defined and need to be regularised. In dimensional regularisation, this is the $\gamma_5$ located in the covariant derivative which entirely carries the ambiguity now.

Before presenting the computation of the Jacobian of Eq. (7.88) and its bosonised form, one should notice that starting from Eq. (7.85), no additional computation is needed, if one is only interested in the result. Indeed, from

$$\partial_\mu J_5^\mu = \partial_\mu J_R^\mu - \partial_\mu J_L^\mu = \frac{-i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta(F^L_{\mu\nu}F^L_{\rho\sigma} + F^R_{\mu\nu}F^R_{\rho\sigma}) \,, \tag{7.89}$$

one can identify,

$$\partial_\mu J_R^\mu = \frac{-i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta(F^L_{\mu\nu}F^L_{\rho\sigma}), \quad \partial_\mu J_L^\mu = \frac{i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta(F^R_{\mu\nu}F^R_{\rho\sigma}) \,, \tag{7.90}$$

with $\theta \in SU(N)$. Hence,

$$\begin{aligned}\partial_\mu J_V^\mu = \partial_\mu J_R^\mu + \partial_\mu J_L^\mu &= \frac{-i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta\big(F^R_{\mu\nu}F^R_{\rho\sigma} - F^L_{\mu\nu}F^L_{\rho\sigma}\big) \\ &= -\frac{i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta(F^V_{\mu\nu}F^A_{\rho\sigma} + F^A_{\mu\nu}F^V_{\rho\sigma}) \,. \end{aligned} \tag{7.91}$$

We are however more interested in presenting an explicit and transparent evaluation of this version of the covariant anomaly.

**Fermionic expansion with free parameters** We proceed in a similar fashion as for the covariant anomaly in the axial current, except there is no mass term to compute according to Eq.(7.88).

The derivative term is ambiguous because of the presence of $\gamma_5$ in the covariant derivative, and of the divergent integrals. As explained in section 7.3.3.2, it can thus be written as,

$$\mathcal{A} = \mathcal{A}_{\not{\partial}} = \frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta^a T^a \left(\sum_i a_i X_{i,\,\mu\nu\rho\sigma}\right) \,, \tag{7.92}$$

where $X_i$ are all the possible operators of the form $\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4$ with $\mathcal{O}_{1\leq i\leq 4} \in \{V, A, \partial\}$ that can be formed. Contrary to the case of the anomaly in the axial current, the operators $X_i$ that appear now have an odd number of $A$ (because there must be an odd number of $\gamma_5$). Note that the operators with a partial derivative to the right vanish, and those with consecutive partial derivatives vanish due to the contraction with the $\epsilon$ tensor. This leaves us with again 22 possible operators, with 22 free parameters $a_i$.

The result should be covariant under the gauge transformation Eq. (7.69). Once again, we only need to enforce the gauge covariance with respect to $\epsilon_A$ to fix the free parameters. The covariance of the result requires that its gauge variation has the form Eq. (7.70), we therefore enforce on the free parameters that the derivatives of $\epsilon_A$ and the operators that have $\epsilon_A$ neither at the beginning nor at the end of the operator vanish. This fixes again 21 free parameters out of 22, and leaves us with a result of the form,

$$\mathcal{A}_{\not{\partial}} = \alpha\,\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta\left(F^V_{\mu\nu}F^A_{\rho\sigma} + F^A_{\mu\nu}F^V_{\rho\sigma}\right) \,, \tag{7.93}$$

where $\alpha$ is the remaining free parameter.

For the anomaly in the axial current, we compared our result to the ABJ anomaly by setting the gauge field $A = 0$ to fix the normalisation. Unfortunately, this is not possible here because setting $A = 0$ makes the whole term vanish.

In the case of the Abelian anomaly, we happen to have the same issue, where the result is gauge covariant but there is a normalisation freedom that remains. In Ref. [2], they deal with the free parameters for the Abelian anomaly to fix the normalisation factor by enforcing the conservation of the axial current (up to the mass term)[15]. In Eq. (7.93), we set the gauge fields and $\theta$ as Abelian[16] and apply the technique from Ref. [2].

We consider Eq. (7.93) with Abelian gauge fields and Abelian $\theta$, which is gauge covariant independently of the remaining free parameter. To break down this gap, we re-organise Eq. (7.93) in terms of Generalised Chern-Simons (GCS) forms using integration by parts, then we introduce an auxiliary background field $\xi_\mu$ associated to the deformation of $(\partial_\mu \theta)$ [17] as follows,

$$\mathcal{A}_{\not\theta} = \beta\, \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\big[\xi_\mu - (\partial_\mu \theta)\big](iA_\mu) F^V_{\rho\sigma}\,, \tag{7.94}$$

where $\beta = 4\alpha$. At this stage, Eq. (7.94) is no longer gauge invariant under the axial gauge transformation. The conservation of the axial current (up to the mass term) can be enforced non-trivially if the axial gauge field obtains its mass after spontaneous symmetry breaking. By introducing the Goldstone boson $\pi_A$ associated to the longitudinal component of the axial gauge field $A_\mu$, we obtain,

$$\tilde{\mathcal{A}}_{\not\theta} = \beta\, \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\big[\xi_\mu - (\partial_\mu \theta)\big](iA_\mu) F^V_{\rho\sigma} - \frac{i}{8\pi^2}\, \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\left[\frac{\pi_A}{v}(\partial_\mu \xi_\nu) F^V_{\rho\sigma}\right]. \tag{7.95}$$

Requiring the quantity $\tilde{\mathcal{A}}_{\not\theta}$ to be gauge invariant, implies that $\beta = -i/(4\pi^2)$, or equivalently $\alpha = -i/(16\pi^2)$. Additional details about the GCS terms and the Goldstone terms are provided in the appendix D. Eventually, going back to non-Abelian gauge fields and $\theta$, we obtain the non-Abelian covariant anomaly in the vector current,

$$\mathcal{A}_{\not\theta} = \frac{-i}{16\pi^2} \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\, \theta \big(F^V_{\mu\nu} F^A_{\rho\sigma} + F^A_{\mu\nu} F^V_{\rho\sigma}\big)\,. \tag{7.96}$$

**Bosonisation method.** We bosonise the Jacobian from Eq. (7.88). Following the method detailed in Eq. (7.78)

$$J[\theta]^2 = \frac{\det(-(i\not{D})^\dagger i\not{D} + m^2)}{\det(-(i\not{D})^\dagger i\not{D} + m^2 + m(i\not{D} - (i\not{D})^\dagger) + f(\theta))}\,, \tag{7.97}$$

and we have defined,

$$f(\theta) = i[\theta, D^2] - \frac{1}{2}[\sigma.F^V, \theta] - \frac{1}{2}[\sigma.F^A \gamma_5, \theta]\,. \tag{7.98}$$

However, this regularisation yields the covariant anomaly in the axial current as seens in the above, it thus comes with no surprise that the vector current is conserved. That is to say, if we supplement the theory with a global axial and a global vector symmetries, then this regularisation puts all the anomaly in the global axial symmetry and conserves the global vector symmetry. Therefore, as expected, the Jacobian in Eq. (7.97) is equal to one and then is unable, from the start, to deal with an anomalous vector transformation.

[15] For the reader interested in anomaly from the global axial(vector) transformation, see Ref. [2] for the detail of computations and also the applications in axion phenomenology.

[16] Eq. (7.93) can be separated in an Abelian part, and a non-Abelian part (formed uniquely of commutators of $\theta$, $V$ and $A$). Since the free parameter is common to both these parts, we can set the non-Abelian part to zero to fix the free parameter as in the Abelian case.

[17] The auxiliary vector field $\xi_\mu$ will be set to zero at the end of the computation.

### 7.3.4 Consistent anomaly

For the covariant anomaly, we have showed that we can bosonise the Jacobian in a gauge covariant way. However, the anomalous operator $\mathcal{A}$ may not be gauge invariant and in that case, one should rather make sure that the determinants in Eq. (7.34) are regularised in a non-gauge invariant way. Now, what we may ask is that the anomaly satisfies the algebra of the gauge group i.e the anomaly can be required to satisfy a consistency relation (also called an integrability condition or Wess-Zumino condition [183]). In that case, the anomaly is more accurately called the consistent anomaly.

As a remark, it has been shown that the Wess-Zumino condition corresponds to the Bose symmetry with respect to the vertices of the one-loop Feynman diagrams. If the covariant anomaly collects the effects of the anomaly to only one of the vertices, this does not satisfy the Bose symmetry and thus the so-called integrability condition. Notice that the leading terms of consistent anomaly and covariant anomaly (e.g the anomaly corresponding to the $AAA$ triangle diagrams) are related by the Bose symmetry factor. In 4 dimensions, the symmetry factor is $1/3$. For arbitrary $2n$ dimensions, the symmetry factor is $1/(n+1)$. The reason for these symmetry factors lies in the distribution of anomaly in all vertices when evaluating the consistent anomaly[18].

#### 7.3.4.1 Fermionic expansion with free parameters

As it is done in the previous sections, it is possible to compute the anomaly without bosonising the Jacobian, although it is ambiguous. This ambiguity transpires in certain traces that bear a $\gamma_5$ in $d$ dimensions. Keeping track of the ambiguity requires the introduction of free parameters that need to be fixed under physical constraints. For the covariant anomaly, those physical constraints arise from the expected gauge covariance of the result. However, for the consistent anomaly, it is not gauge covariance or invariance that needs to be enforced, but rather Wess-Zumino consistency conditions. We will outline the method in the following.

The covariant derivative is $i\not{D} = i\not{\partial} - \not{V} - \not{A}\gamma_5$. Under an axial reparametrisation of the fermions, the path integral yields the following Jacobian,

$$J[\theta] = \frac{\det(i\not{D}-m)}{\det\left[e^{i\theta\gamma_5}(i\not{D}-m)e^{i\theta\gamma_5}\right]} = \frac{\det(i\not{D}-m)}{\det(i\not{D}-m-(\not{D}\theta)\gamma_5 - 2im\theta\gamma_5)}\,, \tag{7.99}$$

where $\theta = \theta^a T^a$ is charged under the gauge group $SU(N)$.

The anomalous operator can be expressed as the following expansion,

$$\mathcal{A} = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\left(-2im\theta\gamma_5 - (\not{D}\theta)\gamma_5\right) \sum_{n\geq 0}\left[\left(\frac{-1}{\not{q}+m}\right)(-i\not{D})\right]^n \left(\frac{-1}{\not{q}+m}\right)\,. \tag{7.100}$$

As we will see, the mass term $2im\theta\gamma_5$ gives rise to the anomaly, while the divergent term $(\not{D}\theta)\gamma_5$ does not contribute to the result at order $m^0$. However the derivative term will contribute at higher order to cancel the contributions from the mass term, so that the whole result is proportional to $m^0$.

The computation is the same as the one of the consistent anomaly in section 7.3.3.2. We thus

[18] We remind the reader that the Bose symmetry will play an essential role in the functional bosonisation formalism; for further discussions, see Ref. [187].

have,

$$\begin{aligned}\mathcal{A}^{m\gamma_5} &= \frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta^a T^a \bigg( F^V_{\mu\nu}F^V_{\rho\sigma} + \frac{1}{3}F^A_{\mu\nu}F^A_{\rho\sigma} \\ &\qquad - \frac{8}{3}\left(iA_\mu iA_\nu F^V_{\rho\sigma} + iA_\mu F^V_{\nu\rho} iA_\sigma + F^V_{\mu\nu} iA_\rho iA_\sigma\right) + \frac{32}{3} iA_\mu iA_\nu iA_\rho iA_\sigma \bigg) \\ &= \mathcal{A}^{\mathrm{Bardeen}}\,,\end{aligned} \tag{7.101}$$

and,

$$\mathcal{A}_{\not{\partial}\gamma_5} = \frac{-1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta^a T^a \left(\sum_i a_i X_{i\mu\nu\rho\sigma}\right)\,, \tag{7.102}$$

where $X_i$ are all the possible operators of the form $\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4$ with $\mathcal{O}_{1\leq i\leq 4} \in \{V, A, \partial\}$ that can be formed, provided it has an even number of $A$ fields (the number of $\gamma_5$ must be odd). Note that the operators with a partial derivative to the right vanish, and those with consecutive partial derivatives vanish due to the contraction with the $\epsilon$ tensor. This leaves us with 22 possible operators, with 22 free parameters $a_i$.

