D-Brane Chan-Paton Factors and Orientifolds

by

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Abstract

In this thesis, we mainly study the structure of D_9-brane Chan-Paton factors in Type II orientifold $O_p^{\pm}$ theories. An explicit structure is found through a thorough and systematic derivation. For this purpose, we give a complete analysis of the worldsheet parity square action for all p-q open strings (i.e, the strings starting from D_p-branes and ending on D_q-branes), which is an extension of Gimon-Polchinski's work on the parity square action for p-q strings with $p-q$ even. We also formulate the rules for computing the scattering amplitudes of open strings ending on D-branes. These are important to build phenomenological models in string theory. As an application to the mathematical aspects of string theory, we confirm the proposal that the D-brane charges in Type II orientifolds are classified by the KR-theory groups. All these results will appear in our paper [1].

We also show some preliminary results on the B-type D-branes in the orientifolds of linear sigma models. A typical linear sigma model has both a geometric phase and a Landau-Ginzberg phase over its Kähler moduli space. The B-type D-branes are described by suitable categories in a linear sigma model and its phases. The orientifold projection induces parity mappings on the various categories. By looking at the transportation of invariant branes in the moduli space, we find the relations among the parity mappings of different phases in the quintic model.

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Chapter 1

Introduction

The theoretical particle physics has been extremely successful. A lot of phenomena have been understood very well in terms of a small number of underlying principles and building blocks. The principles that underly our current understanding of nature are quantum mechanics and quantum field theory. There are four known fundamental interactions in nature: the electromagnetic, the weak, the strong and the gravitational interactions. By the late 1970s, it was found that the first three fundamental interactions can be described by the so-called Standard Model(SM) of particle physics, which is a renormalizable quantum gauge field theory with the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group. In the SM, the fundamental interactions are mediated by the so-called gauge bosons which are spin-1 particles. Photons mediate the electromagnetic interaction. The $W^{\pm}$ and Z bosons mediate the weak interactions. The gluons mediate the strong interaction. All visible matters in nature are described by spin-$\frac{1}{2}$ elementary particles, called fermions. In the SM, there exist three generations of fermions, called quarks and leptons. Quarks take part in all three interactions, while leptons only take part in the electromagnetic and the weak interactions. The SM also includes the spin-0 Higgs boson which is needed for the spontaneous symmetry breaking. The SM has passed a variety of experimental tests and is consistent with virtually all physics down to the energy scale of 100GeV. The only unobserved particle in the SM is the Higgs boson. Searching for it is one of the main challenges of the Large Hadron Collider(LHC), which is being built at CERN and is going to operate in 2009.

Despite all the success of the SM, it is surely not a complete theory of particle physics. First, the gravity does not seem to be able to fit into the framework of a local quantum field theory. The gravitational interactions are mediated by spin-2 particles, called gravitons. Einstein's general relativity theory provides a good description of the gravity. If we try to quantize it in the spirit of quantum field theory, we would end up with a nonrenormalizable quantum field theory. In other words, the general relativity and the SM are incompatible. Second, the SM contains nineteen parameters: 9 fermion masses, 3 gauge couplings, 4 CKM matrix entries, 2 Higgs couplings and one QCD vacuum angle. The values of these parameters are unrelated and arbitrary. What determines these parameters? Third, the SM has the so-called hierarchy problem, i.e, some parameters in the SM Lagrangian are much smaller than one would expect. Put the problem in another way. We can ask why the gravity is so weak. Let us compare the weak interaction with the gravitational interaction. The typical ratio of the gravitational force to the weak force is $(m_W/m_{pl})^2$, where the W-boson mass m_W is about 100GeV, and the Planck mass m_{pl} is about 10^{18}GeV. We can easily see that the gravitational force is 10^{-32} times weaker than the weak force. Adding up all the evidence, it is widely believed that the SM is only an effective theory valid up to the energy scale of 100GeV.

Several proposals have been put forward to solve the problems of the SM. Each proposal has its own attractive features. One is the grand unification theory(GUT). The gauge group of the SM is a direct product of three Lie groups. It contains three gauge coupling constants, corresponding to the three interactions, respectively. Thus, it is not a real unification model of the three interactions. On the other hand, the GUT unifies the three fundamental interactions into one, i.e, embedding the SM gauge group into one larger simple Lie group such as SO(10) or SU(5). There is only one gauge coupling constant. The GUT also predicts relations among the fermion masses. These are the nice aspects of the GUT.

The second proposal is the extra dimensions. The idea is that our spacetime has more than four dimensions, and the extra ones are so highly curved as to be unobservable at current experimental probe. This is logically consistent. Actually, Einstein's general relativity theory does not determine the dimension of our spacetime. It makes the unification of gauge interactions and the gravity possible, through the so-called Kaluza-Klein mechanism. The original Kaluza-Klein model was proposed to unify the electromagnetic and the gravitational interactions. The model extended Einstein's general relativity theory to a five-dimensional spacetime. The extra spatial dimension was compactified on a circle of very small radius. The resulting effect theory in the noncompact four-dimensional spacetime contained Einstein's general relativity for the gravity and Maxwell's theory for the electromagnetism. It is straightforward to generalize the Kaluza-Klein mechanism to higher extra dimensions.

The third proposal is called supersymmetry. Supersymmetry has been widely studied both in physics and mathematics [2, 4, 5]. Unlike spacetime symmetries and gauge symmetries, supersymmetry is a symmetry that relates two particles whose spins differ by $\frac{1}{2}$. The two particles are superpartners to each other. A quantum field theory with supersymmetry is called a supersymmetric quantum field theory. There are many supersymmetric extensions of the SM in literature. The simplest one is just to add the superpartner of each SM particle into the Lagrangian. The superpartners of gauge bosons are called gauginos. The superpartners of quarks and leptons are called squarks and sleptons, respectively. The resulting theory is called the minimal supersymmetric standard model (MSSM). Technically speaking, the MSSM has 4d $\mathcal{N}$=1 supersymmetry. If supersymmetry exists at the energy scale of 1TeV, it will solve two major problems in the SM: the hierarchy problem and the unification of the electromagnetic, the weak and the strong interactions. On the other hand, up to now, there is no direct evidence that supersymmetry is a symmetry of nature. The superpartners of the SM particles have not been observed. This suggests that if supersymmetry exists, it must be a broken symmetry. Searching for the superpartners of the SM particles is also one of the main challenges of the LHC.

String theory starts from a totally different viewpoint from the local quantum field theories [3, 4, 6]. The fundamental objects in string theory are one-dimensional strings. There are two types of strings, open strings and closed strings. Both of them can be either oriented or unoriented. An oriented string is a string with an internal orientation so that the string is distinguished from the string with the opposite orientation. While for the unoriented string, there is no such internal orientation. The open string has two endpoints. One can attach additional degrees of freedom to both endpoints, called the Chan-Paton factors. When moving in the spacetime, the strings will sweep out a two-dimensional surface, called the worldsheet. The graviton and all the other elementary particles can be regarded as the oscillation modes of the closed and open strings, respectively. The only input parameter in string theory is the string's tension, $T = \frac{1}{2\pi\alpha'}$, where α' is the Regge slope. (The Regge slope was first introduced in particle physics. In the 1960s, it was found that the spin of a hadron is proportional to its mass squared, $J = \alpha' m^2$. α' is called the Regge slope, with a value of about $1(GeV)^{-2}$. In string theory, α' becomes an input parameter.) This defines a characteristic length of string, $l_s := \sqrt{\alpha'}$, which means that the interaction in string theory does not happen at a point, but is smeared out into a region on the string worldsheet. On the other hand, the characteristic energy scale $1/l_s$ of string theory is very big compared to the scale of particle physics. Then at energies much below $1/l_s$ (corresponding to the limit $\alpha' \to 0$), the string excitation modes can be ignored and the strings can be well described as point-like states. The string Feynman diagrams will reduce to the quantum field theory ones.

There are two classes of string theories: bosonic string theories and superstring theories. The bosonic string theory is phenomenologically uninteresting because it does not contain spacetime fermions. The superstring theory contains all the attractive features of the grand unification, the extra dimensions and supersymmetry. It is believed to be the most promising candidate for a unified theory of all the four fundamental interactions in nature. So we will focus on superstring theories in the following.

Let us first look at the open superstring theory. The worldsheet action in the conformal gauge is:

$$S_{bulk} = \frac{1}{\pi}\int_{\Sigma} d^2\sigma \left(\frac{1}{\alpha'}\partial_+ X \cdot \partial_- X + i\psi_+ \cdot \partial_- \psi_+ + i\psi_- \cdot \partial_+ \psi_-\right),$$

where the ψ_+^μ and ψ_-^μ are the left-moving and right-moving worldsheet fermions with $\mu = 0, 1, \cdots, d-1$, respectively. Σ is the open string worldsheet, parameterized by $-\frac{\pi}{2} \le \sigma \le \frac{\pi}{2}$ and $-\infty \le t \le \infty$. At the boundaries of Σ, the Poincaré invariant boundary condition on the bosonic field X^μ is the Neumann condition, $\partial_n X^\mu = 0$. There exist two types of boundary conditions on the worldsheet fermion fields which are consistent with the Lorentz invariance. They are denoted by the Ramond(R) and the Neveu-Schwarz(NS) sectors:

$$\begin{array}{lll} \text{R}: & \psi_+^\mu(-\frac{\pi}{2}, t) = \pm\psi_-^\mu(-\frac{\pi}{2}, t) & \psi_+^\mu(\frac{\pi}{2}, t) = \pm\psi_-^\mu(\frac{\pi}{2}, t) \\ \text{NS}: & \psi_+^\mu(-\frac{\pi}{2}, t) = \pm\psi_-^\mu(-\frac{\pi}{2}, t) & \psi_+^\mu(\frac{\pi}{2}, t) = \mp\psi_-^\mu(\frac{\pi}{2}, t). \end{array}$$

The mode expansion of the worldsheet fields is:

$$X^{\mu}(\sigma,t) = x^{\mu} + (2\alpha')^{1/2}\alpha_0^{\mu}t + i(2\alpha')^{1/2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{\mu}cos[n(\sigma-\frac{\pi}{2})]\mathrm{e}^{-int}, \tag{1.1}$$

$$\psi_{-}^{\mu}(\sigma,t) = \sum_{r}\psi_r^{\mu}\mathrm{e}^{ir(\sigma-\frac{\pi}{2}-t)}, \quad r\in\mathbb{Z}\ \ (\mathrm{R})\ \ \text{and}\ \ r\in\mathbb{Z}+\frac{1}{2}\ \ (\mathrm{NS}), \tag{1.2}$$

where $\alpha_0^{\mu} = \sqrt{2\alpha'}p^{\mu}$ is the zero mode for the bosonic field. Then the bosonic and fermionic parts of the energy-momentum tensor can be written as

$$\begin{aligned} T_B(z) &= \sum_{m=-\infty}^{\infty}\frac{L_m}{z^{m+2}}, \\ T_F(z) &= \sum_{r}\frac{G_r}{z^{r+3/2}}\ ,\ \text{where } r\in\mathbb{Z}\ (\mathrm{R})\ \text{ or } r\in\mathbb{Z}+\frac{1}{2}\ (\mathrm{NS}). \end{aligned} \tag{1.3}$$

We have an enlarged Virasoro algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{\mathrm{c}}{12}(m^3-m)\delta_{m+n}, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{\mathrm{c}}{12}(4r^2-1)\delta_{r+s}, \\ [L_m, G_r] &= \frac{1}{2}(m-2r)G_{m+r}\ , \end{aligned} \tag{1.4}$$

$$\tag{1.5}$$

with

$$\begin{aligned} L_m &= \frac{1}{2}\sum_{m} :\alpha_{m-n}\cdot\alpha_m: + \frac{1}{4}\sum_{r}(2r-m):\psi_{m-r}\cdot\psi_r: + a\delta_{m,0}, \\ G_r &= \sum_{n}\alpha_n\cdot\psi_{r-n}\ . \end{aligned} \tag{1.6}$$

In the above, c is the central charge term which gives the conformal anomaly. Its value is d + d/2, where d is from the d worldsheet bosons and d/2 is from the d worldsheet fermions. It turns out that the superconformal ghost fields contribute another -15 to the conformal anomaly term. Then the demand for the conformal anomaly cancellation determines that $d = 10$. This explains why superstring theories can only be consistently defined on 10d spacetimes.

In order to achieve spacetime supersymmetry, the spectrum is projected to states with even fermion number. This is called the GSO-projection. Massless states in ten-dimensional spacetimes are classified by representations of the little group SO(8) of the 10d Lorentz group SO(1,9). They are the $\mathbf{8_v}$ and the $\mathbf{8_s}$ representations for the NS and R sectors, respectively. (Actually, there is a physically equivalent representation $\mathbf{8_c}$ for the R sector.) The NS-sector corresponds to the spacetime gauge bosons and the R-sector corresponds to the superpartners of the gauge bosons. Including the Chan-Paton factors at the ends of the open string, one can get a U(N) gauge theory for the oriented open strings and an SO(N) or an Sp(N) gauge theory for the unoriented open strings.

For the closed superstring, the discussion is very similar to the open string case. The difference is that the closed string worldsheet Σ (parameterized by $0 \leq \sigma \leq 2\pi$ and $-\infty \leq t \leq \infty$ with $\sigma \equiv \sigma + 2\pi$) is closed and does not have boundary. Its spectrum is just the product of two copies of the open string spectrum. One can take the GSO-projection on the two copies independently. Take the same GSO-projection on both copies, we get the Type IIB string theory which is chiral. Taking opposite projections gives the Type IIA string theory which is non-chiral. The massless states in the two theories are

$$\begin{aligned} \text{Type IIA:} &\qquad (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}), \\ \text{Type IIB:} &\qquad (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) \ . \end{aligned}$$

The NS-NS sector is the same in Type IIA and IIB theories,

$$\mathbf{8_v} \otimes \mathbf{8_v} = \Phi \oplus B_{\mu\nu} \oplus \mathrm{g}_{\mu\nu} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35},$$

where Φ, $B_{\mu\nu}$ and $\mathrm{g}_{\mu\nu}$ are the dilaton, the antisymmetric tensor and the graviton, respectively. The R-R sectors in Type IIA and IIB theories are

$$\begin{aligned}
\text{Type IIA}: &\qquad \mathbf{8_s} \otimes \mathbf{8_c} = [1] \oplus [3] = \mathbf{8_v} \oplus \mathbf{56_t}, \\
\text{Type IIB}: &\qquad \mathbf{8_s} \otimes \mathbf{8_s} = [0] \oplus [2] \oplus [4]_+ = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+.
\end{aligned}$$

The $[n]$ denotes the n-times antisymmetric representation of SO(8) and is also called the R-R C_n form. In particular, $[4]_+$ is self–dual. The NS-NS and R-R sectors together form the bosonic components of 10d Type IIA and IIB supergravities, respectively. The NS-R and R-NS sectors are the fermionic components. In Type I and Type II theories, there exist higher-dimensional objects, called D_p-branes, which have $p+1$-dimensional worldvolumes. The open strings can end on them, giving rise to the gauge field theories on the D-brane worldvolume.

There are two more closed superstring theories, called the heterotic string theories. As we know, we can treat the left-moving and the right-moving sectors of the closed string independently. The heterotic string theory is a hybrid of superstring and bosonic string theories. The left-moving sector is a purely bosonic string, and the right-moving sector is a pure superstring. To achieve the conformal anomaly cancellation, we need to introduce 32 fermionic fields in the left-moving sector. The worldsheet action is

$$S = \frac{1}{4\pi} \int_\Sigma d^2\sigma \, \{ \frac{1}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \lambda^A \bar{\partial} \lambda^A + \psi_-^\mu \partial \psi_{-,\mu} \},$$

Type	String	Gauge group	Chirality	10d SUSY	Massless bosons
IIA	Closed Oriented	None	Non-chiral	$\mathcal{N}$=2	NS-NS: Φ, $B_{\mu\nu}$, $g_{\mu\nu}$ R-R: C_1, C_3
IIB	Closed Oriented	None	Chiral	$\mathcal{N}$=2	NS-NS: Φ, $B_{\mu\nu}$, $g_{\mu\nu}$ R-R: C_0, C_2, C_4
I	Open & Closed Unoriented	$SO(32)$	Chiral	$\mathcal{N}$=1	Φ, $g_{\mu\nu}$,C_2, A_μ in Adj. of $SO(32)$
heterotic $SO(32)$	Closed Oriented	$SO(32)$	Chiral	$\mathcal{N}$=1	Φ, $B_{\mu\nu}$, $g_{\mu\nu}$, A_μ in Adj. of $SO(32)$
heterotic $E_8 \times E_8$	Closed Oriented	$E_8 \times E_8$	Chiral	$\mathcal{N}$=1	Φ, $B_{\mu\nu}$, $g_{\mu\nu}$, A_μ in Adj. of $E_8 \times E_8$

Table 1.1: The five 10d consistent superstring theories.

where μ runs from 0 to 9, and 'A' runs from 1 to 32. Depending on the choice of boundary conditions to the 32 left-moving fermions, there are two heterotic string theories with spacetime supersymmetry. In the SO(32) heterotic string theory, all the left-moving fermions have the same boundary conditions. In the $E_8 \times E_8$ heterotic string theory, the fermions are split into two groups with two different boundary conditions. We summarize the five superstring theories in Table(1.1). From the beginning of the 1990s, a lot of research progress has been made. One of the most important findings is the various dualities in string theory. The study of the strong-weak dualities suggested that there exists an 11-dimensional theory, called M-theory [8, 9, 10]. All the five superstring theories can be regarded as a limit of it.

Since superstring theory is only consistently defined in ten-dimensional spacetimes, we have to explain what happens to the six extra dimensions to make contact with the real world particle physics. The idea is again based on the Kaluza-Klein mechanism. The six extra dimensions are so tiny that they are undetectable, and the visible spacetime is effectively four-dimensional. This procedure is also called the string theory compactification. In other words, in string compactifications, we assume the ten-dimensional spacetime to be of the form $\mathbb{R}^{1,3} \times \mathbb{M}^6$, where $\mathbb{R}^{1,3}$ is the four-dimensional Minkowski spacetime and $\mathbb{M}^6$ is the six-dimensional compact internal space. Depending on $\mathbb{M}^6$, the resulting theory can have different supersymmetries in the four-dimensional spacetime. By far, the most promising candidates for realistic models are from the so-called 4d $\mathcal{N}$=1 string compactifications, i.e, the string phenomenological models with $\mathcal{N}$=1 supersymmetry in the four-dimensional spacetime. There are two main methods to perform 4d $\mathcal{N}$=1 string compactifications.

The first method is to compactify the heterotic string on a Calabi-Yau threefold. In mathematical language, a Calabi-Yau manifold is a K$\ddot{a}$hler manifold with a Ricci-flat metric. This is the traditional way to achieve 4d $\mathcal{N}$=1 supersymmetry. After compactification, the massless particle states in the physical spacetime are the 4d $\mathcal{N}$=1 supergravity multiplet ($g_{\mu\nu}$ and the gravitino), the 4d $\mathcal{N}$=1 vector superfield (gauge bosons and gauginos in the adjoint representation of gauge group), and other chiral superfields (scalars and fermions). The low energy effective field theory, describing the massless fields at tree level, is very complicated. Depending on the choice of the Calabi-Yau threefolds, one can build many models (For example, see [25, 26, 27, 28]). When we consider the corrections due to the finite α', a lot of hard problems will show up [58].

Another method is called Type II orientifold compactifications with branes and fluxes. When we compactify Type II string theory on Calabi-Yau threefolds, we will get the 4d $\mathcal{N}$=2 supergravity. If we add D-branes into Type II string theory compactifications, we will break the 4d $\mathcal{N}$=2 supersymmetry down to the 4d $\mathcal{N}$=1 supersymmetry and obtain the gauge fields from the open string sector. To cancel the R-R charges carried by D-branes, we need to add the orientifold. An orientifold in string theory is obtained by gauging a discrete symmetry involving the worldsheet parity. In string compactifications, there usually exist a large number of unobserved neutral scalars (called the moduli fields).

Geometrically, their expectation values determine the size and shape of the compact internal space, the gauge coupling, the masses of the SM particles, etc. Their existence creates a big problem. Unless we have some controllable mechanism to determine the expectation values of the moduli fields, the string models have no predictive power. This is the moduli problem of Calabi-Yau compactifications. In recent years, introducing the R-R fluxes into the string compactifications provides a controllable way towards solving the moduli problem [30, 31]. The two classes of 4d $\mathcal{N}$=1 string models can be related to each other through a web of dualities.

In this thesis, we focus on some physical and mathematical problems in Type II orientifold compactifications. We first study the structure of D_9-brane Chan-Paton factors in Type II orientifold $O_p^{\pm}$ theories systematically. We find that the D_9-brane Chan-Paton factors and the orientifold projection operators should satisfy certain constraints which only depend on the type and the dimension of the orientifold plane $O_p^{\pm}$. The rule of writing down the open string scattering amplitudes is also discussed. All of these are necessary and important ingredients towards building the 4d $\mathcal{N}$=1 string phenomenological models. As applications of the above results, we study some mathematical aspects of D-branes. Witten showed that the D-brane charges in Type II theories are classified by the K-theory groups and those in Type I theory are classified by the KO-theory groups [17]. Later on, it was proposed in [21, 22] that the D-brane charges in Type II orientifolds are classified by the KR-theory groups. We confirm the proposal by showing that the set of invariant tachyon configurations under the orientifold projection is isomorphic to the classifying space of the KR-theory group. We also discuss the B-type D-branes in the orientifolds of linear sigma models. The linear sigma model provides a unifying description of various 2d $\mathcal{N}$=(2,2) theories on the worldsheet. In mathematical language, there exist different categories associated to the various phases of a linear sigma model. The orientifold projection induces parity mappings of the various categories. By looking at the transportation of invariant branes in the moduli space, we find the relations of parity mappings among the different phases in the quintic model. Of course, there exist many unsolved problems in building the 4d $\mathcal{N}$=1 string models, which are beyond the scope of this thesis.

The rest of this thesis is organized as follows. First, we give some necessary physical background in chapter 2. In chapter 3, we discuss the orientifold projection, the D-brane Chan-Paton factors, and the scattering amplitudes. In chapter 4, we discuss the isomorphism between the space of invariant tachyon configurations and the classifying space of the KR-theory group. In chapter 5, we discuss the orientifolds in linear sigma models and the orientifold projection as the equivalent mappings among the different categories. Chapter 6 is the conclusion. Some necessary mathematical background and detailed calculations are put in appendix A.

Chapter 2

Review of D-branes and Orientifolds

2.1 D-branes

There are various ways to discuss D-branes in string theory (e.g., see [4, 6, 11, 12]). Here we describe them from the spacetime point of view. When an open string moves in the ten-dimensional spacetime, it will spread a two-dimensional worldsheet with boundaries. The boundaries will be mapped to some submanifolds of spacetime. These submanifolds are called D-branes. In other words, D-branes are submanifolds of spacetime where open strings can end.

Suppose we have a D_p-brane spanning the directions X^μ, $\mu = 0, 1, \cdots, p$ and transverse to the directions X^μ, $\mu = p+1, \cdots, 9$. Consider an open string with both endpoints on the D_p-brane. For the bosonic worldsheet fields, we have the Neumann boundary condition, $\partial_\sigma X^\mu(\sigma, t)|_{\sigma=\pm\pi/2} = 0$, in the directions along the D-brane worldvolume, and the Dirichlet boundary condition, $\partial_t X^\mu(\sigma, t)|_{\sigma=\pm\pi/2} = 0$, in the transverse directions. This is the reason why it is called the D_p-brane.[1] For the fermionic worldsheet fields, there also exist two types of boundary conditions, called the (+)-type and (−)-type. Denote the normal component of the supercurrent by $G^1_{\pm} = \mp\psi_{\pm} \cdot (\partial_t \pm \partial_\sigma)X$. At the right boundary ($\sigma = \frac{\pi}{2}$), the (±)-type boundary conditions are chosen to be $G^1_+ \pm G^1_- = 0$. While at the left boundary ($\sigma = -\frac{\pi}{2}$), the (±)-type boundary conditions are $G^1_+ \mp G^1_- = 0$. In the NS-sector, the boundary conditions at the two boundaries are of the same type, called the (++) and (−−) NS-sector in short. The (++) and (−−) NS-sector are related by $(-1)^{F_R}$ (or by $(-1)^{F_L}$) and hence are gauge equivalent to each other. In the R-sector, the conditions at the two boundaries are of opposite types, called the (−+) or (+−) R-sector. The spectrum of this open string is slightly different from what we have before. In particular, for the massless states in the NS-sector, instead of an $\mathbf{8_V}$ representation, we have a U(1) gauge field A^μ and $9 - p$ real scalar fields propagating in the D-brane worldvolume. In general, we can have N D-branes at different positions. When n of them overlap, we get the gauge group $\mathrm{U(n)}\times\mathrm{U(1)}^{N-n}$. If all of them come together, we get the gauge group U(N).

Actually, D-branes are dynamical objects. They interact with both open strings and closed strings. The open strings and D-branes interact through various open string modes. The closed strings interact with D-branes gravitationally. Away from the D-branes, only closed strings propagate. Of course, in an interacting string theory, the closed strings can break into open strings and open strings can rejoin into closed strings. The closed string has two 10d spacetime supersymmetries, while the open string boundary conditions (i.e, the D-branes) are invariant under at most one 10d spacetime supersymmetry. The D-branes preserving 10d spacetime supersymmetry are called BPS D-branes. The BPS D-branes must carry conserved charges. In string theory, there exist various antisymmetric

[1] In string theory T-duality will map the Neumann boundary condition to the Dirichlet boundary condition and vice versa. Depending on the T-dual direction, a D_p-brane can be transformed to either a D_{p-1} or a D_{p+1} brane.

R-R forms. The worldvolume of a BPS D_p-brane naturally couples to a R-R $C_{(p+1)}$ form,

$$i\mu_p \int_{D_P-\text{brane worldvolume}} C_{p+1}, \tag{2.1}$$

where μ_p denotes the R-R $(p+1)$-form charge of the D_p-brane. It turns out that each BPS D_p-brane carries one unit of the corresponding R-R charge. In this convention, the anti-D_p-brane (denoted by $\bar{D}_p$) has -1 R-R charge. Type IIA theory has BPS D_p-branes with $p = 0, 2, 4, 6, 8$. The higher rank BPS D_p-branes exist because there are R-R C_{10-n} forms, magnetic dual to the C_n forms, in Type IIA theory. Similarly, Type IIB theory has BPS D_p-branes with $p = -1, 1, 3, 5, 7, 9$. A BPS D_{-1}-brane is a Dirichlet instanton, defined by Dirichlet conditions in the time direction as well as all spatial directions.

There also exist non-BPS D-branes in string theory [13, 14, 15]. They are found by studying the coincident D–D̄-brane system. We know that the D-brane and D̄-brane preserve different halves of the 10d supersymmetry. So the D–D̄-brane system is non-supersymmetric and unstable. In other words, its spectrum contains tachyonic modes. This will cause the system to decay. Naively, the D_p–$\bar{D}_p$-brane system (p even for Type IIA and odd for Type IIB) would decay into a closed string vacuum. No interesting thing would happen. In fact, the tachyon field of the system has non-trivial field configurations. The potential for the tachyon mode is of a Mexican-hat type (See Fig(2.1)). It is argued that the negative energy density associated with the tachyon potential at its minimum ($|T| = T_0$) cancels the tension of D_p and $\bar{D}_p$ branes. Although the tachyon field is complex, one can still consider a kink solution with $Im(T) = 0$. From Fig(2.1), it is easy to see that this real kink configuration is unstable and may decay into the vacuum state. This tachyonic kink configuration can be identified as the unstable non-BPS D_{p-1}-brane of the D_p–$\bar{D}_p$-brane system [14]. There exists a real tachyon field in the non-BPS D_{p-1}-brane worldvolume theory. The kink configuration of this real tachyon field can be interpreted as

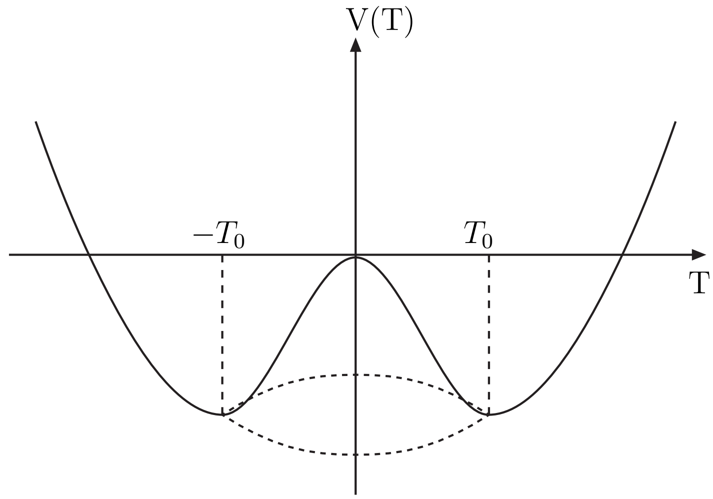

Figure 2.1: The tachyon potential of the D_p–$\bar{D}_p$-brane pair system.

a D_{p-2}-brane. This kink configuration is topologically stable. Thus it gives a stable BPS D_{p-2}-brane. Actually, it can be shown that this two-step kink construction is equivalent to the vortex solution of the D_p–$\bar{D}_p$-brane system. To summarize, Type IIA theory has non-BPS D_p-branes with $p = -1$, 1, 3, 5, 7,9; while Type IIB theory has non-BPS D_p-branes with $p = 0$, 2, 4, 6, 8. Similarly, Type I theory has non-BPS D_p-branes with $p = -1$, 0, 2, 3, 4, 6, 7, 8.

2.2 Orientifolds

An orientifold in string theory is obtained by gauging a discrete symmetry which involves the worldsheet parity [23, 24]. The worldsheet parity Ω is a projection on the string worldsheet Σ that reverses the orientation,

$$\begin{aligned} \text{Closed string}: \quad & \Omega : (\sigma, t) \mapsto (2\pi - \sigma, t); \\ \text{Open string}: \quad & \Omega : (\sigma, t) \mapsto (-\sigma, t). \end{aligned} \tag{2.2}$$

The worldsheet parity will exchange the left-moving and right-moving sectors of a closed string and flip the two endpoints of an open string. In particular, the worldsheet parity action on open string fermionic worldsheet fields is,

$$\Omega : \ \psi^{\mu}_{\pm}(\sigma, t) \rightarrow \mp\psi^{\mu}_{\mp}(-\sigma, t). \tag{2.3}$$

It is clear that the type of open string boundary conditions is preserved by $(\pm 1)^F\Omega$. That means the NS-sector is invariant under $(\pm 1)^F\Omega$ while the R-sector is not. Let us define a dressed worldsheet parity $\tilde{\Omega}$,

$$\tilde{\Omega} = (-1)^{F_R}\Omega : \ \psi^{\mu}_{\pm}(\sigma, t) \rightarrow \psi^{\mu}_{\mp}(-\sigma, t). \tag{2.4}$$

The type of open string boundary conditions is reversed by $(\pm 1)^F\tilde{\Omega}$. So the R-sector is invariant under $(\pm 1)^F\tilde{\Omega}$ and the NS-sector is not. Note that $(\pm 1)^F\Omega$ squares to $(-1)^F$ while $(\pm 1)^F\tilde{\Omega}$ squares to the identity,

$$\Omega^2 = ((-1)^F\Omega)^2 = (-1)^F, \tag{2.5}$$

$$\tilde{\Omega}^2 = ((-1)^F\tilde{\Omega})^2 = id. \tag{2.6}$$

In general, the worldsheet parity Ω can be combined with other symmetries to form a larger symmetry group of a string theory which can be written into the form $G = G_1 \cup \Omega G_2$. G_1 is a subgroup of G consisting of purely internal symmetries of the string worldsheet theory, and G_2 is a collection of target spacetime symmetries. Given a string theory A with a symmetry group G. When gauging the group G, one can construct a new theory $A' = A/G$, called an orientifold of theory A. The group G is called the orientifold group. The states of A' are from the states of A which are invariant under the symmetry group G. The spacetime points which are invariant under the spacetime symmetries G_2 are called fixed points. The set of fixed points under G_2 is called the orientifold plane (denoted by O-plane). An O_p-plane is a $(p+1)$-dimensional hyperplane in the spacetime, with p even for Type IIA orientifolds and p odd for Type IIB orientifolds. Like a BPS D_p-brane, it carries the corresponding R-R charges. The O_p-plane with negative R-R charges is called the O_p^--plane, while the one with positive R-R charges is called the O_p^+-plane. The O-planes are not dynamical objects, i.e, they have no open strings ending on them.

