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Product-Type Classes for Vertex Algebra Cohomology of Foliations on Complex Curves

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Abstract: We introduce the vertex algebra cohomology of foliations on complex curves. Generalizing the classical case, the orthogonality condition with respect to a product of elements of the double complexes associated to a grading-restricted vertex algebra matrix elements leads to the construction of cohomology invariants of codimension one foliations.

1. Introduction

The theory of foliations involves a variety of approaches reflected in [1–3,10–13,15,27–30,51,54,58] and in many other publications. The cohomological techniques applied to smooth manifolds are represented both by algebraic [32–36] and geometrical [18,31,46,48,52,54,62–64,69] approaches to characterization of the space of leaves of foliations. The theory of vertex algebras [7,7,9,17,19,24,25,47,55,61] is a rapidly developing field of studies. Algebraic nature of methods applied in this field is a powerful tool to compute correlation functions in the conformal field theory [4–6,16,21,23,26,49,50,67,68]. On the other hand, the geometrical side of the vertex algebra correlation function theory is related to the behavior of vertex operator formal parameters associated to local coordinates on complex manifolds.

In this paper we develop algebraic and functional-analytic methods of the cohomology theory of foliations on complex curves. The algebraic part is based on the cohomology theory of grading-restricted vertex algebras [43]. The analytic part stems from the theory of vertex algebra correlation functions on complex curves [7] as well as from the geometrical constructions of sewing of Riemann surfaces [8,20,77]. As a result of a combination of the above mentioned techniques we are able to introduce the chain-cochain double complexes associated to the description of foliations in terms of families of transversal sections. Similarly to the conformal field theory considerations [7,23,45,70,78], the algebraic structure of vertex algebra matrix elements leads to a characterization of the space of leaves of a foliation in terms of rational functions with specific properties. The geometrical origin of vertex algebra matrix elements allows us to define a product of elements of double complexes. Properties of such product, in particular, the orthogonality condition, turn to be useful in computation of cohomology invariants for codimension one foliations on complex curves [60,71–74]. In particular, we are able to determine the vertex algebra counterparts generalizing the classical cohomology classes [31] and invariants of codimension one foliations.

The main result of the paper consists in the construction of the vertex algebra cohomology of codimension one foliations of complex curves. In contrast to the classical Lie-algebraic approach [27,28] we use vertex algebras as a structure generalizing Lie algebras. This allows us to involve deep algebraic properties of vertex algebras to establish new and finer cohomology invariants with respect to Čech-de Rham cohomology of foliations [15]. Let V be a grading-restricted vertex algebra, and W be its grading-restricted generalized module. For the algebraic completion $\overline{W}$ of W we introduce in Sects. 4, 5 the chain-cochain double complexes $C^n_m(V, \mathcal{W}, \mathcal{F})$, $n, m \geq 0$, (5.18) and $C^k_{ex}(V, \mathcal{W}, \mathcal{F})$, $0 \leq k \leq 2$, (5.19) associated to a codimension one foliation $\mathcal{F}$ on a complex curve with the coboundary operators (5.7), (5.11). Here $\mathcal{W}$ denotes the space of $\overline{W}$-valued differential forms with specific properties. The orthogonality condition $F_1 \cdot \delta^n_m F_2 = 0$ (8.4) for elements F_1, F_2 of the double complex spaces is defined with respect to the product (8.1). Let $F \in C^n_m(V, \mathcal{W}, \mathcal{F})$. The main statement of this paper consists in the following Theorem proven in Sect. 8 and generalizing classical results of [31] on codimension one foliation invariants:

Theorem 1. *The product* (8.1)*, the coboundary operators* (5.7), (5.11)*, and the orthogonality condition* (8.4) *applied to the double complexes* (5.18) *and* (5.19) *generate non-vanishing cohomology classes* $\left[\left(\delta^n_m F\right) \cdot F\right]$ *independent on the choice of* $F \in C^n_m(V, \mathcal{W}, \mathcal{F})$ *for pairs* $(n, m) = (1, 2)$, $(0, 3)$, $(1, t)$, $0 \leq t \leq 2$.

The content of this paper is subject to multiple possible generalizations. There exist a few approaches to definition and computation of cohomology of vertex algebras [14,22,40,41,43,44,57,76,80]. The most natural direction to generalize results of this paper is to develop a vertex algebra characteristic classes theory for regular and singular foliations of arbitrary codimensions. It would be important to enlarge the theory presented in this paper to find higher non-vanishing invariants. Such invariants would allow to distinguish [1,2,29] types of compact and non-compact leaves of foliations. It worths to mention a possibility to derive differential equations [42,75] for vertex algebra correlation functions considered on leaves of foliations. One would be interested in combining the techniques of [58] with our approach. In order to apply the same methods as for cohomology of a manifold $\mathcal{M}$, a smooth structure on the space of leaves $M/\mathcal{F}$ of a foliation $\mathcal{F}$ of codimension n on $\mathcal{M}$ was introduced in [58]. In that case, the foliation characteristic classes become elements of the cohomology of certain bundles over the space of leaves $M/\mathcal{F}$. It would be interesting to develop also intrinsic (i.e., purely coordinate independent) foliation cohomology of smooth manifolds which would involve vertex algebra bundles [7]. The idea of an auxiliary bundle construction to compute the cohomology of foliations establishes relations with the classical Bott-Segal approach [13]. It is important to determine connections to the chiral de-Rham complex on smooth manifolds developed in [59]. The structure of foliations can be also studied from the automorphic function theory point of view originated from vertex algebra correlation functions [47]. A consideration of the cohomology theory of vertex algebra associated bundles [79] and arbitrary codimension foliations on smooth manifolds will be given elsewhere.

The plan of the paper is the following. Section 2 contains a description of the transversal basis for foliations and the definition of the space $\mathcal{W}_{z_1,\ldots,z_n}$ of $\overline{W}_{z_1,\ldots,z_n}$-valued differential forms. In Sect. 2.3 the definition and properties of maps composable with a number of vertex operators are given. In Sect. 2.4 we provide a vertex algebra interpretation of the local geometry for foliations on smooth manifolds. In Sect. 3 we introduce a product of elements of $\mathcal{W}_{z_1,\ldots,z_n}$-spaces and study its properties. Section 3.1 contains motivations, a geometrical interpretation, and the definition of the product of elements of spaces of differential forms. First we prove that the product maps to another $\mathcal{W}$-space. In Sect. 3.2 the absolute convergence of the result of the product is shown. We then define the action of the symmetric group on the product, and prove that the product satisfies the symmetry property (2.9). Next, the action of partial derivatives on the product with respect to formal parameters is defined. We then show that the product satisfies $L_V(-1)$-derivative and $L_V(0)$-conjugation properties. In Sect. 3.3 we continue to study properties of the product. It is shown that the product is invariant with respect to the group of independent transformations of coordinates. We show also that the product does not depend on a distribution of formal parameters among two initial forms that are being multiplied. The spaces for a double chain-cochain complex associated to a vertex algebra on a foliation are introduced in Sect. 4. We prove that such spaces are well-defined. Namely, the spaces are non-empty, do not depend on the choice of the transversal basis for a foliation, and are canonical, i.e., are independent on the foliation-preserving choice of local coordinates. In Sect. 5 the coboundary operators and the vertex algebra cohomology of codimension one foliations on complex curves are defined. In Sect. 5.1 the cohomology in terms of multi-point connections in described. Section 5.2 introduces the coboundary operators for the double complex spaces. We show that the coboundary operators are expressed in terms of multipoint connections. In Sect. 5.3 we prove that the coboundary operators provide double chain-cochain complexes. The vertex algebra cohomology and its relation to Čech-de Rham cohomology in the Crainic and Moerdijk formulation [15] are discussed in Sect. 5.4. The product of elements of double complex spaces is defined in Sect. 6. In Sect. 6.1 the geometrical adaptation of the this product to a foliation is discussed. We show that the product of elements of double complex spaces maps to another space of the double complex and it is composable with the appropriate number of vertex operators. The properties of the product are studied in Sect. 7. In Sect. 7.1 we show that the original coboundary operators apply to the product of elements of the double complex spaces. It satisfied to an analog of Leibniz rule. Section 8 contains the proof of the main result of this paper. It describes the product-type cohomological invariants for a codimension one foliation on a smooth complex curve. In "Appendixes" we provide the material required for the construction of the vertex algebra cohomology of foliations. In "Appendix 8.2" we recall the notion of a quasi-conformal grading-restricted vertex algebra and its modules. In "Appendix .3" properties of matrix elements for the space $\mathcal{W}$ are listed.

2. Transversal Basis Description for Foliations and Vertex Algebra Interpretation and $\overline{W}_{z_1,\ldots,z_n}$-Valued Forms

In this Section we recall [15] the notion of a basis of transversal sections for foliations, and provide its vertex algebra setup.

2.1. The basis of transversal sections for a foliation. Let $\mathcal{M}$ be a complex curve equipped with a foliation $\mathcal{F}$ of codimension one.

Definition 1. A transversal section of a foliation $\mathcal{F}$ is an embedded one-dimensional submanifold $U \subset M$ which is everywhere transverse to the leaves of $\mathcal{F}$.

Definition 2. If α is a path between two points p_1 and p_2 on the same leaf of $\mathcal{F}$, and U_1 and U_2 are transversal sections through p_1 and p_2, then α defines a transport along the leaves from a neighborhood of p_1 in U_1 to a neighborhood of p_2 in U_2. I.e., it gives a germ of a diffeomorphism

$$hol\ (\alpha) : (U_1, p_1) \hookrightarrow (U_2, p_2),$$

which is called the holonomy of the path α.

Two homotopic paths always define the same holonomy.

Definition 3. If the above transport along α is defined in all of U_1 and embeds U_1 into U_2, this embedding

$$h : U_1 \hookrightarrow U_2,$$

is called the holonomy embedding.

A composition of paths induces a composition of holonomy embeddings. Transversal sections U through p as above should be thought of as neighborhoods of the leaf through p in the space of leaves. Then we have

Definition 4. A transversal basis for the space of leaves $\mathcal{M}/\mathcal{F}$ of a foliation $\mathcal{F}$ is a family $\mathcal{U}$ of transversal sections $U \subset \mathcal{M}$ with the following property. If U_p is any transversal section through a given point $p \in \mathcal{M}$, then there exists a holonomy embedding

$$h : U \hookrightarrow U_p,$$

with $U \in \mathcal{U}$ and $p \in h(U)$.

A transversal section is a one-dimensional disk given by a chart of $\mathcal{F}$. Accordingly, we can construct a transversal basis $\mathcal{U}$ out of a basis $\widetilde{\mathcal{U}}$ of $\mathcal{M}$ by domains of foliation charts

$$\phi_U : \widetilde{U} \tilde{\hookrightarrow} \mathbb{R} \times U,$$

$\widetilde{U} \in \widetilde{\mathcal{U}}$, with $U = \mathbb{R}$.

In the next two Subsections we provide several definitions and properties from [43].

2.2. The space $\overline{W}_{z_1,\dots,z_n}$ of $\overline{W}$-valued rational functions. First, let us recall the notion of shuffles.

Definition 5. Let S_q be the permutation group. For $l \in \mathbb{N}$ and $1 \leq s \leq l-1$, let $J_{l;s}$ be the set of elements of S_l which preserves the order of the first s numbers and the order of the last $l-s$ numbers, that is,

$$J_{l,s} = \{\sigma \in S_l \mid \sigma(1) < \cdots < \sigma(s),\ \sigma(s+1) < \cdots < \sigma(l)\}.$$

The elements of $J_{l;s}$ are called shuffles, and we use the notation

$$J_{l;s}^{-1} = \{\sigma \mid \sigma \in J_{l;s}\}.$$

Definition 6. We define the configuration spaces:

$$F_n\mathbb{C} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\},$$

for $n \in \mathbb{Z}_+$.

Recall the definition and related notations (given in "Appendix 8.2") of a grading-restricted vertex algebra V, and its grading-restricted generalized V-module W. By $\overline{W}$ we denote the algebraic completion of W,

$$\overline{W} = \prod_{n\in\mathbb{C}} W_{(n)} = (W')^*.$$

Definition 7. A $\overline{W}$-valued rational function in $(z_1, \dots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$, is a map

$$f : F_n\mathbb{C} \to \overline{W},$$
$$(z_1, \dots, z_n) \mapsto f(z_1, \dots, z_n),$$

such that for any $w' \in W'$, the bilinear pairing $\langle w', f(z_1, \dots, z_n)\rangle$ is a rational function $R(f(z_1, \dots, z_n))$ in $(z_1, \dots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$. The space of $\overline{W}$-valued rational functions is denoted by $\overline{W}_{z_1,\dots,z_n}$.

Remark 1. Note that though such functions are called $\overline{W}_{z_1,\dots,z_n}$-valued, corresponding element f of the algebraic completion $\overline{W}$ is inserted into the complex-valued bilinear pairing. Thus, $\overline{W}$-valued rational functions are characterized by this pairing.

Definition 8. One defines the following action of the symmetric group S_n on the space $\mathrm{Hom}(V^{\otimes n}, \overline{W}_{z_1,\dots,z_n})$ of linear maps from $V^{\otimes n}$ to $\overline{W}_{z_1,\dots,z_n}$ by

$$\sigma(\Phi)(v_1, z_1; \dots; v_n, z_n) = \Phi(v_{\sigma(1)}, v_{\sigma(1)}; \dots v_{\sigma(n)}, z_{\sigma(n)}), \quad (2.1)$$

for $\sigma \in S_n$, and $v_i \in V$, $1 \leq i \leq n$.

We will use the notation $\sigma_{i_1,\dots,i_n} \in S_n$, to denote the permutation given by $\sigma_{i_1,\dots,i_n}(j) = i_j$, for $j = 1, \dots, n$.

Definition 9. For $n \in \mathbb{Z}_+$, a linear map

$$\Phi(v_1, z_1; \dots; v_n, z_n) = V^{\otimes n} \to \mathcal{W}_{z_1,\dots,z_n},$$

is said to have the $L(-1)$-derivative property if

$$\text{(i)}\quad \langle w', \partial_{z_i}\Phi(v_1, z_1; \dots; v_n, z_n)\rangle = \langle w', \Phi(v_1, z_1; \dots; L_V(-1)v_i, z_i; \dots; v_n, z_n)\rangle, \quad (2.2)$$

and

$$\text{(ii)}\quad \sum_{i=1}^{n} \partial_{z_i}\langle w', \Phi(v_1, z_1; \dots; v_n, z_n)\rangle = \langle w', L_W(-1).\Psi(v_1, z_1; \dots; v_n, z_n)\rangle, \quad (2.3)$$

for $i = 1, \dots, n$, $v_1, \dots, v_n \in V$, $w' \in W$.

Note that since $L_W(-1)$ is a weight-one operator on W, for any $z \in \mathbb{C}$, $e^{zL_W(-1)}$ is a well-defined linear operator on $\overline{W}$.

Proposition 1. *Let Φ be a linear map having the $L(-1)$-derivative property. Then for $v_1, \ldots, v_n \in V$, $w' \in W'$, $(z_1, \ldots, z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$ such that $(z_1 + z, \ldots, z_n + z) \in F_n\mathbb{C}$,*

$$\langle w', e^{zL_W(-1)} \Phi\left(v_1, z_1; \ldots; v_n, z_n\right)\rangle = \langle w', \Phi(v_1, z_1 + z; \ldots; v_n, z_n + z)\rangle, \quad (2.4)$$

and for $v_1, \ldots, v_n \in V$, $w' \in W'$, $(z_1, \ldots, z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$, and $1 \le i \le n$ such that

$$(z_1, \ldots, z_{i-1}, z_i + z, z_{i+1}, \ldots, z_n) \in F_n\mathbb{C},$$

the power series expansion of

$$\langle w', \Phi(v_1, z_1; \ldots; v_{i-1}, z_{i-1}; v_i, z_i + z; v_{v+1}, z_{i+1}; \ldots v_n, z_n)\rangle, \quad (2.5)$$

in z is equal to the power series

$$\langle w', \Phi(v_1, z_1; \ldots; v_{i-1}, z_{i-1}; e^{zL(-1)} v_i, z_i; v_{i+1}, z_{i+1}; \ldots; v_n, z_n)\rangle, \quad (2.6)$$

in z. In particular, the power series (2.6) in z is absolutely convergent to (2.5) in the disk $|z| < \min_{i \neq j}\{|z_i - z_j|\}$.

Next, we have

Definition 10. A linear map

$$\Phi : V^{\otimes n} \to \mathcal{W}_{z_1,\ldots,z_n}$$

has the $L(0)$-conjugation property if for $v_1, \ldots, v_n \in V$, $w' \in W'$, $(z_1, \ldots, z_n) \in F_n\mathbb{C}$ and $z \in \mathbb{C}^\times$ so that $(zz_1, \ldots, zz_n) \in F_n\mathbb{C}$,

$$\langle w', z^{L_W(0)} \Phi\left(v_1, z_1; \ldots; v_n, z_n\right)\rangle = \langle w', \Phi(z^{L(0)} v_1, zz_1; \ldots; z^{L(0)} v_n, zz_n)\rangle. \quad (2.7)$$

In order to introduce the spaces for the double complexes describing the vertex algebra cohomology of foliations on complex curves, we have to define the space of $\overline{W}_{z_1,\ldots,z_n}$-valued differential forms for a quasi-conformal grading-restricted vertex algebra V. Recall the notion of the weight $\mathrm{wt}(v)$ of a vertex algebra element v with respect to the Virasoro algebra $L_V(0)$-mode given in "Appendix 8.2". Following ideas of [7], we consider the space of $\overline{W}_{z_1,\ldots,z_n}$ of functions Φ where each vertex algebra element entry v_i, $1 \le i \le n$ is tensored with the $\mathrm{wt}(v_i)$-power differential $dz_i^{\mathrm{wt}(v_i)}$ of corresponding formal parameter z_i. Namely, we consider the space of forms

$$\Phi\left(dz_1^{\mathrm{wt}(v_1)} \otimes v_1, z_1; \ldots; dz_n^{\mathrm{wt}(v_n)} \otimes v_n, z_n\right). \quad (2.8)$$

Abusing notations, we denote such forms as $\Phi(v_1, z_1; \ldots; v_n, z_n)$ in what follows.

Definition 11. We define the space $\mathcal{W}_{z_1,\ldots,z_n}$ of forms (2.8) satisfying $L_V(-1)$-derivative (2.2), $L_V(0)$-conjugation (2.7) properties, and the symmetry property

$$\sum_{\sigma \in J_{l;s}^{-1}} (-1)^{|\sigma|} \left(\Phi(v_{\sigma(1)}, z_{\sigma(1)}; \ldots; v_{\sigma(l)}, z_{\sigma(1)})\right) = 0, \quad (2.9)$$

with respect to the action of the symmetric group S_n.

In Sect. 4 we prove that (2.8) is invariant with respect to the action of the group of independent changes of formal parameters $(z_1, \ldots, z_n)$.

2.3. Maps composable with vertex operators. In the construction of the double complexes in Sect. 5 we will use linear maps from tensor powers of V to the space $\mathcal{W}_{z_1,\ldots,z_n}$. For that purpose, in particular, to define the coboundary operator, we have to compose cochains with vertex operators. However, as mentioned in [43], the images of vertex operator maps do not in general belong to algebras or their modules but rather to corresponding algebraic completions. Due to this reason, we might not be able to compose vertex operators directly. In order to overcome this problem [45], we consider series obtained by projecting elements of the algebraic completion of an algebra or a module to their homogeneous components. Then we compose these homogeneous components with vertex operators and take formal sums. If such formal sums are absolutely convergent, then these operators can be composed and used in constructions.

Another problem that appears is the question of associativity. Compositions of maps are usually associative. But for compositions of maps defined by sums of absolutely convergent series the existence does not provide associativity in general. Nevertheless, the requirement of analyticity provides the associativity. Recall definitions and notations of "Appendix .3". Then we have

Definition 12. For a gerneralized grading-restricted V-module

$$W = \coprod_{n\in\mathbb{C}} W_{(n)},$$

and $m \in \mathbb{C}$, let

$$P_m : \overline{W} \to W_{(m)},$$

be the projection from $\overline{W}$ to $W_{(m)}$. Let

$$\Phi : V^{\otimes n} \to \mathcal{W}_{z_1,\ldots,z_n},$$

be a linear map. For $m \in \mathbb{N}$, Φ is called [43,65] to be composable with m vertex operators if the following conditions are satisfied:

1) Let $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1+\cdots+l_n = m+n$, $v_1, \ldots, v_{m+n} \in V$ and $w' \in W'$. Set

$$\Xi_i = E_V^{(l_i)}(v_{k_1}, z_{k_1} - \zeta_i; v_{k_i}, z_{k_i} - \zeta_i; \mathbf{1}_V), \tag{2.10}$$

where

$$k_1 = l_1 + \cdots + l_{i-1} + 1, \quad \ldots, \quad k_i = l_1 + \cdots + l_{i-1} + l_i, \tag{2.11}$$

for $i = 1, \ldots, n$. Then there exist positive integers $N_m^n(v_i, v_j)$ depending only on v_i and v_j for $i, j = 1, \ldots, k$, $i \neq j$ such that the series

$$\mathcal{I}_m^n(\Phi) = \sum_{r_1,\ldots,r_n\in\mathbb{Z}} \langle w', \Phi(P_{r_1}\Xi_1; \zeta_1; \ldots; P_{r_n}\Xi_n, \zeta_n)\rangle, \tag{2.12}$$

is absolutely convergent when

$$|z_{l_1+\cdots+l_{i-1}+p} - \zeta_i| + |z_{l_1+\cdots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|, \tag{2.13}$$

for $i, j = 1, \ldots, k$, $i \neq j$, and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$. The sum must be analytically extended to a rational function in $(z_1, \ldots, z_{m+n})$, independent of $(\zeta_1, \ldots, \zeta_n)$, with the only possible poles at $z_i = z_j$, of order less than or equal to $N^n_m(v_i, v_j)$, for i, $j = 1, \ldots, k$, $i \neq j$.

2) For $v_1, \ldots, v_{m+n} \in V$, there exist positive integers $N^n_m(v_i, v_j)$, depending only on v_i and v_j, for i, $j = 1, \ldots, k$, $i \neq j$, such that for $w' \in W'$, and

$$\mathbf{v}_{n,m} = (v_{1+m} \otimes \cdots \otimes v_{n+m}),$$
$$\mathbf{z}_{n,m} = (z_{1+m}, \ldots, z_{n+m}),$$

such that

$$\mathcal{J}^n_m(\Phi) = \sum_{q \in \mathbb{C}} \langle w', E^{(m)}_W \Big(v_1 \otimes \cdots \otimes v_m;\, P_q(\Phi(\mathbf{v}_{n,m})(\mathbf{z}_{n,m})\Big) \rangle, \tag{2.14}$$

is absolutely convergent when

$$z_i \neq z_j, \quad i \neq j,$$
$$|z_i| > |z_k| > 0, \tag{2.15}$$

for $i = 1, \ldots, m$, and $k = m+1, \ldots, m+n$, and the sum can be analytically extended to a rational function in $(z_1, \ldots, z_{m+n})$ with the only possible poles at $z_i = z_j$, of orders less than or equal to $N^n_m(v_i, v_j)$, for i, $j = 1, \ldots, k$, $i \neq j$.