Let's take all the operators from the derivative term when the axial field $A$ goes to zero. There remains only the operators that do not depend on $A$ (they only bear $V$ and $\partial$) and each have a free parameter. Setting $A$ to zero amounts to considering an axial reparametrisation of the fermion in a vector gauge theory, and if we want for example to conserve the vector current we know the result should then be vector gauge invariant. Therefore, the free parameters are fixed under this requirement, and the terms that do not depend on $A$ combine together to form,

$$\alpha\frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta\, F^V_{\mu\nu}\big|_{A=0}\, F^V_{\rho\sigma}\big|_{A=0}\,, \tag{7.103}$$

with $F^V_{\mu\nu}\big|_{A=0} = i\left((\partial_\mu V_\nu) - (\partial_\nu V_\mu) + i[V_\mu, V_\nu]\right)$, and $\alpha$ is a remaining free parameter that cannot be fixed by the sole requirement of gauge invariance (while $A = 0$).

We rewrite this term as,

$$\begin{aligned}&\alpha\frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta\, F^V_{\mu\nu}\big|_{A=0}\, F^V_{\rho\sigma}\big|_{A=0} \\ &= \alpha\frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta\bigg[F^V_{\mu\nu}F^V_{\rho\sigma} \\ &\qquad - F^V_{\mu\nu}\big|_{A=0}\, i^2[A_\rho, A_\sigma] - i^2[A_\mu, A_\nu]\, F^V_{\rho\sigma}\big|_{A=0} - i^2[A_\mu, A_\nu] i^2[A_\rho, A_\sigma]\bigg]\,,\end{aligned} \tag{7.104}$$

where we have made the Bardeen curvature of Eq. (7.50) appear.

Now consider the remaining terms with free parameters, that is to say, the terms that vanished when we set $A$ to zero. Among those operators, we can identify the same operators as those in the last line of equation Eq. (7.104), with different free parameters $\beta_i$. Therefore, they will combine together, and only change the free parameters $\beta_i$ to new free parameters $\beta'_i$.

We will now enforce the Wess-Zumino consistency conditions. No calculation is needed, we will only use the well known fact that the Wess-Zumino consistency conditions fix the coefficients of all the operators with respect to the coefficient of the term $F^V_{\mu\nu}F^V_{\rho\sigma}$ (as explained in Ref. [183]). Therefore, among all the remaining operators, all the free parameters will be fixed with respect to one, $\alpha$ in Eq. (7.104), such that the whole operator respects the integrability conditions. This unequivocally leaves us with,

$$\begin{aligned}\mathcal{A}^{\not{\partial}\gamma_5} = \alpha\frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta^a T^a\Big(F^V_{\mu\nu}F^V_{\rho\sigma} + \frac{1}{3}F^A_{\mu\nu}F^A_{\rho\sigma} \\ - \frac{8}{3}\left(iA_\mu iA_\nu F^V_{\rho\sigma} + iA_\mu F^V_{\nu\rho} iA_\sigma + F^V_{\mu\nu} iA_\rho iA_\sigma\right) + \frac{32}{3} iA_\mu iA_\nu iA_\rho iA_\sigma\Big) \\ = \alpha\mathcal{A}^{\mathrm{Bardeen}}\,,\end{aligned} \tag{7.105}$$

where we still have one free parameter left, $\alpha$.

**Final result.** Now let's put together the contributions from the mass term and the derivative term, i.e Eqs. (7.101) and (7.105) , we obtain,

$$\mathcal{A} = (1+\alpha)\mathcal{A}^{\mathrm{Bardeen}}\,. \tag{7.106}$$

Basically, the Wess-Zumino consistency conditions allow us to fix the coefficients of all the operators with respect to the coefficient of the term $F^V_{\mu\nu}F^V_{\rho\sigma}$, this is why we still have a remaining freedom at the end. The coefficient of $F^V_{\mu\nu}F^V_{\rho\sigma}$ can be fixed by comparing the result with the anomaly in a vector gauge theory as suggested in [183]. That is to say, in our result, we set again $A$ to zero, therefore we can identify our result with the ABJ anomaly (with $\theta \in SU(N)$) of Eq. (7.26), which immediately sets $\alpha$ to zero, leaving the expected result.

Note that there is no need to introduce counter terms in our computation to obtain the minimal Bardeen anomaly, because the vector current conservation has been enforced to fix the free parameters.

As a remark, notice that by comparing our result with the ABJ anomaly to fix the last free parameter, we restrain ourselves to the consistent anomaly with the vector current being conserved. If we want for example to conserve the axial current, we need to compare with the anomaly in a vector gauge theory where the anomaly is in the vector current. Besides, the Wess-Zumino consistency conditions have to be adapted. Indeed, they correspond to enforcing the Lie algebra of the gauge group and the Ward identities as well. Changing the current that remains conserved at the quantum level amounts to changing the Ward identities, hence changing the Wess-Zumino consistency conditions.

The procedure presented in this section can thus also be applied while enforcing the conservation of the axial current. It can even be used to obtain a generic expression where the consistent anomaly is distributed between the vector and the axial currents.

As far as we know, this is the only method that allows to tune which current bears the anomaly from a path integral approach.

**Calculation in BMHV's scheme.** Alternatively, it is possible to obtain the Bardeen anomaly without relying on free parameters. It is known that Pauli-Villars regularisation satisfies the Wess-Zumino consistency conditions, as well as enforcing conservation of the vector current [120]. Besides, as showed in Refs. [119, 159, 188], BMVH scheme in dimensional regularisation is equivalent to a "continuous superposition" of Pauli-Villars regularisations, and thus respects the Wess-Zumino consistency conditions, and vector current conservation as well. We can therefore avoid the introduction of free parameters and significantly simplify the calculation by making use of BMHV scheme to obtain Bardeen's consistent anomaly in the axial current. Although it strips us of the freedom to chose which current should bear the anomaly, as opposed to the free parameters approach discussed above.

#### 7.3.4.2 Bosonisation method

The bosonisation presented in section 7.3.3.2 defines a finite and non-ambiguous Jacobian. But it only allows us to get a gauge covariant result. The same procedure thus cannot be used to compute the consistent anomaly.

Nonetheless, we can try to bosonise with the operator $\not{D}^2$. It has the same spectrum as $i\not{D}$ (aside squaring it). We circumvent the problem of the non-hermitianity using the analytic continuation $A_\mu \to iA_\mu$ [119, 120].

We showed that bosonising using $(i\not{D})^\dagger i\not{D}$ as a regulator enforces the gauge covariance of the result, leaving us with only the possiblity to get the covariant anomaly. However, there is no reason to think that bosonising with the analytic continuation $A_\mu \to iA_\mu$ and $\not{D}^2$ would enforce all the conditions to get the consistent anomaly, namely the Wess-Zumino (integrability) consistency conditions.

Let's now see in the computation why the $\not{D}^2$ bosonisation along with the analytic continuation is still ambiguous.

After the analytic continuation, we have $i\not{D} = i\not{\partial} - \not{V} - i\not{A}\gamma_5$. The Jacobian of the axial field reparametrisation is the following,

$$J[\theta] = \frac{\det(i\not{D} - m)}{\det\left(e^{i\theta\gamma_5}(i\not{D} - m)e^{i\theta\gamma_5}\right)} = \frac{\det(i\not{D} - m)}{\det(i\not{D} - m - 2im\theta\gamma_5 - (\not{D}\theta)\gamma_5)} \ , \tag{7.107}$$

where $\theta = \theta^a T^a$ is charged under the $SU(N)$ gauge group.

Now we perform the bosonisation,

$$\begin{aligned} J^2[\theta] &= \frac{\det(i\not{D} - m)}{\det\left(e^{i\theta\gamma_5}(i\not{D} - m)e^{i\theta\gamma_5}\right)} \frac{\det(i\not{D} - m)}{\det\left(e^{i\theta\gamma_5}(i\not{D} - m)e^{i\theta\gamma_5}\right)} \\ &= \frac{\det(i\not{D} - m)}{\det\left(e^{i\theta\gamma_5}(i\not{D} - m)e^{i\theta\gamma_5}\right)} \frac{\det(-i\not{D} - m)}{\det\left(e^{i\theta\gamma_5}(-i\not{D} - m)e^{i\theta\gamma_5}\right)} \\ &= \frac{\det(\not{D}^2 + m^2)}{\det(\not{D}^2 + m^2 + [2im\theta\gamma_5, i\not{D}] + \{i\not{D}, (\not{D}\theta)\gamma_5\} + 4im^2\theta\gamma_5)} \ , \end{aligned} \tag{7.108}$$

where we have use the fact that the determinant is invariant under the change $\gamma^\mu \to -\gamma^\mu$ (see footnote 14). We expand it following the prescription described in section 7.2,

$$\log J[\theta]^2 = \int \mathrm{d}^4x \frac{\mathrm{d}^d q}{(2\pi)^d} e^{-iq\cdot x} \mathrm{tr}\ \left([2im\theta\gamma_5, i\not{D} - \not{q}] + \{i\not{D} - \not{q}, (\not{D}\theta)\gamma_5\} + 4im^2\theta\gamma_5\right) \frac{1}{\not{D}^2 + m^2} e^{iqx} \ . \tag{7.109}$$

We can straightforwardly see that the term $[2im\theta\gamma_5, i\not{D} - \not{q}]/(\not{D}^2 + m^2)$ has an odd number of Dirac matrices, therefore it vanishes under the trace. For simplicity we extract the $\gamma_5$ in $D_\mu$ using the notations,

$$iD_\mu = iD_\mu^V - iA_\mu\gamma_5 \ \text{ where } \ iD_\mu^V = i\,\partial_\mu - V_\mu \ . \tag{7.110}$$

We have,

$$\begin{aligned} e^{-iq\cdot x}\not{D}^2 e^{iq\cdot x} &= (\not{D} + i\not{q})(\not{D} + i\not{q}) \\ &= \not{D}^2 - q^2 + 2iq\cdot D^V - [\gamma^\mu, \gamma^\nu]\gamma_5 iq_\mu A_\nu \ . \end{aligned} \tag{7.111}$$

Finally, we can expand the Jacobian as usual,

$$2\mathcal{A} = -\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathrm{tr}\ \left(\{i\not{D} - \not{q}, (\not{D}\theta)\gamma_5\} + 4im^2\theta\gamma_5\right) \sum_{n\geq 0} \left[\Delta(\not{D}^2 + 2iq\cdot D^V - [\gamma^\mu,\gamma^\nu]\gamma_5 iq_\mu A_\nu)\right]^n \Delta \ , \tag{7.112}$$

where $\Delta = 1/(q^2 - m^2)$.
The key point is that because of the term $-[\gamma^\mu, \gamma^\nu]\gamma_5 i q_\mu A_\nu$, there are terms with several momenta $q$ contracted with a Dirac matrix as $\gamma^\mu q_\mu$. Combined with divergent integrals, they lead to traces traces such as,

$$\mathrm{tr}\left(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^a \gamma_5 \gamma_a\right), \tag{7.113}$$

which are ambiguous in $d$ dimensions. When bosonising using $(i\not{D})^\dagger i\not{D}$, the term $-[\gamma^\mu, \gamma^\nu]\gamma_5 i q_\mu A_\nu$ does not appear (see Eqs. (E.17) and (E.18) in the appendix E).

Therefore, the bosonisation method does not offer any appealing simplification regarding the calculation of the consistent anomaly. One could of course, proceed with the computation with free parameters or get rid of those ambiguities using the BMHV scheme which satisfies the Wess-Zumino conditions but also enforces vector current conservation. We refrain from doing so as ultimately, this does not offer any insights compared to the fermionic expansion already discussed.

## 7.4 Axial-gravitational anomaly

In this section we aim at deriving the axial-gravitational anomaly, which stems from the gravitational contribution to the Jacobian associated with the axial reparametrisation as defined in Eq. (7.3).

In curved space-time, the covariant derivative does not only bear the gauge fields. Diffeomorphism invariance requires the presence of the Christoffel connection, and Lorentz invariance requires the presence of the spin-connection for fermions. For simplicity, we consider a theory without gauge sector (in any case we know that we do not expect cross terms between the gravity sector and the gauge sector). We denote by $D_\mu$ the general covariant derivative, it includes both the spin-connection when applied to non-trivial element of the Dirac space, and the Christoffel symbols when applied to a Lorentz tensor. We follow [189] for conventions for the spin-connection. We have,

$$D_\mu \Psi = (\partial_\mu + \omega_\mu)\Psi\,, \tag{7.114}$$

with the the spin-connection defined as $\omega_\mu = \frac{1}{8}[\gamma^a, \gamma_b] e_a^\nu (D_\mu e_\nu^b)$, with $e_\mu^a$ the tangent frame vielbein such that $g_{\mu\nu} = e_\mu^a e_\nu^b g_{ab}$ (latin indices referring to the tangent frame).

