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THE HIERARCHIES OF IDENTITIES AND CLOSED PRODUCTS FOR MULTIPLE COMPLEXES

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Abstract. We consider infinite $\mathbb{Z}_{\mathbb{Z}}$-index complexes $\mathcal{C}$ of spaces with elements depending on a number of parameters, complete with respect to a linear associative regular inseparable multilinear product. The existence of nets of vanishing ideals of orders of and powers of differentials is assumed for subspaces of $\mathcal{C}$-spaces. In the polynomial case of orders and powers of the differentials, we derive the hierarchies of differential identities and closed multiple products. We prove that a set of maximal orders and powers for differentials, differential conditions, together with coherence conditions on indices of a complex $\mathcal{C}$ elements generate families of multi-graded differential algebras.

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The authors state that:
1.) The paper does not contain any potential conflicts of interests.
2.) The paper does not use any datasets. No dataset were generated during and/or analysed during the current study.
3.) The paper includes all data generated or analysed during this study.
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1. Introduction

The main purpose of this paper is derive hierarchies of differential identities and closed products following from sets of natural conditions on orders and powers of differentials applied to elements of a multiple complex. In contrast to ordinary wedge-product case for differential forms reflected in [12, 15] we work with the universal enveloping algebra constituted by $\mathbb{N}$-valued powers of elements of complexes, in particular, given by differentials applied to such elements. Note that in general we do not specify commutation relations for elements of $\mathcal{C}$. Nevertheless, the structure given in the paper reproduces the structure of a differential algebra. The differential identities for elements of complexes with multiple indices endowed with regular associative products is an important way to study various algebraic and geometric structures. In particular, they are extremely useful to find closed products in cohomology classes computations of invariants associated to a multiple complex. Differential conditions

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applied to elements of a multiple complex $\mathcal{C}$ provide a system of multiplication rules and form the resulting algebra. The full algebra associated to all possible choices of differential relations is quite huge. For differential forms considered on smooth manifolds, the Frobenius theorem for a distribution leads to orthogonality conditions on forms. Motivated by the notion of integrability for differential forms on foliated manifolds [12, 15], we study systems of differential identities with respect to a multiple product of elements of a multiple-index complex. We then show that differential conditions endow a multiple complex with the structure of a graded differential algebra. Assuming existence of a net of differential power/order vanishing ideals in subspaces of a multiple complex $\mathcal{C}$, and requiring natural orthogonality conditions for completions of a set of $\mathcal{C}$ elements, the hierarchies of differential conditions arise in terms of closed products.

Ideologically, closed products containing powers of multiple action of mixed differentials represent a geometrical idea of "codimension" one products. Namely, there are two possibility. First, a product contains a combination of differentials which vanishes due to the critical orders or powers of differentials as the result of application of differentials. Second, an initial candidate to closed products, does not contain maximal orders or powers but after an application of a differential identity, the total maximal powers of elements appear, and a product can be reduced to the first case. The hierarchies of differential identities are of non-trivial nature since we assume that maximal orders of differentials as well as maximal powers of $\mathcal{C}$-elements depend on elements themselves (see explanations in Section 2). Certain non-trivial examples of multiple product identities are given in Sections 3–5. We provide examples of closed products with respect to multiple products of differentials d_a and $d_{\overline{a}} = \overline{d}_a$

The hierarchies we derive from the conditions on a complex are useful in the theory of continual Lie algebras [19, 20, 21, 23, 1] and in the theory of completely integrable [2] and exactly solvable [16, 18, 3, 9] dynamical systems. In particular, similar to invariants associated to foliations, one is able to use the identities from a hierarchy to prove integrability as well as to find invariants of corresponding dynamical systems. It is important to mention that the hierarchies of identities and closed products (i.e., products annihilating by a single differential) constitute the tools for direct computation and classification of cohomology invariants of the corresponding complex. Therefore, we are interested in generating all possible closed products. We will present such a classification in a forthcoming paper. The next step in finding invariants is to prove their independence with respect to replacements of a complex elements. The classification problem of cohomology invariants associated to a complex endowed with a multiple product will be treated in a separate paper.

As for possible applications of the material presented in this paper, we would like to mention computations of higher cohomology for grading-restricted vertex algebras [14], search for more complicated cohomology invariants, and applications in differential geometry and algebraic topology. It would be interesting to study possible applications of invariants we constructed to cohomology of manifolds. In differential geometry there exist various approaches to the construction of cohomology classes (cf., in particular, [17]). We hope to use these techniques to derive counterparts in the cohomology theory of vertex algebras. The results proven in this paper are also useful in computations of cohomology of foliations [4, 6].

2. Multiple complexes and the main result

Introduce a system of families of multiple horizontal and vertical complexes $\mathcal{C} = \left(C\left(\Theta^{\boldsymbol{n}}_{\boldsymbol{m}}\right), d^{\boldsymbol{n}'}_{\boldsymbol{m}'}, \overline{d}^{\boldsymbol{n}''}_{\boldsymbol{m}''} \right)$ with $\boldsymbol{n} \in \mathbb{Z}_{\mathbb{Z}}$ and $\boldsymbol{m} \in \mathbb{Z}_{\mathbb{Z}}$, i.e., an infinite number of up and down $\mathbb{Z}$-valued indices, and $\boldsymbol{n} = \boldsymbol{n}' \cup \boldsymbol{n}''$, $\boldsymbol{m} = \boldsymbol{m}' \cup \boldsymbol{m}''$. These indices correspond to increasing and decreasing indices of parameters $\Theta^{\boldsymbol{n}}_{\boldsymbol{m}}$ for spaces $C\left(\Theta^{\boldsymbol{n}}_{\boldsymbol{m}}\right)$ under actions of $\mathcal{C}$-differentials correspondingly. Here $d^{\boldsymbol{n}'}_{\boldsymbol{m}'}$ denotes a family of differentials for each pair of entries n'_i of $\boldsymbol{n}'$ and m'_i of $\boldsymbol{m}'$, $i \in \mathbb{Z}$. The action of a differential $d^{n'_i}_{m'_i}$: $C\left(\Theta^{\boldsymbol{n}}_{\boldsymbol{m}}\right) \to C\left(\Theta^{\ldots, n'_i+1, \ldots,}_{\ldots, m'_i-1, \ldots}\right)$, $i \in \mathbb{Z}$, for $\ldots, n'_i+1, \ldots$ and $\ldots, m'_i-1, \ldots$ means that all entries of the multiindices $\boldsymbol{n}'$ and $\boldsymbol{m}' - \boldsymbol{1}$ remain the same except for $n'_i \mapsto n'_i + 1$ and $m'_i \mapsto m_i - 1$, that increases or decreases by one correspondingly. Other indices $\boldsymbol{n}''$ and $\boldsymbol{m}''$ remain unchanged.

The family of vertical differentials $\overline{d}^{\boldsymbol{n}''}_{\boldsymbol{m}''} : C\left(\Theta^{\boldsymbol{n}}_{\boldsymbol{m}}\right) \to C\left(\Theta^{\ldots, n''_j+1, \ldots}_{\ldots, m''_j-1, \ldots}\right)$, $j \in \mathbb{Z}$, change n''_j and m''_j indices similarly, while the indices $\boldsymbol{n}'$ and $\boldsymbol{m}'$ remain unchanged. A subset of horizontal and vertical complexes corresponding to subsets of upper indices in $\boldsymbol{n}$ and lower indices in $\boldsymbol{m}$ are supposed to be chain, cochain, or chain-cochain corresponding to the differentials $(d)^{\boldsymbol{n}}_{\boldsymbol{m}}$ and $\overline{d}^{\boldsymbol{n}}_{\boldsymbol{m}}$. When we write $\chi \in \mathcal{C}$ that means that an element χ belongs to a subspace of $\mathcal{C}$. The $i \in \mathbb{Z}$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$-th slice of the full diagram of a complex $\mathcal{C}$ is described by the following diagram

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow \overline{d}^{\ldots, n'_i-1, \ldots, n''_j-1, \ldots}_{\ldots, m'_i, \ldots, m''_j, \ldots} & & \downarrow \overline{d}^{\ldots, n_i-1, \ldots, n_j-1, \ldots}_{\ldots, m'_i-1, \ldots, m''_j, \ldots} & & \\
\ldots & \overset{d^{\ldots, n_i-1, \ldots, n_j, \ldots}_{\ldots, m_k+1, \ldots}}{\longrightarrow} & C^{\ldots, n_i, \ldots, n_j, \ldots}_{\ldots, m_k, \ldots} & \overset{d^{\ldots, n_i, \ldots, n_j, \ldots}_{\ldots, m_k, \ldots}}{\longrightarrow} & C^{\ldots, n_i+1, \ldots, n_j, \ldots}_{\ldots, m_k-1, \ldots} & \overset{d^{\ldots, n_i+1, \ldots, n_j, \ldots}_{\ldots, m_k-1, \ldots}}{\longrightarrow} & \ldots \\
 & & \downarrow \overline{d}^{\ldots, n_i, \ldots, n_j, \ldots}_{\ldots, m_k, \ldots} & & \downarrow \overline{d}^{\ldots, n_i+1, \ldots, n_j, \ldots}_{\ldots, m_k-1, \ldots} & & \\
\ldots & \overset{d^{\ldots, n_i-1, \ldots, n_j+1, \ldots}_{\ldots, m_k+1, \ldots}}{\longrightarrow} & C^{\ldots, n_i, \ldots, n_j+1, \ldots}_{\ldots, m_k, \ldots} & \overset{d^{\ldots, n_i, \ldots, n_j+1, \ldots}_{\ldots, m_k, \ldots}}{\longrightarrow} & C^{\ldots, n_i+1, \ldots, n_j+1, \ldots}_{\ldots, m_k-1, \ldots} & \overset{d^{\ldots, n_i+1, \ldots, n_j+1, \ldots}_{\ldots, m_k-1, \ldots}}{\longrightarrow} & \ldots \\
 & & \downarrow \overline{d}^{\ldots, n_i, \ldots, n_j+1, \ldots}_{\ldots, m_k, \ldots} & & \downarrow \overline{d}^{\ldots, n_i+1, \ldots, n_j+1, \ldots}_{\ldots, m_k-1, \ldots} & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