The following useful propositions were proven in [43]:

Proposition 2. *Let $\Phi : V^{\otimes n} \to \mathcal{W}_{z_1,\ldots,z_n}$ be composable with m vertex operators. Then we have:*

(1) *For $p \leq m$, Φ is composable with p vertex operators and for $p, q \in \mathbb{Z}_+$ such that $p+q \leq m$ and $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1+\cdots+l_n = p+n$, $(E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}}).\Phi$ and $E^{(p)}_W.\Phi$ are composable with q vertex operators.*

(2) *For $p, q \in \mathbb{Z}_+$ such that $p+q \leq m$, $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = p+n$ and $k_1, \ldots, k_{p+n} \in \mathbb{Z}_+$ such that $k_1 + \cdots + k_{p+n} = q+p+n$, we have*

$$(E^{(k_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(k_{p+n})}_{V;\,\mathbf{1}}).(E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}})).\Phi$$
$$= (E^{(k_1+\cdots+k_{l_1})}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(k_{l_1+\cdots+l_{n-1}+1}+\cdots+k_{p+n})}_{V;\,\mathbf{1}}).\Phi.$$

(3) *For p, $q \in \mathbb{Z}_+$ such that $p+q \leq m$ and $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1+\cdots+l_n = p+n$, we have*

$$E^{(q)}_W.((E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}}).\Phi) = (E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}}).(E^{(q)}_W.\Phi).$$

(4) *For p, $q \in \mathbb{Z}_+$ such that $p+q \leq m$, we have*

$$E^{(p)}_W.(E^{(q)}_W.\Phi) = E^{(p+q)}_W.\Phi.$$

Proposition 3. *The subspace of linear maps* $\mathrm{Hom}(V^{\otimes n}, \mathcal{W}_{z_1,\ldots,z_n})$ *possessing the $L(-1)$-derivative and the $L(0)$-conjugation properties or being composable with m vertex operators is invariant under the action of S_n.*

2.4. Vertex algebra interpretation of the local geometry for a foliation on a smooth manifold. Let $\mathcal{U}$ be a basis of transversal sections of $\mathcal{F}$. We consider a (n,k)-set of points, $n \geq 1$, $k \geq 1$,

$$\left(p_1, \ldots, p_n; p'_1, \ldots, p'_k\right), \tag{2.16}$$

on a smooth complex curve $\mathcal{M}$. Let us denote the set of corresponding local coordinates by

$$\left(c_1(p_1), \ldots, c_n(p_n); c'_1(p'_1), \ldots, c'_k(p'_k)\right).$$

In what follows we consider points (2.16) as points on either the space of leaves $\mathcal{M}/\mathcal{F}$ of $\mathcal{F}$, or on transversal sections U_j of a transversal basis $\mathcal{U}$. Since the space of leaves $\mathcal{M}/\mathcal{F}$ for $\mathcal{F}$ is not in general a manifold, one has to be careful in considerations of charts of local coordinates along leaves of $\mathcal{F}$ [46,58]. In order to associate formal parameters of vertex operators taken at points on $\mathcal{M}/\mathcal{F}$ with local coordinates we will use either local coordinates on $\mathcal{M}$ or local coordinates on sections U of a transversal basis $\mathcal{U}$ which are submanifolds of $\mathcal{M}$ of dimension equal to the codimension of $\mathcal{F}$. Note that the complexes considered below are constructed in such a way that one can always use coordinates on transversal sections only, avoiding any possible problems with localization of coordinates on leaves of $\mathcal{M}/\mathcal{F}$.

For the first n grading-restricted vertex algebra V elements of

$$\left(v_1, \ldots, v_n; v'_1, \ldots, v'_k\right), \tag{2.17}$$

we consider the linear maps

$$\Phi : V^{\otimes n} \rightarrow \mathcal{W}_{c_1(p_1), \ldots, c_n(p_n)}, \tag{2.18}$$

$$\Phi\left(dc_1(p_1)^{\mathrm{wt}(v_1)} \otimes v_1, c_1(p_1); \ldots; dc_n(p_n)^{\mathrm{wt}(v_n)} \otimes v_n, c_n(p_n)\right), \tag{2.19}$$

where we identify (as they usually do in the theory of correlation functions for vertex algebras on curves [45,70,77,78]) formal parameters $(z_1, \ldots, z_n)$ of $\mathcal{W}_{z_1,\ldots,z_n}$, with local coordinates $(c_1(p_1), \ldots, c_n(p_n))$ in vicinities of points p_i, $0 \leq i \leq n$, on $\mathcal{M}$.

Elements $\Phi \in \mathcal{W}_{c_1(p_1),\ldots,c_n(p_n)}$ can be seen as coordinate-independent $\overline{W}$-valued rational sections of a vertex algebra bundle. Note that, according to [7], they can be treated as $\left(\mathrm{Aut}\ \mathcal{O}^{(1)}\right)^{\times n}_{c_1(p_1),\ldots,c_n(p_n)} = \mathrm{Aut}_{c_1(p_1)}\ \mathcal{O}^{(1)} \times \ldots \times \mathrm{Aut}_{c_n(p_n)}\ \mathcal{O}^{(1)}$-torsors of the groups of independent coordinate transformations.

In what follows, according to the definitions of Sect. 2.2, when we write an element Φ of the space $\mathcal{W}_{z_1,\ldots,z_n}$, we actually have in mind corresponding matrix element $\langle w', \Phi\rangle$ that absolutely converges (on a certain domain) to a rational function

$$\langle w', \Phi\rangle = R(\langle w', \Phi\rangle). \tag{2.20}$$

In notations, we will keep tensor products of vertex algebra elements with weight-valued powers of differentials when it is only necessary.

In Sect. 4 we prove, that for arbitrary sets of vertex algebra elements v_i, $v'_j \in V$, $1 \leq i \leq n$, $1 \leq j \leq k$, arbitrary sets of points p_i with local coordinates $c_i(p_i)$ on $\mathcal{M}$, and arbitrary sets of points p'_j with local coordinates $c'_j(p'_j)$ on transversal sections $U_j \in \mathcal{U}$ of $\mathcal{M}/\mathcal{F}$, the element (2.19) as well as the vertex operators

$$\omega_W\left(dc'_j(p'_j)^{\mathrm{wt}(v'_j)} \otimes v'_j, c'_j(p'_j)\right) = Y_W\left(d(c'_j(p'_j))^{\mathrm{wt}(v'_j)} \otimes v'_j, c'_j(p'_j)\right), \tag{2.21}$$

are invariant with respect to the action of the group of independent transformations of coordinates. Then the construction of the spaces for the double complexes does not depend on the choice of coordinates.

In Sect. 4 we construct the spaces for the double complexes associated to a grading-restricted vertex algebra and defined for codimension one foliations on complex curves. In that construction, we consider sections U_j, $j \geq 0$ of a transversal basis $\mathcal{U}$ of $\mathcal{F}$, and mappings Φ that belong to the space $\mathcal{W}_{c(p_1),\ldots,c(p_n)}$ for local coordinates $(c(p_1), \ldots, c(p_n))$ on $\mathcal{M}$ at points $(p_1, \ldots, p_n)$ of intersection of U_j with leaves of $\mathcal{M}/\mathcal{F}$ of $\mathcal{F}$. We then consider a collection of k sections U_j, $1 \leq j \leq k$ of $\mathcal{U}$. In order to define the vertex algebra cohomology of $\mathcal{M}/\mathcal{F}$, we assume that mappings Φ are composable with k vertex operators. On each transversal section U_j, $1 \leq j \leq k$ one point p'_j is chosen with a local coordinate $c'_j(p'_j)$. Let us assume that Φ is composable with k vertex operators. The formal parameters of k vertex operators a map Φ is composable with is taken to be $c'_j(p'_j)$, $1 \leq j \leq k$. The composability of a map Φ with a number of vertex operators consists of two conditions on Φ. The first condition requires the existence of limiting positive integers $N^n_m(v_i, v_j)$ depending on vertex algebra elements v_i and v_j only, while the second condition restricts orders of poles of corresponding sums (2.12) and (2.14). Taking into account these conditions, we will see that the construction of spaces (4.2) depends on the choice of vertex algebra elements (2.17).

3. The Product of $\mathcal{W}_{z_1,\ldots,z_n}$-Spaces

In this Section we introduce a product of elements of $\mathcal{W}_{z_1,\ldots,z_n}$-spaces and study its properties.

3.1. Geometrical interpretation and definition of the ϵ-product for $\mathcal{W}_{z_1,\ldots,z_n}$-spaces. Recall Definition 11 of $\mathcal{W}_{z_1,\ldots,z_n}$-spaces given in Sect. 2.2. The structure of $\mathcal{W}_{z_1,\ldots,z_n}$-spaces is quite complicated and it is a problem to introduce a product of elements of such spaces algebraically. In order to define an appropriate product of two $\mathcal{W}$-spaces we first have to interpret it geometrically. Let us associate a $\mathcal{W}$-space with a certain model space. Then a geometrical product of such model spaces should be defined, and, finally, an algebraic product of $\mathcal{W}$-spaces should be introduced.

For two $\mathcal{W}_{x_1,\ldots,x_k}$- and $\mathcal{W}_{y_1,\ldots,y_n}$-spaces we first associate formal complex parameters in sets $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ to parameters of two auxiliary spaces. Then we describe the geometrical procedure to form a resulting model space by combining two original model spaces. The formal parameters of the algebraic product $\mathcal{W}_{z_1,\ldots,z_{k+n}}$ of $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$ should be then identified with parameters of resulting auxiliary space. Note that according to our assumption, $(x_1, \ldots, x_k) \in F_k\mathbb{C}$, and $(y_1, \ldots, y_n) \in F_n\mathbb{C}$, i.e., belong to corresponding configuration space (Definition 6, Sect. 2.2). As it follows from the definition of $F_n\mathbb{C}$, any coincidence of two formal parameters should be excluded from $F_{k+n}\mathbb{C}$. In general, it might happens that some r formal parameters of $(x_1, \ldots, x_k)$ coincide with formal parameters of $(y_1, \ldots, y_n)$, i.e., $x_{i_l} = y_{j_l}$, $1 \leq i_l, j_l \leq r$.

In Definition 13 of the product of $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$ below we keep only one of two coinciding formal parameters, We require that the set of formal parameters

$$(z_1, \ldots, z_{k+n-r}) = (x_1, \ldots, x_{i_1}, \ldots, x_{i_r}, \ldots, x_k; y_1, \ldots, \widehat{y}_{j_1}, \ldots, \widehat{y}_{j_r}, \ldots, y_n), \quad (3.1)$$

55 would belong to $F_{k+n-r}\mathbb{C}$ where $\widehat{.}$ denotes the exclusion of values $x_{i_l} = y_{j_l}$, $1 \le l \le r$ from the domain of definition for corresponding differential forms that belong to $\mathcal{W}_{z_1,\ldots,z_{k+n}}$ and characterized by the bilinear pairing. We denote this operation of formal parameters exclusion by $\widehat{R}\ \Phi(x_1, \ldots, x_k; y_1, \ldots, y_n; \epsilon)$. Thus, we require that the set of formal parameters $(z_1, \ldots, z_{k+n-r})$ for the resulting product would belong to $F_{k+n-r}\mathbb{C}$. Note that instead of exclusion given by the right hand side of (3.1), we could equivalently omit elements from $(x_1, \ldots, x_k)$-part coinciding with some elements of $(y_1, \ldots, y_n)$.

In our particular case of the space of differential forms $\mathcal{W}$ obtained from matrix elements (2.20), we take two Riemann spheres $\Sigma_a^{(0)}$, $a = 1, 2$ as our two initial auxiliary spaces/geometrical models. The resulting auxiliary/model space is formed by the Riemann sphere $\Sigma^{(0)}$ obtained by the ϵ-sewing procedure of two initial spheres where ϵ is a complex parameter. The formal parameters $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ are identified with local coordinates of k and n points on two initial spheres $\Sigma_a^{(0)}$, $a = 1, 2$ correspondingly. In the ϵ-sewing procedure, some r points among $(p_1, \ldots, p_k)$ may coincide with points among $(p'_1, \ldots, p'_n)$ when we identify the annuluses (3.4). This corresponds to the singular case of coincidence of r formal parameters.

Consider the sphere formed by sewing together two initial spheres in the sewing scheme referred to as the ϵ-formalism in [77]. Let $\Sigma_a^{(0)}$, $a = 1, 2$ be to initial spheres. Introduce a complex sewing parameter ϵ where

$$|\epsilon| \le r_1 r_2,$$

Consider k distinct points on $p_i \in \Sigma_1^{(0)}$, $i = 1, \ldots, k$, with local coordinates $(x_1, \ldots, x_k) \in F_k\mathbb{C}$, and distinct points $p_j \in \Sigma_2^{(0)}$, $j = 1, \ldots, n$, with local coordinates $(y_1, \ldots, y_n) \in F_n\mathbb{C}$, with

$$|x_i| \ge |\epsilon|/r_2,$$
$$|y_i| \ge |\epsilon|/r_1.$$

Choose a local coordinate $z_a \in \mathbb{C}$ on $\Sigma_a^{(0)}$ in the neighborhood of points $p_a \in \Sigma_a^{(0)}$, $a = 1, 2$. Consider the closed disks

$$|\zeta_a| \le r_a,$$

and excise the disk

$$\{\zeta_a,\ |\zeta_a| \le |\epsilon|/r_{\overline{a}}\} \subset \Sigma_a^{(0)}, \tag{3.2}$$

to form a punctured sphere

$$\widehat{\Sigma}_a^{(0)} = \Sigma_a^{(0)} \backslash \{\zeta_a, |\zeta_a| \le |\epsilon|/r_{\overline{a}}\}.$$

We use the convention

$$\overline{1} = 2, \quad \overline{2} = 1. \tag{3.3}$$

Define the annulus

$$\mathcal{A}_a = \{\zeta_a, |\epsilon|/r_{\overline{a}} \le |\zeta_a| \le r_a\} \subset \widehat{\Sigma}_a^{(0)}, \tag{3.4}$$

and identify $\mathcal{A}_1$ and $\mathcal{A}_2$ as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$\zeta_1 \zeta_2 = \epsilon. \tag{3.5}$$

In this way we obtain a genus zero compact Riemann surface

$$\Sigma^{(0)} = \left\{ \widehat{\Sigma}_1^{(0)} \backslash \mathcal{A}_1 \right\} \cup \left\{ \widehat{\Sigma}_2^{(0)} \backslash \mathcal{A}_2 \right\} \cup \mathcal{A}.$$

This sphere forms a suitable geometrical model for the construction of a product of elements of $\mathcal{W}$-spaces. A multiply sewn sphere model is considered in [81].

Recall the notion of an intertwining operator (9.14) given in "Appendix 8.2". Let us now give a formal algebraic definition of the product of $\mathcal{W}$-spaces.

Definition 13. For $\Phi(v_1, x_1; \ldots; v_k, x_k) \in \mathcal{W}_{x_1,\ldots,x_k}$, and $\Psi(v'_1, y_1; \ldots; v'_n, y_n) \in \mathcal{W}_{y_1,\ldots,y_n}$ the ϵ-product

$$\begin{aligned} &\Phi(v_1, x_1; \ldots; v_k, x_k) \cdot_\epsilon \Psi(v'_1, y_1; \ldots; v'_n, y_n) \\ &\qquad \mapsto \widehat{R}\, \Theta\left(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon\right), \end{aligned} \tag{3.6}$$

is defined by the bilinear pairing via (2.20)

$$\begin{aligned} &\langle w', \widehat{R}\, \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\ &\quad = \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y^W_{WV}\left(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right)\, u\rangle \\ &\quad \langle w', Y^W_{WV}\left(\Psi(v'_1, y_1; \ldots; v'_{i_1}, \widehat{y}_{i_1}; \ldots; \ldots; v'_{j_r}, \widehat{y}_{j_r}; \ldots; v'_n, y_n), \zeta_2\right)\, \overline{u}\rangle, \end{aligned} \tag{3.7}$$

parametrized by $\zeta_1, \zeta_2 \in \mathbb{C}$ related by the sewing relation (3.5). The sum is taken over any V_l-basis $\{u\}$, where $\overline{u}$ is the dual of u with respect to the non-degenerate bilinear pairing $\langle .. \rangle_\lambda$, (9.28) over V, (see "Appendix 8.2"). The coinciding values of the formal parameters in $\Phi(v_1, x_1; \ldots; v_k, x_k)$ and $\Psi(v'_1, y_1; \ldots; v'_n, y_n)$ are excluded from the domain of definition of the right hand side of (3.7).

By the standard reasoning [24,78], (3.7) does not depend on the choice of a basis of $u \in V_l$, $l \in \mathbb{Z}$. In the case when the forms Φ and Ψ that we multiply do not contain V-elements, (3.7) defines the following product $\Phi \cdot_\epsilon \Psi$

$$\langle w', \Theta(\epsilon)\rangle = \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y^W_{WV}\left(\Phi, \zeta_1\right)\, u\rangle \langle w', Y^W_{WV}\left(\Psi, \zeta_2\right)\, \overline{u}\rangle. \tag{3.8}$$

As we will see, Definition 13 is also supported by Lemma 3. Recall Remark 1. The right hand side of (3.7) is given by a formal series of bilinear pairings summed over a vertex algebra basis. To complete this definition we have to show that the right hand side of (3.7) defines a differential form that belongs to the space $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$. The main statement of this Section is given by

Proposition 4. *The product* (3.7) *provides a map*

$$\cdot_\epsilon : \mathcal{W}_{x_1,\ldots,x_k} \times \mathcal{W}_{y_1,\ldots,y_n} \to \mathcal{W}_{z_1,\ldots,z_{k+n-r}}.$$

The rest of this Section is devoted to the proof of Proposition 4. We show that the wight hand side of (3.7) belongs to the space $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$.

Remark 2. Note that due to (9.14), in Definition 13, it is assumed that $\Phi(v_1, x_1; \ldots; v_k, x_k)$ and $\Psi(v'_1, y_1; \ldots; v'_n, y_n)$ are composable with the grading-restricted generalized V-module W vertex operators $Y_W(u, -\zeta_1)$ and $Y_W(\overline{u}, -\zeta_2)$ correspondingly (cf. Sect. 2.3 for the definition of composability). The product (3.7) is actually defined by the sum of products of matrix elements of generlized grading-resticted V-module W vertex operators acting on $\mathcal{W}_{z_1,\ldots,z_n}$ elements. The vertex algebra elements $u \in V$ and $\overline{u} \in V'$ are related by (9.29), and ζ_1 and ζ_2 satisfy (3.5). The form of the product defined above is natural in terms of the theory of correlation functions for vertex operator algebras [23,70,78].

3.2. Convergence of the ϵ-product. In order to prove convergence of the product (3.7) of elements of two spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$, we have to use a geometrical interpretation [45,77]. Recall that a $\mathcal{W}_{z_1,\ldots,z_n}$-space is defined by means of matrix elements of the form (2.20). For a vertex algebra V, this corresponds [24] to matrix element of a number of V-vertex operators with formal parameters identified with local coordinates on Riemann sphere. Geometrically, each space $\mathcal{W}_{z_1,\ldots,z_n}$ can be also associated to Riemann sphere with a few marked points, and local coordinates vanishing at these points [45]. An additional point is identified to the center of an annulus used in order to sew the sphere with another sphere. The product (3.7) has then a geometrical interpretation. The resulting model space is then Riemann sphere formed in the sewing procedure.

Matrix elements for a number of vertex operators are usually associated [23,24,70] with a vertex algebra correlation functions on the sphere. We extrapolate this notion to the case of $\mathcal{W}_{z_1,\ldots,z_n}$ spaces. In order to supply an appropriate geometrical construction of the product, we use the ϵ-sewing procedure for two initial spheres to obtain a matrix element associated with (3.6).

Remark 3. In addition to the ϵ-sewing procedure of two initial spheres, one can alternatively use the self-sewing procedure [77] for Riemann sphere to get, at first, the torus, and then by sending parameters to appropriate limit by shrinking genus to zero. As a result, one obtains again a sphere but with a different parameterization [53].

Let us identify (as in [7,23,45,70,77,78]) two sets $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ of complex formal parameters, with local coordinates of two sets of points on the first and the second Riemann spheres correspondingly. Complex parameters ζ_1 and ζ_2 of (3.7) play then the roles of coordinates (3.2) of the annuluses (3.4). On identification of annuluses $\mathcal{A}_a$ and $\mathcal{A}_{\overline{a}}$, r coinciding coordinates may occur.

The product (3.7) describes a differential form that belongs to the space $\mathcal{W}$ defined on a sphere formed as a result of the ϵ-parameter sewing [77] of two initial spheres. Since two initial spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$ $\overline{W}_{x_1,\ldots,x_k}$- and $\overline{W}_{y_1,\ldots,y_n}$-valued differential forms expressed by matrix elements of the form (2.20), it is then proved (see Proposition 5 below), that the resulting product defines a elememts of the space $\mathcal{W}_{x_1,\ldots,x_k;y_1,\ldots,y_n}$ by means of an absolute convergent matrix element on the resulting sphere. The complex sewing parameter ϵ parametrizes the moduli space of the resulting sewn sphere as well as of the product of $\mathcal{W}$-spaces.

Proposition 5. *The product* (3.7) *of elements of the spaces* $\mathcal{W}_{x_1,\ldots,x_k}$ *and* $\mathcal{W}_{y_1,\ldots,y_n}$ *corresponds to a bilinear pairing absolutely converging in* ϵ *with only possible poles at* $x_i = x_j$, $y_{i'} = y_{j'}$, *and* $x_i = y_j$, $1 \le i, i' \le k$, $1 \le j, j' \le n$.