Considering a spinless Lorentz vector $v$ we have,

$$D_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho \tag{7.115}$$

$$D_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu} v_\rho\,. \tag{7.116}$$

The following expressions will be useful later,

$$F_{\mu\nu}\Psi = [D_\mu, D_\nu]\Psi = \frac{1}{4}\gamma^\rho\gamma^\sigma R_{\mu\nu\rho\sigma}\Psi \tag{7.117}$$

$$\not{D}^2 = D^2 - \frac{i}{2}\sigma^{\mu\nu}F_{\mu\nu} \text{ where } \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\,, \tag{7.118}$$

with

$$\frac{i}{2}\sigma.F\,\Psi = \frac{1}{4}R\,\mathbb{1}_{\text{Dirac}}\Psi\,, \tag{7.119}$$

where $\mathbb{1}_{\text{Dirac}}$ is the identity in Dirac space.

Finally, the covariant derivative commutes with the Dirac matrices,

$$(D_\mu \gamma^\nu) = (\partial_\mu \gamma^\nu) + \Gamma^\nu_{\mu\rho}\gamma^\rho + [\omega_\mu, \gamma^\nu] = 0 \tag{7.120}$$

$$(D_\mu \gamma_5) = [\omega_\mu, \gamma_5] = 0\,. \tag{7.121}$$

### 7.4.1 Covariant Derivative Expansion in curved space-time

The CDE in curved space-time requires extra care that significantly complexifies the expansion. The main point is that the commutativity between the covariant derivatives $D$ and the propagators $\Delta = 1/(q^2 - m^2)$ that appear in our expansion is lost. Indeed, $q^2 = g^{\mu\nu}(x) q_\mu q_\nu$ is space-time dependent. Therefore $[D_\mu, \Delta] = -(\partial_\mu q^2)\Delta^2$. The CDE can still be performed in an extended framework that includes the curvature of space-time, as it has been done for example in Refs.[190, 191]. However we propose here a different way of conducting the expansion, that we believe to be simpler in the formalism. Defining the expansion in curved space-time is not trivial, and this is out of the scope of this paper (see Ref. [192]), so we just give the outline of the method without delving too much into the details.

First of all, there is no trivial definition of the Fourier transform in curved space-time. However, following Refs. [191, 193] we can define the Fourier transform using Riemann Normal Coordinates (RNC). We take the momentum $q_\mu$ to be the covariant variable conjugate to the contravariant variable $x^\mu$, so that $\mathrm{d}^D q \mathrm{d}^D x$ is diffeomorphism-invariant. We then have,

$$(\partial_\mu q_\nu) = \frac{\partial q_\nu}{\partial x^\mu} = 0\,, \tag{7.122}$$

but,

$$(\partial_\mu q^\nu) = (\partial_\mu g^{\nu\rho} q_\rho) = (\partial_\mu g^{\nu\rho}) q_\rho \neq 0\,. \tag{7.123}$$

We thus have the standard Fourier transform of the covariant derivative,

$$e^{-iq\cdot x} D_\mu e^{iq\cdot x} = D_\mu + (\partial_\mu i q_\nu x^\nu) = D_\mu + i q_\nu (\partial_\mu x^\nu) = D_\mu + i q_\mu\,. \tag{7.124}$$

### 7.4.2 Computation of the gravitational anomaly

According to the previous results, we know that at order $m^0$ the derivative coupling does not contribute in the bosonised form (and we can show it ), for simplicity we drop it. The anomaly is thus fully encompassed (at order $m^0$) by the following Jacobian [19],

$$J[\theta]^2 = \frac{\det\left(\sqrt{-g}^2(\not{D}^2 + m^2)\right)}{\det\left(\sqrt{-g}^2(\not{D}^2 + m^2 + 4im^2\theta\gamma_5)\right)}\,. \tag{7.125}$$

Since we discarded the derivative term, this Jacobian is trivially finite, thus well-defined.

Using Eq. (7.124), we have,

$$\begin{aligned} e^{-iq\cdot x}\not{D}^2 e^{iq\cdot x} &= g^{\mu\nu}(D_\mu + iq_\mu)(D_\nu + iq_\nu) - e^{-iq\cdot x}\frac{i}{2}\sigma.F e^{iq\cdot x} \\ &= D^2 - q^2 + 2iq\cdot D + ig^{\mu\nu}(D_\mu q_\nu) - \frac{i}{2}\sigma.F \\ &= D^2 - q^2 + 2iq\cdot D - i\Gamma^\rho_{\mu\nu} q_\rho - \frac{i}{2}\sigma.F\,. \end{aligned} \tag{7.126}$$

Hence we can expand the Jacobian as,

$$\log J[\theta]^2 = \int \mathrm{d}^4x \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\left(4im^2\theta\gamma_5 \sum_{n\geq 0}\left[\Delta(D^2 - \frac{i}{2}\sigma.F + 2iq\cdot D - i\Gamma^\rho_{\mu\nu} q_\rho)\right]^n \Delta\right)\,. \tag{7.127}$$

[19] We have decided to work within the bosonised form of the Jacobian, but one could have equivalently chosen to carry the computation with the original Jacobian.

Since all the Lorentz indices are contracted, the field strength that appears in Eq. (7.127) is in the fermion representation[20], hence we can use Eq. (7.119) to simplify,

$$\log J[\theta]^2 = \int \mathrm{d}^4x \frac{\mathrm{d}^dq}{(2\pi)^d} \mathrm{tr} \left( 4im^2\theta\gamma_5 \sum_{n\geq 0} \left[ \Delta(D^2 - \frac{R}{4} + 2iq\cdot D - i\Gamma^\rho_{\mu\nu}q_\rho) \right]^n \Delta \right) . \tag{7.128}$$

We remark that the space-time measure $\sqrt{-g}$ does not come into play in the expansion. This is because it appears both in the numerator and the denominator.

Now, we are interested in the terms that are proportional to $m^0$. In each of these terms, the propagators $\Delta = 1/(q^2 - m^2)$ have to be commuted to the left in order to perform the integration over momentum. Therefore, each of these terms will yield several terms where the open covariant derivatives will be localised or not on a propagator.

For example, consider the following term of order $m^0$,

$$\begin{aligned} &\int \frac{\mathrm{d}^dq}{(2\pi)^d} \mathrm{tr} \left[ 4im^2\theta\gamma_5 \Delta \left( -\frac{R}{4} \right) \Delta D^2 \Delta \right] \\ &= \int \frac{\mathrm{d}^dq}{(2\pi)^d} \Delta^2 \mathrm{tr} \left[ 4im^2\theta\gamma_5 \left( -\frac{R}{4} \right) (\Delta D^2 + (D^2\Delta) + 2(D^\mu\Delta)D_\mu) \right] . \end{aligned} \tag{7.129}$$

The only terms that can contribute in the end are gauge and diffeomorphism invariant. That is to say the remaining open covariant derivatives that are not localised on a propagator have to combine together to form field strengths. For example in Eq. (7.129), the term involving $(D_\mu\Delta)D_\nu$ has a single open derivative, it is impossible to form an invariant term with it, thus it cannot contribute to the final result (besides it vanishes in Riemann Normal Coordinates).

Secondly, notice that whenever a covariant derivative is localised on a propagator, it bears no spin-connection since $\Delta$ is a scalar in Dirac space: $(D_\mu\Delta) = (\partial_\mu \Delta)\mathbb{1}_{\mathrm{Dirac}}$.

Keeping those last two points in mind, one can easily isolate the few terms that will contribute to the gravitational anomaly, making use of,

$$\mathrm{tr}\,\gamma_5 = \mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu = 0\,, \quad \mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \neq 0\,. \tag{7.130}$$

In Eq. (7.128), the only terms that bear Dirac matrices are the covariant derivatives via the spin-connection. The remaining open covariant derivatives will combine in the end to form field strengths that have two Dirac matrices (see Eq. (7.117)), therefore the only way to have enough Dirac matrices so that the trace does not vanish is by having 4 open covariant derivatives, thus two field strengths.

In the end, the only terms that can contribute are the following,

- At $n = 2$: $\int \frac{\mathrm{d}^dq}{(2\pi)^d} \Delta^3 \mathrm{tr} \left[ 4im^2\theta\gamma_5 D^2 D^2 \right]$
- At $n = 3$: $\int \frac{\mathrm{d}^dq}{(2\pi)^d} \Delta^4 \mathrm{tr} \left[ 4im^2\theta\gamma_5 \left( D^2(2iq\cdot D)^2 + 2iq\cdot DD^2 2iq\cdot D + (2iq\cdot D)^2 D^3 \right) \right]$
- At $n = 4$: $\int \frac{\mathrm{d}^dq}{(2\pi)^d} \Delta^5 \mathrm{tr} \left[ 4im^2\theta\gamma_5 (2iq\cdot D)^4 \right]$.

[20]Indeed, recall that the trace in internal space (*ie* Dirac space and gauge space) is defined as $\mathrm{tr}A = \sum_n \Psi_n^\dagger A \Psi_n$, where $\{\Psi_n\}$ is a basis of internal space (constant vectors: $(D_\mu\Psi_n) = \omega_\mu\Psi_n + V_\mu\Psi_n$). Therefore, for any operator $\mathcal{O}$ that is a matrix in internal space without free Lorentz indices (and it can bear open covariant derivatives) we have: $\mathrm{tr}D_\mu\mathcal{O} = \sum_n \Psi_n^\dagger D_\mu \mathcal{O}\Psi_n$. Since $\mathcal{O}$ is a matrix in internal space, then $\mathcal{O}\Psi_n$ is a vector in internal space, and all the derivatives in $\mathcal{O}$ are localised because they act on $\Psi_n$. Therefore, in $\sum_n \Psi_n^\dagger D_\mu\mathcal{O}\Psi_n$, $D_\mu$ acts on a vector in internal space, hence can be written in the fermion representation.

In each of these terms, the momenta can be freely commuted to the left for the integration for the same reasons as before. Indeed, if one of the covariant derivatives were localised on a momentum $q$, it would bear no Dirac matrix since $q$ is a Dirac scalar, hence the term would vanish under the trace. The sum of the different contributions yields,

$$\begin{aligned}2\mathcal{A}^{\text{grav}} =& \operatorname{tr} 4im^2\theta\gamma_5\Big(\mathcal{I}[q^0]^3 D^2D^2 + \mathcal{I}[q^2]^4 g^{\mu\nu}(2i)^2(D^2D_\mu D_\nu + D_\mu D^2 D_\nu + D_\mu D_\nu D^2)\\ &+ \mathcal{I}[q^4]^5(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})(2i)^4 D_\mu D_\nu D_\rho D_\sigma\Big)\\ =& \frac{-i}{16\pi^2}\frac{2}{6}\operatorname{tr} i\theta\gamma_5\Big(0D^2D^2 + 2D^\mu D^\nu D_\mu D_\nu - 2D^\mu D^2 D_\mu\Big)\\ =& \frac{-i}{16\pi^2}\frac{2}{6}\operatorname{tr} i\theta\gamma_5 F^{\mu\nu}F_{\mu\nu}\,. \end{aligned} \tag{7.131}$$

In the last line we have not used any integration by parts nor trace cyclicity, it is only algebra. Note that when computing a gauge anomaly, the contributing term is of the form $\operatorname{tr}\gamma_5\sigma.F\sigma.F$, which vanishes in gravity thanks to the use of Eq. (7.119) earlier (and because we discarded the gauge sector).

Now one must pay some attention to the last line of Eq. (7.131). The field strength on the right is in the fermion representation, we can thus write,

$$\frac{-i}{16\pi^2}\frac{2}{6}\operatorname{tr} i\theta F^{\mu\nu}F^{\psi}_{\mu\nu}\,. \tag{7.132}$$

However, the field strength on the left will contract the indices of $F^{\psi}_{\mu\nu}$ because of the Christoffel connection,

$$F^{\mu\nu}F^{\psi}_{\mu\nu} = F^{\psi,\mu\nu}F^{\psi}_{\mu\nu} + \gamma^\alpha\gamma^\beta[RR]\,, \tag{7.133}$$

where the last term is a sum of Riemann tensors contracted together and with the two Dirac matrices. It vanishes using the symmetries of the tensors (and also vanishes under the Dirac trace).

Using Eq. (7.117), we obtain,

$$\mathcal{A}^{\text{grav}} = \frac{-i}{16\pi^2}\frac{1}{6}\operatorname{tr}\left(i\theta\gamma_5\frac{1}{4}\gamma^\alpha\gamma^\beta R^{\mu\nu}{}_{\alpha\beta}\gamma^\mu\gamma^\nu\frac{1}{4}R_{\rho\sigma\mu\nu}\right) = \frac{-i}{384\pi^2}\bar{\epsilon}^{\mu\nu\rho\sigma}R^{\alpha\beta}{}_{\mu\nu}R_{\alpha\beta\rho\sigma}\,, \tag{7.134}$$

which is the so-called axial-gravitational anomaly. We have $\bar{\epsilon}^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}/\sqrt{-g}$.