Type I string theory can be regarded as an orientifold theory of Type IIB string theory with orientifold group $G = \mathbb{Z}_2 = \{1, \Omega\}$. The invariant R-R forms are C_2, C_6 and C_{10}. So the BPS D_p-branes in Type I theory are D_1, D_5 and D_9 branes. All other branes are non-BPS branes. Because the spacetime symmetry in G is trivial, the whole spacetime can be regarded as an O_9^--plane. The O_9^--plane carries -32 R-R charges. We need 32 D_9-branes to cancel the R-R tadpole. The 32 D_9-branes give the SO(32) gauge group. By T-duality, a D_p-brane in Type I string theory is equivalent to a D_9-brane in Type II orientifold O_p^- theory. The next chapter will focus on the structure of D_9-brane Chan-Paton factors in Type II orientifold theory.

Chapter 3

Worldsheet Parity and Chan-Paton Factors

In this chapter, we will discuss the structure of D_9-brane Chan-Paton factors[1] in Type II orientifold O_p theories. Of course, by T-duality, it is equivalent to the study of D_p-brane Chan-Paton factors in Type I string theory. The idea is to use the so-called boundary fermions (See [17, 44, 45]) living on the one dimensional worldsheet boundaries of the open string ending on D-branes (See the left part of Fig(3.1)). After Witten's work, the boundary fermions are widely discussed in the boundary (open) string field theory (See [40, 41, 42, 43]). The boundary string field theory focuses on the open string sector. That is, the closed strings are treated as an on-shell background, while one doesn't require the

[1]In open string theory, one can put additional degrees of freedom to open string endpoints. When we quantize the theory, the states of the open string will carry more labels. Each end is labeled by i or j running from 1 to N. In other words, a generic open string state Φ can be written into the form,

$$|\Phi; a\rangle = \sum_{i,j=1}^{N} |\Phi, ij\rangle \lambda^a_{ij}. \tag{3.1}$$

Normally, the $N \times N$ matrices λ^a_{ij} are called the Chan-Paton factors [12].

equation of motion on the open strings. A lot of studies on tachyon condensation and string scattering amplitudes have been done in the framework of boundary string field theory.

This chapter is organized as follows. We first summarize the main results of this chapter: the structure of D_9-brane Chan-Paton factors and orientifold projection operators in Type II string theories in section 3.1. Then in section 3.2 and 3.3, we derive our results in terms of boundary fermions. After that, we discuss how to write down the string scattering amplitudes in section 3.4.

3.1 Summary of The Main Results

3.1.1 Some Linear Algebra

To fix the notation, we first need to give a brief introduction on the duality operations in $\mathbb{Z}_2$-graded vector spaces.

Let V be a ($\mathbb{Z}_2$-)graded finite dimensional complex vector space, i.e, $V = V^0 \oplus V^1$. If v is in V^0($or\ V^1$), the degree of v is $|v|$=0 ($|v|$=1). Suppose V and W are both graded vector spaces. The direct sum of the two is again a graded vector space: $(V \oplus W)^0 = V^0 \oplus W^0$ and $(V \oplus W)^1 = V^1 \oplus W^1$. The tensor product of two graded vector spaces is constructed as $(V \otimes W)^0 = (V^0 \otimes W^0) \oplus (V^1 \otimes W^1)$ and $(V \otimes W)^1 = (V^0 \otimes W^1) \oplus (V^1 \otimes W^0)$. The dual vector space to a graded vector space V is defined by $V^* := Hom(V, \mathbb{C})$, with the natural bracket $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{C}$. Then the double dual vector space to V is defined by

$V^{**} := Hom(V^*, \mathbb{C})$. There is a canonical isomorphism $\iota : V \to V^{**}$, which is defined by

$$\langle \iota(v), w \rangle = (-1)^{|v||w|} \langle w, v \rangle, \ \forall \ v \in V, \ w \in V^*.$$

Consider a graded linear map $A : V_1 \to V_2$ between two graded vector spaces V_1 and V_2. A is called even if $A : V_1^i \to V_2^i$ (the degree of A is $|A|$=0 mod 2). While A is called odd if $A : V_1^i \to V_2^{i+1 \bmod 2}$ (we have $|A|$=1 mod 2). The ($\mathbb{Z}_2$-)graded transpose of A as a linear map $A^T : V_2^* \to V_1^*$ is defined by

$$\langle A^T w_2, w_1 \rangle = (-1)^{|A||w_2|} \langle w_2, A w_1 \rangle, \forall \ w_1 \in V_1, \ w_2 \in V_2^*. \tag{3.2}$$

Note that we use A^t to denote the ordinary transpose of a linear map defined as usual $\langle A^t w_2, w_1 \rangle = \langle w_2, A w_1 \rangle$. If we choose the bases of V_1 and V_2 in such a way that $V_1 = V_1^0 \oplus V_1^1$ and $V_2 = V_2^0 \oplus V_2^1$, then A can be written into the block matrix form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is easy to find that

$$A^T = \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix}.$$

The graded transpose has some different properties compared to the ordinary transpose

$$(A^{-1})^T = (-1)^{|A|} (A^T)^{-1}, \tag{3.3}$$

$$A^{TT} = \iota_2 A \iota_1^{-1}, \tag{3.4}$$

$$(BA)^T = (-1)^{|A||B|} A^T B^T, \tag{3.5}$$

where A and B are linear maps $A : V_1 \to V_2$ and $B : V_2 \to V_3$. The tensor product of the graded transposed maps is constructed as follows. Suppose we have linear maps $A_1 : V_1 \to W_1$ and $A_2 : V_2 \to W_2$, then the tensor product map $A_1 \otimes A_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ is defined by

$$(A_1 \otimes A_2)(v_1 \otimes v_2) = (-1)^{|A_2||v_1|}(A_1 v_1) \otimes (A_2 v_2). \tag{3.6}$$

It is easy to show that the graded transposed map $(A_1 \otimes A_2)^T : W_1^* \otimes W_2^* \to V_1^* \otimes V_2^*$ satisfies

$$(A_1 \otimes A_2)^T = A_1^T \otimes A_2^T. \tag{3.7}$$

3.1.2 The D_9-brane Chan-Paton Factors in Type II Orientifolds

D_9-branes are important in Type II string theories. It is known that any lower dimensional D-brane in Type IIB string theory can be realized as a configuration of tachyon and gauge fields on a system of D_9-$\bar{D}_9$-brane pairs. D_9-branes support a gauge bundle E^0, and $\bar{D}_9$-branes support another gauge bundle E^1. It is convenient to write them into a $\mathbb{Z}_2$-graded vector bundle on $\mathbb{X}$,

$$\mathbf{E} = E^0 \oplus E^1,$$

with the $\mathbb{Z}_2$-grading operator

$$\sigma_{\mathbf{E}} = \begin{pmatrix} \mathrm{id}_{E^0} & 0 \\ 0 & -\mathrm{id}_{E^1} \end{pmatrix}.$$

The tachyon becomes an odd endomorphism of $\mathbf{E}$. It is assumed to be hermitian, $\mathcal{T}^\dagger = \mathcal{T}$. The gauge field $\mathcal{A}$ is an even unitary connection of $\mathbf{E}$. They can be written as

$$\mathcal{T} = \begin{pmatrix} 0 & T^\dagger \\ T & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} A^0 & 0 \\ 0 & A^1 \end{pmatrix}. \tag{3.8}$$

Any lower dimensional D-brane in Type IIA string theory can be realized as a configuration of tachyon and gauge fields on a non-BPS D_9-brane system. Non-BPS D_9-branes support a gauge bundle E without a $\mathbb{Z}_2$-grading. The tachyon T is an endomorphism of E, which is hermitian $T^\dagger = T$. The gauge field A is a unitary connection of E. It is convenient to introduce a Z_2-graded double of E,

$$\mathbf{E} = E \oplus E,$$

and an odd operator

$$\xi := i \begin{pmatrix} 0 & -\mathrm{id}_E \\ \mathrm{id}_E & 0 \end{pmatrix}.$$

The gauge field $\mathcal{A}$ commutes with ξ and the tachyon $\mathcal{T}$ anticommutes with ξ. Then the tachyon and gauge field on $\mathbf{E}$ can be written as

$$\mathcal{T} = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}. \tag{3.9}$$

The main result of this chapter is about the structure of D_9-brane Chan-Paton factors in Type II orientifolds on $\mathbb{R}^{10}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on the coordinates as

$$\mathcal{I} : x^\mu \mapsto \begin{cases} x^\mu, & \mu = 0, 1, \cdots, p, \\ -x^\mu, & \mu = p+1, \cdots, 9. \end{cases} \tag{3.10}$$

The set of fixed points under this involution is the O_p-plane. The tachyon and gauge field configurations should be invariant under the orientifold projection, which is the combination of $\mathcal{I}$ and the worldsheet parity Ω. This means that there exists a linear isomorphism $U : \mathcal{I}^*\mathbf{E}^* \to \mathbf{E}$ such that

$$\mathcal{T} = \zeta(-1)^{|U|} U \mathcal{I}^* \mathcal{T}^T U^{-1}, \tag{3.11}$$

$$\mathcal{A} = U(-\mathcal{I}^* \mathcal{A}^T) U^{-1} + i^{-1} U dU^{-1}. \tag{3.12}$$

The factor ζ in eq(3.11) comes from the time reversal operation on the worldsheet fermions, which is $\mp i$ for the $(\pm)$-type boundary conditions, respectively.

It turns out that the isomorphism U must obey a constraint that depends on the value of p as well as the type of the orientifold p-plane. Take k=$(9-p)$ modulo 8 for an O_p^- plane and k=$(5-p)$ mod 8 for an O_p^+ plane. The constraint is

$$U(\mathcal{I}^* U^T)^{-1} \iota = \zeta^{n_k} \sigma_{\mathbf{E}}, \tag{3.13}$$

in which

$$n_k = \begin{cases} \frac{k}{2}, & k \text{ even (IIB)}, \\ \frac{k+1}{2}, & k \text{ odd (IIA)}. \end{cases} \tag{3.14}$$

In addition, for Type IIA orientifolds, we require that U obeys

$$(-1)^{|U|} U \xi^T U^{-1} = \zeta \xi. \tag{3.15}$$

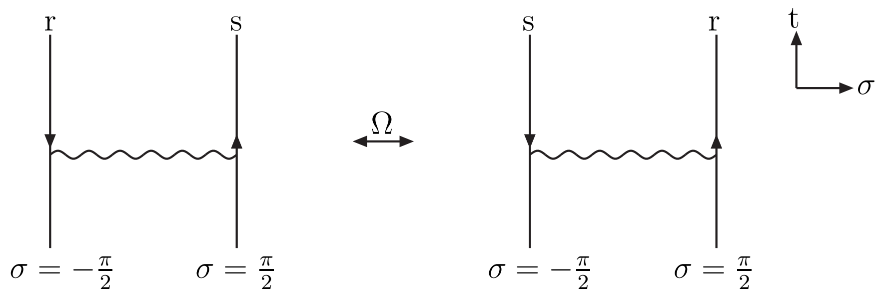

Figure 3.1: The parity action on the open string with boundary fermions $\eta^L_{i'}$ and η^R_i.

One can show that eq(3.13) leads to

$$U\mathcal{I}^*\sigma_{\mathbf{E}}^T U^{-1} = (-1)^{n_k}\sigma_{\mathbf{E}}. \tag{3.16}$$

In other words,

$$(-1)^{|U|} = (-1)^{n_k} = \begin{cases} 1, & k = 0, 4, 3, 7, \\ -1, & k = 1, 5, 2, 6. \end{cases} \tag{3.17}$$

3.2 Boundary Fermions and Parity Actions

Consider an open string with r real boundary fermions $\eta^L_{i'}$ ($i' = 1, 2, \cdots, r$) on the left boundary ($\sigma = -\frac{\pi}{2}$), and s real boundary fermions η^R_i ($i = 1, 2, \cdots, s$) on the right boundary ($\sigma = \frac{\pi}{2}$). Take the orientation of the worldsheet to be upward (t increasing) on the right and downward (t decreasing) on the left (See Fig(3.1)). The action on the right boundary takes the form

$$S_{\text{right bdry}} = \int_{-\infty}^{\infty} dt \left\{ \frac{i}{4} \sum_{i=1}^{s} \eta^R_i \frac{d}{dt}\eta^R_i + ... \right\}, \tag{3.18}$$

where the ellipses stand for the interaction terms which will be discussed in detail when we study the scattering amplitude. The action on the left boundary is of the form

$$S_{\text{left bdry}} = \int_{-\infty}^{\infty} dt' \left\{ \frac{i}{4} \sum_{i'=1}^{r} \eta_{i'}^{L} \frac{d}{dt'} \eta_{i'}^{L} + ... \right\} = \int_{-\infty}^{\infty} dt \left\{ -\frac{i}{4} \sum_{i'=1}^{r} \eta_{i'}^{L} \frac{d}{dt} \eta_{i'}^{L} + ... \right\}, \qquad (3.19)$$

where t' is the time coordinate in the opposite orientation. If we quantize this boundary fermion system in the orientation of t, the canonical commutation relations are

$$\{\eta_{i'}^{L}, \eta_{j'}^{L}\} = -2\delta_{i',j'}, \quad \{\eta_{i}^{R}, \eta_{j}^{R}\} = 2\delta_{i,j}, \quad \{\eta_{i}^{R}, \eta_{j'}^{L}\} = 0, \qquad (3.20)$$

and the hermiticity is

$$\eta_{i}^{R\,\dagger} = \eta_{i}^{R}, \qquad \eta_{i'}^{L\,\dagger} = -\eta_{i'}^{L} \qquad (3.21)$$

The relations (3.20) are those of the Clifford algebra $Cl_{r,s}$. If we quantized the system in the orientation of t', the commutation relation and the herniticity would be inverted.

The space of open string states is the product of the Chan-Paton factor space and the bulk Hilbert space,

$$\mathcal{H}^{tot}_{(r,\alpha),(s,\beta)} = \mathcal{H}^{cp}_{r,s} \otimes \mathcal{H}^{bulk}_{\alpha,\beta} \qquad (3.22)$$

Here α and β are labels for boundary conditions on the bulk fields. One important problem is to define a $\mathbb{Z}_2$-grading operator $(-1)^F$ on the space of open string states that anticommutes with $\eta_{i'}^{L}, \eta_{i}^{R}$ as well as the bulk fermions. Actually, if $r+s$ is even, we can define a $\mathbb{Z}_2$-grading operator on the Chan-Paton part and the bulk part separately. If $r+s$ is odd, the bulk part must have an unpaired fermionic mode so that the total system has a unitary irreducible representation with a $\mathbb{Z}_2$-grading operator.

Let us now look at the worldsheet parity action on the boundary fermions. Take a trivial involution on the flat space $\mathbb{R}^{10}$. The parity acts on the bulk fields as $X^{\mu}(\sigma,t) \rightarrow$

$X^\mu(-\sigma,t)$ and $\psi^\mu_\pm(\sigma,t) \to \varepsilon_\pm \psi^\mu_\mp(-\sigma,t)$, where $\varepsilon_\pm$ are signs. This orientifold theory is nothing but the Type I string theory. We are interested in D-branes which are invariant under this orientifold projection. This requires that the parity action on the boundary fermions maps the boundary action on the left to the one on the right, vice versa. Since η_i^R's are hermitian and η_i^L's are antihermitian, we find that the transformation is $i = \sqrt{-1}$ times an orthogonal transformation. First consider the case of a single real fermion, $s = 1$, for which the transformation must be of the form

$$\eta^R \longrightarrow \zeta\eta^L, \quad \eta^L \longrightarrow \zeta\eta^R, \tag{3.23}$$

where $\zeta = \pm i$. The phase ζ is determined by the type of the boundary condition. The phase for the parity on the (+)-type boundary condition must be opposite to the one for the parity on the (−)-type boundary condition. This is to ensure the relations (2.5) and (2.6). As a simple example, it is easy to check that Ω^2 acts on η as a minus sign

$$\eta^{R,L}|_\pm \stackrel{\Omega}{\longrightarrow} \pm\zeta\eta^{L,R}|_\pm \stackrel{\Omega}{\longrightarrow} \pm\zeta\cdot(\pm\zeta)\eta^{R,L}|_\pm = -\eta^{R,L}|_\pm, \tag{3.24}$$

and $\tilde{\Omega}^2$ acts on η as the identity

$$\eta^{R,L}|_\pm \stackrel{\tilde{\Omega}}{\longrightarrow} \pm\zeta\eta^{L,R}|_\mp \stackrel{\tilde{\Omega}}{\longrightarrow} \pm\zeta\cdot(\mp\zeta)\eta^{R,L}|_\pm = \eta^{R,L}|_\pm. \tag{3.25}$$

For a general number s, the parity action must be of the form $\eta_i^{R,L} \to \zeta\sum_j O_{ij}\eta_i^{L,R}$ for an orthogonal matrix O_{ij} which squares to the identity. With a change of basis, it can be written as

$$\eta_i^R \longrightarrow \zeta o_i \eta_i^L, \quad \eta_i^L \longrightarrow \zeta o_i \eta_i^R, \tag{3.26}$$

for some signs $o_i = \pm 1$. The numbers s_+ and s_- of $o_i = 1$ and $o_i = -1$ eigenvalues are the invariants of the parity transformation.

Once the above notation for the parity action on the boundary fermions is fixed, let us proceed to discuss the parity action on open string states for an open string that carries s and r fermions on the right and left boundaries. We will consider three different cases.

(i) r and s even

In this case, we can quantize the boundary fermions on the left and right boundaries separately. The boundary fermions on the right boundary generates a unitary irreducible representation $\mathcal{H}_s$ of the Clifford algebra $Cl_{0,s}$. The construction is the standard one. Take the complex combinations of the generators $\chi_i = (\eta_{2i-1} + i\eta_{2i})/2$, $\bar{\chi}_i = (\eta_{2i-1} - i\eta_{2i})/2$ for $i = 1, 2, \cdots, s/2$, which obey the commutation relations

$$\{\chi_i, \bar{\chi}_j\} = \delta_{i,j}, \quad \{\chi_i, \chi_j\} = \{\bar{\chi}_i, \bar{\chi}_j\} = 0.$$

The $\mathcal{H}_s$ is spanned by the vectors $|0\rangle, \bar{\chi}_i|0\rangle, \cdots, \bar{\chi}_1 \cdots \bar{\chi}_{s/2}|0\rangle$, where $|0\rangle$ is a vector annihilated by all χ_i's. This representation has dimension $2^{s/2}$. It has a $\mathbb{Z}_2$-grading. For example, even multiples of $\bar{\chi}_i$ on $|0\rangle$ are even and odd multiples are odd. Similarly, the boundary fermions on the left boundary, when quantized with respect to the time t', generate the representation $\mathcal{H}_r$ of $Cl_{0,r}$.

The open string Chan-Paton factor space is then the space of linear homomorphisms

$$\mathcal{H}^{cp}_{r,s} = Hom(\mathcal{H}_r, \mathcal{H}_s). \tag{3.27}$$

Since both $\mathcal{H}_r$ and $\mathcal{H}_s$ are $\mathbb{Z}_2$-graded vector spaces, so is $\mathcal{H}^{cp}_{r,s}$. The boundary fermion operators act on the Chan-Paton factor space as

$$\eta_i^R h_{cp} = \eta_i \cdot h_{cp}, \quad \eta_{i'}^L h_{cp} = (-1)^{|h_{cp}|} h_{cp} \cdot \eta_{i'}, \tag{3.28}$$

for any $h_{cp} \in \mathcal{H}_{r,s}^{cp}$. It is easy to check that this operator action satisfies the commutation relation (3.20) and the hermiticity (3.21).

The parity transforms the boundary fermions as $\eta_i^R \to \zeta o_i \eta_i^L$ for $i = 1, \cdots, s$ and $\eta_{i'}^L \to \zeta' o'_{i'} \eta_{i'}^R$ for $i' = 1, \cdots, r$. We can write the parity operator into the factorized form

$$P := P(\Omega): \ h_{cp} \otimes h_{bk} \mapsto (-1)^{|h_{cp}||P_{bk}|} P_{cp}(h_{cp}) \otimes P_{bk}(h_{bk}). \tag{3.29}$$

We require that it is even,

$$(-1)^{|P|} = (-1)^{|P_{cp}|}(-1)^{|P_{bk}|} = 1. \tag{3.30}$$

This means that we have $|P_{cp}| = |P_{bk}|$. Focus on the Chan-Paton part, $P_{cp} : Hom(\mathcal{H}_r, \mathcal{H}_s) \to Hom(\mathcal{H}_s, \mathcal{H}_r)$. We assume the form

$$P_{cp}(h_{cp}) = (-1)^{|h_{cp}||U|} U' \cdot h_{cp}^T \cdot U^{-1}, \tag{3.31}$$

for linear maps

$$U' : \mathcal{H}_r^* \longrightarrow \mathcal{H}_r \ \text{ and } \ U : \mathcal{H}_s^* \longrightarrow \mathcal{H}_s. \tag{3.32}$$

These linear maps can be determined by the requirement that the transformation rule of the variables η_i^R and $\eta_{i'}^L$ must be realized for the operator action (3.28). That is,

$$U \eta_i^T U^{-1} = (-1)^{|U|} \zeta o_i \eta_i, \tag{3.33}$$

and similarly for U'. This condition can fix U up to an overall constant multiplication. Let us explicitly construct such U for the case where all the signs o_i are $+1$ (i.e, $s_+ = s$ and $s_- = 0$). Rewrite the condition (3.33) in terms of the complex combinations $\chi_i = (\eta_{2i-1} + i\eta_{2i})/2$ and $\bar{\chi}_i = (\eta_{2i-1} - i\eta_{2i})/2$ for $i = 1, \cdots, \frac{s}{2} := \bar{s}$,

$$U\chi_i^T U^{-1} = (-1)^{|U|}\zeta\chi_i, \quad U\bar{\chi}_i^T U^{-1} = (-1)^{|U|}\zeta\bar{\chi}_i. \tag{3.34}$$

From the first set of equations, we find $\chi_i^T U^{-1}|0\rangle = 0$, which means that $U^{-1}|0\rangle$ is proportional to $\langle 0|\chi_1 \cdots \chi_{\bar{s}}$. This shows that $(-1)^{|U|} = (-1)^{\bar{s}} = (-1)^{\frac{s}{2}}$. Using the rest of the conditions, we find that

$$U^{-1}\bar{\chi}_{i_1} \cdots \bar{\chi}_{i_a}|0\rangle = (-\zeta)^a k\langle 0|\chi_1 \cdots \chi_{\bar{s}}\bar{\chi}_{i_1} \cdots \bar{\chi}_{i_a}, \tag{3.35}$$

for some constant k. Construction of U for a more general o_i is equally straightforward. One important result is

$$(-1)^{|U|} = (-1)^{\frac{s_+ - s_-}{2}}. \tag{3.36}$$

The most important constraint comes from the algebra of the parity operators. Note that the algebraic relations (2.5) and (2.6) for the transformations $(\pm 1)^F\Omega$ and $(\pm 1)^F\tilde{\Omega}$ will be promoted to conditions on the corresponding parity operators

$$P^2 = P(\Omega)^2 = (-1)^F, \tag{3.37}$$

$$\tilde{P}^2 = P(\tilde{\Omega})^2 = id. \tag{3.38}$$

Let us first check the condition (3.37). The P^2 acts on a state $h_{cp} \otimes h_{bk} \in \mathcal{H}^{cp}_{r,s} \otimes \mathcal{H}^{bulk}_{\alpha,\beta}$ as

$$\begin{aligned} h_{cp} \otimes h_{bk} &\stackrel{P}{\longmapsto} (-1)^{|h_{cp}||P_{bk}|}(-1)^{|h_{cp}||U|}U' h_{cp}^T U^{-1} \otimes P_{bk}(h_{bk}) \\ &\stackrel{P}{\longmapsto} (-1)^{|h_{cp}|(|P_{bk}|+|U|)}(-1)^{|U' h_{cp}^T U^{-1}|(|P_{bk}|+|U'|)}U(U' h_{cp}^T U^{-1})^T U'^{-1} \otimes P_{bk}^2(h_{bk}). \end{aligned}$$

Using $(U' h_{cp}^T U^{-1})^T = (-1)^{|U||U'|+|h_{cp}||U|+|h_{cp}||U'|}(U^{-1})^T h_{cp}^{TT} U'^T$, we find that

$$\begin{aligned} P^2(h_{cp} \otimes h_{bk}) &= (-1)^{|U'|+|P_{bk}|}U(U^{-1})^T h_{cp}^{TT} U'^T U'^{-1} \otimes P_{bk}^2(h_{bk}) \\ &= U(U^T)^{-1} h_{cp}^{TT} U'^T U'^{-1} \otimes P_{bk}^2(h_{bk}). \end{aligned}$$

In the second equality, we used the general identity $(U^{-1})^T = (-1)^{|U|}(U^T)^{-1}$ and $(-1)^{|P_{bk}|} = (-1)^{|P_{cp}|} = (-1)^{|U|+|U'|}$. Finally, using the relation $h_{cp}^{TT} = \iota h_{cp} \iota'^{-1}$ for the canonical isomorphisms $\iota : \mathcal{H}_s \to \mathcal{H}_s^{**}$ and $\iota' : \mathcal{H}_r \to \mathcal{H}_r^{**}$, we find that

$$P^2(h_{cp} \otimes h_{bk}) = U(U^T)^{-1} \iota h_{cp} (U'(U'^T)^{-1}\iota')^{-1} \otimes P_{bk}^2(h_{bk}). \tag{3.39}$$

We can further fix $U(U^T)^{-1}\iota$. It turns out that

$$U(U^T)^{-1}\iota = \zeta^{\frac{s_+ - s_-}{2}} \sigma_{\mathcal{H}_s}. \tag{3.40}$$

The computation is the following. To be explicit, take all the o_i's to be $+1$. Compute the pairing $\langle (U^T)^{-1}\iota v, w\rangle$ for $v = \bar{\chi}_{i_1} \cdots \bar{\chi}_{i_a}|0\rangle$ and $w = \bar{\chi}_{j_1} \cdots \bar{\chi}_{j_b}|0\rangle$:

$$\begin{aligned} \langle (U^T)^{-1}\iota v, w\rangle &= (-1)^{|U|}\langle (U^{-1})^T \iota v, w\rangle \\ &= (-1)^{a|U|+|U|}\langle \iota v, U^{-1} w\rangle \\ &= (-1)^{a|U|+|U|}(-1)^{a(b+|U|)}\langle U^{-1} w, v\rangle \\ &= (-1)^{ab+|U|}(-\zeta)^b k \langle 0|\chi_1 \cdots \chi_{\bar{s}} \bar{\chi}_{j_1} \cdots \bar{\chi}_{j_b} \bar{\chi}_{i_1} \cdots \bar{\chi}_{i_a}|0\rangle \end{aligned}$$

In the last equality, we used eq(3.35). On the other hand, we have

$$\begin{aligned}\langle U^{-1}\sigma_{\mathcal{H}_s}v, w\rangle &= (-1)^a\langle U^{-1}v, w\rangle\\ &= (-1)^a(-\zeta)^a k\langle 0|\chi_1\cdots\chi_{\bar{s}}\bar{\chi}_{i_1}\cdots\bar{\chi}_{i_a}\bar{\chi}_{j_1}\cdots\bar{\chi}_{j_b}|0\rangle\\ &= (-1)^{a+ab}(-\zeta)^a k\langle 0|\chi_1\cdots\chi_{\bar{s}}\bar{\chi}_{j_1}\cdots\bar{\chi}_{j_b}\bar{\chi}_{i_1}\cdots\bar{\chi}_{i_a}|0\rangle\end{aligned}$$

Note that these are non-zero only when $a+b=\bar{s}$. We find that

$$\langle (U^T)^{-1}\iota v, w\rangle = (-1)^{a+|U|}\zeta^{a-b}\langle U^{-1}\sigma_{\mathcal{H}_s}v, w\rangle = \zeta^{\bar{s}}\langle U^{-1}\sigma_{\mathcal{H}_s}v, w\rangle.$$

This gives eq(3.40) for the case $s_+ = 2\bar{s}$ and $s_- = 0$. Computation for other cases is easy to carry out. One important remark is that the result (3.40) does not depend on the choice of the $\mathbb{Z}_2$-grading. If the grading is inverted $\sigma_{\mathcal{H}_s} \to -\sigma_{\mathcal{H}_s}$, then we find $\iota \to -\iota$ as well as $U^T \to (-1)^{|U|}U^T$. And moreover, s_+ and s_- are exchanged since the graded transpose of odd elements η_i^T appears in the condition (3.33). All together, the eq(3.40) remains invariant.

Using the result (3.40), we find that P^2 acts on the state as

$$P^2(h_{cp}\otimes h_{bk}) = \zeta^{\frac{\Delta s}{2}}\zeta'^{-\frac{\Delta r}{2}}(-1)^{F_{cp}}h_{cp}\otimes P_{bk}^2(h_{bk}), \tag{3.41}$$

where $\Delta s = s_+ - s_-$, $\Delta r = r_+ - r_-$ and $(-1)^{F_{cp}}$ is the $\mathbb{Z}_2$-grading operator of the Chan-Paton factor space, $(-1)^{F_{cp}}h_{cp} = \sigma_{\mathcal{H}_s}h_{cp}\sigma_{\mathcal{H}_r}^{-1}$. Especially, the phase factor is

$$\zeta^{\frac{\Delta s}{2}}\zeta'^{-\frac{\Delta r}{2}} = \zeta^{\frac{\Delta s-\Delta r}{2}} \tag{3.42}$$

in the NS-sector in which $\zeta = \zeta'$. In order to have the relation (3.37), we need P_{bk}^2 to be equal to $(-1)^{F_{bk}}$ times a phase that compensates $\zeta^{\frac{\Delta s}{2}}\zeta'^{-\frac{\Delta r}{2}}$. This imposes a strong constraint on the boundary conditions for given values of Δs and Δr (mod 4).