Proof. In order to prove this proposition we use the geometrical interpretation of the product (3.7) in terms of Riemann spheres with marked points. We consider two sets of vertex algebra elements $(v_1, \ldots, v_k)$ and $(v'_1, \ldots, v'_k)$, and two sets of formal complex parameters $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_n)$. The formal parameters are identified with the local coordinates of k points on Riemann sphere $\widehat{\Sigma}_1^{(0)}$, and n points on $\widehat{\Sigma}_2^{(0)}$, with excised annuluses $\mathcal{A}_a$. Recall the sewing parameter condition (3.5)

$$\zeta_1 \zeta_2 = \epsilon,$$

of the sewing procedure. Then, for (3.7) we obtain

$$\begin{aligned}
&\langle w', \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y_{WV}^W \left(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1\right) \; u\rangle \\
&\langle w', Y_{WV}^W \left(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\right) \; \overline{u}\rangle \\
&= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', e^{\zeta_1 \, L_W(-1)} \; Y_W\left(u, -\zeta_1\right) \; \Phi(v_1, x_1; \ldots; v_k, x_k)\rangle \\
&\qquad \langle w', e^{\zeta_2 \, L_W(-1)} \; Y_W\left(\overline{u}, -\zeta_2\right) \; \Psi(v'_1, y_1; \ldots; v'_n, y_n)\rangle.
\end{aligned}$$

Recall from (3.2) that in the two spheres ϵ-sewing formulation, the complex parameters ζ_a, $a = 1, 2$ are the coordinates inside the identified annuluses $\mathcal{A}_a$, and $|\zeta_a| \le r_a$. Therefore, due to Proposition 1, the matrix elements

$$\widetilde{\mathcal{R}}(x_1, \ldots, x_k; \zeta_1) = \langle w', e^{\zeta_1 \, L_W(-1)} \; Y_W\left(u, -\zeta_1\right) \; \Phi(v_1, x_1; \ldots; v_k, x_k)\rangle, \quad (3.9)$$

$$\widetilde{\mathcal{R}}(y_1, \ldots, y_n; \zeta_2) = \langle w', e^{\zeta_2 \, L_W(-1)} \; Y_W\left(\overline{u}, -\zeta_2\right) \; \Psi(v'_1, y_1; \ldots; v'_n, y_n)\rangle, \quad (3.10)$$

are absolutely convergent in powers of ϵ with some radia of convergence $R_a \le r_a$, with $|\zeta_a| \le R_a$. The dependence of (3.9) and (3.10) on ϵ is expressed via ζ_a, $a = 1, 2$. Let us rewrite the product (3.7) as

$$\begin{aligned}
&\langle w', \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle \\
&= \sum_{l \in Z} \epsilon^l \left(\langle w', \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n)\rangle\right)_l \\
&= \sum_{l \in \mathbb{Z}} \sum_{u \in V_l} \sum_{m \in \mathbb{C}} \epsilon^{l-m-1} \; \widetilde{\mathcal{R}}_m(x_1, \ldots, x_k; \zeta_1) \; \widetilde{\mathcal{R}}_m(y_1, \ldots, y_n; \zeta_2), \qquad (3.11)
\end{aligned}$$

as a formal series in ϵ for $|\zeta_a| \le R_a$, where and $|\epsilon| \le r$ for $r < r_1 r_2$. Then we apply Cauchy's inequality to the coefficient forms (3.9) and (3.10) to find

$$\left|\widetilde{\mathcal{R}}_m(x_1, \ldots, x_k; \zeta_1)\right| \le M_1 R_1^{-m}, \qquad (3.12)$$

with

$$M_1 = \sup_{|\zeta_1| \le R_1, |\epsilon| \le r} \left|\widetilde{\mathcal{R}}(x_1, \ldots, x_k; \zeta_1)\right|.$$

Similarly,

$$\left|\widetilde{\mathcal{R}}_m(y_1, \ldots, y_n; \zeta_2)\right| \le M_2 R_2^{-m}, \qquad (3.13)$$

for

$$M_2 = \sup_{|\zeta_2| \le R_2, |\epsilon| \le r} \left|\widetilde{\mathcal{R}}(y_1, \ldots, y_n; \zeta_2)\right|.$$

Using (3.12) and (3.13) we obtain for (3.11)

$$\begin{aligned}&\left|\left(\langle w', \mathcal{F}(v_1, x_1; \ldots; v_k, x_k; v_1', y_1; \ldots; v_n', y_n\rangle\right)_l\right| \\ &\quad \le \left|\widetilde{\mathcal{R}}_m(x_1, \ldots, x_k; \zeta_1)\right| \, \left|\widetilde{\mathcal{R}}_m(y_1, \ldots, y_n; \zeta_2)\right| \\ &\quad \le M_1 \; M_2 \; \left(R_1 R_2\right)^{-m}. \end{aligned} \tag{3.14}$$

Thus, for $M = \min\{M_1, M_2\}$ and $R = \max\{R_1, R_2\}$, one has

$$\left|\mathcal{R}_l(x_1; \ldots, x_k; y_1, \ldots, y_n'; \zeta_1, \zeta_2)\right| \le M R^{-l+m+1}. \tag{3.15}$$

We see that (3.7) is absolute convergent as a formal series in ϵ and defined for $|\zeta_a| \le r_a$, $|\epsilon| \le r$ for $r < r_1 r_2$, with extra poles only at $x_i = y_j$, $1 \le i \le k$, $1 \le j \le n$. □

Next, we formulate

Definition 14. We define the action of an element $\sigma \in S_{k+n-r}$ on the product of $\Phi(v_1, x_1; \ldots; v_k, x_k) \in \mathcal{W}_{x_1,\ldots,x_k}$, and $\Psi(v_1', y_1; \ldots; v_n', y_n) \in \mathcal{W}_{y_1,\ldots,y_n}$, as

$$\begin{aligned}&\langle w', \sigma(\widehat{R}\,\Theta)(v_1, x_1; \ldots; v_k, x_k; v_1', y_1; \ldots; v_n', y_n; \epsilon)\rangle \\ &\quad = \langle w', \Theta(\widetilde{v}_{\sigma(1)}, z_{\sigma(1)}; \ldots; \widetilde{v}_{\sigma(k+n-r)}, z_{\sigma(k+n-r)}; \epsilon)\rangle \\ &\quad = \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y_{WV}^W \left(\Phi(\widetilde{v}_{\sigma(1)}, z_{\sigma(1)}; \ldots; \widetilde{v}_{\sigma(k)}, z_{\sigma(k)}), \zeta_1\right) \; u\rangle \\ &\langle w', Y_{WV}^W \left(\Psi(\widetilde{v}_{\sigma(k+1)}, z_{\sigma(k+1)}; \ldots; \widetilde{v}_{\sigma(k+n-r)}, z_{\sigma(k+n-r)}), \zeta_2\right) \; \overline{u}\rangle, \end{aligned} \tag{3.16}$$

where by $(\widetilde{v}_{\sigma(1)}, \ldots, \widetilde{v}_{\sigma(k+n-r)})$ we denote a permutation of vertex algebra elements

$$(\widetilde{v}_1, \ldots, \widetilde{v}_{k+n-r}) = (v_1, \ldots; v_k; \ldots, \widehat{v}'_{j_1}, \ldots, \widehat{v}'_{j_r}, \ldots). \tag{3.17}$$

Next, we have

Lemma 1. *The product* (3.7) *satisfies* (2.9) *for* $\sigma \in S_{k+n-r}$, *i.e.,*

$$\sum_{\sigma\in J^{-1}_{k+n-r;s}} (-1)^{|\sigma|} \widehat{R}\;\Theta\left(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)}; v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)}; \epsilon\right) = 0.$$

Proof. For arbitrary $w' \in W'$, we have

$$\begin{aligned}&\sum_{\sigma\in J^{-1}_{k+n;s}} (-1)^{|\sigma|} \langle w', \Theta\left(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)}; v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)})\right)\rangle \\ &\quad = \sum_{\sigma\in J^{-1}_{k+n;s}} (-1)^{|\sigma|} \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y_{WV}^W \left(\Phi(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)}), \zeta_1\right) u\rangle \\ &\langle w', Y_{WV}^W \left(\Psi(v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)}), \zeta_2\right) \overline{u}\rangle \\ &\quad = \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \sum_{\sigma\in J^{-1}_{k+n;s}} (-1)^{|\sigma|} \langle w', e^{\zeta_1 L_W(-1)} \; Y_W(u, -\zeta_1) \; \Phi(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)})\rangle \\ &\langle w', e^{\zeta_2 L_W(-1)} \; Y_W(\overline{u}, -\zeta_2) \; \Psi(v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)})\rangle \\ &\quad = \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', e^{\zeta_1 L_W(-1)} \; Y_W(u, -\zeta_1) \sum_{\sigma\in J^{-1}_{k;s}} (-1)^{|\sigma|} \Phi(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)})\rangle \end{aligned}$$

$$\langle w', e^{\zeta_2 L_W(-1)}\ Y_W(\overline{u}, -\zeta_2)\ \Psi(v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)})\rangle$$
$$+\sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', e^{\zeta_1 L_W(-1)}\ Y_W(u, -\zeta_1)\ \Phi(v_{\sigma(1)}, x_{\sigma(1)}; \ldots; v_{\sigma(k)}, x_{\sigma(k)})\rangle$$
$$\langle w', e^{\zeta_2 L_W(-1)}\ Y_W(\overline{u}, -\zeta_2) \sum_{\sigma\in J^{-1}_{n;s}} (-1)^{|\sigma|} \Psi(v'_{\sigma(1)}, y_{\sigma(1)}; \ldots; v'_{\sigma(n)}, y_{\sigma(n)})\rangle = 0,$$

since, $J^{-1}_{k+n;s} = J^{-1}_{k;s} \times J^{-1}_{n;s}$, and due to the fact that $\Phi(v_1, x_1, \ldots, v_k, x_k)$ and $\Psi(v_1, y_1, \ldots, v'_k, y_n)$ satisfy (2.1). □

Next we prove the existence of an appropriate differential form that belongs to $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$ corresponding to an absolute convergent bilinear pairing $\mathcal{R}(z_1, \ldots, z_{k+n-r})$ defining the ϵ-product of elements of the spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$.

Lemma 2. *For all choices of elements of the spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$ there exists a differential form characterized by the bilinear pairing $\widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon) \in \mathcal{W}_{z_1,\ldots,z_{k+n-r}}$ such that the product (3.7) converges to*

$$R(x_1, \ldots, x_k; y_1, \ldots, y_n; \epsilon) = \langle w', \widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle.$$

Proof. In the proof of Proposition 5 we showed the absolute convergence of the product (3.7) to a bilinear pairing $R(x_1, \ldots, x_k; y_1, \ldots, y_n; \epsilon)$. The lemma follows from the completeness of $\overline{W}_{x_1,\ldots,x_k;y_1,\ldots,y_n}$ and the density of the space of rational differential forms. □

We formulate

Definition 15. For $\Phi(v_1, x_1; \ldots; v_k, x_k) \in \mathcal{W}_{x_1,\ldots,x_k}$ and $\Psi(v'_1, y_1; \ldots; v'_n, y_k) \in \mathcal{W}_{y_1,\ldots,y_n}$, with r coinciding formal parameters $x_{i_q} = y_{j_q}$, $1 \le q \le r$, we define the action of $\partial_s = \partial_{z_s} = \partial/\partial_{z_s}$, $1 \le s \le k+n-r$, $1 \le i \le k$, $1 \le j \le n$ on $\widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ with respect to the sth entry of $(z_1, \ldots, z_{k+n-r})$, as follows

$$\langle w', \partial_s \widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', \partial^{\delta_{s,i}}_{x_i} Y^W_{WV}\ (\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1)\ u\rangle$$
$$\langle w', \partial^{\delta_{s,j}-\delta_{i_q,j_q}}_{y_j} Y^W_{WV} \left(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\right) \overline{u}\rangle. \tag{3.18}$$

Remark 4. As we see in the last expressions, the $L_V(0)$-conjugation property (2.7) for the product (3.7) includes the action of the $z^{L_V(0)}$-operator on the complex parameters ζ_a, $a = 1, 2$.

Proposition 6. *The product (3.7) satisfies the $L_V(-1)$-derivative (2.2) and $L_V(0)$-conjugation (2.7) properties.*

Proof. By using (2.2) for $\Phi(v_1, x_1; \ldots; v_k, x_k)$ and $\Psi(v'_1, y_1; \ldots; v'_n, y_n)$, we consider

$$\langle w', \partial_l \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', \partial^{\delta_{l,i}}_{x_i} Y^W_{WV}\ (\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1)\ u\rangle$$
$$\langle w', \partial^{\delta_{l,j}}_{y_j} Y^W_{WV} \left(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\right) \overline{u}\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', \partial_{x_i}^{\delta_{l,i}} Y_W(u, -\zeta_1)\, \Phi(v_1, x_1; \dots; v_k, x_k))\ u\rangle$$

$$\langle w', \partial_{y_j}^{\delta_{l,j}} Y_W(\overline{u}, -\zeta_2)\, \Psi(v'_1, y_1; \dots; v'_n, y_n)\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\left(\partial_{x_i}^{\delta_{l,i}} \Phi(v_1, x_1; \dots; v_k, x_k), \zeta_1\right)\ u\rangle$$

$$\langle w', Y^W_{WV}\left(\partial_{y_j}^{\delta_{l,j}} \Psi(v'_1, y_1; \dots; v_n, y_n), \zeta_2\right) \overline{u}\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\left(\Phi(v_1, x_1; \dots; (L_V(-1))^{\delta_{l,i}}\ v_i, x_i; \dots; v_k, x_k), \zeta_1\right)\ u\rangle$$

$$\langle w', Y^W_{WV}\left(\Psi(v'_1, y_1; \dots; (L_V(-1))^{\delta_{l,j}}\ v'_j, y_j; \dots; v'_n, y_n), \zeta_2\right) \overline{u}\rangle$$

$$= \langle w', \Theta(v_1, x_1; \dots; (L_V(-1))_l\, ; \dots; v'_n, y_n; \epsilon)\rangle, \tag{3.19}$$

where $(L_V(-1))_l$ acts on the lth entry of $(v_1, \dots; v_k; v'_1, \dots, v'_n)$. Summing over l we obtain

$$\sum_{l=1}^{k+n} \partial_l \Theta(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon))\rangle$$

$$= \sum_{l=1}^{k+n} \langle w', \Theta(v'_1, x_1; \dots; (L_V(-1))\, ; \dots; v'_n, y_n; \epsilon)\rangle$$

$$= \langle w', L_W(-1).\Theta(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)\rangle. \tag{3.20}$$

Due to (2.7), (9.9), (9.29), (9.30), and (9.15), we have

$$\langle w', \Theta(z^{L_V(0)} v_1, z\ x_1; \dots; z^{L_V(0)} v_k, z\ x_k; z^{L_V(0)} v'_1, z\ y_1; \dots; z^{L_V(0)} v'_n, z\ y_n; \epsilon)\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\left(\Phi(z^{L_V(0)} v_1, z\ x_1; \dots; z^{L_V(0)} v_k, z\ x_k), \zeta_1\right)\ u\rangle$$

$$\langle w', Y^W_{WV}\left(\Psi(z^{L_V(0)} v'_1, z\ y_1; \dots; z^{L_V(0)} v'_n, z\ y_n), \zeta_2\right) \overline{u}\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\left(z^{L_V(0)} \Phi(v_1, x_1; \dots; v_k, x_k), \zeta_1\right)\ u\rangle$$

$$\langle w', Y^W_{WV}\left(z^{L_V(0)} \Psi(v'_1, y_1; \dots; v'_n, y_n), \zeta_2\right) \overline{u}\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\, z^{L_V(0)} \Phi(v_1, x_1; \dots; v_k, x_k)\rangle$$

$$\langle w', e^{\zeta_2 L_W(-1)}\ Y_W(\overline{u}, -\zeta_2)\, z^{L_V(0)} \Psi(v'_1, y_1; \dots; v'_n, y_n)\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', e^{\zeta_1 L_W(-1)} z^{L_V(0)} Y_W\left(z^{-L_V(0)} u, -z\ \zeta_1\right) \Phi(v_1, x_1; \dots; v_k, x_k)\rangle$$

$$\langle w', e^{\zeta_2 L_W(-1)}\ z^{L_W(0)}\ Y_W\left(z^{-L_V(0)} \overline{u}, -z\ \zeta_2\right) \Psi(v'_1, y_1; \dots; v'_n, y_n)\rangle$$

$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', e^{\zeta_1 L_W(-1)} z^{L_W(0)} z^{-\mathrm{wt} u}\ Y_W(u, -z\ \zeta_1)\, \Phi(v_1, x_1; \dots; v_k, x_k)\rangle$$

$$
\begin{aligned}
&\langle w', e^{\zeta_2 L_W(-1)}\, z^{L_W(0)}\, z^{-\mathrm{wt}\overline{u}}\, Y_W\, (\overline{u}, -z\,\zeta_2)\, \Psi(v'_1, y_1; \ldots; v'_n, y_n)\rangle \\
&\quad = \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', z^{L_W(0)} e^{\zeta_1 L_W(-1)} Y_W\,(u, -z\zeta_1)\, \Phi(v_1, x_1; \ldots; v_k, x_k)\rangle \\
&\langle w', z^{L_W(0)} e^{\zeta_2 L_W(-1)} Y_W\,(\overline{u}, -z\zeta_2)\, \Psi(v'_1, y_1; \ldots; v'_n, y_n), \rangle \\
&\quad = \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', z^{L_W(0)}\; Y^W_{WV}\, (\Phi(v_1, x_1; \ldots; v_k, x_k), z\zeta_1)\;\; u\rangle \\
&\langle w', z^{L_W(0)}\; Y^W_{WV}\, \left(\Psi(v'_1, y_1; \ldots; v'_n, y_n), z\zeta_2\right) \overline{u}\rangle \\
&\quad = \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', z^{L_W(0)}\; Y^W_{WV}\, \left(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta'_1\right)\;\; u\rangle \\
&\langle w', z^{L_W(0)}\; Y^W_{WV}\, \left(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta'_2\right) \overline{u}\rangle \\
&\quad = \langle w', \left(z^{L_W(0)}\right).\Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle.
\end{aligned}
$$

With (3.5), we obtain (2.7) for (3.7). □

Summing up the results of Proposition (5), Lemmas (1), (2), and Proposition (6), we obtain the proof of Proposition 4. We then have

Definition 16. For the fixed sets $v_1, \ldots, v_k \in V$ and $v'_1, \ldots, v'_n \in V$, $x_1, \ldots, x_k \in \mathbb{C}$, $y_1, \ldots, y_n \in \mathbb{C}$, we call the set of all differential form of $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$ described by $\widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k;\ v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ with the parameter ϵ exhausting all possible values, the complete product of the spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$.

3.3. Properties of the $\mathcal{W}_{z_1,\ldots,z_n}$-product. In this Subsection we study properties of the product $\widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k;\ v'_1, y_1; \ldots; v'_n, y_n; \epsilon)$ of (3.7). We have

Proposition 7. *For generic elements v_i, $v'_j \in V$, $1 \le i \le k$, $1 \le j \le n$, of a quasi-conformal grading-restricted vertex algebra V, the product (3.7) is canonical with respect to the action of the group $(\mathrm{Aut}\ \mathcal{O})^{\times(k+n-r)}_{z_1,\ldots,z_{k+n-r}}$ of independent $k+n-r$-dimensional changes*

$$
(z_1, \ldots, z_{k+n-r}) \mapsto (z'_1, \ldots, z'_{k+n-r}) = (\rho(z_1), \ldots, \rho(z_{k+n-r})), \tag{3.21}
$$

of formal parameters.

Proof. Note that due to Proposition 8

$$
\begin{aligned}
\Phi(v_1, x'_1; \ldots; v_k, x'_k) &= \Phi(v_1, x_1; \ldots; v_k, x_k), \\
\Psi(v_1, y'_1; \ldots; v_n, y'_n) &= \Psi(v_1, y_1; \ldots; v_n, y_n).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\langle w', \Theta(v_1, x'_1; \ldots, ; v_k, x'_k;\ v'_1, y'_1; \ldots; v'_n, y'_n; \epsilon)\rangle \\
&\quad = \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{WV}\, \left(\Phi(v_1, x'_1; \ldots; v_k, x'_k), \zeta_1\right)\;\; u\rangle \\
&\langle w', Y^W_{WV}\, \left(\Psi(v'_1, y'_1; \ldots; v'_n, y'_n), \zeta_2\right) \overline{u}\rangle
\end{aligned}
$$

$$= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y^W_{WV} \left(\Phi(v_1, x_1; \dots; v_k, x_k), \zeta_1\right) \, u\rangle$$

$$\langle w', Y^W_{WV} \left(\Psi(v'_1, y_1; \dots; v'_n, y_n), \zeta_2\right) \overline{u}\rangle$$

$$= \langle w', \Theta(v_1, x_1; \dots, ; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)\rangle.$$

Therefore, the product (3.7) is invariant under (4.6). □

In the geometrical interpretation in terms of auxiliary Riemann spheres, the definition (3.7) depends on the choice of the insertion points p_i, $1 \le i \le k$, with local the coordinated x_i on $\widehat{\Sigma}^{(0)}_1$, and p'_i, $1 \le j \le k$, with the local coordinates y_j on $\widehat{\Sigma}^{(0)}_2$. Suppose we change the distribution of points among two initial Riemann spheres. We then formulate the following

Lemma 3. *In the setup above, for a fixed set* $(\widetilde{v}_1, \dots, \widetilde{v}_{k+n})$, $v_l \in V$, $1 \le l \le k+n$ *of vertex algebra elements, the splittings* $(\widetilde{v}_1, \dots, \widetilde{v}_s)$, $(\widetilde{v}_{s+1}, \dots, \widetilde{v}_{k+n})$ *for elements* $\Phi(\widetilde{v}_1, z_1; \dots; \widetilde{v}_s, z_s) \in \mathcal{W}_{z_1,\dots,z_s}$ *and* $\Psi(\widetilde{v}_{s+1}, z_{s+1}; \dots; \widetilde{v}_{k+n}, z_{k+n}) \in \mathcal{W}_{z_{k+1},\dots,z_{k+n}}$, *bring about the same* ϵ*-product* $\Theta(\widetilde{v}_1, z_1; \dots; \widetilde{v}_{k+n}, z_{k+n}; \epsilon) \in \mathcal{W}_{z_1,\dots,z_{k+n}}$,

$$\cdot_\epsilon : \mathcal{W}_{z_1,\dots,z_s} \times \mathcal{W}_{z_{s+1},\dots,z_{k+n}} \to \mathcal{W}_{z_1,\dots,z_{k+n}}, \tag{3.22}$$

for any s, $0 \le s \le k+n$.

Remark 5. This Lemma is important for the formulation of cohomological invariants associated to grading-restricted vertex algebras on smooth manifolds. In the case $s = 0$, we obtain from (3.23),

$$\cdot_\epsilon : \mathcal{W} \times \mathcal{W}_{z_1,\dots,z_{k+n}} \to \mathcal{W}_{z_1,\dots,z_{k+n}}. \tag{3.23}$$

Now we give a proof of Lemma 3.

Proof. Let $\widetilde{v}_i \in V$, $1 \le i \le k$, $\widetilde{v}_j \in V$, $1 \le j \le k$, and z_i, z_j are corresponding formal parameters. We show that the ϵ-product of $\Phi(\widetilde{v}_1, z_1; \dots; \widetilde{v}_k, z_k)$ and $\Psi(\widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_n, z_n)$, i.e., the differential form that belongs to $\mathcal{W}_{z_1,\dots,z_{k+n-r}}$

$$\Theta((\widetilde{v}_1, z_1; \dots; \widetilde{v}_k, z_k); (\widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_n, z_n); \zeta_1, \zeta_2; \epsilon), \tag{3.24}$$

is independent of the choice of $0 \le k \le n$. Consider

$$\langle w', \Theta(\widetilde{v}_1, z_1; \dots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_n, z_n; \zeta_1, \zeta_2; \epsilon)\rangle$$

$$= \sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y^W_{WV} \left(\Phi(\widetilde{v}_1, z_1; \dots; \widetilde{v}_k, z_k), \zeta_1\right) \, u\rangle$$

$$\langle w', Y^W_{WV} \left(\Psi(\widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_n, z_n), \zeta_2\right) \overline{u}\rangle. \tag{3.25}$$

On the other hand, for $0 \le m \le k$, consider

$$\sum_{l \in \mathbb{Z}} \epsilon^l \sum_{u \in V_l} \langle w', Y^W_{WV} \left(\Phi(\widetilde{v}_1, z_1; \dots; \widetilde{v}_m, z_m), \zeta_1\right) \, u\rangle$$

$$\langle w', Y^W_{WV} \left(\Psi(\widetilde{v}_{m+1}, z'_{m+1}; \dots; \widetilde{v}_k, z'_k; \widetilde{v}_{k+1}, z_1; \dots; \widetilde{v}_n, z_n), \zeta_2\right) \overline{u}\rangle$$

$$= \langle w', \Theta(\widetilde{v}_1, z_1; \dots; \widetilde{v}_m, z_m; \widetilde{v}_{m+1}, z'_{m+1}; \dots; \widetilde{v}_k, z'_k; \widetilde{v}_{k+1}, z_{k+1}; \dots; \widetilde{v}_n, z_n)\rangle.$$

The last is the ϵ-product (3.7) of $\Phi(\widetilde{v}_1, z_1; \dots; \widetilde{v}_m, z_m) \in \mathcal{W}_{z_1,\dots,z_m}$ and $\Psi(\widetilde{v}_{m+1}, z'_{m+1}; \dots; \widetilde{v}_k, z'_k; \widetilde{v}_{k+1}, z_1; \dots; \widetilde{v}_n, z_n) \in \mathcal{W}_{z'_{m+1},\dots,z'_k; z_1,\dots,z_n}$. Let us apply the invariance with respect to a subgroup of $\left(\mathrm{Aut}\, \mathcal{O}^{(1)}\right)^{\times (k+n)}_{z_1,\dots,z_{k+n}}$, with $(z_1, \dots, z_m)$ and $(z_{k+1}, \dots, z_n)$ remaining unchanged. Then we obtain the same product (3.25). □

4. Spaces for the Double Complexes

In this Section we introduce the definition of spaces for double complexes suitable for the construction the grading-restricted vertex algebra cohomology for codimension one foliations on complex curves. We first introduce

Definition 17. Let $(v_1, \ldots, v_n)$ and $(v'_1, \ldots, v'_k)$ be two sets of vertex algebra V elements, and $(p_1, \ldots, p_n)$ be points with the local coordinates $(c_1(p_1), \ldots, c_n(p_n))$ taken on the same transversal section $U_j \in \mathcal{U}$, $j \geq 1$ of the foliation $\mathcal{F}$ transversal basis $\mathcal{U}$ on a complex curve. Assuming $k \geq 1$, $n \geq 0$, we denote by $C^n(V, \mathcal{W}, \mathcal{F})(U_j)$, $0 \leq j \leq k$, the space of all linear maps (2.18)

$$\Phi : V^{\otimes n} \to \mathcal{W}_{c_1(p_1), \ldots, c_n(p_n)}, \tag{4.1}$$

composable with k of vertex operators (2.21) with the formal parameters identified with the local coordinates $c'_j(p'_j)$ around the points p'_j on each of the transversal sections U_j, $1 \leq j \leq k$.

