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BKT PHASE TRANSITION IN THREE-DIMENSIONAL Z(N) GAUGE THEORY

O. A. Borisenko[a], I. E. Surzhikov[b], V. O. Chelnokov[c]

Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine

We perform numerical study of phase transitions in $(2+1)d$ $Z(5)$ lattice gauge theory in the strong coupling limit simulating effective two-dimensional spin model. Critical points of this model are obtained. Critical indices are calculated in the vicinity of these points.

1 Introduction

The Berezinskii-Kosterlitz-Thouless (BKT) phase transition is known to take place in a variety of two-dimensional ($2d$) systems: certain spin models, two-dimensional Coulomb gas, sine-Gordon model, Solid-on-Solid model, etc. The most elaborated case is the two-dimensional XY model [1, 2, 3]. There are several indications that this type of phase transition is not a rare phenomenon in gauge models at finite temperature - one can argue that in some three-dimensional lattice gauge models the deconfinement phase transition is of BKT type as well.

Some details of the critical behavior of $2d$ $Z(N)$ spin models are well known – see the review in Ref. [4]. The $Z(N)$ spin model in the Villain formulation has been studied analytically in Refs. [5, 6, 7, 8, 9]. It was shown that the model has at least two phase transitions when $N \geq 5$. The intermediate phase is a massless phase with power-like decay of the correlation function. The critical index η has been estimated both from the renormalization group (RG) approach of the Kosterlitz-Thouless type and from the weak-coupling series for susceptibility. It turns out that $\eta(\beta_c^{(1)}) = 1/4$ at the transition point from the strong coupling (high-temperature) phase to the massless phase, *i.e.* the behavior is similar to that of the XY model. At the transition point $\beta_c^{(2)}$ from the massless phase to the ordered low-temperature phase one has $\eta(\beta_c^{(2)}) = 4/N^2$. A rigorous proof that the BKT phase transition does take place, and so that the massless phase exists, has been constructed in Ref. [10] for both Villain and standard formulations (with one non-vanishing coupling β_1). Monte-Carlo simulations of the standard version with $N = 6, 8, 12$ were performed in Ref. [11]. Results for the critical index η agree well with the analytical predictions obtained from the Villain formulation of the model.

Swetitsky-Yaffe conjecture is known to connect critical properties of three-dimensional lattice gauge theory ($3d$ LGT) with corresponding properties of $2d$ spin models if they share the same global symmetry of the action. According to it, the phase transitions in $3d$ $Z(N)$ models for $N > 4$ should be of BKT type and have the critical indices equal to those of corresponding $2d$ $Z(N)$ spin models.

The fact that BKT transition has infinite order makes it hard to study its properties using analytical methods. For $2d$ XY model a renormalization group providing critical indices of the phase transition was developed, but even in $2d$ $Z(N)$ spin model this approach was shown in [5] to lead to an involved system of differential equations, which are too hard to be solved. To study this phase transition we need numerical simulations.

We are studying the phase transitions in $3d$ $Z(5)$ LGT at the finite temperature, since it is the lowest value of N for which the model undergoes two infinite-order transitions. We are simulating $2d$ $Z(5)$ spin model with modified action, which is shown to be equivalent to $3d$ $Z(5)$ LGT in the strong-coupling limit ($\beta_s = 0$).

The action for the model (in general $Z(N)$ case) is written as follows

$$S = \sum_{l\in\Lambda} \sum_{k=1}^{N-1} \beta_k \cos\left(\frac{2\pi k}{N}(r_x - r_{x+e_n})\right). \tag{1}$$

In our case $N = 5$, simulations are made on lattice Λ with dimensions $L \times L \times N_T$, where $N_T = 2$.

Modified coupling constants β_i are derived from $\beta = \beta_t$ coupling constant of the $Z(N)$ LGT using following equations:

$$B_k = \sum_{p=0}^{N-1} \exp\left(\beta \cos\left(\frac{2\pi p}{N}\right)\right), A_k = \left(\frac{B_k}{B_0}\right)^{N_t}, Q_k = \sum_{p=0}^{N-1} A_p \cos\left(\frac{2\pi pk}{N}\right), \beta_k = \frac{1}{5}\sum_{p=0}^{N-1} \ln(Q_p)\cos\left(\frac{2\pi pk}{N}\right). \tag{2}$$