Let us next consider the parity square, $\tilde{P}^2 = P(\tilde{\Omega})^2$. The calculation is slightly different from the one for P^2. Note that the U's in the second operation of $\tilde{P}$ must be the ones for the opposite values of ζ's. Looking at the condition (3.33), we find that the U for the opposite ζ may be obtained by multiplication by the $\mathbb{Z}_2$-grading operator $\sigma_{\mathcal{H}_s}$

$$\zeta \to -\zeta \Longleftrightarrow U \to \sigma_{\mathcal{H}_s} U. \tag{3.43}$$

A straightforward computation shows that

$$\begin{aligned} \tilde{P}^2(h_{cp} \otimes h_{bk}) &= \sigma_{\mathcal{H}_s} U (U^T)^{-1} \iota h_{cp} (\sigma_{\mathcal{H}_r} U' (U'^T)^{-1} \iota')^{-1} \otimes \tilde{P}^2_{bk}(h_{bk}) \\ &= \zeta^{\frac{\Delta s}{2}} \zeta'^{-\frac{\Delta r}{2}} h_{cp} \otimes \tilde{P}^2_{bk}(h_{bk}). \end{aligned} \tag{3.44}$$

Thus, the relation (3.38) holds if $\tilde{P}^2_{bk}$ is the identity times a phase that compensates $\zeta^{\frac{\Delta s}{2}} \zeta'^{-\frac{\Delta r}{2}}$.

(ii) r and s odd

In this case, if we try to quantize the boundary fermion system, we would get a representation of the Clifford algebra $Cl_{0,n}$ with odd n. However, it is known that the Clifford algebra $Cl_{0,n}$ with odd n does not have a $\mathbb{Z}_2$-graded irreducible representation. On the other hand, it would be convenient to have a $\mathbb{Z}_2$-graded Chan-Paton vector space. For this purpose, we introduce auxiliary boundary fermions, one at each boundary—η^R_{aux} on the right and η^L_{aux} on the left. We assume that the signs of the auxiliary fermions' kinetic terms are natural with respect to the orientations of the respective boundaries. Quantizing the system of boundary fermions and auxiliary fermions, we will get a space

$Hom(\mathcal{H}_{r+1}, \mathcal{H}_{s+1})$ which is an irreducible representation of the Clifford algebra $Cl_{r+1,s+1}$. The Chan-Paton factor space is enlarged by the auxiliary fermions. To find an irreducible representation of the original fermion algebra (3.20), we need to impose a projection condition on $Hom(\mathcal{H}_{r+1}, \mathcal{H}_{s+1})$,

$$u := \eta^R_{aux}\eta^L_{aux} = 1. \tag{3.45}$$

Using the Clifford algebra relations, it is easy to see that $u^2 = 1$ and that u commutes with all the original boundary fermions η^R_i and $\eta^L_{i'}$, but anticommutes with the auxiliary boundary fermions η^R_{aux} and η^L_{aux}. In particular, the $u = 1$ subspace of $Hom(\mathcal{H}_{r+1}, \mathcal{H}_{s+1})$ is isomorphic to the irreducible representation of the algebra (3.20). In other words, the Chan-Paton factor space can be realized as

$$\mathcal{H}^{cp}_{r,s} = Hom(\mathcal{H}_{r+1}, \mathcal{H}_{s+1})|_{u=1}. \tag{3.46}$$

Note that the $u = 1$ condition amounts to

$$\eta_{aux} h_{cp} = (-1)^{|h_{cp}|} h_{cp} \cdot \eta_{aux}. \tag{3.47}$$

Further more, $u = 1$ has a distinguished meaning only for the NS sector of the open string stretched between the same D-brane. For a string stretched between different branes or between different types of boundary conditions, we may as well replace $u = 1$ by $u = -1$, since the two conditions can be exchanged by a sign flip of one of the η_{aux}'s. This seems to imply that there is an ambiguity in choosing the projection condition for an open string. Actually, there is no such ambiguity, which will be discussed in detail in section 3.4.2.

We can extend the parity transformation to auxiliary fermions as $\eta^R_{aux} \to \zeta\eta^L_{aux}$ and $\eta^L_{aux} \to \zeta'\eta^R_{aux}$ where ζ and ζ' depend on the types of boundary conditions. In the NS-sector ($\zeta = \zeta'$) the $u = 1$ condition is invariant under the parity while in the R-sector ($\zeta = -\zeta'$) the $u = 1$ condition is mapped to the $u = -1$ condition. With this remark in mind, the parity operator can be constructed in the similar way as in case **(i)**. In particular, we see that the square of parity $P = P(\Omega)$ is

$$P^2(h_{cp} \otimes h_{bk}) = \zeta^{\frac{\Delta s+1}{2}} \zeta'^{-\frac{\Delta r+1}{2}} (-1)^{F^{ext}_{cp}} h_{cp} \otimes P^2_{bk}(h_{bk}), \tag{3.48}$$

where $(-1)^{F^{ext}_{cp}}$ is the $\mathbb{Z}_2$-grading operator of the extended Chan-Paton factor space. Again, in order to obey the relation (3.37), we need P^2_{bk} to be equal to $(-1)^{F_{bk}}$ times a phase that compensates $\zeta^{\frac{\Delta s+1}{2}} \zeta'^{-\frac{\Delta r+1}{2}}$.

(iii) r even and s odd (resp. r odd and s even)

As we know, the Chan-Paton part does not have a $\mathbb{Z}_2$-grading in this case. There must exist an unpaired fermionic mode in the bulk part so that the total system can have a $\mathbb{Z}_2$-grading operator $(-1)^F$. Just as before, it is convenient to have a $\mathbb{Z}_2$-graded Chan-Paton vector space at each boundary. We may introduce auxiliary fermions and impose some projection condition. Put one real auxiliary fermion η_{aux} on the right boundary with odd s (resp. on the left boundary with odd r) and introduce another auxiliary fermion η'_{aux} to the bulk part. We assume that η_{aux} and η'_{aux} have opposite orientations, i.e, have kinetic terms of opposite signs. Quantizing the system of the original boundary fermions and the auxiliary one η_{aux}, we get an extended and $\mathbb{Z}_2$-graded Chan-Paton factor space

$$\mathcal{H}^{cp+aux}_{r,s} = Hom(\mathcal{H}_r, \mathcal{H}_{s+1}) \quad (resp.\ Hom(\mathcal{H}_{r+1}, \mathcal{H}_s)). \tag{3.49}$$

Likewise, the bulk part extended by η'_{aux} also has a $\mathbb{Z}_2$-grading. Again, the total space of open string states is enlarged. To get rid of the extra degrees of freedom coming from the auxiliary fermions, we need to impose the projection condition $u' := \eta_{aux}\eta'_{aux} = 1$ on the extended space. After the projection, the original total space of states can be written as

$$\mathcal{H}^{tot}_{(r,\alpha),(s,\beta)} = \mathcal{H}^{cp+aux}_{r,s} \otimes \mathcal{H}^{bulk+aux}_{\alpha,\beta}|_{u=1}. \tag{3.50}$$

We may replace the condition $u' = 1$ by $u' = -1$. Again, there is no arbitrariness in choosing the projection condition for an open string. The determination of the projection condition will be discussed in section 3.4.2.

We will discuss the parity action of this case in the next section. As a rule, in the NS-sector ($\zeta = \zeta'$) the $u' = 1$ condition is invariant under the parity while in the R-sector ($\zeta = -\zeta'$) the $u' = 1$ condition is mapped to the $u' = -1$ condition.

3.3 D-branes and Parity Actions

We have discussed the worldsheet parity action on boundary fermions. Of course, what we are interested in is the worldsheet parity action on D-branes. So we have to find out a relation between boundary fermions and D-branes.

This was first discussed in Witten's paper [17]. It is known that there exist non-BPS D-branes in Type I theory. There are also 32 D_9-branes, giving rise to the $SO(32)$ gauge group. Witten considered how to quantize the open strings spanned between D_0-branes and D_9-branes. (For briefness, we denote the open string spanned between D_p and D_q branes by the p-q string.) Suppose the D_0-branes are located at $x^1 = \cdots x^9 = 0$. In the NS sector of the 0-9 string, there are 9 bulk fermion zero modes, ψ^i_0, $i = 1, \cdots, 9$. There is no satisfactory quantization of a system with odd number of fermion zero modes because of the failure of defining a GSO-projection operator $(-1)^F$. To solve this problem, Witten made a proposal that there exist a boundary fermion field $\eta(t)$ on the boundary of an open string that ends on D_0-branes. We will generalize Witten's proposal by a careful study of the parity action on various p-q open strings.

Let us first discuss the parity action on a 9-8 open string[2]. Of course, D_9-brane fills the whole spacetime. The worldsheet bosonic fields, $X^\mu(\sigma,t)$ for $\mu=0,\cdots,9$, obey the Neumann boundary condition at the left boundary $\sigma=-\frac{\pi}{2}$. Suppose the D_8-brane is located at $x^9=0$. Then the fields, $X^\mu(\sigma,t)$ for $\mu=0,\cdots,8$, obey the Neumann boundary condition and $X^9(\sigma,t)$ obey the Dirichlet boundary condition at the right boundary $\sigma=\frac{\pi}{2}$. For short, let us call x^9 the ND direction and x^μ $(\mu=0,\cdots,8)$ the NN directions. The T-duality will change a ND direction to a DN direction, and a NN direction to a DD direction. Focus on the worldsheet fermionic field in the ND direction, $\psi_\pm := \psi^9_\pm$. In the $(++)$ NS-sector, the mode expansion for $\psi_\pm$ is

$$\psi_\pm(\sigma,t)=\pm\sum_{n\in\mathbb{Z}}\psi_n^{(9-8)}\mathrm{e}^{\mp in(\sigma-\frac{\pi}{2}\pm t)} \tag{3.51}$$

The modes obey $\psi_n^{(9-8)\,\dagger}=\psi_{-n}^{(9-8)}$ and $\{\psi_n^{(9-8)},\psi_m^{(9-8)}\}=\delta_{n+m,0}$. We need to consider how to define a GSO-projection operator $(-1)^{F^{(9-8)}}$ for the $\psi_\pm$ system of the 9-8 string, which should anticommutate with all the fermionic fields. It is easy to find that $(-1)^{F^{(9-8)}_{nz}} = \mathrm{e}^{\pi i\sum_{n\geq 1}\psi_{-n}^{(9-8)}\psi_n^{(9-8)}}$ is the right one for the non-zero modes. However, there is no way to define $(-1)^F$ for the single real zero mode $\psi_0^{(9-8)}$. The solution is to associate a free real fermion η to the D_8-brane, which satisfies $\{\eta,\psi_n^{(9-8)}\}=0$ and $\eta^2=1$. The full GSO-projection $(-1)^{F^{(9-8)}}$ is now defined to be $\sqrt{2}i\eta\psi_0^{(9-8)}(-1)^{F^{(9-8)}_{nz}}$. Once we have a consistent definition of $(-1)^F$, it is a standard procedure to do the GSO projection [46].

[2]By T-duality, it is equivalent to a p-q string with $|p-q|$=1.

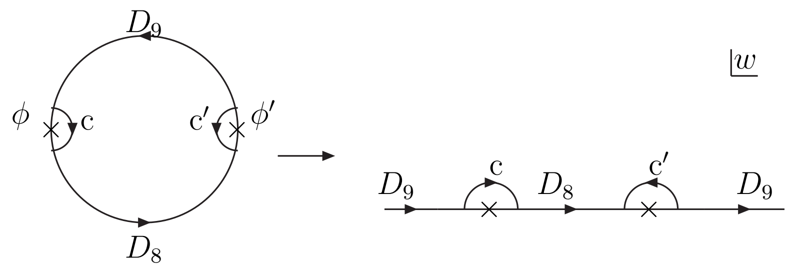

Figure 3.2: The determination of the GSO-projection for the 9-8 string.

Note that once the GSO-projection for the 9-8 string is fixed as above, the GSO-projection for the 8-9 string is also fixed. The derivation is as follows. Consider an amplitude, shown in the left part of Fig(3.2). Suppose the inserted vertex operator ϕ of the 9-8 string is the $(-1)^{F^{(9-8)}} = 1$ ground state $|+\rangle_{9-8}$. There is also an operator $\mathcal{O} = (\sqrt{2}\psi_0^{(9-8)} - i\eta)$ on the contour c surrounding ϕ. Then $\mathcal{O} \cdot \phi$ is proportional to the $(-1)^{F^{(9-8)}} = -1$ ground state $|-\rangle_{9-8}$. In order for the amplitude to be non-vanishing, the vertex operator ϕ' must be proportional to the $(-1)^{F^{(8-9)}} = -1$ ground state $|-\rangle_{8-9}$. On the other hand, the operator $\mathcal{O}$ can be continuously deformed into an operator $\mathcal{O}'$ on the contour c′ surrounding ϕ'. It is convenient to conformally map the amplitude into the upper half plane, shown in the right part of Fig(3.2). Now the insertion points locate at $w = \pm 1$. In order to give a globally defined mode expansion for the $\psi_\pm$ fields, we need to write them into half-forms,

$$\begin{aligned} \psi_- &= \sum_{n\in\mathbb{Z}} \psi_n \left(\frac{w+1}{1-w}\right)^{-n-1/2} (w-1)^{-1}\sqrt{dw}, \\ \psi_+ &= -\sum_{n\in\mathbb{Z}} \psi_n \left(\frac{\bar{w}+1}{1-\bar{w}}\right)^{-n-1/2} (\bar{w}-1)^{-1}\sqrt{d\bar{w}}. \end{aligned} \tag{3.52}$$

In the neighborhood of $w = -1$, the above mode expansion should be identical to the canonical one,

$$\begin{aligned}
\psi_-^{(9-8)} &= \sum_{n\in\mathbb{Z}} \psi_n^{(9-8)}(w+1)^{-n-1/2}\sqrt{dw}, \\
\psi_+^{(9-8)} &= -\sum_{n\in\mathbb{Z}} \psi_n^{(9-8)}(\bar{w}+1)^{-n-1/2}\sqrt{d\bar{w}}.
\end{aligned}$$

Then we have the relation $\psi_n^{(9-8)} = -2^{n-1/2}\psi_n$. In the neighborhood of $w=1$, the $\psi_\pm$ should be identical to the following,

$$\begin{aligned}
\psi_-^{(8-9)} &= \sum_{n\in\mathbb{Z}} \psi_n^{(8-9)}(w-1)^{-n-1/2}\sqrt{dw}, \\
\psi_+^{(8-9)} &= \sum_{n\in\mathbb{Z}} \psi_n^{(8-9)}(\bar{w}-1)^{-n-1/2}\sqrt{d\bar{w}}.
\end{aligned}$$

Thus we have the relation $\psi_n^{(8-9)} = -i(-1)^n 2^{n-1/2}\psi_{-n}$. Construct a one form $b = \omega^-\psi_- + \omega^+\psi_+$, where

$$\begin{aligned}
\omega^- &= \left(\frac{w+1}{1-w}\right)^{1/2} \frac{1}{\pi i(w+1)}\sqrt{dw}, \\
\omega^+ &= \left(\frac{\bar{w}+1}{1-\bar{w}}\right)^{1/2} \frac{1}{\pi i(\bar{w}+1)}\sqrt{d\bar{w}}.
\end{aligned}$$

It is easy to see that b is closed and vanishes on the boundary. So b can be parallelly transported from the neighborhood of $w=-1$ to the neighborhood of $w=1$. This shows that the operator $\mathcal{O}$ is conformally deformed into the operator $\mathcal{O}' = -i\sqrt{2}\psi_0^{(8-9)} - i\eta$. Using the condition that $\mathcal{O}'\cdot\phi'$ should be proportional to the ground state $|+\rangle_{8-9}$, we find that the GSO-projection of the 8-9 string is $(-1)^{F^{(8-9)}} = \sqrt{2}\eta\psi_0^{(8-9)}\mathrm{e}^{\pi i\sum_{n\geq 1}\psi_{-n}^{(8-9)}\psi_n^{(8-9)}}$.

The above mode expansion (3.52) is also useful to determine the parity action on the modes $\psi_n^{(9-8)}$. The subtle point is how to determine the parity action on $\sqrt{dw}$ and $\sqrt{d\bar{w}}$. We know that far from the poles $w=\pm 1$, the (3.52) should reproduce the mode expansion for the 9-9 string. The parity action on the 9-9 string modes is clear. In the upper half plane, the standard mode expansion for the 9-9 string is,

$$\begin{aligned}\psi_-^{(9-9)} &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^{(9-9)} w^{-r-1/2}\sqrt{dw},\\ \psi_+^{(9-9)} &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^{(9-9)} \bar{w}^{-r-1/2}\sqrt{d\bar{w}}.\end{aligned}$$

The worldsheet parity action (2.3) on the 9-9 string is generalized to $\Omega : \psi_\pm^{(9-9)} \mapsto \mp\Omega^*\psi_\mp^{(9-9)}$ and $\psi_r^{(9-9)} \mapsto \mathrm{e}^{\pi i r}\psi_r^{(9-9)}$. Thus we get

$$\Omega^*(\sqrt{dw}) = i\sqrt{d\bar{w}}, \quad \Omega^*(\sqrt{d\bar{w}}) = -i\sqrt{dw}. \tag{3.53}$$

If we require the same parity action on $\psi_\pm$, that is, $\Omega : \psi_\pm \mapsto \mp\Omega^*\psi_\mp$, we find that

$$\psi_n^{(9-8)} \mapsto (-1)^n \psi_n^{(8-9)}. \tag{3.54}$$

Now, the parity action on the 9-8 string depends on how we extend the worldsheet parity action to η. We have two options, $\Omega :\ \eta \mapsto -i\eta$ and $\Omega :\ \eta \mapsto i\eta$. Using the above result, one can show that the two options transform $(-1)^{F^{(9-8)}}$ to $(-1)^{F^{(8-9)}}$ and $-(-1)^{F^{(8-9)}}$, respectively. In other words, the first option does not change the $(-1)^F$ representation, while the second option changes a $(-1)^F$ representation into an opposite one. So only $\Omega :\ \eta \mapsto -i\eta$ is allowed. A similar computation shows that the parity action is $\Omega :\ \eta \mapsto i\eta$ for the $(--)$ NS-sector. In summary, the parity action is $\Omega :\ \eta \mapsto \zeta\eta$, where $\zeta = \mp i$ for the $(\pm)$-type boundary conditions, respectively.

Next, let us discuss the worldsheet parity square action on the p-q string with $p-q$ even (i.e, p and q are both even or odd). We can use Gimon and Polchinski's argument [23]. Suppose the D_p-brane is located at $x^{p+1} = \cdots = x^9 = 0$, and the D_q-brane is located at $x^{q+1} = \cdots = x^9 = 0$. In the NS-sector, the worldsheet fermionic fields $\psi^\mu_\pm$ have integer modes in the DN directions $\mu = p+1, \cdots, q$ if $p < q$ and in the ND directions $\mu = q+1, \cdots, p$ if $p > q$. They contribute a factor $\phi_{bk} = \mathrm{e}^{i(\varphi_1 + \cdots + \varphi_{|q-p|/2})/2}$ to the bulk vertex operator, where φ_i with $i = 1, \cdots, |q-p|/2$ are the bosonization fields of ψ^μ in the DN or ND directions. Then $P_{bk}(\phi_{bk}) \cdot \phi_{bk}$ will be a bulk vertex operator corresponding to the state $(\psi^{p+1} + i\psi^{p+2})_{-1/2} \cdots (\psi^{q-1} + i\psi^q)_{-1/2}|0\rangle$ and $(\psi^{q+1} + i\psi^{q+2})_{-1/2} \cdots (\psi^{p-1} + i\psi^p)_{-1/2}|0\rangle$ of the p-p string for $p < q$ and $p > q$, respectively. Using the parity action (2.3), it is easy to find that

$$P_{bk}(P_{bk}(\phi_{bk}) \cdot \phi_{bk}) = \begin{cases} (+i)^{-(p-q)/2} P_{bk}(\phi_{bk}) \cdot \phi_{bk}, & \text{for the } (++) \text{ NS sector,} \\ (-i)^{-(p-q)/2} P_{bk}(\phi_{bk}) \cdot \phi_{bk}, & \text{for the } (--) \text{ NS sector.} \end{cases} \tag{3.55}$$

Here, we have to be careful with the expansion of P_{bk} action on the bulk operator product in the left hand side of eq(3.55).

Let us do a little digression. Consider a general operator product, $\phi^{b\to c} \cdot \phi^{a\to b}$, where $\phi^{a\to b} = \phi^{a\to b}_{cp} \otimes \phi^{a\to b}_{bk}$ is an operator from brane a to brane b, and $\phi^{b\to c} = \phi^{b\to c}_{cp} \otimes \phi^{b\to c}_{bk}$ is an operator from brane b to brane c. We require that $P^{a\to c}(\phi^{b\to c} \cdot \phi^{a\to b}) = (-1)^{|\phi^{a\to b}||\phi^{b\to c}|} P^{a\to b}(\phi^{a\to b}) \cdot P^{b\to c}(\phi^{b\to c})$ and $|P_{bk}| = |P_{cp}|$, which have already been used in the previous discussion. Using eq(3.31), it is easy to derive that $P^{a\to c}_{cp}(\phi^{b\to c}_{cp} \cdot \phi^{a\to b}_{cp}) = (-1)^{|\phi^{a\to b}_{cp}||\phi^{b\to c}_{cp}| + |\phi^{a\to b}_{cp}||P^{b\to c}_{cp}|} P^{a\to b}_{cp}(\phi^{a\to b}_{cp}) \cdot P^{b\to c}_{cp}(\phi^{b\to c}_{cp})$. Then the above requirement together with eq(3.29) lead to the following bulk parity action expansion,

$$P^{a\to c}_{bk}(\phi^{b\to c}_{bk} \cdot \phi^{a\to b}_{bk}) = (-1)^{|\phi^{a\to b}_{bk}||\phi^{b\to c}_{bk}| + |\phi^{a\to b}_{bk}||P^{b\to c}_{bk}| + |P^{a\to b}_{bk}||P^{b\to c}_{bk}|} P^{a\to b}_{bk}(\phi^{a\to b}_{bk}) \cdot P^{b\to c}_{bk}(\phi^{b\to c}_{bk}).$$

Back to our current case, the left hand side of eq(3.55) is then expanded as $P_{bk}(P_{bk}(\phi_{bk}) \cdot$

$\phi_{bk}) = (-1)^{|\phi_{bk}|+|P_{bk}|} P_{bk}(\phi_{bk}) \cdot P_{bk}^2(\phi_{bk})$. Thus, we get the result

$$P_{bk}^2 = (-1)^{|P_{bk}|} \zeta_{bk}^{-(p-q)/2} (-1)^{F_{bk}}, \tag{3.56}$$

where $\zeta_{bk} = i$ for the $(++)$ NS-sector and $\zeta_{bk} = -i$ for the $(--)$ NS-sector. On the other hand, we have the relation (3.41) from the previous section. Insert eq(3.56) into (3.41), we find that

$$P^2 = \zeta^{(\Delta r - \Delta s)/2} \zeta_{bk}^{-(p-q)/2} (-1)^F. \tag{3.57}$$

In order to obey the relation $P^2 = (-1)^F$, we get the condition $\zeta^{(\Delta r - \Delta s)/2} \zeta_{bk}^{-(p-q)/2} = 1$. It is also known that D_9-branes carry an SO-type Chan-Paton factor in Type I theory(which can be regarded as an O_9^- orientifold theory) and a Sp-type Chan-Paton factor in the O_9^+ orientifold theory [47]. This means that $|U|$=0, $U = U^t$ in the O_9^- theory and $|U|$=0, $U = -U^t$ in the O_9^+ theory. Compare with the result (3.40), one can see that the D_9-brane carries $\Delta r = 0$ mod 8 boundary fermions in the O_9^- orientifold and $\Delta r = 4$ mod 8 boundary fermions in the O_9^+ orientifold. It is natural to put forward a proposal that the D_p-brane carries Δr=$(9 - p)$ mod 8 boundary fermions in Type I string theory and Δr=$(5 - p)$ mod 8 boundary fermions in the O_9^+ orientifold theory. The phase ζ is determined to be $\mp i$ for the $(\pm)$-type boundary conditions, respectively. Then eq(3.40) becomes

$$U(U^T)^{-1}\iota = \begin{cases} \zeta^{(9-p)/2}\sigma_{\mathcal{H}}, & \text{for Type } I \text{ theory,} \\ \zeta^{(5-p)/2}\sigma_{\mathcal{H}}, & \text{for the } O_9^+ \text{ theory,} \end{cases} \quad \text{for } p \text{ odd.} \tag{3.58}$$

Although the above result (3.58) is obtained by using boundary fermions, it is more general than that. In fact, the result applies to any dimensional $\mathbb{Z}_2$-graded Chan-Paton vector space $\mathcal{H}$, not just to the one generated by the boundary fermions. In other words, the result (3.58) only depends on p. By T-duality, a D_p-brane in Type I theory and the O_9^+ theory is equivalent to a D_9-brane in the $O_p^{\mp}$ orientifold theory, respectively, whose spacetime involution is given by eq(3.10). So a more precise statement is the following: the U operator for the D_9-brane in Type IIB orientifold theory satisfies

$$U(\mathcal{I}^* U^T)^{-1}\iota = \zeta^{n_p/2}\sigma_{\mathcal{H}}, \quad \text{for} \quad p \quad \text{odd}, \tag{3.59}$$

where n_p is $(9-p)$ mod 8 for the O_p^- theory and $(5-p)$ mod 8 for the O_p^+ theory.

For a general p-q string with $p-q$ odd, there exist $|p-q|$ worldsheet fermionic fields which have integer modes in the NS-sector. r boundary fermions are associated to the D_p-brane and s boundary fermions are put to the D_q-brane. The constraint is that $r+s$ is odd. Of course, we can quantize the bulk fermionic modes and boundary fermions together. The parity action on this p-q string is treated in the way similar to the 9-8 string case. As we mentioned in the last section, it is convenient to have a $\mathbb{Z}_2$-graded bulk Hilbert space and $\mathbb{Z}_2$-graded Chan-Paton vector spaces on both boundaries by introducing auxiliary fermionic fields. Then the question would be how we determine the parity action on this extended system.

To be explicit, let us again take the 9-8 string as an example. We introduce one auxiliary boundary fermion field η_{aux} and one auxiliary bulk fermionic zero mode η'_{aux}. The projection operator is $u' = \eta_{aux}\eta'_{aux}$. In the $(++)$ NS-sector, we have an extended bulk fermionic mode system, η'_{aux}, $\psi_0^{(9-8)}$ and $\psi_n^{(9-8)}$ with non-zero integer n. Quantizing it, we get an extended $\mathbb{Z}_2$-graded bulk Hilbert space. The GSO-projection on this extended Hilbert space is $(-1)^{F_{bk}^{ext,\,(9-8)}} = \sqrt{2}i\eta'_{aux}\psi_0^{(9-8)}(-1)^{F_{nz}^{(9-8)}}$. Note that by a similar consideration as before, the GSO-projection on the extended 8-9 string bulk Hilbert space turns out to be $(-1)^{F_{bk}^{ext,\,(8-9)}} = \sqrt{2}\eta'_{aux}\psi_0^{(8-9)}(-1)^{F_{nz}^{(8-9)}}$. The extended boundary fermion system consists of η and η_{aux}. Quantizing it, we get an extended $\mathbb{Z}_2$-graded Chan-Paton space. The GSO-projection on it is chosen to be $(-1)^{F_{cp}^{ext}} = i\eta_{aux}\eta$. To recover the original space of states, we need to apply the projection condition $u' = 1$ to the extended systems of both 9-8 string and 8-9 string, which will be explained in section 3.4.2.

To study the parity action on the extended fermion system, we first need to know how to extend the worldsheet parity action to η'_{aux}. There are two options,

$$a)\ \Omega :\ \eta'_{aux} \mapsto -i\eta'_{aux} \quad \text{and} \quad b)\ \Omega :\ \eta'_{aux} \mapsto i\eta'_{aux}.$$

Using the relation(3.54), one can show that the two options transform $(-1)^{F_{bk}^{ext,\,(9-8)}}$ to $(-1)^{F_{bk}^{ext,\,(8-9)}}$ and $-(-1)^{F_{bk}^{ext,\,(8-9)}}$, respectively. The bulk parity square action on the extended bulk system is determined by looking at the property of the bulk parity square action on the bulk vertex operators of the 9-8 string. Suppose ϕ_{bk} denotes some bulk vertex operator of the 9-8 string, and $P_{bk}^{ext}(\phi_{bk})$ is the parity image of ϕ_{bk}. Then $P_{bk}^{ext}(\phi_{bk}) \cdot \phi_{bk}$ is in the $(-1)^{F_{bk}} = \pm 1$ representation of the 9-9 string for parity action a) and b), respectively. On the other hand, all the states in the $(-1)^{F_{bk}} = 1$ representation of the 9-9 string can be written into the form $\psi_{-r_1} \cdots \psi_{-r_s}|0\rangle_{9-9}$ with even s, while all the states in the $(-1)^{F_{bk}} = -1$ representation are of the form $\psi_{-r_1} \cdots \psi_{-r_{s'}}|0\rangle_{9-9}$ with odd s'. It can be shown that a pair of fermionic operators, $\psi_{-r_i}\psi_{-r_j}$, gives a trivial phase contribution under the parity action. Thus, non-trivial phase contributions to the operator products in a) and b) are from $|0\rangle_{9-9}$ and $\psi_{-1/2}|0\rangle_{9-9}$, respectively. Using the fact that $P_{bk}(|0\rangle_{9-9}) = |0\rangle_{9-9}$ and $P_{bk}(\psi_{-1/2}^{(9-9)}|0\rangle_{9-9}) = -i\psi_{-1/2}^{(9-9)}|0\rangle_{9-9}$, we can do a simple computation for the two different parity options on η'_{aux}

$$\text{a)}: \qquad P_{bk}^{ext}(P_{bk}^{ext}(\phi_{bk}) \cdot \phi_{bk}) = (-1)^{|\phi_{bk}|+|P_{bk}^{ext}|} P_{bk}^{ext}(\phi_{bk}) \cdot (P_{bk}^{ext})^2(\phi_{bk}) = P_{bk}^{ext}(\phi_{bk}) \cdot \phi_{bk},$$

$$\text{b)}: \qquad P_{bk}^{ext}(P_{bk}^{ext}(\phi_{bk}) \cdot \phi_{bk}) = (-1)^{|\phi_{bk}|+|P_{bk}^{ext}|} P_{bk}^{ext}(\phi_{bk}) \cdot (P_{bk}^{ext})^2(\phi_{bk}) = -iP_{bk}^{ext}(\phi_{bk}) \cdot \phi_{bk}.$$

From the computation, we find that

$$\begin{aligned} \text{Option a)}: \qquad & (P_{bk}^{ext})^2 = (-1)^{|P_{bk}^{ext}|}(-1)^{F_{bk}^{ext,\,(9-8)}}, \\ \text{Option b)}: \qquad & (P_{bk}^{ext})^2 = -i(-1)^{|P_{bk}^{ext}|}(-1)^{F_{bk}^{ext,\,(9-8)}}. \end{aligned} \tag{3.60}$$

For the parity action on the boundary fermion, we have $\Omega : \eta \mapsto \zeta\eta$. For the auxiliary boundary fermion, we have two choices for η_{aux}: i) $\Omega : \eta_{aux} \mapsto \zeta\eta_{aux}$ and ii) $\Omega : \eta_{aux} \mapsto -\zeta\eta_{aux}$. By a similar derivation as above, we can find that $(P_{cp}^{ext})^2 = \zeta^{-1}(-1)^{F_{cp}^{ext,\,(9-8)}}$ for option i) and $(P_{cp}^{ext})^2 = (-1)^{F_{cp}^{ext,\,(9-8)}}$ for option ii).