The set of vertex algebra elements (2.17) plays the role of non-commutative parameters in our further construction of the vertex algebra cohomology associated with a foliation $\mathcal{F}$. According to the considerations of Sect. 2.1, we assume that each transversal section of a transversal basis $\mathcal{U}$ has a coordinate chart which is induced by a coordinate chart of $\mathcal{M}$ (cf. [15]).

Recall the notion of a holonomy embedding (cf. Sect. 2.1, cf. [15]) which maps a section into another section of a transversal basis, and a coordinate chart on the first section into a coordinate chart on the second transversal section. Motivated by the definition of the spaces for Čech-de Rham complex in [15] (see Sect. 2.1), let us now introduce the following spaces:

Definition 18. For $n \geq 0$, and $1 \leq m \leq k$, with Definition 17, we define the space

$$C^n_m(V, \mathcal{W}, \mathcal{U}, \mathcal{F}) = \bigcap_{\substack{U_1 \overset{h_1}{\hookrightarrow} \ldots \overset{h_{m-1}}{\hookrightarrow} U_m \\ 1 \leq j \leq m}} C^n(V, \mathcal{W}, \mathcal{F})(U_j), \tag{4.2}$$

where the intersection ranges over all possible $(m-1)$-tuples of holonomy embeddings h_j, $j \in \{1, \ldots, m-1\}$, between transversal sections of a basis $\mathcal{U}$ for $\mathcal{F}$.

First, we have the following

Lemma 4. (4.2) *is non-empty.*

Proof. From the construction of the spaces for the double complex of the grading-restricted vertex algebra cohomology, it is clear that the spaces $C^n(V, \mathcal{W}, \mathcal{U}, \mathcal{F})(U_j)$, $1 \leq s \leq m$ in Definition 17 are non-empty. On each transversal section U_s, $1 \leq s \leq m$, $\Phi(v_1, c_j(p_1); \ldots; v_n, c_j(p_n))$ belongs to the space $\mathcal{W}_{c_j(p_1), \ldots, c_j(p_n)}$, and satisfies the $L(-1)$-derivative (2.2) and $L(0)$-conjugation (2.7) properties. The map $\Phi(v_1, c_j(p_1);$ $\ldots; v_n, c_j(p_n))$ is composable with m vertex operators with the formal parameters identified with the local coordinates $c_j(p'_j)$, on each transversal section U_j. Note that on each transversal section, the spaces (4.2) remain the same for fixed n and m. The only difference may be constituted by the composability conditions (2.12) and (2.14) for Φ.

In particular, for $l_1, \ldots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = n + m$, $v_1, \ldots, v_{m+n} \in V$ and $w' \in W'$, recall (6.13) that

$$\Xi_i = \omega_V(v_{k_1}, c_{k_1}(p_{k_1}) - \zeta_i) \ldots \omega_V(v_{k_i}, c_{k_i}(p_{k_i}) - \zeta_i) \, \mathbf{1}_V, \tag{4.3}$$

where k_i is defined in (6.18), for $i = 1, \ldots, n$, depend on the coordinates of points on the transversal sections. At the same time, in the first composability condition (2.12) depends on the projections $P_r(\Xi_i), r \in \mathbb{C}$, of $\mathcal{W}_{c(p_1),\ldots,c(p_n)}$ to W, and on the arbitrary variables ζ_i, $1 \le i \le m$. On each transversal connection U_s, $1 \le s \le m$, the absolute convergence is assumed for the series (2.12) (cf. Sect. 2.3). The positive integers $N^n_m(v_i, v_j)$, (depending only on v_i and v_j) as well as ζ_i, for $i, j = 1, \ldots, k$, $i \ne j$, may vary for the transversal sections U_s. Nevertheless, the domains of convergence determined by the conditions (6.12) which have the form

$$|c_{m_i}(p_{m_i}) - \zeta_i| + |c_{n_i}(p_{n_i}) - \zeta_i| < |\zeta_i - \zeta_j|, \tag{4.4}$$

for $m_i = l_1 + \cdots + l_{i-1} + p$, $n = l_1 + \cdots + l_{j-1} + q$, $i, j = 1, \ldots, k$, $i \ne j$ and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$, are limited by $|\zeta_i - \zeta_j|$ in (4.4) from above. Thus, for the intersection variation of the sets of homology embeddings in (4.2), the absolute convergence condition for (2.12) is still fulfilled. Under the intersection in (4.2), by choosing appropriate $N^n_m(v_i, v_j)$, one can analytically extend (2.12) to a rational function in $(c_1(p_1), \ldots, c_{n+m}(p_{n+m}))$, independent of $(\zeta_1, \ldots, \zeta_n)$, with the only possible poles at $c_i(p_i) = c_j(p_j)$, of order less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \ne j$.

As for the second condition in the definition of composability, we note that, on each transversal section, the domains of absolute convergensy $c_i(p_i) \ne c_j(p_j)$, $i \ne j$

$$|c_i(p_i)| > |c_k(p_j)| > 0,$$

for $i = 1, \ldots, m$, and $k = 1 + m, \ldots, n + m$, for

$$\mathcal{J}^n_m(\Phi) = \sum_{q \in \mathbb{C}} \langle w', \omega_W(v_1, c_1(p_1)) \ldots \omega_W(v_m, c_m(p_m)) \\ P_q(\Phi(v_{1+m}, c_{1+m}(p_{1+m}); \ldots; v_{n+m}, c_{n+m}(p_{n+m}))\rangle, \tag{4.5}$$

are limited from below by the same set of the absolute values of the local coordinates on the transversal section. Thus, under the intersection in (4.2), this condition is preserved, and the sum (2.14) can be analytically extended to a rational function in $(c_1(p_1), \ldots, c_{m+n}(p_{m+n}))$ with the only possible poles at $c_i(p_i) = c_j(p_j)$, of orders less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \ne j$. Thus, we proved the lemma. □

Lemma 5. *The double complex* (4.2) *does not depend on the choice of a transversal basis* $\mathcal{U}$.

Proof. Suppose we consider another transversal basis $\mathcal{U}'$ for $\mathcal{F}$. According to the definition, for each transversal section U_i which belongs to the original basis $\mathcal{U}$ in (4.2) there exists a holonomy embedding

$$h'_i : U_i \hookrightarrow U'_j,$$

i.e., it embeds U_i into a section U'_j of our new transversal basis $\mathcal{U}'$. Then consider the sequence of holonomy embeddings $\{h'_k\}$ such that

$$U'_0 \overset{h'_1}{\hookrightarrow} \ldots \overset{h'_k}{\hookrightarrow} U'_k.$$

For the combination of the embeddings $\left\{h'_i, i \geq 0\right\}$ and

$$U_0 \overset{h_1}{\hookrightarrow} \ldots \overset{h_k}{\hookrightarrow} U_k,$$

we obtain commutative diagrams. Since the intersection in (4.2) is performed over all sets of homology mappings, then it is independent on the choice of a transversal basis. □

Thus we then denote $C^n_m(V, \mathcal{W}, \mathcal{U}, \mathcal{F})$ as $C^n_m(V, \mathcal{W}, \mathcal{F})$. Recall the notation of a quasi-conformal grading-restricted vertex algebra given in "Appendix 8.2". The main statement of this section is contained in the following

Proposition 8. *For a quasi-conformal grading-restricted vertex algebra V and its grading-restricted generalized module W, the construction* (4.2) *is canonical, i.e., does not depend on the foliation preserving choice of local coordinates on* $\mathcal{M}/\mathcal{F}$.

Proof. Here we prove that for the generic elements of a quasi-conformal grading-restricted vertex algebra V, the maps Φ (2.19) and the operators $\omega_W \in \mathcal{W}_{z_1,\ldots,z_n}$ (2.21) are canonical, i.e., independent on changes

$$z_i \mapsto w_i = \rho(z_i), \quad 1 \leq i \leq n, \tag{4.6}$$

of the local coordinates of $c_i(p_i)$ and $c_j(p'_j)$ at points p_i and p'_j, $1 \leq i \leq n$, $1 \leq j \leq k$. Thus the construction of the double complex spaces (4.2) is proved to be canonical too. Let us denote

$$\xi_i = \left(\beta_0^{-1}\, dw_i\right)^{\mathrm{wt}(v_i)}.$$

Recall the linear operator (2.21) (cf. "Appendix 8.2"). Introduce the action of the transformations (4.6) as

$$\begin{aligned} &\Phi\left(dw_1^{\mathrm{wt}(v_1)} \otimes v_1, w_1; \ldots; dw_n^{\mathrm{wt}(v_n)} \otimes v_n, w_n\right) \\ &\quad = \left(\frac{df(\zeta)}{d\zeta}\right)^{-L_W(0)} P(f(\zeta))\, \Phi\left(\xi_1 \otimes v_1, z_1; \ldots; \xi_n \otimes v_n, z_n\right). \end{aligned} \tag{4.7}$$

We then obtain

Lemma 6. *An element* (2.8)

$$\Phi\left(dz_1^{\mathrm{wt}(v_1)} \otimes v_1, z_1; \ldots; dz_n^{\mathrm{wt}(v_n)} \otimes v_n, z_n\right),$$

of $\mathcal{W}_{z_1,\ldots,z_n}$ *is invariant under the transformations* (4.6) *of* $\left(\mathrm{Aut}\, \mathcal{O}^{(1)}\right)^{\times n}_{z_1,\ldots,z_n}$.

Proof. Consider (4.7). First, note that

$$f'(\zeta) = \frac{df(\zeta)}{d\zeta} = \sum_{m \geq 0} (m+1)\, \beta_m \zeta^m.$$

By using the identification (9.19) and the $L_W(-1)$-properties (2.2) and (2.7) we obtain

$$\langle w', \Phi\left(dw_1^{\mathrm{wt}(v_1)} \otimes v_1, w_1; \ldots; dw_n^{\mathrm{wt}(v_n)} \otimes v_n, w_n\right)\rangle$$

$$= \langle w', f'(\zeta)^{-L_W(0)} \; P(f(\zeta)) \; \Phi \left(\xi_1 \otimes v_1, z_1; \ldots; \xi_n \otimes v_n, z_n\right)\rangle$$

$$= \langle w', \left(\frac{df(\zeta)}{d\zeta}\right)^{-L_W(0)} \; \Phi \left(dw_1^{\mathrm{wt}(v_1)} \otimes v_1, \sum_{m \geq 0} (m+1) \; \beta_m z_1^{m+1}; \ldots; \right.$$

$$\left. dw_n^{\mathrm{wt}(v_n)} \otimes v_n, \sum_{m \geq 0} (m+1) \; \beta_m z_n^{m+1} \right)\rangle$$

$$= \langle w', \left(\frac{df(\zeta)}{d\zeta}\right)^{-L_W(0)} \; \Phi \left(dw_1^{\mathrm{wt}(v_1)} \otimes v_1, \left(\frac{df(z_1)}{dz_1}\right) z_1; \right.$$

$$\left. \ldots; dw_n^{\mathrm{wt}(v_n)} \otimes v_n, \left(\frac{df(z_n)}{dz_n}\right) z_n \right)\rangle$$

$$= \langle w', \Phi \left(\left(\frac{df(z_1)}{dz_i} \; dw_1\right)^{-\mathrm{wt}(v_1)} \otimes v_1, z_1; \right.$$

$$\left. \ldots; \left(\frac{df(z_n)}{dz_n} \; dw_n\right)^{-\mathrm{wt}(v_n)} \otimes v_n, z_n \right)\rangle$$

$$= \langle w', \Phi \left(dz_1^{\mathrm{wt}(v_1)} \otimes v_1, z_1; \ldots; dz_n^{\mathrm{wt}(v_n)} \otimes v_n, z_n \right)\rangle.$$

Thus we proved the lemma. □

The elements $\Phi(v_1, z_1; \ldots; v_n, z_n)$ of $C^n_k(V, \mathcal{W}, \mathcal{F})$ belong to the space $\mathcal{W}_{z_1,\ldots,z_n}$ and are assumed to be composable with a set of vertex operators $\omega_W(v'_j, c_j(p'_j))$, $1 \leq j \leq k$. The vertex operators $\omega_W(dc(p)^{\mathrm{wt}(v')} \otimes v'_j, c_j(p'_j))$ constitute the particular examples of the mapping of $C^1_\infty(V, \mathcal{W}, \mathcal{F})$ and, therefore, are invariant with respect to (4.6). Thus, the construction of the spaces (4.2) is invariant under the action of the group $(\mathrm{Aut}\ \mathcal{O})^{\times n}_{z_1,\ldots,z_n}$. □

Remark 6. The condition of quasi-conformality is necessary in the proof of invariance of elements of the space $\mathcal{W}_{z_1,\ldots,z_n}$ with respect to a vertex algebraic representation (cf. "Appendix 8.2") of the group $\left(\mathrm{Aut}\ \mathcal{O}^{(1)}\right)^{\times n}_{z_1,\ldots,z_n}$. In what follows, when it concerns the spaces (4.2) we will always assume the quasi-conformality of V.

The proofs of generalizations of Lemmas 4, 5, 7 and Proposition 8 for the case of an arbitrary codimension foliation on a smooth complex manifold of arbitrary dimension will be given elsewhere.

Let W be a grading-restricted generalized V module. Since for $n = 0$, maps Φ do not include variables, and due to Definition 12 of the composability, we can put:

$$C^0_k(V, \mathcal{W}, \mathcal{F}) = W, \tag{4.8}$$

for $k \geq 0$. Nevertheless, according to Definition 4.2, mappings that belong to (4.8) are assumed to be composable with a number of vertex operators depending on local coordinates of k points on k transversal sections.

We observe

Lemma 7.

$$C^n_m(V, \mathcal{W}, \mathcal{F}) \subset C^n_{m-1}(V, \mathcal{W}, \mathcal{F}). \tag{4.9}$$

Proof. Since n is the same for both spaces in (4.9), it only remains to check that the conditions for (2.12) and (2.14) for $\Phi(v_1, c_j(p_1); \ldots; v_n, c_j(p_n))$ of the composability Definition 2.3 with vertex operators are stronger for $C^n_m(V, \mathcal{W}, \mathcal{U}, \mathcal{F})$ then for $C^n_{m-1}(V, \mathcal{W}, \mathcal{U}, \mathcal{F})$. In particular, in the first condition for (2.12) in Definition 12 the difference between the spaces in (4.9) is in indexes. Consider (4.3). For $C^n_{m-1}(V, \mathcal{W}, \mathcal{U}, \mathcal{F})$, the summations in indexes

$$k_1 = l_1 + \cdots + l_{i-1} + 1, \ \ldots, \ k_i = l_1 + \cdots + l_{i-1} + l_i,$$

for the coordinates $c_j(p_1)$,..., $c_j(p_n)$ with $l_1, \ldots, l_n \in \mathbb{Z}_+$, such that $l_1 + \cdots + l_n = n + (m-1)$, and the vertex algebra elements $v_1, \ldots, v_{n+(m-1)}$ are included in the summation for the indexes for $C^n_m(V, \mathcal{W}, \mathcal{U}, \mathcal{F})$. The conditions for the domains of absolute convergence, i.e.,

$$|c_{l_1+\cdots+l_{i-1}+p} - \zeta_i| + |c_{l_1+\cdots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|,$$

for $i, j = 1, \ldots, k$, $i \neq j$, and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$, for the series (2.12) are more restrictive then for $(m-1)$ vertex operators. The conditions for $\mathcal{I}^n_{m-1}(\Phi)$ to be extended analytically to a rational function in $(c_1(p_1), \ldots, c_{n+(m-1)}(p_{n+(m-1)}))$, with positive integers $N^n_{m-1}(v_i, v_j)$, depending only on v_i and v_j for $i, j = 1, \ldots, k$, $i \neq j$, are included in the conditions for $\mathcal{I}^n_m(\Phi)$.

Similarly, the second condition for (2.14), of the absolute convergence and analytical extension to a rational function in $(c_1(p_1), \ldots, c_{m+n}(p_{m+n}))$, with the only possible poles at $c_i(p_i) = c_j(p_j)$, of orders less than or equal to $N^n_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$, for (2.14) when

$$c_i(p_i) \neq c_j(p_j), \ i \neq j, \ |c_i(p_i)| > |c_k(p_k)| > 0,$$

for $i = 1, \ldots, m$, and $k = m+1, \ldots, m+n$ includes the same condition for $\mathcal{J}^n_{m-1}(\Phi)$. Thus we obtain the conclusion of Lemma. □

5. Coboundary Operators and Cohomology of Codimension One Foliations

In this Section we introduce the coboundary operators acting on the double complex spaces.

5.1. Cohomology in terms of connections. In various geometrical configurations, it is sometimes effective to use the interpretation of cohomology in terms of connections [38,39]. That applies, in particular, to our supporting example of the vertex algebra cohomology of codimension one foliations. It is convenient to introduce the multi-point connections over a graded space and to express the coboundary operators and cohomology in terms of connections:

$$\delta^n \phi \in G^{n+1}(\phi),$$
$$\delta^n \phi = G(\phi).$$

Then the cohomology is defined as the factor space

$$H^n = \mathcal{C}on^n_{cl} / G^{n-1},$$

of closed multi-point connections with respect to the space of connection forms defined below.

We continue this Section with the definition of holomorphic multi-point connections on a smooth complex variety. Let $\mathcal{X}$ be a smooth complex variety and $\mathcal{V} \to \mathcal{X}$ a holomorphic vector bundle over $\mathcal{X}$. Let E be the sheaf of holomorphic sections of $\mathcal{V}$. Denote by Ω the sheaf of differentials on $\mathcal{X}$. A holomorphic connection ∇ on E is a $\mathbb{C}$-linear map

$$\nabla : E \to E \otimes \Omega,$$

satisfying Leibniz formula

$$\nabla(f\phi) = \nabla f \phi + \phi \otimes dz,$$

for any holomorphic function f. Motivated by the definition of the holomorphic connection ∇ defined for a vertex algebra bundle (cf. Sect. 6, [7]) over a smooth complex variety $\mathcal{X}$, we introduce the definition of the multiple point holomorphic connection over $\mathcal{X}$.

Definition 19. Let $\mathcal{V}$ be a holomorphic vector bundle over $\mathcal{X}$, and let $\mathcal{X}_0$ be its subvariety. A holomorphic multi-point connection $\mathcal{G}$ on $\mathcal{V}$ is a $\mathbb{C}$-multi-linear map

$$\mathcal{G} : E \to E \otimes \Omega,$$

such that for any holomorphic function f, and two sections $\phi(p)$ and $\psi(p')$ at points p and p' on $\mathcal{X}_0$ correspondingly, we have

$$\sum_{q, q' \mathcal{X}_0 \subset \mathcal{X}} \mathcal{G}\left(f(\psi(q)).\phi(q')\right) = f(\psi(p'))\, \mathcal{G}(\phi(p)) + f(\phi(p))\, \mathcal{G}\left(\psi(p')\right), \quad (5.1)$$

where the summation on left hand side is performed over a locus of points q, q' on $\mathcal{X}_0$. We denote by $\mathcal{C}on_{\mathcal{X}_0}(\mathcal{S})$ the space of such connections defined over a smooth complex variety $\mathcal{X}$. We will call $\mathcal{G}$ satisfying (5.1), a closed connection, and denote the space of such connections by $\mathcal{C}on^n_{\mathcal{X}_0;cl}$.

Geometrically, for a vector bundle $\mathcal{V}$ defined over a complex variety $\mathcal{X}$, a multi-point holomorphic connection (5.1) relates two sections ϕ and ψ of E at points p and p' with a number of sections at a subvariety $\mathcal{X}_0$ of $\mathcal{X}$.

Definition 20. We call

$$G(\phi, \psi) = f(\phi(p))\, \mathcal{G}\left(\psi(p')\right) + f(\psi(p'))\, \mathcal{G}(\phi(p)) - \sum_{q, q' \mathcal{X}_0 \subset \mathcal{X}} \mathcal{G}\left(f(\psi(q')).\phi(q)\right), \quad (5.2)$$

the form of a holomorphic connection $\mathcal{G}$. The space of n-point holomorphic connection forms will be denoted by $G^n(p, p', q, q')$.

Let us formulate another definition which we use in what follows:

Definition 21. We call a multi-point holomorphic connection $\mathcal{G}$ the transversal connection, i.e., when it satisfies

$$f(\psi(p'))\ \mathcal{G}(\phi(p)) + f(\phi(p))\ \mathcal{G}(\psi(p')) = 0. \tag{5.3}$$

We call

$$G_{tr}(p, p') = (\psi(p'))\ \mathcal{G}(\phi(p)) + f(\phi(p))\ \mathcal{G}(\psi(p')), \tag{5.4}$$

the form of a transversal connection. The space of such connections is denoted by G^2_{tr}.

The construction of the vertex algebra cohomology of foliations in terms of connections is parallel to ideas of [13]. Such a construction will be explained elsewhere.

5.2. Coboundary operators. Recall the definitions of the E-operators given in "Appendix .3". Consider the vector of E-operators:

$$\mathcal{E} = \left(E^{(1)}_{W}.,\ \sum_{i=1}^{n} (-1)^i\ E^{(2)}_{V;\mathbf{1}_V}.,\ E^{W;(1)}_{WV}. \right). \tag{5.5}$$

As we see from the definition of the E-operators given in "Appendix .3", when acting on a map $\Phi \in C^n_m(V, \mathcal{W}, \mathcal{F})$, each entry of (5.5) increases the set of the vertex algebra elements $(v_1, \ldots, v_n)$ with a vertex algebra element v_{n+1}. On the other hand, according to Proposition 2, the action of each entry of (5.5) on Φ is composable with $(m-1)$-vertex operators with the vertex algebra elements $(v'_1, \ldots, v'_m)$. Then we formulate

Definition 22. The coboundary operator δ^n_m acting on elements $\Phi \in C^n_m(V, \mathcal{W}, \mathcal{F})$ of the spaces (4.2), is defined by

$$\delta^n_m \Phi = \mathcal{E}.\Phi, \tag{5.6}$$

where . denotes the action given by the vector of each element of $\mathcal{E}$ acting on Φ. A vertex operator added by δ^n_m has a formal parameter associated with an extra point p_{n+1} on $\mathcal{M}$ with a local coordinate $c_{n+1}(p_{n+1})$.

