

Gromov-Witten/Hilbert versus $\text{AdS}_3/\text{CFT}_2$ correspondence

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ABSTRACT: We consider the boundary dual of $\text{AdS}_3 \times S^3 \times K3$ for NS5-flux $Q_5^{NS} = 1$, which is described by a sigma model with target space given by the d -fold symmetric product of $K3$. Building on results in algebraic geometry, we address the problem of deforming it away from the orbifold point from the viewpoint of topological strings. We propose how the 't Hooft expansion can be geometrized in terms of Gromov-Witten invariants and, in favorable settings, how it can be summed up to all orders in closed form. We consider an explicit example in detail for which we discuss the genus expansion around the orbifold point, as well as the divergence in the strong coupling regime. We find that within the domain of convergence, scale separation does not occur. However, in order for the mathematical framework to be applicable in the first place, we need to consider “reduced” Gromov-Witten invariants that fit, as we argue, naturally to topologically twisted $N = 4$ strings. There are some caveats and thus to what extent this toy model captures the physics of strings on $\text{AdS}_3 \times S^3 \times K3$ remains to be seen.

KEYWORDS: AdS-CFT Correspondence, String Duality, Topological Strings

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1 Introduction

1.1 Motivation

One of the best understood and tested instances of the AdS/CFT Correspondence [1] is the near horizon limit of the F1NS5 (resp. the dual D1D5) system. It corresponds to a Type IIB string compactification on $\text{AdS}_3 \times S^3 \times M_4$, where $M_4 = T^4$ or $K3$. It was found that the system drastically simplifies¹ for one unit of NS5 flux, ie., $Q_5^{NS} = 1$ [2–9]. The continuous spectrum of long strings collapses to the bottom, and strings nestled against the boundary $\Sigma = \partial \text{AdS}_3$ become tensionless. Also, the world-sheet theory becomes free and is to a large extent solvable. On the other hand, the boundary theory is described by a two dimensional sigma model with target space $\text{Sym}^N(M_4)$, where $N = Q_1^{NS} Q_5^{NS}$ for $M_4 = T^4$ and $N = Q_1^{NS} Q_5^{NS} + 1$ for $M_4 = K3$ [10, 11]. Numerous tests, mainly for $M_4 = T^4$, confirmed the equality of correlation functions [12–44] in the large N limit.

Most of the work done so far deals with computations in the tensionless regime which is characterized by the singular orbifold $\text{Sym}^N(M_4)$. There is a universal marginal operator, given by the twist field σ_2 , which is associated with the blow-up mode of the orbifold

¹Most results were found for $M_4 = T^4$ but one expects the theory to behave analogously for $M_4 = K3$.

singularity. One can perturb the theory by²

$$\delta S \sim u \int G_{-1}^- \bar{G}_{-1}^- \sigma_2, \tag{1.1}$$

which desingularizes $\text{Sym}^N(M_4)$ by deforming it into the Hilbert scheme of N points on M_4 , $\text{Hilb}^N(M_4)$ (for an extensive review of the role of Hilbert schemes in the context of D-brane physics and black holes, see [45]; for general aspects of this deformation, in particular from the supergravity point of view, see eg. [10, 46–51]). Importantly, it has been known since long that u has characteristic features of a string coupling, and in particular that it should lead to a genus expansion [52]. It cannot however coincide with the string coupling proper, as there is a non-trivial genus expansion already at $u = 0$. One of our goals is to clarify the role of u , and address its all-order expansion globally over the moduli space.

This problem is interesting from several perspectives, and was the initial motivation for this work. For example, the extrapolation from $u = 0$ to $u \rightarrow \infty$ touches upon the question of scale separation [53–56]. The AdS and string scales are tied together by the supergravity formula [47–49, 57]:

$$\frac{R_{\text{AdS}}^2}{\alpha'} \sim Q_5^{NS} \sqrt{1 + (g_{10} c_2)^2}, \tag{1.2}$$

where $g_{10} = e^\Phi$ is the ten dimensional string coupling and $C_2^+ =: c_2(e^6 \wedge e^7 + e^8 \wedge e^9)$ is a self-dual RR two-form. Assuming validity of the formula in this regime, it implies that the scales are of the same order at the orbifold point where $Q_5^{NS} = 1$ and $c_2 = 0$. Switching on u amounts to switching on c_2 [10, 48, 49], so that the size of AdS would increase as $\frac{R_{\text{AdS}}^2}{\alpha'} \propto u$ for large u , but can one make it larger than the size of M_4 in order to reach the supergravity regime?

Relatedly one could ask how the deformation from $u = 0$ to $u \rightarrow \infty$ fits to the general picture of infinite distance limits [58] within the Swampland Program (see eg. [59–61] for reviews). It is well established that near $u = 0$ there is an infinite tower of weakly coupled tensionless strings [3, 62–66], while the general expectation is that at large u , the theory should enter a strongly coupled supergravity regime [67]. However, since geometrically u corresponds to the blow-up mode of a \mathbb{Z}_2 orbifold singularity, the point $u = 0$ corresponds to a finite distance singularity³ at the center of the moduli space. Thus there seems a bit of tension with the lore that it is not at small but at large distance limits where asymptotically massless towers of higher spin string or KK states should appear that are weakly coupled to gravity (in some suitable duality frame) [53–55, 68, 69]. A viewpoint to reconcile this could be that the tensionless strings at $u = 0$ are in this respect analogous to the non-gravitational tensionless strings that arise at ADE singularities in the interior of the moduli space of string compactifications; indeed the geometry at $u = 0$ is strongly curved from the viewpoint of supergravity, and is not a weakly coupled gravitational theory *despite* having a perturbative genus expansion in an appropriate duality frame.

²For now, u is a real parameter. Later we will complexify it by adding a theta-angle, such that $u = 0$ corresponds to $\theta/2\pi = 1/2$. Note that $u = 0$ then corresponds precisely to the theory with $Q_5^{NS} = 1$, with no further shifts of background moduli.

³Due to supersymmetry, singularities due to massless states ought to appear only in corrections to higher derivative terms.

Clearly, in order to address such kind of questions one would like to have analytical control over the whole of the u -moduli space, including the regime where u is large. There have been numerous computations in conformal perturbation theory, see for example refs. [70–92]. However, standard CFT methods are typically limited to the regime near $u = 0$, and the situation is aggravated by the complicated combinatorics of twist fields, which makes an all-genus expansion a tall order.

On the other hand, all-order expansions are routine in geometrical setups, such as moduli spaces of topological strings on Calabi-Yau manifolds. Mirror symmetry and/or localization techniques allow (in BPS protected sectors) to sum up infinite orders in conformal perturbation theory in closed form, including all contact terms. Moreover, a geometrical modulus can turn into a string coupling constant in another duality frame, which opens the door to all-genus computations. This is exactly what we propose to be the case here as well, namely that the geometrical blow-up modulus u on the CFT side will map to a perturbative 't Hooft coupling λ on the AdS side.

Certainly we expect such a geometric approach, if at all, to capture the physics of the geometry $\text{AdS}_3 \times S^3 \times M_4$ at best for $Q_5^{NS} = 1$, where there is no continuum of long strings and the theory drastically simplifies [2–9]. Indeed the theory has characteristic features of a topological $N = 4$ string [8, 9, 31, 39, 93], for which the sectors $\text{AdS}_3 \times S^3$ and $K3$ have separately vanishing central charges. In the hybrid formulation, the first factor can be described by a $\mathfrak{psu}(1, 1|2)_1$ WZW model with extra ghost fields, while the latter is described by a topologically twisted $N = 4$ sigma model [47, 94] on $K3$. It is not clear to us to what degree this describes the full theory, but we expect it to capture at least a protected BPS subsector of it [17, 20, 21, 26, 40, 42–44, 51, 80], thereby realizing what may be loosely called topological AdS/CFT Correspondence.⁴

In the next section we will outline how such a topological AdS/CFT Correspondence could work by making use of some mathematical results for $M_4 = K3$.

1.2 $\text{AdS}_3/\text{CFT}_2$ and GW/Hilb correspondences

The mathematical key statement on which we will base our discussion is a correspondence between the quantum cohomology of the Hilbert scheme of d points on $K3$,⁵ and the Gromov-Witten theory on $K3 \times \Sigma$, which originates from the works of Oberdieck and Pandharipande [100–103]. Our purpose is to see whether this correspondence can offer new insights into $\text{AdS}_3/\text{CFT}_2$ Correspondence in terms of algebraic geometry. A schematic representation is shown in figure 1.

One is tempted to associate the “Hilbert” box to the far right of figure 1 with the boundary CFT, and relate the “Gromov-Witten” box via the dotted arrow to the topological string theory at the boundary. The curve Σ would thus play a dual role, namely as the world-sheet of the 2d CFT sigma-model on $\text{Hilb}^d(K3)$, as well as (part of) the target space on the gravity side, namely as boundary of anti-de-Sitter space: $\Sigma = \partial\text{AdS}_3$; we will focus on $\Sigma = \mathbb{P}^1$ here. The string world-sheets on the AdS side would then arise via branched

⁴Note that several notions of a topological AdS/CFT Correspondence exist, see eg. [15, 42, 95–98].

⁵Analogous correspondences exist also for geometries other than $K3$, see [99] for a list of references.

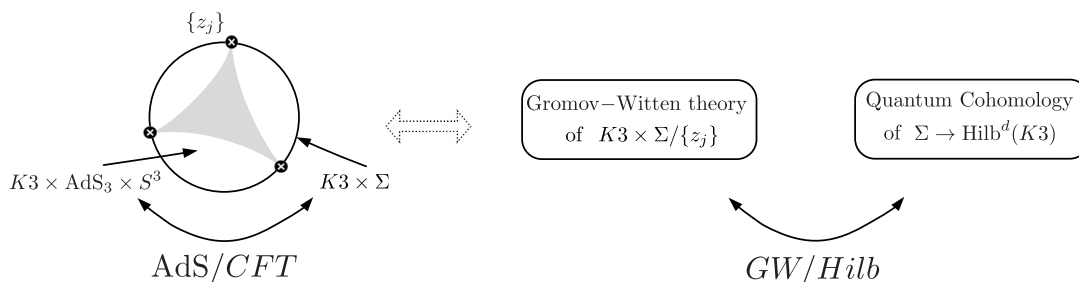


Figure 1. Connecting AdS/CFT and GW/Hilb Correspondences, where $\Sigma = \partial\text{AdS}_3$.

coverings of degree d of Σ , so that the sum over branched coverings yields a perturbative genus expansion, precisely in the spirit of the literature that we cited in the previous section.

Note that the mathematical GW/Hilb Correspondence is an exact duality and holds for any $d \geq 1$, whereas the physical $\text{AdS}_3/\text{CFT}_2$ Correspondence, which relates the boundary theory based on $K3 \times \mathbb{P}^1$ to the bulk of AdS_3 , holds primarily in the large N limit. An immediate question is whether these two correspondences could fit together at their intersection $K3 \times \mathbb{P}^1$, and what concrete benefits this connection could provide. Clearly, one new perspective the geometric approach could offer is that the counting of covering maps is phrased in the language of GW invariants, thereby providing a link to topological gravity and Hurwitz theory, which in turn has the potential for solving the genus expansion to all orders.

In order to address these questions, there are several but related issues to be sorted out. One issue deals with the enumerative counting of holomorphic covering maps from the world-sheet to \mathbb{P}^1 , another with the counting of holomorphic curves in $K3$ (an analogous problem exists for $\text{Hilb}^d(K3)$ if we talk about the ‘‘Hilbert’’ side of the correspondence). Their common problem is that Gromov-Witten theory is naturally based on holomorphic geometry while the $N = (4, 4)$ superconformal AdS background is based on hyperkähler geometry and so canonically has a harmonic rather than a holomorphic structure. Thus, connecting these two correspondences requires at least projecting to a suitable holomorphic structure embedded in the quaternionic hyperkähler geometry, or in physical terms, singling out a suitable $N = (2, 2)$ superconformal subsector of the topologically twisted $N = (4, 4)$ string on the boundary of $\text{AdS}_3 \times S^3 \times K3$.

Let us first consider $\mathbb{P}^1 = \partial\text{AdS}_3$, which is a boundary in the full geometry so there is an interior into which world-sheets can in principle collapse. Thus an enumerative counting of holomorphic maps can make only sense if the string world-sheets are tied to the boundary so that there is a meaningful notion of them ‘‘wrapping’’ it. In fact it has been argued [5, 8, 39, 104, 105] that world-sheets of long strings are pinned to the boundary at the branch points $\{z_j\}$, but otherwise, especially for large twists, may contract to the interior; see figure 1 for a sketchy visualization. One may expect that in the present framework where one counts holomorphic coverings of \mathbb{P}^1 of degree d , all what matters is the topology of the branch points at the boundary.

There is indeed strong evidence that for $Q_5^{NS} = 1$, the world-sheets effectively do localize at the boundary via holomorphic maps. This mechanism has been recently understood [8, 9, 31, 39] by reformulating the $\mathfrak{psu}(1, 1|2)_1$ WZW model that underlies the $N = (4, 4)$

superconformal quantum geometry of $\text{AdS}_3 \times S^3$ in terms of free twistor variables, and showing that the associated holomorphic twistorial maps localize at the boundary. Moreover very recently it has been argued [93] that precisely for $Q_5^{NS} = 1$ there appears a radial global symmetry which more or less guarantees this to be true. One thus expects that the theory on $\text{AdS}_3 \times S^3$ reduces to some version of topological strings on \mathbb{P}^1 (whose precise nature needs to be clarified), and this opens the possibility to count holomorphic covering maps via the Gromov-Witten/Hurwitz Correspondence [106], thereby producing a non-trivial genus expansion despite $\mathbb{P}^1 = \partial\text{AdS}_3$ being a boundary.

The other issue is that standard Gromov-Witten invariants of $K3$ (and of its associated Hilbert schemes) vanish identically, as a consequence of its hyperkähler structure. Indeed for a generic choice of complex structure, there are no holomorphic curves of pure Hodge type $(1,1)$ at all that could be counted. In the mathematical literature one defines instead *reduced* Gromov-Witten invariants [107–111], for example by considering twistor families of complex structures on $K3$. For each such twistor family there exists a fixed choice of complex structure for which curves become holomorphic so that they can be counted in terms of non-vanishing invariants [107].

Thus for both issues there is a common underlying structure that provides the desired connection between the (not naturally holomorphic) hyperkähler geometry and the counting of holomorphic maps from the world-sheet to $\partial\text{AdS}_3 \times K3$. We will propose later in section 2.3 how to translate the mathematical concept of reduced Gromov-Witten invariants to the language of topologically twisted $N = 4$ strings [47, 94].

While these reduced invariants count certain BPS states, the precise relationship between the correlators of the topological toy model and those of the physical theory on the AdS space remains unclear. This also touches upon the compatibility with non-renormalization theorems [21, 26] that are known for extremal correlators [112]. While we will make some comments in this respect in sections 2.3 and 5, we leave a more detailed analysis to future work. For now we stick in this paper to the right half of figure 1 and simply discuss generating functions of certain BPS invariants, without caring about their interpretation in terms of space-time correlators on the AdS side. Rather our focus will be on the moduli space of the blow-up mode of the orbifold singularity and its genus expansion.

In the next section we will collect a few underlying mathematical concepts on which we will base our discussion. We won't add anything new here, but will in turn interpret some of these from a physical perspective. The informed reader is invited to jump to the key section 3 which is devoted to a discussion of the Gromov-Witten/Hilbert Correspondence. Subsequently, in section 3.3 we will give an interpretation of the genus expansion in terms of a 't Hooft expansion on top of the large- N expansion. In section 4 we discuss a simple toy model and discuss in some detail the nature of the genus expansion, as well comment on the strong coupling regime. Finally in section 5 we recapitulate our discussion and mention some loopholes of our proposal.

2 Background material

We recall some facts that are well-known in the mathematical literature, with focus on the cohomology of the Hilbert scheme of d points on $K3$, $\text{Hilb}^d(K3)$, and on what are

called reduced Gromov-Witten invariants. Moreover, in order to make this accessible from a physicist’s perspective, we propose an interpretation of the latter in terms of $N = 4$ topological strings. For details on the relevant algebraic geometry we refer eg. to [100–102, 113, 114].