Introduce the following notations. Since indices of differentials and indices of $\mathcal{C}$-subspaces are taken to be always coherent, let us denote $d_a = (d_a)^{\boldsymbol{n}}_{\boldsymbol{m}}$, where the index a denotes either of the differentials d or $\overline{d}$, $\overline{a}$ denotes the opposite choice, and $d_{\overline{a}} = \overline{d}_a$. A combination of differentials is notated by $d^{r_1}_{a_1} \ldots d^{r_k}_{a_k}$, where $r_1, \ldots, r_k$ are orders of corresponding differentials. We call q of $(\phi)^q$ the power of an element ϕ. For all possible combinations of l elements of $\mathcal{C}$ we assume the existence of a formal multi-linear associative inseparable $\cdot_l$-product

$$\cdot_l = \cdot_{i=1}^{l} : \times_{i=1}^{l} C^{\boldsymbol{n}_i}_{\boldsymbol{m}_i} \to C^{\widetilde{\boldsymbol{n}}}_{\widetilde{\boldsymbol{m}}}, \tag{2.1}$$

with $\widetilde{\boldsymbol{n}}(\phi) = \sum_{i=1}^{l} \boldsymbol{n}_i(\phi_i)$, $\widetilde{\boldsymbol{m}}(\phi) = \sum_{i=1}^{l} \boldsymbol{m}_i(\phi_i)$ where the summations are componentwise. We set that the result of a $\cdot_l$-product which is an element of a $\mathcal{C}$-space, is regular in the domains of definition of all parameters $\Theta^{\boldsymbol{n}}_{\boldsymbol{m}}$ defining the product. We assume that a $\cdot_l$-product is inseparable in pairs of two elements of $\mathcal{C}$ in general. Let us introduce the notations. In the arguments of a multiple product we denote by $(\phi)_i$ a $\mathcal{C}$-element placed at the i-th position. The notation $\widehat{s}$ means that the value s of the index is omitted. We assume the completeness of a complex $\mathcal{C}$-spaces with respect to the multiple product (2.1). That means that for ϕ, $\psi \in \mathcal{C}$, placed in the product at the i-th and j-th positions, from $0 = \cdot_l(\ldots, (\phi)_i, \ldots, (\psi)_j, \ldots)$ it follows that there exists a $\mathcal{C}$-element χ, such that $\psi = \cdot_l(\ldots, (\phi)_{i'}, \ldots, (\chi)_k, \ldots)$, with some i' and k. For a $\cdot_l$-product we set $\cdot_l(\ldots, 0, \ldots) = 0$. One can also set that $\cdot_l(\ldots, \mathrm{Id}_{\mathcal{C}}, \ldots) = \cdot_{l-1}(\ldots, 0, \ldots) = 0$, where $\mathrm{Id}_{\mathcal{C}}$ is the identity element in the corresponding $\mathcal{C}$-subspace. In certain cases, a product (2.1) may have coincidences of parameters of multiplied elements of a multiple complex $\mathcal{C}$ spaces. The result of a product may not allow such coincidences leading to possible overcounting of the number of parameters. In order avoid such a possibility, we take into account one coinciding elements/parameters only in the result of a product. For certain $\cdot_l$-products of $\mathcal{C}$-elements depending on parameters, (e.g., for a product of vertex operator algebra complexes [14]), it is needed to exclude a number of coinciding arguments and count them only once in the resulting product. Examples of such products can be found in [24, 13] and others. For an element $\phi_i \in C^{\boldsymbol{n}}_{\boldsymbol{m}}$ in a particular $\cdot_l$-product let $\boldsymbol{r}$ and $\boldsymbol{t}$ be the numbers of common parameters corresponding to upper and lower indices for ϕ with other elements in a product. In that case, the conditions on indices for a resulting element of a $\cdot_l$-product ϕ are $\boldsymbol{n}(\phi) = \sum_{i=1}^{l} \boldsymbol{n}_i(\phi_i) - \boldsymbol{r}_i(\phi_i)$, and $\boldsymbol{m}(\phi) = \sum_{i=1}^{l} \boldsymbol{n}_i(\phi_i) - \boldsymbol{t}_i(\phi_i)$. The associativity of a $\cdot_l$-product means $(a \cdot_l \ldots \cdot_{l'} b) \cdot_{l''} \ldots \cdot_{l'''} c = a \cdot_l \ldots \cdot_{l'} (b \cdot_{l''} \ldots \cdot_{l'''} c)$. for all elements a, b, $c \in \mathcal{C}$. Note that l can be taken infinite if we assure that a product $\times_{i \geqslant 1}$ is converging with infinite l. We call the product formal since elements of a complex $\mathcal{C}$ spaces can be formal (in particular, geometric) objects. Then the result of a $\cdot_l$-product is a superposition of formal objects (e.g., Riemann surfaces [24]). In that case, convergence of a product means that the corresponding superposition leads to a well-defined formal object.

In this paper, all the constructions are independent of actual commutation relations for elements for elements inside a $\cdot_l$-product. In addition to all that above we assume that in a multiple complex $\mathcal{C}$ spaces there exist a net of subspaces constituting vanishing products of exactly q its elements. That we call an order q ideal $\mathcal{I}(q) \subset \mathcal{C}$. We define a distributed ideal $\mathcal{I}(q)$ of order $q \leqslant l$ as a union of subsets of $\mathcal{C}$ such that for any set of q elements $\theta_1, \ldots, \theta_q \in \mathcal{I}(q)$, distributed in a product, $\cdot_l(\ldots, \theta_1, \ldots, \theta_2, \ldots, \theta_{q-1}, \ldots, \theta_q, \ldots) = 0$. A product (2.1) vanishes if at least one entry of its arguments belonging to $\mathcal{C}$ is zero. Now, let us explain how we understand powers of an element $\phi \in \mathcal{C}$. We denote $(\ldots, (\phi)^r, \ldots) = (\ldots, (\phi)_{j_1}, \ldots, (\phi)_{j_2}, \ldots, (\phi)_{j_r}, \ldots)$ for ϕ placed at the positions $(j_1, \ldots, j_r)$ with $r \leqslant q(\phi)$ where $q(\phi)$ is the maximal power of an element ϕ, i.e., for $r \geqslant q(\phi)$, $(\phi)^r = 0$.

It is taken that a product (2.1) is coherent in indices with respect to all differentials of a complex $\mathcal{C}$. That means that for every set of $\phi_i \in C\left(\Theta^{\boldsymbol{n_i}}_{\boldsymbol{m_i}}\right)$, $1 \leqslant i \leqslant l$, $\boldsymbol{n}(\phi) = \sum\limits_{i=1}^{l} \boldsymbol{n_i}(\phi_i)$, and $\boldsymbol{m}(\phi) = \sum\limits_{i=1}^{l} \boldsymbol{m_i}(\phi_i)$, and the resulting $\phi \in C\left(\Theta^{\boldsymbol{n}}_{\boldsymbol{m}}\right)$, $\phi = \cdot_l(\phi_1, \ldots, \phi_l)$, the action of corresponding differential d satisfies Leibniz formula

$$d_a \phi = d_a \cdot_l (\ldots, (\phi)_i, \ldots) = \sum_{i=1}^{l} \cdot_l \left(\ldots, (d_a \phi)_i, \ldots\right). \tag{2.2}$$

Note that in general, being applied to the result of a $\cdot_l$-product, Leibniz formula represents a map $\mathcal{C} \to \times\mathcal{C}$. Therefore a result of Leibniz formula application vanishes if in each element of the sum (2.2) an element is zero.