Then we obtain

Lemma 8. *The definition* (5.6) *is equivalent to a multi-point vertex algebra connection*

$$\delta^n_m \Phi = G(p_1, \ldots, p_{n+1}), \tag{5.7}$$

where

$$\begin{aligned} G(p_1, \ldots, p_{n+1}) = \langle w', &\sum_{i=1}^{n} (-1)^i\ \Phi\left(\omega_V\left(v_i, c_i(p_i) - c_{i+1}(p_{i+1})\right)v_{i+1}\right)\rangle, \\ &+\langle w', \omega_W\left(v_1, c_1(p_1)\right)\ \Phi(v_2, c_2(p_2); \ldots; v_{n+1}, c_n(p_{n+1}))\rangle \\ &+(-1)^{n+1} \langle w', \omega_W(v_{n+1}, c_{n+1}(p_{n+1}))\ \Phi(v_1, c_1(p_2); \ldots; v_n, c_n(p_n))\rangle, \end{aligned} \tag{5.8}$$

for arbitrary $w' \in W'$ (dual to W).

Proof. The statement follows from the intertwining operator (cf. "Appendix 8.2") representation of the definition (5.7) in the form

$$\delta_m^n \Phi = \sum_{i=1}^{3} \langle w', e^{\xi_i L_W(-1)} \, \omega_{WV}^W \, (\Phi_i) \, u_i \rangle,$$

for some $\xi_i \in \mathbb{C}$, and $u_i \in V$, and Φ_i obvious from (5.7). Namely,

$$\begin{aligned}
\delta_m^n \Phi &= \langle w', e^{c_1(p_1) L_W(-1)} \, \omega_{WV}^W \, (\Phi \, (v_2, c_2(p_2); \ldots; v_n, c_{n+1}(p_{n+1}) - c_1(p_1)) \, v_1) \rangle \\
&\quad + \sum_{i=1}^{n} (-1)^i e^{\zeta L_W(-1)} \langle w', \omega_{WV}^W \, (\Phi \, (\omega_V \, (v_i, c_i(p_i) - c_{i+1}(p_{i+1})) \, , -\zeta)) \, \mathbf{1}_V \rangle \\
&\quad + \langle w', e^{c_{n+1}(p_{n+1}) L_W(-1)} \, \omega_{WV}^W \, (\Phi \, (v_1, c_1(p_1); \ldots; v_n, c_n(p_n) - c_{n+1}(p_{n+1})) \, v_{n+1}) \rangle,
\end{aligned}$$

for an arbitrary $\zeta \in \mathbb{C}$. □

Remark 7. Inspecting the construction of the double complex spaces (4.2) we see that the action (5.8) of the δ_m^n on an element of $C_m^n(V, \mathcal{W}, \mathcal{F})$ provides a coupling (in terms of differential forms of $\mathcal{W}_{z_1,\ldots,z_n}$) of the vertex operators taken at the local coordinates $c_i(z_{p_i})$, $0 \le i \le k$, at the vicinities of the same points p_i taken on transversal sections for $\mathcal{F}$, with elements of $C_{m-1}^n(V, \mathcal{W}, \mathcal{F})$ taken at the points with the local coordinates $c_i(z_{p_i})$, $0 \le i \le n$ on $\mathcal{M}$ for the points p_i considered on the leaves of $\mathcal{M}/\mathcal{F}$.

5.3. Complexes on transversal connections. In addition to the double complex $(C_m^n(V, \mathcal{W}, \mathcal{F}), \delta_m^n)$ provided by (4.2) and (5.7), there exists an exceptional short double complex which we call the transversal connection complex. We have

Lemma 9. *For $n = 2$, and $k = 0$, there exists a subspace $C_{ex}^0(V, \mathcal{W}, \mathcal{F})$*

$$C_m^2(V, \mathcal{W}, \mathcal{F}) \subset C_{ex}^0(V, \mathcal{W}, \mathcal{F}) \subset C_0^2(V, \mathcal{W}, \mathcal{F}),$$

for all $m \ge 1$, with the action of the coboundary operator δ_m^2 defined by (5.7).

Proof. Let us consider the space $C_0^2(V, \mathcal{W}, \mathcal{F})$. It consists of $\mathcal{W}_{c_1(p_1), c_2(p_2)}$-elements with zero vertex operators composable. The space $C_0^2(V, \mathcal{W}, \mathcal{F})$ contains elements of $\mathcal{W}_{c_1(p_1), c_2(p_2)}$ so that the action of δ_0^2 is zero. Nevertheless, as for $\mathcal{J}_m^n(\Phi)$ in (2.14), Definition 12, let us consider the sum of the projections

$$P_r : \mathcal{W}_{z_i, z_j} \to W_r,$$

for $r \in \mathbb{C}$, and $(i, j) = (1, 2), (2, 3)$, so that the condition (2.14) is satisfied for some connections similar to the action (2.14) of δ_0^2. Separating the first two and the second two summands in (5.8), we find that for a subspace of $C_0^2(V, \mathcal{W}, \mathcal{F})$, which we denote as $C_{ex}^2(V, \mathcal{W}, \mathcal{F})$, consisting of three-point connections Φ such that for $v_1, v_2, v_3 \in V$, $w' \in W'$, and arbitrary $\zeta \in \mathbb{C}$, the following forms of connections

$$\begin{aligned}
&G_1(c_1(p_1), c_2(p_2), c_3(p_3)) \\
&\quad = \sum_{r \in \mathbb{C}} \Big(\langle w', E_W^{(1)} \, (v_1, c_1(p_1); \, P_r \, (\Phi \, (v_2, c_2(p_2) - \zeta; \, v_3, c_3(p_3) - \zeta)) \, \rangle
\end{aligned}$$

$$
\begin{aligned}
&+\langle w', \Phi\left(v_1, c_1(p_1);\, P_r\left(E^{(2)}_V\left(v_2, c_2(p_2)-\zeta;\, v_3, c_3(z_3)-\zeta;\, \mathbf{1}_V\right), \zeta\right)\rangle\right) \\
&= \sum_{r\in\mathbb{C}} \big(\langle w', \omega_W\left(v_1, c_1(p_1)\right)\ P_r\left(\Phi\left(v_2, c_2(p_2)-\zeta;\, v_3, c_3(p_3)-\zeta\right)\right)\rangle \\
&\quad +\langle w', \Phi\left(v_1, c_1(p_1);\, P_r\left(\omega_V\left(v_2, c_2(p_2)-\zeta\right)\omega_V\left(v_3, c_3(z_3)-\zeta\right)\mathbf{1}_V\right), \zeta\right)\rangle\big),
\end{aligned} \tag{5.9}
$$

and

$$
\begin{aligned}
&G_2(c_1(p_1), c_2(p_2), c_3(p_3)) \\
&= \sum_{r\in\mathbb{C}} \Big(\langle w', \Phi\left(P_r\left(E^{(2)}_V\left(v_1, c_1(p_1)-\zeta, v_2, c_2(p_2)-\zeta;\, \mathbf{1}_V\right)\right), \zeta;\, v_3, c_3(p_3)\right)\rangle \\
&\quad +\langle w', E^{W;(1)}_{WV}\left(P_r\left(\Phi\left(v_1, c_1(p_1)-\zeta;\, v_2, c_2(p_2)-\zeta\right), \zeta;\, v_3, c_3(p_3)\right)\right)\rangle\Big) \\
&= \sum_{r\in\mathbb{C}} \big(\langle w', \Phi\left(P_r(\omega_V\left(v_1, c_1(p_1)-\zeta\right)\omega_V\left(v_2, c_2(p_2)-\zeta\right)\mathbf{1}_V, \zeta)\right);\, v_3, c_3(p_3))\rangle \\
&\quad +\langle w', \omega_V\left(v_3, c_3(p_3)\right)\ P_r\left(\Phi\left(v_1, c_1(p_1)-\zeta;\, v_2, c_2(p_2)-\zeta\right)\right)\rangle\big),
\end{aligned} \tag{5.10}
$$

are absolutely convergent in the regions

$$
\begin{aligned}
&|c_1(p_1)-\zeta| > |c_2(p_2)-\zeta|, \\
&|c_2(p_2)-\zeta| > 0, \\
&|\zeta - c_3(p_3)| > |c_1(p_1)-\zeta|, \\
&|c_2(p_2)-\zeta| > 0,
\end{aligned}
$$

where c_i, $1 \le i \le 3$ are coordinate functions, respectively, and can be analytically extended to rational functions in $c_1(p_1)$ and $c_2(p_2)$ with the only possible poles at $c_1(p_1), c_2(p_2) = 0$, and $c_1(p_1) = c_2(p_2)$. Note that (5.9) and (5.10) constitute the first two and the last two terms of (5.8) correspondingly. According to Proposition 2 (cf. Sect. 2.3), $C^2_m(V, \mathcal{W}, \mathcal{F})$ is a subspace of $C^2_{ex}(V, \mathcal{W}, \mathcal{F})$, for $m \ge 0$, and $\Phi \in C^2_m(V, \mathcal{W}, \mathcal{F})$ are composable with m vertex operators. Note that (5.9) and (5.10) represent the sums of the forms $G_{tr}(p, p')$ of the transversal connections (5.4) (cf. Sect. 5.1). □

Remark 8. It is important to mention that, according to the general principle observed in [1], for the non-vanishing connection $G(c(p), c(p'), c(p''))$, there exists an invariant structure, e.g., a cohomological class. In our case, it appears as a non-empty subspace $C^2_m(V, \mathcal{W}, \mathcal{F}) \subset C^2_{ex}(V, \mathcal{W}, \mathcal{F})$ in $C^2_0(V, \mathcal{W}, \mathcal{F})$.

Then we have

Definition 23. The coboundary operator

$$
\delta^2_{ex} : C^2_{ex}(V, \mathcal{W}, \mathcal{F}) \to C^3_0(V, \mathcal{W}, \mathcal{F}), \tag{5.11}
$$

is defined by the three point connection of the form

$$
\delta^2_{ex}\Phi = \mathcal{E}_{ex} \cdot \Phi = G_{ex}(p_1, p_2, p_3), \tag{5.12}
$$

where

$$
\mathcal{E}_{ex} = \left(E^{(1)}_W .,\ \sum_{i=1}^{2}(-1)^n E^{(2)}_{V;\mathbf{1}_V} .,\ E^{W;(1)}_{WV} .\right), \tag{5.13}
$$

$$\begin{aligned} G_{ex}(p_1, p_2, p_3) = \langle w', \omega_W\,(v_1, c_1(p_1))\;\; \Phi\,(v_2, c_2(p_2); v_3, c_3(p_3))\rangle \\ -\langle w', \Phi\,(\omega_V(v_1, c_1(p_1))\;\; \omega_V(v_2, c_2(p_2))\mathbf{1}_V; v_3, c_3(p_3))\rangle \\ +\langle w', \Phi(v_1, c_1(p_1);\;\; \omega_V(v_2, c_2(p_2))\; \omega_V(v_3, c_3(p_3))\mathbf{1}_V)\rangle \\ +\langle w', \omega_W(v_3, c_3(p_3))\; \Phi\,(v_1, c_1(p_1); v_2, c_2(p_2))\rangle, \end{aligned} \tag{5.14}$$

for $w' \in W'$, $\Phi \in C^2_{ex}(V, \mathcal{W}, \mathcal{F})$, $v_1, v_2, v_3 \in V$ and $(z_1, z_2, z_3) \in F_3\mathbb{C}$.

Then we have

Proposition 9. *The operators* (5.7) *and* (5.11) *provide the chain-cochain complexes*

$$\delta^n_m : C^n_m(V, \mathcal{W}, \mathcal{F}) \to C^{n+1}_{m-1}(V, \mathcal{W}, \mathcal{F}), \tag{5.15}$$

$$\delta^{n+1}_{m-1} \circ \delta^n_m = 0, \tag{5.16}$$

$$\delta^2_{ex} \circ \delta^1_2 = 0, \tag{5.17}$$

$$0 \longrightarrow C^0_m(V, \mathcal{W}, \mathcal{F}) \xrightarrow{\delta^0_m} C^1_{m-1}(V, \mathcal{W}, \mathcal{F}) \xrightarrow{\delta^1_{m-1}} \cdots \xrightarrow{\delta^{m-1}_1} C^m_0(V, \mathcal{W}, \mathcal{F}) \longrightarrow 0, \tag{5.18}$$

$$0 \longrightarrow C^0_3(V, \mathcal{W}, \mathcal{F}) \xrightarrow{\delta^0_3} C^1_2(V, \mathcal{W}, \mathcal{F}) \xrightarrow{\delta^1_2} C^2_{ex}(V, \mathcal{W}, \mathcal{F}) \xrightarrow{\delta^2_{ex}} C^3_0(V, \mathcal{W}, \mathcal{F}) \longrightarrow 0, \tag{5.19}$$

on the spaces (4.2).

Since

$$\delta^1_2\; C^1_2(V, \mathcal{W}, \mathcal{F}) \subset C^2_1(V, \mathcal{W}, \mathcal{F}) \subset C^2_{ex}(V, \mathcal{W}, \mathcal{F}),$$

it follows that

$$\delta^2_{ex} \circ \delta^1_2 = \delta^2_1 \circ \delta^1_2 = 0.$$

Proof. The proof of this proposition is analogous to that of Proposition (4.1) in [43] for the chain-cochain complex of a grading-restricted vertex algebra. The only difference is that we work with the space $\mathcal{W}_{c_1(p_1),\ldots,c_n(p_n)}$ instead of $\overline{W}_{z_1,\ldots,z_n}$. Let $\Phi \in C^n_m(V, \mathcal{W}, \mathcal{F})$. First, we show that $\delta^n_m \Phi \in C^{n+1}_{m-1}(V, \mathcal{W}, \mathcal{F})$. Indeed, it is clear from (5.5), the definitions of the E-operators given in "Appendix .3", and Definitions (4.2) and 22 that the action of the coboundary operator δ^n_m increases the number of the vertex operators for Φ with the vertex algebra elements $(v_1, \ldots, v_{n+1})$ with the local coordinates $c_i(p_i)$ around the points p_i, $1 \le i \le n+1$. Simultaneously, according to Proposition 2, it decreases the number of the vertex operators with the vertex algebra elements $(v'_1, \ldots, v'_k)$ and with the local coordinates $c_j(p'_j)$, $1 \le j \le k$, with which Φ is composable. Note that $\delta^n_m \Phi$ has the $L(-1)$-derivative property and the $L(0)$-conjugation property. Therefore, it decreases the number of the transversal sections for $\delta^n_m \Phi$ according to the definition (4.2). Thus, $\delta^n_m \Phi \in C^{n+1}_{m-1}(V, \mathcal{W}, \mathcal{F})$. Then consider

$$\begin{aligned} &\delta^{n+1}_{m-1}\; G(p_1, \ldots, p_{n+1}) \\ &= \left(E^{(1)}_W .,\; \sum_{i=1}^{n} (-1)^i\; E^{(2)}_{V;\mathbf{1}_V} .,\;\; E^{W;(1)}_{WV} . \right) .G(p_1, \ldots, p_{n+2}) \\ &= E^{(1)}_W .(E^{(1)}_W .\Phi) + \sum_{j=1}^{n} (-1)^j E^{(1)}_W .(\mathrm{Ins}_j (E^{(2)}_{V;\mathbf{1}} .)\Phi) \end{aligned}$$

$$+(-1)^{n+1} E^{(1)}_{W}.(\sigma_{n+1,1,\dots,n}(E^{(1)}_{W}.\Phi)) - E^{(2)}_{V;\mathbf{1}}.(E^{(1)}_{W}.\Phi)$$
$$+\sum_{i=1}^{n+1}(-1)^{i} \sum_{j=i+2}^{n+1} (-1)^{j-1}\mathrm{Ins}_{i}(E^{(2)}_{V;\mathbf{1}}.)(\mathrm{Ins}_{j}(E^{(2)}_{V\mathbf{1}}.)\Phi)$$
$$+\sum_{i=1}^{n+1}(-1)^{i} \sum_{j=1}^{i-2}(-1)^{j}\mathrm{Ins}_{i}(E^{(2)}_{V;\mathbf{1}}.)(\mathrm{Ins}_{j}(E^{(2)}_{V;\mathbf{1}}.)\Phi)$$
$$+(\sigma_{n+1,1,\dots,n}\mathrm{Ins}_{n+1}(E^{(2)}_{V;\mathbf{1}}.)(E^{(1)}_{W}.\Phi)) - \sigma_{n+2,1,\dots,n+1}(E^{(1)}_{W}.\sigma_{n+1,1,\dots,n}(E^{(1)}_{W}.\Phi)),$$

where $\mathrm{Ins}_i(E)$ denotes the insertion of the action of the E-operator at the ith position. In [43] it was proven that

$$E^{(1)}_{W}.(E^{(1)}_{W}.\Phi) = E^{(2)}_{V;\mathbf{1}}.(E^{(1)}_{W}.\Phi),$$
$$\sum_{j=1}^{n}(-1)^{j} E^{(1)}_{W}.(\mathrm{Ins}_{j}(E^{(2)}_{V;\mathbf{1}}.)\Phi) = -\sum_{i=2}^{n+1}(-1)^{i}\mathrm{Ins}_{i}(E^{(2)}_{V;\mathbf{1}}.)(E^{(1)}_{W}.\Phi),$$
$$E^{(1)}_{W}.(\sigma_{n+1,1,\dots,n}(E^{(1)}_{W}.\Phi)) = \sigma_{n+2,1,\dots,n+1}(E^{(1)}_{W}.(E^{(1)}_{W}.\Phi)),$$
$$\sum_{i=1}^{n+1}(-1)^{i} \sum_{j=1}^{i-2}(-1)^{j}\mathrm{Ins}_{i}(E^{(2)}_{V;\mathbf{1}}.)(\mathrm{Ins}_{j}(E^{(2)}_{V;\mathbf{1}}.)\Phi)$$
$$= -\sum_{i=1}^{n+1}(-1)^{i} \sum_{j=i+2}^{n+1} (-1)^{j-1}\mathrm{Ins}_{i}(E^{(2)}_{V;\mathbf{1}}.)(\mathrm{Ins}_{j}(E^{(2)}_{V\mathbf{1}}.)\Phi),$$
$$(\sigma_{n+1,1,\dots,n}\mathrm{Ins}_{n+1}(E^{(2)}_{V;\mathbf{1}}.)(E^{(1)}_{W}.\Phi)) = \sigma_{n+2,1,\dots,n+1}(E^{(1)}_{W}.\sigma_{n+1,1,\dots,n}(E^{(1)}_{W}.\Phi)).$$

Therefore, $\delta^n_m G(p_1, \dots, p_{n+1})$ vanishes and one has (5.16). Similar considerations are applicable to (5.19) and (5.17). □

5.4. The vertex algebra cohomology and its relation to the Crainic and Moerdijk construction. Now let us define the cohomology associated with a grading-restricted vertex algebra V of the space of leaves $\mathcal{M}/\mathcal{F}$ for a codimension one foliation $\mathcal{F}$.

Definition 24. We define the nth cohomology $H^n_k(V, \mathcal{W}, \mathcal{F})$ of $\mathcal{M}/\mathcal{F}$ with coefficients in $\mathcal{W}_{z_1,\dots,z_n}$ (containing maps composable with k vertex operators on k transversal sections) to be the factor space of closed multi-point connections by the space of connection forms:

$$H^n_k(V, \mathcal{W}, \mathcal{F}) = \mathcal{C}on^n_{k;\ cl}/G^{n-1}_{k+1}. \tag{5.20}$$

Note that due to (5.8), (5.14), and Definitions 5.1 and 5.2 (cf. Sect. 5), it is easy to see that (5.20) is equivalent to the standard cohomology definition

$$H^n_k(V, \mathcal{W}, \mathcal{F}) = \ker \delta^n_k / \mathrm{Im}\ \delta^{n-1}_{k+1}. \tag{5.21}$$

Recall the construction of Čech-de Rham cohomology of a foliation [15]. Consider a foliation $\mathcal{F}$ of codimension one defined on a smooth complex curve $\mathcal{M}$. Consider the double complex

$$C^{k,l} = \prod_{U_0 \overset{h_1}{\hookrightarrow} \cdots \overset{h_k}{\hookrightarrow} U_k} \Omega^l(U_0), \tag{5.22}$$

where $\Omega^l(U_0)$ is the space of differential l-forms on U_0, and the product ranges over all k-tuples of holonomy embeddings between transversal sections from a fixed transversal basis $\mathcal{U}$. Components of $\varpi \in C^{k,l}$ are denoted by $\varpi(h_1, \ldots, h_l) \in \Omega^l(\mathcal{U}_0)$. The vertical differential is defined as

$$(-1)^k d : C^{k,l} \to C^{k,l+1},$$

where d is the usual de Rham differential. The horizontal differential

$$\delta : C^{k,l} \to C^{k+1,l},$$

is given by

$$\delta = \sum_{i=1}^{k} (-1)^i \delta_i,$$

$$\delta_i \varpi(h_1, \ldots, h_{k+1}) = G(h_1, \ldots, h_{k+1}), \tag{5.23}$$

where $G(h_1, \ldots, h_{k+1})$ is the multi-point connection of the form (5.1), i.e.,

$$\delta_i \varpi(h_1, \ldots, h_{p+1}) = \begin{cases} h_1^* \varpi(h_2, \ldots, h_{p+1}), & \text{if } i = 0, \\ \varpi(h_1, \ldots, h_{i+1} h_i, \ldots, h_{p+1}), & \text{if } 0 < i < p+1, \\ \varpi(h_1, \ldots, h_p), & \text{if } i = p+1. \end{cases} \tag{5.24}$$

This double complex is actually a bigraded differential algebra, with the usual product

$$(\varpi \cdot \eta)(h_1, \ldots, h_{k+k'}) = (-1)^{kk'} \varpi(h_1, \ldots, h_k)\ h_1^* \ldots h_k^* . \eta(h_{k+1}, \ldots h_{k+k'}), \tag{5.25}$$

for $\varpi \in C^{k,l}$ and $\eta \in C^{k',l'}$, thus $(\varpi \cdot \eta)(h_1, \ldots, h_{k+k'}) \in C^{k+k',l+l'}$.

Definition 25. The cohomology $\check{H}^*_{\mathcal{U}}(M/\mathcal{F})$ of this complex is called Čech-de Rham cohomology of the leaf space $\mathcal{M}/\mathcal{F}$ with respect to the transversal basis $\mathcal{U}$. It is defined by

$$\check{H}^*_{\mathcal{U}}(M/\mathcal{F}) = \mathcal{C}on^{k+1}_{cl}(h_1, \ldots, h_{k+1}) / G^k(h_1, \ldots, h_k),$$

where $\mathcal{C}on^{k+1}_{cl}(h_1, \ldots, h_{k+1})$ is the space of closed multi-point connections, and G^k $(h_1, \ldots, h_k)$ is the space of k-point connection forms.

