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# Covariant electromagnetic field lines

Y Hadad<sup>1</sup>, E Cohen<sup>2</sup>, I Kaminer<sup>3</sup> and A C Elitzur<sup>4</sup>

<sup>1</sup> Department of Mathematics, University of Arizona, Tucson, AZ, 85721, USA

<sup>2</sup> H.H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol, BS8 1TL, U.K

<sup>3</sup> Department of Physics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139, USA

<sup>4</sup> Iyar, The Israeli Institute for Advanced Research, POB 651, Zichron Ya'akov 3095303, Israel

E-mail: yaron.hadad@gmail.com

**Abstract.** Faraday introduced electric field lines as a powerful tool for understanding the electric force, and these field lines are still used today in classrooms and textbooks teaching the basics of electromagnetism within the electrostatic limit. However, despite attempts at generalizing this concept beyond the electrostatic limit, such a fully relativistic field line theory still appears to be missing. In this work, we propose such a theory and define covariant electromagnetic field lines that naturally extend electric field lines to relativistic systems and general electromagnetic fields. We derive a closed-form formula for the field lines curvature in the vicinity of a charge, and show that it is related to the world line of the charge. This demonstrates how the kinematics of a charge can be derived from the geometry of the electromagnetic field lines. Such a theory may also provide new tools in modeling and analyzing electromagnetic phenomena, and may entail new insights regarding long-standing problems such as radiation-reaction and self-force. In particular, the electromagnetic field lines curvature has the attractive property of being non-singular everywhere, thus eliminating all self-field singularities without using renormalization techniques.

## 1. Introduction

Mathematical formalism is usually preferred over pictorial explanations in physics. It is the equations that enable us to predict the evolution of physical systems with nearly any desired precision. When we think of atoms as tiny solar systems, or draw spacetime as two-dimensional we are well aware that these pictorial methods are limited, leaving it to mathematics to give the “right” description. The compatibility between the two methods is indeed a source of wonder [1].

The visual aspect, however, goes far beyond being an intuitive aid to the formalism. Physical reality, after all, is often richer and more complex than what mathematics captures. Moreover, since mathematics is based on abstract definitions, in general it is compatible with many possible realities. This leaves open the question why this particular set of assumptions rather than another is compatible with our universe. Gödel’s theorem [2], sometimes phrased as “truth transcends theoremhood” [3], stresses this incompleteness.

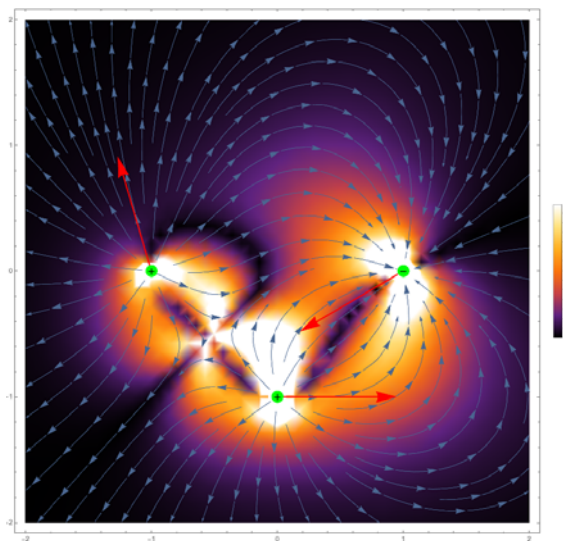
Indeed, geometry-based physical models not only preceded mathematical ones, but often kept guiding their development. The evolution of general relativity from the special one is a case in point. It was highly pictorial almost naive thought-experiments which enabled the two theories to generalize their insights from inertial to non-inertial motions.



This is also the case of electromagnetism, and the nature of the electromagnetic field in particular. It was Faraday's highly pictorial description of the field lines [4] as carriers of the electromagnetic field through their tension and pressure which later led to Maxwell's concise mathematical theory yielding the new predictions of this unique field representation [5]. Intriguingly, Maxwell himself kept heavily employing Faraday's "strings" and "rubber bands" alongside with his abstract equations, stressing time and again their mechanical properties.

In the current paper we resume this line of investigation by exploring the notion of fully relativistic electromagnetic field lines. We ascribe the field-lines mechanical properties and prove quantitatively that their curvature is not only the source of the charge's acceleration but also its origin.