To satisfy the condition $P^2 = (-1)^F$ on the extended fermion system and make the u'=1 projection condition invariant for both 9-8 string and 8-9 string, we find that there are only two choices of the parity action. The first one is the parity action b) on η'_{aux} combined with the parity action i) on η_{aux}. The second one is the parity action a) on η'_{aux} combined with the parity action ii) on η_{aux}. In both choices, the phase ζ is found to be $-i$ for the (+)-type boundary condition. In other words, the two choices are physically equivalent. For convenience, we pick the first choice in our discussion. Repeat the same calculation, we can find that $\zeta = i$ for the ($-$)-type boundary condition.

For a general p-q string with $p - q$ odd, the analysis is the following. Consider the case where p is odd and q is even. (The discussion for the p even and q odd case is similar.) In the NS-sector, there exist $|p - q|$ bulk fermionic zero modes, $\psi_0^{(\mathrm{p-q}),i}$ with $i = 1, \cdots, |p - q|$. Suppose we put even number of boundary fermions, $\eta_{i'}^L$ with $i' = 1, \cdots, r$, on the left boundary and odd number of boundary fermions, η_i^R with $i = 1, \cdots, s$, on the right boundary. In order to have a $\mathbb{Z}_2$-graded bulk space of states and Chan-Paton factor space, we introduce one auxiliary bulk fermionic zero mode η'_{aux} and one auxiliary boundary fermion η_{aux} to the right boundary. The projection operator is $u' = \eta_{aux}\eta'_{aux}$. Then, the GSO-projection on the extended bulk fermion system is $(-1)^{F_{bk}^{ext,\,(\mathrm{p-q})}} = \sqrt{2}i^{|p-q|(|p-q|+1)/2}\eta'_{aux}\psi_0^{(\mathrm{p-q}),1}\cdots\psi_0^{(\mathrm{p-q}),|p-q|}(-1)^{F_{nz}^{(\mathrm{p-q})}}$, where the phase $i^{|p-q|(|p-q|+1)/2}$ is needed to make $((-1)^{F_{bk}^{ext,\,(\mathrm{p-q})}})^2 = 1$. The $\mathbb{Z}_2$-grading on the Chan-Paton vector space of the left boundary is $(-1)^{F_{cp}^{(L)}} = i^{r(r-1)/2}\eta_1^L\cdots\eta_r^L$. The $\mathbb{Z}_2$ grading on the Chan-Paton vector space of the right boundary is $(-1)^{F_{cp}^{(R)}} = i^{s(s+1)/2}\eta_{aux}\eta_1^R\cdots\eta_s^R$. Thus the GSO-projection on the extended Chan-Paton factor space is $(-1)^{F_{cp}^{ext,\,(\mathrm{p-q})}}h_{cp} = (-1)^{F_{cp}^{(R)}}h_{cp}(-1)^{F_{cp}^{(L)}}$. There exist several physically equivalent choices of the parity action on the extended system. To be explicit, we choose the following one, $\Omega:\ \eta'_{aux}\mapsto \pm i\eta'_{aux}$, $\eta_{aux}\mapsto\mp i\eta_{aux}$, $\eta_{i'}^L\mapsto\mp i\eta_{i'}^R$ and $\eta_i^R\mapsto\mp i\eta_i^L$, where the upper and lower signs are for the $(++)$ and $(--)$ NS-sector respectively. By the similar derivation to the 9-8 string case, we get

$$\begin{aligned}(P_{bk}^{ext})^2 &= (\pm i)^{-(p-q+1)/2}(-1)^{|P_{bk}^{ext}|}(-1)^{F_{bk}^{ext,\,(\mathrm{p-q})}},\\ (P_{cp}^{ext})^2 &= (\mp i)^{(r-s-1)/2}(-1)^{F_{cp}^{ext,\,(\mathrm{p-q})}}.\end{aligned}$$

To obey the condition $P^2 = (-1)^F$, we have the relation that $r{=}(9-p)$ mod 8, $s{=}(9-q)$ mod 8 and $r{=}(5-p)$ mod 8, $s{=}(5-q)$ mod 8 for Type I theory and the O_9^+ theory, respectively. For the D_p-brane with p even, eq(3.40) changes to

$$U(U^T)^{-1}\iota = \begin{cases}\zeta^{(10-p)/2}\sigma_{\mathcal{H}}, & \text{for Type I theory,}\\ \zeta^{(6-p)/2}\sigma_{\mathcal{H}}, & \text{for the } \mathrm{O}_9^+ \text{ theory,}\end{cases}\quad \text{for } p \text{ even,} \tag{3.61}$$

where $\zeta = \mp i$ for the $(\pm)$-type boundary conditions, respectively. As argued before, the above result (3.61) only depends on p and thus applies to a more general Chan-Paton vector space $\mathcal{H}$. Then, after T-dualizing the D_p-brane in Type I theory and the O_9^+ theory to the D_9-brane in Type IIA $O_p^{\mp}$ orientifold theory, respectively, the precise statement is that the U operator for the D_9-brane in Type IIA orientifold theory satisfies

$$U(\mathcal{I}^* U^T)^{-1}\iota = \zeta^{(n_p+1)/2}\sigma_{\mathcal{H}}, \quad \text{for} \quad p \quad \text{even}, \tag{3.62}$$

where n_p is $(9-p)$ mod 8 for the O_p^- theory and $(5-p)$ mod 8 for the O_p^+ theory. Then, eq(3.59) and eq(3.62) together produce the result (3.13). Furthermore, the auxiliary field η_{aux} takes the role of the odd operator ξ, defined in section 3.1. The parity action on η_{aux} leads to eq(3.15).

Actually, from the discussion in this section, we have extended Gimon-Polchinski's analysis of the parity square action on the states of the p-q open string with $p-q$ even case to the p-q string with $p-q$ odd case. Thus, a complete analysis of the parity square action on the states of all p-q strings is also obtained. This complete our discussion on the parity action on D-branes.

3.4 The Scattering Amplitude

After finding out the relation between boundary fermions and D-branes, we can use it to study the open string scattering amplitudes in the framework of boundary string field theory.

3.4.1 The Full Boundary Action

We will use the worldsheet metric g $= diag(+1, -1)$, and $\sigma_\pm = t \pm \sigma$. The bulk open string worldsheet action can be written as

$$S_{bulk} = \frac{1}{\pi}\int_\Sigma d^2\sigma(\frac{1}{\alpha'}\partial_+ X \cdot \partial_- X + i\psi_+ \cdot \partial_-\psi_+ + i\psi_- \cdot \partial_+\psi_-). \tag{3.63}$$

Under the bulk $\mathcal{N}$=1 supersymmetry transformation

$$\begin{aligned}
\delta X^\mu &= i(\epsilon_+\psi_-^\mu - \epsilon_-\psi_+^\mu), \\
\delta\psi_-^\mu &= -\frac{1}{\alpha'}\epsilon_+\partial_- X^\mu, \\
\delta\psi_+^\mu &= \frac{1}{\alpha'}\epsilon_-\partial_+ X^\mu,
\end{aligned}$$

the S_{bulk} varies by

$$\delta S_{bulk} = \frac{-i}{2\pi\alpha'}\int_{\partial\Sigma} dt(\epsilon_-\partial_- X \cdot \psi_+ + \epsilon_+\partial_+ X \cdot \psi_-).$$

It is easy to see that a diagonal $\mathcal{N}$=1 SUSY, $\epsilon_+ = c\epsilon_- = \epsilon$ with $c = \pm 1$, is preserved by a modified action S'_{bulk} which is

$$S'_{bulk} = S_{bulk} + \frac{-ic}{2\pi}\int_{\partial\Sigma} dt\psi_- \cdot \psi_+. \tag{3.64}$$

The coefficient c is determined by the type of boundary conditions, defined in section 2.1. For example, in the NN direction, $c = \pm 1$ for the $(\pm)$-type boundary conditions at the left boundary and $c = \mp 1$ for the $(\pm)$-type boundary conditions at the right boundary.

Let us first derive the general boundary interaction for a system of 2^{m-1} D_9–$\bar{D}_9$ pairs in Type IIB string theory, which preserves the diagonal $\mathcal{N}$=1 SUSY. The method is to use $n = 2m$ real boundary fermions. It is convenient to work in the Euclidean metric after Wick rotation: $t \to -i\tau$. We introduce the worldsheet boundary superfield $\mathbf{X}^\mu(\tau) = X^\mu + \sqrt{\alpha'}\theta\psi^\mu$(where $\psi^\mu = \psi^\mu_+ - c\psi^\mu_-$), the boundary fermion superfield $\Gamma_i(\tau) = \eta_i + \theta F_i$ with $i = 1, \cdots, n$ and the derivative $\mathbf{D} = \partial_\theta - \theta\partial_\tau$ with conformal dimension 0,0 and 1, respectively. The F_i's are auxiliary fields which will be eliminated after quantization. The $\mathcal{N} = 1$ invariant boundary interaction term is

$$S_{int} = -\int_{\partial\Sigma} d\tau d\theta \{f_0(\mathbf{X}) + f_i(\mathbf{X})\Gamma_i + f_{ij}(\mathbf{X})\Gamma_i\Gamma_j + \cdots + f_{12\cdots n}(\mathbf{X})\Gamma_i \cdots \Gamma_n\}, \qquad (3.65)$$

where the f's are functions of $\mathbf{X}$ and are antisymmetric with respect to the indices. We only keep the terms of the lowest conformal dimension. That is,

$$\begin{aligned} f_0(\mathbf{X}) &= \mathbf{D}\mathbf{X}^\mu \mathcal{A}_{\mu,0}(\mathbf{X}) + \cdots, \\ f_i(\mathbf{X}) &= \sqrt{\alpha'} T_i(\mathbf{X}) + \cdots, \\ f_{ij}(\mathbf{X}) &= \mathbf{D}\mathbf{X}^\mu \mathcal{A}_{\mu ij}(\mathbf{X}) + \cdots, \\ f_{ijk}(\mathbf{X}) &= \sqrt{\alpha'} T_{ijk}(\mathbf{X}) + \cdots, \\ &\cdots \end{aligned}$$

So the full boundary action for $n = 2m$ boundary fermions is

$$S_{bdry} = -\int_{\partial\Sigma} d\tau d\theta \{\frac{1}{4}\Gamma_i \mathbf{D}\Gamma_i + \sum_{k=0}^{2m} \frac{1}{2k!}\mathbf{M}^{i_1\cdots i_k}(\mathbf{X})\Gamma_{i_1}\cdots\Gamma_{i_k}\}. \qquad (3.66)$$

We know that the Γ_i's will become SO($2m$) gamma matrices after quantization. So the coefficients $\mathbf{M}^{i_1\cdots i_k}$'s can be regarded as the coefficients of a $2^m \times 2^m$ matrix $\mathbf{M}(\mathbf{X})$

expanded in SO($2m$) gamma matrices. $\mathbf{M}(\mathbf{X})$ can be written into the form

$$\mathbf{M}(\mathbf{X}) = \begin{pmatrix} iA^0_\mu(\mathbf{X})\mathbf{D}\mathbf{X}^\mu & \sqrt{\alpha'}T(\mathbf{X})^\dagger \\ \sqrt{\alpha'}T(\mathbf{X}) & iA^1_\mu(\mathbf{X})\mathbf{D}\mathbf{X}^\mu \end{pmatrix} \tag{3.67}$$

For convenience, we decompose $\mathbf{M}$ into $\mathbf{M}(\mathbf{X}) = \mathbf{M}_0(X,\psi) + \theta\mathbf{M}_1(X,\psi)$. As discussed in [41], carrying out the θ-integral and integrating out the F_i's, we get

$$S_{bdry} = -\int_{\partial\Sigma} d\tau \{\frac{1}{4}\dot{\eta}_i\eta_i + \sum_{k=0}^{2m} \frac{1}{2k!}(\mathbf{M}_1 - \mathbf{M}_0^2)^{i_1\cdots i_k}\eta_{i_1}\cdots\eta_{i_k}\}. \tag{3.68}$$

The second term is the interaction term which was omitted in section 3.2.

Doing the path integral of the η_i's on a closed worldsheet boundary $\partial\Sigma$, depending on the anti-periodic or periodic boundary condition, we get a trace over a path ordered exponential,

$$\mathrm{tr}\left[\mathrm{P}\exp\left(-\int_{\partial\Sigma} d\tau \mathbf{A}_\tau\right)\right] \quad \text{or} \quad \mathrm{tr}\left[\sigma_{\mathcal{H}_{2m}}\mathrm{P}\exp\left(-\int_{\partial\Sigma} d\tau \mathbf{A}_\tau\right)\right], \tag{3.69}$$

where

$$\mathbf{A}_\tau = i\mathcal{A}\cdot\dot{X} - \frac{i}{2}\alpha'\mathcal{F}_{\mu\nu}\psi^\mu\psi^\nu + i\alpha'\psi\cdot\mathcal{D}\mathcal{T} + \alpha'\mathcal{T}^2, \tag{3.70}$$

$$\mathcal{A}_\mu(X) = \begin{pmatrix} A^0_\mu(X) & 0 \\ 0 & A^1_\mu(X) \end{pmatrix}, \tag{3.71}$$

$$\mathcal{T}(X) = \begin{pmatrix} 0 & T(X)^\dagger \\ T(X) & 0 \end{pmatrix}, \tag{3.72}$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu,\mathcal{A}_\nu], \tag{3.73}$$

$$\mathcal{D}_\mu\mathcal{T} := \partial_\mu\mathcal{T} + i[\mathcal{A}_\mu,\mathcal{T}]. \tag{3.74}$$

$\mathcal{A}_\mu$ is the gauge field, which is an even operator on the Chan-Paton factor space. While, $\mathcal{T}$ is called the tachyon field, which is an odd operator on the Chan-Paton factor space. For the open boundary, we get a path ordered exponential,

$$G_{i,f} = \mathrm{P}\exp\left(-\int_{\tau_i}^{\tau_f} d\tau \mathbf{A}_\tau\right). \tag{3.75}$$

Although the above result eq(3.70) is obtained in terms of boundary fermions, the generalization to any system of D_9 and $\bar{D}_9$ branes is straightforward. For a system of N D_9-branes and M $\bar{D}_9$-branes, the boundary interaction $\mathbf{A}_\tau$ is still given by eq(3.70), with the change that the gauge field $\mathcal{A}_\mu$ in eq(3.71) and the tachyon field $\mathcal{T}$ in eq(3.72) are now regarded as $(N+M)\times(N+M)$ matrices. In other words, in Type IIB theory, D_9-branes support an N-dimensional vector bundle E^0 and $\bar{D}_9$-branes support an M-dimensional vector bundle E^1 on the spacetime $\mathbb{X}$. They combine into a $\mathbb{Z}_2$-graded Chan-Paton vector bundle $\mathbf{E} = E^0 \oplus E^1$, with the $\mathbb{Z}_2$-grading operator

$$\sigma_{\mathbf{E}} = \begin{pmatrix} \mathrm{id}_{E^0} & 0 \\ 0 & -\mathrm{id}_{E^1} \end{pmatrix}.$$

The tachyon $\mathcal{T}$ becomes an odd hermitian endomorphism of $\mathbf{E}$. The gauge field $\mathcal{A}$ is an even unitary connection of $\mathbf{E}$.

As for the non-BPS D_9 branes in Type IIA string theory, the derivation is slightly different. The difference is from the fact that there is no $\mathbb{Z}_2$ grading operator defined on the Chan-Paton factor space of non-BPS D_9 branes. Let us start with 2^{m-1} non-BPS D_9 branes. They can be described by $2m-1$ boundary fermions. There is no satisfying way to quantize this system. So it is convenient to introduce an auxiliary boundary fermion η_{aux}. This amounts to double the number of non-BPS D_9 branes. To recover the original system, we have to apply the projection (3.45). For a closed worldsheet boundary, introducing the auxiliary boundary fermion is by multiplying 1 to the path integral of the $2m-1$ boundary fermion system. For the anti-periodic boundary condition, we multiply

$$1 = \frac{1}{\sqrt{2}} \int D\eta_{aux} \exp\left(- \int_{\partial\Sigma} \frac{1}{4} \dot{\eta}_{aux} \eta_{aux} d\tau \right), \tag{3.76}$$

where the factor $\frac{1}{\sqrt{2}}$ is explained in [17]. For the periodic boundary condition, we multiply

$$1 = \frac{1}{\sqrt{2}} \int D\eta_{aux} \eta_{aux}(\tau_0) \exp\left(- \int_{\partial\Sigma} \frac{1}{4} \dot{\eta}_{aux} \eta_{aux} d\tau \right), \tag{3.77}$$

where τ_0 is any point on the closed boundary. Then we can mimic the above calculation, and find that

$$\frac{1}{\sqrt{2}} \mathrm{tr} \left[\mathrm{P} \exp\left(- \int_{\partial\Sigma} d\tau \mathbf{A}_\tau \right) \right] \quad \text{or} \quad \frac{1}{\sqrt{2}} \mathrm{tr} \left[\sigma_{\mathcal{H}_{2m}} \eta_{aux} \, \mathrm{P} \exp\left(- \int_{\partial\Sigma} d\tau \mathbf{A}_\tau \right) \right], \tag{3.78}$$

where the first expression is for the anti-periodic condition and the second one is for the periodic condition. The boundary interaction $\mathbf{A}_\tau$ is again given by (3.70). The gauge field and tachyon field are now in the $\mathbb{Z}_2$-graded form. But they have to subject to the projection condition $u = 1$. That is,

$$\mathcal{A} = \eta_{aux} \mathcal{A} \eta_{aux}, \quad \mathcal{T} = -\eta_{aux} \mathcal{T} \eta_{aux}.$$

By a suitable choice of the basis of Chan-Paton vector space, we can write η_{aux} into

$$\eta_{aux} = i \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{3.79}$$

Then the gauge field and the tachyon field are found to be of the form

$$\mathcal{T} = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \tag{3.80}$$

where $T = T^\dagger$ and $A = A^\dagger$.

Once the result for the system of 2^{m-1} non-BPS D_9 branes is obtained, we can easily generalize it to the case of arbitrary number of non-BPS D_9-branes. For N non-BPS D_9-branes in Type IIA theory, the boundary interaction is still given by (3.70). The gauge and tachyon fields in eq(3.80) are viewed as $2N \times 2N$ matrices. The operator η_{aux} becomes the odd operator ξ introduced in section 3.1. The projection condition is generalized to the statement that the gauge field $\mathcal{A}$ commutes with ξ and the tachyon field $\mathcal{T}$ anticommutes with ξ. The boundary interaction $\mathbf{A}_\tau$ commutes with ξ. In the vector bundle language, non-BPS D_9-branes in Type IIA theory support an ungraded vector bundle E on the spacetime $\mathbb{X}$. The gauge field A is a unitary connection of E and the tachyon field T is a hermitian endomorphism on E. In order to write down the boundary interaction, it is convenient to introduce a $\mathbb{Z}_2$-graded double of E, $\mathbf{E} = E \oplus E$, with an odd operator ξ. We require that the boundary interaction $\mathbf{A}_\tau$ commutes with ξ.

In Type II orientifold theory, the orientifold projection is the combination of the spacetime involution $\mathcal{I}$ and the worldsheet parity Ω. Ω reverses the orientation of the time coordinate τ. The worldsheet fields are transformed as: $X^\mu(\sigma,\tau) \to X^\mu(\sigma,-\tau)$ and $\psi^\mu_\pm(\sigma,\tau) \to -i\psi^\mu_\mp(\sigma,-\tau)$. Then the boundary interaction will be transformed as

$$\mathbf{A}_\tau \mapsto \mathcal{I}^*\Omega^*(\mathbf{A}_\tau)^T = -i\mathcal{A}(\tilde{X})^T \cdot \dot{\tilde{X}} + \frac{i}{2}\alpha' \mathcal{F}^T_{\mu\nu}\psi^\mu\psi^\nu \pm \alpha'\psi \cdot \mathcal{D}\mathcal{T}(\tilde{X})^T + \alpha'(\mathcal{T}^2)^T, \tag{3.81}$$

where $\tilde{X} = \mathcal{I}(X)$ and the $(\pm)$ signs are for the $(\pm)$-type boundary conditions, respectively.

The invariance of a D-brane under the orientifold projection means that the infrared limit of the boundary interaction $\mathbf{A}_\tau$ is isomorphic to the infrared limit of the boundary interaction $\mathcal{I}^*\Omega^*(\mathbf{A}_\tau)^T$. Obviously, if there exists a gauge transformation between $\mathbf{A}_\tau$ and $\mathcal{I}^*\Omega^*(\mathbf{A}_\tau)^T$, it will naturally induce an isomorphism between the infrared limits. In fact, there exist other isomorphisms between the infrared limits, which will be discussed in chapter 5.

Here, we focus on the case where there exists a gauge transformation between $\mathbf{A}_\tau$ and $\mathcal{I}^*\Omega^*(\mathbf{A}_\tau)^T$. This gauge transformation is a linear isomorphism $U : \mathcal{I}^*\mathbf{E}^* \to \mathbf{E}$. The invariance condition of a brane leads to the following conditions

$$\mathcal{A}(X) = -U\mathcal{A}(\tilde{X})^T U^{-1} + i^{-1} U dU^{-1}, \tag{3.82}$$

$$\mathcal{T}(X) = \zeta(-1)^{|U|} U\mathcal{T}(\tilde{X})^T U^{-1}. \tag{3.83}$$

This explains the reason why we have the relations (3.11) and (3.12). The discussion in the last section shows that the operator U must satisfy the constraint (3.13) for all Type II orientifolds. For Type IIA orientifolds, it has to obey one more condition (3.15).

3.4.2 The Scattering Amplitude

According to the classification of open string states in Type I string theory in section 3.2, we will discuss the scattering amplitudes of three different cases. By T-duality, it is straightforward to transform the scattering amplitudes of Type I theory to those of Type II orientifold theories.

- **The scattering amplitude involving open string states of type (i) only**

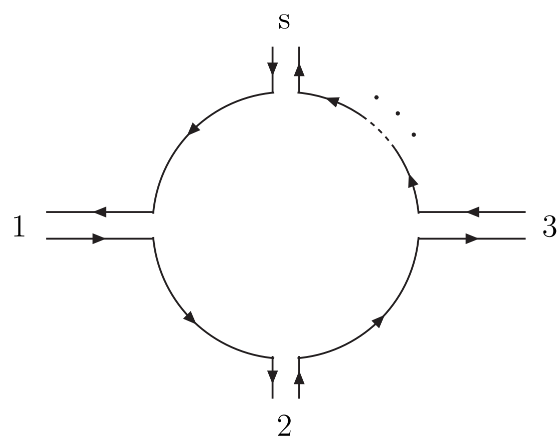

Figure 3.3: The scattering amplitude involving type **(i)** states only.

A typical s-point scattering amplitude is shown in Fig(3.3). The open string states $V_1, \cdots, V_s$ are inserted for the open strings, $1, \cdots, s$. For the segment between open string i and j along the boundary $\partial\Sigma$, we put the path ordered exponential $G_{i,j}$ defined in (3.78). Then the scattering amplitude is given by

$$\mathcal{M} = \int DXD\psi e^{-S_{bulk}} tr(V_1 G_{1,2} V_2 G_{2,3} \cdots V_s G_{s,1}). \tag{3.84}$$

• The scattering amplitude involving open string states of type (ii) only

As discussed before, we have to introduce an auxiliary boundary fermion η_{aux} to the scattering amplitude by multiplying 1, defined in (3.76) for the anti-periodic boundary condition. Then we define a projection operator (3.45) and apply projection conditions to the extended system to recover the original system. We have to consider what projection condition to apply at each open string state of the extended scattering amplitude.

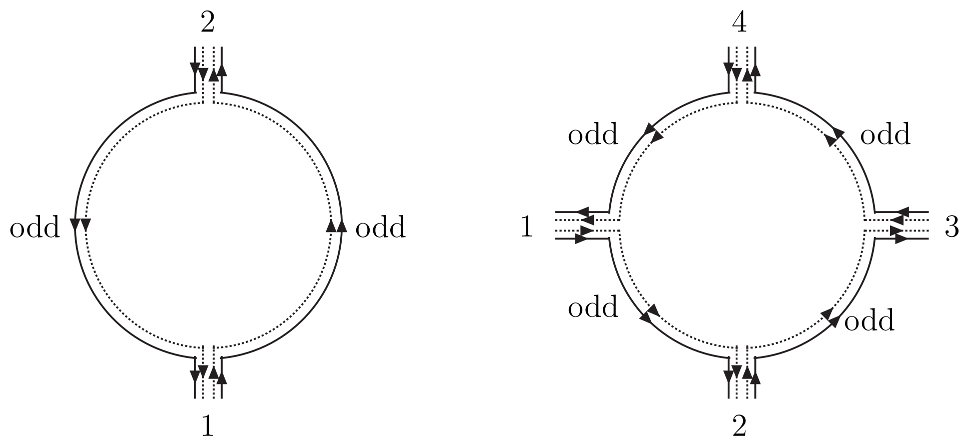

Figure 3.4: The scattering amplitudes involving type **(ii)** states only.

Let us start with the scattering amplitude involving two open string states of type **(ii)**, shown in the left part of Fig(3.4). The dotted lines denote the flow of the auxiliary boundary fermion η_{aux} along the boundary of the worldsheet Σ. The boundary of the worldsheet is broken into two segments by the insertion of two open string states, V_1 and V_2. The η_{aux} is extended to the left and right sides of the strip-like end of each open string state. We have to consider how to connect the flow of η_{aux} at each strip to form a closed flow. Suppose we take the projection condition to be $u = \eta^R_{aux}\eta^L_{aux} = 1$ for the state V_1. If η_{aux} flows continuously along the the two segments, then η^R_{aux} and η^L_{aux} deform to $\tilde{\eta}^L_{aux}$ and $\tilde{\eta}^R_{aux}$ for the second state. The $u = 1$ condition for V_1 deforms into the $u = \tilde{\eta}^R_{aux}\tilde{\eta}^L_{aux} = \eta^L_{aux}\eta^R_{aux} = -\eta^R_{aux}\eta^L_{aux} = -1$ condition for V_2. On the other hand, if η_{aux} flows discontinuously along one of the segments, say along the right segment, a minus sign will be generated. η^R_{aux} and η^L_{aux} deform to $-\tilde{\eta}^L_{aux}$ and $\tilde{\eta}^R_{aux}$ for the second insertion. Then the $u = 1$ condition for V_1 deforms to the $u = 1$ condition for V_2. We thus obtain a rule: if the two insertion points have opposite projection conditions, then η_{aux} flows continuously along the boundary; if the two insertion points have the same projection condition, then there must be a minus sign produced when η_{aux} flows along the boundary of the worldsheet.

We can generalize this rule to all scattering amplitudes involving open string states of type **(ii)**. For example, the right part of Fig(3.4) shows a 4-point scattering amplitude with the flow of η_{aux}. We have the following rule. If the number of $u = -1$ projection conditions is odd, then η_{aux} flows continuously along the boundary. While if the number of $u = -1$ conditions is even, η_{aux} flows discontinuously somewhere along the boundary and a minus sign is generated.

For an s-point scattering amplitude involving only type **(ii)** states with anti-periodic condition, the scattering amplitude is given by

$$\mathcal{M} = \frac{1}{\sqrt{2}} \int DXD\psi e^{-S_{bulk}} tr(V_1 G_{1,2} V_2 G_{2,3} \cdots V_s G_{s,1}), \tag{3.85}$$

where the factor $\frac{1}{\sqrt{2}}$ comes from the path integral of the auxiliary fermion.

• The scattering amplitude involving open string states of type (iii)

For this case, we have to introduce both auxiliary boundary fermion η_{aux} and auxiliary bulk fermionic mode η'_{aux} to the scattering amplitude by multiplying 1, defined in (3.76) for the anti-periodic boundary condition. For example, the 2-point and 5-point scattering amplitudes are shown in the left and right parts of Fig(3.5), respectively. The dotted line denotes the flow of η_{aux} and the dashed line denotes the flow of η'_{aux}. We need to explain a bit about how to draw the scattering amplitudes. At the strip-like end of a type **(iii)** state, we have a pair of auxiliary fermions, η_{aux} and η'_{aux} at the boundary with odd number of fermions, with opposite orientation. We extend both of the pair along lines near the boundary component with odd fermions. If the component ends with another strip of type **(iii)**, we let the pair end there. If the component ends with a strip of type **(ii)**, then η_{aux} ends there and the η'_{aux} line continues to the next boundary component, without going into the strip. In this way, a unique diagram can be drawn for every scattering amplitude involving type **(iii)** states.

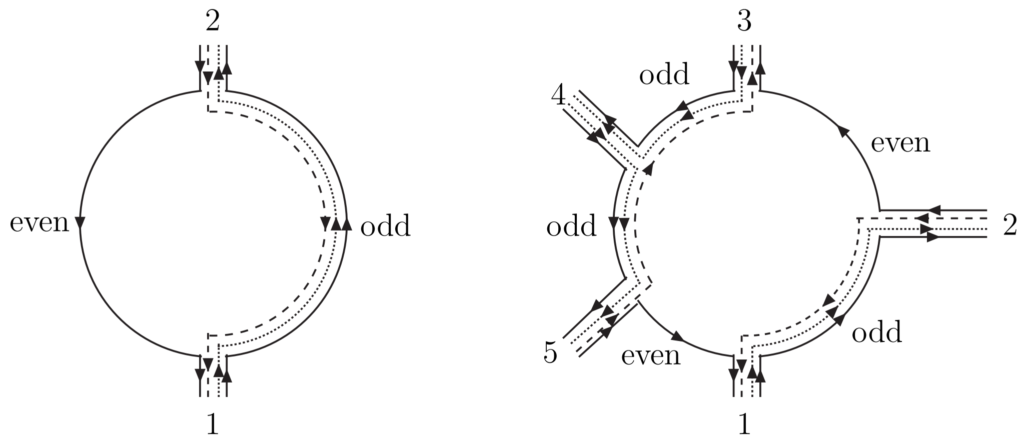

Figure 3.5: The scattering amplitudes involving type **(iii)** states.

Again, we have to apply projection conditions to the extended scattering amplitude to recover the original one. The projection operator for a type **(ii)** state is the same as before $u = \eta^R_{aux}\eta^L_{aux}$. For a type **(iii)** state, the projection operator is defined to be $u' = \eta_{aux}\eta'_{aux}$. We have to determine the projection condition at each open string state in an extended scattering amplitude.