We can notice that in our computation, the only contribution to the gravitational anomaly is in the end the spin-connection via the field strengths, although there are many terms with covariant derivatives that are localised on propagators that can yield Riemann squared terms via the Christoffel connection. This translates the fact that a fermion in curved space-time is not subject to diffeomorphism invariance, but only to Lorentz invariance. The spin-connection only ensures that Lorentz invariance is preserved in curved space-time. Therefore, it is expected that one can get the gravitational anomaly only considering the spin-connection, and not minding the Christoffel connection.

In the end, we could have had the correct result in a very simple framework where space-time is considered flat, but the covariant derivatives acting on a spinor bears the spin-connection (the covariant derivative acting on a Dirac matrix would be zero since we would consider it "uncharged" under the spin-connection).

## 7.5 Scale anomaly

It is well-known that there are two main categories of symmetries which are broken by the quantisation of a theory. The first is the axial symmetry associated with Dirac's $\gamma_5$, the chiral anomaly, that we have just treated in details. The other is the Weyl transformation, which changes the length scale of space-time, keeping the local angle invariant, this is called the Weyl anomaly or conformal or trace or scale anomaly [174–178]. We then propose to evaluate the Weyl anomaly always following the prescription described in section 7.2 and for pedagogical reasons we stick to the case of QED,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - V\!\!\!/ - m)\psi - \frac{1}{4e^2}F^2 . \tag{7.135}$$

Scale invariance is classically broken by the fermion mass term. The divergence of the Noether current $J^\mu$ associated to the scale transformation, i.e the trace of the symmetric energy-momentum tensor $\tilde{T}^\mu_\mu$, reads,

$$\partial_\mu J^\mu = \tilde{T}^\mu_\mu = m\bar{\psi}\psi . \tag{7.136}$$

This relation is also broken at the quantum level by the renormalisation of the coupling $e$.

The scale transformation $x_\mu \to x'_\mu = e^\sigma x_\mu$ induces,

$$\begin{aligned} \frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = e^{-\sigma}\frac{\partial}{\partial x^\mu} , \\ \mathrm{d}^d x \to \mathrm{d}^d x' = e^{d\sigma}\mathrm{d}^d x , \end{aligned} \tag{7.137}$$

and the fields transform with their canonical mass dimension,

$$\begin{aligned} \psi(x) \to \psi'(x') &= e^{-(d-1)\sigma/2}\psi(x) , \\ \bar{\psi}(x) \to \bar{\psi}'(x') &= e^{-(d-1)\sigma/2}\bar{\psi}(x) , \\ A_\mu(x) \to A'_\mu(x') = A_\mu(x') &= e^{-\sigma}A_\mu(x) , \end{aligned} \tag{7.138}$$

where $d$ is the dimension of space-time. Notice that the gauge field does not transform by itself, it only transforms due to its dependence on $x$ [120].

Using the invariance of the path integral under the relabelling of the path integral variables, and the invariance of the space-time integral under relabelling the space-time variable, we can write,

$$\int (\mathcal{D}\psi)'(\mathcal{D}\bar{\psi})' \exp\left(i\int \mathrm{d}^d x' \mathcal{L}[x', \psi'(x'), A_\mu(x')]\right) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(i\int \mathrm{d}^d x \mathcal{L}[x, \psi(x), A_\mu(x)]\right) . \tag{7.139}$$

On the other hand we know how the action transforms, and we can assume that the transformation of the measure produces a Jacobian,

$$\begin{aligned} &\int (\mathcal{D}\psi)'(\mathcal{D}\bar{\psi})' \exp\left(i\int \mathrm{d}^d x' \mathcal{L}[x', \psi'(x'), A_\mu(x')]\right) \\ &= \int J[\sigma]\mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(i\int \mathrm{d}^d x \mathcal{L}[x, \psi(x), A_\mu(x)] + \int \mathrm{d}^d x \left(\bar{\psi}(-i\frac{d-1}{2}(\partial\!\!\!/\sigma) - m\sigma)\psi\right)\right) . \end{aligned} \tag{7.140}$$

One can take advantage of equating the two path integrals to express the Jacobian as,

$$J[\sigma] = \frac{\det(iD\!\!\!\!/ - m)}{\det(iD\!\!\!\!/ - m - \sigma m - i\frac{d-1}{2}(\partial\!\!\!/\sigma))} . \tag{7.141}$$

The term proportional to $(\partial\!\!\!/\sigma)$ requires to be regularised, we use the BMHV scheme of dimensional regularisation [105, 106] since the calculation does not involve any $\gamma_5$ matrices. At order $m^0$, the divergent contribution from $-i\frac{d-1}{2}(\partial\!\!\!/\sigma)$ vanishes, only the finite contribution from $\sigma m$ remains and yields the scale anomaly,

$$\mathcal{A}_{\text{scale}} = \frac{\sigma}{24\pi^2}\text{tr}\,(F_{\mu\nu})^2\,. \tag{7.142}$$

More details of the calculation are provided in appendix F. However, higher order terms (terms of order $1/m^k$, with $k > 0$) involve contributions from both $\sigma m$ and $\frac{d-1}{2}(\partial\!\!\!/\sigma)$ which cancel one another.

We can now relate the anomaly to the $\beta$ function. At tree level the coupling $e$ does not transform. It however transforms at one loop level, and by definition of the $\beta$ function we have,

$$e \to e + \sigma\beta(e)\,. \tag{7.143}$$

The following action is invariant under the scale transformation up to the mass term,

$$S = \int \mathrm{d}^d x \mathcal{L}[x, \psi(x), A_\mu(x)] = \int \mathrm{d}^d x \left(\bar{\psi}(iD\!\!\!\!/ - m)\psi - \frac{1}{4e^2}F^2\right)\,. \tag{7.144}$$

By definition of the $\beta$ function, the gauge sector transforms at one loop like,

$$-\frac{1}{4e^2}(F_{\mu\nu})^2 \to -\frac{1}{4e^2}(F_{\mu\nu})^2 + \sigma\frac{\beta(e)}{2e^3}(F_{\mu\nu})^2. \tag{7.145}$$

By identification with the term produced at one loop by the Jacobian, we deduce following expression for the one loop $\beta$ function,

$$\beta(e) = \frac{e^3}{12\pi^2}\,, \tag{7.146}$$

which corresponds to the well-known QED $\beta$ function.

A derivation of the scale anomaly has been proposed by Fujikawa in Ref. [176]. This is interesting to point out an important difference between our procedure and Fujikawa's procedure for computing the scale anomaly. In Fujikawa's method, we temper directly with the path integral measure, therefore is it necessary to isolate the field transformation from the space-time transformation. This is achieved by introducing the curvature of space-time and defining a diffeomorphism invariant path integral measure. Because of this redefinition of the fields, they do not transform with their canonical mass dimension anymore. It is even emphasised in [120] that doing the transformation with their canonical mass does not yield the correct result.

However, in our procedure we can use the invariance of the space-time integral under relabeling the space-time variable, along with transforming the fields with their canonical mass dimension. As we showed it provides the correct result, without having to introduce the curvature of space-time, nor redefining the fields in a diffeomorphism invariant way.

## 7.6 Comparison with Fujikawa's method

Let us consider the simple case of a vector gauge theory and an axial fermion reparametrisation. The covariant derivative is,

$$D_\mu = \partial_\mu + iV_\mu\,. \tag{7.147}$$

Under the infinitesimal Abelian field reparametrisation,

$$\psi \to e^{i\theta(x)\gamma_5}\psi\,, \quad \bar{\psi} \to \bar{\psi}e^{i\theta(x)\gamma_5}\,, \tag{7.148}$$

the path integral,

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{\int \mathrm{d}^4x\bar{\psi}(i\not{D}-m)\psi} \,, \tag{7.149}$$

produces the following Jacobian,

$$J[\theta] = \frac{\det(i\not{D}-m)}{\det(i\not{D}-m-(\not{\partial}\theta)\gamma_5 - 2im\theta\gamma_5)} \,. \tag{7.150}$$

As emphasised before, the initial theory is ill-defined, leading to an ambiguous Jacobian. A well-known way of dealing with the ambiguity is to bosonise it [119] (it enforces the conservation of the vector current as explained in section 7.3.1.2),

$$J[\theta]^2 = \frac{\det(\not{D}^2+m^2)}{\det(\not{D}^2+m^2+\{i\not{D},(\not{\partial}\theta)\gamma_5\}+4im^2\theta\gamma_5)} \,. \tag{7.151}$$

Note also that since the theory is non-chiral we have $(i\not{D})^\dagger = i\not{D}$, hence there is no difference between the bosonisation with $\not{D}^\dagger\not{D}$ or $\not{D}^2$.

As detailed in Eq. (C.2) of appendix C, the *log* of this ratio of determinant can be written as follows,

$$\log J[\theta] = -\mathrm{Tr}\left(\frac{1}{2}\frac{\{i\not{D},(\not{\partial}\theta)\gamma_5\}}{m^2}\frac{1}{1+\frac{\not{D}^2}{m^2}}\right) - \mathrm{Tr}\left(2i\theta\gamma_5\frac{1}{1+\frac{\not{D}^2}{m^2}}\right) \,, \tag{7.152}$$

where Tr is the trace over both space-time and internal spaces.

Now let's recall Fujikawa's procedure to compute the anomaly. The Dirac operator $i\not{D}$ is hermitian thus provides a complete and orthonormal set of eigenfunctions $\{\varphi_n\}$ with real eigenvalues $\lambda_n$, such that $i\not{D}\varphi_n = \lambda_n\varphi_n$. We use this basis to decompose the fermion field, this will enable us to define the path integral measure,

$$\psi(x) = \sum_n a_n\varphi_n(x) \,, \quad \bar{\psi}(x) = \sum_n \varphi_n^\dagger(x)\bar{b}_n \,. \tag{7.153}$$

The measure is then defined as,

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \prod_n da_n d\bar{b}_n \,. \tag{7.154}$$

Now the fermion undergoes an axial reparametrisation as in Eq. (7.148), the reparametrised field can be decomposed in the eigenbasis as well, with coefficients $a'_n$ and $\bar{b}'_n$. They are related to $a_n$ and $\bar{b}_n$ by the transformation matrix $C_{nm}$,

$$a'_n = \sum_m C_{nm}a_m \,, \quad \bar{b}'_m = \sum_n C_{nm}\bar{b}_n \,, \tag{7.155}$$

where,

$$C_{nm} = \delta_{nm} + i\int \mathrm{d}^4x\theta(x)\varphi_n^\dagger(x)\gamma_5\varphi_m(x) \,. \tag{7.156}$$

Now we know that the Grassmann measure transforms with the inverse determinant of the transformation operators,

$$\begin{aligned} \prod_n da'_n &= (\det C)^{-1}\prod_n da_n \\ \prod_m d\bar{b}'_m &= (\det C)^{-1}\prod_m d\bar{b}_m \,, \end{aligned} \tag{7.157}$$

whence,

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = (\det C)^{-2}\mathcal{D}\psi\mathcal{D}\bar{\psi}\,. \tag{7.158}$$

Finally, using $\det = \exp \mathrm{Tr} \log$ and expanding the *log* in first order in $\theta$ infinitesimal, the Jacobian reads,

$$\log J[\theta] = -2i \int \mathrm{d}^4x\theta(x) \sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x)\,. \tag{7.159}$$

Since $\{\varphi_n\}$ defines a complete set of operators, it is in fact,

$$\log J[\theta] = -\mathrm{Tr}\, 2i\theta\gamma_5\,, \tag{7.160}$$

where the trace *Tr* is over both internal indices and space-time.

This quantity needs to be regularised, hence Fujikawa introduces a regulator that will work as a cut-off. This regulator needs to preserve the spectrum of the theory. In the simple case of a vector gauge theory, a good choice is,

$$\log J[\theta] = -\lim_{\Lambda\to\infty} \mathrm{Tr}\, 2i\theta\gamma_5\, f\left(\frac{\not{D}^2}{\Lambda^2}\right)\,. \tag{7.161}$$

The function $f$ that was introduced, has to be a smooth function such that $f$ and all its derivatives vanish at infinity, and respect the requirement $f(0) = 1$. It can for example be,

$$f(x) = \frac{1}{1+x}\,. \tag{7.162}$$

This leaves us with,

$$\log J[\theta] = -\lim_{\Lambda\to\infty} \mathrm{Tr}\left(2i\theta\gamma_5\frac{1}{1+\frac{\not{D}^2}{\Lambda^2}}\right)\,. \tag{7.163}$$

Now let's compare Eq. (7.152) and Eq. (7.163). If one takes the infinite mass limit in Eq. (7.152), it is clear that one recovers Eq. (7.163) with $\Lambda$ identified as the physical fermion mass. It thus appears that the procedure presented in this paper (the bosonic CDE) not only amounts to Fujikawa's procedure in the infinite mass limit, but moreover generalises it to a finite and physical mass.