In [6] the authors found a relationship between the curvature of electric field lines and the acceleration of a non-relativistic charge (Fig. 1). While external fields and acceleration of charges bend the electric field lines, non-relativistic electric charges always accelerate in the direction of straight electric field lines. Another advantage of the field lines curvature framework is its lack of singularities. It is well known that the total electric field is singular nearby point charges due to the notorious self field problem [7]. However, the field lines curvature is well-defined and everywhere non-singular, and provides a potential route to study electrodynamics without the use of renormalization. There are reasons to believe that similar results hold in a more general setting, and this paper is attempting to convey this.



**Figure 1.** Electromagnetic field lines are represented by curves, whose curvature is computed explicitly below in Eq. (22). Bright colors in the figure represent high scalar curvature and black areas have zero electromagnetic field lines curvature. The red arrows are the acceleration of the charges, and the figure shows one of the main conclusions of this work - charges always accelerate in the direction of straight electromagnetic field lines. It is also proven that while the total electromagnetic field is singular due to the self fields of the charges, the electromagnetic field lines curvature is always non-singular (even for point charges).

Geometrically, the four-acceleration of a particle is the curvature vector of its world line [8]. This is an equivalence between the kinematics of a charge and the geometry of its world line, which links acceleration (of a particle) and curvature (of its world line). This paper defines a covariant notion of field lines for the electromagnetic tensor in Minkowski spacetime. We study the curvature of the electromagnetic field lines and show that it is closely related to the four-

acceleration of the charge. This gives another equivalence, relating the kinematics of charges to the geometry of field lines. Therefore, at least in the framework of the classical theory of electrodynamics there is a sequence of three equivalences between four-acceleration, curvature of world lines and curvature of electromagnetic field lines.

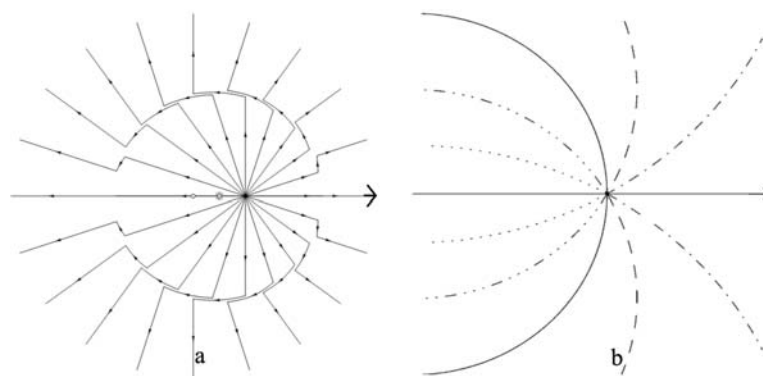
Classical field theories suffer from the problem of self-field singularities when point-like particles are introduced. This is the notorious *self-force (or self-field) problem*. For example, Maxwell's equations predict the electromagnetic field to be singular at the location of point-like electrons [7]. One of the consequences of this work is that the curvature of the electromagnetic field lines is well-defined and non-singular, even in the vicinity of a point charge. This means that the framework studied in this work does not suffer from the problem of self-force. In other words, the electromagnetic field lines interpretation may shed light on how charges interact with their own electromagnetic field.

In this conference proceedings paper we summarize some of our recent results regarding the relativistic electromagnetic curvature. Many derivations and further examples and implications were omitted for brevity. A more detailed analysis can be found in [9].

The outline of the paper is as follows: in section 2 we motivate the equivalence between field lines curvature and acceleration. This is done by studying the special case of non-relativistic charges in an external electric field. In section 3 we provide a covariant definition of electromagnetic field lines. Section 4 studies the curvature of covariant field lines and its relationship to four-acceleration. Section 5 summarizes the results of this work.

## 2. Acceleration implies curvature

Consider a charge in empty (flat) space. Such a charge generates an electric field, which is commonly depicted by electric field lines. If the charge's velocity changes abruptly from zero to a constant velocity, it will be emitting an electromagnetic wave. For a sufficiently far observer the charge still appears to be at rest and its field-lines isotropic outside of the forward light-cone of the event of instantaneous acceleration. Inside the light-cone the charge is already moving, hence the field-lines appear denser in the direction perpendicular to the direction of motion. To ensure continuity of the field-lines, they appear to be partially "broken-and-redrawn" on the light-cone's surface, as shown in Fig. 2a.