For a set of elements ϕ_k, $1 \leqslant k \leqslant l$, in a $\cdot_l$-product

$$\cdot_l \left(\Phi_1, \phi_1, \Phi_2, \ldots, \Phi_i, \phi_i, \Phi_{i+1}, \ldots, \Phi_k, \phi_k, \Phi_{k+1}\right), \tag{2.3}$$

we call the union of sets of $\mathcal{C}$-elements $\{\Phi_j\}$, $1 \leqslant j \leqslant k+1$, the completion of a set of elements ϕ_k in a $\cdot_l$-product. Due to the associativity of a $\cdot_l$-product we can view each set Φ_j as a single element. The completion (2.3) for a set of elements ϕ_i, $1 \leqslant i \leqslant k$, with respect to a differential d_a is called closed if

$$\sum_{j=1}^{p+1} d_{a,j}. \cdot_l \left(\Phi_1, \phi_1, \Phi_2, \ldots, \Phi_i, \phi_i, \Phi_{i+1}, \ldots, \Phi_k, \phi_k, \Phi_{k+1}\right) = 0, \tag{2.4}$$

where $d_{a,j}$, $j \leqslant k+1$, acts on Φ_j-elements of the completion only. In what follows we will always assume that all completions in $\cdot_l$-products are closed with respect to corresponding differentials. Thus, we will skip the completion elements $\Phi_i, \ldots, \Phi_{k+1}$ of $\mathcal{C}$-elements ϕ_i, $1 \leqslant i \leqslant k$, so that a $\cdot_l$-product will be denoted as $(\ldots, \phi_1, \ldots, \phi_k \ldots)$. Due to the associativity property of the $\cdot_l$-product mention above, we will skip the notation $\cdot_l$ and denote all completion closed products for various l as $(\ldots, \ldots, \ldots)$. Since all actions of $\mathcal{C}$ differentials are coherent with the indexing of $\mathcal{C}$-spaces, we will skip also upper and lower indices from the notations of $d^{n,\kappa}_m$.

In general, an arbitrary element $\phi \in \mathcal{C}$ is characterizes by its maximal power $q(\phi) \in \mathbb{Z}_+ \cup \{0\}$, i.e., $\left(\ldots, (\phi)_{i_1}, \ldots, (\phi)_{i_{q(\phi)}}, \ldots\right) = 0$ with q identical elements ϕ distributed in a product (2.1). Next, according to values of maximal orders and powers of a collection of differentials $d^{p_i}_{r_i}$, $1 \leqslant i \leqslant k$, a $\mathcal{C}$-element ϕ satisfies certain conditions with respect to subsequent actions of differentials $d^{p_i}_{r_i}$ on ϕ.

One can also introduce the rules (possible commutation relations, chain-cochain property, maximal orders and powes) for actions of $\mathcal{C}$-differentials d_b, with $b = a$ or $\overline{a}$, i.e.,

$$d_a d_b = A_{a,b}\, d_b d_a, \tag{2.5}$$

with some $A_{a,b}$. Note that $A_{a,b}$ can be either a complex number (in particular, zero), a map $\mathcal{C} \to \mathcal{C}$, or undefined at all. We assume that for an element $\phi \in \mathcal{C}$ might exists the maximal order $q(\phi) \in \mathbb{R}$ depending on ϕ, such that for $r \in \mathbb{R}$, $r \geqslant q(\phi)$, $(\phi)^r = 0$. Similarly, for both differentials d_b, $b = a$ or $\overline{a}$, and any element $\phi \in \mathcal{C}$, might exists $p(\phi) \in \mathbb{R}$, depending on ϕ, such that for any $s \geqslant p(\phi)$, $d^s \phi = 0$.

We assume always that when we apply a differential d_a to a product (γ), then there exists $p(\gamma) \in \mathbb{R}$ such that for all $s \geqslant p(\gamma)$ $d_a^s(\gamma) = 0$, even if we do not know exact value of $p(\gamma)$. In deriving identities for $\mathcal{C}$-elements, we should always keep in mind that the power of a collection of differentials applied to a $\mathcal{C}$-element can overcome corresponding maximal value and, therefore, a result of such applications vanish at some iteration. Note again that an element $d_a^{p_i}(\gamma)$, $0 \leqslant p_i < p(d_a(\gamma))$ is characterized by $q(d_a^{p_i}(\gamma))$ such that $(d_a^{p_i}(\gamma))^{q(d_a^{p_i}(\gamma))} = 0$. For $p_i = 0$, $q(d_a^{p_i}(\gamma)) = q(\gamma)$. The property of having maximal orders of differentials and powers of elements can be seen as a redefinition of an initial complex $\mathcal{C}$ to resulting complexes with differentials defined by all possible partitions of all $\mathcal{C}$ differentials into pairs of differentials d_σ and $\overline{d}_\sigma$ where σ is a particular element of a partition.

Let us introduce some further notations. For a set of double indices $\boldsymbol{J} = (J_1, \ldots, J_n)$, $J_i = \begin{pmatrix} r_i \\ a_i \end{pmatrix}$, define

$$\mathcal{D}_{\boldsymbol{J}} \phi = d_{a_1}^{r_1} \ldots d_{a_n}^{r_n} \phi. \tag{2.6}$$

The indices a_i denote the type of differentials. The indices r_j are orders of differentials satisfying the recurrence conditions

$$r_j < p\left(d_{a_{j-1}}^{r_{j-1}} \ldots d_{a_1}^{r_1} \phi \right), \tag{2.7}$$

with the corresponding maximal order for the differential $d_{a_j}^{r_j}$ acting on a $\mathcal{C}$-element $d_{a_{j-1}}^{r_{j-1}} \ldots d_{a_1}^{r_1} \phi$, $1 \leqslant j \leqslant n_i$. Note, that due to the property (2.2) and the definition of powers of an element distributed in a $\cdot_l$-product, we do not take into account conditions on $\phi \in \mathcal{C}$ of the form $(d_{a_n}^{r_n}(\ldots(d_{a_2}^{r_2}(d_{a_1}^{r_1}\phi)^{q_1}))^{q_2}\ldots)^{q_n}$ since, according to (2.2) it is equivalent to a finite sum of powers of orders of differentials. Note that the definitions of maximal orders and powers above characterize elements of a complex $\mathcal{C}$, and are given in the form of differential/orthogonality relations (6.1) of Section 6. In what follows, we deal with multiple products completion of a number of $\mathcal{C}$-elements closed with respect to corresponding differentials.

Let us now formulate the main result of this paper:

Theorem 1. *The conditions* (2.5) *together with a set of maximal orders and powers of differentials for a multiple complex $\mathcal{C}$ result in a hierarchy of closed products in terms of differential identities on elements of $\mathcal{C}$ given by the general formula*

$$0 = \sum_{\boldsymbol{J}_1, \ldots, \boldsymbol{J}_k} \left(\ldots, (\mathcal{D}_{\boldsymbol{J}_1} \phi_1)^{q_1}, \ldots, (\mathcal{D}_{\boldsymbol{J}_k} \phi_k)^{q_k}, \ldots \right), \tag{2.8}$$

with $1 \leqslant q_i < q(\mathcal{D}_{\boldsymbol{J}_i})$, $1 \leqslant i \leqslant k$, where $q(\mathcal{D}_{\boldsymbol{J}_i})$ are the maximal powers for the corresponding differentials. The differentials $\mathcal{D}_{\boldsymbol{J}_{s,i}}$ are determined by all vanishing products closed with respect to single differentials d_a.

Now let us give a proof of Theorem 4.3.

Proof. The idea of the proof is related to geometry. Namely, we show that the identities (2.8) are defined by vanishing products closed with respect a differential d_a. Geometrically, this corresponds to a separation of "codimension one" differential forms [15] expressed in terms of multiple products. The main idea to generate differential identities for multiple products of elements of $\mathcal{C}$ is to pick products vanishing due to maximal powers of differentials action. It is easy to see that in order to find a differential identity, one has to pick elements given by multiple products which would give "almost" the rule (2.7), and would vanish under actions of the differentials d_a or $d_{\overline{a}}$. Thus, the set of closed products consists of such "almost" rule (2.7) elements having "almost" maximal powers. There are two general ways how to obtain an identity starting from a vanishing product closed under a differential. Let us start with a product containing k entries of $q_i(\phi_i)$-powers of arbitrary $\mathcal{C}$-elements ϕ_i, i.e., $(\ldots,(\phi_i)^{q_i},\ldots,(\phi_j)^{q_j},\ldots)$, for $0 \leqslant q_i < q(\phi_i)$, with the maximal power $q(\phi_i)$ of elements ϕ_i for $1 \leqslant i \neq j \leqslant k$, and, $q_j = q(\phi_j)$. That product vanishes. Recall that for elements ϕ_i of $(\phi_i)^{q_i}$ may be distributed as fingle ϕ_i entries among arguments of the product. Acting by a differential d_a according to the formula (2.2) we obtain two types of summands. The action of a differential on the distributed power of $(\phi_j)^{q(\phi_j)}$ gives $(d_a\phi_j)$ as one entry, and remaining distributed entries $(\phi_j)^{q(\phi_j)-1}$. As we see, since the maximal power $q(\phi_j)-1$ is dropped now by one, that particular summand does not vanish. There are all together $q(\phi_j)$ non-vanishing summands of this type. The second type of summands with a differential acting on $(\phi_i)^{q_i}$, $i \neq j$, contains the maximal power of $(\phi_j)^{q(\phi_j)}$, and, therefore, vanishes. Another possibility for a vanishing product to be closed is to collect the maximal power of an element at some iteration of differentials application. We thus obtain the first identity of the hierarchy:

$$0 = \sum_{s=1}^{q(\phi_j)} \Big(\ldots,\phi_i^{q_i},\ldots,(d_a\phi_j)_s,\ldots,(\phi_j)_{\hat{s}}^{q(\phi_j)-1},\ldots\Big). \tag{2.9}$$