2.1 (Co)homology of Hilbert schemes

The Hilbert scheme, $\text{Hilb}^d(K3)$, can be viewed as a deformation of the orbifold symmetric product $\text{Sym}^d(K3) = K3^{\otimes d}/S_d$ (see eg., [45, 115]). It is a hyperkähler manifold of complex dimension $2d$. For $d > 1$ its second cohomology is well-known to be of dimension 23, which is one larger than that of a single $K3$:

$$H^2(\text{Hilb}^d(K3), \mathbb{Q}) = H^2(K3, \mathbb{Q}) \oplus \mathbb{C}[\gamma^{(\pi_A)}], \quad d > 1. \quad (2.1)$$

The extra piece corresponds to the class of the blow-up mode of the \mathbb{Z}_2 orbifold singularity on the small diagonal in $\text{Sym}^d(K3)$ where two points collide. An explicit representative will be given below in eq. (2.10). It has been recognized since long that the corresponding modulus, u , plays the role of a string coupling constant (see eg., ref. [52]). This will be our main theme to which we will return later.

The full (co-)homology of $\text{Hilb}^d(K3)$ can be given a convenient Fock space representation. We will be brief, details can be found eg. in ref. [101]. The basic ingredients are the Nakajima creation operators

$$p_{-a}(\alpha) : H^*(\text{Hilb}^b(K3), \mathbb{Q}) \longrightarrow H^*(\text{Hilb}^{a+b}(K3), \mathbb{Q}), \quad \alpha \in H^*(K3), \quad (2.2)$$

which lift cohomology elements γ to $p_{-a}(\alpha)\gamma$ while adding cycles of length a to α . In physical terms, the $p_{-a}(_)$ correspond to twist fields σ_a that will create “long” strings of length a on the AdS side. In this way, one can systematically generate the full cohomology of $\text{Hilb}^d(K3)$ by repeatedly acting with $p_{-a}(\alpha)$ on the Fock vacuum, $\mathbf{1}$. An arbitrary cohomology representative can be assembled by pairing some (non-ordered) partition $\mu = [\mu_1, \dots, \mu_\ell]$ of size $d = \sum \mu_i$ and length $\ell(\mu) = d - \sum(\mu_i - 1)$ with cohomology elements $\alpha_i \in H^{q_i, q_i}(K3, \mathbb{Q})$. Altogether this forms what is called a cohomology weighted partition:

$$\pi = [(\mu_i, \alpha_i)], \quad i = 1, \dots, \ell(\mu). \quad (2.3)$$

Specifically:

$$\gamma^{(\pi)} = \mathfrak{z}(\mu)^{-1} \prod_i p_{-\mu_i}(\alpha_i) \mathbf{1} \in H^{q, q}(\text{Hilb}^d(K3), \mathbb{Q}), \quad (2.4)$$

with symmetry factor

$$\mathfrak{z}(\pi) = |\text{Aut}(\pi)| \prod \mu_i \quad (2.5)$$

that we don’t write explicitly in the following. As indicated, one can associate with $\gamma^{(\pi)}$ a total degree, or R -charge given by

$$q = \sum_{i=1}^{\ell(\pi)} (q_i + (\mu_i - 1)). \quad (2.6)$$

Alternatively, one can consider the Poincaré duals of (2.4) which we use interchangeably with the same notation (except with subscripts) when it fits better.

The notation should become more clear when considering examples of cohomology/curve classes that will be relevant for us. The first originates from a generic curve class $\beta \in H_2(K3, \mathbb{Z})$, which will eventually be linked to a Kähler modulus. Its image in the Hilbert scheme

$$C(\beta) \equiv \gamma_{(\pi_\beta)} = p_{-1}(\beta) p_{-1}(\text{pt})^{d-1} \mathbf{1} \in H_2(\text{Hilb}^d(K3), \mathbb{Z}), \quad (2.7)$$

is a curve class whose associated Gromov-Witten invariants we will consider later. It is of partition type $\pi_\beta = [(1, \beta), (1, \text{pt})^{d-1}]$, where pt denotes the point class which is dual to $H^4(K3, \mathbb{Q})$; note that we implicitly sum here and in the following over cyclic permutations. The curve $C(\beta)$ involves only one non-trivial length one piece, so it will eventually correspond to a string that starts and ends at the same copy of $K3$.

The following class will be more interesting for us: it corresponds to the exceptional curve that arises as blow-up of the orbifold singularity of $\text{Sym}^d(K3)$:

$$A \equiv \gamma_{(\pi_A)} = p_{-2}(\text{pt}) p_{-1}(\text{pt})^{d-2} \mathbf{1} \in H_2(\text{Hilb}^d(K3), \mathbb{Z}), \quad d \geq 2. \quad (2.8)$$

Its partition type is $\pi_A = [(2, \text{pt}), (1, \text{pt})^{d-2}]$. In physics language this would correspond to a string in the twisted sector that links two copies of $K3$. We combine these two classes by writing

$$C_{\beta,k} \equiv C(\beta) + kA \in H_2(\text{Hilb}^d(K3), \mathbb{Z}), \quad \beta \in H_2(K3, \mathbb{Z}), k \in \mathbb{Z}, \quad (2.9)$$

where we understand that $k \equiv 0$ if $d = 1$.

Moreover, we will need a few more cohomology classes. First, the cohomology dual of A is

$$\gamma^{(\pi_A)} = p_{-2}(\mathbb{1}) p_{-1}(\mathbb{1})^{d-2} \mathbb{1} \in H^2(\text{Hilb}^d(K3), \mathbb{Q}), \quad (2.10)$$

which corresponds in CFT language to the universal \mathbb{Z}_2 twist field, σ_2 , that couples to the blow-up modulus of the orbifold singularity. Here, $\mathbb{1} \in H^0(K3, \mathbb{Q})$ denotes the unit. Moreover we define

$$\gamma^{(\pi_{\alpha^d})} = p_{-1}(\alpha)^d \mathbb{1} \in H^{d,d}(\text{Hilb}^d(K3), \mathbb{Q}), \quad (2.11)$$

which is of partition type $\pi_{\alpha^d} = [(1, \alpha)^d]$ with Kähler class $\alpha \in H^{1,1}(K3, \mathbb{Q})$. Finally we need the class

$$\gamma^{(\pi_\alpha)} = p_{-1}(\alpha) p_{-1}(\mathbb{1})^{d-1} \mathbb{1} \in H^{1,1}(\text{Hilb}^d(K3), \mathbb{Q}), \quad (2.12)$$

with $\pi_\alpha = [(1, \alpha), (1, \mathbb{1})^{d-1}]$, which will correspond to a marginal Kähler deformation.

2.2 Reduced Gromov-Witten invariants

We now introduce Gromov-Witten invariants that in physics serve as building blocks for correlation functions. Naively these are given by counting maps $\varphi : \Sigma_{g,n} \rightarrow X_N$ of genus g curves with n punctures into a given Calabi-Yau N -fold in question, which we keep generic for the moment. One may in addition specify the homology class of the image, which we

denote by $C = \varphi_*[\Sigma_{g,n}] \in H_2(X_N, \mathbb{Z})$. However what one really considers are integrals over the suitably compactified moduli spaces, $\overline{\mathcal{M}}_{g,n}$, of stable maps. More generally one considers *relative* Gromov-Witten invariants counting curves that pass through cycles dual to given cohomology classes $\gamma_j \in H^{q_j q_j}(X_N, \mathbb{Q})$:

$$\langle \gamma_1 \dots \gamma_n \rangle_{g,n;C}^{X_N} = \int_{[(\overline{\mathcal{M}}_g(X_N, C), \{\gamma_j\})^{vir}]} ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n). \tag{2.13}$$

Here ev^* denotes the pullback by the evaluation map, $ev : \overline{\mathcal{M}}_{g,n}(X_N, C) \rightarrow X_N$. In order for these invariants not to vanish identically, the expected, or virtual dimension, given by

$$vdim[\overline{\mathcal{M}}_{g,n}(X_N, C), \{\gamma_j\}]^{vir} = (N - 3)(1 - g) - \sum (q_j - 1), \tag{2.14}$$

must vanish.

So far this is the well-known story designed for Calabi-Yau threefolds, for which $N = 3$ so that the genus dependence of the virtual dimension disappears. However, in the present situation where we deal with $2d$ complex-dimensional hyperkähler manifolds, $X_{2d} = \text{Hilb}^d(K3)$, the story needs to be refined, because the GW invariants as defined above vanish identically. This can be understood in various ways. In physics this mirrors the fact that due to the $N = (4, 4)$ supersymmetry, non-renormalization theorems protect certain correlation functions. Mathematically, the problem originates [107–111] from the fact that a generic, non-algebraic deformation moves a curve C of Hodge type $(1, 1)$ to one with a type $(2, 0)$ or $(0, 2)$ component, so that it is not holomorphic any more. Therefore it does not contribute to the counting of holomorphic maps, and by deformation invariance the Gromov-Witten invariants must then vanish even for holomorphic curves of type $(1, 1)$.

Nevertheless, one can define generalized *reduced* GW invariants, abstractly speaking by modifying the obstruction theory such that the $(2, 0)$ and $(0, 2)$ components do not appear [108, 109]. Concretely, one can construct reduced invariants by considering $X_{2d} = \text{Hilb}^d(K3)$ as fiber of some $(2d + 1)$ -fold [107–111]. This ensures that the fibral classes stay at Hodge type $(1, 1)$ under algebraic deformations. There are various, basically equivalent possibilities to implement this, and in the present context it is most natural to view X_{2d} as fiber of its twistor family $T_{2d+1} \rightarrow \mathbb{P}^1$, where \mathbb{P}^1 stands for the unit sphere of complex structures on hyperkähler manifolds.

The crucial point [107] is now that while for generic complex structure there are no holomorphic curves, there exists a unique, fixed choice of complex structure for which holomorphic curves in class $[C]$ with $C^2 \geq -2$ exist, so that they can be meaningfully counted in terms of non-vanishing invariants. Moreover, the modified obstruction theory leads to a modified, reduced version of the virtual dimension,

$$vdim[\overline{\mathcal{M}}_{g,n}(\text{Hilb}^d(K3), C), \{\gamma_j\}]^{red} = (2d - 3)(1 - g) - \sum_{j=1}^n (q_j - 1) + 1, \tag{2.15}$$

which is one dimension larger than the unreduced one. This allows to abstractly define non-vanishing reduced GW invariants for hyperkähler Hilbert schemes with $d \geq 1$ as follows:

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g;C}^{\text{Hilb}^d(K3)} := \int_{[(\overline{\mathcal{M}}_{g,n}(\text{Hilb}^d(K3), C), \{\gamma_j\})^{red}]} ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n). \tag{2.16}$$

In fact, most combinations of $d > 1$ and $g > 1$ still lead to vanishing invariants as these are genuinely zero (the exception being for $d = g = 2$), but for physics only $g = 0, 1$ is relevant if $d > 1$. In this paper we focus on $g = 0$ only, which corresponds to $\Sigma = \partial AdS_3 = \mathbb{P}^1$ on the gravity side.

2.3 Interpretation in terms of the topological $N = 4$ string

We now suggest how the foregoing abstract mathematics can be translated to a more tangible physical language as follows. The key structure is the topologically twisted $N = (4, 4)$ superconformal symmetry of the sigma model on $\text{Hilb}^d(K3)$. In the left-moving sector, the relevant “small” $N = 4$ superconformal algebra [116] is generated by the familiar holomorphic currents J, T, G^\pm that form an $N = 2$ subalgebra, extended by the (super-)currents J^{++}, J^{--} and $\tilde{G}^\pm = \oint J^{\pm\pm} G^\mp, G^\mp = -\oint J^{\mp\mp} \tilde{G}^\pm$. Its central charge is given by $\hat{c} = 2d$, which vanishes after topologically twisting. The bosonic currents extend the $U(1)$ symmetry to $SU(2)$, under which the supercharges transform as two doublets. The same applies to the right-moving sector that we don’t write explicitly.

The $SU(2)$ symmetry reflects in CFT that in hyperkähler geometry, there is no a priori notion of holomorphicity. That is, any chiral primary field, satisfying

$$\hat{G}_0^+ \phi = \tilde{\hat{G}}_0^+ \phi = \left(L_0 - \frac{1}{2} J_0 \right) \phi = 0, \tag{2.17}$$

with charge $q = 1$, can be transformed to an equivalent anti-chiral field via a local $SU(2)$ transformation,

$$\int J^{--} \phi_j =: \sum_{\bar{k}} M_j^{\bar{k}} \bar{\phi}_{\bar{k}}, \tag{2.18}$$

where M is some unspecified numerical mixing matrix [47, 94]. Thus, when combining this with the right-moving counterparts, the (chiral, chiral) ring, which represents elements of $H^{1,1}(K3)$, becomes identified not only with the (anti-chiral, anti-chiral) ring, but also with the twisted (anti-chiral, chiral) ring, which formally represents elements of $H^{-1,1}(K3)$, as well with its complex conjugate. In geometry, this reflects the equivalence of holomorphic and complex structure deformations, implemented by contraction with the holomorphic $(2, 0)$ -form.

A crucial new feature of $N = 4$ superconformal theories is that there is an additional, “global” $\widehat{SU}(2)$, which acts as outer automorphism by mixing the supercurrents of the same charge and so is compatible with the “local” $SU(2)$ symmetry generated by $J^{\pm\pm}$. It is however not a symmetry, and accordingly one can form the following family of supercurrents [47, 94],

$$\begin{aligned} \hat{G}^+(\zeta) &= \zeta_1 G^+ + \zeta_2 \tilde{G}^+ \\ \hat{G}^-(\zeta) &= \zeta_1^* G^- + \zeta_2^* \tilde{G}^- \\ \tilde{\hat{G}}^-(\zeta) &= -\zeta_2 G^- + \zeta_1 \tilde{G}^- \\ \tilde{\hat{G}}^+(\zeta) &= -\zeta_2^* G^+ + \zeta_1^* \tilde{G}^+ \end{aligned} \tag{2.19}$$

for which the inequivalent choices are parametrized by $\widehat{SU}(2)/U(1) \simeq \mathbb{P}_\zeta^1$ defined by $|\zeta_1|^2 + |\zeta_2|^2 = 1$. It represents a family of inequivalent embeddings of the $N = 2$ subalgebra spanned by $\hat{G}^\pm(\zeta)$, within the full $N = 4$ superconformal algebra. Translated back into the

mathematical terms of the foregoing section, \mathbb{P}_ζ^1 corresponds to the unit sphere of possible complex structures within the hyperkähler structure. Fixing a point on \mathbb{P}_ζ^1 corresponds to picking a particular member of the family of inequivalent $N = (2, 2)$ subalgebras, relative to which one can define a notion of holomorphic, or primary chiral fields.⁶

Concretely, in order to gain some intuition, consider as an example a topologically twisted, $N = 4$ superconformal sigma model on an algebraic $K3$ surface (ie, one that is described by a holomorphic embedding via an algebraic equation $W = 0$ in some weighted projected space). Writing down an equivalent $N = 2$ supersymmetric Landau-Ginzburg model with superpotential W gives a canonical definition of the $N = 2$ supercharges, which can be identified with $\widehat{G}^\pm(\zeta)$ in (2.19) with $\zeta_1 = 1, \zeta_2 = 0$. Algebraic curves embedded in this $K3$ then provide holomorphic representatives of Hodge type $(1, 1)$ that are aligned with this fixed complex structure. Moreover, algebraic deformations of W , which are characterized by the Picard lattice $Pic(K3) := H^2(K3, \mathbb{Z}) \cap H^{1,1}(K3)$, will preserve the holomorphicity of these curves [45, 117]. Thus, restricting to algebraic surfaces and their deformations provides a way to define reduced invariants and count holomorphic curves.

This brings us to correlation functions, for which we write only the left-moving part for notational simplicity. The four supersymmetries impose constraints which tend to make correlation functions vanish unless one modifies them in a suitable way. Our purpose here is to connect this modification to the reduced invariants of the previous section. The strategy is to adopt some ideas of $N = 4$ topological strings [47, 94] and apply them to the $N = (4, 4)$ superconformal sigma model on $\text{Hilb}^d(K3)$ under consideration.