**Figure 2.** (a) The electric field lines generated by a charge after an instantaneous velocity change. The field lines appear to be partially "broken-and-redrawn" on the light-cone due to the instantaneous acceleration of the charge. (b) Electric field lines of a constantly accelerating charge bend. The charge always accelerates in the direction of straight field lines.

When the velocity of the charge changes at a constant rate (constant acceleration) the field lines appear to be "peeling off" the charge (see Fig. 2b). The only field-lines that do not curve

are the two that are directed along the charge's acceleration vector. The lines remain straight because they are on the only axis on which symmetric constraints act from both sides.

This implies that as a charge accelerates, the electric field lines around the charge curve. At least for purely electric field lines we may say that charge acceleration implies that the field lines are (locally) curved, a fact that was proven in [6].

The goals of the rest of this paper are to (a) extend the notion of electric field lines to covariant electromagnetic field lines, (b) study the curvature of electromagnetic field lines, and (c) show that the electromagnetic field lines curvature is fundamentally related to the notion of charge acceleration.

### 3. Covariant field lines

Electric (or magnetic) field lines are defined to be curves in three-dimensional space which are everywhere tangent to the electric (magnetic) field, while their density is proportional to the magnitude of the field. Since the tangent is defined using the derivative (a first order operator), a point in space with a non-singular electric (magnetic) field has a unique field line that passes through it.

Studying the electromagnetic field lines in a fully relativistic manner requires a covariant definition of field lines that generalizes the notions of electric and magnetic field lines. It was previously argued [10] that field lines are intrinsically non-relativistic for two reasons. First, since electric and magnetic fields transform into each other under Lorentz transformations one cannot expect to find a covariant notion of a pure electric field line or a pure magnetic field line. Secondly, points on electric (or magnetic) field line all have the same time coordinate and are therefore inherently non-local.

In Minkowski spacetime the electromagnetic field is represented by the Faraday tensor  $F^{\alpha\beta}$ . The Faraday tensor is a covariant tensor that treats the electric and magnetic fields as a single physical object. One would expect that the first issue raised in [10] can be addressed by studying electromagnetic field lines using the Faraday electromagnetic tensor. Therefore, we are not expecting to split the electric field lines and magnetic field lines, but maintaining them as a single entity. The second issue can be overcome by refraining to use the notion of an absolute time in the definition of the electromagnetic field lines. Instead, electromagnetic field lines must be defined relatively to the observer who is measuring them.

The electromagnetic field  $F^{\alpha\beta}$  as measured by an observer with four-velocity  $u^\alpha$  can be reduced to a (four-)vector field. Consider an electromagnetic field line  $e^\alpha(\lambda)$  with a parameterization variable  $\lambda$ . We require the electromagnetic field line to satisfy *the electromagnetic field line equation*

$$\frac{de^\alpha}{d\lambda} = E^\alpha, \quad (1)$$

where

$$E^\alpha(x^\beta) = F^{\alpha\gamma}(x^\beta)u_\gamma(\tau_{\text{ret}}), \quad (2)$$

and  $\tau_{\text{ret}}$  is the retarded time from the observer to the spacetime event  $x^\beta$ , namely, the time it takes a ray of light emitted from the observer to reach  $x^\beta$ . Here and in the rest of this work we use a spacetime metric with signature  $(-+++)$ . We also employ the following convention: space-like four-vectors with no indices are to be interpreted as their Lorentz norm ( $E = \sqrt{E^\beta E_\beta}$ ). Note that this is well-defined as space-like vectors have a positive Minkowski norm.