Let us first look at the 2-point scattering amplitude, shown in the left part of Fig(3.5). There are two type **(iii)** open string states, V_1 and V_2. The boundary of the worldsheet is divided into two segments by the insertion of two open string states. We have to consider how to connect the flow of η_{aux} and η'_{aux} at each strip to form a closed flow. Suppose $u' = \eta_{aux}\eta'_{aux} = 1$ for V_1. If η_{aux} and η'_{aux} flow continuously, then η_{aux} and η'_{aux} deform to $\tilde{\eta}_{aux}$ and $\tilde{\eta}'_{aux}$ at the second insertion. Thus, the $u' = 1$ condition for V_1 deforms into the $u' = \tilde{\eta}_{aux}\tilde{\eta}'_{aux} = 1$ condition for V_2. (This explains the reason why we need to keep the $u' = 1$ condition invariant under the parity action for the 9-8 string in the last section.) On the other hand, if either η_{aux} or η'_{aux} flows discontinuously to the second insertion, then the $u' = 1$ condition for V_1 deforms to the $u' = -1$ condition for V_2. So we have a rule: if the projection conditions are the same at both insertions, then η_{aux} and η'_{aux} connect continuously at one insertion and discontinuously at another one. If the projection conditions are opposite for V_1 and V_2, then η_{aux} and η'_{aux} connect continuously at both insertions and there is a discontinuity in the flow of η_{aux} and η'_{aux}. A minus sign is generated during the flow.

We can easily generalize this rule to all scattering amplitudes involving type **(iii)** states. For example, a 5-point scattering amplitude is shown in the right part of Fig(3.5), containing two closed flows of auxiliary fermions. The rule for the projection condition is the following. For each closed flow of auxiliary fermions in a scattering amplitude, if the number of $u = -1$ and $u' = -1$ conditions is odd, there is a discontinuity in the flow of η_{aux} or η'_{aux} somewhere along the segments with odd number of boundary fermions, and a minus sign is generated. While if the number of $u = -1$ and $u' = -1$ conditions is even, η_{aux} and η'_{aux} flow continuously along the segments with odd number of boundary fermions.

For an s-point scattering amplitude involving type **(iii)** states with the anti-periodic condition, the scattering amplitude is given by

$$\mathcal{M} = N_0 \int DXD\psi D\eta'_{aux} e^{-S_{bulk}} tr(V_1 G_{1,2} V_2 G_{2,3} \cdots V_s G_{s,1}), \tag{3.86}$$

where the normalization coefficient $N_0 = (\frac{1}{\sqrt{2}})^n$ and n is the number of closed flows of auxiliary fermions in the scattering amplitude. Sen discussed similar things for the Type I non-BPS D_0-branes [48]. Our results generalize Sen's results to scattering amplitudes involving more general non-BPS D-branes.

Note that the structure of Chan-Paton factors of D-branes in Type I theory was also discussed in [19, 20]. They claimed that they obtained the result of the D-brane Chan-Paton structure and a subset of our results of the parity action on tachyon configurations in Type I theory. We find explicit results of the D-brane Chan-Paton structure and parity actions on gauge and tachyon fields, not only for Type I theory but also for Type II orientifold theory, through a thorough and systematic derivation motivated by using boundary fermions.

Chapter 4

Orientifolds and K-theory

It is known that in Type II string theories there exist both BPS D-branes and non-BPS D-branes. Witten argued that the D-brane charge takes value in the K-theory groups of the spacetime $\mathbb{X}$, that is, $K(\mathbb{X})$ for Type IIB theory and $K^{-1}(\mathbb{X})$ for Type IIA theory [17]. Motivated by Witten's work, Hořava [18] gave a natural explanation of why the Type IIA D-brane charge takes value in the $K^{-1}(\mathbb{X})$ group.

The mathematics underlying Witten's work is basically the use of vector bundles to define the K-theory group. The D-branes located at a submanifold of the spacetime $\mathbb{X}$ support a complex vector bundle E, and the $\bar{D}$-branes support another vector bundle F. We label this configuration by the pair (E, F). The tachyon field is a map $T : E \to F$. If E and F are isomorphic, the brane-anti-brane system will decay into vacuum. That is, the pair (E, F) is said to be equivalent to the pair $(E \oplus H, F \oplus H)$ for any H. The D-brane charge is classified by the equivalence classes of the pairs (E, F) which form nothing but the $K(\mathbb{X})$ group. Hořava's paper used a different definition for $K^{-1}(\mathbb{X})$ which is given in Karoubi's book [34]. The idea is that one starts with the pair (E, α), where E is a complex vector bundle on $\mathbb{X}$, and α is an automorphism on E. A pair (F, β) is called elementary if β is homotopic to the identity on F within automorphisms of F. Two pairs (E_1, α_1) and (E_2, α_2) are said to be equivalent to each other if there exist two elementary pairs (F_1, β_1) and (F_2, β_2) such that $(E_1 \oplus F_1, \alpha_1 \oplus \beta_1) \cong (E_2 \oplus F_2, \alpha_2 \oplus \beta_2)$. The set of all such equivalence classes of pairs (E, α) on $\mathbb{X}$ forms the group $K^{-1}(\mathbb{X})$. Hořava showed that the bundle E is the bundle supported by the Type IIA non-BPS D_9-branes and the role of the automorphism α is played by $\alpha = -e^{i\pi T}$, where T is the tachyon field.

In the paper [21], Hori used the operator algebra to discuss the classification of D-brane charges. In mathematics, this corresponds to a third way to define the K-theory group. Suppose $\mathcal{H}_{\mathbb{C}}$ is a separable infinite dimensional complex Hilbert space. Let $F(\mathcal{H}_{\mathbb{C}})$ denote the space of Fredholm operators on $\mathcal{H}_{\mathbb{C}}$ and $\hat{F}_*(\mathcal{H}_{\mathbb{C}})$ denote the the space of neither positive nor negative skew-adjoint Fredholm operators on $\mathcal{H}_{\mathbb{C}}$. Mathematically (See [32, 33]), it is known that $F(\mathcal{H}_{\mathbb{C}})$ is the classifying space for the K-theory group and $\hat{F}_*(\mathcal{H}_{\mathbb{C}})$ is the classifying space for the K^{-1}-theory group. That is,

$$\begin{aligned} Index &: [\mathbb{X}, F(\mathcal{H}_{\mathbb{C}})] \xrightarrow{\cong} K(\mathbb{X}); \\ Index &: [\mathbb{X}, \hat{F}_*(\mathcal{H}_{\mathbb{C}})] \xrightarrow{\cong} K^{-1}(\mathbb{X}). \end{aligned}$$

Based on the study of the transformation of D-brane charges under T-duality, it was proposed in [21, 22] that the D-brane charges of Type II orientifold on a spacetime $\mathbb{X}$ with involution (3.10) are classified by $KR^{-(9-p)}(\mathbb{X})$ if there are O_p^--planes only and by $KR^{-(5-p)}(\mathbb{X})$ if there are O_p^+-planes only. Our purpose here is to confirm the statement by using the results in the previous chapter. We will show that the space of invariant tachyon configurations is isomorphic to the classifying space of the KR-theory group.

4.1 The Orientifold O_p Theory and The KR^{-n_p}-theory Group

Consider the D_9-branes in Type II orientifolds O_p on $\mathbb{R}^{10}/\mathbb{Z}_2$, where the involution is given in eq(3.10). As before, define n_p=$(9-p)$ mod 8 and $(5-p)$ mod 8 for the O_p^- and O_p^+ theories, respectively.

N D_9–$\bar{D}_9$ pairs in Type IIB orientifold theory support a 2N-dimensional $\mathbb{Z}_2$-graded vector bundle $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. The Chan-Paton factor space $\mathcal{H}^{cp}$ for the 9-9 open strings is $Hom(\mathcal{H}, \mathcal{H})$. The gauge field $\mathcal{A}$ is an even connection of $\mathcal{H}$ and the tachyon $\mathcal{T}$ is an odd endomorphism of $\mathcal{H}$.

$$\mathcal{T} = \begin{pmatrix} 0 & b^\dagger \\ b & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} a^0 & 0 \\ 0 & a^1 \end{pmatrix}. \tag{4.1}$$

where $a^0 = (a^0)^\dagger$, $a^1 = (a^1)^\dagger$ and b is a map $b : \mathcal{H}^0 \to \mathcal{H}^1$.

N non-BPS D_9-branes in Type IIA orientifold theory only support an ungraded N-dimensional vector bundle $\mathcal{H}'$. As mentioned before, it is convenient to introduce a $\mathbb{Z}_2$-graded double $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'$ with an odd operator ξ. The gauge field $\mathcal{A}$ commutes with ξ and the tachyon $\mathcal{T}$ anticommutes with ξ. They can be written into the form

$$\mathcal{T} = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}. \tag{4.2}$$

In Type II orientifold theory, the gauge field and the tachyon field have to satisfy the invariance conditions (3.82) and (3.83), respectively. Define the set of tachyon field configurations,

$$\tilde{\mathfrak{F}} := \{\mathcal{T} \in [Hom(\mathcal{H},\mathcal{H})]^{odd}\}. \tag{4.3}$$

For the orientifold O_p theory on a spacetime $\mathbb{X}$ with involution $\mathcal{I}$, we have the following result

$$Index : [\mathbb{X}, \tilde{\mathfrak{F}}]^{\mathbb{Z}_2} \xrightarrow{\cong} KR^{-n_p}(\mathbb{X}), \tag{4.4}$$

where the $\mathbb{Z}_2$ action on the operators is given by eq(3.83). This confirms the proposal in [21, 22]. When restricted to the set of fixed points $\mathbb{X}_0$ of $\mathbb{X}$ under $\mathcal{I}$, the KR^{-n_p} group reduces to the KO^{-n_p} group as it should be. Next, we will confirm this statement by showing that the classifying space and the symmetry group match case by case.

For the constant operators U and fixed points $\tilde{x} = x$, eq(3.82) and eq(3.83) become

$$\mathcal{A} = -U\mathcal{A}^T U^{-1}, \tag{4.5}$$

$$\mathcal{T} = \zeta(-1)^{|U|} U \mathcal{T}^T U^{-1}. \tag{4.6}$$

Denote the set of invariant tachyon field configurations by

$$\mathfrak{T}_{n_p} := (\tilde{\mathfrak{F}})^{\mathbb{Z}_2} = \{\mathcal{T} \in [Hom(\mathcal{H},\mathcal{H})]^{odd} \,|\, \mathcal{T} = \zeta(-1)^{|U|} U \mathcal{T}^T U^{-1}\}. \tag{4.7}$$

We will show that $\mathfrak{T}_{n_p} \simeq \hat{\hat{\mathfrak{F}}}_{n_p}$ and that the symmetry group matches.

We first consider the n_p even cases. Motivated by the result (3.33) of the parity action on the boundary fermions, we have a generalized relation,

$$U \begin{pmatrix} 0 & -iI_N \\ iI_N & 0 \end{pmatrix}^T U^{-1} = (-1)^{|U|}\zeta \begin{pmatrix} 0 & -iI_N \\ iI_N & 0 \end{pmatrix}. \tag{4.8}$$

$\underline{n_p = \mathbf{0 \ mod \ 8}}$

Eq(3.13) determines U into the form

$$U = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \tag{4.9}$$

where $A^t = A$ and $D^t = D$. Using eq(4.8), we can find that $D = -\zeta A$. Using eq(4.6), we find that

$$\begin{aligned} b &= -\zeta A (b^\dagger)^t D^{-1} \\ &= -\zeta^2 A \bar{b} A^{-1} \\ &= A \bar{b} A^{-1}. \end{aligned}$$

Because A is symmetric, it can be made into the identity matrix $\mathrm{I_N}$. Then we have $b = \bar{b}$. That is, $b : \mathcal{H}^0|_{\mathbb{R}} \to \mathcal{H}^1|_{\mathbb{R}}$ is $\mathbb{R}$-linear. So $\mathfrak{T}_0 \simeq \hat{\hat{\mathfrak{F}}}_0$.

Let us determine the symmetry group. From eq(4.5), we find that $a^0 = -A(a^0)^t A^{-1}$ and $a^1 = -A(a^1)^t A^{-1}$. Then the symmetry group is determined to be SO(N)×SO(N) which matches the result in Table(A.3).

$\underline{n_p = \mathbf{2 \ mod \ 8}}$

Eq(3.13) determines U into the form

$$U = \begin{pmatrix} 0 & B \\ \zeta B^t & 0 \end{pmatrix}. \tag{4.10}$$

Using eq(4.8), we can find that $B = B^t$. Define $F := U^{-1}\mathcal{T}|_{\mathcal{H}^0} : \mathcal{H}^0 \to \mathcal{H}^{*0}$. From eq(4.6), we can show that $F = -F^t$. If we define $\overline{T} := c.c.F : \mathcal{H}^0 \to \mathcal{H}^0$. Then $\overline{T}$ is

$\mathbb{C}$-antilinear. It is easy to see that $\overline{T}^{\dagger} = -\overline{T}$. So $\mathfrak{T}_2 \simeq \hat{\hat{\mathfrak{F}}}_2$. From eq(4.5), we find that $a^0 = -B(a^1)^t B^{-1}$. This means $tr(\mathcal{A}) = 0$. Together with the condition $\mathcal{A} = \mathcal{A}^{\dagger}$, the symmetry group is found to be U(N).

n_p = 4 mod 8

Eq(3.13) determines U into the form

$$U = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \tag{4.11}$$

where $A^t = -A$ and $D^t = -D$. Using eq(4.8), we can find that $D = -\zeta A$. From eq(4.6), we find that $b = A\bar{b}A^{-1}$. Because A is antisymmetric, it can be made into $A = J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$. Suppose,

$$b = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}. \tag{4.12}$$

By the condition $b = J\bar{b}J^{-1}$, b is determined into the form

$$b = \begin{pmatrix} B_1 & B_2 \\ -\bar{B}_2 & \bar{B}_1 \end{pmatrix}. \tag{4.13}$$

Then $b : \mathcal{H}^0 \to \mathcal{H}^1$ is $\mathbb{H}$-linear. So $\mathfrak{T}_4 \simeq \hat{\hat{\mathfrak{F}}}_4$. From eq(4.5), we find that $a^0 = -A(a^0)^t A^{-1} = -J(a^0)^t J^{-1}$ and $a^1 = -A(a^1)^t A^{-1} = -J(a^1)^t J^{-1}$. Then the symmetry group is determined to be Sp(N)×Sp(N).

n_p = 6 mod 8

Eq(3.13) determines U into the form

$$U = \begin{pmatrix} 0 & B \\ -\zeta B^t & 0 \end{pmatrix}. \tag{4.14}$$

Using eq(4.8), we can find that $B = -B^t$. Define $F := U^{-1}\mathcal{T}|_{\mathcal{H}^0} : \mathcal{H}^0 \to \mathcal{H}^{*0}$. From eq(4.6), we can show that $F^t = F$. If we define $A' := c.c.F : \mathcal{H}^0 \to \mathcal{H}^0$, then A' is $\mathbb{C}$-linear. It is easy to see that $(A')^\dagger = A'$. So $\mathfrak{T}_6 \simeq \hat{\hat{\mathfrak{F}}}_6$. From eq(4.5), we find that $a^0 = -B(a^1)^t B^{-1}$. This means $tr(\mathcal{A}) = 0$. Together with the condition $\mathcal{A} = \mathcal{A}^\dagger$, the symmetry group is found to be U(N).

Next, we discuss the n_p odd cases. The difference from the even cases is that the U operator has to satisfy the condition (3.15). The gauge field commutes with the operator ξ and the tachyon field anticommutes with it.

n_p = 1 mod 8

The matrix U is given in eq(4.10) with $B = B^t$. From eq(4.6), we find that

$$b = -B^t b^t B^{-1} = -B\bar{b}B^{-1}.$$

Because B is symmetric, b is purely imaginary. Define $F := \xi\mathcal{T}|_{\mathcal{H}'} = -ib : \mathcal{H}' \to \mathcal{H}'$. Then F is $\mathbb{R}$-linear and skew-adjoint. So $\mathfrak{T}_1 \simeq \hat{\hat{\mathfrak{F}}}_1$. Next, we determine the symmetry group. Eq(4.5) gives the condition $a = -Ba^t B^{-1}$. Then the symmetry group is SO(N).

n_p = 3 mod 8

The matrix U is given in eq(4.11) with $D = -\zeta A$. From eq(4.6), we find that $b = A\bar{b}A^{-1}$.

Because A is antisymmetric, it can be made into J. Suppose

$$b = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}. \tag{4.15}$$

By the condition $b = J\bar{b}J^{-1}$, b is determined into the form

$$b = \begin{pmatrix} B_1 & B_2 \\ -\bar{B}_2 & \bar{B}_1 \end{pmatrix}. \tag{4.16}$$

Then b is $\mathbb{H}$-linear. Define $F := i\xi\mathcal{T}|_{\mathcal{H}'} = b : \mathcal{H}' \to \mathcal{H}'$. We can show that F is $\mathbb{H}$-linear and self-adjoint. So $\mathfrak{T}_3 \simeq \hat{\hat{\mathfrak{F}}}_3$. Eq(4.5) gives the condition $a = -Aa^tA^{-1} = -Ja^tJ^{-1}$. Then the symmetry group is Sp(N).

n_p **= 5 mod 8**

The matrix U is given in eq(4.14) with $B = -B^t$. From eq(4.6), we find that

$$b = B^t b^t B^{-1} = -B\bar{b}B^{-1}.$$

Define $b' := ib$, then we can show that b' is $\mathbb{H}$-linear. Define $F := \xi\mathcal{T}|_{\mathcal{H}'} = -b' : \mathcal{H}' \to \mathcal{H}'$. We can show that F is $\mathbb{H}$-linear and skew-adjoint. So $\mathfrak{T}_5 \simeq \hat{\hat{\mathfrak{F}}}_5$. Eq(4.5) gives the condition $a = -Ba^tB^{-1}$. Then the symmetry group is Sp(N).

n_p **= 7 mod 8**

The matrix U is given in eq(4.9) with $D = -\zeta A$. From eq(4.6), we find that

$$b = -iD\bar{b}A^{-1} = A\bar{b}A^{-1}. \tag{4.17}$$

Because A is symmetric, it can be made into the identity matrix I_N. So we can show that b is $\mathbb{R}$-linear. Define $F := i\xi\mathcal{T}|_{\mathcal{H}^0_{r+1}} = b : \mathcal{H}' \to \mathcal{H}'$. We can show that F is $\mathbb{R}$-linear and self-adjoint. So $\mathfrak{T}_7 \simeq \hat{\hat{\mathfrak{F}}}_7$. Eq(4.5) gives the condition $a = -Aa^tA^{-1} = -a^t$. Then the symmetry group is SO(N).

Chapter 5

D-branes in Orientifolds of Linear Sigma Models

In this chapter, we will study the D-branes in orientifolds of the 2d $\mathcal{N}$=(2,2) linear sigma model(LSM). In Witten's paper [52], the LSM is used to provide an unified description of the closed string worldsheet theory over its moduli space. This property makes the LSM become more and more important in Type II orientifold model buildings [54, 55]. Recently, Herbst, Hori and Page [57] used the LSM on the worldsheet with boundary to study the B-type D-brane transformation in the moduli space. This chapter is to extend their work to the B-type D-branes in orientifolds of the LSM.

5.1 The Linear Sigma Model and Its Phases

Before introducing the LSM, we first discuss a little about the supersymmetry in two-dimensional manifolds. The supercharges associated with the supersymmetry are Weyl-

Majorana spinors. We can have N left-moving supercharges Q_L^A and $\tilde{N}$ right-moving supercharges Q_R^A. The supersymmetry algebra is called the 2d $\mathcal{N}$=(N,$\tilde{N}$) algebra. We focus on the 2d $\mathcal{N}$=(2,2) supersymmetry. Introduce a 2d $\mathcal{N}$=(2,2) superspace, denoted by the coordinates $(t, s, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)$. There exist the so-called vector and axial R-rotations on the superspace coordinates:

$$\begin{aligned} \text{Vector R} - \text{rotation}: \quad & (t, s, \theta^\pm, \bar{\theta}^\pm) \longmapsto (t, s, \mathrm{e}^{-i\alpha}\theta^\pm, \mathrm{e}^{i\alpha}\bar{\theta}^\pm), \\ \text{Axial R} - \text{rotation}: \quad & (t, s, \theta^\pm, \bar{\theta}^\pm) \longmapsto (t, s, \mathrm{e}^{\mp i\alpha}\theta^\pm, \mathrm{e}^{\pm i\alpha}\bar{\theta}^\pm). \end{aligned}$$

Suppose we have a classical 2d $\mathcal{N}$=(2,2) field theory. There are four supercharges: $Q_+, \bar{Q}_+, Q_-$ and $\bar{Q}_-$. From the Poincaré invariance, we obtain three conserved charges, the Hamiltonian(H), the momentum(P) and the angular momentum(M). If the theory also has the vector and the axial R-rotation invariance (R-symmetry), there are corresponding conserved charges: F_V and F_A. The 2d $\mathcal{N}$=(2,2) supersymmetry algebra is the following,

$$\begin{aligned} Q_+^{\,2} = Q_-^{\,2} &= \bar{Q}_+^{\,2} = \bar{Q}_-^{\,2} = 0, \\ \{Q_\pm, \bar{Q}_\pm\} &= H \pm P, \end{aligned}$$

$$\begin{aligned} [iM, Q_\pm] = \mp Q_\pm, &\quad [iM, \bar{Q}_\pm] = \mp \bar{Q}_\pm, \\ [iF_V, Q_\pm] = -iQ_\pm, &\quad [iF_V, \bar{Q}_\pm] = i\bar{Q}_\pm, \\ [iF_A, Q_\pm] = \mp iQ_\pm, &\quad [iF_A, \bar{Q}_\pm] = \pm i\bar{Q}_\pm. \end{aligned}$$

All the other commutation relations vanish.

There are several widely studied 2d $\mathcal{N}$=(2,2) theories. One is called the non-linear sigma model(NLSM). Suppose we have N chiral superfields Φ_i. The Lagrangian is given by

$$\mathcal{L} = \int d^4\theta K(\Phi_i, \bar{\Phi}_i), \tag{5.1}$$

where $K(\Phi_i, \bar{\Phi}_i)$ is a general real function. The field space spanned by Φ_i's is called the target space of the NLSM. Of course, we can add a F-term to the above model. The total Lagrangian will become

$$\mathcal{L} = \int d^4\theta K(\Phi_i, \bar{\Phi}_i) + \frac{1}{2}(\int d^2\theta W(\Phi_i) + c.c.), \tag{5.2}$$

where $d^2\theta = d\theta^- d\theta^+$. $W(\Phi_i)$ is called superpotential which is a holomorphic function of $\Phi_i, \cdots, \Phi_N$. This theory is called the Landau-Ginzburg model(LG) in literature.

We are interested in the LSM with abelian gauge group $\Xi = U(1)_1 \times \cdots \times U(1)_k$ and chiral superfields $\Phi = (\Phi_1, \cdots, \Phi_N)$, where Φ_i has the charge Q_i^a under the a^{th} gauge group $\text{U}(1)_a$. The vector superfield for $\text{U}(1)_a$ is denoted by V_a and its field strength $\Sigma_a = \bar{D}_+ D_- V_a$ is a twisted chiral superfield. The bulk part of the Lagrangian is

$$\begin{aligned} \mathcal{L}_{bulk} &= \int d^4\theta \left(- \sum_{a=1}^{k} \frac{1}{2e^2} \overline{\Sigma}_a \Sigma_a + \sum_{i=1}^{N} \bar{\Phi}_i e^{Q_i \cdot V} \Phi_i \right) \\ &\quad + Re \int d^2\widetilde{\theta} \left(- \sum_{a=1}^{k} t^a \Sigma_a \right) + Re \int d^2\theta \, W(\Phi), \end{aligned} \tag{5.3}$$

where

$$t^a = r^a - i\theta^a \tag{5.4}$$

is a complex combination of the Fayet-Iliopoulos (FI) parameter r^a and the theta angle θ^a for the a^{th} gauge group $\text{U}(1)_a$. The first term in eq(5.3) is the kinetic term for the vector superfields and e is the gauge coupling. The second term is the kinetic term for the chiral superfields with minimal coupling to the gauge fields. The third term is called the twisted superpotential term (note that $d^2\tilde{\theta} = d\bar{\theta}^- d\theta^+$) and the last term is the superpotential term where $W(\Phi)$ is a gauge invariant holomorphic polynomial of $\Phi_1, \cdots, \Phi_N$. There is an axial $U(1)_A$ R-symmetry in the classical LSM:

$$U(1)_A : \Phi_i(t, s, \theta^{\pm}, \bar{\theta}^{\pm}) \longmapsto \mathrm{e}^{i\alpha R_A^i} \Phi_i(t, s, \mathrm{e}^{\mp i\alpha}\theta^{\pm}, \mathrm{e}^{\pm i\alpha}\bar{\theta}^{\pm}). \tag{5.5}$$

The classical LSM will also have a vector $U(1)_V$ R-symmetry

$$U(1)_V : \Phi_i(t, s, \theta^{\pm}, \bar{\theta}^{\pm}) \longmapsto \mathrm{e}^{i\alpha R_i} \Phi_i(t, s, \mathrm{e}^{-i\alpha}\theta^{\pm}, \mathrm{e}^{i\alpha}\bar{\theta}^{\pm}), \tag{5.6}$$

if the superpotential $W(\Phi)$ is of a quasi-homogeneous form,

$$W(\lambda^{R_1}\Phi_1, \ldots, \lambda^{R_N}\Phi_N) = \lambda^2 W(\Phi_1, \ldots, \Phi_N), \tag{5.7}$$

where $R_1, ..., R_N$ are called the (vector) R-charges of the fields $\Phi_1, ..., \Phi_N$.

The classical potential for the scalar fields ϕ_i and σ_a, which are the scalar components of Φ_i and V_a respectively, is

$$\mathrm{U} = \sum_{i=1}^{N} \left| \sum_{a=1}^{k} Q_i^a \sigma_a \phi_i \right|^2 + \frac{e^2}{2} \sum_{a=1}^{k} \left(\sum_{i=1}^{N} Q_i^a |\phi_i|^2 - r^a \right)^2 + \sum_{i=1}^{N} \left| \frac{\partial W}{\partial \phi_i}(\phi) \right|^2. \tag{5.8}$$

The first term is from the coupling of Φ and V. The second term is called the D-term potential and the last term is called the F-term potential which is from the superpotential W. We want to find out the vacuum locus, U=0. It depends very much on the value of the FI parameters r^a through the D-term equations,

$$\sum_{i=1}^{N} Q_i^a |\phi_i|^2 - r^a = 0 \qquad \forall a = 1, \ldots, k. \tag{5.9}$$

The space parameterized by FI parameters will be divided into a finite number of chambers by the solutions to U=0. The chambers are called the *phases* and the walls separating them are called the *phase boundaries.*

A *geometric* phase is a phase in which Ξ is completely broken and all modes transverse to U=0 are massive. The low energy theory is a non-linear sigma model with target space $(\mathrm{U}=0)/\Xi$. For the superpotential $W=0$ case, the space $(\mathrm{U}=0)/\Xi$ can be regarded as the quotient by the complexified gauge group $\Xi_{\mathbb{C}}$,

$$X_r = (\mathbb{C}^N - \Delta_r)/\Xi_{\mathbb{C}}, \tag{5.10}$$

where Δ_r consists of points whose $\Xi_{\mathbb{C}}$ orbits do not pass through solutions to (5.9). X_r is a so-called *toric manifold.* If $W \neq 0$, the vacuum locus is a submanifold of X_r defined by the F-term equations

$$\frac{\partial W}{\partial \phi_i} = 0 \qquad \forall i = 1, \ldots, N. \tag{5.11}$$

There are various phases in which some of the transverse modes to U=0 are massless. The extreme cases are the so-called *Landau-Ginzburg* phases in which the vacuum locus $(\mathrm{U}=0)/\Xi$ is one point and all modes transverse to the $\Xi_{\mathbb{C}}$ orbit are massless. In the limit where r is scaled up to infinity, the modes tangent to the $\Xi_{\mathbb{C}}$ orbit decouple and the theory reduces to the Landau-Ginzburg model for the transverse modes. In some cases, there possibly exists a residual discrete gauge symmetry, then the theory is called the Landau-Ginzburg orbifold(LGO) model.

Next, we want to discuss the quantum behavior of the FI parameters. They run as $r^a(\mu) = r^a(\mu') + \sum_{i=1}^N Q_i^a \log(\mu/\mu')$ under the renormalization group(RG) where μ is some energy scale. In general, this induces a flow between different phases. However, if the condition

$$\sum_{i=1}^{N} Q_i^a = 0 \quad \forall a = 1, \ldots, k, \tag{5.12}$$

is satisfied, the FI parameters do not run and are genuine parameters of the quantum theory. It is also easy to show that all the k theta angles θ^a are also genuine parameters. In other words, if the two conditions (5.7) and (5.12) are satisfied, the quantum LSM has both axial and vector U(1) R-symmetries. From now on, let us assume that the two conditions (called Calabi-Yau conditions) are met. A singularity of the LSM is a point in the moduli space where there exists an unbroken continuous subgroup of Ξ and the corresponding σ_a is unconstrained. The singularity can be determined by computing the effective potential at large values of σ's, which is

$$\mathrm{U}_{\text{eff}} = \frac{1}{2} \sum_{a,b=1}^{k} e_{ab}^2(\sigma) \frac{\partial \overline{\widetilde{W}}_{\text{eff}}}{\partial \overline{\sigma}_a} \frac{\partial \widetilde{W}_{\text{eff}}}{\partial \sigma_b}. \tag{5.13}$$

$e_{ab}^2(\sigma)$ are the effective gauge coupling constants which approach their classical values as $|\sigma| \to \infty$. $\widetilde{W}_{\text{eff}}(\sigma)$ is the effective twisted superpotential whose first derivatives are given by

$$\frac{\partial \widetilde{W}_{\text{eff}}}{\partial \sigma_a}(\sigma) = -t^a - \sum_{i=1}^{N} Q_i^a \log \left(\sum_{b=1}^{k} Q_i^b \sigma_b \right). \tag{5.14}$$

It is easy to see that the effective potential is zero if the derivatives (5.14) vanish modulo $2\pi i$ times integers [52]. In other words, the solution to the vacuum equation is given by

$$t^a = - \sum_{i=1}^{N} Q_i^a \log \left(\sum_{b=1}^{k} Q_i^b \sigma_b \right). \tag{5.15}$$

According to eq(5.12), this equation is invariant under the uniform rescaling of σ_a's. So the existence of one solution means the existence of a non-compact Coulomb branch. After eliminating σ_a's one can get an equation for e^{-t^a}'s that determines the location of a quantum Coulomb branch. Denote the set of singular points by $\mathcal{S}$ and the set of complex parameters $\{(\mathrm{e}^{t^1}, ..., \mathrm{e}^{t^k})\}$ by $(\mathbb{C}^\times)^k$. The Kähler moduli space is the complement

$$\mathfrak{M}_K = (\mathbb{C}^\times)^k \setminus \mathcal{S}.$$

5.2 D-branes and Orientifolds of The LSM

The D-branes in the LSM with a non-zero superpotential W can be regarded as the boundary of the worldsheet and can be described by adding the corresponding boundary interaction to the bulk Lagrangian, which is

$$\mathbf{A}_t = \rho_*(v_0 - \mathrm{Re}(\sigma)) + \frac{1}{2}\{Q, Q^\dagger\} - \frac{1}{2}\sum_{i=1}^{N} \psi^i \frac{\partial}{\partial \phi_i} Q + \frac{1}{2}\sum_{i=1}^{N} \bar{\psi}_i \frac{\partial}{\partial \bar{\phi}_i} Q^\dagger, \tag{5.16}$$

where we have an odd operator $Q(\phi_i)$ on the $\mathbb{Z}_2$-graded Chan-Paton vector space, $\mathcal{V} = \mathcal{V}^{\mathrm{ev}} \oplus \mathcal{V}^{\mathrm{od}}$, and a representation ρ of the gauge group $\mathrm{U}(1)^k$ on $\mathcal{V}$. This is nothing but the Wilson line interaction term, so the brane is also called the Wilson line brane $\mathcal{W}(q)$. Since the Chan-Paton vector space has dimension larger than one, the boundary interaction term is in the path-ordered exponential form

$$U(t_f, t_i) = \mathrm{P}\exp\left(-i\int_{t_i}^{t_f} \mathbf{A}_t \mathrm{d}t\right).$$

The total Lagrangian, the boundary term combined with eq(5.3), is supersymmetric if and only if Q is a matrix factorization of W, i.e., $Q^2 = W \cdot id_{\mathcal{V}}$. Another thing to notice is that the total theory only depends on the combination $\frac{\theta^a}{2\pi} + q^a$. In other words, the theory is invariant under

$$\theta^a \to \theta^a + 2\pi m^a, \text{ and } q^a \to q^a - m^a, \tag{5.17}$$

for integers $m^1, \cdots, m^k$. Because the worldsheet has boundaries, the θ^a's are no longer periodic parameters. The transformation $\theta^a \to \theta^a + 2\pi$ with no change in q^a is a non-trivial operation. Depending on what combination of the $\mathcal{N}$=2 supersymmetry is preserved, we have two types of D-branes in 2d $\mathcal{N}$=2 theories. One is called the A-type D-brane (also called the A-brane) which preserves the supersymmetry generated by $Q_A := \bar{Q}_+ + e^{i\alpha}Q_-$, where α is some phase. The other is called the B-type D-brane (the B-brane for short) which preserves the supersymmetry generated by $Q_B := \bar{Q}_+ + e^{i\alpha}\bar{Q}_-$. The two types of branes can be related by the so-called mirror symmetry [53]. We will focus on B-branes in this thesis (taking α to be π). To make a connection with what we have studied in chapter 3, the tachyon configuration is nothing but $\mathcal{T} = iQ - iQ^\dagger$. After this identification, the two boundary interaction terms (3.70) and (5.16) agree with each other.