For our application where $\Sigma = \mathbb{P}^1$, only $g = 0$ is relevant. The usual procedure to deal with vanishing genus is to consider m -point correlators with $m \geq 3$, and cancel three of the \widehat{G}_{-1}^- of the integrated insertions against the three negative powers associated with the Beltrami differentials. This amounts to mark three points on the sphere to fix its automorphisms. In the end, the modified, genus zero correlator of $N = 4$ strings, as proposed in [47, 94], takes the form

$$\mathcal{F}_{0,m \geq 3}(\zeta) = \left\langle -2d \left| \prod_{i=1}^3 \phi_j \left(\int \widehat{G}^+ \right)^{-1} \int J \prod_{j=1}^{m-3} \int \widehat{G}_{-1}^- \phi_{j+3} \right\rangle_{0,m}^{\text{Hilb}^d(K3)}. \quad (2.20)$$

In the left bracket we have indicated the background $U(1)$ charge $q_0 = \hat{c}(g - 1)$, which arises from the topological twist and which needs to be cancelled by the sum of the charges in the correlator. We also indicated the dependence on the \mathbb{P}_ζ^1 -worth of complex structures of the supercharges (2.19).

The new ingredients, as compared to correlators of standard topological strings, are the insertions of $(\widehat{G}^+)^{-1}$ and J . They are crucial for making non-vanishing correlators of the desired kind possible. In particular, the first insertion implements the modified charge selection rule (2.15) of the reduced Gromov-Witten invariants (2.16). That is, if we take for the remaining integrated insertions moduli fields with $q_{j+3} = 1$, the condition on the unintegrated insertions is $\sum_{i=1}^3 q_j = 2d + 1$, just like for a Calabi-Yau $(2d + 1)$ -fold. However, one needs to make sense of the formal inverse power of \widehat{G}_0^+ (which, when adopting a field

⁶Much more could be said about these matters, but this would lead us too far away. We refer the reader to refs. [21, 45, 47, 94, 117] for more details.

theoretical language, one may view as propagator of the form $\widehat{G}_0^- / \{\widehat{G}_0^+, \widehat{G}_0^-\}$. There are at least two, essentially equivalent ways to deal with that.

First, in many instances the cohomology of \widehat{G}_0^+ is trivial; this applies to the examples for topological $N = 4$ strings of refs. [47, 94], which include NSR strings on $K3$. This means that we can pick one of the un-integrated insertions and write it as $\phi_j = \int \widehat{G}^+ V_j$, where V_j has a $U(1)$ charge reduced by one. Then $\int \widehat{G}^+$ can be moved to J , which gives it back and in the second step one can cancel its inverse power. In the end the genus zero correlator takes the form

$$\mathcal{F}_{0,m \geq 3}(\zeta) = \left\langle -2d \left| \phi_1 \phi_2 V_3 \prod_{j=1}^{m-3} \int \widehat{G}_{-1}^- \phi_{j+3} \right\rangle_{0,m}^{\text{Hilb}^d(K3)}. \quad (2.21)$$

If we were to consider topological $N = 4$ strings in NSR formulation, we can take $\zeta_1 = 1$, $\zeta_2 = 0$ in (2.19), and $\widehat{G}^+ = \eta$, so that it can be easily inverted: $(\widehat{G}^+)^{-1} = \xi$. Here η and ξ are fermions with charges $q = \pm 1$ that arise from bosonizing the familiar (β, γ) super-ghost system. Hence $V_3 = \xi \phi_3$ and the correlator takes the form $\langle \phi \phi \phi \xi_0 \prod \int G^- \phi \rangle$. Thus the zero mode of ξ not only ensures the correct charge balance, but also provides a modified, enlarged Hilbert space along the lines of ref. [118]. We expect this interpretation of the formal insertion $(\widehat{G}^+)^{-1}$ to be morally valid also beyond this suggestive example.

Alternatively, if $m \geq 4$ one can make use of the local $SU(2)$ symmetry transformations (2.18) and $\widehat{G}^- = \int J^- \widehat{G}^+$, which together imply:

$$\widehat{G}_{-1}^- \phi_j = \widehat{G}_0^+ \sum_{\bar{k}} M_j^{\bar{k}} \bar{\phi}_{\bar{k}}. \quad (2.22)$$

Picking an integrated insertion and transforming it into an anti-chiral field, we can move as before the contour of $\int \widehat{G}^+$ first on J and then on top of its inverse to cancel it. This leads to

$$\mathcal{F}_{0,m \geq 4}(\zeta) = \left\langle -2d \left| \phi_1 \phi_2 \phi_3 \sum_{\bar{k}} M_4^{\bar{k}} \int \bar{\phi}_{\bar{k}} \prod_{j=1}^{m-4} \int \widehat{G}_{-1}^- \phi_{j+4} \right\rangle_{0,m}^{\text{Hilb}^d(K3)}. \quad (2.23)$$

This can also be directly obtained from (2.21) by applying (2.22) if $m \geq 4$. Note that in the twisted theory, $\bar{\phi}$ has conformal dimension equal to one and so can be integrated over. Being charged it is not an ordinary, symmetry preserving modulus, but rather its role is to provide the correct charge balance for the reduced invariant.

We thus see that the inverse power of \widehat{G}_0^+ is not only crucial for obtaining non-vanishing correlators in the first place, but also for obtaining non-vanishing deformations by marginal operators. The non-renormalization theorems of refs. [21, 26] apply when this insertion is not there: the argument uses the symmetry transformation (2.22) to map a chiral field to an equivalent anti-chiral one, and moving $\int \widehat{G}_0^+$ to the three un-integrated insertions yields zero by the physical state condition (2.18). More specifically, refs. [21, 26] consider the untwisted theory which means that the background charge is carried by an anti-chiral field with charge $-2d$. Thus altogether two anti-chiral fields appear in the correlator and for this configuration the non-renormalization theorem does not apply.

Note that both correlators (2.21) and (2.23) can be defined just within conformal field theory, which is what is relevant at the boundary of AdS₃. This lets us to propose, tentatively, the following correspondence between the reduced invariants (2.16) and topologically twisted CFT correlation functions:

$$\langle \gamma_1, \dots, \gamma_n \rangle_{0,n}^{\text{Hilb}^d(K3)} \simeq \langle -2d \mid \left(\prod_{i=1}^n \gamma_j \right) \left(\int \tilde{G}^+ \right)^{-1} \left(\prod \int \tilde{G}_{-1}^- \phi \right) \rangle_{0,n}^{\text{Hilb}^d(K3)}, \quad (2.24)$$

where \tilde{G}^+ refers to the corresponding supercharge of the sigma model on $\text{Hilb}^d(K3)$. Note that we talk here about a topological CFT and not of topological strings, which means that the n relative insertions γ_j are *not* integrated over. On the other hand, marginal deformations involving (1, 1) form operators with charge one, ϕ , are integrated over the genus zero world-sheet. Thus, the relevant selection rule for general n is given by

$$2d + 1 \stackrel{!}{=} \sum_{j=1}^n q^{(j)}, \quad (2.25)$$

where $q^{(j)}$ are the charges (2.6) of the γ_j . Moreover also note, as discussed in section 2.2, that the definition of reduced invariants implicitly involves an appropriate choice of complex structure in order to be nonzero, and this corresponds to a definite choice of the ζ_i for the family of supercharges (2.19).

Concluding, we have outlined how the mathematical notion of reduced Gromov-Witten invariants could be translated into the physical language of topological $N = (4, 4)$ strings, which employ the specific definition (2.20) of correlation functions [47, 94]. This definition was designed to generate non-vanishing correlation functions, in a similar spirit as reduced GW invariants of hyperkähler manifolds were introduced as a modification of ordinary ones. While it would be certainly desirable to have a better understanding of this tentative connection, the following considerations are largely independent from it; the aim of this section was to motivate the reduced invariants from a physical perspective.

The main question is clearly whether the prescription (2.24) on the CFT side leads to physically meaningful correlation functions. Encouraging is that the exact free-field formulation of strings on the AdS side, as developed in [8, 9, 31, 39, 93], does realize a topological $N = 4$ string, which suggests that considering reduced invariants is natural. See however the concluding section 5 for some important caveats.

For now we adopt the relevance of reduced invariants as a working hypothesis and leave a more detailed analysis for later work. With this in mind we now proceed to the central theme of this paper.

3 The Gromov-Witten/Hilbert correspondence

The link between the “Gromov-Witten” and “Hilbert” boxes in figure 1 can now be given a concrete meaning in terms of the following key formula.⁷ It has been derived and discussed

⁷While we focus here on the Hilb/GW Correspondence for $\Sigma = \partial AdS_3 = \mathbb{P}^1$, let us note that for thermal AdS₃ with torus boundary, $\Sigma = \mathbb{E}$, there is a noteworthy difference in that the relation between invariants of the Hilbert and Gromov-Witten sides has an extra term [102, 119] as compared to eq. (3.1). In a broader context, it has the interpretation of a wall crossing term [120–122]. Such wall crossing is absent for $\Sigma = \mathbb{P}^1$ that we consider here.

in refs. [100–103, 123] where it has been proven for $n \leq 3$ and conjectured to be more generally valid. For $d \geq 1$ and assuming that β be a primitive⁸ and effective class, we write it in the following form:

$$\sum_{k \in \mathbb{Z}} y^k \langle \gamma_1, \dots, \gamma_n \rangle_{0; C_{\beta, k}}^{\text{Hilb}^d(K3)} = u^{N_0(d, \mu)} \sum_{g \geq 0} u^{2g-2} \langle \gamma_1, \dots, \gamma_n \rangle_{g; (\beta, d)}^{K3 \times \mathbb{P}^1 / \{z_1, \dots, z_n\}}, \quad (3.1)$$

where, as will be explained later: $N_0(d, \mu) := 2d - \sum_j^n \sum_i^{\ell(\mu^{(j)})} (\mu_i^{(j)} - 1)$, and the brackets stand for connected, reduced invariants (2.16). Here we use the same symbol, γ_i , for insertions of primary chiral fields on both sides, because they encode equivalent mathematical data despite representing cohomology classes on different spaces.

Moreover, the map between the expansion parameters on both sides is given by

$$y = e^{2\pi i(z+1/2)} \equiv -e^{-u}. \quad (3.2)$$

Thus the identity eq. (3.1) amounts to a non-trivial resummation between two a priori very different expansions, namely an exponential, instanton type expansion in the blow-up modulus of $\text{Sym}^d(K3)$ on the left hand side, and a power type, perturbative genus expansion on the right. This is analogous to the Donaldson-Thomas/Gromov-Witten correspondences as discussed for example in refs. [113, 125, 126]. In fact (3.1) is a special example within a much more general framework of correspondences as discussed in [120–122].

A generic feature is the symmetry $y \leftrightarrow 1/y$ which is not manifest in a series or Laurent expansion in y or $1/y$. Also note that u is taken here to be complex, and its imaginary part plays the role of a theta-angle. The shift of origin in (3.2) reflects a constant B -field background in the sigma-model on $\text{Hilb}^d(K3)$ at the orbifold point in the moduli space, which is a familiar phenomenon in string geometry. On the AdS-side the role of the theta-angle is played by a linear combination of RR zero- and four-forms, C_0 and C_4 , see for example [10, 49].

Let us now focus on the left- and right-hand sides of the formula eq. (3.1) separately, for more mathematical details see eg. [100–103, 123].

3.1 The Hilbert side

The ingredients of the left-hand side of (3.1) have been extensively discussed in the preceding sections: the reduced GW invariants count holomorphic maps of genus zero curves into the class $C_{\beta, k} \equiv C(\beta) + kA \in H_2(\text{Hilb}^d(K3), \mathbb{Z})$, subject to incidence conditions imposed by the cohomology elements $\gamma_j \equiv \gamma^{\pi^{(j)}} \in H^{*,*}(\text{Hilb}^d(K3), \mathbb{Q})$ (which just means that the curves intersect cycles Poincaré dual to the γ_j). Recall that these cohomology elements are defined in terms of cohomology weighted partitions $\pi^{(j)} = [(\mu_i^{(j)}, \alpha_i^{(j)})]$ of size $d = \sum \mu_i$. Recall also that the condition (2.25) for non-vanishing, reduced invariants (2.24) reads

$$2d + 1 \stackrel{!}{=} \sum_{j=1}^n q^{(j)} + \sum_{j=n+1}^m (q^{(j)} - 1). \quad (3.3)$$

if we add extra integrated insertions. Clearly marginal operators based on primary chiral fields with $q^{(j)} = 1$, $j = n + 1, \dots, m$, do not contribute.

⁸Multicover contributions are fully determined by Hecke transforms [103, 124] and thus do not provide independent extra information.

Let us focus on the dependence on the most interesting of such marginal deformations, namely the blow-up mode u , which we keep here real for simplicity. It parametrizes the deformation by the marginal twist field,

$$\left\langle \gamma_1, \dots, \gamma_n \exp \left(-u \int_{\mathbb{P}^1} \gamma^{(\pi_A)[2]} \right) \right\rangle_{0; C_{\beta, k}}^{\text{Hilb}^d(K3)}, \quad (3.4)$$

where $\gamma^{(\pi_A)[2]} \equiv \widehat{G}_{-1}^- \widetilde{G}_{-1}^- \gamma^{(\pi_A)}$ and $\gamma^{(\pi_A)} \in H^{1,1}(\text{Hilb}^d(K3), \mathbb{Q})$ is a holomorphic representative of the two-form defined in (2.10). It represents the Kähler form of the exceptional curve A (2.8), such that $u = 0$ amounts to zero volume. It appears on the left-hand side of (3.1) exponentiated via $y = -e^{-u}$, and the powers of y count holomorphic maps from \mathbb{P}^1 into the curve kA . In the language of the topological sigma model on $\text{Hilb}^d(K3)$, such maps correspond to holomorphic world-sheet instantons whose action is proportional to the volume u of the exceptional curve. This connection can be made more explicit by first expanding (3.4) and then writing:

$$\begin{aligned} \sum_m \frac{(-u)^m}{m!} \left\langle \prod_{j=1}^n \gamma_j \left(\int_{\mathbb{P}^1} \gamma^{(\pi_A)[2]} \right)^m \right\rangle_{0; C_{\beta, k}}^{\text{Hilb}^d(K3)} &= \sum_m \frac{(-u)^m}{m!} \left(\int_{C_{\beta, k}} \gamma^{(\pi_A)} \right)^m \left\langle \prod_{j=1}^n \gamma_j \right\rangle_{0; C_{\beta, k}}^{\text{Hilb}^d(K3)} \\ &= e^{-u \int_{kA} \gamma^{(\pi_A)}} \langle \gamma_1, \dots, \gamma_n \rangle_{0; C_{\beta, k}}^{\text{Hilb}^d(K3)} \\ &= (-1)^k y^k \langle \gamma_1, \dots, \gamma_n \rangle_{0; C_{\beta, k}}^{\text{Hilb}^d(K3)}. \end{aligned} \quad (3.5)$$

In the first line we used the divisor axiom [127], the second line uses $C_{\beta, k} \equiv C(\beta) + kA$ as per (2.9), and the third $\int_A \gamma^{(\pi_A)} = 1$. Analogous arguments hold for holomorphic Kähler moduli of $K3$ as defined in (2.12).

Note that in theories with $N = (2, 2)$ supersymmetry, such expansions are a priori well defined for large Kähler moduli only, which corresponds to a weakly coupled geometric sigma model regime. In the present situation with $N = (4, 4)$ supersymmetry there is no obstacle against tuning the blow-up modulus to $u = 0$, and in fact it will turn out that the expansion is indeed best defined near $u = 0$. This brings us to the right-hand side of the correspondence (3.1).

3.2 The Gromov-Witten side

The right-hand side of (3.1) is structurally more complicated. Here we sum over relative, reduced Gromov-Witten invariants counting genus g maps into class (β, d) , where $\beta \in H_2(K3, \mathbb{Z})$ as before and d is the degree of the map to \mathbb{P}^1 . Imposed on these maps are incidence conditions implemented by the cohomology elements $\gamma_j \equiv \gamma^{(\pi^{(j)})} \in H^{*,*}(K3 \times \mathbb{P}^1 / \{z_j\})$, which are tied to fixed positions $\{z_j\}$ on \mathbb{P}^1 . The partition types $\mu^{(j)}$ on the Hilbert side translate into ramification profiles over $\{z_j\}$ of the same types, now pertaining to the d -fold cover of \mathbb{P}^1 . This is in line with the familiar viewpoint from AdS/CFT, where one counts covering maps from the world-sheet to $\mathbb{P}^1 = \partial \text{AdS}_3$ with the incidence conditions that they pass through the preimages of the branch points $\{z_j\}$. Indeed this is precisely what is meant by relative invariants, as far as \mathbb{P}^1 is concerned.