Let  $\kappa^\alpha$  denote the curvature vector of the electromagnetic field line. In general, the curvature vector of a curve is defined to be the curve's acceleration with respect to unit-length parameterization. We may therefore compute the curvature of an electromagnetic field line as a curve in Minkowski spacetime by calculating

$$\kappa^\alpha = \frac{d^2 e^\alpha}{ds^2}, \quad (3)$$

where  $s$  is a unit-length parameterization satisfying

$$\frac{de^\alpha}{ds} = \frac{E^\alpha}{E}. \quad (4)$$

By using Eq. (4) the curvature vector may be written explicitly as

$$\kappa^\alpha = \kappa_1^\alpha - \kappa_2^\alpha, \quad (5)$$

where we used the chain-rule  $\frac{d}{ds} = \frac{dx^\gamma}{ds} \frac{\partial}{\partial x^\gamma}$ , and introduce the notation

$$\kappa_1^\alpha = \frac{1}{E^2} E^\beta \frac{\partial E^\alpha}{\partial x^\beta}, \quad (6)$$

and

$$\kappa_2^\alpha = \frac{1}{E^4} (E_\gamma E^\beta \frac{\partial E^\gamma}{\partial x^\beta}) E^\alpha \quad (7)$$

which will be used in the next section.

Many questions may rise regarding the relationship between the definition of covariant electromagnetic field lines (4) and field lines of electric or magnetic fields in the electrostatic limit. Unfortunately this is beyond the scope of this paper and such questions will be answered in details elsewhere [9]. The reader can easily verify that for a charge at rest, the spatial components of Eq. (4), the spatial components of the curvature four-vector (5), and the scalar curvature  $\kappa = \sqrt{\kappa^\alpha \kappa_\alpha}$  restore their definition in this electrostatic limit<sup>1</sup>. Therefore, the form (4) of covariant electromagnetic field lines naturally extends its non-relativistic form used in [6].

Now, we have a covariant notion of electromagnetic field lines that passes the concerns raised in [10]. Having introduced this natural generalization, we proceed to studying its implications.

#### 4. Covariant field lines curvature

Consider a charge  $q$  with mass  $m$  traveling along a world line  $z^\alpha(\tau)$  with proper time  $\tau$ . The charge generates an electromagnetic field on its forward light cone. A point  $x^\alpha$  is on the light cone of the charge if

$$k^\alpha k_\alpha = 0 \quad (8)$$

where the null vector  $k^\alpha$  is

$$k^\alpha = x^\alpha - z^\alpha(\tau_{\text{ret}}). \quad (9)$$

The (self) electromagnetic field produced by the charge is given by the retarded Liénard-Wiechert equation [11]

$$F_{\text{self}}^{\alpha\beta} = \frac{q}{R^2} \left( U^\alpha k^\beta - U^\beta k^\alpha \right), \quad (10)$$

where  $R$  is the retarded distance

$$R = -k^\alpha u_\alpha \quad (11)$$

and  $U^\alpha$  is the Synge vector [12]

$$U^\alpha = B u^\alpha + a^\alpha. \quad (12)$$

$B$  is the so-called Plebański invariant [13]

$$B = \frac{1 - W}{R} \quad (13)$$

<sup>1</sup> One may study another form of (dual) electromagnetic field lines, by considering the dual Faraday tensor. Here the spatial components of the dual electromagnetic field lines for a charge which is at rest give the magnetic field lines. The study of the dual electromagnetic field lines is straightforward and will not be discussed here.

and  $W$  is the contraction

$$W = -k^\alpha a_\alpha. \quad (14)$$

The four-velocity and four-acceleration used in Eqs. (11), (12) and (14) are the retarded four-velocity  $u^\alpha = u^\alpha(\tau_{\text{ret}})$  and the retarded four-acceleration  $a^\alpha = a^\alpha(\tau_{\text{ret}})$  respectively.

We shall consider first the problem of a free charge in empty space. Since  $q$  is a free charge, the only contribution to the field lines is the self field produced by the charge  $q$  itself. Substituting Eq. (10) into Eq. (2) gives

$$E_{\text{self}}^\alpha = -\frac{q}{R}U^\alpha + \frac{q}{R^3}(1-W)k^\alpha, \quad (15)$$

with squared Minkowski norm

$$(E_{\text{self}})^\alpha{}_\alpha = \frac{q^2}{R^4}(1-W^2 + a^2 R^2). \quad (16)$$