e-mail: [a]oleg@bitp.kiev.ua, [b]i_van_go@inbox.ru, [c]yyyaduren@rambler.ru

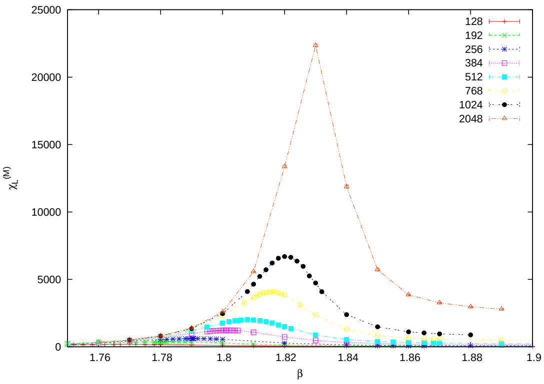

Figure 1. Dependence of magnetization suspectibility χ_M on β

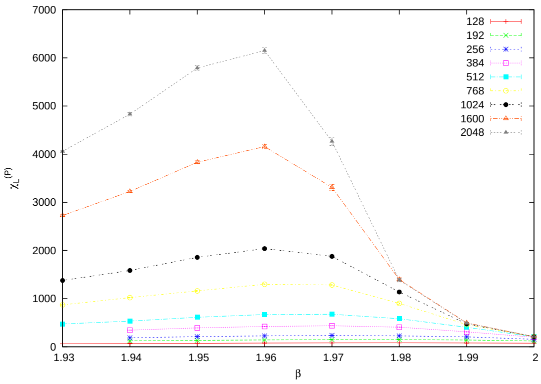

Figure 2. Dependence of population suspectibility χ_P on β

These equations can be obtained in few steps

- Fourier expansion of the original Boltzmann weight.
- Integration over spatial gauge fields. This leads to an effective $2d$ model for the Polyakov loops.
- Exponentiating and re-expansion in a new Fourier series.

To study these transitions we used a cluster algorithm described in [12]. The model is studied on a square $L \times L$ lattice Λ with periodic boundary conditions. Simulations were made for $N_t = 2$ but can easily be conducted for other values of N_t (it appears only in definition of the couplings (2)).

We expect that in our model there are two infinite-order phase transitions, which have critical indices $\nu = 1/2,\ \eta = 1/4$ (first transition) and $\nu = 1/2,\ \eta = 4/N^2$ (second transition).

In the simulations we measure the following observables

- complex magnetization $M_L = |M_L| e^{i\psi}$

$$M_L \ = \ \sum_{x \in \Lambda} \exp\left(\frac{2\pi i}{N} r_x \right) \tag{3}$$

- population S_L

$$S_L \ = \ \frac{N}{N-1} \left(\frac{\max_{i=0,N-1} n_i}{L^2} - \frac{1}{N} \right), \tag{4}$$

  where n_i is number of r_x equal to i.

- rotated magnetization $M_R = |M_L| e^{5i\psi}$
- susceptibilities of M_L, S_L and M_R: $\chi_L^{(M)}$, $\chi_L^{(S)}$, $\chi_L^{(M_R)}$

$$\chi_\cdot \ = \ L^2 \left(\left\langle \cdot^2 \right\rangle - \left\langle \cdot \right\rangle^2 \right) \tag{5}$$

- Binder cumulants $U_L^{(M)}$ and $B_4^{(M_R)}$

$$U_L^{(M)} = 1 - \frac{\left\langle |M_L|^4 \right\rangle}{3 \left\langle |M_L|^2 \right\rangle^2}, B_4^{(M_R)} = \frac{\left\langle |M_R - \langle M_R \rangle|^4 \right\rangle}{\left\langle |M_R - \langle M_R \rangle|^2 \right\rangle^2}, \tag{6}$$

2 Simulation results

On Figure 1 one can see a clear peak of magnetization susceptibility for various lattice sizes. It showed us the location of the first phase transition.

Finding the best overlapping of the binder cumulants U_L for various values of L (see Figure 3) gives us another estimate

$$\beta_c \ = \ 1.8692$$

Fitting $|M_L|$ at $\beta_c^{(1)}$ (see Table 1)with formula

$$|M_L| \ = \ A L^{-\beta/\nu} \tag{7}$$

gives us

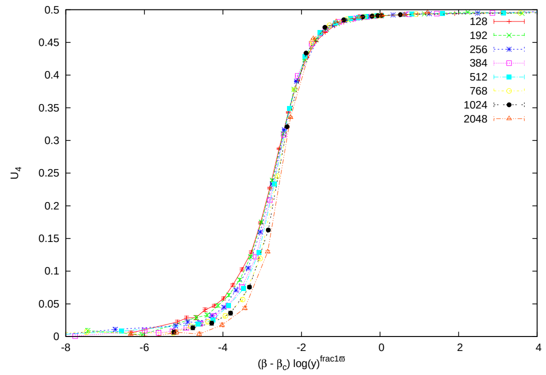

Figure 3. Binder cumulant U_L plotted versus $(\beta - \beta_c^{(1)})(\log(L))^{\frac{1}{\nu}}$.