More specifically, for a given cohomology weighted partition $\pi^{(j)} = [(\mu_i^{(j)}, \alpha_i^{(j)})]$, $i = 1, \dots, \ell(\pi^{(j)})$, we have an insertion of $\prod_i \alpha_i^{(j)}$, where each $\alpha_i^{(j)} \in H^{*,*}(K3, \mathbb{Q})$ is tied to the

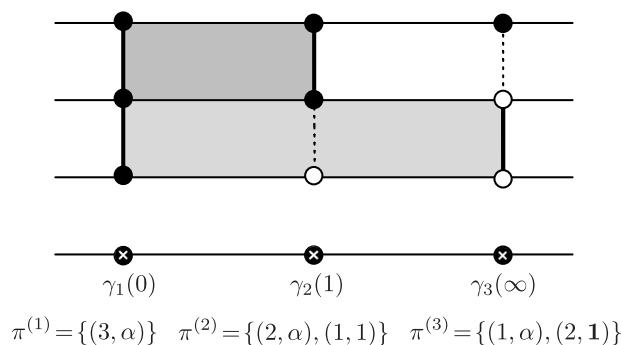


Figure 2. Cohomology weighted partitions $\pi^{(j)} = \{(\mu_i^{(j)}, \alpha_i^{(j)})\}$ defining a $d = 3$ sheeted covering of \mathbb{P}^1 (horizontal lines). Solid vertical lines link μ_i sheets at the ramification points, dashed lines indicate unramified coverings, and the shaded areas signal branch cuts that may be viewed as bridges between the sheets. Solid dots signify insertions of $\alpha \in H^{1,1}(K3)$, while open dots indicate trivial insertions. The three shown ramification profiles with in total 4 non-trivial cycles correspond to the connected genus zero invariant $\langle \sigma_3(\alpha)\sigma_2(\alpha)\sigma_1(\alpha)\sigma_2(1) \rangle_0^{K3 \times \mathbb{P}^1 / \{0,1,\infty\}}$, where the twists σ_{μ_i} are as defined in (3.6) and we denote trivial twists by σ_1 . In physical language, the first two insertions correspond to massive, “spectrally flowed” Kähler classes, while the last two correspond to one massless Kähler modulus plus one massless orbifold blow-up modulus.

corresponding cycle over z_j that links $\mu_i^{(j)}$ sheets together. One should consider this “product” as short-hand for repeated entries. Thus, if individual partitions involve multiple cycles, this may correspond to higher than n -point functions (which are possibly, but not necessarily disconnected — see figure 2 for an example). Putting this together, we can thus write explicit representations of the γ_j on the right-hand side of (3.1) in term of twist fields $\sigma_{\mu_i^{(j)}}$ as follows:

$$\gamma_j \equiv \gamma^{(\mu^{(j)}, \alpha^{(j)})} = \prod_{i=1}^{\ell(\mu^{(j)})} (\sigma_{\mu_i^{(j)}} \alpha_i^{(j)}) \tag{3.6}$$

(for notational convenience, we dropped here the symmetry factor $\mathfrak{z}(\pi)$). The twist fields $\sigma_{\mu_i^{(j)}}$ are the analogs of the Nakajima operators, $p_{-\mu_i^{(j)}}$ in (2.2), and create branch points of order $\mu_i^{(j)}$ at the locations z_j on \mathbb{P}^1 . In (3.6) we indicated by the parentheses that the insertions of the twist fields and of the α_i are correlated and tied to particular sheets if there are multiple cycles over a given branch point.

A priori, the charge selection rule for non-zero correlators, resp. the reduced virtual dimension of the moduli space, looks different for the non-Calabi-Yau threefold $K3 \times \mathbb{P}^1$, for which the genus dependence drops. However, there is an extra contribution of $2d$ from the non-zero first Chern class so that:

$$\begin{aligned} \text{vdim}[\overline{\mathcal{M}}_{g,n}(K3 \times \mathbb{P}^1, (\beta, d)), \{\gamma_j\}]^{\text{red}} &= \left(\dim_{\mathbb{C}}(K3 \times \mathbb{P}^1) - 3 \right) (1-g) + \int_{\beta} c_1(\mathbb{P}^1) + 1 - \sum_{j=1}^n q^{(j)} \\ &= 2d + 1 - \sum_{j=1}^n q^{(j)} \stackrel{!}{=} 0. \end{aligned} \tag{3.7}$$

This reproduces the charge selection rule (2.25) for reduced invariants on the “Hilbert” side.

Let us momentarily focus on maps to branched coverings of \mathbb{P}^1 for some given degree d and ramification profile $\{\mu^{(j)}\}$, the $K3$ going along for the ride. This data defines via the Riemann-Hurwitz formula a curve $\Sigma^{(0)}$ with arithmetic genus

$$g_0 = 1 - d + \frac{1}{2}\Delta(\mu), \tag{3.8}$$

where

$$\Delta(\mu) := \sum_{j=1}^n \sum_{i=1}^{\ell(\pi^{(j)})} (\mu_i^{(j)} - 1). \tag{3.9}$$

We call $\Sigma^{(0)}$ the “bare” curve as it is defined for the unperturbed theory at $u = 0$. Depending on d and $\{\mu^{(j)}\}$, its arithmetic genus can be negative, in which case the curve is necessarily disconnected. For example, in the completely unramified case where all $\mu_i^{(j)} = 1$ so that $\Delta(\mu) = 0$, $\Sigma^{(0)}$ consists of d disconnected curves $C_i \sim \mathbb{P}^1$ with total genus $g_0 = \sum(g(C_i) - 1) + 1 = 1 - d$. We will come back to this example in more detail in section 4.

Since we aim for connected invariants for arbitrary $g \geq 0$, a certain number $N(g, d, \mu) \geq 0$ of extra insertions of the twist field σ_2 will be generally needed in order to obtain non-vanishing invariants. Morally speaking, in the language of conformal field theory we have

$$\left\langle \left(\int_{\mathbb{P}^1} \sigma_2^{[2]} \right)^{N(g,d,\mu)} \prod_{j=1}^n \left(\prod_{i=1}^{\ell(\mu^{(j)})} (\sigma_{\mu_i^{(j)}} \alpha_i^{(j)}) \right) (z_j) \right\rangle_{g;(\beta,d)}^{K3 \times \mathbb{P}^1 / \{z_j\}}, \tag{3.10}$$

where $\sigma_2^{[2]} \equiv \widehat{G}_{-1}^- \widetilde{G}_{-1}^- \sigma_2$ and where we suppressed any other insertions; we will present an abstract, but precise mathematical expression momentarily. Importantly,

$$\begin{aligned} N(g, d, \mu) &:= 2(g - g_0) = 2g - 2 + 2d - \Delta(\mu) \\ &= 2g - 2 + N_0(d, \mu), \end{aligned} \tag{3.11}$$

is the genus deficit between the bare curve $\Sigma^{(0)}$ and the genus g curves we aim for. Remembering that each power of the twist insertion comes with a factor of u , this reproduces the u -dependence on the right-hand side of (3.1).

So far we have neglected $K3$. In general, if we count maps from the world-sheet to $K3 \times \mathbb{P}^1$, there will be extra contributions from maps into $K3$, which can be non-zero because we consider reduced GW invariants. Thus, the genus of the world-sheet will distribute over both $K3$ and ramified degree d covers of \mathbb{P}^1 . Non-trivial maps to curves β on $K3$ will be weighted by exponentiated Kähler moduli. There can also be degenerate, degree zero maps to points on $K3$, an example of which we will meet at the end of section 4.1.

In (3.10) we were writing down just a schematic expression for the Gromov-Witten invariant, but there is a mathematically precise way of writing it [101], namely as an integral over the relative, reduced moduli space:

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g;(\beta,d)}^{K3 \times \mathbb{P}^1 / \{z_j\}} = \int_{[\overline{\mathcal{M}}_{g,n}(K3 \times \mathbb{P}^1, (\beta, d), \{\gamma_j\})]^{red}} \sum_{l_j \geq 0} \prod_{j=1}^n \left(\psi_j^{l_j} \cup \prod_{i=1}^{\ell(\mu^{(j)})} ev_i^{*(j)}(\alpha_i^{(j)}) \right), \tag{3.12}$$

up to symmetry factors. The new ingredient is the Psi-class, which is the first Chern class of the cotangent line bundle at the i -th point, $\psi_i = c_1(L_i)$. More specifically, the canonical interpretation of these invariants is in terms of topological gravity [128, 129], which provides a connection between Gromov-Witten and Hurwitz theory [106] (we will make use of this connection later in section 4.1). In order not to disrupt the current flow of arguments, we delegated a few remarks to appendix A. We just mention here that in this formulation, the role of the twist fields is played by gravitational descendants:⁹ $\sigma_{\mu_i} \simeq \tau_{\mu_i-1}$, and in particular, the insertions of σ_2 in (3.10) correspond to in total $N(d, g, \mu)$ insertions of τ_1 . The Psi-class in (3.12) is nothing but the representative of τ_1 under the integral, distributed in all ways over the n branch points.

Evidently, the question arises how to actually evaluate abstract integrals such as above for concrete examples. Generically they can be expressed in terms of complicated, ELSV-type Hodge integrals [130, 131], which are hard to solve. However, using localization and degeneration methods, explicit results have been obtained for a variety of examples in refs. [100–103], and in section 4 we make good use of one of these when making contact with the AdS perspective.

3.3 't Hooft expansion

Note that depending on the ramification profiles $\{\mu^{(j)}\}$ of the insertions γ_j , non-zero connected invariants for genus $g \geq 0$ curves can arise also without extra insertions of the twist field, ie., $N(g, d, \mu) = 0$ for particular combinations of g, d and $\{\mu^{(j)}\}$. This is the “undeformed” situation at $u = 0$ most often discussed in the physics literature, for which only a finite number of possible bare curves $\Sigma^{(0)}$ can contribute for given d , ie., g_0 is bounded from above [5, 13, 22, 29]. Deforming the theory by switching on u can thus have two effects: namely connecting initially disconnected sheets to produce non-vanishing connected invariants, and/or inflicting higher genus corrections to $\Sigma^{(0)}$ to arbitrarily high orders.

Besides u we have in general also Kähler moduli from the $K3$. To be more specific, let us specialize to curves $\beta = \beta_h \in H_2(K3, \mathbb{Z})$ with self-intersection $\langle \beta_h, \beta_h \rangle = 2h - 2$. This is canonical for elliptic $K3$'s where we can take $\beta_h = b + h f$, with b, f being base and fiber curve classes, respectively (generic $K3$'s can be always deformed to this situation). Denoting $q = e^{2\pi i \tau}$, where τ is the complexified Kähler parameter of the elliptic fiber, we can then sum also over h and turn the r.h.s. of (3.1) into the following generating function:

$$\mathcal{F}_{\{\pi^{(j)}\}}^{(d)}(q, u) = \sum_{g, h \geq 0} u^{N(g, d, \mu)} q^{h-1} \text{GW}_{g, \beta_h, \pi} \tag{3.13}$$

where $\text{GW}_{g, \beta, \pi} := \langle \gamma_1, \dots, \gamma_n \rangle_{g; (\beta, d)}^{K3 \times \mathbb{P}^1 / \{z_j\}}$.

This may be viewed as a generating function in the superselection sector specified by a fixed background geometry $\{\pi^{(j)}\}$, or bare curve $\Sigma^{(0)}$ (possibly with extra operator insertions such as shown in figure 2), around which we expand in q and u .

Let us make the proposed connection between the GW/Hilb and the AdS/CFT Correspondences more concrete, by addressing how the blow-up modulus u relates to the string

⁹A connection between twist fields in the AdS/CFT Correspondence and gravitational descendants has been emphasized early in ref. [15].

coupling constant on the AdS side. They can't be identical, because, as said above, one can have background geometries $\{\pi^{(j)}\}$ with non-trivial correlators even at the orbifold point, $u = 0$. These couple to the six dimensional string coupling g_s in the usual way. Let us therefore weigh the generating function (3.13) by the appropriate power of g_s and write

$$\begin{aligned}
 g_s^{2g_0-2+p} \mathcal{F}_{\{\pi^{(j)}\}}^{(d)}(q, u) &= \sum_{g \geq g_0^*, h \geq 0} g_s^{2g_0-2+p} u^{N(g,d,\mu)} q^{h-1} \text{GW}_{g,\beta_h,\pi} \\
 &= \sum_{g \geq g_0^*, h \geq 0} g_s^{2g_0-2+p} u^{2g-2g_0} q^{h-1} \text{GW}_{g,\beta_h,\pi} \tag{3.14} \\
 &= \sum_{g \geq g_0^*, h \geq 0} N^{1-g-\frac{1}{2}p} \lambda^{2g-2g_0} q^{h-1} \text{GW}_{g,\beta_h,\pi} \\
 &=: \mathcal{F}_{\{\pi^{(j)}\}}^{(N)}(q, \lambda).
 \end{aligned}$$

with $g_0^* = \max\{0, g_0\}$. If $N(g, d, \mu) = 2g - 2g_0 < 0$, the degree constraint cannot be satisfied and the Gromov-Witten invariant must vanish. Moreover, if the bare curve is disconnected so that $g_0 < 0$, g starts at zero since we consider connected invariants only. This is why we wrote the genus expansion as starting at $g = g_0^*$. Finally, $p = nd - \Delta$ denotes the number of cycles, or external legs.

In the third line we made the following substitutions:

$$d = N, \quad g_s = \frac{1}{\sqrt{N}}, \quad u = \frac{\lambda}{\sqrt{N}}, \tag{3.15}$$

where λ plays the rôle of a 't Hooft parameter which is taken to be constant as $N \rightarrow \infty$. Thus $\mathcal{F}_{\{\pi^{(j)}\}}^{(N)}(q, \lambda)$ has the correct asymptotics for large N [22]. The new feature is the extra dependence on λ in terms of GW invariants relative to the bare curve $\Sigma^{(0)}$: switching on λ adds arbitrarily many extra handles to $\Sigma^{(0)}$.

Note that there is such a 't Hooft expansion on top of each possible bare curve; even initially disconnected ones, as will be exemplified in the next section. Each corresponds to a choice of fixed background geometry determined by a collection $\{\pi^{(j)}\}$ of partitions of N . One may sum over such partitions, or bare curves for given N , and for $N \rightarrow \infty$ this turns into a double genus expansion in both the string and 't Hooft couplings, the latter being subleading in $1/N$.

4 A basic example

So far we have discussed the formal structure of generating functions with generic ramification profiles. We now focus on the simplest possible example, which is also the most significant one, since the expansion starts from the most degenerate configuration. The practical bonus is that this setup been already discussed in the mathematical literature [100–102].

For this we consider again an elliptic $K3$ whose fiber and base classes we denote by $f, b \in H_2(K3, \mathbb{Z})$, respectively, and count curves in the primitive class

$$\beta_h = b + h f, \quad \langle \beta_h, \beta_h \rangle = 2h - 2, \quad h \geq 0. \tag{4.1}$$

Concretely we will consider invariants (3.1) involving two insertions of $\gamma^{(\pi_{\alpha d})}$ and one of $\gamma^{(\pi_{\alpha})}$ as defined in eqs. (2.11) and (2.12), for $\alpha = f$. These cohomology weighted partitions

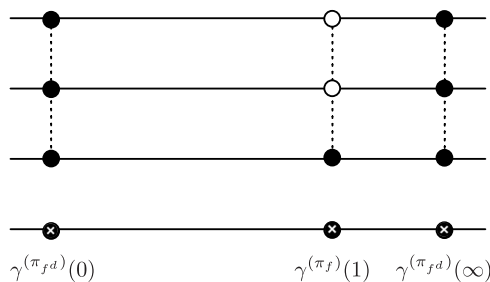


Figure 3. Maximally degenerate, unramified bare geometry $\{\pi_{fd}, \pi_f, \pi_{fd}\}$ for $d = 3$, with notation as for figure 2. It corresponds to a disconnected bare curve $\Sigma^{(0)}$ of arithmetic genus $g = 1 - d = -2$. It represents three separate string world-sheets at the tensionless point $u = 0$ in moduli space, with in total $2d + 1 = 7$ insertions of massless Kähler moduli.

correspond to trivial ramifications. Thus all branched coverings will arise from insertions of twist fields σ_2 , so that the genus expansion is not irritated by initial ramifications over $\{z_j\} = \{0, 1, \infty\}$. See figure 3 and the next sub-section.