In this Subsection we show the following

Lemma 10. *In the case of a codimension one foliation on a smooth complex curve, the construction of the double complex* $\left(C^{k,l}, \delta\right)$, (5.22), (5.23) *results from the construction of the double complexes* $\left(C^n_m(V, \mathcal{W}, \mathcal{F}), \delta^n_m\right)$ *of* (5.18) *and* (5.19).

Proof. One constructs the space of differential forms of degree k

$$\langle w', \Phi\left(dc_1(p_1)^{\mathrm{wt}(v_1)} \otimes v_1, c_1(p_1); \ldots; dc_n(p_n)^{\mathrm{wt}(v_n)} v_n, c_n(p_n)\right)\rangle, \tag{5.26}$$

by the elements Φ of $C^n_m(V, \mathcal{W}, \mathcal{F})$ such that $n = k$ and the total degree

$$\sum_{i=1}^{n} \mathrm{wt}(v_i) = l,$$

$v_i \in V$. The condition of the composability of Φ with m vertex operators allows us make the association of the differential form $\varpi(h_1, \ldots, h_n)$ with (5.26), and $(h^*_1, \ldots, h^*_k)$ with $(v_i, \ldots, v_k)$, and to represent a sequence of the holomorphic embeddings $h_1, \ldots, h_p$ for $U_0, \ldots, U_p$ in (5.22) by the vertex operators ω_W, i.e,

$$\left(h(h^*_1) \ldots h(h^*_n)\right)(z_1, \ldots, z_n)) = \omega_W\left(v_1, t_1(p_1)\right) \ldots \omega_W\left(v_l, t(p_n)\right).$$

Then, by using the definition of the coboundary operator (5.7), we see that the definition of the coboundary operator of [15] is parallel to the definition (5.7). ◻

6. The Product of $C^n_m(V, \mathcal{W}, \mathcal{F})$-Spaces

In this Section we consider the application of the material of Sect. 3 to the double complex spaces $C^n_m(V, \mathcal{W}, \mathcal{F})$ (Definition 18, Sect. 4) for a foliation $\mathcal{F}$ on a complex curve. We introduce the product of two double complex spaces with the image in another double complex space coherent with respect to the original coboundary operators (5.7) and (5.11), and the symmetry property (2.9). We prove the canonicity of the product, and derive an analogue of Leibniz formula.

6.1. The geometrical adaptation of the ϵ-product to a foliation. In this Subsection we show how the definition of the product of $\mathcal{W}_{z_1,\ldots,z_n}$-spaces can be extended to the case of $C^k_n(V, \mathcal{W}, \mathcal{F})$-spaces for a codimension one foliation of a complex curve. Recall the definition (4.2) of $C^k_n(V, \mathcal{W}, \mathcal{F})$-spaces in Sect. 4. We use again the geometrical scheme of the sewing of two Riemann surfaces in order to introduce the product of two elements $\Phi \in C^k_m(V, \mathcal{W}, \mathcal{F})$ and $\Psi \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$ which belong to two double complex spaces (4.2) for a foliation $\mathcal{F}$. The construction is again local, thus we assume that both spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ and $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$ are considered on the same fixed transversal basis $\mathcal{U}$. Moreover, we assume that the marked points used in the definition (4.2) of the spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ and $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$ are chosen on the same transversal section.

Let us recall again the setup for two double complex spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ and $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$. Let $(p_1, \ldots, p_k)$, $(\widetilde{p}_1, \ldots, \widetilde{p}_n)$ be two sets of points with the local coordinates $(c_1(p_1), \ldots, c_k(p_k))$ and $(\widetilde{c}_1(\widetilde{p}_1), \ldots, \widetilde{c}_n(\widetilde{p}_n))$ taken on the jth transversal section $U_j \in \mathcal{U}$, $j \geq 1$, of the transversal basis $\mathcal{U}$. For $k \geq 0$, $n \geq 0$, let $C^k(V, \mathcal{W}, \mathcal{F})(U_j)$ and $C^n(V, \mathcal{W}, \mathcal{F})(U_j)$, $0 \leq j \leq l$, be as before the spaces of all linear maps (2.18)

$$\begin{aligned} \Phi &: V^{\otimes k} \to \mathcal{W}_{c_1(p_1),\ldots,c_k(p_k)}, \\ \Psi &: V^{\otimes n} \to \mathcal{W}_{\widetilde{c}_1(\widetilde{p}_1),\ldots,\widetilde{c}_n(\widetilde{p}_n)}, \end{aligned} \tag{6.1}$$

composable with l_1 and l_2 vertex operators (2.21) with the formal parameters identified with the local coordinate functions $c'_j(p'_j)$ and $\widetilde{c}'_j(p'_{j'})$ around points p_j, $p'_{j'}$, on each of the transversal sections U_j, $1 \le j \le l_1$, and $U_{j'}$, $1 \le j' \le l_2$, correspondingly. Then, for $k \ge 0$, $1 \le m \le l_1$, and $n \ge 0$, and $1 \le m' \le l_2$,

according to the definition (4.2), the spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ and $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$ are:

$$C^k_m(V, \mathcal{W}, \mathcal{F}) = \bigcap_{\substack{U_1 \overset{h_1}{\hookrightarrow} \dots \overset{h_{m-1}}{\hookrightarrow} U_m \\ 1 \le i \le m}} C^k(V, \mathcal{W}, \mathcal{F})(U_i), \tag{6.2}$$

$$C^n_{m'}(V, \mathcal{W}, \mathcal{U}, \mathcal{F}) = \bigcap_{\substack{U_1 \overset{h'_1}{\hookrightarrow} \dots \overset{h'_{m'-1}}{\hookrightarrow} U_{m'} \\ 1 \le i' \le m'}} C^n(V, \mathcal{W}, \mathcal{F})(U_{i'}), \tag{6.3}$$

where the intersection ranges over all possible m- and m'-tuples of the holonomy embeddings h_i, $i \in \{1, \dots, m-1\}$, and $h'_{i'}$, $i' \in \{1, \dots, m'-1\}$, between the transversal sections $(U_1, \dots, U_m)$ and $(U_1, \dots, U_{m'})$ of the basis $\mathcal{U}$ for $\mathcal{F}$.

Let t be the number of the common vertex operators for the mappings $\Phi(v_1, x_1; \dots; v_k, x_k) \in C^k_m(V, \mathcal{W}, \mathcal{F})$ and $\Psi(v'_1, y_1; \dots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$ are composable with. We then have the main proposition of this Section

Proposition 10. *For* $\Phi(v_1, x_1; \dots; v_k, x_k) \in C^k_m(V, \mathcal{W}, \mathcal{F})$ *and* $\Psi(v'_1, y_1; \dots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$*, the product* $\widehat{R}\ \Theta\left(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon\right)$ (3.16) *belongs to the space* $C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F})$*, i.e.,*

$$\cdot_\epsilon : C^k_m(V, \mathcal{W}, \mathcal{F}) \times C^n_{m'}(V, \mathcal{W}, \mathcal{F}) \to C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F}). \tag{6.4}$$

Proof. In Proposition 5 we proved that $\widehat{R}\ \Theta\left(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon\right) \in \mathcal{W}_{x_1; ,\dots, x_k; y_1, \dots, y_n}$. Namely, the differential form corresponding to the ϵ-product $\widehat{R}$ $\Theta(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon)$ converges in ϵ, and satisfies (2.9), the $L_V(0)$-conjugation (2.7) and the $L_V(-1)$-derivative (2.2) properties. The action of $\sigma \in S_{k+n-r}$ on the product $\Theta(v_1, x_1; \dots; v_k, x_k; v'_{k+1}, y_1; \dots; v'_n, y_n; \epsilon)$ (3.16) is given by (2.1). Then we see that for the sets of points $(p_1, \dots, p_k; p'_1, \dots, p'_n)$, taken on the same transversal section $U_j \in \mathcal{U}$, $j \ge 1$, by Proposition 5 we obtain a map

$$\begin{aligned} &\widehat{R}\ \Theta\left(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon\right) \\ &\qquad : V^{\otimes(k+n)} \to \mathcal{W}_{c''_1(p''_1), \dots, c''_1(p''_{k+n-r})}, \end{aligned} \tag{6.5}$$

with the formal parameters $(z_1, \dots, z_{k+n-r})$ identified with the local coordinates $(c''_1(p''_1), \dots, c''_1(p''_{k+n-r}))$ of the points

$$(p''_1, \dots, p''_{k+n-r}) = (p_1, \dots, p_k; p_1, \dots, \widehat{p'_{i_l}}, \dots, p'_n),$$

for the coinciding points $p_{i_l} = p'_{j_l}$, $1 \le l \le r$. Next, we prove

Proposition 11. *The product* $\Theta\left(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n; \epsilon\right)$ (3.16) *is composable with* $m + m' - t$ *vertex operators.*

First we note

Lemma 11.

$$\sum_{q\in\mathbb{C}} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(\Theta(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}\Big)\Big)\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E_W^{(m)}\Big(v_{k+1}, x_{k+1}; \ldots; v_{k+m}, x_{k+m};$$
$$P_q\Big(Y_{WV}^W\,(\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1)\ u\Big)\Big)\rangle$$
$$\langle w', E_W^{(m')}\Big(v'_{n+1}, y_{n+1}; \ldots; v'_{n+m'}, y_{n+m'};$$
$$P_q\Big(Y_{WV}^W\,\big(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\big)\ \overline{u}\Big)\Big)\rangle.$$

Proof. Consider

$$\sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(Y_{WV}^W\,\big(\Phi(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k}), \zeta_1\big)\ u\Big)\Big)\rangle$$
$$\langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(Y_{WV}^W\,\big(\Psi\,\big(v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}), \zeta_2\big)\ \overline{u}\big)\Big)\Big)\rangle$$
$$= \sum_{q\in\mathbb{C}} \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(e^{\zeta_1 L_W(-1)}\ Y_W\,(u, -\zeta_1)\ \ \Phi(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k})\Big)\Big)\rangle$$
$$\langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(e^{\zeta_2 L_W(-1)}\ Y_W\,(\overline{u}, -\zeta_2)\ \ \Psi(v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots;$$
$$v''_{m+m'+k+n}, z_{m+m'+k+n})\Big)\Big)\rangle.$$

The action of the exponentials $e^{\zeta_a L_W(-1)}$, $a = 1, 2$, of the differential operator $L_W(-1)$, and a grading-restricted generalized V-module W vertex operators $Y_W\,(u, -\zeta_1)$, $Y_W\,(u, -\zeta_2)$ shifts the grading index q of the W_q-subspaces by $\alpha \in \mathbb{C}$ which can be later rescaled to q. Thus, we can rewrite the last expression as

$$\sum_{q\in\mathbb{C}} \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$e^{\zeta_1 L_W(-1)}\ Y_W\,(u, -\zeta_1)\ \ P_{q+\alpha}\,\big(\Phi(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k})\big)\Big)\rangle$$
$$\langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};\ e^{\zeta_2 L_W(-1)}\ Y_W\,(\overline{u}, -\zeta_2)$$
$$P_{q+\alpha}\Big(\Psi(v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n})\Big)\rangle$$

$$= \sum_{q\in\mathbb{C}} \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$

$$Y_{WV}^W \left(P_{q+\alpha}\Big(\Phi(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k})\Big), \zeta_1\right)\ u\rangle$$

$$\langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$

$$Y_{WV}^W \left(P_{q+\alpha}\Big(\Psi(v''_{m+m'+k+1}, z_{m+m'+k+1};\right.$$

$$\left. \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}), -\zeta_2\right)\ \overline{u}\Big)\rangle$$

$$= \sum_{q\in\mathbb{C}} \sum_{\widetilde{w}\in W} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'}; \widetilde{w}\Big)\rangle$$

$$\sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y_{WV}^W \Big(\ P_{q+\alpha}\Big(\Phi(v''_{m+m'+1}, z_{m+m'+1};$$

$$\ldots; v''_{m+m'+k}, z_{m+m'+k}), -\zeta_1)\ u\Big)\ \Big)\rangle$$

$$\langle \widetilde{w}', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'}; \widetilde{w}\Big)\rangle$$

$$\langle w', Y_{WV}^W \Big(P_{q+\alpha}\Big(\Psi(v''_{m+m'+k+1}, z_{m+m'+k+1};$$

$$\ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}), -\zeta_2)\ \overline{u}\Big)\ \Big)\rangle$$

$$= \sum_{q\in\mathbb{C}} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$

$$P_{q+\alpha}\Big(\Theta(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k};$$

$$v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n})\Big)\Big)\rangle.$$

Now note that, according to Proposition 7, as an element of $\mathcal{W}_{z_1,\ldots,z_{k+n+m+m'}}$

$$\langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_{q+\alpha}\Big(\Theta(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k};$$
$$v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n})\Big)\Big)\rangle, \tag{6.6}$$

is invariant with respect to the action of $\sigma \in S_{k+n+m+m'}$. Thus we are able to use this invariance to show that (6.6) is reduced to

$$\langle w', E_W^{(m+m')}\Big(v''_{k+1}, z_{k+1}; \ldots; v''_{k+1+m}, z_{k+1+m}; v''_{n+1}, z_{n+1}; \ldots; v''_{n+1+m'}, z_{n+1+m'};$$

$$P_{q+\alpha}\Big(\Theta(v''_1, z_1; \ldots; v''_k, z_k; v''_{k+1}, z_{k+1}; \ldots; v''_{k+n}, z_{k+n})\Big)\Big)\rangle$$

$$= \langle w', E_W^{(m+m')}\Big(v_{k+1}, x_{k+1}; \ldots; v_{k+1+m}, x_{k+1+m}; v'_{n+1}, y_{n+1}; \ldots; v'_{n+1+m'}, y_{n+1+m'};$$

$$P_{q+\alpha}\Big(\Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n)\Big)\rangle.$$

Similarly, since

$$\langle w', E_W^{(m)}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(Y_{WV}^W\ \big(\mathcal{F}(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k}, z_{m+m'+k}), \zeta_1\big)\ u\Big)\Big)\rangle,$$
$$\langle w', E_W^{(m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(Y_{WV}^W\ \big(\mathcal{F}(v''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}), \zeta_2\big)\ \overline{u}\Big)\Big)\rangle.$$

correspond to the elements of $\mathcal{W}_{z_1,\ldots,z_{m+m'+k}}$ and $\mathcal{W}_{z_{m+m'+k+1},\ldots,z_{m+m'+k+n}}$, we use Proposition 7 again and obtain

$$\langle w', E_W^{(m)}\Big(v_{k+1}, x_{k+1}; \ldots; v_{k+m}, x_{k+m}; P_q\Big(Y_{WV}^W\ (\mathcal{F}(v_1, x_1; \ldots; v_k, x_k), \zeta_1)\ u\Big)\Big)\rangle$$
$$\langle w', E_W^{(m')}\Big(v'_{n+1}, y_{n+1}; \ldots; v'_{n+m'}, y_{n+m'}; P_q\Big(Y_{WV}^W\ \big(\mathcal{F}(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\big)\ \overline{u}\Big)\Big)\rangle,$$

correspondingly. Thus, the assertion of Lemma follows. □

Under conditions

$$\begin{aligned} z_{i''} &\neq z_{j''}, \quad i'' \neq j'', \\ |z_{i''}| &> |z_{k'''}| > 0, \end{aligned} \tag{6.7}$$

for $i'' = 1, \ldots, m + m'$, and $k''' = m + m' + 1, \ldots, m + m' + k + n$, let us introduce

$$\mathcal{J}_{m+m'}^{k+n}(\Theta) = \sum_{q\in\mathbb{C}} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(\Theta(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}); \epsilon)\Big)\rangle. \tag{6.8}$$

Using Lemma 11 we obtain

$$|\mathcal{J}_{m+m'}^{k+n}(\Theta)|$$
$$= \Bigg|\sum_{q\in\mathbb{C}} \langle w', E_W^{(m+m')}\Big(v''_1, z_1; \ldots; v''_{m+m'}, z_{m+m'};$$
$$P_q\Big(\Theta(v''_{m+m'+1}, z_{m+m'+1}; \ldots; v''_{m+m'+k+n}, z_{m+m'+k+n}); \epsilon)\Big)\rangle\Bigg|$$
$$= \Bigg|\sum_{q\in\mathbb{C}} \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', E_W^{(m)}\Big(v_{k+1}, x_{k+1}; \ldots; v_{k+m}, x_{k+m};$$
$$P_q\Big(Y_{WV}^W\ (\Phi(v_1, x_1; \ldots; v_k, x_k), \zeta_1)\ u\Big)\Big)\rangle$$
$$\langle w', E_W^{(m')}\Big(v'_{n+1}, y_{n+1}; \ldots; v'_{n+m'}, y_{n+m'};$$
$$P_q\Big(Y_{WV}^W\ \big(\Psi(v'_1, y_1; \ldots; v'_n, y_n), \zeta_2\big)\ \overline{u}\Big)\Big)\rangle\Bigg|$$
$$\leq \left|\mathcal{J}_m^k(\mathcal{F})\right|\ \left|\mathcal{J}_{m'}^n(\mathcal{F})\right|,$$

where we have used the invariance of (3.16) with respect to $\sigma \in S_{m+m'+k+n}$. In the last expression, according to Definition 12 $\mathcal{J}^k_m(\Phi)$ and $\mathcal{J}^n_{m'}(\Psi)$ are absolute convergent. Thus, we infer that $\mathcal{J}^{k+n}_{m+m'}(\Theta)$ is absolutely convergent, and the sum (6.19) is analytically extendable to a rational function in $(z_1, \ldots, z_{k+n+m+m'})$ with the only possible poles at $x_i = x_j$, $y_{i'} = y_{j'}$, and at $x_i = y_{j'}$, i.e., the only possible poles at $z_{i''} = z_{j''}$, of orders less than or equal to $N^{k+n}_{m+m'}(v''_{i''}, v''_{j''})$, for $i'', j'' = 1, \ldots, k'''$, $i'' \neq j''$. □

Now, we are on a position to prove Proposition 11.

Proof. Recall that $\Phi(v_1, x_1; \ldots; v_k, x_k)$ is composable with m vertex operators, and $\Psi(v'_1, y_1; \ldots; v'_n, y_n)$ is composable with m' vertex operators. For $\Phi(v_1, x_1; \ldots; v_k, x_k)$ we have:

1) Let $l_1, \ldots, l_k \in \mathbb{Z}_+$ such that $l_1 + \ldots + l_k = k + m$, and $v_1, \ldots, v_{k+m} \in V$, and arbitrary $w' \in W'$. Set

$$\Xi_i = E^{(l_i)}_V(v_{k_1}, x_{k_1} - \zeta_i; \ldots; v_{k_i}, x_{k_i} - \zeta_i; \mathbf{1}_V), \tag{6.9}$$

where

$$k_1 = l_1 + \ldots + l_{i-1} + 1, \quad \cdots, \quad k_i = l_1 + \ldots + l_{i-1} + l_i, \tag{6.10}$$

for $i = 1, \ldots, k$. Then the series

$$\mathcal{I}^k_m(\Phi) = \sum_{r_1, \ldots, r_k \in \mathbb{Z}} \langle w', \Phi(P_{r_1} \Xi_1; \zeta_1; \ldots; P_{r_k} \Xi_k, \zeta_k)\rangle, \tag{6.11}$$

is absolutely convergent when

$$|x_{l_1+\ldots+l_{i-1}+p} - \zeta_i| + |x_{l_1+\ldots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|, \tag{6.12}$$

for $i, j = 1, \ldots, k$, $i \neq j$ and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$. There exist positive integers $N^k_m(v_i, v_j)$, depending only on v_i and v_j for $i, j = 1, \ldots, k$, $i \neq j$, such that the sum is analytically extended to a rational function in $(x_1, \ldots, x_{k+m})$, independent of $(\zeta_1, \ldots, \zeta_k)$, with the only possible poles at $x_i = x_j$, of order less than or equal to $N^k_m(v_i, v_j)$, for $i, j = 1, \ldots, k$, $i \neq j$.

For $\Psi(v'_1, y_1; \ldots; v'_n, y_n)$ we have:

1') Let $l'_1, \ldots, l'_n \in \mathbb{Z}_+$ such that $l'_1 + \ldots + l'_n = n + m'$, $v'_1, \ldots, v_{n+m'} \in V$ and arbitrary $w' \in W'$. Set

$$\Xi'_{i'} = E^{(l'_{i'})}_V(v'_{k'_1}, y_{k'_1} - \zeta'_{i'}; \ldots; v'_{k'_{i'}}, y_{k'_{i'}} - \zeta'_{i'}; \mathbf{1}_V), \tag{6.13}$$

where

$$k'_1 = l'_1 + \cdots + l'_{i'-1} + 1, \quad \ldots, \quad k'_{i'} = l'_1 + \ldots + l'_{i'-1} + l'_{i'}, \tag{6.14}$$

for $i' = 1, \ldots, n$. Then the series

$$\mathcal{I}^n_{m'}(\Psi) = \sum_{r'_1, \ldots, r'_n \in \mathbb{Z}} \langle w', \Psi(P_{r'_1} \Psi'_1; \zeta'_1; \ldots; P_{r'_n} \Psi'_n, \zeta'_n)\rangle, \tag{6.15}$$

is absolutely convergent when

$$|y_{l'_1+\ldots+l'_{i'-1}+p'} - \zeta'_{i'}| + |y_{l'_1+\cdots+l'_{j'-1}+q'} - \zeta'_{i'}| < |\zeta'_{i'} - \zeta'_{j'}|, \tag{6.16}$$

for i', $j' = 1, \ldots, n$, $i' \neq j'$ and for $p' = 1, \ldots, l'_i$ and $q' = 1, \ldots, l'_j$. There exist positive integers $N^n_{m'}(v'_{i'}, v'_{j'})$, depending only on $v'_{i'}$ and $v'_{j'}$ for $i, j = 1, \ldots, n, i' \neq j'$, such that the sum is analytically extended to a rational function in $(y_1, \ldots, y_{n+m'})$, independent of $(\zeta'_1, \ldots, \zeta'_n)$, with the only possible poles at $y_{i'} = y_{j'}$, of order less than or equal to $N^n_{m'}(v'_{i'}, v'_{j'})$, for $i', j' = 1, \ldots, n, i' \neq j'$.

Now let us consider the first condition of Definition 12 of the composability for the product (3.16) of $\Phi(v_1, x_1; \ldots; v_k, x_k)$ and $\Psi(v'_1, y_1; \ldots; v'_n, y_n)$ with a number of vertex operators. Then we obtain for $\Theta\left(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon\right)$ the following. We redefine the notations for the set

$$\begin{aligned}
&(v''_1, \ldots, v''_k; v''_{k+1}, \ldots, v''_{k+m}; v''_{k+m+1}, \ldots, v''_{k+n+m+m'}; v_{n+1}, \ldots, v'_{n+m'}) \\
&\qquad = (v_1, \ldots, v_k; v_{k+1}, \ldots, v_{k+m}; v'_1, \ldots, v'_n; v'_{n+1}, \ldots, v'_{n+m'}), \\
&(z_1, \ldots, z_k; z_{k+1}, \ldots, z_{k+n-r}) = (x_1, \ldots, x_k; y_1, \ldots, y_n),
\end{aligned}$$

of the vertex algebra V elements. Introduce $l''_1, \ldots, l''_{k+n} \in \mathbb{Z}_+$, such that $l''_1 + \cdots + l''_{k+n} = k + n + m + m'$. Define

$$\Xi''_i = E_V^{(l''_{i''})}(v''_{k''_1}, z_{k''_1} - \zeta''_{i''}; \ldots; v''_{k''_{i''}}, z_{k''_{i''}} - \zeta''_{i''}; \mathbf{1}_V), \tag{6.17}$$

where

$$k''_1 = l''_1 + \cdots + l''_{i''-1} + 1, \quad \ldots, \quad k''_{i''} = l''_1 + \cdots + l''_{i''-1} + l''_{i''}, \tag{6.18}$$

for $i'' = 1, \ldots, k + n$, and we take

$$(\zeta''_1, \ldots, \zeta''_{k+n}) = (\zeta_1, \ldots, \zeta_k; \zeta'_1, \ldots, \zeta'_n).$$

Then we consider

$$\mathcal{I}^{k+n}_{m+m'}(\Theta) = \sum_{r''_1, \ldots, r''_{k+n} \in \mathbb{Z}} \langle w', \Theta(P_{r''_1} \Psi''_1; \zeta''_1; \ldots; P_{r''_{k+n}} \Psi''_{k+n}, \zeta''_{k+n}) \rangle, \tag{6.19}$$

and prove it is absolutely convergent with some conditions.