A B-brane in the LSM can be represented as $\mathcal{B} = (\mathcal{V}, \rho, Q(\phi), R(\lambda))$, satisfying

$$\rho(g)^{-1}Q(g \cdot \phi)\rho(g) = Q(\phi), \tag{5.18}$$

$$R(\lambda)Q(R_\lambda \cdot \phi)R(\lambda)^{-1} = \lambda Q(\phi), \tag{5.19}$$

$$\rho(g)R(\lambda) = R(\lambda)\rho(g), \tag{5.20}$$

$$R(e^{\pi i}) = \sigma_{\mathcal{V}}, \tag{5.21}$$

where $R(\lambda)$ is the R-symmetry operator of the model(λ is a phase), $\sigma_{\mathcal{V}}$ is the grading operator on $\mathcal{V}$ and $g \cdot \phi$ is the gauge transformation of $\phi = (\phi_1, \cdots, \phi_N)$. Eq(5.18) is required because we want $\mathbf{A}_t$ to transform under the gauge transformation as

$$i\mathbf{A}_t \longrightarrow \rho(g) i\mathbf{A}_t \rho(g)^{-1} + \rho(g)\partial_t \rho(g)^{-1}. \tag{5.22}$$

Eq(5.19) is from the fact that the superpotential $W(\Phi)$ has R-charge 2 and $Q^2 = W \cdot id_{\mathcal{V}}$. Eq(5.20) is because we want the R-symmetry action to commute with the gauge group action. Eq(5.21) is from the following consideration. Suppose we have two B-branes, $\mathcal{B}_1 = (\mathcal{V}_1, \rho_1, Q_1(\phi), R_1(\lambda))$ and $\mathcal{B}_2 = (\mathcal{V}_2, \rho_2, Q_2(\phi), R_2(\lambda))$. Consider any polynomial φ of $\phi_1, \cdots, \phi_N$ in $Hom(\mathcal{V}_1, \mathcal{V}_2)$. We have the equation

$$R_2(-1)\varphi(R_{e^{\pi i}}(\phi))R_1(-1)^{-1} = (-1)^{|\varphi|}\varphi(\phi).$$

This requires that $R(-1) = \sigma_{\mathcal{V}}$. The collection of all B-branes (i.e, matrix factorizations) in a LSM with gauge group Ξ and superpotential $W(\Phi)$ forms a category, called the category of matrix factorizations. We denote it by $MF_{\Xi_{\mathbb{C}}}(W)$.

The 2d $\mathcal{N}$=2 superspace has A- and B-type involutive parity symmetries: $\Omega_A(t, s, \theta^\pm, \bar{\theta}^\pm)$ $=(t, -s, -\bar{\theta}^\mp, -\theta^\mp)$ and $\Omega_B(t, s, \theta^\pm, \bar{\theta}^\pm) = (t, -s, \mp\theta^\mp, \mp\bar{\theta}^\mp)$. We focus on the B-type parity in the following. It will induce B-type worldsheet parities $\mathcal{P}^B_\tau = \tau^* \cdot \Omega^*_B$ in the LSM, where τ denotes some involution on the field space. $\mathcal{P}^B_\tau$ transforms the fields and the superpotential as

$$\Phi_i \rightarrow e^{i\theta_i}\Omega^*_B \Phi_{\sigma(i)},\ V_a \rightarrow {\sigma_a}^b \Omega^*_B V_b, \text{ and } W(e^{i\theta_i}\Phi_{\sigma(i)}) \rightarrow W(\Phi_i),$$

where $({\sigma_a}^b)$ denotes permutation on the worldsheet fields.

The parity operation $\mathcal{P}^B_\tau$ can be regarded as a parity functor on the category $MF_{\Xi_{\mathbb{C}}}(W)$.

In other words, it is a contravariant functor

$$\mathcal{P}_\tau^B : MF_{\Xi_\mathbb{C}}(W) \to MF_{\Xi_\mathbb{C}}(W), \tag{5.23}$$

satisfying the condition $\mathcal{P}_\tau^B \circ \mathcal{P}_\tau^B \cong id_{MF_{\Xi_\mathbb{C}}(W)}$. It acts on the B-brane by

$$\mathcal{P}_\tau^B : \mathcal{B} \longmapsto \mathcal{P}_\tau^B(\mathcal{B}) = (\mathcal{V}^*, g^{q_o}\rho(g)^{-T}, -i\tau^* Q(\phi)^T, \lambda^{p_o} R(\lambda)^{-T}). \tag{5.24}$$

A B-brane is called an invariant brane if there exists an isomorphism

$$U : \mathcal{P}_\tau^B(\mathcal{B}) \mapsto \mathcal{B}. \tag{5.25}$$

As we mentioned before, this isomorphism should be regarded as the isomorphism between the infrared limits of the boundary interactions associated to brane $\mathcal{B}$ and brane $\mathcal{P}_\tau^B(\mathcal{B})$. A gauge transformation between the boundary interactions naturally reduces to an isomorphism between the infrared limits. As discussed in [57], there exist other operations on D-branes that lead to isomorphisms in the infrared limit. Two classes of such operations were discussed, the boundary D-term deformations and the brane-antibrane annihilation. They are called D-isomorphisms. In the NLSM, the D-isomorphisms are nothing but the quasi-isomorphisms of complexes.

For the category $MF_{\Xi_\mathbb{C}}(W)$ of the LSM, U is a linear isomorphism in $Hom(\mathcal{V}^*, \mathcal{V})$ such that

$$U(-i\tau^* Q(\phi)^T)U^{-1} = Q(\phi), \tag{5.26}$$

$$U(g^{q_o}\rho(g)^{-T})U^{-1} = \rho(g), \tag{5.27}$$

$$U(\lambda^{p_o} R(\lambda)^{-T})U^{-1} = R(\lambda). \tag{5.28}$$

As we know, a typical LSM contains several different phases. It is interesting to see how a B-brane transports among the different phases and how the worldsheet parity acts on the branes in different phases. In the following we will use the quintic model to show this.

5.3 The Quintic Model

The quintic model is a LSM with a single U(1) gauge group, and 6 worldsheet fields $P, X_1, \ldots, X_5$ with charge $-5, 1, \ldots, 1$. The superpotential is

$$W = PG(X) = P(X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5). \tag{5.29}$$

It has two different phases, the $r \gg 0$ phase and the $r \ll 0$ phase. In the $r \gg 0$ phase, the theory reduces to the non-linear sigma model on the Calabi-Yau hypersurface $X_G = \{G = 0\}$ in the projective space $\mathbf{CP}^4$. In the $r \ll 0$ phase, the vacuum manifold breaks the gauge group down to a discrete group $\Gamma = \{\omega \in \mathrm{U}(1) | \omega^5 = 1\} \cong \mathbb{Z}_5$. In the $r \to -\infty$ limit, the theory reduces to the Landau-Ginzburg orbifold theory of $X_1, \ldots, X_5$ with the superpotential $W_G = G(X)$ and the orbifold group Γ. The singularities are located at $\mathrm{e}^t = -5^5$. The phase diagram of this quintic model is shown in Fig(5.1).

According to our notation, a B-brane in the quintic model is denoted by $\mathcal{B} = (\mathcal{V}, \rho(g), Q(p, x), R(\lambda))$. The category of matrix factorizations in the quintic model is denoted by $MF_{\mathbb{C}^\times}(PG(X))$. Choose a basis of the 2n-dimensional space $\mathcal{V}$ such that $\rho(g)$ and $R(\lambda)$ are simultaneously

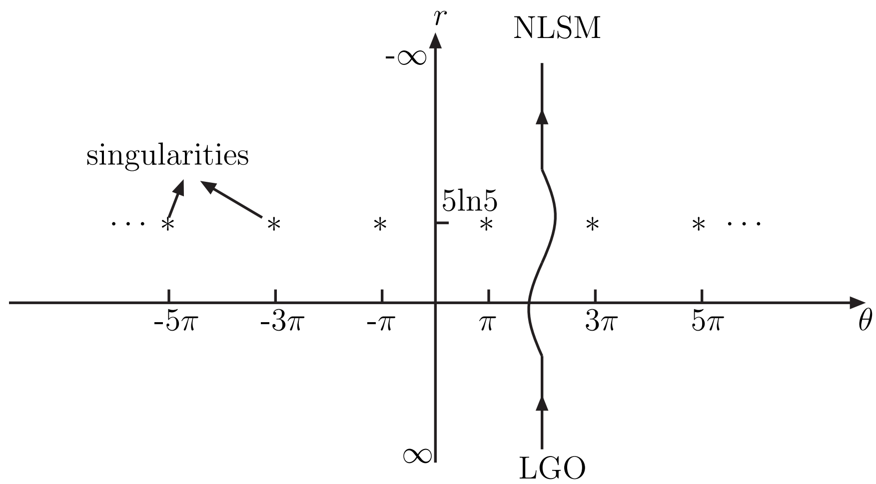

Figure 5.1: The phase diagram of the quintic model.

diagonalized. We can write the data into an explicit matrix form:

$$Q(p,x) = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix},$$

$$\begin{aligned} \rho(g) &= diag(g^{q_1}, \cdots, g^{q_{2n}}), && (5.30) \\ R(\lambda) &= diag(\lambda^{R_1}, \cdots, \lambda^{R_{2n}}), && (5.31) \end{aligned}$$

where $f \cdot g = g \cdot f = pG(x) \cdot id_{\mathcal{V}}$. Denote a B-brane in the NLSM by $\tilde{\mathcal{B}} = (\tilde{\mathcal{V}}, \tilde{\rho}(g), \tilde{Q}(p,x), \tilde{R}(\lambda))$ satisfying

$$\begin{aligned} \tilde{\rho}(g)^{-1}\tilde{Q}(g^{-5}p, gx)\tilde{\rho}(g) &= \tilde{Q}(p,x), && (5.32) \\ \tilde{R}(\lambda)\tilde{Q}(\lambda^2 p, x)\tilde{R}(\lambda)^{-1} &= \lambda\tilde{Q}(p,x), && (5.33) \\ \tilde{\rho}(g)\tilde{R}(\lambda) &= \tilde{R}(\lambda)\tilde{\rho}(g), && (5.34) \end{aligned}$$

where $\tilde{R}(\lambda)$ is the R-symmetry operator of the NLSM phase, $\tilde{\rho}(g)$ is the representation on the space $\tilde{\mathcal{V}}$ and $\tilde{Q}(p,x)^2 = 0$. We can decompose $\tilde{\mathcal{V}}$ into a direct sum of even components (R-charge even) and odd components (R-charge odd) $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}^{ev} \oplus \tilde{\mathcal{V}}^{od}$. Then the data of a brane can be translated into a complex in the NLSM. It is found that the collection of B-branes in the NLSM forms a derived category of coherent sheaves over X_G [58]. Let us denote it by $D^b(Coh\ X_G)$. Denote a B-brane in the LGO by $\bar{\mathcal{B}} = (\bar{\mathcal{V}}, \bar{\rho}(\omega), \bar{Q}(x), \bar{R}(\lambda))$.

$$\bar{\rho}(\omega)^{-1}\bar{Q}(\omega x)\bar{\rho}(\omega) = \bar{Q}(x), \tag{5.35}$$

$$\bar{R}(\lambda)\bar{Q}(\lambda^{2/5}x)\bar{R}(\lambda)^{-1} = \lambda\bar{Q}(x), \tag{5.36}$$

$$\bar{\rho}(\omega)\bar{R}(\lambda) = \bar{R}(\lambda)\bar{\rho}(\omega), \tag{5.37}$$

$$\bar{R}(\mathrm{e}^{\pi i})\bar{\rho}(\mathrm{e}^{2\pi i/5}) = \sigma_{\bar{\mathcal{V}}}, \tag{5.38}$$

where $\bar{R}(\lambda)$ is the R-symmetry operator in the LGO phase, $\bar{\rho}(\omega)$ is a representation of $\mathbb{Z}_5$ on $\bar{\mathcal{V}}$ and $\bar{Q}(x)^2 = G(x)\cdot id_{\bar{\mathcal{V}}}$. Obviously, the collection of B-branes in the LGO phase forms a category of matrix factorizations, denoted by $MF_{\mathbb{Z}_5}(G(X))$.

A brane can transport between the two phases freely along a path (not hitting singular points) in the moduli space of the quintic model. This will induce equivalence relations among the categories $MF_{\mathbb{C}^\times}(PG(X))$, $D^b(Coh\ X_G)$ and $MF_{\mathbb{Z}_5}(G(X))$. In other words, we pick up a brane in a phase, say the LGO phase, lift it to the corresponding brane in the quintic model. Transport the brane inside the quintic model moduli space to another phase, the NLSM phase. And then project the brane down to a brane in the NLSM. The problem is that when we lift a brane in the LGO or the NLSM to a brane in the quintic model the lifting is not unique. In the paper [57], the authors studied the problem and gave a so-called *grade restriction rule*. In the quintic model, the singular points are located at $r = 5\ln 5$ and $\theta = 2\pi\mathbb{Z} + \pi\mathbf{S}$, where $\mathbf{S} := \sum_{Q_i>0} Q_i$ is 5 for the quintic model. We

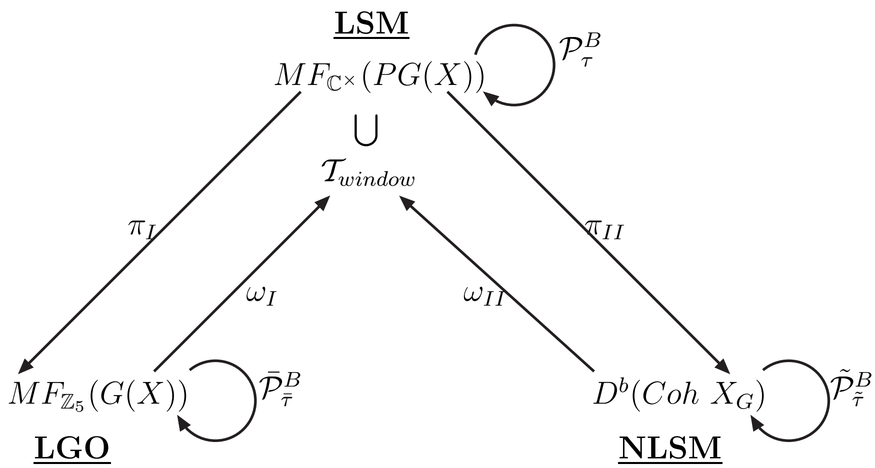

Figure 5.2: The brane transportation in the orientifold of the quintic model.

want to transport branes along a path avoiding singular points. That is, the path must go through the windows between singular points (See a typical path in Fig(5.1)). The grade restriction rule says that the Wilson line brane $\mathcal{W}(q)$ can be transported smoothly between two phases if and only if the following inequality is satisfied

$$-\frac{\mathbf{S}}{2} < \frac{\theta}{2\pi} + q < \frac{\mathbf{S}}{2} \ . \tag{5.39}$$

If we have a complex of Wilson line branes $\mathcal{W}(q_i)$ or a matrix factorization, each q_i must satisfy the inequality (5.39). For a given window, this inequality determines the allowed values of the gauge charges. For example, in the quintic model ($\mathbf{S} = 5$), for the window $\pi < \theta < 3\pi$, the allowed charges are $\{-3, -2, -1, 0, 1\}$. When the window changes, the allowed charges also change. For a general abelian gauge group $\Xi = U(1)^k$, there is a band grade restriction rule:

$$-\frac{\mathbf{S}^a}{2} < \frac{\theta^a}{2\pi} + q^a < \frac{\mathbf{S}^a}{2}, \tag{5.40}$$

for $a = 1, \cdots, k$ and $\mathbf{S}^a = \sum_{Q_i^a > 0} Q_i^a$.

Denote the category of the grade restricted branes of a LSM by $\mathcal{T}_{window}$. The paper

[57] constructed the equivalence relations among the categories explicitly. We will discuss the worldsheet parity functors on the categories (See Fig(5.2)).

5.3.1 The Worldsheet Parity Functor on Categories

Let us start with the worldsheet parity in the LGO phase. From [55] we know that the B-type parity in the LGO is simply $\bar{\mathcal{P}}^B_{\vec{m}} = \tau^*_{\vec{m}}\Omega^*_B$. The action of $\tau_{\vec{m}}$ is now the phase rotation $X_i \to e^{i2\pi m_i/5}X_i = \omega^{m_i}X_i$ where $\omega = e^{2i\pi/5}$, which maps $G(X) \to G(X)$.

According to [55], the RCFT invariant branes are those with $M = 0, 5$. For simplicity, consider the case $\vec{L} = (0,0,0,0,0)$, $M = 0$ and $\vec{m} = (0,0,0,0,0)$. In the matrix factorization language, the invariant brane $\bar{\mathcal{B}} = (\bar{\mathcal{V}}, \bar{\rho}(g), \bar{Q}(x), \bar{R}(\lambda))$ can be obtained as the tensor product of branes of minimal models. For example, we have

$$\bar{Q}(x)_{\vec{L}} = \bar{Q}_{L_1}(x_1) \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} + ... + \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \bar{Q}_{L_5}(x_5)\ , \tag{5.41}$$

where

$$\bar{Q}_{L_i}(x_i) = \begin{pmatrix} 0 & x_i^{L_i+1} \\ x_i^{4-L_i} & 0 \end{pmatrix} \quad , \quad \mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{5.42}$$

A more compact way to write this expression is $\bar{Q}(x)_{\vec{L}} = \sum_{i=1}^{5}(\, x_i^{L_i+1}\eta_i + x_i^{4-L_i}\overline{\eta}_i)$ which is also called the Recknagel-Schomerus brane in literature. The space $\bar{\mathcal{V}}$ is the 32-dimensional representation space spanned by $\{|0\rangle\,, \overline{\eta}_i\,|0\rangle\,, \cdots, \overline{\eta}_1 \cdots \overline{\eta}_5\,|0\rangle\}$. We also find the corresponding representation $\bar{\rho}(\omega)$ and $\bar{R}(\lambda)$

$$\begin{aligned} \bar{R}(\lambda) &= \bar{R}_0(\lambda)^{\otimes 5} = \begin{pmatrix} \lambda^{\frac{2}{5}} & 0 \\ 0 & \lambda^{-\frac{1}{5}} \end{pmatrix}^{\otimes 5} \lambda^{\frac{2}{5}n}, \\ \bar{\rho}(\omega) &= \bar{\rho}_0(\omega)^{\otimes 5} = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^1 \end{pmatrix}^{\otimes 5} \omega^{-n}. \end{aligned} \tag{5.43}$$

In order that $\bar{R}_{|0\rangle} = 0$ is obtained, we can take $\frac{2}{5}n = -2$, or $n = -5$.

The parity will transform the brane $\bar{\mathcal{B}}$ to $\bar{\mathcal{P}}^B_{\vec{0}}(\bar{\mathcal{B}}) = (\bar{\mathcal{V}}^*, g^{\bar{q}_o}\bar{\rho}(g)^{-T}, -i\tau^*_{\vec{0}}\bar{Q}(x)^T, \lambda^{\bar{p}_o}\bar{R}(\lambda)^{-T})$. For an invariant brane, there is an isomorphism between $\bar{\mathcal{P}}^B_{\vec{0}}(\bar{\mathcal{B}})$ and $\bar{\mathcal{B}}$. According to [56], for the category $MF_{\mathbb{Z}_5}(G(X))$ of the LGO, $\bar{U}$ is a linear isomorphism $\bar{U} : \bar{\mathcal{V}}^* \longrightarrow \bar{\mathcal{V}}$ such that

$$\begin{aligned} \bar{U}(-i\tau^*_{\vec{0}}\bar{Q}(x)^{\mathrm{T}})\bar{U}^{-1} &= \bar{Q}(x), \\ \omega^{\bar{q}_o}\bar{U}\bar{\rho}(\omega)^{-\mathrm{T}}\bar{U}^{-1} &= \bar{\rho}(\omega), \\ \lambda^{\bar{p}_o}\bar{U}\bar{R}(\lambda)^{-\mathrm{T}}\bar{U}^{-1} &= \bar{R}(\lambda). \end{aligned} \tag{5.44}$$

By looking at the worldsheet parity operator in the minimal model, we can find the worldsheet parity operator in the quintic model

$$\bar{U} = \bar{\mathcal{U}}_0^{\otimes 5} = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}^{\otimes 5}. \tag{5.45}$$

Then it is easy to find that $\bar{q}_o$=0 mod 5 and $\bar{p}_o$=−3. Following the paper [57], we can lift the brane $\bar{\mathcal{B}}$ in the LGO to a brane $\mathcal{B}$ obeying the grade restriction rule. Similarly, the worldsheet parity operator $\bar{U}$ of the LGO is lifted to the parity operator U in the LSM. The

B-type parity $\mathcal{P}^B_{\vec{0}}$ will transform a brane $\mathcal{B}$ to $\mathcal{P}^B_{\vec{0}}(\mathcal{B}) = (\mathcal{V}^*, g^{q_o}\rho(g)^{-\mathrm{T}}, -i\tau^*_{\vec{0}} Q(p,x)^{\mathrm{T}}, \lambda^{p_o} R(\lambda)^{-\mathrm{T}})$. Looking at Fig(5.2), there are two different ways to lift the brane and apply the worldsheet parity: $\bar{\mathcal{B}} \stackrel{\omega_I}{\to} \mathcal{B} \stackrel{\mathcal{P}^B}{\to} \mathcal{P}^B(\mathcal{B})$ and $\bar{\mathcal{B}} \stackrel{\bar{\mathcal{P}}^B}{\to} \bar{\mathcal{P}}^B(\bar{\mathcal{B}}) \stackrel{\omega_I}{\to} \omega_I(\bar{\mathcal{P}}^B(\bar{\mathcal{B}}))$. The two ways should be equivalent. That is, $\mathcal{P}^B(\mathcal{B}) = \omega_I(\bar{\mathcal{P}}^B(\bar{\mathcal{B}}))$. Then it is easy to find that

$$\bar{q}_o = q_o \mod 5 \quad , \quad \bar{p}_o = p_o - \frac{2}{5} q_o \ , \tag{5.46}$$

The value of q_o determines the window of the grade restriction rule. The reason is the following. Under the worldsheet parity, $\theta \to -\theta$ and $q_i \to q_i + q_o$, the invariance of $\frac{\theta}{2\pi} + q_i$ means that $\frac{\theta}{\pi} = q_o$. This invariant value of θ determines the window. For explicit, we may take the window $-\pi < \theta < \pi$. The grade restriction rule gives the allowed values $q_i \in \{0, \pm 1, \pm 2\}$. Using the relations

$$\bar{q}_i = q_i \mod 5 \quad , \quad \bar{R}_i = R_i - \frac{2}{5} q_i \ , \tag{5.47}$$

and the LGO data

vector	$\vert 0\rangle$	$\overline{\eta}_i\overline{\eta}_j\vert 0\rangle$	$\overline{\eta}_i\overline{\eta}_j\overline{\eta}_k\overline{\eta}_l\vert 0\rangle$	$\overline{\eta}_i\vert 0\rangle$	$\overline{\eta}_i\overline{\eta}_j\overline{\eta}_k\vert 0\rangle$	$\overline{\eta}_1\cdots\overline{\eta}_5\vert 0\rangle$
$\bar{R}_{\mathbf{i}}$	0	$-\frac{6}{5}$	$-\frac{12}{5}$	$-\frac{3}{5}$	$-\frac{9}{5}$	-3
$\bar{q}_{\mathbf{i}}$	0	-2	1	-1	2	0

the charge assignments in the LSM for the basis vectors of the Clifford module are found to be

vector	$\vert 0\rangle$	$\overline{\eta}_i\overline{\eta}_j\vert 0\rangle$	$\overline{\eta}_i\overline{\eta}_j\overline{\eta}_k\overline{\eta}_l\vert 0\rangle$	$\overline{\eta}_i\vert 0\rangle$	$\overline{\eta}_i\overline{\eta}_j\overline{\eta}_k\vert 0\rangle$	$\overline{\eta}_1\cdots\overline{\eta}_5\vert 0\rangle$
$R_{\mathbf{i}}$	0	-2	-2	-1	-1	-3
$q_{\mathbf{i}}$	0	-2	1	-1	2	0
	$\mathcal{W}(0)_{_0}$	$\mathcal{W}(-2)^{\oplus 10}_{-2}$	$\mathcal{W}(1)^{\oplus 5}_{-2}$	$\mathcal{W}(-1)^{\oplus 5}_{-1}$	$\mathcal{W}(2)^{\oplus 10}_{-1}$	$\mathcal{W}(0)_{_{-3}}$

The grade restricted lift of the RS brane $\mathcal{B}_{\vec{0},0,0}$ is given by the complex

$$\begin{array}{ccccccccc}
\mathcal{W}(0)_{-3} & \underset{pX^4}{\overset{X}{\rightleftarrows}} & \mathcal{W}(1)^{\oplus 5}_{-2} & \underset{pX^4}{\overset{X}{\rightleftarrows}} & \mathcal{W}(2)^{\oplus 10}_{-1} & & \\
& & \oplus & \overset{pX}{\swarrow}\,\underset{X^4}{\nearrow} & \oplus & & \\
& & \mathcal{W}(-2)^{\oplus 10}_{-2} & \underset{pX^4}{\overset{X}{\rightleftarrows}} & \mathcal{W}(-1)^{\oplus 5}_{-1} & \underset{pX^4}{\overset{X}{\rightleftarrows}} & \mathcal{W}(0)_{0}
\end{array}$$

where $X = \sum_{i=1}^{5} x_i \eta_i$ and $pX^4 = p \sum_{i=1}^{5} x_i^4 \overline{\eta}_i$. On the other hand, the lift of the worldsheet operator $\bar{U}$ is trivial noticing

$$p_o = -3 = R_{|0\rangle} + R_{\overline{\eta}_1 \ldots \overline{\eta}_5|0\rangle} = R_{\overline{\eta}_i\overline{\eta}_j|0\rangle} + R_{\overline{\eta}_i\overline{\eta}_j\overline{\eta}_k|0\rangle} = R_{\overline{\eta}_i|0\rangle} + R_{\overline{\eta}_i\overline{\eta}_j\overline{\eta}_k\overline{\eta}_l|0\rangle} \ , \tag{5.48}$$

and so

$$U(p) = \bar{U}. \tag{5.49}$$

Following the procedure in [57], we transport the above brane to the NLSM phase and project it down to a brane in the NLSM. We get the following semi-infinite complex

$$\begin{array}{ccccccccccccc}
& & & & \mathcal{O}(-1)^{\oplus 5}_{-1} & \xrightarrow{X} & \mathcal{O}(0)_0 & \xrightarrow{X^4} & \mathcal{O}(4)^{\oplus 5}_{1} & \xrightarrow{X} & \mathcal{O}(5)_0 & \xrightarrow{X^4} & \cdots \\
& & & \overset{X}{\nearrow} & \oplus & \overset{X^4}{\searrow} & \oplus & \overset{X}{\nearrow} & \oplus & \overset{X^4}{\searrow} & \oplus & \overset{X}{\nearrow} & \\
& & \mathcal{O}(-2)^{\oplus 10}_{-2} & \xrightarrow{X^4} & \mathcal{O}(2)^{\oplus 10}_{-1} & \xrightarrow{X} & \mathcal{O}(3)^{\oplus 10}_{0} & \xrightarrow{X^4} & \mathcal{O}(7)^{\oplus 10}_{1} & \xrightarrow{X} & \cdots & & \\
& & \oplus & \overset{X}{\nearrow} & \oplus & \overset{X^4}{\searrow} & \oplus & \overset{X}{\nearrow} & \oplus & \overset{X^4}{\searrow} & & & \\
\mathcal{O}(0)_{-3} & \xrightarrow{X} & \mathcal{O}(1)^{\oplus 5}_{-2} & \xrightarrow{X^4} & \mathcal{O}(5)_{-1} & \xrightarrow{X} & \mathcal{O}(6)^{\oplus 5}_{0} & \xrightarrow{X^4} & \mathcal{O}(10)_{1} & \xrightarrow{X} & \cdots & &
\end{array}$$

It is easy to show that this complex is exact at almost all the entries except the entries $\mathcal{O}(-2)^{\oplus 10}_{-2}$, $\mathcal{O}(-1)^{\oplus 5}_{-1}$ and $\mathcal{O}(0)_0$. Then this semi-infinite complex is quasi-isomorphic to a simpler finite complex $\mathcal{O}(-2)^{\oplus 10} \xrightarrow{X} \mathcal{O}(-1)^{\oplus 5} \xrightarrow{X} \underline{\mathcal{O}(0)}$. The worldsheet parity image of the semi-infinite complex is

$$
\begin{array}{ccccccccccccc}
 & & \cdots & \xleftarrow{X^T} & \mathcal{O}(10)_4 & \xleftarrow{(X^4)^T} & \mathcal{O}(6)_3^{\oplus 5} & \xleftarrow{X^T} & \mathcal{O}(5)_2 & \xleftarrow{(X^4)^T} & \mathcal{O}(1)_1^{\oplus 5} & \xleftarrow{X^T} & \mathcal{O}(0)_0 \\
 & & & \nwarrow^{(X^4)^T} & \oplus & \swarrow^{X^T} & \oplus & \nwarrow^{(X^4)^T} & \oplus & \swarrow^{X^T} & \oplus & & \\
 & & \cdots & \xleftarrow{X^T} & \mathcal{O}(7)_4^{\oplus 10} & \xleftarrow{(X^4)^T} & \mathcal{O}(3)_3^{\oplus 10} & \xleftarrow{X^T} & \mathcal{O}(2)_2^{\oplus 10} & \xleftarrow{(X^4)^T} & \mathcal{O}(-2)_1^{\oplus 10} & & \\
 & & & \nwarrow^{(X^4)^T} & \oplus & \swarrow^{X^T} & \oplus & \nwarrow^{(X^4)^T} & \oplus & \swarrow^{X^T} & & & \\
\cdots & \xleftarrow{(X^4)^T} & \mathcal{O}(5)_5 & \xleftarrow{X^T} & \mathcal{O}(4)_4^{\oplus 5} & \xleftarrow{(X^4)^T} & \mathcal{O}(0)_3 & \xleftarrow{X^T} & \mathcal{O}(-1)_2^{\oplus 5} & & & &
\end{array}
$$

This parity image complex is exact except the entries $\mathcal{O}(-2)_1^{\oplus 10}$, $\mathcal{O}(-1)_2^{\oplus 5}$ and $\mathcal{O}(0)_3$. Then it is quasi-isomorphic to a finite complex $\mathcal{O}(0)_3 \xleftarrow{X^T} \mathcal{O}(-1)_2^{\oplus 5} \xleftarrow{X^T} \mathcal{O}(-2)_1^{\oplus 10}$. This shows that the worldsheet parity image complex is isomorphic to the original one shifted by -3.