The important point is that the connected generating function (3.13)

$$\mathcal{F}_{\{\pi_{fd}, \pi_f, \pi_{fd}\}}^{(d)}(q, u) = u^{2d-2} \sum_{g, h \geq 0} u^{2g} q^{h-1} \left\langle \gamma^{(\pi_{fd})} \gamma^{(\pi_f)} \gamma^{(\pi_{fd})} \right\rangle_{g; (\beta_h, d)}^{K3 \times \mathbb{P}^1 / \{0, 1, \infty\}}, \quad (4.2)$$

can be determined exactly to all orders by localization and degeneration arguments [100–102], and is given in closed form by:

$$\mathcal{F}_{\{\pi_{fd}, \pi_f, \pi_{fd}\}}^{(d)}(q, u) = \frac{1}{(d!)^2} \varphi_{-2,1}(\tau, z)^{d-1} \frac{1}{\Delta(\tau)}, \quad d \geq 1, \quad (4.3)$$

with

$$\varphi_{-2,1}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6} = u^2 + \mathcal{O}(u^4), \quad (4.4)$$

where we substituted $q = e^{2\pi i \tau}$, $u \equiv -2\pi i z$. Before we will discuss some physical implications, let us first make a few brief remarks:

- For $d = 0$, (4.3) turns formally into the well-known KKV formula [132]. It typically arises as denominator of elliptic genera of $\text{Sym}^d(K3)$, where it reflects the orbital contributions of BPS states, and its poles play an important role for black hole physics [132, 133]. However, for us where $d \geq 1$ the physics is different: there aren't any poles, and the blow-up mode u of the symmetric product exists only for $d \geq 2$.

Indeed, for $d = 1$ which corresponds to a single $K3$, all u -dependence disappears from (4.3). It turns into the Yau-Zaslow formula $\sum_{h \geq 0} \langle 1 \rangle_{0,0; \beta_h}^{K3} q^{h-1} = \Delta(q)^{-1}$ which counts genus zero curves in the $K3$ [134]. This reflects that we count here only those higher genus curves that are induced by the perturbation in the blow-up mode u ; in a sense the extra inverse power of $\varphi_{-2,1}(\tau, z)$ in (4.3) cancels out the unwanted higher genus curves in $K3$.

That the u -dependence of the $(2d + 1)$ -point function (4.2) drops for $d = 1$ is also consistent with non-renormalization theorems [21, 26], which state that quadratic and cubic couplings are not renormalized.

- The function in (4.4) is a standard generator of the ring of Jacobi forms, $\varphi_{w,m} : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$, of modular weight $w = -2$ and index $m = 1$. As such it shares the defining feature of Jacobi forms, namely their modular and double periodic transformation properties,

$$\varphi_{w,m} \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^w e^{2\pi i \frac{mc}{c\tau + d} z^2} \varphi_{w,m}(\tau, z) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad (4.5)$$

$$\varphi_{w,m}(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \varphi_{w,m}(\tau, z), \quad \lambda, \mu \in \mathbb{Z}, \quad (4.6)$$

which has have important physical implications as will see further below.

- Given that $\varphi_{-2,1}(\tau, z)$ is essentially a theta function, we can make use of its various canonical expansions, thereby using different parametrizations in terms of y, u or z , etc, whichever fits best. In particular we can write

$$\varphi_{-2,1}(q, u) = u^2 \exp \left(\sum_{g \geq 1} \frac{B_{2g}}{g(2g)!} E_{2g}(q) u^{2g} \right). \quad (4.7)$$

Here $E_{2g}(q) = 1 + \mathcal{O}(q)$ are the familiar Eisenstein series, which are (quasi-)modular forms of weight $w = 2g$, and B_{2g} are the Bernoulli numbers. In the limit of large $K3$ volume, $q = 0$, this simplifies to the ubiquitous sine function

$$\begin{aligned} \varphi_{-2,1}(0, u) &= \mathcal{S}(u)^2 = u^2 + \mathcal{O}(u^4), \quad \text{where} \\ \mathcal{S}(u) &:= 2 \sinh \left(\frac{u}{2} \right) = -i(y^{1/2} + y^{-1/2}), \end{aligned} \quad (4.8)$$

whose enumerative significance in GW/Hurwitz theory is well known. In terms of it, we can express $\varphi_{-2,1}$ and thus the generating function in Gopakumar-Vafa form [135, 136]:

$$\mathcal{F}_{\{\pi_{fd}, \pi_f, \pi_{fd}\}}^{(d)}(q, u) = \frac{1}{(d!)^2} \sum_{g, h \geq 0} \text{GV}_{g, \beta_h, d} q^{h-1} \mathcal{S}(u)^{2g-2+2d}, \quad (4.9)$$

where $\text{GV}_{g, \beta_h, d}$ are integer invariants that count BPS states. This way of writing makes the advertised $y \rightarrow 1/y$ symmetry manifest which is not visible in a Laurent expansion around $y = 0$ (resp. ∞).

- There are the following canonical expansion points, namely the perturbative genus expansion around the orbifold point $u = 0$ resp. $y = -1$, potential strong-coupling expansions around the singularities at $y = \{0, \infty\}$, plus possible scaling limits involving concurrent tuning of the $K3$ Kähler modulus. We will discuss the genus and strong coupling expansions in more detail in the next two sections.
- As a side remark, note that $2d + 1$ -point functions of Kähler moduli being encoded in the quantum cohomology of $\text{Sym}^d(K3)$ fits structurally together with findings made

in refs. [137, 138]. The physical context there was very different, namely it was about computing certain BPS-saturated, quartic gauge and gravitational couplings in $D = 8$ dimensions, which by supersymmetry relate to quartic derivative scalar couplings (or to quintic ones by taking an extra derivative, which fits better to the present discussion). The idea was to compute the couplings in the heterotic string duality frame at one-loop order, and then to translate the result to F-theory to see what geometrical structure of the elliptic $K3$ this would correspond to.

It was found that the couplings were governed by Picard-Fuchs equations pertaining to $\text{Sym}^2(K3)$, with extra source terms that looked like as if coming from a five-fold with $\text{Sym}^2(K3)$ as fiber. Attempts to find such a five-fold were not successful, but with hindsight we would say today that what was computed were in fact *reduced* invariants related to the twistor family of $\text{Sym}^2(K3)$; this notion was not known at the time.

Moreover, based on detailed computations [138] it was conjectured that this pattern is more general, in that formally in $D = 4d$ dimensions, certain protected $2d + 1$ -point amplitudes of gauge/moduli fields are geometrically characterized by $\text{Sym}^d(K3)$. This mirrors the structure that we find here.

4.1 Perturbation theory near the orbifold point

Our example exhibits the highly degenerate nature of the theory at the orbifold point, $u = 0$: all ramifications are trivial, and the “world-sheet” is maximally disconnected (cf., figure 3). It consists of d copies of \mathbb{P}^1 's, which we may interpret as decoupled world-sheets of a collection of tensionless strings that ought to arise in the undeformed theory. Each comes with two insertions of the Kähler form f of the elliptic fiber of $K3$, except one that comes with an extra insertion for stability. In total this gives a completely factorized, $(2d + 1)$ -point function of Kähler moduli, whose connected component vanishes identically.

This is reflected by the overall factor u^{2d-2} of the generating function (4.2): one needs to take $2 \cdot (d - 1)$ u -derivatives in order to arrive at a connected genus $g = 0$ invariant. This amounts to inserting in total $2d - 2$ twist fields such as to glue the d sheets together along $d - 1$ branch cuts. One may view this as a binding process that merges d single tensionless strings wrapped once along $\mathbb{P}^1 = \partial\text{AdS}_3$ into a single, d -times wrapped long string, with $2d + 1$ insertions of massless Kähler moduli. This realizes a picture that had been advocated in [52]. The details how this works are a bit subtle and will be discussed below. Going the other way, ie., approaching $u = 0$ from non-zero u , is nothing but the fragmentation process described in [67] according to which branes can separate without cost in energy when the background fields are switched off.

The combinatorics of twist fields at arbitrary orders in conformal perturbation theory is a priori complicated, but here all is encoded in the u -expansion of the Jacobi form $\varphi_{-2,1}(q, u)$. Since for the expansion near $u = 0$ the Kähler modulus is not important, we set $q = 0$ for now; in the next section we will discuss some extra features that arise when $q \neq 0$.

Dropping q one expects the generating function (4.2) to be governed by Hurwitz theory (for a review, see eg., [139]), as we count branched coverings of \mathbb{P}^1 after all. At first sight, the relevant Hurwitz numbers ought to be the so-called simple Hurwitz numbers $H_{g,d}^{(\mu)}$ that count covering maps of degree d with given ramification $\mu = \{\mu_i\}$ over one fixed point, plus

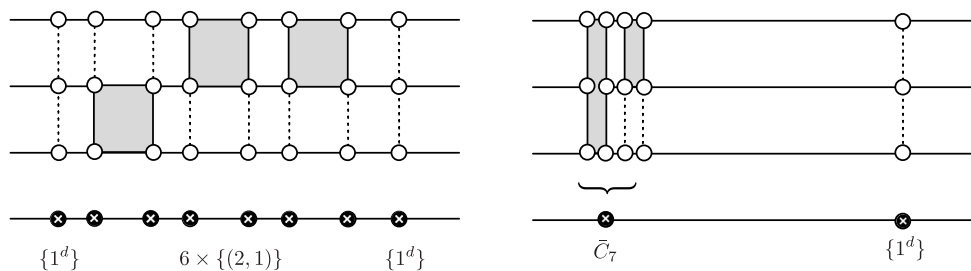


Figure 4. Examples of connected genus $g = 1$, degree $d = 3$ branched covers of \mathbb{P}^1 . Left: genus one curve arising from connecting the sheets by $2g - 2 + 2d = 6$ simple branch points. The number of such coverings is given by the simple Hurwitz number $H_{1,3}^{(1^3)} = \langle \tau_1(\omega)^6 \rangle_{1,3}^{\mathbb{P}^1} = 40$. This is however not the geometry we encounter in the perturbation expansion at order u^6 , rather it corresponds to what we call a bare curve, $\Sigma^{(0)}$, at $u = 0$. Right: crude sketch of the perturbed geometry at order u^6 . It corresponds to an insertion of a completed cycle with associated Hurwitz number $\bar{H}_{1,3}^{(1^3)}(\bar{C}_7) = \langle \tau_6(\omega) \rangle_{1,3}^{\mathbb{P}^1} = \frac{5}{864}$. By misuse of notation, we drew it in terms of slightly separated simple ramification points, but actually they all lie on top of each other so that the genus $g = 1$ curve contracts to a single branch point over the base.

$N(g, d, \mu) = 2g - 2 + d + \ell(\mu)$ simple branch points. As indicated in appendix A, Hurwitz numbers can be written as correlators in topological gravity on \mathbb{P}^1 [140], and concretely for simple Hurwitz numbers we have:

$$H_{g,d}^{(\mu)} = \langle \mu | \tau_1(\omega)^{2g-2+2d} | 1^d \rangle_{g,d}^{\mathbb{P}^1/\{0,\infty\}}, \quad (4.10)$$

which can be summed up to a generating function:

$$H_d^{(\mu)}(u) = u^{2d-2} \sum_{g \geq 0} \frac{u^{2g}}{(2g + 2d - 2)!} H_{g,d}^{(\mu)}. \quad (4.11)$$

However, for our situation with trivial ramifications so that $(\mu) = (1^d)$, this does not match the (q -independent part of the) generating function (4.2) for $K3 \times \mathbb{P}^1$ which reads:

$$\begin{aligned} \mathcal{F}^{(d)}(0, u) &::= q(d!)^2 \mathcal{F}_{\{\pi_{fd}, \pi_f, \pi_{fd}\}}^{(d)}(q, u) \Big|_{q=0} \\ &= \mathcal{S}(u)^{2d-2}. \end{aligned} \quad (4.12)$$

An exception is for $d = 2$ where:

$$\mathcal{S}(u)^2 = 4 H_2^{(1^2)}(u), \quad (4.13)$$

and trivially for $d = 1$. Thus the most direct connection to simple Hurwitz numbers can be stated as $\mathcal{F}^{(d)}(0, u) = 2^{2d-2} (H_2^{(1^2)}(u))^{d-1}$, which reflects the factorized geometry.

This may seem odd as it is well established [23] that certain symmetric orbifold correlators are given by simple Hurwitz numbers $H_{g,d}^{(\mu)}$ (at $g = 0$ for extremal correlators). How does this tie together with our discussion?

In fact, simple Hurwitz numbers (4.10) do appear when counting covering maps for the bare curves $\Sigma^{(0)}$ at $u = 0$ [23]. For these, all insertions are tied to fixed branch points z_j of \mathbb{P}^1 — see the left hand side of figure 4. However, the perturbation theory in u we consider

corresponds to twist field insertions, $\int \widehat{G}^- \widetilde{G}^- \sigma_2$, that are integrated over. Thus degenerate situations occur where twist fields collide with each other or with the relative insertions $\gamma^{(\pi^{(j)})}(z_j)$. It is known [141] that in topological gravity integrated descendant insertions contribute via contact terms, and no extra markings at fixed locations are introduced. Thus it is no surprise that the simple Hurwitz numbers (4.11) do not appear in the perturbation expansion.

What is then the enumerative significance of the genus expansion of $\mathcal{F}^{(d)}(0, u) = \mathcal{S}(u)^{2d-2}$? It is revealed by the appearance of the sine function: for Gromov-Witten theory of \mathbb{P}^1 it is known [106] to arise as generating function for Hurwitz numbers $H_d(\overline{C}_{k+1})$ of “completed” cycles, \overline{C}_{k+1} . Such completed Hurwitz numbers, also called $k + 1$ -spin Hurwitz numbers [142, 143], are generalizations of the simple Hurwitz numbers where simple ramifications are replaced by ramifications with completed cycles.

As briefly mentioned in appendix A, these cycles are labeled by certain distinguished linear combinations of partitions, namely precisely those that are equivalent to insertions of the gravitational descendants: $\tau_k \equiv \tau_1^k \sim \frac{1}{k!} \overline{C}_{k+1}$. The coefficients of the linear combinations are given in terms of the sine function. Specifically, denoting by (μ) the conjugacy class of a partition $\mu = \{\mu_i\}$ of d , the completed cycles are given by linear combinations [106]

$$\overline{C}_{k+1} = \sum_{\mu} \rho_{k+1, \mu}(\mu), \tag{4.14}$$

where

$$\rho_{k+1, \mu} = \frac{k!}{d!} \prod \mu_i \left(\mathcal{S}^{d-1}(u) \prod \mathcal{S}(\mu_i u) \right) \Big|_{u^{k+1}}, \tag{4.15}$$

is the source of the sine function. The completed Hurwitz numbers are then given by

$$\overline{H}_{g,d}^{(\mu)}(\overline{C}_{2g-1+d+\ell(\mu)}) = \langle \mu | \tau_{2g-2+d+\ell(\mu)}(\omega) | 1^d \rangle_{g,d}^{\mathbb{P}^1/\{0,\infty\}}, \tag{4.16}$$

and in our situation with trivial ramification, $(\mu) = (1^d)$, one obtains [140]

$$u^{2d-1} \sum_{g \geq 0} u^{2g} \overline{H}_{g,d}^{(1^d)}(\overline{C}_{2g-1+2d}) = \frac{1}{(d!)^2} \mathcal{S}(u)^{2d-1}. \tag{4.17}$$

To give a geometrical interpretation, recall that the sine function prominently appears in topological strings as summing degenerate (pinched) curves as per the GV formula (4.9). This is what is reflected here, as we can think¹⁰ of a completed cycle as a degenerate configuration where a genus g curve carrying d marked points contracts via the covering map to a single point in the base \mathbb{P}^1 ; see the right part of figure 4 for a crude visualization. We can view this as the result of the transformation (A.4). Recall, however, that the representation of the GW invariants is not unique and that there are in general several different but mathematically equivalent representations. For example one can distribute the genus g over curves contracted over several branch points.