This identity relates powers of elements ϕ_i, $1 \leqslant i \leqslant k$ with the differential $d_a\phi_j$. To obtain further identities of that branch of the hierarchy, we apply differentials d_a or $d_{\overline{a}}$ to (2.9), and take into account (2.5) to derive the next identity in that branch of the hierarchy. Continuing the process, we decrease the powers of elements $(\phi_i)^{q_i}$, and increase powers of differentials $d_{a_1}^{r_1}\ldots d_{a_n}^{r_n}\phi_i^{q(\phi_i)}$. Note that by assumption that the maximal order of each differential (i.e., the power when it vanishes) applied depends on the element it acts on. Therefore, the final form of an identity in the hierarchy depends on a sequence of elements $d_{a_t}^{r_t}(d_{a_{t-1}}^{r_{t-1}}\ldots(d_{a_1}^{r_1}\phi_{i'})))$, $d_{a_{t-1}}^{r_{t-1}}(\ldots(d_{a_1}^{r_1}\phi_{i'}))$, $\ldots$, $d_{a_1}^{r_1}\phi_{i'}$, where $r_p(\phi'_p)$ are lower than the maximal orders for corresponding differentials. A sequence of identities stops when at least one order of differentials reaches its maximal value in each summand. The whole hierarchy depends on a set of initial elements ϕ_i. That elements may initially contain orders of differentials of some powers. □

To finish this Section we formulate

Corollary 1. *The hierarchies* (2.8) *result in the set of closed multiple products.*

Indeed, each a differential identity $(\chi) = 0$ from the system (2.8) can be "integrated" with respect the differentials d or $\overline{d}$ to find a multiple product (Γ) such that $d_a(\Gamma) = (\chi) = 0$. When we say integrated we mean that for an element $\gamma \in \mathcal{C}$ we find an element γ' such that $\gamma' = d_a\gamma$. In addition to that relations specified by the identities (2.8) can be used in order to derive closed products which do not follow explicitly from the integration of identities. Let us underline that we work with a multiple complex spaces as grading subspaces of an associative algebra. In particular, the multiple product $\cdot_l$ can be introduces for geometric objects of general kinds. Thus, we keep the constructions of this paper independent of commutation relations. In the next three Sections we will provide instructive examples of identities and closed products of the first order of differentials actions. The full description and classification of differential identities and closed products at higher orders and powers of differentials action will be given in a forthcoming paper. The identities and closed products given in Sections (3)–5 generalize to symmetrized versions with respect to permutations of indices, positions of $\mathcal{C}$-elements, and orders and powers of differentials.

3. Examples of identities for multiple products

In this Section we illustrate the proof of Theorem 4.3 by complexity-growing examples. In this and next two Sections we always assume that all completions are closed with respect to corresponding differentials.

3.1. Two differentials, and the total maximal power two for two kinds of elements. Consider two elements $(\phi, \psi) \in \mathcal{I}(2)$. Recall that the notation $(\phi, \psi) \in \mathcal{I}(2)$ means that a pair of ϕ and ψ (but not their differentials) placed in a product makes it vanishing. For any two positions $1 \leqslant i \neq j \leqslant l$ in a multiple product

$$\begin{aligned} 0 &= d_a(\ldots, (\phi)_i, \ldots, (\psi)_j, \ldots) \\ &= (\ldots, (d_a\phi)_i, \ldots, (\psi)_j, \ldots) + (\ldots, (\phi)_i, \ldots, (d_a\psi)_j, \ldots), \end{aligned}$$

which is not a trivial identity for $p(\phi)$, $p(\psi) > 1$. Recall that the notation $(\phi)_i$ means that an element ϕ is placed at i-th position in a product. Therefore,

$$(\ldots, (d_a\phi)_i, \ldots, (\psi)_j, \ldots) = -(\ldots, (\phi)_i, \ldots, (d_a\psi)_j, \ldots), \tag{3.1}$$

i.e., one can transfer a differential between $\mathcal{I}(2)$-elements while changing the sign. In particular, with $d_b d_a = 0$, $b = a$ or $b = \overline{a}$, and $\overline{p}(\phi) > 1$, $\overline{p}(\phi) > 1$, we get

$$\Big(\ldots, (d_a\phi)_i, \ldots, (d_b\psi)_j, \ldots\Big) = -\Big(\ldots, (d_b\phi)_i, \ldots, (d_a\psi)_j, \ldots\Big). \tag{3.2}$$

From ϕ, $\psi \in \mathcal{I}(2)$ with $a = b$ it follows $d_a\phi$, $d_a\psi \in \mathcal{I}(2)$. Both with $d_b d_a = 0$ or $d_b d_a \neq 0$, by applying further choices d_c, $c = a$ or $\overline{a}$ of differentials, we receive from (3.1) further relations for higher differentials if the maximal powers $p\,(d_{a_n} \ldots d_{a_1})\phi)$ of sequences $d_{a_n} \ldots d_{a_1}$, (where a_i, $1 \leqslant i \leqslant n$ is a choice of a and $\overline{a}$) of the differentials d_a and $d_{\overline{a}}$ permit.

3.2. Single differential, two kinds of elements. Consider $d_a(\ldots, \phi^r, \ldots, \psi^s, \ldots)$, with separate maximal powers for ϕ and for ψ, or with total maximal power $r + s = k$ for ϕ and ψ together. It is assumed that ϕ- and ψ-entries are mixed and commutation relations are not known. Let $d_a^{\alpha(\phi)}\phi = 0$, $d_a^{\beta(\psi)}\psi = 0$, where $\alpha(\phi) > 1$ and $\beta(\psi) > 1$

are maximal orders of the differential d_a. Then, one has

$$\begin{aligned} 0 = d_a\left(\dots,\phi^r,\dots,\psi^s,\dots\right) &= \sum_{i_1=1}^{r}\Big(\dots,(d_a\phi)_{i_1},\dots,(\phi)^{r-1}_{\widehat{i_1}},\dots,(\psi)^s\dots\Big) \\ &+ \sum_{i_2=1}^{s}\Big(\dots,(\phi)^r\dots,(d_a\psi)_{i_2},\dots,(\psi)^{s-1}_{\widehat{i_2}},\dots\Big). \end{aligned}$$

By continuing the process of n-times application of the differential d_a until the powers of ϕ and ψ become zero, the sequence of identities stops.

3.3. **Two differentials, identical elements of the total maximal power.** For $q(\phi)$ identical elements ϕ, placed at all i-th positions, $1 \leqslant i \leqslant q(\phi) \in \mathbb{N}$, such that $\phi \in \mathcal{I}(q(\phi))$, i.e., $\phi^{q(\phi)} = 0$, equivalently, $(\dots,\ (\phi)_1,\ \dots,\ (\phi)_i,\ \dots,\ (\phi)_{q(\phi)},\dots) = 0$, (i.e., ϕ is placed $q(\phi)$ times in various places in the product). Then,

$$0 = d_a\left(\dots,(\phi)_i,\dots\right) = \sum_{i=1}^{q(\phi)}\Big(\dots,(d_a\phi)_i,\dots,(\phi)^{q(\phi)-1}_{\hat{i}},\dots\Big), \tag{3.3}$$

which is not trivial for $p(\phi) > 1$. In the case when $d_b d_a = 0$, $b = a$ or $\overline{a}$, by applying d_b to (3.3), one finds

$$0 = \sum_{i=1}^{q(\phi)}\sum_{1\leqslant j\neq i}^{q(\phi)-1}\Big(\dots,(d_a\phi)_i,\dots,(d_b\phi)_j,\dots,(\phi)^{q(\phi)-2}_{\hat{i},\hat{j}},\dots\Big). \tag{3.4}$$

With $(d_a\phi, d_b\phi) \in \mathcal{I}(2)$, this identity trivializes. For $b = a$ and $d_a\phi \notin \mathcal{I}(2)$, one gets relations on differentials of ϕ. Both for $d_b d_a = 0$ or $d_b d_a \neq 0$, by applying a sequence $d_{a_n}\dots d_{a_1}$ of the differentials d_a and $d_{\overline{a}}$ to (3.3), the hierarchy of identities arises,

$$0 = \sum_{\substack{i_n,i_{n-1},\dots,i_1=1 \\ i_s\neq j_r,\ 1\leqslant s\neq r\leqslant n}}^{q(\phi)-n,\dots,q(\phi)-1,q(\phi)}\Big(\dots,(d_{a_n}\dots d_{a_1}\phi)_{i_1},\dots,\left(d_{a_{n-1}}\dots d_{a_1}\phi\right)_{i_n},\dots,(\phi)^{q(\phi)-n}_{\widehat{i_1},\dots,\widehat{i_n}},\dots\Big),$$

when maximal orders and powers of corresponding combinations of differentials permit.