The condition

$$|z_{l''_1 + \cdots + l''_{i-1} + p''} - \zeta''_i| + |z_{l''_1 + \cdots + l''_{j-1} + q''} - \zeta''_i| < |\zeta''_i - \zeta''_j|, \tag{6.20}$$

of the absolute convergence for (6.19) for i'', $j'' = 1, \ldots, k + n$, $i \neq j$ and for $p'' = 1, \ldots, l''_i$ and $q'' = 1, \ldots, l''_j$, follows from the conditions (6.12) and (6.24). The action of $e^{\zeta L_W(-1)}\, Y_W(.,.)$, $a = 1, 2$, in

$$\begin{aligned}
&\langle w', e^{\zeta_1 L_W(-1)}\, Y_W(u, -\zeta) \sum_{r_1, \ldots, r_k \in \mathbb{Z}} \Phi(P_{r_1} \Xi_1; \zeta_1; \ldots; P_{r_k} \Xi_k, \zeta_k) \rangle, \\
&\langle w', e^{\zeta_2 L_W(-1)}\, Y_W(\overline{u}, -\widetilde{\zeta}) \sum_{r'_1, \ldots, r'_n \in \mathbb{Z}} \Psi(P_{r'_1} \Xi'_1; \zeta_1; \ldots; P_{r'_k} \Xi'_n, \zeta'_n) \rangle,
\end{aligned}$$

does not affect the absolute convergence of (6.11) and (6.15). We obtain

$$\left| \mathcal{I}^{k+n}_{m+m'}(\Theta) \right|$$

$$= \left| \sum_{r''_1,\dots,r''_{k+n}\in\mathbb{Z}} \langle w', \Theta(P_{r''_1}\Xi''_1;\zeta''_1;\dots;P_{r''_{k+n}}\Xi''_{k+n},\zeta''_{k+n})\rangle \right|$$

$$= \left| \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle w', Y^W_{VW}(\sum_{r_1,\dots,r_k\in\mathbb{Z}} \Phi(P_{r_1}\Xi_1;\zeta_1;\dots;P_{r_k}\Xi_k,\zeta_k),\zeta)u\rangle \right.$$

$$\left. \langle w', Y^W_{VW}(\sum_{r'_1,\dots,r'_n\in\mathbb{Z}} \Psi(P_{r'_1}\Xi'_1;\zeta'_1;\dots;P_{r'_n}\Xi'_n,\zeta'_n),\widetilde{\zeta})\overline{u}\rangle \right|$$

$$\le \left|\mathcal{I}^k_m(\Phi)\right| \left|\mathcal{I}^n_{m'}(\Psi)\right|.$$

Thus, we infer that (6.19) is absolutely convergent. Recall that the maximal orders of possible poles of (6.19) are $N^k_m(v_i, v_j)$, $N^n_{m'}(v'_{i'}, v'_{j'})$ at $x_i = x_j$, $y_{i'} = y_{j'}$. From the last expression we infer that there exist positive integers $N^{k+n}_{m+m'}(v''_{i''}, v''_{j''})$ for $i, j = 1,\dots,k$, $i \ne j, i', j' = 1,\dots,n, i' \ne j$, depending only on $v''_{i''}$ and $v''_{j''}$ for $i'', j'' = 1,\dots,k+n$, $i'' \ne j''$ such that the series (6.19) can be analytically extended to a rational function in $(x_1,\dots,x_k; y_1,\dots,y_n)$, independent of $(\zeta''_1,\dots,\zeta''_{k+n})$, with extra possible poles at and $x_i = y_j$, of order less than or equal to $N^{k+n}_{m+m'}(v''_{i''}, v''_{j''})$, for $i'', j'' = 1,\dots,n$, $i'' \ne j''$.

Let us proceed with the second condition of the composability. For $\Phi(v_1, x_1;\dots;v_k, x_k) \in C^k_m(V, \mathcal{W}, \mathcal{F})$, and $(v_1,\dots,v_{k+m}) \in V$, $(x_1,\dots,x_{k+m}) \in \mathbb{C}$, we have

2) For arbitrary $w' \in W'$, the series

$$\mathcal{J}^k_m(\Phi) = \sum_{q\in\mathbb{C}} \langle w', E^{(m)}_W\Big(v_1, x_1;\dots;v_m, x_m; P_q(\Phi(v_{m+1}, x_{m+1};\dots;v_{m+k}, x_{m+k}\Big)\rangle, \tag{6.21}$$

is absolutely convergent when

$$\begin{aligned} x_i &\ne x_j, \quad i \ne j, \\ |x_i| &> |x_{k'}| > 0, \end{aligned} \tag{6.22}$$

for $i = 1,\dots,m$, and $k' = m+1,\dots,k+m$, and the sum can be analytically extended to a rational function in $(x_1,\dots,x_{k+m})$ with the only possible poles at $x_i = x_j$, of orders less than or equal to $N^k_m(v_i, v_j)$, for $i, j = 1,\dots,k$, $i \ne j$.

2') For $\Psi(v'_1, y_1;\dots;v'_n, y_n) \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, $(v'_1,\dots,v'_{n+m'}) \in V$, and $(y_1,\dots,y_{n+m'}) \in \mathbb{C}$, the series

$$\begin{aligned} \mathcal{J}^n_{m'}(\Psi) = \sum_{q\in\mathbb{C}} \langle w', E^{(m')}_W\Big(& v'_1, y_1;\dots;v'_{m'}, y_{m'}; \\ & P_q(\Psi(v'_{m'+1}, y_{m'+1};\dots;v'_{m'+n}, y_{m'+n}))\Big)\rangle, \end{aligned} \tag{6.23}$$

is absolutely convergent when

$$\begin{aligned} y_{i'} &\ne y_{j'}, \quad i' \ne j', \\ |y_{i'}| &> |y_{k''}| > 0, \end{aligned} \tag{6.24}$$

for $i' = 1, \ldots, m'$, and $k'' = m'+1, \ldots, n+m'$, and the sum can be analytically extended to a rational function in $(y_1, \ldots, y_{n+m'})$ with the only possible poles at $y_{i'} = y_{j'}$, of orders less than or equal to $N^n_{m'}(v'_{i'}, v'_{j'})$, for $i', j' = 1, \ldots, n$, $i' \neq j'$.

2”) Thus, for the product (3.16) we obtain $(v''_1, \ldots, v''_{k+n+m+m'}) \in V$, and $(z_1, \ldots, z_{k+n+m+m'}) \in \mathbb{C}$, we find positive integers $N^{k+n}_{m+m'}(v'_i, v'_j)$, depending only on v'_i and v''_j, for $i'', j'' = 1, \ldots, k+n$, $i'' \neq j''$, such that for arbitrary $w' \in W'$. This finishes the proof of Proposition 11.

Since we have proved that the product $\widehat{R}\ \Theta \left(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon\right)$ is composable with a $m + m' - t$ of vertex operators (2.21) with the formal parameters identified with the local coordinates $c_j(p''_j)$ around the points $(p_1, \ldots, p_k; p'_1, \ldots, p'_n)$ on each of the transversal sections U_j, $1 \leq j \leq l$, we conclude that according to Definition 17, the product $\widehat{R}\ \Theta \left(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon\right)$ belongs to the space

$$C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{U}, \mathcal{F}) = \bigcap_{\substack{U_1 \overset{h_1}{\hookrightarrow} \ldots \overset{h_{m+m'-1}}{\hookrightarrow} U_{m+m'} \\ 1 \leq j \leq m+m'-t}} C^{k+n-r}(V, \mathcal{W}, \mathcal{F})(U_j), \tag{6.25}$$

where the intersection ranges over all possible $m+m'-t$-tuples of holonomy embeddings h_i, $i \in \{1, \ldots, m+m'-t-1\}$, between transversal sections $U_1, \ldots, U_{m+m'-t-1}$ of the basis $\mathcal{U}$ for $\mathcal{F}$. This finishes the proof of Proposition 10. □

7. Properties of the ϵ-Product of $C^k_m(V, \mathcal{W}, \mathcal{F})$-Spaces

Since the ϵ-product of $\Phi(v_1, x_1; \ldots; v_k, x_k) \in C^k_m(V, \mathcal{W}, \mathcal{F})$ and $\Psi(v'_1, y_1; \ldots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$ results in an element of $C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F})$, then, similar to [43], the following corollary follows directly from Proposition (10) and Definition 14:

Corollary 1. *For the spaces $\mathcal{W}_{x_1,\ldots,x_k}$ and $\mathcal{W}_{y_1,\ldots,y_n}$ with the product (3.16) $\Theta \in \mathcal{W}_{z_1,\ldots,z_{k+n-r}}$, the subspace of* $\mathrm{Hom}(V^{\otimes n}, \mathcal{W}_{z_1,\ldots,z_{k+n-r}})$ *consisting of linear maps having the $L_W(-1)$-derivative property, having the $L_V(0)$-conjugation property or being composable with m vertex operators is invariant under the action of S_{k+n-r}.* □

We also have

Corollary 2. *For a fixed set $(v_1, \ldots v_k; v_{k+1}, \ldots, v_{k+n-r})$, $v_i \in V$, $1 \leq i \leq k+n-r$ of vertex algebra elements, and fixed $k+n-r$, and $m+m'-t$, the ϵ-product $\widehat{R}\ \Theta(v_1, z_1; \ldots; v_k, z_k; v_{k+1}, z_{k+1}; \ldots; v_{k+n-r}, y_{k+n-r}; \epsilon)$,*

$$\cdot_\epsilon : C^k_m(V, \mathcal{W}, \mathcal{F}) \times C^n_{m'}(V, \mathcal{W}, \mathcal{F}) \to C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F}),$$

of the spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ and $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, for all choices of k, n, m, $m' \geq 0$, is the same element of $C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F})$ for all possible $k \geq 0$. □

Proof. In Proposition 5 we have proved that the result of the ϵ-product belongs to $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$, for all $k, n \geq 0$, and fixed $k+n-r$. As in the proof of Proposition 10, by checking the conditions for the forms (6.11) and (6.15), we see by Proposition 2 that the product $\widehat{R}\ \Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n)$ is composable with the fixed $m + m' - t$ number of vertex operators. □

By Proposition 7, elements of the space $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$ resulting from the ϵ-product (3.7) are invariant with respect to group $(\text{Aut }\mathcal{O})^{\times(k+n-r)}_{z_1,\ldots,z_{k+n-r}}$ of independent changes of the formal parameters. Now we prove the following

Corollary 3. *For* $\Phi(v_1, x_1; \ldots; v_k, x_k) \in C^k_m(V, \mathcal{W}, \mathcal{F})$ *and* $\Psi(v'_1, y_1; \ldots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$*, the product*

$$\begin{aligned}\widehat{R}\,\Theta\left(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon\right)\\ = \Phi(v_1, x_1; \ldots; v_k, x_k) \cdot_\epsilon \Psi(v'_1, y_1; \ldots; v'_n, y_n),\end{aligned} \tag{7.1}$$

is invariant with respect to the action

$$(z_1, \ldots, z_{k+n-r}) \mapsto (z'_1, \ldots, z'_{k+n-r}) = (\rho(z_1), \ldots, \rho(z_{k+n-r})), \tag{7.2}$$

of the group $(\text{Aut }\mathcal{O})^{\times(k+n-r)}_{z_1,\ldots,z_{k+n-r}}$.

Proof. In Sect. 3.3 we have proved that the product (3.7) belongs to $\mathcal{W}_{z_1,\ldots,z_{k+n-r}}$, and is invariant with respect to the group $(\text{Aut }\mathcal{O})^{\times(k+n-t)}_{z_1,\ldots,z_{k+n-r}}$. Similar as in the proof of Proposition 8, the vertex operators $\omega_V(v_i, x_i)$, $1 \le i \le m$, composable with $\Phi(v_1, x_1; \ldots; v_k, x_k)$, and the vertex operators $\omega_V(v_j, y_j)$, $1 \le j \le m'$, composable with $\Psi(v'_1, y_1; \ldots; v'_n, y_n)$, are also invariant with respect to independent changes of coordinates $(\rho(z_1), \ldots, \rho(z_{k+n-r})) \in \text{Aut }\mathcal{O}^{\times(k+n-r)}_{z_1,\ldots,z_{k+n-r}}$. □

7.1. The coboundary operator acting on the product of elements of $C^n_m(V, \mathcal{W}, \mathcal{F})$*-spaces.* In Proposition 10 we proved that the product (3.16) of elements of spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ and $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$ belongs to $C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F})$. Thus, the product admits the action of the coboundary operators $\delta^{k+n-r}_{m+m'-t}$ and δ^{2-r}_{ex-t} defined in (5.7) and (5.11). The coboundary operators (5.7) and (5.11) possess a variation of Leibniz law with respect to the product (3.16). Indeed, we state here

Proposition 12. *For* $\Phi(v_1, x_1; \ldots; v_k, x_k) \in C^k_m(V, \mathcal{W}, \mathcal{F})$ *and* $\Psi(v'_1, y_1; \ldots; v'_n, y_n) \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$*, the action of the coboundary operator* $\delta^{k+n-r}_{m+m'-t}$ (5.7) *(and* δ^{2-r}_{ex-t} (5.11)*) on the* ϵ*-product* (3.16) *is given by*

$$\begin{aligned}&\delta^{k+n-r}_{m+m'-t}\left(\Phi(v_1, x_1; \ldots; v_k, x_k) \cdot_\epsilon \Psi(v'_1, y_1; \ldots; v'_n, y_n)\right)\\ &= \left(\delta^k_m \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\right) \cdot_\epsilon \Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\\ &\quad + (-1)^k \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k) \cdot_\epsilon \left(\delta^{n-r}_{m'-t} \Psi(\widetilde{v}_1, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\right),\end{aligned} \tag{7.3}$$

where we use the notation as in (3.1) *and* (3.17).

Proof. For the vertex operator $Y_{V,W}(v, z)$ let us introduce the notation

$$\omega_{V,W} = Y_{V,W}(v, z)\; dz^{\text{wt}(v)}.$$

Let us use the notations (3.1) and (3.17). According to (5.7) and (5.8), the action of $\delta^{k+n-r}_{m+m'-t}$ on $\widehat{R}\Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_k, y_n; \epsilon)$ is given by

$$\langle w', \delta^{k+n-r}_{m+m'-t} \widehat{R}\,\Theta(v_1, x_1; \ldots; v_k, x_k; v'_1, y_1; \ldots; v'_n, y_n; \epsilon)\rangle$$

$$= \langle w', \sum_{i=1}^{k} (-1)^i \; \widehat{R}\, \Theta(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_{i-1}, z_{i-1};\; \omega_V(\widetilde{v}_i, z_i - z_{i+1})\widetilde{v}_{i+1}, z_{i+1};\; \widetilde{v}_{i+2}, z_{i+2};$$
$$\ldots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n}, z_{k+n}; \epsilon))\rangle$$
$$+ \sum_{i=1}^{n-r} (-1)^i \; \langle w', \Theta\,(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+i-1}, z_{k+i-1};$$
$$\omega_V\,(\widetilde{v}_{k+i}, z_{k+i} - z_{k+i+1})\; \widetilde{v}_{k+i+1}, z_{k+i+1};$$
$$\widetilde{v}_{k+i+2}, z_{k+i+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}; \epsilon))\rangle$$
$$+\langle w', \omega_W\,(\widetilde{v}_1, z_1)\;\; \Theta(\widetilde{v}_2, z_2; \ldots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}; \epsilon))\rangle$$
$$+\langle w, (-1)^{k+n+1-r} \omega_W(\widetilde{v}_{k+n-r+1}, z_{k+n-r+1})$$
$$\Theta(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k; \widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}; \epsilon))\rangle.$$

Using (3.7) we see that it is equivalent to

$$\sum_{l\in\mathbb{Z}} \epsilon^l \langle w', \sum_{i=1}^{k} (-1)^i \; Y^W_{VW}(\Theta(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_{i-1}, z_{i-1};\; \omega_V(\widetilde{v}_i, z_i - z_{i+1})\widetilde{v}_{i+1}, z_{i+1};$$
$$\widetilde{v}_{i+2}, z_{i+2}; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\Theta(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$+ \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{i=1}^{n-r} (-1)^i \; \langle w', Y^W_{VW}\,(\Phi\,(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\,, \zeta_1)\, u\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+i-1}, z_{k+i-1};$$
$$\omega_V\,(\widetilde{v}_i, z_{k+i} - z_{k+i+1})\;\; \widetilde{v}_{k+i+1}, z_{k+i+1}; \widetilde{v}_{k+i+2}, z_{k+i+2};$$
$$\ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$+ \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\omega_W\,(\widetilde{v}_1, z_1)\;\; \Phi(\widetilde{v}_2, z_2; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$+ \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}((-1)^{k+1} \omega_W\,(\widetilde{v}_{k+1}, z_{k+1})\;\; \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$- \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', (-1)^{k+1} \langle w', Y^W_{VW}(\omega_W\,(\widetilde{v}_{k+1}, z_{k+1})\;\; \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$+ \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\omega_W(\widetilde{v}_{k+n-r+1}, z_{k+n-r+1})$$
$$\Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$- \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)\rangle$$
$$\langle w', Y^W_{VW}(\omega_W(\widetilde{v}_{k+n-r+1}, z_{k+n-r+1})$$

$$\Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\rangle. \tag{7.4}$$

According to the definition (9.14) of the intertwining operator and the locality property (9.6) of vertex operators

$$\sum_{l\in\mathbb{Z}} \epsilon^l \langle w', (-1)^{k+1} Y^W_{VW}(\omega_W\ (\widetilde{v}_{k+1}, z_{k+1})\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', (-1)^{k+1} e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \omega_W\ (\widetilde{v}_{k+1}, z_{k+1})\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', (-1)^{k+1} e^{\zeta_1 L_W(-1)} \omega_W\ (\widetilde{v}_{k+1}, z_{k+1})\, Y_W(u, -\zeta_1)\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', (-1)^{k+1}\ \omega_W\ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)\ \ e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle.$$

By inserting an arbitrary vertex algebra basis and using the definition of the intertwining operator (9.14) we obtain

$$\sum_{v\in V} \sum_{u\in V_l} \sum_{l\in\mathbb{Z}} \epsilon^l \sum_{u\in V_l} \langle v', (-1)^{k+1}\ \omega_W\ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)\ w\rangle$$
$$\langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', e^{\zeta_1 L_W(-1)} Y_W(u, -\zeta_1)\ \ \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\rangle$$
$$\sum_{v\in V} \langle v', (-1)^{k+1}\ \omega_W\ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)\ w\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\ \rangle$$
$$\langle w', (-1)^{k+1}\ \omega_W\ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)$$
$$Y^W_{VW}(\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\ \rangle$$
$$\langle w', (-1)^{k+1}\ \omega_W\ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1)$$
$$e^{\zeta_2 L_W(-1)} Y_W(\overline{u}, -\zeta_2)\ \Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\rangle$$
$$= \sum_{l\in\mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\ \rangle$$
$$\langle w', (-1)^{k+1}\ \ e^{\zeta_2 L_W(-1)}\ Y_W(\overline{u}, -\zeta_2)\ \omega_W\ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1 - \zeta_2)$$
$$\Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r})\rangle$$

$$= \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}(\Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\omega_W(\widetilde{v}_{k+1}, z_{k+1})\ \Psi(\widetilde{v}_{k+2}, z_{k+2}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}$$

due to locality (9.6) of vertex operators, and arbitrariness of $\widetilde{v}_{k+1} \in V$ and z_{k+1}, we can always put

$$\omega_W\ (\widetilde{v}_{k+1}, z_{k+1} + \zeta_1 - \zeta_2) = \omega_W(\widetilde{v}_{k+2}, z_{k+2}),$$

for $\widetilde{v}_{k+1} = \widetilde{v}_{k+2}$, $z_{k+2} = z_{k+1} + \zeta_2 - \zeta_1$. By combining the action of δ^k_m on Φ and $\delta^{n-r}_{m'-t}$ on Ψ according to (5.8), (7.4) gives

$$\sum_{l \in \mathbb{Z}} \epsilon^l \langle w',\ Y^W_{VW}(\delta^k_m \Phi(\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k), \zeta_1)u\rangle$$
$$\langle w', Y^W_{VW}(\Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle$$
$$+(-1)^k \sum_{l \in \mathbb{Z}} \epsilon^l \langle w', Y^W_{VW}\ (\Phi\ (\widetilde{v}_1, z_1; \ldots; \widetilde{v}_k, z_k)\ , \zeta_1)\ u\rangle$$
$$\langle w', Y^W_{VW}(\delta^{n-r}_{m'-t} \Psi(\widetilde{v}_{k+1}, z_{k+1}; \ldots; \widetilde{v}_{k+n-r}, z_{k+n-r}), \zeta_2)\overline{u}\rangle,$$

which gives (7.3) due to (3.7). The statement of the proposition for δ^2_{ex} (5.11) can be checked accordingly. □

Remark 9. Checking (5.7) we see that an extra arbitrary vertex algebra element $v_{n+1} \in V$, as well as corresponding extra arbitrary formal parameter z_{n+1} appear as a result of the action of δ^n_m on $\Phi \in C^n_m(V, \mathcal{W}, \mathcal{F})$ mapping it to $C^{n+1}_{m-1}(V, \mathcal{W}, \mathcal{F})$. In application to the ϵ-product (3.16) these extra arbitrary elements are involved in the definition of the action of $\delta^{k+n-r}_{m+m'-t}$ on $\Phi(v_1, x_1; \ldots; v_k, x_k) \cdot_\epsilon \Psi(v'_1, y_1; \ldots; v'_n, y_n)$.

Note that both sides of (7.3) belong to the space $C^{n+n'-r-1}_{m+m'-t+1}(V, \mathcal{W}, \mathcal{F})$. The coboundary operators δ^n_m and $\delta^{n'}_{m'}$ in (7.3) do not include the number of common vertex algebra elements (and formal parameters), neither the number of common vertex operators corresponding mappings composable with. The dependence on common vertex algebra elements, parameters, and composable vertex operators is taken into account in the mappings multiplying the action of the coboundary operators on Φ.