Note that the foregoing formulas refer to topological gravity with target \mathbb{P}^1 rather than $K3 \times \mathbb{P}^1$. The generating functions are closely related in that there is an extra power of $\mathcal{S}(u)^{-1}$ in (4.12) as compared to (4.17). To understand this, recall that the GW invariants we

¹⁰I thank Vincent Bouchard for explanations.

consider momentarily count maps into curve classes (pt, d) of $K3 \times \mathbb{P}^1$, where d is the degree of a map to \mathbb{P}^1 . These include also constant ($d = 0$) maps from collapsed curve components into $K3$. For example, such a map from a genus $g = g_1 + g_2$ curve can involve a degree d map from the genus g_1 component to \mathbb{P}^1 and a degree zero map from a collapsed genus g_2 component to a point on $K3$. Thus in general there is an extra contribution from degree zero maps to $K3$, which in the present context is given by an integral over an extra, one-pointed component $\overline{\mathcal{M}}_{g_2,1}$ of the compactified moduli space. It can be written as a Hodge integral [144]

$$u^{-1} \left[1 + \sum_{g_2 \geq 1} u^{2g_2} \int_{\overline{\mathcal{M}}_{g_2,1}} \lambda_g \psi^{2g_2-2} \right] = \mathcal{S}(u)^{-1}, \tag{4.18}$$

where $\lambda_i = c_i(\Lambda^\vee) \in H^{2i}(\overline{\mathcal{M}}_{g,n})$ are Chern classes of the dual Hodge bundle, and ψ is the representative of τ_1 under the integral as mentioned before. Together with the contribution (4.17) from \mathbb{P}^1 , this reproduces the generating function (4.12) for $K3 \times \mathbb{P}^1$ at $q = 0$.

Wrapping up: our purpose in this section was to elucidate the genus expansion in the blow-up modulus u near $q = 0$. We argued that a natural interpretation is in terms of degenerate, contracted genus g curves that are attached to the bare curves $\Sigma^{(0)}$ at their branch points. This supports the viewpoint of contact terms: the genus corrections are integrated out and are subsumed by “renormalized” insertions, given by completed cycles. This is reminiscent of flat operator representatives in topological Landau-Ginzburg theory which encode renormalization due to integrating out massive states. This viewpoint also offers an appealing physical picture where the string world-sheet touches the boundary of AdS_3 only at a few fixed branch points and not at extra marked points whose number would depend on the order of perturbation theory.

4.2 Large-distance limit

We now turn to a potentially more interesting limit, namely $y = -e^{-u} = -e^{2\pi iz} \rightarrow 0$ (which is equivalent to $y \rightarrow \infty$ due to the $y \leftrightarrow 1/y$ symmetry), while restoring the dependence of the fibral modulus τ of the $K3$. From the viewpoint of a flat coordinate on the CFT moduli space this corresponds to a large distance limit, which by the lore of the Swampland Distance Conjecture [58] ought to correspond to a weak coupling limit in some proper duality frame.

However, the situation is different as compared to the standard large distance limits in Kähler moduli space. This is because u as elliptic parameter of a Jacobi form lives on the Jacobian torus and thus is essentially a double periodic variable. More precisely, our generating function, which is a Jacobi form with non-zero index, transforms with an “anomalous” prefactor (4.6), that is,

$$\mathcal{F}_{\{\pi_{fd}, \pi_{fd}\}}^{(d)}(\tau, z + \lambda\tau + \mu) \propto \varphi_{-2,1}(\tau, z + \lambda\tau + \mu)^{d-1} = q^{-\lambda^2(d-1)} (-y)^{-2\lambda(d-1)} \varphi_{-2,1}(\tau, z)^{d-1}, \tag{4.19}$$

with $\lambda, \mu \in \mathbb{Z}$. The periodicity in μ corresponds to the familiar shift of the theta-angle, while the quasi-periodicity in λ is less obvious. Such a shift can always be interpreted in terms of spectral flow¹¹ related to a $U(1)$ symmetry whose associated field strength, or fugacity

¹¹The period of this spectral flow is given by the $K3$ fibral Kähler parameter τ so it appears different to the one familiar from the literature [105].

parameter, is given by z . Altogether there seems a whole lattice $\mathbb{C}/(\mathbb{Z}\tau \times \mathbb{Z})$ of zeros that allow for weakly coupled genus expansions in terms of suitably spectrally flowed degrees of freedom that become light there.

However, due to the prefactor in the transformation law (4.19), the situation is not really periodic. That is, if we increase $z = iu/2\pi$ arbitrarily far in the τ -direction, we can always map it back to the fundamental domain

$$F_z = \{z \mid 0 \leq \text{Im } z < \text{Im } \tau\}, \quad (4.20)$$

albeit at the expense of a divergent prefactor. This just reflects the lack of convergence outside the fundamental domain. This can also be seen by considering the limit of (4.9) as y becomes small, after factoring out the leading singularity y^{1-d} :

$$q \mathcal{F}_{\{\pi_{fd}, \pi_f, \pi_{fd}\}}^{(d)}(q, y) \propto y^{1-d} \sum (-1)^g \text{GV}_{g, \beta_h, d} y^{-g} q^h. \quad (4.21)$$

Then one may worry about the convergence of the second factor. It is a property of $\varphi_{-2,1}$ that its Gopakumar-Vafa expansion coefficients obey

$$\text{GV}_{g, \beta_h, 2} = 0 \quad \text{for } g > h, \quad (4.22)$$

so that the leading terms in $q/y = -e^{2\pi i(\tau-z)}$ take the form:

$$q \mathcal{F}_{\{\pi_{fd}, \pi_f, \pi_{fd}\}}^{(d)}(q, y) \propto y^{1-d} \left[\sum (-1)^g \text{GV}_{g, \beta_g, d} \left(\frac{q}{y}\right)^g + \mathcal{O}\left(\frac{q^2}{y}\right) \right] \propto y^{1-d} \left[\left(1 + \frac{q}{y}\right)^{2d-2} + \mathcal{O}\left(\frac{q^2}{y}\right) \right]. \quad (4.23)$$

The second factor indeed converges over the fundamental domain, F_z .

The upshot is that even for large fiber volume¹² of the $K3$, $\text{Im}\tau \rightarrow \infty$, one can take $\text{Im}z$ large only as far as one stays within the domain of convergence F_z . The lack of convergence outside F_z signals that new degrees of freedom should come into play that were not taken care of in the present framework. This calls for a non-perturbative completion of the theory, quite likely in the form of a matrix model [6].

That a string coupling enters as the elliptic argument of a Jacobi form and so exhibits (quasi-)periodicity, is a common feature in topological strings [132, 133, 145], one difference being that we deal here with zeros rather than with poles at the integral lattice points. There has been recent progress in non-perturbative completions (see eg., [146–148]) of topological strings in terms of resurgence, and it would be natural to expect that analogous methods would work here too.

¹²Note that (for the F1NS5 system) the total volume of the $K3$ in the near horizon limit is linked to the ten dimensional string coupling via $g_{10}^2 = d^{-1} \text{Vol}(K3)/\alpha'^2$ [10, 46, 47, 49]. We can choose the volume as a free parameter at the expense of a volume dependent ten dimensional string coupling. Thus with increasing volume the theory runs into a non-perturbative regime at $g_{10} = 1$, beyond which one needs to switch to the S-dual D1D5 frame [51]. If we want to stay in the current framework, this puts a bound on the volume, $\text{Vol}(K3)/\alpha'^2 < d$, and thus morally speaking on $\text{Im}\tau$. Strictly speaking $\text{Im}\tau$ gives the volume of the elliptic fiber of the $K3$ only, and the total volume will depend also on the volume of the base of the fibration; this subtlety is not important for our purposes. We just need to keep this bound in mind when talking about large volume limits.

As for Swampland physics, it is unfortunate that the regime of large $\text{Im}z$ lies outside the domain of validity of the framework we developed here, and the answer to important questions, like about the appearance of higher spin states and scale separation, seems to be buried in the details of the purported non-perturbative completion. On the other hand, if we stick to the regime F_z (4.20) where convergence is under control, $\text{Im}z$ cannot grow larger than $\text{Im}\tau$ and this translates, via (1.2), to the statement that R_{AdS} is bounded from above by the size (of the elliptic fiber) of the $K3$. In other words, scale separation, in the sense we just explained, does not occur within the present framework.

5 Discussion

In this paper we argued how certain results in algebraic geometry [100–103] could provide insights in topological aspects of the AdS/CFT correspondence for $AdS_3 \times S^3 \times K3$ for $Q_5^{NS} = 1$. Our emphasis was on the moduli space and the perturbation expansion in the blow-up deformation parameter, u , of the symmetric orbifold $\text{Sym}^d(K3)$ on the CFT side. We showed that it corresponds to a 't Hooft-like expansion around a given background geometry, defined in terms of a bare curve, $\Sigma^{(0)}$, which is specified by a given set of (in general cohomology weighted) ramification conditions, $\{\pi_j\}$. The effect of this expansion is to add connections between a priori disconnected sheets and/or adding more and more handles to $\Sigma^{(0)}$. Near the orbifold point, $u = 0$, it takes the form of a perturbative genus expansion for the tensionless strings that dominate this regime. On the other hand, in the large distance limit $u \rightarrow \infty$, the framework breaks down and ought to be supplemented by a suitable non-perturbative completion. We argued that within the domain of convergence, scale separation does not occur.

However, as pointed out at various stages of this work, there is an important issue when attempting to connect, as per figure 1, the mathematical Gromov-Witten/Hilbert Correspondence to the physical AdS/CFT Correspondence. The root of the problem lies in the fact that GW theory deals with holomorphic geometry while a hyperkähler manifold such as $\text{Hilb}^d(K3)$ has a natural harmonic rather than holomorphic structure. In order to get non-zero invariants in the first place, one considers in the mathematical literature reduced Gromov-Witten invariants [107–111], which per design count holomorphic maps in hyperkähler geometry. While in physical language these ought to count certain BPS invariants, it is a priori not clear what actual physical correlation functions on the AdS side they would correspond to.

We have argued how to interpret these reduced invariants in the language of topological $N = 4$ strings, as defined in [47, 94]. At first sight, this seems to fit to the recent works on the AdS side (e.g., [8, 9, 31, 39, 93]) which indeed point to topological $N = 4$ strings; their correlators boil down to counting holomorphic covering maps of the boundary, and thus rest on an embedding of holomorphic into hyperkähler geometry as well. If this interpretation bears out, reduced invariants for $m \geq 4$ insertions can be written as physical correlators of the form $\langle \phi\phi\phi \int \bar{\phi} \int \widehat{G}^- \phi \cdots \int \widehat{G}^- \phi \rangle$, for details see around eq. (2.23).

Evidently there are loopholes in this line of thought. In particular, $N = 4$ strings are not unique in the sense that there is a choice of which combinations of $\widehat{G}^\pm(\zeta)$ versus $\widehat{\widehat{G}}^\pm(\zeta)$ are inserted in correlation functions. This can be translated to the language of

picture changing, and the prescription of [47, 94] is to sum over all pictures, or possible combinations of such insertions. Now, in the concrete hybrid formulation of $N = 4$ strings on $\text{AdS}_3 \times S^3 \times M_4$ alluded to above, there are extra constraints on correlators due to ghost number conservation, and these restrict which combinations of insertions of $\widehat{G}^\pm(\zeta)$, $\widetilde{G}^\pm(\zeta)$ can yield non-vanishing contributions. In other words, the issue is whether the localization to the boundary $\mathbb{P}^1 = \partial \text{AdS}_3$ properly aligns the complex structure parametrized by ζ_i such that it leads to non-trivial reduced invariants.

While in ref. [91] marginal deformations of genus zero correlators with an insertion of $(\widetilde{G}^+)^{-1}$ have been considered recently, it is not clear at this point whether our specific interpretation of reduced invariants is compatible with the concrete realization of $N = 4$ strings that is relevant for $\text{AdS}_3 \times S^3$. There might be a subtlety in defining the theory, analogous to the choice in topological gravity whether to consider the “small” or the “large” Hilbert (or phase) space, depending on the definition of physical states in terms of equivariant cohomology [129, 141]; the latter choice includes gravitational descendants, while the former does not. This analogy is suggested by the fact that the genus expansion we consider here is implemented by insertions of the twist field σ_2 , which via the Gromov-Witten/Hurwitz Correspondence (cf., appendix A) is equivalent to insertions of the gravitational descendant τ_1 . In a nutshell, essentially this boils down to the question whether the topological $N = 4$ strings on $\text{AdS}_3 \times S^3 \times K3$ really localize to topological strings on $K3 \times \mathbb{P}^1$ that admit such a “large” Hilbert space, or not.

Clearly, in order to put the suggested connection between the GW/Hilbert and the AdS/CFT Correspondence on a firmer ground, a more detailed analysis of correlation functions and their marginal deformations will be necessary. These matters also have a bearing on non-renormalization theorems such as discussed in [21, 26]. We leave this to future work.

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A GW/Hurwitz correspondence and topological gravity

Let us briefly elucidate some of the underlying mathematical structure on the “Gromov-Witten”, right-hand side of (3.1), which at least in principle may facilitate explicit computations.

Branched coverings of \mathbb{P}^1 are well-known to be the subject of Hurwitz theory, and what is relevant here is its correspondence [106] with Gromov-Witten theory of \mathbb{P}^1 . The underlying mathematical logic is an equivalence of ramification profiles and intersection theory of Riemann surfaces, which in the language of topological gravity [128, 129] is formulated in terms of gravitational descendants, $\tau_k \equiv \tau_1^k$. To leading order these correspond to the twist fields: $\tau_k \sim \frac{1}{k!} \sigma_{k+1}$, but in general there are extra correction terms such that τ_k correspond to

a special basis of “completed cycles” [106]. These reflect degenerations of Hurwitz coverings, which amount to contact terms in physics language; see the end of section 4.1 for comments in the context of an example. There are no corrections for $\tau_1 = \sigma_2$, which is associated with the blow-up mode of the orbifold singularity.

More specifically, what is relevant here is an extension of Hurwitz theory to cohomology weighted partitions that was dubbed “relative/descendent correspondence” in [149]. It maps between the Nakajima Fock space basis (labeled by cohomology weighted partitions $\pi = [(\mu_i, \alpha_i)]$) and descendant insertions as follows:

$$\gamma^{(\pi)} \rightarrow \gamma^{(\pi)}[\tau] = \sum_{\pi'} (M^{-1})_{\pi\pi'} \tau^{(\pi')}, \tag{A.1}$$

$$\text{where } \tau^{(\pi)} := \prod_{i=1}^{\ell(\pi)} \tau_{\pi_i-1}(\alpha_i \omega), \tag{A.2}$$

with $\omega \in H^2(\mathbb{P}^1)$. The linear basis transformation is given by a matrix of relative one-point overlap invariants of the schematic form $M^{\pi'\pi} = \mathfrak{z}(\mu) \langle \gamma^{(\pi)} | \tau^{(\pi')} \rangle_{K3 \times \mathbb{P}^1 / \infty}$, which can be shown to be invertible; details are spelled out in refs. [99, 106, 119, 149]. This allows to express invariants with ramification conditions $\mu^{(j)}$ at locations z_j in terms of correlators of topological gravity and vice versa. Concretely one can represent the invariants (3.12) as linear combinations of

$$\sum_{\substack{l_j \geq 0 \\ \sum l_j = N(d,g,\mu)}} \left\langle \prod_{j=1}^n \tau_{l_j} \left(\prod_{i=1}^{\ell(\mu^{(j)})} \tau_{\mu_i^{(j)}-1}(\alpha_i^{(j)} \omega) \right) \right\rangle_{g;(\beta,d)}^{K3 \times \mathbb{P}^1 / \{z_1, \dots, z_n\}}, \tag{A.3}$$

where $N(g, d, \pi) = 2(g - g_0)$ is the genus deficit as defined in (3.11).

In fact, by repeated application of (A.1) one can map all relative insertions to just one insertion ([119], section 5.2), schematically:

$$\left\langle \prod_{j=1}^n \gamma^{(\pi^{(j)})} \right\rangle_{K3 \times \mathbb{P}^1 / \{z_1, \dots, z_n\}} \longrightarrow \left\langle \gamma^{\tilde{\pi}} \prod_{i=1}^{\ell(\tilde{\pi})} \tau_{\tilde{\pi}_i-1}(\alpha_i \omega) \right\rangle_{K3 \times \mathbb{P}^1 / \infty}. \tag{A.4}$$

This representation is highly degenerate as the whole genus g curve contracts over one point of \mathbb{P}^1 ; see again the end of section 4.1 for an example.