3.4. **Orders of two differentials of several identical elements.** In this subsection we show how to use transfer of differentials for higher order differentials. Let $d_a^{p(\phi)}\phi = 0$, and $\left(d_a^{p'}\phi\right)^{q(p',\phi)} = 0$ for $p' < p(\phi)$ (we do not allow the ambiguity 0^0). Then, for $0 < p_i < p(\phi)$, $1 \leqslant q_i \leqslant q(p_i,\phi)$, $1 \leqslant i \leqslant k$, the identity is

$$\begin{aligned} 0 &= d_a\sum_{i=1}^{k}\Big(\dots,(d_a^{p_1}\phi)^{q_1},\dots,(d_a^{p_i}\phi)^{q(p_i,\phi)},\dots,(d_a^{p_k}\phi)^{q_k},\dots\Big) \\ &= \sum_{i=1}^{k}\sum_{s_1=1}^{q(p_i,\phi)}\Big(\dots,\left(d_a^{p_i+1}\phi\right)_{s_1},\dots,(d_a^{p_i}\phi)^{q(p_i)-1}_{\widehat{s_1}},\dots,\left(d_a^{p_{\hat{i}}}\phi\right)^{q_{\hat{i}}},\dots\Big), \end{aligned} \tag{3.5}$$

which is non-trivial if $p_i + 1 < p(\phi)$, and $q(p_i+1,\phi) > 1$. Applying the differential d_a further times one obtains higher identities. Then we can have a finite number

n applications of the differential and corresponding sequence of identities. Similar products including also powers of the differential $d_{\overline{a}}$ lead to similar identities.

3.5. **Two differentials, multiple elements.** Recall again that the main idea of differential identity generation is to be able to use them to transfer powers of $\mathcal{C}$-elements into orders of corresponding differentials. Let us look at $(\ldots,(\phi_i)^{r_i},\ldots)$, with distributed power of single elements ϕ_i-entries for all i, $1\leqslant i\leqslant k\leqslant l$, $1<r_i\leqslant q(\phi_i)$, $(\phi_i)^{q(\phi_i)}=0$, i.e., where k is the total number of various ϕ_i with individual maximal powers depending on an element $\phi_i\in\mathcal{C}$. Let $d_a^{p(\phi_i)}\phi_i=0$, $p(\phi_i)>1$, $d_{\overline{a}}^{\overline{p}(\phi_i)}\phi_i=0$, $\overline{p}(\phi_i)>1$, and $\left(d_a^{p(\phi_i)}\phi_i\right)^{q(p_i,\phi_i)}=0$, $q(p_i,\phi_i)>1$, $\left(d_{\overline{a}}^{\overline{p}(\phi_i)}\phi_i\right)^{q(\overline{p}_i,\phi_i)}=0$, $q(\overline{p}_i,\phi_i)>1$, be the maximal orders and powers of the differentials d_a, $d_{\overline{a}}$ depending on ϕ_i.

Consider the case of a single differential and multiple elements. In order to form the vanishing product, we include exactly one maximal power $q(\phi_i)$ of an element ϕ_i, and powers $1\leqslant r_{\widehat{i}}<q(\phi_{\widehat{i}})$ of other types of elements $\phi_{\widehat{i}}$, $1\leqslant\widehat{i}\neq i\leqslant k-1$. For d_a as before

$$\begin{aligned} 0 &= d_a\sum_{i=1}^{k}\Big(\ldots,\phi_1^{r_1},\ldots,\phi_i^{q(\phi_i)},\ldots,(\phi_k)^{r_k},\ldots\Big)\\ &= \sum_{i=1}^{k}\sum_{s=1}^{q(\phi_i)}\Big(\ldots,(d_a\phi_i)_s,\ldots,(\phi_i)_{\widehat{s}}^{q(\phi_i)-1},\ldots,\phi_{\widehat{i}}^{q(\phi_{\widehat{i}})},\ldots\Big). \qquad (3.6)\end{aligned}$$

Further several applications of the differential d_a to the last identity results in further identities. It is clear that the orders of differentials $d_a(\phi_i)$ and $d_a(\phi_j)$ are growing until they reach $p(\phi_i)$ and $p(\phi_j)$, and thus corresponding summands vanish. Similarly, the powers of the differentials $d_a^{p_i}\phi_i$ and $d_a^{p_j}\phi_j$ are growing until they reach $q(p_i,\phi_i)$ and $q(p_j,\phi_j)$, and corresponding summands vanish.

3.6. **The most general polynomial case: the identities for orders of two differentials and powers of multiple elements.** In this subsection the identities for the most general polynomical differential-order and power products are computed. Let us first assume that the maximal orders $p(\phi_i)>1$ and $\overline{p}(\phi_i)>1$ of the differentials d_a and $d_{\overline{a}}$ correspondingly, i.e., such that $d_a^{p(\phi_i)}=0$, $d_{\overline{a}}^{\overline{p}(\phi_i)}=0$ and the maximal powers $q(p_i,\phi_i)>1$ and $\overline{q}(\overline{p}_i,\phi_i)>1$ of $d_a^{p_i(\phi_i)}$ and $d_{\overline{a}}^{\overline{p}_i(\phi_i)}$ i.e., such that $\left(d_a^{p_i(\phi_i)}\right)^{q(p_i,\phi_i)}=0$ and $\left(d_{\overline{a}}^{\overline{p}_i(\phi_i)}\right)^{\overline{q}(\overline{p}_i,\phi_i)}=0$, do depend on elements $\phi_i\in\mathcal{C}$, for $1\leqslant i\leqslant k$ types of elements ϕ_i. Take a product with exactly one differential in the maximal power $q(p_i,\phi_i)$, while all orders $p_i(\phi_i)$ of the differentials $d_a^{p_i(\phi_i)}$ are lower than the maximal order $p(\phi_i)$, i.e., $0\leqslant p_i(\phi_i)<p(\phi_i)$,

$$\begin{aligned} 0 &= d_a\sum_{i=1}^{k}\left(\ldots,\left(d_a^{p_1(\phi_1)}\phi_1\right)^{r_1},\ldots,\left(d_a^{p_i(\phi_i)}\phi_i\right)^{q_i(p_i,\phi_i)},\ldots,\left(d_a^{p_k(\phi_k)}\phi_k\right)^{r_k},\ldots\right)\\ &= \sum_{i=1}^{k}\sum_{s_1=1}^{q(p_i,\phi_i)}\left(\ldots,(d_a^{p_i(\phi_i)+1}\phi_i)_{s_1},\ldots,(d_a^{p_i}\phi_i)_{\widehat{s_1}}^{q_i(p_i,\phi_i)-1},\ldots,(d_a^{p_{\widehat{i}}}\phi_{\widehat{i}})^{r_{\widehat{i}}},\ldots\right). \quad (3.7)\end{aligned}$$

It is clear that if we put at least one maximal order $p_i(\phi_i) = p(\phi_i)$ or $p_j(\phi_j) = p(\phi_j)$ of the differential d_a into the first line of (3.7) then all further identities trivialize.

One can also take the case of the total maximal power elements. I.e., $\sum_{i=1}^{t} q_i (p_i, \phi_i) = q$, and $q_i(p_i, \phi_i) < q(p_i, \phi_i)$ i.e., when the sum of powers reaches the critical value, and $\times \left(d_a^{p_i(\phi_i)}\phi\right)^{q_i(\phi_i)} \in \mathcal{I}(q)$. Similarly, $\sum_{i=1}^{t} p_i(\phi_i) = p$, and $p_i(\phi_i) < p(\phi_i)$, i.e., the sum of orders of the differentials is at its maximum, and $\times d_a^{p_i(\phi_i)}\phi_i \in \mathcal{I}(p)$ Then in both cases $\left(\ldots, \left(d_a^{p_i(\phi_i)}\phi_i\right)^{q_i(p_i,\phi_i)}, \ldots\right) = 0$. When we apply the second differential $d_{\overline{a}}$ to the identity (3.7), all branches of the hierarchy of identities arise with appropriate conditions on orders and powers of differentials.

4. Examples of a single element closed products

In this Section we provide examples of closed products arising, in particular, from the identities of Section 3. Closed products and their identities are very useful in the theory of completely integrable [2] and exactly solvable [16, 18] systems. Note also possible applications for operads. Note that in the search of products annihilated by a differential, we take only non-vanishing products. To use the annihilation given by maximal powers of $d_a\phi_i$ in multi-element products after application of a differential d_a once, we have to include not just one power of each element but lower powers also. The main idea of a closed product construction is in taking into account changes both in orders of differentials themselves and powers of their action on $\mathcal{C}$-elements. For that reason we have to include not only powers of elements acted by the differentials, but also powers of elements not acted by them. The balance of raising and lowering powers of elements results in extra conditions producing closed products. In some of the examples below we use the transfer of differentials using certain identities from Section 3. By integrating the identities of the form (3.5) one finds sets of advanced closed products.

Note again that when we write a product in the form $(\ldots, (\phi)_i, \ldots, (\psi)_j, \ldots)$, $1 \leqslant i \neq j \leqslant l$, it is not assumed that all elements ψ are positioned on the right to all ϕ elements, i.e., ϕ and ψ elements can be mixed in the product.