In Fujikawa's procedure, the infinite mass limit ensures that the Jacobian is of order $m^0$, while in the bosonic CDE, it is the derivative coupling that plays this role[21]. Besides, in the bosonic CDE, there is no need to add the regulator by hand. Although, in more complicated cases (when the theory is vector-axial for example), there is still a choice on how to define non-ambiguously the Jacobian, which is in the end making a choice of regulator.

As pointed out in the above, in the bosonic CDE the regulator that naturally appears corresponds to,

$$f(x) = \frac{1}{1+x}\,, \tag{7.164}$$

as in the treatment of Fujikawa. It is known that this specific regulator amounts to doing a Pauli-Villars regularisation [119]. It is also known that Pauli-Villars regularisation enforces the

[21] In the case of a vector-axial theory, we showed in section 7.3.4.2 that the analytic continuation $A_\mu \to iA_\mu$ to make $i\not{D}$ hermitian, and the bosonisation as in Eq. (7.108), hence Eq. (7.152) replacing $(\not{\partial}\theta)$ by $(\not{D}\theta)$, is still ambiguous. However, we also noted that the ambiguity was carried by the derivative term only. In Fujikawa's procedure, the derivative term vanishes because of the infinite mass limit, this is why the analytic continuation is sufficient to bring forth the consistent anomaly.

conservation of the vector current, while the axial one is anomalous. Therefore, in the infinite mass limit, the bosonic CDE amounts to a Pauli-Villars regularisation, hence conserves the vector current. Now since the anomaly is mass independent, we can expect that with a finite mass, the conservation of the vector current will hold.

Another point that is worth noticing is that in the bosonic CDE, the anomaly always arises in the mass term, even if the mass is finite. Likewise, in a Pauli-Villars regularisation, we know that the anomaly is carried by the mass regulating term (see section 6.2 of Ref. [120]).

## 7.7 Conclusion

EFT has always been a pillar in particle physics. Its fundamental reason is to transform a QFT paradigm into a phenomenologically accessible one. In practice, it can be used to "replace" (integrating-out) supposedly directly inaccessible fields by potentially observable distortions (higher dimensional operators effects). Those heavy or weakly coupled fields may or may not be sensible to other symmetries.

If these symmetries are for some reasons anomalous, then the EFT will inherit very specific anomalous operators i.e interactions between the "light" fields. In order to compute these interactions, it exists very efficient and well-known master formulas and techniques to apply on the full UV theory, mainly by computing loop Feynman diagrams or by computing the variance of the fermionic measure in the path integral which may involve quite a different toolkit.

Building successively EFTs from the path integral, i.e integrating out fields successively, offers undeniable advantages. Then, when encountering anomalous symmetries one has to inevitably borrow standard calculations made in substantially different context. This task is not always straightforward.

In this work, we have presented a natural way to build EFTs from the path integral, being able to get the anomalous operators for free and in a self consistent way, i.e without having to rely on any external results and then specific conditions of applicability. Indeed, when building EFTs involving anomalous interactions one may face the choice of deciding in which current this has to go, consequently, one may customise the regularisation of EFTs.

While constructing an EFT from a generic UV theory, this method allows to directly obtain the anomalous interactions. The procedure is simple and based on the well-know fact that an anomalous tranformation means that the path integral measure transforms with a non-trivial Jacobian. Opposed to Fujikawa's prescription which computes directly the transformed measures from accessing the zero modes of Dirac operators, we expressed this Jacobian as a ratio of functional determinants i.e two EFTs. The "comparison" of these two EFT's allows us to access straightforwardly the anomalous operators. This is even more remarkable when this method is combined with the CDE techniques. In practice, one only has to perform basic algebra and power counting to access the expected result. This all the more striking that the CDE, extended to curved space-time, provides so straightforwardly the gravitational anomaly.

The presented methodology is even more impressive since it allows one to obtain all various types of anomalies in vector-axial gauge theory (covariant anomaly involving vector or axial symmetries; consistent anomaly), the axial gravitational anomaly and the Weyl anomaly (even if the last one has a quite different physical nature).

These computations have been presented in details and the heart of those computations are (as expected) the procedure used to regularise ill-define functional determinants. A first method consists in fermionic CDE combined with dimensional regularisation and introducing free parameters to keep track of $\gamma_5$ ambiguities subsequently fixed when imposing gauge invariance (covariant anomaly) or integrability conditions (consistent anomaly). We have also investigated another method based on evaluating a “squared Jacobian” combined with a CDE.

We believe that this is the first time that a method is proposed to evaluate in a general way the covariant and consistent anomaly from the path integral having then the possibility to choose in which current the anomaly has to stand.

We have also presented an enlightening comparison between our work and the seminal work of Fujikawa which allows one to appreciate how an EFT mass expansion can mimic the educated guess regulators used by Fujikawa. Moreover, it appears that the bosonic CDE extends Fujikawa’s procedure by replacing the regulator by a physical and finite mass.

Another interesting comparison with Fujikawa’s method is highlighted in the computation of the scale anomaly, where we are able to compute the scale anomaly without having to introduce the curvature of space-time, nor redefining the path integral measure in a diffeomorphism invariant way, which substantially simplifies the procedure.

Furthermore, it appeared recently several public codes (see for example Refs. [85, 86]) based on analytical and systematic CDE to efficiently build EFTs. They also drastically simplify the so-called *matching* step. Unfortunately, models involving QFT anomalies are out of reach of these codes. The results of this work, could be straightforwardly implemented in these or similar codes and then allow to compute anomalous interactions in a self-consistent manner in the path integral formalism. Incorporating theoretical models involving anomalous features, as appearing in many BSM models, would have important phenomenological implications.

# Conclusion and outlook

We are living in the precision era of particle physics. Many fundamental parameters of the Standard Model have been tested or constrained stringently. The lack of new physics discoveries so far has strongly motivated us to consider the Standard Model as an effective field theory at a low energy scale. The benefits of the EFT framework are twofold. First, allow to hunt for new physics effects in a model-independent way. Second, consistently and effectively connect UV complete models with the theories at low energies and their precision measurements. Thanks to its generality and broad range of applications, the EFT framework is and always will be a cornerstone in the development of particle physics. In this PhD thesis, we have developed functional techniques for calculating the one-loop effective action, which is the central object in the EFT paradigm and many theories.

In chapter 1 and 2 we presented how to construct an effective field theory from both IR and UV point of view. More importantly, we presented the state-of-the-art of the functional techniques for one-loop matching. In chapter 3, we show that the universality of the one-loop effective action obtained by integrating out heavy degrees of freedom has emerged as a byproduct of improvements in path integral methods. The so-called UOLEA makes the repeated evaluation of functional determinants redundant and provides a more efficient way of matching at one loop compared to the Feynman diagrams approach, especially for systematically obtaining an ensemble of operator coefficients at once. Ultimately, our universal results have the advantage of being easier to automatize or can also serve as a reference to cross-check.

In chapter 5, we have briefly presented the strong CP problem and its solutions. In chapter 6, we applied our functional method to derive an effective Lagrangian of axion UV complete models. We showed an alternative way to evaluate axion-gauge bosons effective couplings resulting from integrating out heavy chiral fermions charged under both local and global symmetries. This exercise is then much more involved due to the presence of anomalies in various currents and because the fermions can only have a mass when all the local chiral symmetries are spontaneously broken. Our results show a consistent and generic way to compute effective operators involving Goldstone bosons.

In chapter 4, we have briefly introduced the anomalies in QFT. In chapter 7, we showed an alternative way to derive the non-trivial Jacobian terms of the path integral measure. These Jacobian terms are expressed as a ratio of functional determinants, in other words, the comparison between two EFTs. We have used the covariant derivative expansion (CDE) techniques to explicitly evaluate these Jacobian terms. Our framework provides a comprehensive method to evaluate all various types of anomalies in chiral gauge theory: covariant anomaly, consistent anomaly, axial-gravitational anomaly, and Weyl anomaly.

As discussed in chapter 1, matching a UV model onto SMEFT or HEFT operator bases is ultimately a question of decoupling. The functional and regularization techniques developed in this thesis offer a transparent way to match a given UV model onto several EFTs at a low-energy scale (SMEFT, HEFT,$\cdots$), hence shedding new light on the work of hunting new physics effects.

# Part III

# Appendices

# Appendix A

# Master integrals

In this appendix, we discuss the master integrals and tabulate some of them that are useful in practice. In this paper our results are written in terms of master integrals $\mathcal{I}$, defined by

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^{\mu_1} \cdots q^{\mu_{2n_c}}}{(q^2 - M_i^2)^{n_i}(q^2 - M_j^2)^{n_j} \cdots} = g^{\mu_1 \cdots \mu_{2n_c}} \mathcal{I}[q^{2n_c}]_{ij\ldots}^{n_i n_j \cdots} \tag{A.1}$$

In the mass degenerate case, the master integrals, $\mathcal{I}[q^{2n_c}]_{ij\ldots}^{n_i n_j \cdots}$, reduce to the form $\mathcal{I}[q^{2n_c}]_i^{n_i}$, for which the general expression reads,

$$\mathcal{I}[q^{2n_c}]_i^{n_i} = \frac{i}{16\pi^2}\left(-M_i^2\right)^{2+n_c-n_i} \frac{1}{2^{n_c}(n_i-1)!} \frac{\Gamma(\frac{\epsilon}{2} - 2 - n_c + n_i)}{\Gamma(\frac{\epsilon}{2})}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log \frac{M_i^2}{\mu^2}\right), \tag{A.2}$$

where $d = 4 - \epsilon$ is the space-time dimension, and $\mu$ is the renormalization scale.

In the $\overline{\text{MS}}$ scheme, we replace,

$$\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log \frac{M_i^2}{\mu^2}\right) \rightarrow \left(-\log \frac{M_i^2}{\mu^2}\right) \tag{A.3}$$

in the final result. We factor out the common prefactor, $\mathcal{I} = \frac{i}{16\pi^2}\tilde{\mathcal{I}}$ and present a table of $\tilde{\mathcal{I}}[q^{2n_c}]_i^{n_i}$ for various $n_c$ and $n_i$, needed in our computations, in Table A.1.

| $\tilde{\mathcal{I}}[q^{2n_c}]_i^{n_i}$ | $n_c = 0$ | $n_c = 1$ | $n_c = 2$ |
|---|---|---|---|
| $n_i = 1$ | $M_i^2\left(1 - \log \frac{M_i^2}{\mu^2}\right)$ | $\frac{M_i^4}{4}\left(\frac{3}{2} - \log \frac{M_i^2}{\mu^2}\right)$ | $\frac{M_i^6}{24}\left(\frac{11}{6} - \log \frac{M_i^2}{\mu^2}\right)$ |
| $n_i = 2$ | $-\log \frac{M_i^2}{\mu^2}$ | $\frac{M_i^2}{2}\left(1 - \log \frac{M_i^2}{\mu^2}\right)$ | $\frac{M_i^4}{8}\left(\frac{3}{2} - \log \frac{M_i^2}{\mu^2}\right)$ |
| $n_i = 3$ | $-\frac{1}{2M_i^2}$ | $-\frac{1}{4}\log \frac{M_i^2}{\mu^2}$ | $\frac{M_i^2}{8}\left(1 - \log \frac{M_i^2}{\mu^2}\right)$ |
| $n_i = 4$ | $\frac{1}{6M_i^4}$ | $-\frac{1}{12M_i^2}$ | $-\frac{1}{24}\log \frac{M_i^2}{\mu^2}$ |
| $n_i = 5$ | $-\frac{1}{12M_i^6}$ | $\frac{1}{48M_i^4}$ | $-\frac{1}{96M_i^2}$ |

**Table A.1:** *Commonly-used master integrals with degenerate heavy particle masses.* $\tilde{\mathcal{I}} = \mathcal{I}/\frac{i}{16\pi^2}$ *and the* $\frac{2}{\epsilon} - \gamma + \log 4\pi$ *contributions are dropped.*

In this thesis, we only need the integrals in the mass degenerate case, i.e $M_i = M$ for all $i$.

# Appendix B

# Integrating out heavy Majorana fermions

We give here more details if one would like to integrate out heavy Majorana fermions. In this case, the quadratic operator does not change, and we are still able to use the pre-computed coefficients in the fermionic UOLEA for 1-loop matching. In practice, we can put the Majorana fields, and their conjugate into a doublet (using the four-component spinors for each field), and the quadratic operator will multiply with the factor $\frac{1}{2}$ and the $1_{2\times 2}$ identity matrix.