Thus, in general there is a large degree of ambiguity in representing the reduced GW invariants for $K3 \times \mathbb{P}^1$. Specifically, it can be shown by localization and degeneration arguments [101] that the full Gromov-Witten theory of $K3 \times \mathbb{P}^1$ can be expressed in terms of ELSV-type [130, 131], linear Hodge integrals for $K3$ only. Conversely, Hodge integrals are completely determined in terms of descendant invariants [144].

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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References

- [1] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [[hep-th/9711200](#)] [[INSPIRE](#)].
- [2] G. Giribet et al., *Superstrings on AdS_3 at $|| = 1$* , *JHEP* **08** (2018) 204 [[arXiv:1803.04420](#)] [[INSPIRE](#)].
- [3] M.R. Gaberdiel and R. Gopakumar, *Tensionless string spectra on AdS_3* , *JHEP* **05** (2018) 085 [[arXiv:1803.04423](#)] [[INSPIRE](#)].
- [4] L. Eberhardt, M.R. Gaberdiel and R. Gopakumar, *The worldsheet dual of the symmetric product CFT*, *JHEP* **04** (2019) 103 [[arXiv:1812.01007](#)] [[INSPIRE](#)].
- [5] L. Eberhardt, M.R. Gaberdiel and R. Gopakumar, *Deriving the AdS_3/CFT_2 correspondence*, *JHEP* **02** (2020) 136 [[arXiv:1911.00378](#)] [[INSPIRE](#)].
- [6] M.R. Gaberdiel, R. Gopakumar, B. Knighton and P. Maity, *From symmetric product CFTs to AdS_3* , *JHEP* **05** (2021) 073 [[arXiv:2011.10038](#)] [[INSPIRE](#)].
- [7] L. Eberhardt, *Partition functions of the tensionless string*, *JHEP* **03** (2021) 176 [[arXiv:2008.07533](#)] [[INSPIRE](#)].
- [8] F. Bhat, R. Gopakumar, P. Maity and B. Radhakrishnan, *Twistor coverings and Feynman diagrams*, *JHEP* **05** (2022) 150 [[arXiv:2112.05115](#)] [[INSPIRE](#)].
- [9] M.R. Gaberdiel, K. Naderi and V. Sriprachyakul, *The free field realisation of the BVW string*, *JHEP* **08** (2022) 274 [[arXiv:2202.11392](#)] [[INSPIRE](#)].
- [10] F. Larsen and E.J. Martinec, *$U(1)$ charges and moduli in the $D1$ - $D5$ system*, *JHEP* **06** (1999) 019 [[hep-th/9905064](#)] [[INSPIRE](#)].
- [11] M. Beccaria, G. Macorini and A.A. Tseytlin, *Supergravity one-loop corrections on AdS_7 and AdS_3 , higher spins and AdS/CFT* , *Nucl. Phys. B* **892** (2015) 211 [[arXiv:1412.0489](#)] [[INSPIRE](#)].
- [12] A. Jevicki, M. Mihailescu and S. Ramgoolam, *Gravity from CFT on $S^N(X)$: symmetries and interactions*, *Nucl. Phys. B* **577** (2000) 47 [[hep-th/9907144](#)] [[INSPIRE](#)].
- [13] O. Lunin and S.D. Mathur, *Correlation functions for M^N/S_N orbifolds*, *Commun. Math. Phys.* **219** (2001) 399 [[hep-th/0006196](#)] [[INSPIRE](#)].
- [14] O. Lunin and S.D. Mathur, *Three point functions for M^N/S_N orbifolds with $N=4$ supersymmetry*, *Commun. Math. Phys.* **227** (2002) 385 [[hep-th/0103169](#)] [[INSPIRE](#)].
- [15] L. Rastelli and M. Wijnholt, *Minimal AdS_3* , *Adv. Theor. Math. Phys.* **11** (2007) 291 [[hep-th/0507037](#)] [[INSPIRE](#)].
- [16] M.R. Gaberdiel and I. Kirsch, *Worldsheet correlators in AdS_3/CFT_2* , *JHEP* **04** (2007) 050 [[hep-th/0703001](#)] [[INSPIRE](#)].
- [17] A. Dabholkar and A. Pakman, *Exact chiral ring of AdS_3/CFT_2* , *Adv. Theor. Math. Phys.* **13** (2009) 409 [[hep-th/0703022](#)] [[INSPIRE](#)].
- [18] M. Taylor, *Matching of correlators in AdS_3/CFT_2* , *JHEP* **06** (2008) 010 [[arXiv:0709.1838](#)] [[INSPIRE](#)].

- [19] G. Giribet, A. Pakman and L. Rastelli, *Spectral flow in AdS_3/CFT_2* , *JHEP* **06** (2008) 013 [[arXiv:0712.3046](#)] [[INSPIRE](#)].
- [20] A. Pakman and A. Sever, *Exact $N=4$ correlators of AdS_3/CFT_2* , *Phys. Lett. B* **652** (2007) 60 [[arXiv:0704.3040](#)] [[INSPIRE](#)].
- [21] J. de Boer, J. Manschot, K. Papadodimas and E. Verlinde, *The chiral ring of AdS_3/CFT_2 and the attractor mechanism*, *JHEP* **03** (2009) 030 [[arXiv:0809.0507](#)] [[INSPIRE](#)].
- [22] A. Pakman, L. Rastelli and S.S. Razamat, *Diagrams for symmetric product orbifolds*, *JHEP* **10** (2009) 034 [[arXiv:0905.3448](#)] [[INSPIRE](#)].
- [23] A. Pakman, L. Rastelli and S.S. Razamat, *Extremal correlators and Hurwitz numbers in symmetric product orbifolds*, *Phys. Rev. D* **80** (2009) 086009 [[arXiv:0905.3451](#)] [[INSPIRE](#)].
- [24] C.A. Cardona and I. Kirsch, *Worldsheet four-point functions in AdS_3/CFT_2* , *JHEP* **01** (2011) 015 [[arXiv:1007.2720](#)] [[INSPIRE](#)].
- [25] I. Kirsch and T. Wirtz, *Worldsheet operator product expansions and p -point functions in AdS_3/CFT_2* , *JHEP* **10** (2011) 049 [[arXiv:1106.5876](#)] [[INSPIRE](#)].
- [26] M. Baggio, J. de Boer and K. Papadodimas, *A non-renormalization theorem for chiral primary 3-point functions*, *JHEP* **07** (2012) 137 [[arXiv:1203.1036](#)] [[INSPIRE](#)].
- [27] L. Rastelli, K. Roumpedakis and X. Zhou, *$AdS_3 \times S^3$ tree-level correlators: hidden six-dimensional conformal symmetry*, *JHEP* **10** (2019) 140 [[arXiv:1905.11983](#)] [[INSPIRE](#)].
- [28] A. Dei and L. Eberhardt, *Correlators of the symmetric product orbifold*, *JHEP* **01** (2020) 108 [[arXiv:1911.08485](#)] [[INSPIRE](#)].
- [29] L. Eberhardt, *AdS_3/CFT_2 at higher genus*, *JHEP* **05** (2020) 150 [[arXiv:2002.11729](#)] [[INSPIRE](#)].
- [30] Y. Hikida and T. Liu, *Correlation functions of symmetric orbifold from AdS_3 string theory*, *JHEP* **09** (2020) 157 [[arXiv:2005.12511](#)] [[INSPIRE](#)].
- [31] A. Dei, M.R. Gaberdiel, R. Gopakumar and B. Knighton, *Free field world-sheet correlators for AdS_3* , *JHEP* **02** (2021) 081 [[arXiv:2009.11306](#)] [[INSPIRE](#)].
- [32] B. Knighton, *Higher genus correlators for tensionless AdS_3 strings*, *JHEP* **04** (2021) 211 [[arXiv:2012.01445](#)] [[INSPIRE](#)].
- [33] L. Eberhardt, *Summing over geometries in string theory*, *JHEP* **05** (2021) 233 [[arXiv:2102.12355](#)] [[INSPIRE](#)].
- [34] A.A. Lima, G.M. Sotkov and M. Stanishkov, *On the dynamics of protected ramond ground states in the $D1-D5$ CFT*, *JHEP* **07** (2021) 120 [[arXiv:2103.04459](#)] [[INSPIRE](#)].
- [35] A. Dei and L. Eberhardt, *String correlators on AdS_3 : three-point functions*, *JHEP* **08** (2021) 025 [[arXiv:2105.12130](#)] [[INSPIRE](#)].
- [36] A. Dei and L. Eberhardt, *String correlators on AdS_3 : four-point functions*, *JHEP* **09** (2021) 209 [[arXiv:2107.01481](#)] [[INSPIRE](#)].
- [37] A. Alves Lima, G.M. Sotkov and M. Stanishkov, *Four-point functions with multi-cycle fields in symmetric orbifolds and the $D1-D5$ CFT*, *JHEP* **05** (2022) 106 [[arXiv:2202.12424](#)] [[INSPIRE](#)].
- [38] A. Dei and L. Eberhardt, *String correlators on AdS_3 : analytic structure and dual CFT*, *SciPost Phys.* **13** (2022) 053 [[arXiv:2203.13264](#)] [[INSPIRE](#)].
- [39] B. Knighton, *Classical geometry from the tensionless string*, *JHEP* **05** (2023) 005 [[arXiv:2207.01293](#)] [[INSPIRE](#)].

- [40] M.R. Gaberdiel and B. Nairz, *BPS correlators for AdS_3/CFT_2* , *JHEP* **09** (2022) 244 [[arXiv:2207.03956](#)] [[INSPIRE](#)].
- [41] S. Iguri, N. Kovensky and J.H. Toro, *Spectral flow and string correlators in $AdS_3 \times S^3 \times T^4$* , *JHEP* **01** (2023) 161 [[arXiv:2211.02521](#)] [[INSPIRE](#)].
- [42] S.K. Ashok and J. Troost, *The chiral ring of a symmetric orbifold and its large N limit*, *JHEP* **08** (2023) 004 [[arXiv:2303.09308](#)] [[INSPIRE](#)].
- [43] S. Iguri, N. Kovensky and J.H. Toro, *Spectral flow and the exact AdS_3/CFT_2 chiral ring*, *JHEP* **08** (2023) 034 [[arXiv:2304.08361](#)] [[INSPIRE](#)].
- [44] S.K. Ashok and J. Troost, *The operator rings of topological symmetric orbifolds and their large N limit*, *JHEP* **04** (2024) 039 [[arXiv:2309.17052](#)] [[INSPIRE](#)].
- [45] R. Dijkgraaf, *Instanton strings and hyperKähler geometry*, *Nucl. Phys. B* **543** (1999) 545 [[hep-th/9810210](#)] [[INSPIRE](#)].
- [46] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on AdS_3* , *Adv. Theor. Math. Phys.* **2** (1998) 733 [[hep-th/9806194](#)] [[INSPIRE](#)].
- [47] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux*, *JHEP* **03** (1999) 018 [[hep-th/9902098](#)] [[INSPIRE](#)].
- [48] J.R. David, G. Mandal and S.R. Wadia, *Microscopic formulation of black holes in string theory*, *Phys. Rept.* **369** (2002) 549 [[hep-th/0203048](#)] [[INSPIRE](#)].
- [49] O. Ohlsson Sax and B. Stefański, *Closed strings and moduli in AdS_3/CFT_2* , *JHEP* **05** (2018) 101 [[arXiv:1804.02023](#)] [[INSPIRE](#)].
- [50] L. Eberhardt and K. Ferreira, *Long strings and chiral primaries in the hybrid formalism*, *JHEP* **02** (2019) 098 [[arXiv:1810.08621](#)] [[INSPIRE](#)].
- [51] E.J. Martinec, S. Massai and D. Turton, *On the BPS sector in AdS_3/CFT_2 holography*, *Fortsch. Phys.* **71** (2023) 2300015 [[arXiv:2211.12476](#)] [[INSPIRE](#)].
- [52] R. Dijkgraaf, *Instanton strings and hyperKähler geometry*, *Nucl. Phys. B* **543** (1999) 545 [[hep-th/9810210](#)] [[INSPIRE](#)].
- [53] D. Lüst, E. Palti and C. Vafa, *AdS and the swampland*, *Phys. Lett. B* **797** (2019) 134867 [[arXiv:1906.05225](#)] [[INSPIRE](#)].
- [54] E. Perlmutter, L. Rastelli, C. Vafa and I. Valenzuela, *A CFT distance conjecture*, *JHEP* **10** (2021) 070 [[arXiv:2011.10040](#)] [[INSPIRE](#)].
- [55] F. Baume and J. Calderón Infante, *Tackling the SDC in AdS with CFTs*, *JHEP* **08** (2021) 057 [[arXiv:2011.03583](#)] [[INSPIRE](#)].
- [56] G. Shiu, F. Tonioni, V. Van Hemelryck and T. Van Riet, *AdS scale separation and the distance conjecture*, *JHEP* **05** (2023) 077 [[arXiv:2212.06169](#)] [[INSPIRE](#)].
- [57] J.M. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005 [[hep-th/9804085](#)] [[INSPIRE](#)].
- [58] W. Lerche, S. Stieberger and N.P. Warner, *Prepotentials from symmetric products*, *Adv. Theor. Math. Phys.* **3** (1999) 1613 [[hep-th/9901162](#)] [[INSPIRE](#)].
- [59] E. Palti, *The swampland: introduction and review*, *Fortsch. Phys.* **67** (2019) 1900037 [[arXiv:1903.06239](#)] [[INSPIRE](#)].

- [60] M. van Beest, J. Calderón-Infante, D. Mirfendereski and I. Valenzuela, *Lectures on the swampland program in string compactifications*, *Phys. Rept.* **989** (2022) 1 [[arXiv:2102.01111](#)] [[INSPIRE](#)].
- [61] N.B. Agmon, A. Bedroya, M.J. Kang and C. Vafa, *Lectures on the string landscape and the swampland*, [arXiv:2212.06187](#) [[INSPIRE](#)].
- [62] B. Sundborg, *Stringy gravity, interacting tensionless strings and massless higher spins*, *Nucl. Phys. B Proc. Suppl.* **102** (2001) 113 [[hep-th/0103247](#)] [[INSPIRE](#)].
- [63] K. Ferreira, M.R. Gaberdiel and J.I. Jottar, *Higher spins on AdS_3 from the worldsheet*, *JHEP* **07** (2017) 131 [[arXiv:1704.08667](#)] [[INSPIRE](#)].
- [64] M.R. Gaberdiel and R. Gopakumar, *Higher spins & strings*, *JHEP* **11** (2014) 044 [[arXiv:1406.6103](#)] [[INSPIRE](#)].
- [65] M.R. Gaberdiel and R. Gopakumar, *String theory as a higher spin theory*, *JHEP* **09** (2016) 085 [[arXiv:1512.07237](#)] [[INSPIRE](#)].
- [66] M. Baggio, M.R. Gaberdiel and C. Peng, *Higher spins in the symmetric orbifold of $K3$* , *Phys. Rev. D* **92** (2015) 026007 [[arXiv:1504.00926](#)] [[INSPIRE](#)].
- [67] N. Seiberg and E. Witten, *The $D1/D5$ system and singular CFT*, *JHEP* **04** (1999) 017 [[hep-th/9903224](#)] [[INSPIRE](#)].
- [68] M. Etheredge et al., *Sharpening the distance conjecture in diverse dimensions*, *JHEP* **12** (2022) 114 [[arXiv:2206.04063](#)] [[INSPIRE](#)].
- [69] F. Baume and J. Calderón-Infante, *On higher-spin points and infinite distances in conformal manifolds*, *JHEP* **12** (2023) 163 [[arXiv:2305.05693](#)] [[INSPIRE](#)].
- [70] S.G. Avery, B.D. Chowdhury and S.D. Mathur, *Deforming the $D1$ - $D5$ CFT away from the orbifold point*, *JHEP* **06** (2010) 031 [[arXiv:1002.3132](#)] [[INSPIRE](#)].
- [71] S.G. Avery, B.D. Chowdhury and S.D. Mathur, *Excitations in the deformed $D1$ - $D5$ CFT*, *JHEP* **06** (2010) 032 [[arXiv:1003.2746](#)] [[INSPIRE](#)].
- [72] B.A. Burrington, A.W. Peet and I.G. Zadeh, *Operator mixing for string states in the $D1$ - $D5$ CFT near the orbifold point*, *Phys. Rev. D* **87** (2013) 106001 [[arXiv:1211.6699](#)] [[INSPIRE](#)].
- [73] Z. Carson, S. Hampton, S.D. Mathur and D. Turton, *Effect of the twist operator in the $D1$ - $D5$ CFT*, *JHEP* **08** (2014) 064 [[arXiv:1405.0259](#)] [[INSPIRE](#)].
- [74] Z. Carson, S. Hampton, S.D. Mathur and D. Turton, *Effect of the deformation operator in the $D1$ - $D5$ CFT*, *JHEP* **01** (2015) 071 [[arXiv:1410.4543](#)] [[INSPIRE](#)].
- [75] M.R. Gaberdiel, C. Peng and I.G. Zadeh, *Higgsing the stringy higher spin symmetry*, *JHEP* **10** (2015) 101 [[arXiv:1506.02045](#)] [[INSPIRE](#)].
- [76] Z. Carson, S. Hampton and S.D. Mathur, *Second order effect of twist deformations in the $D1$ - $D5$ CFT*, *JHEP* **04** (2016) 115 [[arXiv:1511.04046](#)] [[INSPIRE](#)].
- [77] Z. Carson, S. Hampton and S.D. Mathur, *Full action of two deformation operators in the $D1$ - $D5$ CFT*, *JHEP* **11** (2017) 096 [[arXiv:1612.03886](#)] [[INSPIRE](#)].
- [78] B.A. Burrington, I.T. Jardine and A.W. Peet, *Operator mixing in deformed $D1$ - $D5$ CFT and the OPE on the cover*, *JHEP* **06** (2017) 149 [[arXiv:1703.04744](#)] [[INSPIRE](#)].
- [79] C.A. Keller and I.G. Zadeh, *Conformal perturbation theory for twisted fields*, *J. Phys. A* **53** (2020) 095401 [[arXiv:1907.08207](#)] [[INSPIRE](#)].