4.1. **A single differential, a single element.**

First, let us look at expressions without differentials, $k < q(\phi)$ distributed position entries in a product, where $q(\phi) > 1$ is the maximal order of ϕ, $\phi^{q(\phi)} = 0$, i.e., $\phi \in \mathcal{I}(q(\phi))$. Let the maximal order of the differential d_a is $p(\phi)$, i.e., $d_a^{p(\phi)}\phi = 0$, and $q(p', \phi) \geqslant 1$ is the maximal power of $d_a^{p'}\phi$.

1.) With $p(\phi) = 1$,

$$d_a(\ldots, (\phi)_1, \ldots, (\phi)_i, \ldots, (\phi)_k, \ldots) = \sum_{i=1}^{k} \left(\ldots, (d_a\phi)_i, \ldots, (\phi)^{k-1}, \ldots\right) = 0, \quad (4.1)$$

due to each summand is zero. Since there is only one differential in each summand the use of the transfer formula (3.1) would only changes the sign.

2.) Suppose $d_a d_a \phi = 0$, i.e., $p(\phi) = 2$, and a distributed product $(\ldots, (d_a\phi)_{q\ times, \ldots}) = 0$. We denote this relation as $(d_a\phi)^{q(1,\phi)} = 0$, i.e., $d_a\phi \in \mathcal{I}(q(1,\phi))$. We continue with a distributed product containing $(d_a\phi)^k$ with $k \leqslant q(1,\phi)$ entries of $d_a\phi$. Then

$$d_a\left(\ldots,(d_a\phi)^k,\ldots\right) = \sum_{i=1}^{k}\Big(\ldots,(d_ad_a\phi)_i,\ldots,(d_a\phi)^{k-1}_{\hat{i}},\ldots\Big) = 0. \tag{4.2}$$

3.) Next, with $d_ad_a\phi = 0$, include $(\phi)^r$, i.e., take $(\ldots,(d_a\phi)^k,\ldots,(\phi)^r,\ldots)$, for $1 \leqslant k+r \leqslant l$, $k \leqslant q(1,\phi)$ entries of $d_a\phi$, $(d_a\phi)^{q(1,\phi)} = 0$, for $d_a\phi$, and $r \leqslant q(\phi)$ entries of ϕ, $\phi^{q(\phi)} = 0$. Then, for $k+1 = q(1,\phi)$,

$$d_a\left(\ldots,(d_a\phi)^k,\ldots,(\phi)^r,\ldots\right) = \sum_{s=1}^{r}\left(\ldots,(d_a\phi)^k,\ldots,(d_a\phi)_s,\ldots,(\phi)^{r-1}_{\hat{s}},\ldots\right) = 0, \tag{4.3}$$

since the total power of $d_a\phi$ in each of summand becomes $q(1,\phi)$ and it, therefore, vanishes.

4.2. **Higher orders of a single differential, single element.** Now let us proceed with higher order closed products. Let $p(\phi)$ be the maximal order of the differential $d_a\phi$, i.e., $d_a^{p(\phi)}\phi = 0$.

1.) With k entries of $d_a^{p_i}\phi$ differentials, $k < q(p_i,\phi)$, $(d_a^{p_i}\phi)^{q(p_i,\phi)} = 0$, $1 \leqslant p_i < p(\phi)$, i.e., one gets the product $(\ldots,(d^{p_i}\phi)_i,\ldots)$. Then, for $p_i = p(\phi)-1$ for all i,

$$d_a\left(\ldots,(d_a^{p_i}\phi)_i,\ldots\right) = \sum_{s=1}^{k}\Big(\ldots,\Big(d_a^{p(\phi)}\phi\Big)_s,\ldots,(d_a^{p_i}\phi)_{\hat{s}},\ldots\Big) = 0. \tag{4.4}$$

2.) Now, let $\phi^{q(\phi)} = 0$. Include $k < q(p_i,\phi)$, entries of $d_a^{p_i}\phi$, $1 \leqslant i \leqslant k$, and $1 \leqslant r < q(\phi)$ entries of ϕ, i.e., $(\ldots,(d_a^{p_i}\phi)_i,\ldots,(\phi)^r,\ldots)$. One has

$$\begin{aligned} d_a\left(\ldots,(d_a^{p_i}\phi)_i,\ldots,(\phi)^r,\ldots\right) &= \sum_{s=1}^{k}\left(\ldots,(d_a^{p_s+1}\phi)_s,\ldots,(d_a^{p_{\hat{s}}}\phi)_{\hat{s}},\ldots,(\phi)^r,\ldots\right) \\ &+ \sum_{t=1}^{r}\left(\ldots,(d_a^{p_i}\phi)_i,\ldots,(d_a\phi)_t,\ldots,(\phi)^{r-1},\ldots\right). \end{aligned} \tag{4.5}$$

For $p_i = 1$, $k+1 = q(1,\phi)$, and $p_i + 1 = 2 = p(\phi)$, $1 \leqslant s \leqslant k$, both groups of summands vanish. Note that that does not depend on r. By inclusion of an extra summation into (4.5) one can use (3.4) to move differentials among $d_a\phi$ and ϕ with changing the sign. The symmetry of allows to distribute the orders of differentials among entries of ϕ.

4.3. **Orders of a single differential, single element, powers of differentials.** 1.) Let $(\ldots,(d_a^{p_i}\phi)_i^{q_i},\ldots)$ be a product containing k various powers of the differential d_a, $1 \leqslant i \leqslant k \leqslant l$, for $0 \leqslant p_i \leqslant p(\phi)$, $1 \leqslant q_i \leqslant q(p_i,\phi)$, where $d_a^{p(\phi)}\phi = 0$, $(d_a^{p_i}\phi)^{q(p_i,\phi)} = 0$. We then obtain

$$d_a\left(\ldots,(d_a^{p_i}\phi)_i^{q_i},\ldots\right) \quad = \quad \sum_{s=1}^{k}\sum_{t=1}^{q_s}\Big(\ldots,\left(d_a^{p_s+1}\phi\right)_t,\ldots,(d_a^{p_s}\phi)^{q_s-1}_{\hat{t}},\ldots,(d_a^{p_{\hat{s}}}\phi)^{q_{\hat{s}}}_{\hat{s}},\ldots\Big),$$

where $\widehat{t}$ denote the omission of t. That product is closed when 1.) $p_i + 1 = p(\phi)$ for all $1 \leqslant i \leqslant k$; 2.) $d_a^{p_s+1}\phi$, $d_a^{p_s}\phi \in \mathcal{I}(2)$; 3.) $d_a^{p_s+1}\phi$, $d_a^{p_s}\phi \in \mathcal{I}(q_s)$; 4.) $d_a^{p_s+1}\phi$, $(d_a^{p_{\widehat{s}}}\phi) \in \mathcal{I}(2)$; 5.) $d_a^{p_s+1}\phi$, $(d_a^{p_{\widehat{s}}}\phi)^{q_{\widehat{s}}} \in \mathcal{I}(q_{\widehat{s}} + 1)$.

2.) The general expression for a closed product in the case of a single element with k types of orders and powers of a single differential is given by $\left(\ldots, (d_a^{p_i}\phi)_i^{q_i}, \ldots, \left(d_a^{p_i-1}\phi\right)_j^{r_i}, \ldots\right)$, where for any pair $p_i = p_j - 1$, $2p_i < p_i(\phi)$, and $q_i + r_j < q(p_i, \phi)$, $1 \leqslant i, j \leqslant k$ (such that the initial product does not vanish).

$$
\begin{aligned}
& d_a\left(\ldots, (d_a^{p_i}\phi)_i^{q_i}, \ldots, \left(d_a^{p_i-1}\phi\right)_j^{r_i}, \ldots\right) \\
&= \sum_{s=1}^{k}\sum_{t=1}^{q_s}(\ldots, (d_a^{p_s+1}\phi)_t, \ldots, (d_a^{p_s}\phi)_{\widehat{t}}^{q_s-1}, \ldots, (d_a^{p_i-1}\phi)_j^{r_j}, \ldots) \\
&+ \sum_{s=1}^{k}\sum_{t=1}^{r_s}(\ldots, (d_a^{p_i}\phi)_i^{q_i}, \ldots, (d_a^{p_s}\phi)_t, \ldots, (d_a^{p_s-1}\phi)_{\widehat{t}}^{r_s-1}, \ldots) = 0,
\end{aligned}
$$

when 1.) $p_i + 1 = p(\phi)$ and $q_i + 1 = q(p_i, \phi)$, for all $1 \leqslant i \leqslant k$; 2.) $p_i + 1 = p(\phi)$ and $p_i = p_s$, $2p_i \geqslant p_i(\phi)$.