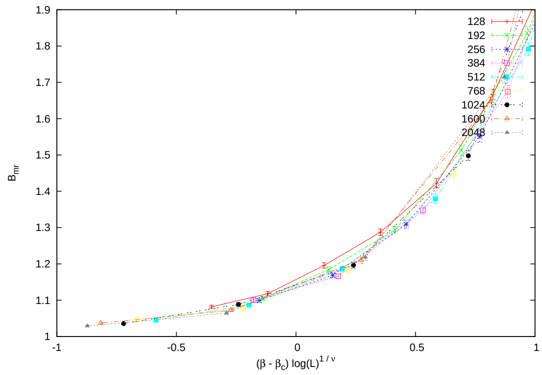

Figure 4. Binder cumulant B_{mr} plotted versus $(\beta - \beta_c^{(2)})(\log(L))^{\frac{1}{\nu}}$.

Table 1. Values of M_L and χ_{M_L} at $\beta = 1.8692$ for various lattice sizes.

L	M_L	Δ_{M_L}	χ_{M_L}	$\Delta_{\chi_{M_L}}$
128	0.5643	0.0002	26.1	0.2
192	0.5373	0.0002	53.4	0.4
256	0.5192	0.0002	87.2	0.8
384	0.4942	0.0002	175	1
512	0.4775	0.0002	285	2
768	0.4549	0.0002	592	5
1024	0.4393	0.0001	961	7
1600	0.4159	0.0001	2148	19
2048	0.4039	0.0001	3250	26

Table 2. Values of M_R and χ_{M_R} at $\beta_c^{(2)} = 1.945$ for various lattice sizes.

L	M_R	Δ_{M_R}	χ_{M_R}	$\Delta_{\chi_{M_R}}$
16	0.30	0.01	65.34	0.07
32	0.25	0.01	238.2	0.1
64	0.21	0.01	862.9	0.2
128	0.19	0.01	3112.5	0.3
192	0.17	0.01	6606.8	0.5
256	0.16	0.01	11256.7	0.6
384	0.16	0.01	23661.2	0.8
512	0.15	0.01	40671	1

$$\beta_c^{(1)} = 1.8692 \qquad \beta/\nu = 0.120635 \qquad \Delta_{\beta/\nu} = 0.000137$$

Fitting χ_L at $\beta_c^{(1)}$ with formula

$$\chi_L = \ AL^{\gamma/\nu} \tag{8}$$

gives us

$$\beta_c^{(1)} = 1.8692 \qquad \gamma/\nu = 1.739745 \qquad \Delta_{\gamma/\nu} = 0.003188 \qquad \eta = 2 - \gamma/\nu = 0.260255$$

On Figure 2 one can see a clear peak of population suspectibility for various lattice sizes. It showed us the location of the second phase transition.

Fitting binder cumulant $B_4^{(M_R)}$ (Figure 4) for the second transition we get

$$\beta_c^{(2)} \ = \ 1.945$$

For this value of $\beta_c^{(2)}$ we extract critical index η from M_R and χ_{M_R} data given in Table 2. The result obtained is shown in Table 3, where $L_{\min}$ is a minimal size of the lattice, which we used for fitting.

3 Conclusions

In this paper we have presented a wealth of numerical data aimed at shedding light on the phase structure of the three-dimensional $Z(N)$ gauge theory at the finite temperature. In the strong coupling limit we find two phase transitions. We have determined the critical points $\beta_c^{(1)}$ and $\beta_c^{(2)}$. For these values we get critical index η, which coincide with critical indices of $2d$ $Z(5)$ spin model. So $(2+1)d$ $Z(5)$ LGT belongs to the universality class of the $2d$ $Z(5)$ spin model.

Table 3. Critical index η for second transition ($\beta_c^{(2)} = 1.945$), calculated for different starting lattice sizes

$L_{\min}$	β/ν	γ/ν	$d = 2\beta/\nu + \gamma/\nu$	$\eta = 2 - \gamma/\nu$
16	0.20 ± 0.01	1.853 ± 0.004	2.26 ± 0.02	0.147 ± 0.004
32	0.18 ± 0.01	1.853 ± 0.004	2.22 ± 0.02	0.147 ± 0.004
64	0.17 ± 0.01	1.853 ± 0.005	2.19 ± 0.03	0.147 ± 0.005
128	0.15 ± 0.02	1.853 ± 0.006	2.15 ± 0.04	0.147 ± 0.006
192	0.12 ± 0.01	1.854 ± 0.009	2.10 ± 0.03	0.146 ± 0.009
256	0.13 ± 0.01	1.86 ± 0.01	2.1 ± 0.2	0.14 ± 0.01

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