Chapter 6

Conclusion

In this thesis, we have studied the D-brane Chan-Paton factor structure in Type II orientifolds. We find the explicit structure for both BPS and non-BPS D-branes, and investigate the properties of orientifold projection operators. We give the rules for computing the scattering amplitudes of open strings ending on D-branes. A direct application of our results is to build string theory phenomenological models. In other words, for a Type II orientifold compactification with branes and fluxes, we know what the gauge group is and how to write down the effective Lagrangian by calculating the string scattering amplitudes. Of course, these results only provide us some necessary information on the string phenomenological model building. We need to do further work to fully build string phenomenological models in the future.

We have shown some preliminary results on the B-type D-branes in orientifolds of the quintic model. The Kähler moduli space of the quintic model is of complex dimension one with a geometric phase and a LGO phase. The worldsheet parity produces O-planes of the same type. It is interesting to study the orientifold projection in more complicate linear sigma models. For example, some orientifold projections in the so-called two-parameter model produce O-planes of different types. The moduli space of the two-parameter model has more complicate phases than the quintic model. So the relations among the worldsheet parity operators of different phases will be more complicated and interesting.

Appendix A

Basic Mathematical Background

A.1 Category Theory

In mathematics, category theory deals with mathematical structures and relationships between them. Categories now appear in most branches of mathematics. We will give a brief introduction to some basic definitions and notations of category theory [49, 50, 51].

Category

A category $\mathcal{C}$ is given by

- A collection of objects A, B, C, $\cdots \in Ob(\mathcal{C})$.
- For each pair of objects A and B, a collection of morphisms $Hom_{\mathcal{C}}(A, B)$.
- Composition: For all A,B,C $\in Ob(\mathcal{C})$, for any $f \in Hom_{\mathcal{C}}(A, B)$ and $g \in Hom_{\mathcal{C}}(B, C)$, then $gf \in Hom_{\mathcal{C}}(A, C)$.
- Identity: $id_A \in Hom_{\mathcal{C}}(A, A)$.

It satisfies the following axioms:

• Associativity: For all A,B,C,D $\in Ob(\mathcal{C})$ and all $f \in Hom_{\mathcal{C}}(A,B)$, $g \in Hom_{\mathcal{C}}(B,C)$ and $h \in Hom_{\mathcal{C}}(C,D)$ we have $h(gf) = (hg)f$.

• Identity: $(id_A \cdot f)$=$(f \cdot id_B)$=f.

An object A is called an *initial object* if $Hom_{\mathcal{C}}(A,B)$ consists of exactly one element for all $B \in \mathcal{C}$. And A is called a *final object* if $Hom_{\mathcal{C}}(B,A)$ consists of exactly one element for all $B \in \mathcal{C}$. An object is called a *zero object* if it is both an initial and a final object.

Functor

Let $\mathcal{B}$ and $\mathcal{C}$ be categories. A functor $\mathcal{F} : \mathcal{B} \to \mathcal{C}$ is defined by

• A map $Ob(\mathcal{B}) \ni A \mapsto \mathcal{F}(A) \in Ob(\mathcal{C})$;

• For all A,B $\in Ob(\mathcal{B}), Hom_{\mathcal{B}}(A,B) \ni f \mapsto \mathcal{F}(f) \in Hom_{\mathcal{C}}(\mathcal{F}(A), \mathcal{F}(B))$;

• $\mathcal{F}(id_A) = id_{\mathcal{F}(A)}$ for all A$\in Ob(\mathcal{B})$;

• $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$ for all $f \in Hom_{\mathcal{B}}(B,C)$ and $g \in Hom_{\mathcal{B}}(A,B)$.

Let $\mathcal{F} : \mathcal{B} \to \mathcal{C}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{C}$ be two functors. A *natural transform* $\varphi : \mathcal{F} \to \mathcal{G}$ is a family of morphisms $\{\varphi(A) : \mathcal{F}(A) \to \mathcal{G}(A)\}$ for all $A \in Ob(\mathcal{B})$ such that we have $\varphi(B)\mathcal{F}(f) = \mathcal{G}(f)\varphi(A)$ for all $f \in Hom_{\mathcal{B}}(A,B)$. A natural transform φ is called a *natural isomorphism* if there exists a natural transformation $\psi : \mathcal{G} \to \mathcal{F}$ such that $\psi\varphi = id_{\mathcal{F}}$ and $\varphi\psi = id_{\mathcal{G}}$.

Two categories $\mathcal{B}$ and $\mathcal{C}$ are said to be equivalent to each other if there exist two functors $\mathcal{F} : \mathcal{B} \to \mathcal{C}$ and $\mathcal{G} : \mathcal{C} \to \mathcal{B}$ such that there are two natural isomorphisms $\varphi : \mathcal{F}\mathcal{G} \to id_{\mathcal{C}}$ and $\psi : \mathcal{G}\mathcal{F} \to id_{\mathcal{B}}$. The categories $\mathcal{B}$ and $\mathcal{C}$ are isomorphic to each other if $\mathcal{F}\mathcal{G} = id_{\mathcal{C}}$ and $\mathcal{G}\mathcal{F} = id_{\mathcal{B}}$. In other words, if two categories are isomorphic, both the objects and morphisms of the two categories are in one-to-one correspondence. While if two categories are equivalent to each other, only the isomorphism classes of the objects and isomorphism classes of morphisms of the two categories are in one-to-one correspondence. We can see that isomorphisms of categories are very strong conditions. Therefore, it is important to study the equivalence relations of categories.

Additive and Abelian Categories

An additive category $\mathcal{C}$ is a category satisfying axioms A1–A3. An abelian category $\mathcal{C}$ is a category satisfying axioms A1–A4.

- Axiom A1: Any set $Hom_{\mathcal{C}}(A, B)$ is an abelian group for all A, B $\in Ob(\mathcal{C})$.
- Axiom A2: There exists a zero object O$\in Ob(\mathcal{C})$.
- Axiom A3: For any two objects A_1 and A_2, there exist an object B and morphisms

$A_1 \xrightarrow{i_1} B$, $A_1 \xleftarrow{p_1} B$, $A_2 \xrightarrow{i_2} B$ and $A_2 \xleftarrow{p_2} B$ such that

$$p_1 i_1 = id_{A_1},\ p_2 i_2 = id_{A_2},\ i_1 p_1 + i_2 p_2 = id_B,\ p_2 i_1 = p_1 i_2 = 0.$$

- Axiom A4: For any morphism $\varphi \in Hom(A, B)$, there exists a sequence

$$C \xrightarrow{k} A \xrightarrow{i} I \xrightarrow{j} B \xrightarrow{c} C'$$

satisfying the following properties: i) $ji = \varphi$; ii) C is the kernel [1] of φ and C' is the cokernel [2] of φ; iii) I is both the kernel of c and the cokernel of k.

A *complex* $C^\bullet$ in an additive category $\mathcal{C}$ is a sequence of objects and morphisms in $\mathcal{C}$

$$C^\bullet : \cdots \xrightarrow{f_{n-1}} C_n \xrightarrow{f_n} C_{n+1} \xrightarrow{f_{n+1}} \cdots ,$$

[1] The kernel of $\varphi : A \to B$ is defined to be a morphism $f : D \to A$ such that $\varphi f = 0$, with the universal property: $\forall\, e : D' \to A$ s.t. $\varphi e = 0$, $\exists$ a unique morphism $e' : D' \to D$ s.t. $e = fe'$. D is called a kernel object.

[2] The cokernel of $\varphi : A \to B$ is defined to be a morphism $f : B \to D$ such that $f\varphi = 0$, with the universal property: $\forall\, g : B \to D'$ s.t. $g\varphi = 0$, $\exists$ a unique morphism $g' : D \to D'$ s.t. $g = g'f$. D is called a cokernel object.

satisfying the condition $f_{n+1} \cdot f_n = 0$, where $n \in \mathbb{Z}$ labels the position of an object C_n in a complex. If $C_i = 0$ for all $i < i_0$, the complex is called a left-sided bounded complex, while if $C_i = 0$ for all $i > i_0$ the complex is called a right-sided bounded complex. If a complex contains finite number of non-zero objects, it is called a bounded complex (or a finite complex). The i-th *cohomology group* of a complex $C^\bullet$ is defined as $H^i(C^\bullet) = Ker(f_i)/Im(f_{i-1})$. The translation operation $T[i]$ on a complex $C^\bullet$ is denoted by $T[i](C^\bullet) := C[i]^\bullet$, defined by $C[i]^\bullet_n := C^\bullet_{n-i}$. A complex is called *exact* at position n if $Ker(f_n) = Im(f_{n-1})$. If a complex is exact everywhere, it is called an exact complex, i.e, $H^i(C^\bullet) = 0$ for all $i \in \mathbb{Z}$. A morphism of two complexes $f : C^\bullet \to D^\bullet$ in an abelian category is called *quasi-isomorphic* if the corresponding cohomology morphism $H^i(f) : H^i(C^\bullet) \to H^i(D^\bullet)$ is an isomorphism for any i.

(Bounded) Derived Category

Let $\mathcal{C}$ be an abelian category and $Kom(\mathcal{C})$ be the category of (bounded) complexes over $\mathcal{C}$. There exists a category $D(\mathcal{C})$ and a functor $Q : Kom(\mathcal{C}) \to D(\mathcal{C})$ with the following properties:

- $Q(f)$ is an isomorphism for any quasi-isomorphism.
- Any functor $\mathcal{F} : Kom(\mathcal{C}) \to \mathcal{D}$ transforming quasi-isomorphisms into isomorphisms can be uniquely factorized through $D(\mathcal{C})$, i.e, there exists a unique functor $G : D(\mathcal{C}) \to \mathcal{D}$ with $\mathcal{F} = G \cdot Q$.

The category $D(\mathcal{C})$ is called the derived category of the abelian category $\mathcal{C}$.

Any object A in an abelian category $\mathcal{C}$ can be considered as an infinite complex $\cdots \to 0 \to 0 \to \underline{A} \to 0 \to 0 \to \cdots$ (with A at the 0-th place, denoted by $\underline{A}$). Similarly we can translate the place of A to the i-th position, and denotes the resulting complex by $A[-i]$. In this way, we define the i-th extension group $\mathrm{Ext}^i_{\mathcal{C}}(A, B) := Hom_{D(\mathcal{C})}(A, B[i])$.

Category of Sheaves

Let $\mathbb{X}$ be a topological space. A presheaf $\mathcal{F}$ on $\mathbb{X}$ is defined by the following data

- For every open set $U \subset \mathbb{X}$ we associate an abelian group $\mathcal{F}(U)$ (sections of the presheaf $\mathcal{F}$).
- If $V \subset U$ are open sets we have a restriction map $\gamma_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$.

This data satisfies the conditions

$$\gamma_{UU} = id, \;\; \gamma_{UW} = \gamma_{VW}\gamma_{UV} \;\text{ for } W \subset V \subset U.$$

A presheaf $\mathcal{F}$ is called a sheaf if the following additional condition is satisfied

- For any open covering $U = \cup_{i \in I} U_i$ and for any family of sections $s_i \in \mathcal{F}(U_i)$, such that for all $i, j \in I$

$$\gamma_{U_i, U_i \cap U_j}(s_i) = \gamma_{U_j, U_i \cap U_j}(s_j),$$

there exists a unique section $s \in \mathcal{F}(U)$ satisfying $s_i = \gamma_{U,U_i}(s)$ for all $i \in I$.

One important example of sheaves is the so-called *structure sheaf* of a complex manifold $\mathbb{X}$, denoted by $\mathcal{O}_{\mathbb{X}}$. Its group $\mathcal{F}(U)$ is the group of holomorphic functions over U under addition. Actually, $\mathcal{O}_{\mathbb{X}}$ has a ring structure under addition and multiplication of functions. Then we can introduce the concept of sheaf of $\mathcal{O}_{\mathbb{X}}$-modules. A sheaf $\mathcal{E}$ on $\mathbb{X}$ is called a *locally free sheaf* of rank n if there is an open covering $U = \cup_{i \in I} U_i$ of $\mathbb{X}$ such that $\mathcal{E}(U_i) \cong \mathcal{O}_{\mathbb{X}}(U_i)^{\oplus n}$ for all i. There is a one-to-one correspondence between holomorphic vector bundles of rank n on $\mathbb{X}$ and locally free sheaves of rank n on $\mathbb{X}$. Unfortunately, the category of all locally free sheaves on $\mathbb{X}$ is additive, not abelian. For this purpose, we have to use *coherent sheaves*. A coherent sheaf on $\mathbb{X}$ is a sheaf of $\mathcal{F}$ of $\mathcal{O}_{\mathbb{X}}$-modules satisfying the following two properties:

- $\mathcal{F}$ is of finite type, i.e, for any point $x \in \mathbb{X}$, there is an open neighborhood U such that there exists a surjective morphism from $\mathcal{O}_{\mathbb{X}}(U)^{\oplus n}$ to $\mathcal{F}(U)$.

• For any open set $U \subset \mathbb{X}$, any natural number n, and any morphism $\varphi : \mathcal{O}_{\mathbb{X}}(U)^{\oplus n} \to \mathcal{F}(U)$, the kernel of φ is of finite type.

The category of coherent sheaves on $\mathbb{X}$, denoted by $Coh\,\mathbb{X}$, is an abelian category. The derived category of $Coh\,\mathbb{X}$ is denoted by $D(Coh\,\mathbb{X})$ which is important in string theory.

A.2 K-theory

K-theory was first introduced by Grothendieck. We will give a brief introduction to the (topological) K-theory [32, 33, 34] which is constructed from the category of vector bundles on the compact space $\mathbb{X}$.

Manifold

An m-dimensional topological space M is called an manifold if

a) M is provided with a family of pairs $\{(U_i, \varphi_i)\}$.

b) $\{U_i\}$ is an open covering of M, i.e, $\bigcup_{i\in I} U_i = M$. φ_i is a homeomorphism from U_i onto an open subset U_i' of $\mathbb{R}^m$.

c) Given U_i and U_j such that $U_i \bigcap U_j \neq \emptyset$, there is a transition function $\psi_{ij} = \varphi_i \cdot \varphi_j^{-1}$. Note that $\psi_{ij} = \psi_{ji}^{-1}$ and $\psi_{ij}\psi_{jk}\psi_{ki} = 1$.

If the transition functions are differentiable, the manifold M is called *differentiable*, and it is called *smooth* if the transition functions are smooth.

Vector bundle

A *fibre bundle* (E, π, M, F, G) consists of the following data [35]:

a) A differentiable manifold E is called the total space.

b) A differentiable manifold M is called the base space.

c) A differentiable manifold F is called the fibre.

d) A surjection $\pi : E \to M$ is called the projection. The inverse image $\pi^{-1}(p) := F_p \cong F$ is called the fibre at p.

e) A Lie group G is called the structure group, which acts on F.

f) An open covering $\{U_i\}$ of M with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi\phi_i(p, f) = p$. The map ϕ_i is called the local trivialization.

g) If we write $\phi_i(p, f) = \phi_{i,p}(f)$, the map $\phi_{i,p} : F \to F_p$ is a diffeomorphism. On $U_i \bigcap U_j \neq \emptyset$, the $t_{ij}(p) := \phi_{i,p}^{-1}\phi_{j,p} : F \to F$ is an element of group G. $\{t_{ij}\}$ are called the transition functions.

If all the transition functions can be taken to be identity maps, the bundle is called a trivial bundle which is simply a direct product $M \times F$. A section of a bundle $s : M \to E$ is a smooth map which satisfies $\pi s = id_M$. The set of sections is denoted by $\Gamma(M, E)$. We often use the notation $E \xrightarrow{\pi} M$ or simply E to denote the fibre bundle (E, π, M, F, G). If the fibre F is a vector space, then the bundle is called a *vector bundle*. The dimension n of the fibre is called the rank of the bundle, denoted by rank(E).

A *principal bundle* (or *gauge bundle*) P over M with group G is defined to be a manifold P with a group G action on it, satisfying the following conditions:

- G acts freely on P on the right: $P \times G \ni (u, a) \to ua \in P$.
- P is locally trivial. In other words, for every point $x \in M$, there is a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times G$ in the sense that there exists a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ s.t. $\psi(u) = (\pi(u), \varphi(u))$, where φ is a map from $\pi^{-1}(U)$ to G satisfying $\varphi(ua) = \varphi(u)a$ for all $u \in \pi^{-1}(U)$ and $a \in G$.

A gauge bundle $P \xrightarrow{\pi} M$ is also denoted by $P(M, G)$ and is often called G-bundle over M.

We can define a lot of operations on bundles. Let $E \xrightarrow{\pi} M$ be a fibre bundle with fibre F, and give a map $f : N \to M$. The pair (E,f) defines a new bundle with fibre F. Let f^*E be a subspace of $N \times E$ which consists of points (p, u) such that $f(p) = \pi(u)$. The f^*E is called the pullback bundle of E by f. Given two vector bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$. The *product bundle* $E \times E' \xrightarrow{\pi\times\pi'} M \times M'$ is a bundle with fibre $F \oplus F'$. If $M' = M$, we can define the so-called *Whitney sum bundle* $E \oplus E'$ which is the pullback bundle of $E \times E'$ by $f : M \to M \times M$ defined by $f(p) = (p, p)$. The *tensor product bundle* $E \otimes E'$ is defined by the tensor product of fibres $F_p \otimes F'_p$ at every point $p \in M$.

K-theory group K($\mathbb{X}$)

Three different ways to define the K-theory group will be discussed here.

• $K(\mathbb{X})$ as the Grothendieck group of a category.

What is a Grothendieck group? Let us consider an *abelian monoid* M, i.e, a set with a composition law (denoted +) which satisfies all the properties of an abelian group except the possible existence of inverses. One can naturally associate an abelian group S(M) to M by the quotient construction $S(M) = M \times M/\sim$, where $\sim$ is an equivalence relation on $M \times M$. There are various possible quotient constructions. They all give the same result up to isomorphism. One choice of $\sim$ is

$$(m, n) \sim (m', n') \Leftrightarrow \exists p \text{ such that } m + n' + p = n + m' + p, \forall m, n, m', n' \in M.$$

Another commonly used $\sim$ is

$$(m, n) \sim (m', n') \Leftrightarrow \exists p, q \text{ such that } (m, n) + (p, p) = (m', n') + (q, q), \forall m, n, m', n' \in M.$$

For example, for $M = \mathbb{N}$ under addition, $S(M) = \mathbb{Z}$; while for $M = \mathbb{Z} - \{0\}$ under multiplication, $S(M) = \mathbb{Q} - \{0\}$. Another important example is the application to an additive category. Let $\mathcal{C}$ be an additive category and $\Phi(\mathcal{C})$ be the isomorphism classes of objects in $E \in \mathcal{C}$ which is denoted by [E]. $\Phi(\mathcal{C})$ becomes an abelian monoid if we define $[E] + [F] := [E \oplus F]$. The Grothendieck group of $\mathcal{C}$ is defined to be $K(\mathcal{C}) := S(\Phi(\mathcal{C}))$. Every element of $K(\mathcal{C})$ can be written as a formal difference $[E] - [F]$ (which is suggestive by the quotient construction).

Let $\mathbb{X}$ be a compact manifold and let $\mathcal{C}_{\mathbb{C}}(\mathbb{X})$ be the category of complex vector bundles on $\mathbb{X}$. (Sometime we omit $\mathbb{C}$ without confusion.) We use $\mathrm{Vect}(\mathbb{X})$ to denote the set of isomorphism classes of complex vector bundles on $\mathbb{X}$. $\mathrm{Vect}(\mathbb{X})$ is an abelian monoid under the Whitney sum of bundles. Define the K-theory group $K(\mathbb{X})$ as the Grothendieck group of $\mathrm{Vect}(\mathbb{X})$, i.e, $K(\mathbb{X}) := S(\mathrm{Vect}(\mathbb{X}))$. If $E \in \mathrm{Vect}(\mathbb{X})$, we shall write $[E]$ as the image of E in $K(\mathbb{X})$. Then every element of $K(\mathbb{X})$ is of the form $[E] - [F]$. Define $\underline{n}$ to be the trivial bundle of rank n over $\mathbb{X}$, i.e, $\underline{n} \cong \mathbb{X} \times \mathbb{C}^n$. Bundle E on $\mathbb{X}$ is said to be *stably equivalent* to bundle F, if there exists a trivial bundle $\underline{n}$ such that

$$E \oplus \underline{n} \cong F \oplus \underline{n} \ . \tag{A.1}$$

In $K(\mathbb{X})$, $[E] = [F]$ if and only if E and F are stably equivalent.

If $\mathbb{X}$=point, then $K(\textit{point}) = \mathbb{Z}$. This motivates us to introduce a reduced K-theory group $\tilde{K}(\mathbb{X})$ such that $\tilde{K}(\textit{point}) = 0$. Take a base point x_0 of $\mathbb{X}$, the reduced K-theory group is defined to be $\tilde{K}(\mathbb{X}) = ker(i_0^* : K(\mathbb{X}) \to K(x_0))$, where the map $i_0 : x_0 \hookrightarrow \mathbb{X}$ is the inclusion map. Suppose $\mathbb{X}$ is a compact topological space and U is a closed subset of $\mathbb{X}$. We define $K(\mathbb{X}, U)$ by $K(\mathbb{X}, U) := \tilde{K}(\mathbb{X}/U)$. In particular, $K(\mathbb{X}, \emptyset) = K(\mathbb{X})$. We now introduce the *smash product* $\wedge$ operation on compact topological spaces X and Y as

$$X \wedge Y := X \times Y / X \vee Y,$$

where $X \vee Y = X \times y_0 \cup x_0 \times Y$, x_0, y_0 being the base-points of X and Y, respectively. The smash product has a property

$$X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z.$$

Especially, for the n-sphere S^n, we have

$$S^n \cong S^1 \wedge S^1 \wedge \cdots \wedge S^1 \ \ (\text{n factors}).$$

Normally, we use $S^n\mathbb{X} := S^n \wedge \mathbb{X}$ to denote the n-th reduced suspension of $\mathbb{X}$. For this notation, we can define several higher K-theory groups:

$$\tilde{K}^{-n}(\mathbb{X}) := \tilde{K}(S^n\mathbb{X}), \ \ K^{-n}(\mathbb{X}, U) := \tilde{K}^{-n}(\mathbb{X}/U), \ \ K^{-n}(\mathbb{X}) := K^{-n}(\mathbb{X}, \emptyset).$$

For the K-theory group, we have the so-called *Bott periodicity theorem*:

$$K^{-n}(\mathbb{X}) = K^{-n-2}(\mathbb{X}).$$

In other words, we only have two inequivalent K-theory groups, $K(\mathbb{X})$ and $K^{-1}(\mathbb{X})$, for a manifold $\mathbb{X}$.

• $K(\mathbb{X})$ as the Grothendieck group of a functor.

This way to define K-theory group was given by Karoubi [34]. We need to give several definitions first. A *Banach space* is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ with a norm $\|\cdot\|$ such that every Cauchy sequence [3] (with respect to the induced metric $d(x, y) = \|x-y\|$) in V has a limit in V. Let $\mathcal{C}$ be an additive category. A Banach structure on $\mathcal{C}$ is given by a Banach

[3] A sequence, $x_1, x_2, x_3, \cdots$, is Cauchy, if for every positive number $\varepsilon > 0$ there exists a natural number N such that for all $m, n > N$, $d(x_m, x_n) < \varepsilon$. The limit of a sequence in V may not always exist within V.

space structure on all the morphism groups $Hom_{\mathcal{C}}(E, F)$, where $E, F \in Ob(\mathcal{C})$. A Banach category is an additive category provided with a Banach structure. Let $\varphi : \mathcal{C} \to \mathcal{C}'$ be an additive functor between two additive categories $\mathcal{C}$ and $\mathcal{C}'$. φ is called *quasi-surjective* if every object of $\mathcal{C}'$ is a direct factor of an object of the form $\varphi(E)$; while φ is called full if the map $Hom_{\mathcal{C}}(E, F) \to Hom_{\mathcal{C}'}(\varphi(E), \varphi(F))$ is surjective for all $E, F \in Ob(\mathcal{C})$. If $\mathcal{C}$ and $\mathcal{C}'$ are Banach categories, the functor φ is called a *Banach functor* if the map $Hom_{\mathcal{C}}(E, F) \to Hom_{\mathcal{C}'}(\varphi(E), \varphi(F))$ is linear and continuous.

Let $\varphi : \mathcal{C} \to \mathcal{C}'$ be a quasi-surjective Banach functor. Let $\Gamma(\varphi)$ denote the set of triples (E, F, α), where $\alpha : \varphi(E) \to \varphi(F)$ is an isomorphism for $E, F \in Ob(\mathcal{C})$. A triple (E, F, α) is called elementary if $E = F$ and α is homotopic to $id_{\varphi(E)}$ within the automorphisms of $\varphi(E)$. Two triples (E, F, α) and (E', F', α') are called isomorphic if there exist isomorphisms $f : E \to E'$ and $g : F \to F'$ such that $\alpha' \cdot \varphi(f) = \varphi(g) \cdot \alpha$. The sum of two triples is defined to be $(E \oplus E', F \oplus F', \alpha \oplus \alpha')$. We can do the quotient construction of $\Gamma(\varphi)$ by the equivalence relation: $\sigma \sim \sigma' \Leftrightarrow \exists$ elementary τ and τ' such that $\sigma + \tau = \sigma' + \tau'$. We denote the quotient by $K(\varphi)$ and use $d(E, F, \alpha)$ to denote the class of (E, F, α) in $K(\varphi)$. If we apply the result to the case $\mathcal{C} = \mathcal{C}(\mathbb{X})$ and $\mathcal{C}' = \mathcal{C}(Y)$, where Y is a closed subspace of the compact space $\mathbb{X}$ and $\varphi : \mathcal{C}(\mathbb{X}) \to \mathcal{C}(\mathcal{Y})$ is the functor induced by the restriction of bundles. Then we get the K-theory group $K(\mathbb{X}, \mathcal{Y}) = K(\varphi)$. When $\mathcal{Y} = \emptyset$, $K(\mathbb{X}) = K(\mathbb{X}, \emptyset)$.

In the similar spirit, we can discuss $K^{-1}(\mathbb{X})$. Let $\mathcal{C}$ be a Banach category. Consider the set of pairs (E, β), where E is an object of $\mathcal{C}$ and β is an automorphism of E. A pair (E, β) is called elementary if β is homotopic to id_E with the automorphisms of E. Two pairs (E, β) and (E', β') are isomorphic if there exists an isomorphism $f : E \to E'$ in the category $\mathcal{C}$ such that $f \cdot \beta = \beta' \cdot f$. The sum of two pairs is defined to be $(e \oplus E', \beta \oplus \beta')$. Define $K^{-1}(\mathcal{C})$ as the quotient of the set of pairs by the equivalence relation: $\sigma \sim \sigma' \Leftrightarrow \exists$

elementary τ and τ' such that $\sigma + \tau = \sigma' + \tau'$. Apply the result to the case $\mathcal{C} = \mathcal{C}(\mathbb{X})$, we have $K^{-1}(\mathbb{X}) = K^{-1}(\mathcal{C}(\mathbb{X}))$. The nice thing of this definition is that we don't need to introduce higher dimensional space $S^1\mathbb{X}$ as in the former discussion.

• The Hilbert space interpretation of K-theory group

There is a third way to discuss the K-theory group in terms of the *Fredholm operators* on Hilbert spaces. Let $\mathcal{H}_{\mathbb{C}}$ be a separable infinite-dimensional complex Hilbert space, and let $\mathbf{a}(\mathcal{H}_{\mathbb{C}})$ be the algebra of all bounded linear operators [4] on $\mathcal{H}_{\mathbb{C}}$. An operator $T \in \mathbf{a}(\mathcal{H}_{\mathbb{C}})$ is a *Fredholm operator* if the $Ker\,T$ and $Coker\,T$ are finite dimensional. Let $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$ denote the set of all Fredholm operators on $\mathcal{H}_{\mathbb{C}}$. For any topological space $\mathbb{X}$, denote the set of homotopy [5] classes of mappings $\mathbb{X} \to \mathcal{F}(\mathcal{H}_{\mathbb{C}})$ by $[\mathbb{X}, \mathcal{F}(\mathcal{H}_{\mathbb{C}})]$. There is a theorem states that we have a natural isomorphism:

$$\text{Index} : [\mathbb{X}, \mathcal{F}(\mathcal{H}_{\mathbb{C}})] \to K(\mathbb{X}).$$

$\mathcal{F}(\mathcal{H}_{\mathbb{C}})$ is called a classifying space for K-theory. Similarly, let $\hat{\mathcal{F}}_*(\mathcal{H}_{\mathbb{C}})$ denote the set of neither essentially positive nor essentially negative skew-adjoint Fredholm operators [6] on $\mathcal{H}_{\mathbb{C}}$. A similar theorem says that there exists a natural isomorphism

$$\text{Index} : [\mathbb{X}, \hat{\mathcal{F}}_*(\mathcal{H}_{\mathbb{C}})] \to K^{-1}(\mathbb{X}).$$

In the above, we have discussed the K-theory group of complex vector bundles on $\mathbb{X}$,

[4] A bounded linear operator L on a Hilbert space $\mathcal{H}$ is a linear transformation satisfying the condition: there exists some $M > 0$ such that $\|Lv\| \leq M\|v\|$ for all $v \in \mathcal{H}$. The smallest such M is called the operator norm of L.

[5] A homotopy between two continuous functions $f : X \to Y$ and $g : X \to Y$ is defined to be a continuous function $H : X \times [0,1] \to Y$ such that, for all points x in X, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Note that [0,1] denotes the unit interval.