For illustration, we present here a mini-example integrating out gluino and wino to clarify our statement. Suppose

$$\begin{aligned} \mathcal{L}_{MSSM} \supset\, & \frac{1}{2}(\tilde{g})^T \mathcal{C}\big[i\left(\partial_\mu - i g_3 t^A G^A_\mu\right)\gamma^\mu - m_{\tilde{g}}\big]\tilde{g} \\ & + \frac{1}{2}(\tilde{W})^T \mathcal{C}\big[i\left(\partial_\mu - i g_2 t^I W^I_\mu - i g_1 Y B_\mu\right)\gamma^\mu - m_{\tilde{w}}\big]\tilde{W}\,, \end{aligned} \tag{B.1}$$

where we used the notations and conventions in Ref. [194]

$$\tilde{g} = \begin{pmatrix}\lambda_g \\ \lambda_g^\dagger\end{pmatrix},\; \tilde{W} = \begin{pmatrix}\lambda_W \\ \lambda_W^\dagger\end{pmatrix},\; \overline{\tilde{g}} = \left(\tilde{g}\right)^T\mathcal{C}, \text{ and } \overline{\tilde{W}} = \left(\tilde{W}\right)^T\mathcal{C}\,.$$

We note that the gluino and wino are denoted as Majorana spinors $\lambda_g$ and $\lambda_W$, respectively. Integrating out the heavy fields, $\tilde{g}$ and $\tilde{W}$, and computing the contributions from the loop with only heavy Majorana fermions to the gauge boson kinetic terms, we obtain

$$\begin{aligned} \mathcal{L}_{eft} &\supset \frac{1}{2}\, C^{(\tilde{g})}_{(P^4)}\, \mathrm{tr}\big[P_\mu, P_\nu\big]\big[P_\mu, P_\nu\big]\delta^{\alpha\beta}1_{2\times 2} + \frac{1}{2}\, C^{(\tilde{W})}_{(P^4)}\, \mathrm{tr}\big[P_\mu, P_\nu\big]\big[P_\mu, P_\nu\big]\delta^{ij}1_{2\times 2} \\ &= -\frac{1}{4}\left(\frac{-1}{16\pi^2}\, 2\log\frac{m^2_{\tilde{g}}}{\mu^2}\right) G^A_{\mu\nu}G^A_{\mu\nu} - \frac{1}{4}\left(\frac{-1}{16\pi^2}\,\frac{4}{3}\log\frac{m^2_{\tilde{W}}}{\mu^2}\right) W^I_{\mu\nu}W^I_{\mu\nu} \\ &\equiv -\frac{1}{4}\,\delta Z^{(\tilde{g})}_G\, G^A_{\mu\nu}G^A_{\mu\nu} - \frac{1}{4}\,\delta Z^{(\tilde{W})}_W\, W^I_{\mu\nu}W^I_{\mu\nu}\,, \end{aligned} \tag{B.2}$$

where $C^{(i)}_{(P^4)} = i\left(-\frac{1}{2}m_i^4\mathcal{I}_i^4 + 4m_i^2\mathcal{I}[q^2]_i^4 + (5\epsilon - 8)\mathcal{I}[q^4]_i^4\right)$ is taken from Tables 3.4 (result of the master integrals are listed in Ref. [77]) and we then sum over all internal indices ($SU(3)$ and $SU(2)$ indices for the loop with gluino and wino respectively). The results of $\delta Z^{(\tilde{g})}_G$ and $\delta Z^{(\tilde{W})}_W$ agree with Eq. (3.66) in Ref. [194].

# Appendix C

# Expansion of a ratio of determinants

The Jacobians that we have to compute are always of the form,

$$J[\theta] = \frac{\det(A)}{\det(A + f(\theta))} \tag{C.1}$$

where $A$ is some operator, and $f(\theta)$ carries all the $\theta$ dependence.

Using $\log\det = \mathrm{Tr}\log$, it can be expanded at first order in $\theta$ (which is infinitesimal) as follows,

$$\log J[\theta] = \mathrm{Tr}\log A - \mathrm{Tr}\log(A + f(\theta)) = -\mathrm{Tr}\log\left(1 + \frac{f(\theta)}{A}\right) = -\mathrm{Tr}\frac{f(\theta)}{A} + \mathcal{O}(\theta)\,. \tag{C.2}$$

If the Jacobian has been bosonised, $A = -\not{P}^\dagger\not{P} + m^2$, whereas if it was kept in the fermionic form, $A = i\not{D} - m$. $f(\theta)$ also depends on whether the Jacobian has been bosonised or not. To fix the ideas consider the fermionic form, although the reasoning holds for both.

We now explicit the trace over space-time, and use the Fourier transform to make momentum appear. We thus have $i\not{D} - \not{q} - m = \Delta^{-1}(1 - \Delta(-i\not{D}))$, with the propagator $\Delta = -1/(\not{q} + m)$ . We denote $f_q(\theta)$ the Fourier transform of $f(\theta)$. We then proceed with,

$$\begin{aligned}
\log J[\theta] &= -\int \mathrm{d}^d x \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\, \frac{f_q(\theta)}{\Delta^{-1}(1 - \Delta(-i\not{D}))} \\
&= -\int \mathrm{d}^d x \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\, f_q(\theta) \sum_{n\geq 0} \left(\Delta(-i\not{D})\right)^n \Delta\,.
\end{aligned} \tag{C.3}$$

It is apparent that the ratio of the two determinants in Eq. (C.1) is proportional to $\theta$. Now, using the cyclicity of the trace, this determinant can be written under the form,

$$\begin{aligned}
\log J[\theta] &= \int \mathrm{d}^d x \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr} \sum_{n\geq 0} \left(\Delta(-i\not{D})\right)^n \left(\Delta(-f_q(\theta))\right) \\
&= \int \mathrm{d}^d x \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr} \sum_{n\geq 0} \frac{1}{n+1} \left[\Delta\big(-i\not{D} - f_q(\theta)\big)\right]^{n+1} \Bigg|_{\mathcal{O}(\theta)} \\
&= \int \mathrm{d}^d x \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr} \sum_{n\geq 1} \frac{1}{n} \left[\Delta\big(-i\not{D} - f_q(\theta)\big)\right]^{n} \Bigg|_{\mathcal{O}(\theta)}
\end{aligned} \tag{C.4}$$

The factor $1/(n+1)$ appears in the second-to-last line to avoid overcounting the number of terms with one $f_q(\theta)$ insertion.

Note that the use of trace cyclicity despite the presence of $\gamma_5$ is not an issue, since we regularise using the free parameters, or using a proper hermitian operator that makes the Jacobian unambiguous ( $\not{D}^\dagger \not{D}$ ).

# Appendix D

# Fermionic expansion with free parameters: case of anomalous vector symmetry

In this appendix, we recast in details the calculation that is presented in section 7.3.3.3. A similar approach have been studied in a different context in Ref. [2]. As derived in section 7.3.3.3, we state here the Jacobian produced by the vector transformation,

$$J[\theta] = \frac{\det\left(i\not{D} - m\right)}{\det\left(i\not{D} - m - (\not{D}\theta)\right)} \,. \tag{D.1}$$

Since we introduced the auxiliary background field $\xi_\mu$, and the longitudinal mode $\pi_A$ of the axial gauge field $A_\mu$, the expansion of the Jacobian now reads,

$$\tilde{\mathcal{A}} = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}\left[\frac{-1}{\not{q}+m}\left(-i\not{D} + [\not{\xi} - (\not{\partial}\theta)] + m\frac{\pi_A}{v} i\gamma_5\right)\right]^n\Bigg|_{\mathcal{O}(\theta,\pi_A)} \,, \tag{D.2}$$

where we restrict our self in the Abelian gauge fields and Abelian $\theta$.

**Evaluating $\mathcal{A}_{\not{\partial}}$.** Since the mass term does not explicitly break the vector transformation, only the term involving $(\not{\partial}\theta)$ will contribute to the anomaly as can be seen in Eq. (D.1) (with $\theta$ Abelian).

**At $\mathbf{n = 4}$**, we obtain the $m^0$ term,

$$\begin{aligned}
\tilde{\mathcal{A}}_{\not{\partial}} &= \Big[ - 8m^4 \mathcal{I}_i^4 + 32 m^2 \mathcal{I}[q^2]_i^4 + \alpha\big(48\varepsilon\big)\mathcal{I}[q^4]_i^4 \Big] \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\big[\xi_\mu - (\partial_\mu\theta)\big](iA_\nu)\big[i\partial_\rho V_\sigma\big] \\
&= \frac{i}{8\pi^2}\big( - 1 + \alpha'\big)\epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\big[\xi_\mu - (\partial_\mu\theta)\big](iA_\nu)\big[i\partial_\rho V_\sigma\big] \\
&= \beta'\, \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\left[\xi_\mu - (\partial_\mu\theta)\right](iA_\nu) F^V_{\rho\sigma} \,,
\end{aligned} \tag{D.3}$$

where we define $\beta' = i(-1+\alpha')/(8\pi^2)$. In this computation, we followed the strategy outlined in section 7.3.1.1 to deal with the ambiguous traces. In $d = 4 - \epsilon$ dimensions, we explicitly have,

$$\begin{aligned}
\mathrm{tr}\, g^{abcd}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\gamma_5\big) \to\ & \alpha_1 \mathrm{tr}\, g^{abcd}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\gamma_5\big) + \alpha_2 \mathrm{tr}\, g^{abcd}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_5\gamma_d\gamma^\sigma\big) \\
& + \alpha_3 \mathrm{tr}\, g^{abcd}\big(\gamma_a\gamma^\mu\gamma_b\gamma^\nu\gamma_5\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\big) + \alpha_4 \mathrm{tr}\, g^{abcd}\big(\gamma_a\gamma^\mu\gamma_5\gamma_b\gamma^\nu\gamma_c\gamma^\rho\gamma_d\gamma^\sigma\big) \\
& = \varepsilon\big(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4\big)\big[i24\,\epsilon^{\mu\nu\rho\sigma}\big] \\
& = \varepsilon\alpha'\big[i24\epsilon^{\mu\nu\rho\sigma}\big] \,.
\end{aligned} \tag{D.4}$$

Even though we enforced the condition $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, the above trace still depends on free parameters and thus is ambiguous. In the last line of Eq. (D.4), we relabeled the sum of free parameters by a new parameter, $\alpha'$. Note that the standard evaluation of BMHV's scheme without free parameters is equivalent to $\alpha' = 1$.

As a remark, Eq. (D.3) is not gauge invariant due to the presence of the Chern-Simons term, $\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,\xi_\mu(iA_\nu)F^V_{\rho\sigma}$.

**At $\mathbf{n = 5}$**, the order $m^0$ terms related to the Goldstone boson $\pi_A$ and the auxiliary field $\xi_\mu$ that we need to enforce the gauge invariance of the axial current are,

$$\tilde{\mathcal{A}}_{\not{\partial}} = -\frac{i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\left[\frac{\pi_A}{v}(\partial_\mu\xi_\nu)F^V_{\rho\sigma}\right]. \tag{D.5}$$

**Enforcing gauge invariance.** In the Abelian case, a complete set of gauge transformations is

$$\begin{cases} V_\mu \to V_\mu + (\partial_\mu\varepsilon_V)\,, \\ A_\mu \to A_\mu + (\partial_\mu\varepsilon_A)\,, \\ \pi_A \to \pi_A - 2v\varepsilon_A\,, \end{cases} \tag{D.6}$$

where $\varepsilon_{V,A}$ are infinitesimal gauge parameters. Under the gauge transformation of Eq. (D.6), we enforce,

$$\delta_G\big(\tilde{\mathcal{A}}_{\not{\partial}}\big) = \delta_G\,\epsilon^{\mu\nu\rho\sigma}\left[\beta\,\mathrm{tr}\,\big[\xi_\mu - (\partial_\mu\theta)\big](iA_\nu)F^V_{\rho\sigma} - \frac{i}{8\pi^2}\,\mathrm{tr}\,\frac{\pi_A}{v}(\partial_\mu\xi_\nu)F^V_{\rho\sigma}\right] = 0\,. \tag{D.7}$$

Hence, we are now able to fix the value of the free parameter,

$$\beta' = \frac{-i}{8\pi^2}\,. \tag{D.8}$$

As a small remark in parallel with the usual Feynman diagrams technique, the gauge invariant combination of the General Chern-Simons term, $\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\big[\xi_\mu - (\partial_\mu\theta)\big](iA_\nu)F^V_{\rho\sigma}$, and the Goldstone term, $\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,(\pi_A/v)(\partial_\mu\xi_\nu)F^V_{\rho\sigma}$, is equivalent to enforcing the classical Ward identity of the axial current in the massive case.