We use Eqs. (15) and (16) in the curvature field vector Eq. (5)

$$\kappa^\alpha = \kappa_1^\alpha - \kappa_2^\alpha. \quad (17)$$

under the assumption that *the field lines are measured by the charge itself*. The first term in (17) defined in Eq. (6) gives

$$\kappa_1^\alpha = \frac{1}{\Delta} \left\{ \left[ \frac{-2(1-W)}{R^2} + \frac{V}{R} - a^2 \right] P^\alpha + \frac{W(1-W)}{R^2} k^\alpha + (1-W)a^\alpha + R\dot{a}^\alpha \right\}, \quad (18)$$

and the second term which was defined in Eq. (7) gives

$$\kappa_2^\alpha = \frac{1}{\Delta^2} [(1-W)(-2+W^2) + WRV - R^2 a^2 - R^3 a \cdot \dot{a}] \left( \frac{1-W}{R^2} P^\alpha - a^\alpha \right). \quad (19)$$

Here we defined the projection four-vector

$$P^\alpha = k^\alpha - Ru^\alpha, \quad (20)$$

the scalar  $\Delta$  to be

$$\Delta = 1 - W^2 + a^2 R^2 \quad (21)$$

and  $V = -k^\alpha \dot{a}_\alpha$  to be the projection of the jerk  $\dot{a}^\alpha = \frac{da^\alpha}{d\tau}$  on the null-vector  $k^\alpha$ . The projection four-vector  $P^\alpha$  defined in Eq. (20) has convenient properties that will be studied elsewhere [9].

Eq. (17) together with Eqs. (18) and (19) give a closed-form expression for the curvature of the covariant electromagnetic field lines produced by a free charge in Minkowski spacetime. The expression is ready to be evaluated at any event in spacetime explicitly. Studying Eq. (17) reveals a couple of interesting properties the electromagnetic field lines curvature possesses.

As is well-known due to the self-field problem, the electromagnetic field tensor (10) is singular at the position of the charge and behaves like  $\frac{1}{R^2}$  in the limit  $R \rightarrow 0$ . Despite that fact, *the electromagnetic field lines curvature is non-singular in the vicinity of the charge*. In order to see this we recall from Eq. (10) that the electromagnetic (self) field at each event  $x^\alpha$  is determined by the four-position of the charge at the retarded four-position  $z^\alpha(\tau_{\text{ret}})$ . This means that we need to study the limit  $x^\alpha \rightarrow z^\alpha$  along the light-cone, and we do so by letting  $k^\alpha = \varepsilon \hat{k}^\alpha$ , where  $\hat{k}^\alpha$  is of order unity and  $\varepsilon$  is a small parameter.

The leading order expansion of the curvature four-vector (17) in  $\varepsilon$  gives

$$\kappa^\alpha = -a^\alpha - \frac{\hat{W}}{\hat{R}^2} \hat{k}^\alpha + \frac{2\hat{W}}{\hat{R}} u^\alpha + O(\varepsilon), \quad (22)$$

where the hats denote normalized quantities of order one in  $\varepsilon$ , namely  $k^\alpha = \varepsilon \hat{k}^\alpha$ ,  $W = \varepsilon \hat{W}$  and  $R = \varepsilon \hat{R}$ . Eq. (22) shows that the field lines curvature does not contain any singular terms in  $\varepsilon$ .

It may appear that Eq. (22) is singular due to the denominators containing the normalized retarded position  $\hat{R}$ , but this is not the case. Let us explicitly write the normalized null-vector  $\hat{k}^\alpha = (1, \hat{k})$  and the four-velocity to be  $u^\alpha = \gamma(1, \vec{v})$ , where  $\hat{k}$  is a unit spatial vector,  $\vec{v}$  is the spatial velocity of the charge in units where the speed of light is unity  $c = 1$ , and  $\gamma = \frac{1}{\sqrt{1-\vec{v}^2}}$  is the relativistic Lorentz factor. We now see that the normalized retarded distance is always non-zero

$$\hat{R} = \gamma(1 - \hat{k} \cdot \vec{v}) \neq 0 \quad (23)$$

as the charge is not massless and is moving slower than the speed of light in vacuum. Therefore we may conclude that the covariant electromagnetic field lines curvature (17) is everywhere non-singular even for point charges. This is a non-trivial property as quantities that are dependent on the self field of point charges are typically singular.