- [80] D. Bufalini, S. Iguri and N. Kovensky, *A proof for string three-point functions in AdS_3* , *JHEP* **02** (2023) 246 [[arXiv:2212.05877](#)] [[INSPIRE](#)].
- [81] B. Guo and S.D. Mathur, *Lifting of level-1 states in the D1-D5 CFT*, *JHEP* **03** (2020) 028 [[arXiv:1912.05567](#)] [[INSPIRE](#)].
- [82] A.A. Lima, G.M. Sotkov and M. Stanishkov, *Microstate renormalization in deformed D1-D5 SCFT*, *Phys. Lett. B* **808** (2020) 135630 [[arXiv:2005.06702](#)] [[INSPIRE](#)].
- [83] A.A. Lima, G.M. Sotkov and M. Stanishkov, *Correlation functions of composite Ramond fields in deformed D1-D5 orbifold SCFT₂*, *Phys. Rev. D* **102** (2020) 106004 [[arXiv:2006.16303](#)] [[INSPIRE](#)].
- [84] B. Guo and S.D. Mathur, *Lifting at higher levels in the D1-D5 CFT*, *JHEP* **11** (2020) 145 [[arXiv:2008.01274](#)] [[INSPIRE](#)].
- [85] A.A. Lima, G.M. Sotkov and M. Stanishkov, *Renormalization of twisted Ramond fields in D1-D5 SCFT₂*, *JHEP* **03** (2021) 202 [[arXiv:2010.00172](#)] [[INSPIRE](#)].
- [86] N. Benjamin, C.A. Keller and I.G. Zadeh, *Lifting 1/4-BPS states in $AdS_3 \times S^3 \times T^4$* , *JHEP* **10** (2021) 089 [[arXiv:2107.00655](#)] [[INSPIRE](#)].
- [87] L. Eberhardt, *A perturbative CFT dual for pure NS-NS AdS_3 strings*, *J. Phys. A* **55** (2022) 064001 [[arXiv:2110.07535](#)] [[INSPIRE](#)].
- [88] L. Apolo et al., *Deforming symmetric product orbifolds: a tale of moduli and higher spin currents*, *JHEP* **08** (2022) 159 [[arXiv:2204.07590](#)] [[INSPIRE](#)].
- [89] B. Guo, M.R.R. Hughes, S.D. Mathur and M. Mehta, *Universal lifting in the D1-D5 CFT*, *JHEP* **10** (2022) 148 [[arXiv:2208.07409](#)] [[INSPIRE](#)].
- [90] B. Guo and S. Hampton, *Bootstrapping the effect of the twist operator in the D1-D5 CFT*, *JHEP* **03** (2024) 030 [[arXiv:2210.07217](#)] [[INSPIRE](#)].
- [91] M.-A. Fiset, M.R. Gaberdiel, K. Naderi and V. Sriprachyakul, *Perturbing the symmetric orbifold from the worldsheet*, *JHEP* **07** (2023) 093 [[arXiv:2212.12342](#)] [[INSPIRE](#)].
- [92] L. Eberhardt and K. Ferreira, *The plane-wave spectrum from the worldsheet*, *JHEP* **10** (2018) 109 [[arXiv:1805.12155](#)] [[INSPIRE](#)].
- [93] N.M. McStay and R.A. Reid-Edwards, *Symmetries and covering maps for the minimal tension string on $AdS_3 \times S^3 \times T^4$* , *JHEP* **04** (2024) 048 [[arXiv:2306.16280](#)] [[INSPIRE](#)].
- [94] N. Berkovits and C. Vafa, *$N=4$ topological strings*, *Nucl. Phys. B* **433** (1995) 123 [[hep-th/9407190](#)] [[INSPIRE](#)].
- [95] Y. Sugawara, *Topological string on $AdS_3 \times N$* , *Nucl. Phys. B* **576** (2000) 265 [[hep-th/9909146](#)] [[INSPIRE](#)].
- [96] K. Costello and N.M. Paquette, *Twisted supergravity and Koszul duality: a case study in AdS_3* , *Commun. Math. Phys.* **384** (2021) 279 [[arXiv:2001.02177](#)] [[INSPIRE](#)].
- [97] S. Li and J. Troost, *The topological symmetric orbifold*, *JHEP* **10** (2020) 201 [[arXiv:2006.09346](#)] [[INSPIRE](#)].
- [98] S. Li and J. Troost, *Twisted string theory in anti-de Sitter space*, *JHEP* **11** (2020) 047 [[arXiv:2005.13817](#)] [[INSPIRE](#)].
- [99] G. Oberdieck, *Marked relative invariants and GW/PT correspondences*, *Adv. Math.* **439** (2024) 109472 [[arXiv:2112.11949](#)] [[INSPIRE](#)].

- [100] G. Oberdieck, *Gromov-Witten invariants of the Hilbert schemes of points of a K3 surface*, *Geometry & Topology* **22** (2017) 323 [[arXiv:1406.1139](#)].
- [101] G. Oberdieck, *Gromov-Witten theory of $K3 \times \mathbb{P}^1$ and quasi-Jacobi forms*, *Int. Math. Res. Not.* **2019** (2017) 4966 [[arXiv:1605.05238](#)].
- [102] G. Oberdieck and R. Pandharipande, *Curve counting on $K3 \times E$, the Igusa cusp form χ_{10} , and descendent integration*, [arXiv:1411.1514](#) [[INSPIRE](#)].
- [103] G. Oberdieck, *Holomorphic anomaly equations for the Hilbert scheme of points of a K3 surface*, *Geom. Topol.* **28** (2024) 3779 [[arXiv:2202.03361](#)] [[INSPIRE](#)].
- [104] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, *String theory on AdS_3* , *JHEP* **12** (1998) 026 [[hep-th/9812046](#)] [[INSPIRE](#)].
- [105] J.M. Maldacena and H. Ooguri, *Strings in AdS_3 and $SL(2, R)$ WZW model 1: the spectrum*, *J. Math. Phys.* **42** (2001) 2929 [[hep-th/0001053](#)] [[INSPIRE](#)].
- [106] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and completed cycles*, *Ann. Math.* **163** (2006) 517 [[math/0204305](#)] [[INSPIRE](#)].
- [107] J. Bryan and N.C. Leung, *The enumerative geometry of K3 surfaces and modular forms*, *J. Amer. Math. Soc.* **13** (1999) 371 [[alg-geom/9711031](#)].
- [108] D. Maulik and R. Pandharipande, *Gromov-Witten theory and Noether-Lefschetz theory*, [arXiv:0705.1653](#).
- [109] D. Maulik, R. Pandharipande, R.P. Thomas and A. Pixton, *Curves on K3 surfaces and modular forms*, *J. Topol.* **3** (2010) 937 [[arXiv:1001.2719](#)] [[INSPIRE](#)].
- [110] A. Klemm, D. Maulik, R. Pandharipande and E. Scheidegger, *Noether-Lefschetz theory and the Yau-Zaslow conjecture*, *J. Amer. Math. Soc.* **23** (2010) 1013 [[arXiv:0807.2477](#)].
- [111] R. Pandharipande and R.P. Thomas, *Notes on the proof of the KKV conjecture*, *Surveys Diff. Geom.* **21** (2016) 289 [[arXiv:1411.0896](#)] [[INSPIRE](#)].
- [112] E. D'Hoker et al., *Extremal correlators in the AdS/CFT correspondence*, in *The many faces of the superworld*, World Scientific, Singapore (2000), p. 332 [[DOI:10.1142/9789812793850_0020](#)] [[hep-th/9908160](#)] [[INSPIRE](#)].
- [113] R. Pandharipande and R.P. Thomas, *13/2 ways of counting curves*, *Lond. Math. Soc. Lect. Note Ser.* **411** (2014) 282 [[arXiv:1111.1552](#)] [[INSPIRE](#)].
- [114] J. Bryan and N.C. Leung, *The enumerative geometry of K3 surfaces and modular forms*, [alg-geom/9711031](#).
- [115] R. Dijkgraaf, *Fields, strings, matrices and symmetric products*, [hep-th/9912104](#) [[INSPIRE](#)].
- [116] T. Eguchi and A. Taormina, *Unitary representations of $N = 4$ superconformal algebra*, *Phys. Lett. B* **196** (1987) 75 [[INSPIRE](#)].
- [117] P.S. Aspinwall, *K3 surfaces and string duality*, in the proceedings of the *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96)*, (1996) [[hep-th/9611137](#)] [[INSPIRE](#)].
- [118] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys. B* **271** (1986) 93 [[INSPIRE](#)].
- [119] G. Oberdieck, *Multiple cover formulas for K3 geometries, wall-crossing, and Quot schemes*, *Geom. Topol.* **28** (2024) 3221 [[arXiv:2111.11239](#)] [[INSPIRE](#)].
- [120] D. Nesterov, *Quasimaps to moduli spaces of sheaves*, *Forum Math. Pi* **13** (2025) e12 [[arXiv:2111.11417](#)] [[INSPIRE](#)].

- [121] D. Nesterov, *Quasimaps to moduli spaces of sheaves on a surface*, *Forum Math. Sigma* **12** (2024) e61 [[arXiv:2111.11425](#)] [[INSPIRE](#)].
- [122] D. Nesterov, *Gromov-Witten/Hurwitz wall-crossing*, [arXiv:2208.00889](#).
- [123] D. Genlik and H.-H. Tseng, *On Hilb/Sym correspondence for K3 surfaces*, [arXiv:2304.06536](#).
- [124] G. Oberdieck, *Gromov-Witten theory and Noether-Lefschetz theory for holomorphic-symplectic varieties*, *Forum Math. Sigma* **10** (2022) e21 [[arXiv:2102.11622](#)] [[INSPIRE](#)].
- [125] D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, I*, *Compos. Math.* **142** (2006) 1263 [[math/0312059](#)] [[INSPIRE](#)].
- [126] D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, II*, *Compos. Math.* **142** (2006) 1286 [[math/0406092](#)] [[INSPIRE](#)].
- [127] K. Hori et al., *Mirror symmetry*, volume 1 of *Clay Mathematics Monographs*, AMS, Providence, RI, U.S.A. (2003).
- [128] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, *Surveys Diff. Geom.* **1** (1991) 243 [[INSPIRE](#)].
- [129] E.P. Verlinde and H.L. Verlinde, *A solution of two-dimensional topological quantum gravity*, *Nucl. Phys. B* **348** (1991) 457 [[INSPIRE](#)].
- [130] T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein, *On Hurwitz numbers and Hodge integrals*, *Comptes Rendus Acad. Sci. Ser. I Math.* **328** (1999) 1175 [[math/9902104](#)].
- [131] T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, *Invent. Math.* **146** (2001) 297 [[math/0004096](#)] [[INSPIRE](#)].
- [132] S.H. Katz, A. Klemm and C. Vafa, *M theory, topological strings and spinning black holes*, *Adv. Theor. Math. Phys.* **3** (1999) 1445 [[hep-th/9910181](#)] [[INSPIRE](#)].
- [133] B. Haghighat, S. Murthy, C. Vafa and S. Vandoren, *F-theory, spinning black holes and multi-string branches*, *JHEP* **01** (2016) 009 [[arXiv:1509.00455](#)] [[INSPIRE](#)].
- [134] S.-T. Yau and E. Zaslow, *BPS states, string duality, and nodal curves on K3*, *Nucl. Phys. B* **471** (1996) 503 [[hep-th/9512121](#)] [[INSPIRE](#)].
- [135] R. Gopakumar and C. Vafa, *M theory and topological strings. 1*, [hep-th/9809187](#) [[INSPIRE](#)].
- [136] R. Gopakumar and C. Vafa, *M theory and topological strings. 2*, [hep-th/9812127](#) [[INSPIRE](#)].
- [137] W. Lerche, S. Stieberger and N.P. Warner, *Quartic gauge couplings from K3 geometry*, *Adv. Theor. Math. Phys.* **3** (1999) 1575 [[hep-th/9811228](#)] [[INSPIRE](#)].
- [138] W. Lerche, S. Stieberger and N.P. Warner, *Prepotentials from symmetric products*, *Adv. Theor. Math. Phys.* **3** (1999) 1613 [[hep-th/9901162](#)] [[INSPIRE](#)].
- [139] S.K. Lando, *Hurwitz numbers: on the edge between combinatorics and geometry*, in *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010)*, <https://www.hse.ru/data/2011/03/19/1211260938/ICMHurwitz10.pdf>, (2010), p. 2444
- [140] R. Pandharipande, *The Toda equations and the Gromov-Witten theory of the Riemann sphere*, *Lett. Math. Phys.* **53** (2000) 59 [[math/9912166](#)] [[INSPIRE](#)].
- [141] R. Dijkgraaf, H.L. Verlinde and E.P. Verlinde, *Notes on topological string theory and 2D quantum gravity*, in the proceedings of the *Cargese study institute: random surfaces, quantum gravity and strings*, (1990) [[INSPIRE](#)].

- [142] I. Goulden, D. Jackson and R. Vakil, *Towards the geometry of double Hurwitz numbers*, [math/0309440](#).
- [143] P. Dunin-Barkowski, R. Kramer, A. Popolitov and S. Shadrin, *Loop equations and a proof of Zvonkine's qr -ELSV formula*, *Ann. Sci. École Normale Supérieure* (2023) [[arXiv:1905.04524](#)] [[INSPIRE](#)].
- [144] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, *Invent. Math.* **139** (2000) 173 [[math/9810173](#)] [[INSPIRE](#)].
- [145] M.-X. Huang, S. Katz and A. Klemm, *Topological string on elliptic CY 3-folds and the ring of Jacobi forms*, *JHEP* **10** (2015) 125 [[arXiv:1501.04891](#)] [[INSPIRE](#)].
- [146] M. Marino, *Spectral theory and mirror symmetry*, *Proc. Symp. Pure Math.* **98** (2018) 259 [[arXiv:1506.07757](#)] [[INSPIRE](#)].
- [147] M. Alim, A. Saha, J. Teschner and I. Tulli, *Mathematical structures of non-perturbative topological string theory: from GW to DT invariants*, *Commun. Math. Phys.* **399** (2023) 1039 [[arXiv:2109.06878](#)] [[INSPIRE](#)].
- [148] J. Gu, A.-K. Kashani-Poor, A. Klemm and M. Marino, *Non-perturbative topological string theory on compact Calabi-Yau 3-folds*, *SciPost Phys.* **16** (2024) 079 [[arXiv:2305.19916](#)] [[INSPIRE](#)].
- [149] R. Pandharipande and A. Pixton, *Descendent theory for stable pairs on toric 3-folds*, *J. Math. Soc. Jpn.* **65** (2013) 1337 [[arXiv:1011.4054](#)].