[6] A Fredholm operator T is skew-adjoint if $T^\dagger = -T$. A skew-adjoint Fredholm operator T is called essentially positive (negative) if iT is positive (negative) on some invariant subspace of $\mathcal{H}_{\mathbb{C}}$ of finite codimension.

denoted by $K(\mathbb{X})$. There are many variants and refinements of the K-theory. We can extend our discussion to the K-theory of real vector bundles on $\mathbb{X}$, denoted by $KO(\mathbb{X})$ in literature. The higher KO-theory groups are defined as:

$$\widetilde{KO}^{-n}(\mathbb{X}) := \widetilde{KO}(S^n\mathbb{X}), \;\; KO^{-n}(\mathbb{X}, U) := \widetilde{KO}^{-n}(\mathbb{X}/U), \;\; KO^{-n}(\mathbb{X}) := KO^{-n}(\mathbb{X}, \emptyset).$$

The *Bott periodicity theorem* says:

$$KO^{-n}(\mathbb{X}) = KO^{-n-8}(\mathbb{X}).$$

We can discuss K-theory of a space with involution. An involution on a topological space $\mathbb{X}$ is a homeomorphism $\tau : \mathbb{X} \to \mathbb{X}$ of period 2, i.e, $\tau^2 = id$. A point $x \in \mathbb{X}$ is called a fixed point if $\tau(x) = x$. A space with involution is also called a *Real* space. A *Real* vector bundle on a *Real* space $\mathbb{X}$ is defined to be a complex vector bundle E with an antilinear involution $\tau : E \to E$ which commutes with the involution of $\mathbb{X}$. The Grothendieck group of the category of all *Real* vector bundles on a *Real* space $\mathbb{X}$ is called $KR(\mathbb{X})$. The relative group $KR(\mathbb{X}, U)$ is defined in the usual way as $\widetilde{KR}(\mathbb{X}/U)$, where U is an invariant subspace under the involution τ. The higher KR-theory groups are defined as follows. Let $S^{p,q}$ (respectively $B^{p,q}$) denote the sphere (the ball) in $\mathbb{R}^{p+q}$, provided with the involution $(x, y) \mapsto (-x, y)$, for $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$. For a pair $(\mathbb{X}, \mathcal{Y})$ of *Real* compact spaces, we define $KR^{p,q}(\mathbb{X}, \mathcal{Y}) = KR(\mathbb{X} \times B^{p,q}, \mathbb{X} \times S^{p,q} \cup \mathcal{Y} \times B^{p,q})$. If $\mathcal{Y} = \emptyset$, $KR^{p,q}(\mathbb{X}) = KR^{p,q}(\mathbb{X}, \emptyset)$. The usual suspension groups KR^{-q} are given by

$$KR^{-q}(\mathbb{X}) = KR^{0,q}(\mathbb{X}).$$

The *Bott periodicity theorem* for KR-theory is:

$$KR^{-n}(\mathbb{X}) = KR^{-n-8}(\mathbb{X}).$$

There is a natural isomorphism:

$$KR(\mathbb{X} \times S^{1,0}) = K(\mathbb{X}).$$

If the involution on $\mathbb{X}$ is trivial, we have $KR^{-n}(\mathbb{X}) = KO^{-n}(\mathbb{X})$. The classifying spaces for KO- and KR-theory will be discussed in the following sections.

A.3 Clifford Algebra and Index Theory

The Clifford algebra $Cl_{p,q}$ over $\mathbb{R}$-field is generated by the following fundamental generators

$$\begin{aligned} e_i^2 &= -1, \ i = 1, ..., p \\ e_i^2 &= +1, \ i = p+1, ..., p+q. \end{aligned}$$

We adopt the convention that $Cl_k := Cl_{k,0}$. It is known that

$$Cl_k = Cl_k^0 \oplus Cl_k^1,$$

where Cl_k^0 is generated by $e_i^0 := e_k e_i, \ i = 1, ..., k-1$. The Clifford algebras over $\mathbb{R}$-field and $\mathbb{C}$-field are given in Table(A.1).

Here we discuss the index theory for Fredholm operators briefly. Much more details

k	$Cl_{k,0}$	$Cl_{0,k}$	$\mathbb{C}l_k$
0	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$
1	$\mathbb{C}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{C}\oplus\mathbb{C}$
2	$\mathbb{H}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$
3	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{C}(4)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$
7	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$

Table A.1: The Clifford algebras over $\mathbb{R}$ and $\mathbb{C}$

can be found in [37, 38, 39]. We mainly discuss the Cl_k-valued Fredholm operators. The story for the complex Clifford algebra $\mathbb{C}l_k$-valued Fredhold operators is similar. Let $\mathcal{H}$ be a separable infinite dimensional real Hilbert space and let $F(\mathcal{H})$ denote the space of all Fredholm operators on $\mathcal{H}$ and $\hat{F}(\mathcal{H})$ denote the space of skew-adjoint Fredholm operators on $\mathcal{H}$. $\mathcal{H}$ is said to be a Hilbert module for Cl_k if we have a representation $\rho : Cl_k \to$ bounded operators on $\mathcal{H}$ with $J_i = \rho(e_i)$ and

$$J_i^2 = -1,\ i = 1, \cdots, k \quad \text{and} \quad J_iJ_j = -J_jJ_i,\ i \neq j.$$

For simplicity, we shall usually omit the symbol ρ. A Cl_k-module $\mathcal{H}$ is called a $\mathbb{Z}_2$-graded module for Cl_k if it has a decomposition $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ such that

$$Cl_k^i \cdot \mathcal{H}^j = \mathcal{H}^{(i+j)\ mod\ 2} \quad \text{for } 0 \leqslant i, j \leqslant 1.$$

A Fredholm operator T on $\mathcal{H}$ is called Cl_k-linear, if it satisfies

$$T(av) = aT(v), \quad \forall\ a \in Cl_k,\ v \in \mathcal{H}.$$

A Fredholm operator T on $\mathcal{H}$ is called Cl_k-antilinear, if it satisfies

$$T(av) = \alpha(a)T(v), \quad \forall\ a \in Cl_k,\ v \in \mathcal{H},$$

where α is the canonical parity operation on Cl_k defined by $\alpha(e_i) = -e_i,\ i = 1, \cdots, k$. We define the following spaces of Fredholm operators

$$\begin{aligned} F_k &:= \{T \in F(\mathcal{H})|\ T \text{ is } \mathbb{Z}_2\text{−graded},\ Cl_k\text{−linear and self−adjoint.}\}, \\ \hat{F}_k &:= \{T \in F(\mathcal{H})|\ T \text{ is ungraded},\ Cl_{k-1}\text{−antilinear and skew−adjoint.}\}, \end{aligned}$$

where Cl_{k-1} is generated by $e_i^0 := e_k e_i,\ i = 1, ..., k-1$ and the convension is $F_0 = F(\mathcal{H}),\ \hat{F}_0 = \hat{F}(\mathcal{H})$. Any $T \in F_k$ can be written into the following form

$$T = \begin{pmatrix} 0 & {T_0}^\dagger \\ T_0 & 0 \end{pmatrix},$$

where $T_0 : \mathcal{H}^0 \to \mathcal{H}^1$ is a Cl_{k-1}-linear Fredholm operator. It is easy to show that $F_k \cong \hat{F}_k$.

Because the Clifford algebra Cl_k is not simple for $k = -1$ mod 4, we refine our definitions as follows. For $k \neq -1$ mod 4, we define $\mathfrak{F}_k := F_k$ and $\hat{\mathfrak{F}}_k := \hat{F}_k$. For $k = -1$ mod 4, We define

$$\mathfrak{F}_k = \{T \in F_k|\ T \text{ is neither essentially positive nor essentially negative.}\}; \quad \text{(A.2)}$$

$$\hat{\mathfrak{F}}_k = \{T \in \hat{F}_k|\ T \text{ is neither essentially positive nor essentially negative.}\}. \quad \text{(A.3)}$$

k	$\hat{\hat{\hat{\mathfrak{F}}}}_k$
0	$\{T_0 : \mathcal{H}^0 \to \mathcal{H}^1$ is $\mathbb{R}$-linear; $\mathcal{H}^0$ and $\mathcal{H}^1$ are $\mathbb{R}$-vector spaces.$\}$
1	$\{\overline{T} : \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ is $\mathbb{R}$-linear skew-adjoint; $\overline{\mathcal{H}}$ is a $\mathbb{R}$-vector space.$\}$
2	$\{\overline{T} : \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ is $\mathbb{C}$-antilinear skew-adjoint; $\overline{\mathcal{H}}$ is a $\mathbb{C}$-vector space.$\}$
3	$\{\widetilde{\overline{T}} : \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ is $\mathbb{H}$-linear, neither essentially positive nor essentially negative, skew-adjoint. $\overline{\mathcal{H}}$ is a $\mathbb{H}$-vector space.$\}$
4	$\{T^+ : \mathcal{H}^+ \to \mathcal{H}^-$ is $\mathbb{H}$-linear; $\mathcal{H}^+$ and $\mathcal{H}^-$ are $\mathbb{H}$-vector spaces.$\}$
5	$\{T_+ : \mathcal{H}^+ \to \mathcal{H}^+$ is $\mathbb{H}$-linear skew-adjoint; $\mathcal{H}^+$ is a $\mathbb{H}$-vector space.$\}$
6	$\{A' : \mathcal{H}' \to \mathcal{H}'$ is $\mathbb{C}$-antilinear self-adjoint; $\mathcal{H}'$ is a $\mathbb{C}$-vector space.$\}$
7	$\{A' : \mathcal{H}' \to \mathcal{H}'$ is $\mathbb{R}$-linear, neither essentially positive nor essentially negative, self-adjoint. $\mathcal{H}'$ is a $\mathbb{R}$-vector space.$\}$

Table A.2: The classifying space $\hat{\hat{\hat{\mathfrak{F}}}}_k$ for the KO^{-k}-theory group

It is known that $\mathfrak{F}_k$ and $\hat{\hat{\mathfrak{F}}}_k$ are the classifying spaces for KO-theory groups. That is,

$$Index \ : \ [\mathbb{X}, \mathfrak{F}_k] \xrightarrow{\cong} KO^{-k}(\mathbb{X}); \tag{A.4}$$

$$Index \ : \ [\mathbb{X}, \hat{\hat{\mathfrak{F}}}_k] \xrightarrow{\cong} KO^{-k}(\mathbb{X}). \tag{A.5}$$

In the next section, we will discuss a third space of Fredholm operators $\hat{\hat{\hat{\mathfrak{F}}}}_k$ in detail. (The definition is given in Table(A.2).) We will show that $\hat{\hat{\hat{\mathfrak{F}}}}_k$ is also a classifying space for KO-theory groups by showing that the three spaces are homotopic

$$\mathfrak{F}_k \simeq \hat{\hat{\mathfrak{F}}}_k \simeq \hat{\hat{\hat{\mathfrak{F}}}}_k.$$

A.4 The Classifying Space $\hat{\hat{\hat{\mathfrak{F}}}}_k$

Because the periodicity of the KO-theory group is 8, we have 8 cases to discuss.

<u>$k = \mathbf{0}$</u>

Take $T \in \mathfrak{F}_0$, and a $\mathbb{Z}_2$-graded $\mathbb{R}$-module $\mathcal{H}$. Then $\mathcal{H} = \mathcal{H}^0_{\mathbb{R}} \oplus \mathcal{H}^1_{\mathbb{R}}$ and T can be written into the following form when acting on $\mathcal{H}$

$$T = \begin{pmatrix} 0 & {T_0}^\dagger \\ T_0 & 0 \end{pmatrix},$$

where $T_0 : \mathcal{H}^0_{\mathbb{R}} \to \mathcal{H}^1_{\mathbb{R}}$ is a $\mathbb{R}$-linear Fredholm operator. Clearly, there is a one-to-one correspondence between T and T_0. Thus, $\mathfrak{F}_0 \simeq \hat{\hat{\hat{\mathfrak{F}}}}_0$.

<u>$k = \mathbf{1}$</u>

Take $T \in \mathfrak{F}_1$, and a $\mathbb{Z}_2$-graded Cl_1-module $\mathcal{H}$. Then $\mathcal{H} = \overline{\mathcal{H}}_{\mathbb{R}} \oplus i\overline{\mathcal{H}}_{\mathbb{R}}$. Define

$$\overline{T} := e_1 T|_{\overline{\mathcal{H}}_{\mathbb{R}}} : \quad \overline{\mathcal{H}}_{\mathbb{R}} \to \overline{\mathcal{H}}_{\mathbb{R}}. \tag{A.6}$$

Then

$$\begin{aligned} (\overline{T})^\dagger &= T^\dagger e_1^\dagger|_{\overline{\mathcal{H}}_{\mathbb{R}}} \\ &= -Te_1|_{\overline{\mathcal{H}}_{\mathbb{R}}} \\ &= -e_1 T|_{\overline{\mathcal{H}}_{\mathbb{R}}} \\ &= -\overline{T}. \end{aligned}$$

This shows that $\overline{T} \in \hat{\hat{\hat{\mathfrak{F}}}}_1$. So we have $\mathfrak{F}_1 \simeq \hat{\hat{\hat{\mathfrak{F}}}}_1$.

<u>$k = \mathbf{2}$</u>

Take $T \in \mathfrak{F}_2$, and a $\mathbb{Z}_2$-graded Cl_2-module $\mathcal{H}$. Then $\mathcal{H}$ can be written as

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{\mathbb{R}}^{'} \otimes Cl_2 \\ &= (\mathcal{H}_{\mathbb{R}}^{'} \otimes Cl_2^0) \oplus (\mathcal{H}_{\mathbb{R}}^{'} \otimes Cl_2^1) \\ &:= \overline{\mathcal{H}}_{\mathbb{C}} \oplus e_2 \overline{\mathcal{H}}_{\mathbb{C}}. \end{aligned}$$

Define

$$\overline{T} := e_2 T|_{\mathcal{H}_C} : \quad \overline{\mathcal{H}}_{\mathbb{C}} \to \overline{\mathcal{H}}_{\mathbb{C}}.$$

It is easy to show that $\overline{T} \in \hat{\hat{\mathfrak{F}}}_2$. Then $\mathfrak{F}_2 \simeq \hat{\hat{\mathfrak{F}}}_2$.

<u>k **= 3**</u>

Take $T \in \mathfrak{F}_3$, and a $\mathbb{Z}_2$-graded Cl_3-module $\mathcal{H}$. $\mathcal{H}$ can be written as

$$\mathcal{H} = \mathcal{H}_{\mathbb{H}}^0 \oplus \mathcal{H}_{\mathbb{H}}^1.$$

Define

$$T^{'} := e_1 e_2 e_3 T|_{\mathcal{H}_{\mathbb{H}}^0} : \quad \mathcal{H}_{\mathbb{H}}^0 \to \mathcal{H}_{\mathbb{H}}^0.$$

It is easy to show that $T^{'} \in \hat{\hat{\mathfrak{F}}}_3$. Thus, $\mathfrak{F}_3 \simeq \hat{\hat{\mathfrak{F}}}_3$.

<u>k **= 4**</u>

Take $T \in \hat{\mathfrak{F}}_4$, and an ungraded Cl_3-module $\mathcal{H}$. The volume element of Cl_3 is $\omega = e_1 e_2 e_3$ and $\omega^2 = 1$. So $\mathcal{H}$ can be decomposed into the submodules

$$\mathcal{H}_{\mathbb{H}}^{\pm} = (1 \pm \omega)\mathcal{H}.$$

Because ω commutes with Cl_3, $\mathcal{H}_{\mathbb{H}}^+$ and $\mathcal{H}_{\mathbb{H}}^-$ are both Cl_3-modules. Also $T\omega = -\omega T$, so we can define

$$T^+ := T|_{\mathcal{H}^+_{\mathbb{H}}} : \quad \mathcal{H}^+_{\mathbb{H}} \to \mathcal{H}^-_{\mathbb{H}}.$$

T^+ is Cl_3^0-linear. Then T can be written as

$$T = \begin{pmatrix} 0 & -(T^+)^\dagger \\ T^+ & 0 \end{pmatrix},$$

This shows that $T^+ \in \hat{\hat{\mathfrak{F}}}_4$. So $\hat{\mathfrak{F}}_4 \simeq \hat{\hat{\mathfrak{F}}}_4$.

<u>$k = \mathbf{5}$</u>

Take $T \in \hat{\mathfrak{F}}_5$ and an ungraded Cl_4-module $\mathcal{H}$. $\mathcal{H}$ can be written into the form

$$\mathcal{H} = \mathcal{H}'_{\mathbb{R}} \otimes \mathbb{H}^2.$$

The volume element of Cl_4 is $\omega = e_1e_2e_3e_4$ and $\omega^2 = 1$. According to $\omega = \pm 1$, $\mathcal{H}$ can be decomposed into $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Because $\omega T = T\omega$, we can define

$$T^\pm := T|_{\mathcal{H}_\pm} : \quad \mathcal{H}_\pm \to \mathcal{H}_\pm.$$

T^+ is Cl_4-linear skew-adjoint, so $T^+ \in \hat{\mathfrak{F}}_4$. Follow the same thing we did for $k = 4$ case. The $\mathcal{H}_+$ can be decomposed into subspaces according to the volume element $\omega^0 = e_1^0e_2^0e_3^0$ of Cl_3

$$\mathcal{H}_+ = \mathcal{H}^+_{\mathbb{H}} \oplus \mathcal{H}^-_{\mathbb{H}}.$$

Because $T^+\omega^0 = \omega^0T^+$, we can define

$$T_+ := T^+|_{\mathcal{H}^+_{\mathbb{H}}} : \quad \mathcal{H}^+_{\mathbb{H}} \to \mathcal{H}^+_{\mathbb{H}}.$$

T_+ is Cl_3^0-linear and skew-adjoint, so $T_+ \in \hat{\hat{\mathfrak{F}}}_5$. Thus, $\hat{\mathfrak{F}}_5 \simeq \hat{\hat{\mathfrak{F}}}_5$.

$\underline{k = 6}$

Take $T \in \hat{\mathfrak{F}}_6$ and an ungraded Cl_5-module $\mathcal{H}$. Then $\mathcal{H}$ can be written as

$$\mathcal{H} = \mathcal{H}'_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}^4.$$

We know that there is an involution α on Cl_5 defined by $\alpha(e_i) = -e_i, i = 1, ..., 5$. We can show that there exists an operator J on Cl_5 such that

$$\alpha(\phi) = J\bar{\phi}J^{-1}, \forall \phi \in Cl_5,$$

where $\bar{\phi}$ means the complex conjugate. J also satisfies

$$\bar{J} = J, \quad J^\dagger = -J.$$

Then

$$T(\phi v) = \alpha(\phi)T(v) = J\bar{\phi}J^{-1}T(v).$$

Define a $\mathbb{C}$-antilinear operator $C : \mathcal{H} \to \mathcal{H}$ by

$$C : a \otimes w \mapsto \bar{a} \otimes J^{-1}(\bar{w}), \quad \forall a \in \mathcal{H}'_{\mathbb{C}}, w \in \mathbb{C}^4.$$

Define

$$\tilde{T} := CT : \quad \mathcal{H} \to \mathcal{H}.$$

We can show that $\tilde{T}$ is $\mathbb{C}(4)$-linear,

$$\begin{aligned}
\tilde{T}(a \otimes \phi w) &= CT(a \otimes \phi w) \\
&= C(1 \otimes J\bar{\phi}J^{-1})T(a \otimes w) \\
&= C(1 \otimes J\bar{\phi}J^{-1})\sum_i a_i \otimes w_i \\
&= C\sum_i a_i \otimes J\bar{\phi}J^{-1}(w_i) \\
&= \sum_i \bar{a}_i \otimes J^{-1}\overline{J\bar{\phi}J^{-1}w_i} \\
&= \sum_i \bar{a}_i \otimes \phi J^{-1}\bar{w}_i \\
&= (1 \otimes \phi)CT(a \otimes w) \\
&= (1 \otimes \phi)\tilde{T}(a \otimes w).
\end{aligned}$$

So $\tilde{T}$ can be written into $\tilde{T} = A \otimes Id_{4\times 4}$, where A is a $\mathbb{C}$-linear operator on $\mathcal{H}'_{\mathbb{C}}$. Thus,

$$\begin{aligned}
T &= C^{-1} \cdot (A \otimes Id_{4\times 4}) \\
&= (\bar{A} \cdot c.c.) \otimes (c.c. \cdot J),
\end{aligned}$$

where *c.c.* denotes the complex conjugate operation. We define $A' := \bar{A} \cdot c.c.$ and A' is $\mathbb{C}$-antilinear. Next, we will show that A' is $\mathbb{C}$-antilinear self-adjoint on $\mathcal{H}'_{\mathbb{C}}$.

First, let us discuss the definition of the $\mathbb{C}$-antilinear hermitian conjugation. On $\mathcal{H}$, we have inner products defined over both $\mathbb{R}$ and $\mathbb{C}$. They are related by

$$\langle h_1 \otimes c_1, h_2 \otimes c_2\rangle_{\mathbb{R}} = Re\langle h_1 \otimes c_1, h_2 \otimes c_2\rangle_{\mathbb{C}}, \quad \forall h_1, h_2 \in \mathcal{H}'_{\mathbb{C}}, c_1, c_2 \in \mathbb{C}^4. \tag{A.7}$$

For a $\mathbb{C}$-antilinear operator F, in the $\mathbb{R}$ inner product sense,

$$\langle Fh_1 \otimes c_1, h_2 \otimes c_2\rangle_{\mathbb{R}} = \langle h_1 \otimes c_1, F^{\dagger}h_2 \otimes c_2\rangle_{\mathbb{R}}. \tag{A.8}$$

That is,

$$Re\langle Fh_1 \otimes c_1, h_2 \otimes c_2\rangle_{\mathbb{C}} = Re\langle h_1 \otimes c_1, F^{\dagger}h_2 \otimes c_2\rangle_{\mathbb{C}}. \tag{A.9}$$

We also have

$$\begin{aligned}
Im\langle Fh_1 \otimes c_1, h_2 \otimes c_2\rangle_{\mathbb{C}} &= Re(-i\langle Fh_1 \otimes c_1, h_2 \otimes c_2\rangle_{\mathbb{C}}) \\
&= Re\langle Fh_1 \otimes c_1, -ih_2 \otimes c_2\rangle_{\mathbb{C}} \\
&= \langle Fh_1 \otimes c_1, -ih_2 \otimes c_2\rangle_{\mathbb{R}} \\
&= \langle h_1 \otimes c_1, F^{\dagger}(-ih_2 \otimes c_2)\rangle_{\mathbb{R}} \\
&= Re\langle h_1 \otimes c_1, F^{\dagger}(-ih_2 \otimes c_2)\rangle_{\mathbb{C}} \\
&= Re\langle h_1 \otimes c_1, iF^{\dagger}h_2 \otimes c_2\rangle_{\mathbb{C}} \\
&= Re(i\langle h_1 \otimes c_1, F^{\dagger}h_2 \otimes c_2\rangle_{\mathbb{C}}) \\
&= -Im\langle h_1 \otimes c_1, F^{\dagger}h_2 \otimes c_2\rangle_{\mathbb{C}}
\end{aligned}$$

So we get

$$\begin{aligned}
\langle Fh_1 \otimes c_1, h_2 \otimes c_2\rangle_{\mathbb{C}} &= \overline{\langle h_1 \otimes c_1, F^{\dagger}h_2 \otimes c_2\rangle_{\mathbb{C}}} \\
&:= \langle F^{\dagger}h_2 \otimes c_2, h_1 \otimes c_1\rangle_{\mathbb{C}}. \qquad \text{(A.10)}
\end{aligned}$$

This is the definition of the hermitian conjugate of a $\mathbb{C}$-antilinear operator F.

Then go back to the operator T. T is Cl_5-antilinear skew-adjoint

$$\begin{aligned}\langle T^\dagger h_2 \otimes c_2, h_1 \otimes c_1\rangle_{\mathbb{C}} &= -\langle T h_2 \otimes c_2, h_1 \otimes c_1\rangle_{\mathbb{C}} \\ &= -\langle (A^{'} \otimes c.c. \cdot J) h_2 \otimes c_2, h_1 \otimes c_1\rangle_{\mathbb{C}} \\ &= -\langle A^{'} h_2, h_1\rangle_{\mathbb{C}} \cdot \langle c.c. \cdot J c_2, c_1\rangle_{\mathbb{C}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle T^\dagger h_2 \otimes c_2, h_1 \otimes c_1\rangle_{\mathbb{C}} &:= \langle T h_1 \otimes c_1, h_2 \otimes c_2\rangle_{\mathbb{C}} \\ &= \langle (A^{'} \otimes c.c. \cdot J) h_1 \otimes c_1, h_2 \otimes c_2\rangle_{\mathbb{C}} \\ &= \langle A^{'} h_1, h_2\rangle_{\mathbb{C}} \cdot \langle c.c. \cdot J c_1, c_2\rangle_{\mathbb{C}}.\end{aligned}$$

$$\begin{aligned}\langle c.c. \cdot J c_1, c_2\rangle_{\mathbb{C}} &= \langle \bar{J c}_1, c_2\rangle_{\mathbb{C}} = \overline{\langle J c_1, \bar{c}_2\rangle_{\mathbb{C}}} \\ &= \langle \bar{c}_2, J c_1\rangle_{\mathbb{C}} = \langle J^\dagger \bar{c}_2, c_1\rangle_{\mathbb{C}} \\ &= -\langle (J \cdot c.c.) c_2, c_1\rangle_{\mathbb{C}} = -\langle c.c. \cdot J c_2, c_1\rangle_{\mathbb{C}}.\end{aligned}$$

Finally, we get

$$\langle A^{'} h_1, h_2\rangle_{\mathbb{C}} = \langle A^{'} h_2, h_1\rangle_{\mathbb{C}}. \tag{A.11}$$

Thus, $A^{'}$ is $\mathbb{C}$-antilinear self-adjoint on $\mathcal{H}^{'}_{\mathbb{C}}$. That is, $\hat{\mathfrak{F}}_6 \simeq \hat{\hat{\mathfrak{F}}}_6$.

<u>k **= 7**</u>

Take $T \in \hat{\mathfrak{F}}_7$ and an ungraded Cl_6-module $\mathcal{H}$. Then $\mathcal{H}$ can be written as

$$\mathcal{H} = \mathcal{H}^{'}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{R}^8.$$

Define

$$T^{'} := e_1e_2e_3e_4e_5e_6T : \quad \mathcal{H} \to \mathcal{H}.$$

It is easy to show that $T^{'}$ is Cl_6-linear and self-adjoint on $\mathcal{H}$. Then $T^{'}$ can be written into

$$T^{'} = A^{'} \otimes Id_{8\times 8}, \tag{A.12}$$

where $A^{'}$ is $\mathbb{R}$-linear and self-adjoint on $\mathcal{H}^{'}_{\mathbb{R}}$. So $\hat{\mathfrak{F}}_7 \simeq \hat{\hat{\mathfrak{F}}}_7$.

A.5 The Maximal Symmetry Group Preserved by $\hat{\hat{\mathfrak{F}}}_k$

This section is a further discussion of the classifying space $\hat{\hat{\mathfrak{F}}}_k$. We will study the maximal symmetry group (denoted by M) preserved by $\hat{\hat{\mathfrak{F}}}_k$. Again, we have 8 different cases.

<u>k = **0**</u>

Suppose $\mathcal{H}^0$ and $\mathcal{H}^1$ are N-dimensional $\mathbb{R}$-linear spaces. The symmetric condition is $g_1^{-1}T_0g_0 = T_0$, where $g_0 \times g_1 \in$ SO(N)×SO(N). Take an invertible $T_0 \in \hat{\hat{\mathfrak{F}}}_0$, and we find that $g_1 = T_0g_0(T_0)^{-1}$. That is, g_1 is totally determined by g_0. So M=SO(N).

<u>k = **1**</u>

Suppose $\overline{\mathcal{H}}$ is a 2N-dimensional $\mathbb{R}$-linear vector space. (For the odd dimensional $\overline{\mathcal{H}}$ case, it is totally determined by the even case.) The symmetric condition is

$$g^{-1}\overline{T}g = \overline{T}, \quad g \in SO(2N).$$

To determine M, take

$$\overline{T} = J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}.$$

For an element F in the Lie algebra $so(2N)$, $F^t = -F$. Writes F into the form

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We have $A^t = -A, B^t = -B, C^t = -C, D^t = -D$. The symmetric condition determines that

$$F = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Define $X := A + iB$. Then $X^\dagger = X$. X generates the Lie algebra $u(N)$. So M=U(N).

<u>k **= 2**</u>

Suppose $\overline{\mathcal{H}}$ is a N-dimensional $\mathbb{C}$-linear vector space. The symmetric condition is

$$g^{-1}\overline{T}g = \overline{T}, \quad g \in U(N).$$

Take $\overline{T} = c.c.J$. We get conditions $g^tJG = J$ and $g^\dagger g = gg^\dagger = I$. So M=Sp(N/2).

<u>k **= 3**</u>

Suppose $\overline{\mathcal{H}}$ is an N-dimensional $\mathbb{H}$-linear vector space. The symmetric condition is

$$g^{-1}\tilde{T}g = \tilde{T}, \quad g \in Sp(N).$$

Take

$$\tilde{T} = \begin{pmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{pmatrix}.$$

Write g as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The symmetric condition determines that $C = B = 0$. Because g is in Sp(N), we find that M =Sp(N/2)×Sp(N/2).

$\underline{k = \mathbf{4}}$

Suppose $\mathcal{H}^+$ and $\mathcal{H}^-$ are N-dimensional $\mathbb{H}$-linear vector spaces. The symmetric condition is $g_-^{-1}T^+g_+ = T^+, \;\; g_+ \times g_- \in$ Sp(N) ×Sp(N). If T^+ is invertible, g_- is totally determined by g_+. So M=Sp(N).

$\underline{k = \mathbf{5}}$

Suppose $\mathcal{H}^+$ is an N-dimensional $\mathbb{H}$-linear vector space. The symmetric condition is $g^{-1}T_+g = T_+, \;\; g \in$ Sp(N). Take

$$T_+ = i \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}$$

and write g as

$$g = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}.$$

The symmetric condition determines that $B = 0$. From the condition $g^\dagger g = gg^\dagger = I$, we get $A^\dagger A = AA^\dagger = I$. So M=U(N).

$\underline{k = \mathbf{6}}$

k	G	M
0	SO(N) $\times$ SO(N)	SO(N)
1	SO(2N)	U(N)
2	U(N)	Sp(N/2)
3	Sp(N)	Sp(N/2) $\times$ Sp(N/2)
4	Sp(N) $\times$ Sp(N)	Sp(N)
5	Sp(N)	U(N)
6	U(N)	SO(N)
7	SO(2N)	SO(N) $\times$ SO(N)

Table A.3: The symmetric group G for a Cl_k-module and the maximal subgroup M preserved by $\hat{\hat{\hat{\mathfrak{F}}}}_k$

Suppose $\mathcal{H}'$ is an N-dimensional $\mathbb{C}$-linear vector space. The symmetric condition is $g^{-1}A'g = A', \quad g \in$ U(N). Take $A' = c.c.I_N$, then we get $\bar{g}^{-1}g = I_N$ and $g^\dagger g = gg^\dagger = I_N$. So $g^t g = gg^t = I$. Then M=SO(N).

<u>k = **7**</u>

Suppose $\mathcal{H}$ is a 2N-dimensional $\mathbb{R}$-linear vector space. The symmetric condition is $g^{-1}A'g = A', \quad g \in$ SO(2N). Take

$$A' = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}$$

and write g as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The symmetric condition determines that $B = C = 0$. Because g is in SO(N), we get $A \in$ SO(N), $D \in$ SO(N). So M=SO(N)$\times$SO(N).

We summarize our results in Table(A.3).

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