Eventually, we substitute the value of $\beta'$ into Eq. (D.3), we then set $\xi_\mu \to 0$, and perform integration by parts. Going back to non-Abelian gauge fields and $\theta$, we obtain the non-Abelian covariant anomaly in the vector current,

$$\mathcal{A} = \mathcal{A}_{\not{\partial}} = \frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\theta\big(F^V_{\mu\nu}F^A_{\rho\sigma} + F^A_{\mu\nu}F^V_{\rho\sigma}\big)\,. \tag{D.9}$$

# Appendix E

# Covariant anomaly: bosonised form

In this appendix, we detail the bosonisation and the computation of the covariant anomaly in the bosonised form as discussed in sections 7.3.3.2 and 7.3.3.3 . For more details about bosonisation, we refer the reader to Refs. [119, 186].

## Bosonisation

The operator $\not{P} = i\not{D}$ is not hermitian,

$$\not{P}^\dagger = (i\not{\partial} - \not{V} - \not{A}\gamma_5)^\dagger = i\not{\partial} - \not{V} + \not{A}\gamma_5, \tag{E.1}$$

therefore, it does not have a well defined eigenvalue problem. However, $\not{P}^\dagger\not{P}$ and $\not{P}\not{P}^\dagger$ are hermitian, hence they admit two orthogonal eigenbasis with real eigenvalues,

$$\not{P}^\dagger\not{P}\phi_n = \lambda_n^2\phi_n, \quad \not{P}\not{P}^\dagger\varphi_n = \lambda_n^2\varphi_n, \quad n \in \mathbb{N}, \quad \lambda_n \in \mathbb{R}. \tag{E.2}$$

where

$$\not{P}\phi_n = \lambda_n\varphi_n\,, \;\; \not{P}^\dagger\varphi_n = \lambda_n\phi_n \;\text{ with }\; \lambda_n \in \mathbb{R}\;. \tag{E.3}$$

By decomposing $\psi$ on the orthonormal basis $\{\phi_n\}$, and $\bar{\psi}$ on the orthonormal basis $\{\varphi_n\}$, we can show that,

$$\det(\not{P} - m) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{\int \mathrm{d}^4x\bar{\psi}(\not{P}-m)\psi} = \mathcal{N}\Pi_n(\lambda_n - m), \tag{E.4}$$

and similarly,

$$\det(-\not{P}^\dagger - m) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{\int \mathrm{d}^4x\bar{\psi}(-\not{P}^\dagger-m)\psi} = \mathcal{M}\Pi_n(-\lambda_n - m)\;. \tag{E.5}$$

$\mathcal{N}$ and $\mathcal{M}$ depend on the determinant of the matrices that relate $\psi$ to $\phi_n$ and $\bar{\psi}$ to $\varphi_n$. They play no role in the computation of the anomaly [119].

We can therefore conclude that,

$$\begin{aligned}
|\det(\not{P} - m)|^2 &= \det(\not{P} - m)\det(\not{P}^\dagger - m) \\
&= \det(\not{P} - m)\det(-\not{P}^\dagger - m) \\
&= \mathcal{N}\mathcal{M}\Pi_n(\lambda_n - m)(-\lambda_n - m) \\
&= \mathcal{N}\mathcal{M}\Pi_n(-\lambda_n^2 + m^2) \\
&= \mathcal{N}\mathcal{M}\det(-\not{P}^\dagger\not{P} + m^2) = \mathcal{N}\mathcal{M}\det(-\not{P}\not{P}^\dagger + m^2)\;.
\end{aligned} \tag{E.6}$$

In the second line, we have used the fact that the determinant is invariant under the change of sign of the Dirac matrices. This can be understood by writing the determinant as a trace using $\log\det = \mathrm{Tr}\log$, and the fact that a trace of an odd number of Dirac matrices vanishes, hence allowing us to flip their sign. Besides, note that under this sign flip, $\gamma_5$ does not change since it's composed of an even number of Dirac matrices.

The sign flip of the Dirac matrices has two purposes. Firstly, it rids us of the cross term between $m$ and $\not{P}$. Indeed, without the change of sign, we would have,

$$\det(\not{P}^\dagger - m)\det(\not{P} - m) = \mathcal{N}\mathcal{M}\det(\not{P}^\dagger\not{P} + m^2 - m\not{P} - m\not{P}^\dagger)\,. \tag{E.7}$$

Secondly, after the Fourier transform, it provides the good relative sign between the $q^2$ and the $m^2$ terms, which allows us to factorise the propagator $\Delta = 1/(q^2 - m^2)$ instead of $\Delta = 1/(q^2 + m^2)$ without sign flip.

Now, for a determinant of the form,

$$\det(\not{P} - m + A), \tag{E.8}$$

where $A$ is a non-diagonal matrix in Dirac and in gauge space, we cannot easily write the determinant in terms of eigenvalues of the Dirac operator, because $A$ is non-diagonal. Therefore, we use naively the product of determinants[1] to get,

$$\det\left[e^{i\theta\gamma_5}(\not{P}^\dagger - m)e^{i\theta\gamma_5}\right]\det\left[e^{i\theta\gamma_5}(-\not{P} - m)e^{i\theta\gamma_5}\right] = \mathcal{N}\mathcal{M}\det\left[-\not{P}^\dagger\not{P} + m^2 + m(\not{P} - \not{P}^\dagger) + f(\theta)\right], \tag{E.9}$$

where

$$\begin{aligned} f(\theta) =& 4im^2\theta\gamma_5 - i[\theta, P^2]\gamma_5 - \frac{1}{2}[\sigma.F^V, \theta]\gamma_5 - \frac{1}{2}[\sigma.F^A\gamma_5, \theta]\gamma_5 \\ &+ 2im\left(\theta\gamma_5\not{P} - \not{P}\theta\gamma_5\right) + im\left((\not{P}\theta) - (\not{P}^\dagger\theta)\right)\gamma_5\ . \end{aligned} \tag{E.10}$$

We have used the convenient formulae,

$$\not{P}^\dagger\not{P} = \not{D}^\dagger\not{D} = -P^2 + \frac{i}{2}\sigma^{\mu\nu}[D_\mu, D_\nu] \tag{E.11}$$

and

$$[D_\mu, D_\nu] = F^V_{\mu\nu} + F^A_{\mu\nu}\gamma_5\ , \tag{E.12}$$

with the Bardeen curvatures as defined in Eqs. (7.50) and (7.51). As a result we get the Jacobian in the bosonised form,

$$J^2[\theta] = \frac{\det(-\not{P}^\dagger\not{P} + m^2)}{\det(-\not{P}^\dagger\not{P} + m^2 + m(\not{P} - \not{P}^\dagger) + f(\theta))}\ . \tag{E.13}$$

Expanding this ratio of determinants we obtain,

$$2\mathcal{A} = \int \frac{\mathrm{d}^d q}{(2\pi)^d} e^{iqx}\mathrm{tr}\left(f(\theta) + m(\not{P} - \not{P}^\dagger)\right)\frac{1}{-\not{P}^\dagger\not{P} + m^2}e^{-iqx}\ . \tag{E.14}$$

[1]In general the group homomorphism property: $\det(A.B) = \det(A)\det(B)$ is not correct for regularised determinants. However, the determinants we temper with are not regularised yet. Nonetheless, it is generally accepted that for non-regularised determinants one has $\log\det A = \mathrm{Tr}\log A$, which trivially implies the group homomorphism property of the determinant on non-regularised matrices.

At this point, one can notice that the term $-m(\not{P}+\not{P}^\dagger)$, and the term $-2im\left(\not{P}^\dagger\theta-\theta i\not{D}\right)\gamma_5$ in $f(\theta)$ can be dropped since they produce terms with an odd number of Dirac matrices, which will vanish under the Dirac trace.

Finally, let's compute the Fourier transform of $\not{P}^\dagger\not{P}$,

$$e^{-iqx}\not{P}^\dagger\not{P}e^{iqx}=\left(e^{-iqx}\not{P}e^{iqx}\right)^\dagger\left(e^{-iqx}\not{P}e^{iqx}\right)=\not{P}^\dagger\not{P}-\not{P}^\dagger\not{q}-\not{q}\not{P}+\not{q}^2 \tag{E.15}$$

Because of the presence of the axial field, $P_\mu$ does not commute with the Dirac matrices. In order to proceed with the computation let's define,

$$P_\mu=P_\mu^V-A_\mu\gamma_5 \text{ where } P_\mu^V=i\,\partial_\mu-V_\mu\,, \tag{E.16}$$

with $(\not{P}^V)^\dagger=\not{P}^V$ and $(\not{A}\gamma_5)^\dagger=-\not{A}\gamma_5$.

Therefore we can write,

$$-\not{P}^\dagger\not{q}-\not{q}\not{P}=-\{\not{P}^V,\not{q}\}+\not{A}\gamma_5\not{q}-\not{q}\not{A}\gamma_5=-2q\cdot P^V-2\{\not{A},\not{q}\}\gamma_5=-2q\cdot P^V-2q.A\gamma_5=-2q\cdot P \tag{E.17}$$

Finally we have,

$$e^{-iqx}\not{P}^\dagger\not{P}e^{iqx}=\not{P}^\dagger\not{P}-2q\cdot P+q^2\,. \tag{E.18}$$

## Computation

We expand the Jacobian using the mass expansion:

$$2\mathcal{A}=\int\frac{\mathrm{d}^dq}{(2\pi)^d}\mathrm{tr}\,h(\theta)\sum_{n\geq 0}\left[\Delta\left(-P^2+\frac{i}{2}\sigma.F^V+\frac{i}{2}\sigma.F^A\gamma_5+2q\cdot P\right)\right]^n\Delta\,, \tag{E.19}$$

where

$$h(\theta)=-i[\theta,(P-q)^2]-\frac{1}{2}[\sigma.F^V,\theta]\gamma_5-\frac{1}{2}[\sigma.F^A\gamma_5,\theta]\gamma_5+4im^2\theta\gamma_5\,, \tag{E.20}$$

and $\Delta=\frac{1}{q^2-m^2}$.

Now we gather the terms of order $m^0$, and that have an odd number of $\gamma_5$. Firstly, consider the contributions from the term $-i[\theta,(P-q)^2]$,

$$\int\frac{\mathrm{d}^dq}{(2\pi)^4}\mathrm{tr}\,(-i[\theta,(P-q)^2])\Big(\Delta^2(-P^2+\frac{i}{2}\sigma.F^V+\frac{i}{2}\sigma.F^A\gamma_5)+\Delta^3(2q\cdot P)^2\Big)\,. \tag{E.21}$$

These contributions vanish under the Dirac trace by lack of Dirac matrices. Secondly the contributions from the terms $-\frac{1}{2}[\sigma.F^V,\theta]\gamma_5-\frac{1}{2}[\sigma.F^A\gamma_5,\theta]\gamma_5$,

$$\int\frac{\mathrm{d}^dq}{(2\pi)^4}\mathrm{tr}\,(-\frac{1}{2}[\sigma.F^V,\theta]\gamma_5-\frac{1}{2}[\sigma.F^A\gamma_5,\theta]\gamma_5)\Big(\Delta^2(-P^2) \tag{E.22}$$

$$+\Delta^2(\frac{i}{2}\sigma.F^V+\frac{i}{2}\sigma.F^A\gamma_5) \tag{E.23}$$

$$+\Delta^3(2q\cdot P)^2\Big)\,. \tag{E.24}$$

The terms from Eq. (E.22) and (E.24) both vanish under the Dirac trace by lack of Dirac matrices. The term in Eq. (E.23) vanishes too using trace cyclicity in gauge space (all the operators are local). In the last two contributions, some of the integrals are divergent, but we do not need to compute them to see that the terms vanish, it is the operator itself that vanishes. Therefore, no ambiguity related to the $\gamma_5$ in dimensional regularisation can arise. Finally, we consider the contributions from the mass term $4im^2\theta\gamma_5$. Note that for this term, all the integrals are finite.