4.4. Two differentials, single element. For the product $(\ldots, d_{\overline{a}}\phi, \ldots, d_a\phi, \ldots, (\phi)^r, \ldots)$, with $1 < r < q(\phi)$ one has:

1.) For $d_a\phi \in \mathcal{I}(2)$, i.e., $(d_a\phi)^2 = 0$, and $d_a d_{\overline{a}}\phi = 0$, $d_a^2\phi = 0$

$$
\begin{aligned}
& d_a(\ldots, d_{\overline{a}}\phi, \ldots, d_a\phi, \ldots, (\phi)^r, \ldots) = (\ldots, d_a(d_{\overline{a}}\phi), \ldots, d_a\phi, \ldots, (\phi)^r, \ldots) \\
& +(\ldots, d_{\overline{a}}\phi, \ldots, d_a d_a\phi, \ldots, (\phi)^r, \ldots) + (\ldots, d_{\overline{a}}\phi, \ldots, d_a\phi, \ldots, d_a\phi, \ldots, (\phi)^{r-1}, \ldots) = 0.
\end{aligned}
$$

2.) For $d_{\overline{a}}d_a\phi \neq 0$, $d_a d_{\overline{a}} d_a\phi = 0$, $d_a^2\phi = 0$, and $d_a\phi \in \mathcal{I}(2)$, the product vanishes:

$$
\begin{aligned}
& d_a(\ldots, d_{\overline{a}}d_a\phi, \ldots, d_a\phi, \ldots, (\phi)^r, \ldots) = (\ldots, d_a d_{\overline{a}} d_a\phi, \ldots, d_a\phi, \ldots, (\phi)^r, \ldots) \\
& + (\ldots, d_{\overline{a}}d_a\phi, \ldots, d_a d_a\phi, \ldots, (\phi)^r, \ldots) + \left(\ldots, d_{\overline{a}}d_a\phi, \ldots, d_a\phi, \ldots, d_a\phi, \ldots, (\phi)^{r-1}, \ldots\right).
\end{aligned}
$$

Note that both closed products do not depend on r.

When commutation rules for elements in a product are known, we are able to move them around. In a distributed case, we first gather the powers of the same element together. Then the classical formula $d_b\left(d_a^p\phi\right)^q = q d_b\left(d_a^p\phi\right)\left(d_a^p\phi\right)^{q-1}$ (we agree to put the extra derivative in front of the power of the derivative) for a differential applied to a power of a $\mathcal{C}$-element is true, and we can write invariant elements in the explicit form.

5. Examples of multi-element closed products

In this Section we give examples of multi-element closed product.

5.1. Two-elements closed product. Here we have the case of dependence on two elements ϕ and ψ. That case does not fall in the general idea [12, 15] of one-parameter-element invariants. Let us see if it is possible to transpose d_a-differential.

1.) For $(d_a\phi, d_a\psi) \in \mathcal{I}(2)$, and $d_a d_a\phi = 0$,

$$
d_a(\ldots, d_a\phi, \ldots, \psi, \ldots) = (\ldots, d_a d_a\phi, \ldots, \psi, \ldots) + (\ldots, d_a\phi, \ldots, d_a\psi, \ldots) = 0.
$$

2.) With $(d_a\phi,\, d_{\overline{a}}\psi) \in \mathcal{I}(2)$, $d_a d_{\overline{a}} = 0$, then

$$d_a(\ldots, d_{\overline{a}}\phi, \ldots, \psi, \ldots) = (\ldots, d_a d_{\overline{a}}\phi, \ldots, \psi, \ldots) + (\ldots, d_a\phi, \ldots, d_{\overline{a}}\psi, \ldots) = 0;$$

3.) $d_a\phi,\, \psi \in \mathcal{I}(2)$, $d_a d_{\overline{a}}\phi = d_{\overline{a}} d_a\phi = 0$,

$$\begin{aligned} d_a(\ldots, d_{\overline{a}}\phi, \ldots, \psi, \ldots) &= (\ldots, d_{\overline{a}} d_a\phi, \ldots, \psi, \ldots) + (\ldots, d_a\phi, \ldots, d_{\overline{a}}\psi, \ldots) \\ &= -(\ldots, d_a\phi, \ldots, d_{\overline{a}}\psi, \ldots) + (\ldots, d_a\phi, \ldots, d_{\overline{a}}\psi, \ldots) = 0. \end{aligned}$$

due to property (3.1).

5.2. **Two differentials, three elements.** Take the product $(\ldots, d_{\overline{a}}\phi, \ldots, d_a\psi, \ldots, (\chi)^r, \ldots)$ with $d_a d_a\psi = 0$, $d_a d_{\overline{a}}\phi = 0$, and 1.) $d_{\overline{a}}\phi$, $d_a\psi \in \mathcal{I}(2)$; 2.) $d_{\overline{a}}\phi$, $d_a\chi \in \mathcal{I}(2)$; 3.) $d_a\psi$, $d_a\chi \in \mathcal{I}(2)$; 4.) ψ, $d_a\chi \in \mathcal{I}(2)$, $d_a d_a\chi = 0$; 5.) ϕ, $d_a\chi \in \mathcal{I}(2)$, and $d_{\overline{a}} d_a\chi = 0$, 6.) ϕ, $d_a\psi \in \mathcal{I}(2)$, and $d_{\overline{a}} d_a\psi = 0$. Then the product vanishes

$$d_a(\ldots, d_{\overline{a}}\phi, \ldots, d_a\psi, \ldots, (\chi)^r, \ldots) = (\ldots, d_a(d_{\overline{a}}\phi), \ldots, d_a\psi, \ldots, (\chi)^r, \ldots)$$
$$+(\ldots, d_{\overline{a}}\phi, \ldots, d_a d_a\psi, \ldots, (\chi)^r, \ldots) + (\ldots, d_{\overline{a}}\phi, \ldots, d_a\psi, \ldots, d_a\chi, \ldots, (\chi^{r-1}), \ldots),$$

according to property (3.1).

5.3. **Orders of a single differential, powers of several elements.** Consider the following case: $\big(\ldots, (d_a^{p_i}\phi_i)^{q_i(p_i)}, \ldots, (\phi_j)^{q_j}, \ldots\big)$, with $0 \leqslant p_i(\phi_i) < p(\phi_i)$, and $1 \leqslant q_i(p_i) < q(p_i, \phi_i)$, $1 \leqslant q_j(p_j) < q(p_j, \phi_j)$, $1 \leqslant q_i < q(\phi_i)$, $1 \leqslant q_j < q(\phi_j)$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k'$, the maximal order and power (here it starts from 1 to preserve l, or keep the identical element in the product). Note that the powers are lower than $p(\phi_i) - 1$ and $q(\phi_j) - 1$ correspondingly to insure that the corresponding product does not vanish. Next, by applying conditions on $p_i(\phi_i)$ and $q(\phi_i)$, we find their combinations such that the product is closed.

1.) It is clear that for all $p_i + 1 = p(\phi_i)$, $1 \leqslant i \leqslant k$, $p_i = 1$, and $q_i(1) + 1 = q(1)$; 2.) for all $p_i + 1 = p(\phi_i)$, $1 \leqslant i \leqslant k$, and with $p_i \neq 1$, $d_a\phi_i$, $\phi_j \in \mathcal{I}(2)$; 3.) for all $p_i + 1 = p(\phi_i)$, $1 \leqslant i \leqslant k$, and with $p_i \neq 1$, $d_a\phi_i$, $(\phi_j)^{q_j - 1} \in \mathcal{I}(q_j)$ the product is closed.

$$\begin{aligned} & d_a\left(\ldots, (d_a^{p_i}\phi_i)^{q_i(p_i)}, \ldots, (\phi_i)^{q_i}, \ldots\right) \\ &= \sum_{s=1}^{k} \sum_{t=1}^{q_s(p_s)} \left(\ldots, (d_a^{p_s+1}\phi_s)_t, \ldots, (d_a^{p_s}\phi_s)_{\widehat{t}}^{q(p_s)-1}, \ldots, (\phi_i)^{q_i}, \ldots\right) \\ &+ \sum_{s'=1}^{k'} \sum_{t=1}^{q_{s'}} \left(\ldots, (d_a^{p_i}\phi_i)^{q_i(p_i)}, \ldots, (d_a\phi_{s'})_t, \ldots, (\phi_{s'})_{\widehat{t}}^{q_{s'}-1}, \ldots\right), \end{aligned}$$

where $\widehat{t}$ denotes the omission of the index t.

5.4. **Powers of orders of multiple differentials, multiple elements.** Consider the product $\left(\ldots, (d_{a_{n,i}}^{p_{n,i}} \ldots d_{a_{1,i}}^{p_{1,i}}\phi_i)_i^{q(\mathbf{p}_{n,i}, \mathbf{a}_{n,i})}, \ldots\right)$, for some $1 \leqslant i \leqslant k \leqslant l$, and where we use the notation $\mathbf{x}_{n,i} = (x_{n,i}, x_{n-1,i}, \ldots, x_{1,i})$. For the product not to vanish, all orders $p_{s,i}$, $1 \leqslant s \leqslant n$ satisfy $p_{s,i}\left(d_{a_{s-1,i}}^{p_{s-1,i}} \ldots d_{a_{1,i}}^{s,i}\phi\right) < p\left(d_{a_{s-1,i}}^{p_{s-1,i}} \ldots d_{a_{1,i}}^{p_{1,i}}\phi\right)$, where $p\left(d_{a_{s-1,i}}^{p_{s-1,i}} \ldots d_{a_{1,i}}^{p_{1,i}}\phi\right)$ is the maximal order of the corresponding multiple differentials, and the power of the corresponding multiple differentials $d^{p_{n,i}}_{a_{n,i}} \ldots d^{p_{1,i}}_{a_{1,i}} \phi_i$ is lower than $q(\mathbf{p}_{n,i}, \mathbf{a}_{n,i}) < q(\mathbf{p}_{n,i}, \mathbf{a}_{n,i}, \phi)$ is lower than the maximal ones for all i. Then

$$d^{p_{n+1}}_{a_{n+1}} \left(\ldots, \left(d^{p_{n,i}}_{a_{n,i}} \ldots d^{p_{1,i}}_{a_{1,i}} \phi_i \right)_i^{q(\mathbf{p}_{n,i}, \mathbf{a}_{n,i})}, \ldots \right) = 0, \tag{5.1}$$

when 1.) there exists i, $1 \leqslant i \leqslant k$, such that $p_{n+1,i} \left(d^{p_{n,i}}_{a_{n,i}} \ldots d^{p_{1,i}}_{a_{1,i}} \phi \right) \geqslant p \left(d^{p_{n+1,i}}_{a_{n,i}} \ldots d^{p_{1,i}}_{a_{1,i}} \phi \right)$; 2.) if $k \geqslant 1$, and $(\mathbf{p}_{n+1,i}, \mathbf{a}_{n+1,i}) = (\mathbf{p}_{n,j}, \mathbf{a}_{n,j})$, for some $1 \leqslant i \neq j \leqslant k$.