Contracting Eq. (22) with the normalized null-vector  $\hat{k}_\alpha$  gives

$$\hat{k}_\alpha \kappa^\alpha = -\hat{W} \quad (24)$$

which vanishes when  $\hat{W} = -\hat{k} \cdot a = 0$  and defines a two-dimensional surface in spacetime that intersects the four-position of the charge. On this surface Eq. (22) reveals that the four-curvature is just the negative four-acceleration

$$\kappa^\alpha = -a^\alpha. \quad (25)$$

Eq. (25) implies that the four-acceleration of the charge is precisely (minus) the curvature of the electromagnetic field lines that it is producing. By virtue of the perpendicularity condition (24), this relationship holds whenever the field line curvature itself is perpendicular to the light cone. Eq. (25) also implies that we may interchangeably translate a property related to the kinematics of the charge (acceleration) and the geometry of the field lines (curvature).

We can observe other properties of Eq. (17) by considering the motion of the charge in its instantaneous rest frame. The four-velocity now is  $u^\alpha = (1, 0, 0, 0)$  and the four-acceleration is  $a^\alpha = (0, \vec{a})$ , where  $\vec{a}$  is the spatial acceleration of the charge. The electromagnetic field lines four-curvature (25) is

$$\kappa^\alpha = |\vec{a}|(-\cos\theta, -\hat{a} + \hat{k} \cos\theta). \quad (26)$$

The scalar curvature is given by the squared norm of the curvature four-vector,

$$\kappa^2 = -\vec{a}^2 \cos(2\theta), \quad (27)$$

where  $\theta$  is the angle between the spatial acceleration  $\vec{a}$  and the retarded position of the particle  $\vec{k} = \vec{x} - \vec{z}(\tau_{\text{ret}})$ . If we separate the four-curvature  $\kappa^\alpha$  to its temporal part (0-component) and spatial part (components 1, 2, 3) by writing  $\kappa^\alpha = (\kappa^0, \vec{\kappa})$ , we see that the *scalar spatial curvature* is

$$|\vec{\kappa}| = |\vec{a}| |\sin\theta|. \quad (28)$$

Therefore, locally the field lines of an accelerating charge ( $\vec{a} \neq \vec{0}$ ) are spatially straight only in the direction of the charge acceleration  $\theta = 0$ . In this case the temporal component of the field lines curvature yields the magnitude of the acceleration

$$\kappa^0 = -|\vec{a}|. \quad (29)$$

In other words, nearby a freely accelerating charge all of its kinematical properties are determined by the field lines curvature. This generalizes the result derived in [6].

The next obvious step for this work is to study the curvature of arbitrary covariant electromagnetic field lines. In the electrostatic limits, this was done in [6], where the local behavior of the electric field lines curvature and the dynamics of the charge were shown to be related. A similar analysis can be provided in this fully relativistic case.

The total electromagnetic field can be written as a superposition

$$F^{\alpha\beta} = F_{\text{ext}}^{\alpha\beta} + F_{\text{self}}^{\alpha\beta} \quad (30)$$

of the external field  $F_{\text{ext}}^{\alpha\beta}$  and the self field  $F_{\text{self}}^{\alpha\beta}$  produced by the charge as defined in (10). Even when there is an external electromagnetic, in the leading order the covariant electromagnetic field lines curvature satisfies Eq. (22). This implies that also in the most general case, the electromagnetic field lines curvature is a non-singular physical object. The next order correction to (22) is directly related to the external electromagnetic field  $F_{\text{ext}}^{\alpha\beta}$ , and reveals great insights on the relationship between curvature of electromagnetic field lines and the full dynamics of accelerating electric charges. This result will be studied in details in [9].

## 5. Summary

In this paper we returned to the origins of the electromagnetic theory as outlined by Faraday and Maxwell. Following the founding fathers we brought the field lines back into the heart of discussion, and also elevated them to be fully covariant objects.

To summarize our main results: we defined a notion of covariant electromagnetic field lines (4) and calculated their four-curvature at an arbitrary event  $x^\alpha$  for a free charge. We derived an explicit closed-form formula for the four-curvature of self fields (22). The field lines curvature was shown to be everywhere non-singular. Most importantly, the field lines curvature seems to completely determine the kinematics of the electric charges. Specifically and as depicted in Fig. 1, *charges always accelerate in the direction of straight electromagnetic field lines*. Further results are presented in [9].

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