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathrm{tr}\, 4im^2\theta\gamma_5 \Bigg[ \Delta^5 (2q\cdot P)^4 + \Delta^3(-P^2)^2 + \Delta^3(\frac{i}{2}\sigma.F^V + \frac{i}{2}\sigma.F^A\gamma_5)^2 \tag{E.25}$$

$$+ \Delta^4 \left( (\frac{i}{2}\sigma.F^V + \frac{i}{2}\sigma.F^A\gamma_5)(2q\cdot P)^2 + 2q\cdot P(\frac{i}{2}\sigma.F^V + \frac{i}{2}\sigma.F^A\gamma_5)2q\cdot P + (2q\cdot P)^2(\frac{i}{2}\sigma.F^V + \frac{i}{2}\sigma.F^A\gamma_5) \right) \Bigg] . \tag{E.26}$$

The first two terms from Eq. (E.25) vanish under the Dirac trace. The third term from Eq. (E.25) does contribute and actually yields the covariant anomaly. The term from Eq. (E.26) vanishes under the Dirac trace, by lack of Dirac matrices. Therefore, among all the possible combinations, in the end only one term contribute:

$$\begin{aligned} 2\mathcal{A} &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \Delta^3 \mathrm{tr}\, 4im^2\theta\gamma_5 (\frac{i}{2}\sigma.F^V + \frac{i}{2}\sigma.F^A\gamma_5)^2 \\ &= \frac{-i}{16\pi^2}\frac{1}{2m^2} \mathrm{tr}\, 4im^2\theta\gamma_5 \left( (\frac{i}{2}\sigma.F^V)^2 + (\frac{i}{2}\sigma.F^A\gamma_5)^2 \right) \\ &= \frac{-i}{16\pi^2}\frac{1}{8} \mathrm{tr}\, i\theta\gamma_5 \left( [\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma]F^V_{\mu\nu}F^V_{\rho\sigma} + [\gamma^\mu,\gamma^\nu]\gamma_5[\gamma^\rho,\gamma^\sigma]\gamma_5 F^A_{\mu\nu}F^A_{\rho\sigma} \right) \\ &= 2\frac{-i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\, \theta (F^V_{\mu\nu}F^V_{\rho\sigma} + F^A_{\mu\nu}F^A_{\rho\sigma}) \end{aligned} \tag{E.27}$$

where we have discarded terms with even number of $\gamma_5$ matrices (they cannot yield a boundary term so cannot contribute to the final result). The remaining trace in the last line is the trace over the gauge space.

# Appendix F

# Scale anomaly

In this appendix, we detail the computation of the following Jacobian,

$$J[\sigma] = \frac{\det(i\not{D} - m)}{\det(i\not{D} - m - \sigma m - i\frac{d-1}{2}(\not{\partial}\sigma))}\,, \tag{F.1}$$

where $d$ is the dimension of space-time, and $\sigma$ is a local scalar function. No $\gamma_5$ is involved in the computation therefore the Jacobian is well-defined, and the computation in $d$ dimensions is performed using BMHV scheme. This Jacobian can be expanded following the usual procedure described in this paper,

$$\mathcal{A} = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\left(-i\frac{d-1}{2}(\not{\partial}\sigma) - \sigma m\right) \sum_{n\geq 0} \left[\Delta(-i\not{D})\right]^n \Delta\,. \tag{F.2}$$

$\Delta = -1/(\not{q}+m) = \Delta_f + \Delta_b$, where we label fermionic propagator $\Delta_f = -\not{q}/(q^2+m^2)$, and bosonic propagator $\Delta_b = m/(q^2-m^2)$. We focus on the terms of order $m^0$ only. The higher order terms see a cancellation between the mass term and the derivative term.

**Derivative term $-i\frac{d-1}{2}(\not{\partial}\sigma)$.** The only term of order $m^0$ is,

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\left(-i\frac{d-1}{2}(\not{\partial}\sigma)\right) \left(\Delta(-i\not{D})\right)^3 \Delta = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\left(\frac{d-1}{2}(\not{\partial}\sigma)\right) \left(\Delta\not{D}\right)^3 \Delta\,. \tag{F.3}$$

The propagators $\Delta$ now have to be decomposed in terms of fermionic and bosonic propagators $\Delta = \Delta_f + \Delta_b$, keeping in mind that the integral over momentum vanishes if the integrand bears an odd power in momentum. That is to say, the non-vanishing terms have an even number of fermionic propagators.

Note that there is a factor $\frac{i}{16\pi^2}$ from the master integrals that we discard for now for clarity. It will be accounted for at the end.

- The term with only bosonic propagators is,

$$\frac{d-1}{2} m^4 \mathcal{I}[q^0]^4 \mathrm{tr}\left((\not{\partial}\sigma)\not{D}^3\right) = \frac{1}{4}\left(4g^{\mu\sigma}g^{\nu\rho} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\nu}g^{\rho\sigma}\right) \mathrm{tr}\,(\partial_\mu\sigma)D_\nu D_\rho D_\sigma\,. \tag{F.4}$$

- For the term with two fermionic propagators, we have to account for all the possible positions

(2 fermionic propagators among 4 propagators, thus 6 possibilities),

$$\frac{d-1}{2}m^2\mathcal{I}[q^2]^4 g_{ab}\mathrm{tr}\left((\partial\!\!\!/\sigma)\left(\gamma^a D\!\!\!\!/\gamma^b D\!\!\!\!/^2+\gamma^a D\!\!\!\!/^2\gamma^b D\!\!\!\!/+\gamma^a D\!\!\!\!/^3\gamma^b\right.\right.$$
$$\left.\left.+D\!\!\!\!/\gamma^a D\!\!\!\!/\gamma^b D\!\!\!\!/+D\!\!\!\!/\gamma^a D\!\!\!\!/^2\gamma^b+D\!\!\!\!/^2\gamma^a D\!\!\!\!/\gamma^b\right)\right)$$
$$=-\frac{1}{8}\left(-16g^{\mu\sigma}g^{\nu\rho}+32g^{\mu\rho}g^{\nu\sigma}-16g^{\mu\nu}g^{\rho\sigma}\right)\mathrm{tr}\,(\partial_\mu\,\sigma)D_\nu D_\rho D_\sigma\,, \tag{F.5}$$

- Finally, there is a term with four fermionic propagator insertions. This is the only term that is divergent and that is computed in $d=4-\epsilon$ dimensions,

$$\frac{d-1}{2}\mathcal{I}[q^4]^4 g_{abcd}\mathrm{tr}\left((\partial\!\!\!/\sigma)\gamma^a D\!\!\!\!/\gamma^b D\!\!\!\!/\gamma^c D\!\!\!\!/\gamma^d\right) \tag{F.6}$$
$$=\left(-\frac{13}{3}g^{\mu\sigma}g^{\nu\rho}+\frac{23}{3}g^{\mu\rho}g^{\nu\sigma}-\frac{13}{3}g^{\mu\nu}g^{\rho\sigma}\right. \tag{F.7}$$
$$\left.+\log\left(\frac{m^2}{\mu^2}\right)\left(-2g^{\mu\sigma}g^{\nu\rho}+4g^{\mu\rho}g^{\nu\sigma}-2g^{\mu\nu}g^{\rho\sigma}\right)\right)\mathrm{tr}\,(\partial_\mu\,\sigma)D_\nu D_\rho D_\sigma\,,$$

where $g_{abcd}=g_{ab}g_{cd}+g_{ac}g_{bd}+g_{ad}g_{bc}$.

Those three contributions come together to yield,

$$\left(-\frac{4}{3}g^{\mu\sigma}g^{\nu\rho}+\frac{8}{3}g^{\mu\rho}g^{\nu\sigma}-\frac{4}{3}g^{\mu\nu}g^{\rho\sigma}+\log\left(\frac{m^2}{\mu^2}\right)\left(-2g^{\mu\sigma}g^{\nu\rho}+4g^{\mu\rho}g^{\nu\sigma}-2g^{\mu\nu}g^{\rho\sigma}\right)\right)\mathrm{tr}\,(\partial_\mu\,\sigma)D_\nu D_\rho D_\sigma$$
$$=\left(-g^{\mu\sigma}g^{\nu\rho}+2g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma}\right)\left(\frac{4}{3}+2\log\left(\frac{m^2}{\mu^2}\right)\right)\mathrm{tr}\,(\partial_\mu\,\sigma)D_\nu D_\rho D_\sigma\,. \tag{F.8}$$

Now let's explicit the operator,

$$\mathrm{tr}\,(\partial_\mu\,\sigma)D_\nu D_\rho D_\sigma=-\mathrm{tr}\,\sigma\,\partial_\mu\,D_\nu D_\rho D_\sigma\,, \tag{F.9}$$

up to a boundary term. Writing explicitly $D_\mu=\partial_\mu+V_\mu$, and distributing the derivatives we can write[1],

$$-\,\mathrm{tr}\,\sigma\,\partial_\mu\,D_\nu D_\rho D_\sigma=-\sigma\mathrm{tr}\left[(\partial_{\mu\nu\rho}\,V_\sigma)+(\partial_\mu\,V_\nu)(\partial_\rho\,V_\sigma)+V_\nu(\partial_{\mu\rho}\,V_\sigma)+(\partial_{\mu\nu}\,V_\rho V_\sigma)+(\partial_\mu\,V_\nu V_\rho V_\sigma)\right]. \tag{F.10}$$

where the parenthesis on the derivatives means that it acts locally in everything inside the parenthesis. It is then simple algebra to show that,

$$\left(-g^{\mu\sigma}g^{\nu\rho}+2g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma}\right)\mathrm{tr}\,\partial_\mu\,D_\nu D_\rho D_\sigma=0\,, \tag{F.11}$$

using Eq. (F.10) and trace cyclicity. As a result Eq. (F.8) vanishes, and the derivative coupling has no contribution at order $m^0$.

[1] Recall that a partial derivative to the right vanishes.

**Mass term $\sigma m$.** The only term of order $m^0$ is,

$$-\int \frac{\mathrm{d}^d q}{(2\pi)^d} \mathrm{tr}\,(-\sigma m)\left(\Delta(-i\not{D})\right)^4 \Delta = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \sigma m \mathrm{tr}\,\left(\Delta \not{D}\right)^4 \Delta\,. \tag{F.12}$$

Once again, the only non-vanishing terms are those that involve an even number of fermionic propagators. The term with only bosonic propagators is,

$$\sigma m^6 \mathcal{I}[q^0]^5 \mathrm{tr}\,\not{D}^4 = -\sigma \frac{1}{12}\left(4g^{\mu\sigma}g^{\nu\rho} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\nu}g^{\rho\sigma}\right)\mathrm{tr}\,D_\mu D_\nu D_\rho D_\sigma\,. \tag{F.13}$$

The term with two fermionic propagators has $\binom{5}{2} = 10$ contributions. We do not write them all for clarity,

$$\begin{aligned}
&\sigma m^4 \mathcal{I}[q^2]^5 g_{ab} \mathrm{tr}\,\left(\gamma^a\gamma^\mu\gamma^b\gamma^\nu\gamma^\rho\gamma^\sigma + \gamma^a\gamma^\mu\gamma^\nu\gamma^b\gamma^\rho\gamma^\sigma + ...\right)\mathrm{tr}\,D_\mu D_\nu D_\rho D_\sigma \\
&= \sigma \frac{1}{48}\left(32g^{\mu\rho}g^{\nu\sigma} - 16g^{\mu\sigma}g^{\nu\rho}\right)\mathrm{tr}\,D_\mu D_\nu D_\rho D_\sigma\,.
\end{aligned} \tag{F.14}$$

Finally, the term with four fermionic propagators has $\binom{5}{4} = 5$ contributions. Again, we do not write them all for clarity,

$$\begin{aligned}
&\sigma m^2 \mathcal{I}[q^4]^5 g_{abcd}\,\mathrm{tr}\,\left(\gamma^a\gamma^\mu\gamma^b\gamma^\nu\gamma^c\gamma^\rho\gamma^d\gamma^\sigma + \gamma^a\gamma^\mu\gamma^b\gamma^\nu\gamma^c\gamma^\rho\gamma^\sigma\gamma^d + ...\right) D_\mu D_\nu D_\rho D_\sigma \\
&= -\sigma \frac{1}{96}\left(64g^{\mu\sigma}g^{\nu\rho} - 32g^{\mu\rho}g^{\nu\sigma} - 32g^{\mu\nu}g^{\rho\sigma}\right)\mathrm{tr}\,D_\mu D_\nu D_\rho D_\sigma\,.
\end{aligned} \tag{F.15}$$

Note that the three contributions are finite, hence the computation is performed in 4 dimensions. Putting together the three contributions, we obtain,

$$\sigma \mathrm{tr}\,\left(-\frac{4}{3}D^\mu D^2 D_\mu + \frac{4}{3}D^\mu D^\nu D_\mu D_\nu + 0 D^2 D^2\right) = \sigma \frac{2}{3}\mathrm{tr}\,F^{\mu\nu}F_{\mu\nu}\,. \tag{F.16}$$

**Final result.** Finally, the only term contributing to the scale anomaly at order $m^0$ is the mass term. Recovering the factor $\frac{i}{16\pi^2}$ from the master integrals, we obtain the scale anomaly,

$$\mathcal{A} = \sigma \frac{i}{24\pi^2}\mathrm{tr}\,F^2\,. \tag{F.17}$$

The remaining trace is in gauge space only.

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