6. Multiple-graded differential algebras

Let $\gamma_s \in \mathcal{C}$, $s \in \mathbb{Z}$, and $\boldsymbol{J}_s = (J_{s,1}, \ldots, J_{s,n_s})$ be a set of choices $J_{s,j} = \binom{a_{s,j}}{r_{s,j}}$, $1 \leqslant j \leqslant n_s$. Let us require that for l chain-cochain spaces $C\left(\Theta^{\boldsymbol{n}_i}_{\boldsymbol{m}_i}\right)$ of the multiple complex $\mathcal{C}$, there exist subspaces $C\left(\Theta'^{\boldsymbol{n}_i}_{\boldsymbol{m}_i}\right) \subset C\left(\Theta^{\boldsymbol{n}_i}_{\boldsymbol{m}_i}\right)$ such that for all $\phi_{s,i} \in C\left(\Theta'^{\boldsymbol{n}_{s,i}}_{\boldsymbol{m}_{s,i}}\right)$, $1 \leqslant i \leqslant k$,

$$\mathcal{D}_{\boldsymbol{J}_s} \gamma_s = \cdot_l \left(\ldots, \left(\mathcal{D}_{\boldsymbol{J}_{s,i}} \phi_{s,i} \right)^{q_{s,i}}, \ldots \right), \tag{6.1}$$

where it is clear that $q_{s,i} < q\left(\mathcal{D}_{\boldsymbol{J}_{s,i}} \phi_{s,i}\right)$ are less than the maximal powers of the corresponding elements $\mathcal{D}_{\boldsymbol{J}_{s,i}} \phi_{s,i}$, and the maximal orders of differentials constituting (as in (2.6)) $\mathcal{D}_{\boldsymbol{J}_{s,i}} \phi_{s,i}$ satisfy the recurrence conditions (2.7). We call this a differential condition. Symmetrizing with respect to all choices of $\boldsymbol{J}_s$ for a fixed l, we obtain a set of differential conditions

$$\left\{ Symm_{\boldsymbol{J}_s, l} \left\{ \mathcal{D}_{\boldsymbol{J}_s} \gamma_s = \cdot_l \left(\ldots, \left(\mathcal{D}_{\boldsymbol{J}_{s,i}} \phi_{s,i} \right)^{q(i,s)}, \ldots \right) \right\} \right\}, \tag{6.2}$$

for $\gamma_s \in \mathcal{C}$, $s \in \mathbb{Z}$. With $\mathcal{D}_{\boldsymbol{J}_s} \gamma_s = 0$ for some s, we call (6.2) an orthogonality condition. For each differential condition above of the set (6.2) we have the coherence condition for corresponding upper and lower indices of $\mathcal{C}$-spaces, i.e., $\boldsymbol{n}\left(\mathcal{D}_{J_s} \gamma_s\right) = \sum_{i=1}^l \left(\boldsymbol{n}_i \left(\mathcal{D}_{\boldsymbol{J}_{s,i}} \phi_{s,i} \right) - r_i \left(\mathcal{D}_{\boldsymbol{J}_{s,i}} \phi_{s,i} \right) \right)$, $\boldsymbol{m}\left(\mathcal{D}_{\boldsymbol{J}_s} \gamma_s\right) = \sum_{i=1}^l \left(\boldsymbol{m}_i \left(\mathcal{D}_{\boldsymbol{J}_{s,i}} \phi_{s,i} \right) - \boldsymbol{t}_i \left(\mathcal{D}_{\boldsymbol{J}_{s,i}} \phi_{s,i} \right) \right)$. We will skip these coherence relations for all differential conditions below. The notion of a set of differential and orthogonality conditions (6.2) generalizes corresponding orthogonality conditions assumed in [12, 15]. One could like to understand which part of differential conditions (6.2) are independent. Indeed, in the case with known commutation rules for $\mathcal{C}$-elements and differentials, we use (2.5) to normalize its sequence in the definition (2.6) and in the conditions (6.2) by sending all $d_{\overline{a}}$ to the left with respect to d_a. Then we use known vanishing rules for powers of differentials. In the case when commutation rules are not known, one might use the independence of powers of differentials. The next result of this paper is the following.

Lemma 1. *The rules* (2.5)*, the differential conditions* (6.2)*, and a set of maximal orders and powers for all $\mathcal{C}$-elements endow $\mathcal{C}$ with the structure of a multiply graded infinite-dimensional differential algebra with respect to a $\cdot_l$-multiplication, $l \geqslant 0$.*

Note that, apart from the value of l for a multiple product, the parameters characterizing a differential algebra above are the distributions of the net of mixed or non-mixed ideals $\mathcal{I}(p)$, $\mathcal{I}(q) \in \mathcal{C}$; the distributions of orders and powers of differentials bounded by their maximal values when applied to $\mathcal{C}$-elements; domains of values of indices for $\mathcal{C}$-spaces; the set of rules (2.5) for differentials; conditions on completions with respect to differentials; commutation relations for $\mathcal{C}$-elements (when known); the domains of values for $\Theta^{\boldsymbol{n}}_{\boldsymbol{m}}$ parameters for $C\left(\Theta^{\boldsymbol{n}}_{\boldsymbol{m}}\right)$, and distribution of horizontal and vertical indices for the corresponding differentials. Now let us give a proof of the Lemma.

Proof. The way of operation with sequences of differential and orthogonality conditions follows from the rules (2.5), Leibniz rule (2.2), as well as from taking intro account maximal orders and powers of differentials and elements, i.e., the assumption that some of $\mathcal{C}$ belong to the ideals with respect to orders $\mathcal{I}(p)$ or powers $\mathcal{I}(q)$. Starting from a particular orthogonality or differential condition, we act consequently by the differentials d_a and $d_{\overline{a}}$. Secondly, using the $\mathcal{I}(p)$, $\mathcal{I}(q)$ ideal vanishing properties applied to orthogonality conditions, and by using the completeness property of the complex $\mathcal{C}$ with respect to the $\cdot_l$-product, we express particular elements $d^{j_i}\phi_i$ in terms of other $\mathcal{C}$ elements. With the maximal orders and powers for each particular $\mathcal{C}$-element, a differential or an $\mathcal{C}$-element reaches its maximal order or power, and corresponding summand vanishes. Continuing the process, we finally obtain the full structure of differential conditions. A sequence of relations does not not stop as long as coherence conditions on indices are fulfilled, of until the sequence gives the both side identical zero. In some cases, due to the completeness condition for the $\cdot_l$-product, a sequence of differential conditions becomes infinite in a certain branch of the hierarchy. Let us reproduce the general structure of relations following from differential conditions for an expression in the form of a multiple product (2.1). The most general configuration associated to a $\cdot_l$-multiple product where several $\mathcal{C}$-elements is with elements that are situated at various places and mixed with the corresponding completions. In our setup, we work with differential conditions containing powers of elements of $\mathcal{C}$ as well as orders of the differentials. The general form of the differential condition is given by (6.2), where $\mathcal{D}_{\boldsymbol{J}_{s,i}}\phi_{s,i}$ is of the form (2.6), $1 \leqslant i \leqslant k \leqslant l$. By applying the differentials to (6.2) one arrives at further differential and orthogonality relations. Recall, that for each differential condition in this proof, there exist a coherence relation for corresponding $\mathcal{C}$-indices. The sequences of differential and orthogonality conditions derived above together with corresponding coherences relations for indices, taken for all choices of $\boldsymbol{J}_s$, provide the full set of differential and orthogonality relations defining the multiply graded differential algebra. The structure of a resulting differential algebra relations as well as the structure of corresponding closed products given by Theorem 4.3 depends on which differential or orthogonality condition of a hierarchy we started from. In practice, one can start with a particular set of differential or/and orthogonality conditions. Then the resulting multiple graded differential algebra is a reduction of the full algebra. In addition to that, one can also restrict domain of definitions for gradings for some families of chain-cochain complexes. E.g., one can set $n \geqslant 0$ instead of $\mathbb{Z}$. This will affect the coherence conditions for corresponding identities. □

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