We have the following

Corollary 4. *The product* (3.16) *and the coboundary operators* (5.7), (5.11) *endow the space* $C^k_m(V, \mathcal{W}, \mathcal{F}) \times C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, *k*, $n \geq 0$, *m*, $m' \geq 0$, *with the structure of a double graded differential algebra* $\mathcal{G}\left(V, \mathcal{W}, \cdot_\epsilon, \delta^{k+n-r}_{m+m'-t}\right)$. □

Finally, we prove the following

Proposition 13. *The product* (3.16) *extends the property* (5.16) *of the chain-cochain complexes* (5.18) *and* (5.19) *to all products* $C^k_m(V, \mathcal{W}, \mathcal{F}) \cdot_\epsilon C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, *k*, $n \geq 0$, *m*, $m' \geq 0$.

Proof. For $\Phi \in C^k_m(V, \mathcal{W}, \mathcal{F})$ and $\Psi \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$ we proved in Proposition 10 that the product $\Phi \cdot_\epsilon \Psi$ belongs to the space $C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F})$. Using (7.3) and chain-cochain property for Φ and Ψ we also check that

$$\delta^{k+n+1-r}_{m+m'-1-t} \circ \delta^{k+n-r}_{m+m'-t}\ (\Phi \cdot_\epsilon \Psi) = 0.$$

$$\delta^{2-r}_{ex-t} \circ \delta^{1-r}_{2-t} \left(\Phi \cdot_\epsilon \Psi\right) = 0. \tag{7.5}$$

Thus, the chain-cochain property extends to the ϵ product $C^k_m(V, \mathcal{W}, \mathcal{F}) \cdot_\epsilon C^n_{m'}(V, \mathcal{W}, \mathcal{F})$. □

7.2. The exceptional complex. For elements of the spaces $C^2_{ex}(V, \mathcal{W}, \mathcal{F})$ we have the following

Corollary 5. *The product of elements of the spaces $C^2_{ex}(V, \mathcal{W}, \mathcal{F})$ and $C^n_m(V, \mathcal{W}, \mathcal{F})$ is given by* (3.16)*,*

$$\cdot_\epsilon : C^2_{ex}(V, \mathcal{W}, \mathcal{F}) \times C^n_m(V, \mathcal{W}, \mathcal{F}) \to C^{n+2-r}_m(V, \mathcal{W}, \mathcal{F}), \tag{7.6}$$

and, in particular,

$$\cdot_\epsilon : C^2_{ex}(V, \mathcal{W}, \mathcal{F}) \times C^2_{ex}(V, \mathcal{W}, \mathcal{F}) \to C^{4-r}_0(V, \mathcal{W}, \mathcal{F}).$$

Proof. The fact that the number of formal parameters is $n + 2 - r$ in the product (3.16) follows from Proposition (5). Consider the product (3.16) for $C^2_{ex}(V, \mathcal{W}, \mathcal{F})$ and $C^n_m(V, \mathcal{W}, \mathcal{F})$. It is clear that, similar to the considerations of the proof of Proposition 10, the total number m of vertex operators the product Θ is composable with remains the same. □

8. The Product-Type Cohomological Classes

In this Section we provide the main results of this paper. In particular, the invariants for the first and the second vertex algebra cohomology for codimension one foliations are found.

8.1. The commutator multiplication. In this Subsection we define a further product of a pair of elements of the spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ and $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, suitable for the formulation of cohomological invariants. Let us consider the mappings

$$\Phi(v_1, z_1; \ldots; v_n, z_k) \in C^k_m(V, \mathcal{W}, \mathcal{F}),$$
$$\Psi(v_{k+1}, z_{k+1}; \ldots; v_{k+n}, z_{k+n}) \in C^n_{m'}(V, \mathcal{W}, \mathcal{F}),$$

with r common vertex algebra elements (and, correspondingly, r formal variables), and t common vertex operators mappings Φ and Ψ are composable with. Note that when applying the coboundary operators (5.7) and (5.11) to a map $\Phi(v_1, z_1; \ldots; v_n, z_n) \in C^n_m(V, \mathcal{W}, \mathcal{F})$,

$$\delta^n_m : \Phi(v_1, z_1; \ldots; v_n, z_n) \to \Phi(v'_1, z'_1; \ldots; v'_{n+1}, z'_{n+1}) \in C^{n+1}_{m-1}(V, \mathcal{W}, \mathcal{F}),$$

one does not necessary assume that we keep the same set of vertex algebra elements/formal parameters and vertex operators composable with for $\delta^n_m \Phi$, though it might happen that some of them could be common with Φ. Then we have

Definition 26. Let us define extra product of Φ and Ψ,

$$\Phi \cdot \Psi : V^{\otimes(k+n-r)} \to \mathcal{W}_{z_1,\dots,z_{k+n-r}}, \tag{8.1}$$
$$\Phi \cdot \Psi = [\Phi, \cdot_\epsilon \Psi] = \Phi \cdot_\epsilon \Psi - \Psi \cdot_\epsilon \Phi, \tag{8.2}$$

where brackets denote the ordinary commutator on $\mathcal{W}_{z_1,\dots,z_{k+n-r}}$.

Due to the properties of the maps $\Phi \in C^k_m(V, \mathcal{W}, \mathcal{F})$ and $\Psi \in C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, we obtain

Lemma 12. *The product $\Phi \cdot \Psi$ belongs to the space $C^{k+n-r}_{m+m'-t}(V, \mathcal{W}, \mathcal{F})$.* □

For $k = n$ and

$$\Psi(v_{n+1}, z_{n+1}; \dots ; v_{2n}, z_{2n}) = \Phi(v_1, z_1; \dots ; v_n, z_n),$$

we obtain from (3) and (3.16) that

$$\Phi(v_1, z_1; \dots ; v_n, z_n) \cdot \Phi(v_1, z_1; \dots ; v_n, z_n) = 0. \tag{8.3}$$

The product (8.1) will be used in the next Subsection in order to introduce cohomological invariants.

8.2. The cohomological invariants. In this Subsection, using the vertex algebra double complex construction (5.15)–(5.16), we provide invariants for the grading-restricted vertex algebra cohomology of codimension one foliations on complex curves. Let us introduce cohomological classes associated to grading-restricted vertex algebras. We describe here certain classes associated to the first and the second vertex algebra cohomology for codimension one foliations. Let us give some further definitions. Usually, the cohomology classes for codimension one foliations [15,31,52] are introduced by means of an extra condition (in particular, the orthogonality condition) applied to differential forms, and leading to the integrability condition. As we mentioned in Sect. 5, it is a separate problem to introduce a product defined on one or among various spaces $C^n_m(V, \mathcal{W}, \mathcal{F})$ of (4.2). Note that elements of $\mathcal{E}$ in (5.7) and $\mathcal{E}_{ex}$ in (5.13) can be seen as elements of spaces $C^1_\infty(V, \mathcal{W}, \mathcal{F})$, i.e., maps composable with an infinite number of vertex operators. Though the actions of coboundary operators δ^n_m and δ^2_{ex} in (5.7) and (5.11) are written in form of a product (as in Frobenius theorem [31]), and, in contrast to the case of differential forms, it is complicated to use these products for further formulation of cohomological invariants and derivation of analogues of the product-type invariants. Nevertheless, even with such a product yet missing, it is possible to introduce the lower-level cohomological classes of the form $[\delta\eta]$ which are counterparts of the Godbillon class [29]. Let us give some further definitions. By analogy with differential forms, let us introduce

Definition 27. We call a map

$$\Phi \in C^n_k(V, \mathcal{W}, \mathcal{F}),$$

closed if it is a closed connection:

$$\delta^n_k \Phi = G(\Phi) = 0.$$

For $k \geq 1$, we call it exact if there exists

$$\Psi \in C^{n+1}_{k-1}(V, \mathcal{W}, \mathcal{F}),$$

such that

$$\Psi = \delta^n_k \Phi,$$

i.e., Ψ is a form of a connection.

Definition 28. For $\Phi \in C^n_k(V, \mathcal{W}, \mathcal{F})$ we call the cohomology class of mappings $[\Phi]$ the set of all closed forms that differ from Φ by an exact mapping, i.e., for $\Lambda \in C^{n-1}_{k+1}(V, \mathcal{W}, \mathcal{F})$,

$$[\Phi] = \Phi + \delta^{n-1}_{k+1} \Lambda.$$

As we will see in this Section, there are cohomological classes, (i.e., $[\Phi]$, $\Phi \in C^1_m(V, \mathcal{W}, \mathcal{F})$, $m \geq 0$), associated with two-point connections and the first cohomology $H^1_m(V, \mathcal{W}, \mathcal{F})$, and classes (i.e., $[\Phi]$, $\Phi \in C^2_{ex}(V, \mathcal{W}, \mathcal{F})$), associated with transversal connections and the second cohomology $H^2_{ex}(V, \mathcal{W}, \mathcal{F})$, of $\mathcal{M}/\mathcal{F}$. The cohomological classes we obtain are vertex algebra cohomology counterparts of the Godbillon class [29,52] for codimension one foliations.

Remark 10. As it was discovered in [1,2], it is a usual situation when the existence of a connection (affine or projective) for codimension one foliations on smooth manifolds prevents corresponding cohomology classes from vanishing. Note also, that for a few examples of codimension one foliations, the cohomology class $[d\eta]$ is always zero.

Remark 11. In contrast to [1], our cohomological class is a functional of $v \in V$. That means that the actual functional form of $\Phi(v, z)$ (and therefore $\langle w', \Phi \rangle$, for $w' \in W'$) varies with various choices of $v \in V$. That allows one to use it in order to distinguish types of leaves of $\mathcal{M}/\mathcal{F}$.

In this Subsection we consider the general classes of cohomological invariants which arise from the definition of the product of pairs of $C^n_m(V, \mathcal{W}, \mathcal{F})$-spaces. Under a natural extra condition, the double complexes (5.18) and (5.19) allow us to establish relations among elements of $C^n_m(V, \mathcal{W}, \mathcal{F})$-spaces. By analogy with the notion of the integrability for differential forms [31], we use here the notion of the orthogonality for the spaces of a complex.

Definition 29. For the double complexes (5.18) and (5.19) let us require that for a pair of double complex spaces $C^k_m(V, \mathcal{W}, \mathcal{F})$ and $C^n_{m'}(V, \mathcal{W}, \mathcal{F})$, there exist subspaces

$$\widetilde{C}^k_m(V, \mathcal{W}, \mathcal{F}) \subset C^k_m(V, \mathcal{W}, \mathcal{F}),$$
$$\widetilde{C}^n_{m'}(V, \mathcal{W}, \mathcal{F}) \subset C^n_{m'}(V, \mathcal{W}, \mathcal{F}),$$

such that for all $\Phi \in \widetilde{C}^k_m(V, \mathcal{W}, \mathcal{F})$ and all $\Psi \in \widetilde{C}^n_{m'}(V, \mathcal{W}, \mathcal{F})$,

$$\Phi \cdot \delta^n_{m'} \Psi = 0, \tag{8.4}$$

namely, Φ is supposed to be orthogonal to $\delta^n_{m'} \Psi$ with respect to the product (3). We call this the orthogonality condition for mappings of the double complexes (5.18) and (5.19).

Note that in the case of differential forms considered on a smooth manifold, the Frobenius theorem for a distribution provides the orthogonality condition [31]. The fact that both sides of (8.6) (see below) belong to the same double complex space, applies limitations to possible combinations of (k, m) and (n, m'). Below we derive the algebraic relations occurring from the orthogonality condition on the double complexes (5.18) and (5.19). Taking into account the correspondence (see Sect. 24) with Čech-de Rham complex due to [15], we reformulate the derivation of the product-type invariants in the vertex algebra terms. Recall that the Godbillon–Vey cohomological class [31] is considered on codimension one foliations of three-dimensional smooth manifolds. In this paper, we supply its analogue for complex curves. According to the definition (4.2) we have k-tuples of one-dimesional transversal sections. In each section we attach one vertex operator $Y_W(u_k, w_k)$, $u_k \in V$, $w_k \in U_k$. Similarly to the differential forms setup, a mapping $\Phi \in C_k^m(V, \mathcal{W}, \mathcal{F})$ defines a codimension one foliation. As we see from (5.7), (8.3), and (7.3) it satisfies the properties similar as differential forms do.

Now we show that the analog of the integrability condition provides the generalizations of the product-type invariants for codimension one foliations on complex curves. Here we give a proof of the main statement of this paper, Theorem 1 formulated in the Introduction.

Proof. Let us consider two maps $\Phi(v_1) \in C_2^1(V, \mathcal{W}, \mathcal{F})$ and $\Lambda \in C_3^0(V, \mathcal{W}, \mathcal{F})$. We require them to be orthogonal, i.e.,

$$\Phi \cdot \delta_3^0 \Lambda = 0. \tag{8.5}$$

Thus, there exists $\Psi(v_2) \in C_m^n(V, \mathcal{W}, \mathcal{F})$, such that

$$\delta_3^0 \Lambda = \Phi \cdot \Psi, \tag{8.6}$$

and $1 = 1+n-r$, $2 = 2+m-t$, i.e., $n = r$, which leads to $r = 1$; $m = t$, $0 \le t \le 2$, i.e., $\Psi \in C_t^1(V, \mathcal{W}, \mathcal{F})$. Here r and t are numbers of common vertex algebra elements/formal parameters and correspondingly of vertex operators a map composable with. All other orthogonality conditions for the short sequence (5.19) does not allow relations of the form (8.6).

Consider now (8.5). We obtain, using (7.3)

$$\delta_{4-t'}^{2-r'} (\Phi \cdot \delta_3^0 \Lambda) = \left(\delta_2^1 \Phi\right) \cdot \delta_3^0 \Lambda + \Phi \cdot \delta_2^1 \delta_3^0 \Lambda = \left(\delta_2^1 \Phi\right) \cdot \delta_3^0 \Lambda = \left(\delta_2^1 \Phi\right) \cdot \Phi \cdot \Psi.$$

Thus

$$0 = \delta_{3-t'}^{3-r'} \delta_{4-t'}^{2-r'} (\Phi \cdot \delta_3^0 \Lambda) = \delta_{3-t'}^{3-r'} \left(\left(\delta_2^1 \Phi\right) \cdot \Phi \cdot \Psi.\right),$$

and $\left(\left(\delta_2^1 \Phi\right) \cdot \Phi \cdot \Psi\right)$ is closed. At the same time, from (8.5) it follows that

$$0 = \delta_2^1 \Phi \cdot \delta_3^0 \Lambda - \Phi \cdot \delta_2^1 \delta_3^0 \Lambda = \left(\Phi \cdot \delta_3^0 \Lambda\right).$$

Thus

$$\delta_2^1 \Phi \cdot \delta_3^0 \Lambda = \delta_2^1 \Phi \cdot \Phi \cdot \Psi = 0.$$

Consider (8.6). Acting by δ_2^1 and substituting back we obtain

$$0 = \delta_2^1 \delta_3^0 \Lambda = \delta_2^1 (\Phi \cdot \Psi) = \delta_2^1 (\Phi) \cdot \Psi - \Phi \cdot \delta_t^1 \Psi.$$

thus

$$\delta_2^1(\Phi) \cdot \Psi = \Phi \cdot \delta_t^1 \Psi.$$

The last equality trivializes on applying δ^3_{t+1} to both sides.

Let us show now the non-vanishing property of $\left(\left(\delta_2^1 \Phi\right) \cdot \Phi\right)$. Indeed, suppose

$$\left(\delta_2^1 \Phi\right) \cdot \Phi = 0.$$

Then there exists $\Gamma \in C_m^n(V, \mathcal{W}, \mathcal{F})$, such that

$$\delta_2^1 \Phi = \Gamma \cdot \Phi.$$

Both sides of the last equality should belong to the same double complex space but one can see that it is not possible. Thus, $\left(\delta_2^1 \Phi\right) \cdot \Phi$ is non-vanishing. One proves in the same way that $\left(\delta_3^0 \Lambda\right) \cdot \Lambda$ and $\left(\delta_t^1 \Psi\right) \cdot \Psi$ do not vanish too.

Now let us show that $\left[\left(\delta_2^1 \Phi\right) \cdot \Phi\right]$ is invariant, i.e., it does not depend on the choice of $\Phi \in C_2^1(V, \mathcal{W}, \mathcal{F})$. Substitute Φ by $(\Phi + \eta) \in C_2^1(V, \mathcal{W}, \mathcal{F})$. We have

$$\left(\delta_2^1 (\Phi + \eta)\right) \cdot (\Phi + \eta) = \left(\delta_2^1 \Phi\right) \cdot \Phi + \left(\left(\delta_2^1 \Phi\right) \cdot \eta - \Phi \cdot \delta_2^1 \eta\right) + \left(\Phi \cdot \delta_2^1 \eta + \delta_2^1 \eta \cdot \Phi\right) + \left(\delta_2^1 \eta\right) \cdot \eta. \tag{8.7}$$

Since

$$\left(\Phi \cdot \delta_2^1 \eta + \left(\delta_2^1 \eta\right) \cdot \Phi\right) = \Phi \cdot_\epsilon \delta_2^1 \eta - (\delta_2^1 \eta) \cdot_\epsilon \Phi + \left(\delta_2^1 \eta\right) \cdot_\epsilon \Phi - \Phi \cdot_\epsilon \delta_2^1 \eta = 0,$$

then (8.7) represents the same cohomology class $\left[\left(\delta_2^1 \Phi\right) \cdot \Phi\right]$. The same folds for $\left[\left(\delta_3^0 \Lambda\right) \cdot \Lambda\right]$, and $\left[\left(\delta_t^1 \Psi\right) \cdot \Psi\right]$. □

Remark 12. In this paper we provide results concerning complex curves, i.e., the case $n \le 1$, $n_0 \le 1$, $n_i \le 1$. They generalize to the case of higher dimensional complex manifolds.

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9. Appendix: Grading-Restricted Vertex Algebras and Their Modules

In this "Appendix", following [43], we recall basic properties of grading-restricted vertex algebras and their grading-restricted generalized modules, useful for our purposes of the paper. We work over the base field $\mathbb{C}$ of complex numbers.

9.1. Grading-restricted vertex algebras.

Definition 30. A vertex algebra $(V, Y_V, \mathbf{1}_V)$, (cf. [47]), consists of a $\mathbb{Z}$-graded complex vector space

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)}, \quad \dim V_{(n)} < \infty,$$

for each $n \in \mathbb{Z}$, and linear map

$$Y_V : V \to \text{End }(V)[[z, z^{-1}]],$$

for a formal parameter z and a distinguished vector $\mathbf{1}_V \in V$. The evaluation of Y_V on $v \in V$ is the vertex operator

$$Y_V(v) \equiv Y_V(v, z) = \sum_{n\in\mathbb{Z}} v(n) z^{-n-1}, \tag{9.1}$$

with components $(Y_V(v))_n = v(n) \in \text{End }(V)$, where $Y_V(v, z)\mathbf{1}_V = v + O(z)$.

Definition 31. A grading-restricted vertex algebra satisfies the following conditions:

(1) Grading-restriction condition: $V_{(n)}$ is finite dimensional for all $n \in \mathbb{Z}$, and $V_{(n)} = 0$ for $n \ll 0$;
(2) Lower-truncation condition: For $u, v \in V$, $Y_V(u, z)v$ contains only finitely many negative power terms, that is,

$$Y_V(u, z)v \in V((z)),$$

(the space of formal Laurent series in z with coefficients in V);
(3) Identity property: Let Id_V be the identity operator on V. Then

$$Y_V(\mathbf{1}_V, z) = \text{Id}_V;$$

(4) Creation property: For $u \in V$,

$$Y_V(u, z)\mathbf{1}_V \in V[[z]],$$

and

$$\lim_{z\to 0} Y_V(u, z)\mathbf{1}_V = u;$$

(5) Duality: For $u_1, u_2, v \in V$,

$$v' \in V' = \coprod_{n \in \mathbb{Z}} V^*_{(n)},$$

where $V^*_{(n)}$ denotes the dual vector space to $V_{(n)}$ and $\langle .,. \rangle$ the evaluation pairing $V' \otimes V \to \mathbb{C}$, the series

$$\langle v', Y_V(u_1, z_1) Y_V(u_2, z_2) v \rangle, \tag{9.2}$$

$$\langle v', Y_V(u_2, z_2) Y_V(u_1, z_1) v \rangle, \tag{9.3}$$

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2) u_2, z_2) v \rangle, \tag{9.4}$$

are absolutely convergent in the regions

$$|z_1| > |z_2| > 0,$$
$$|z_2| > |z_1| > 0,$$
$$|z_2| > |z_1 - z_2| > 0,$$

respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$;

(6) $L_V(0)$-bracket formula: Let $L_V(0) : V \to V$, be defined by

$$L_V(0) v = n v, \qquad n = \mathrm{wt}(v),$$

for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, z)] = Y_V(L_V(0) v, z) + z \frac{d}{dz} Y_V(v, z),$$

for $v \in V$.

(7) $L_V(-1)$-derivative property: Let

$$L_V(-1) : V \to V,$$

be the operator given by

$$L_V(-1) v = \mathrm{Res}_z z^{-2} Y_V(v, z) \mathbf{1}_V = Y_{(-2)}(v) \mathbf{1}_V,$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dz} Y_V(u, z) = Y_V(L_V(-1) u, z) = [L_V(-1), Y_V(u, z)]. \tag{9.5}$$

In addition to that, we recall here the following definition (cf. [7]):

Definition 32. A grading-restricted vertex algebra V is called conformal of central charge $c \in \mathbb{C}$, if there exists a non-zero conformal vector (Virasoro vector) $\omega \in V_{(2)}$ such that the corresponding vertex operator

$$Y_V(\omega, z) = \sum_{n \in \mathbb{Z}} L_V(n) z^{-n-2},$$

is determined by modes of Virasoro algebra $L_V(n) : V \to V$ satisfying

$$[L_V(m), L_V(n)] = (m - n) L(m + n) + \frac{c}{12}(m^3 - m) \delta_{m+b,0} \, \mathrm{Id_V}.$$

Definition 33. A vector A which belongs to a module W of a quasi-conformal grading-restricted vertex algebra V is called primary of conformal dimension $\Delta(A) \in \mathbb{Z}_+$ if

$$L_W(k)A = 0, \ k > 0,$$
$$L_W(0)A = \Delta(A)A.$$

.1. Grading-restricted generalized V-module. In this Subsection we describe the grading-restricted generalized V-module for a grading-restricted vertex algebra V.

Definition 34. A grading-restricted generalized V-module is a vector space W equipped with a vertex operator map

$$Y_W : V \otimes W \to W[[z, z^{-1}]],$$
$$u \otimes w \mapsto Y_W(u, w) \equiv Y_W(u, z)w = \sum_{n\in\mathbb{Z}} (Y_W)_n(u, w)z^{-n-1},$$

and linear operators $L_W(0)$ and $L_W(-1)$ on W satisfying the following conditions:

(1) Grading-restriction condition: The vector space W is $\mathbb{C}$-graded, that is,

$$W = \coprod_{\alpha\in\mathbb{C}} W_{(\alpha)},$$

such that $W_{(\alpha)} = 0$ when the real part of α is sufficiently negative;
(2) Lower-truncation condition: For $u \in V$ and $w \in W$, $Y_W(u, z)w$ contains only finitely many negative power terms, that is, $Y_W(u, z)w \in W((z))$;
(3) Identity property: Let Id_W be the identity operator on W. Then

$$Y_W(\mathbf{1}_V, z) = \mathrm{Id}_W;$$

(4) Duality: For $u_1, u_2 \in V$, $w \in W$,

$$w' \in W' = \coprod_{n\in\mathbb{Z}} W^*_{(n)},$$

W' denotes the dual V-module to W. The locality and associativity properties in terms of the bilinear pairing $\langle ., .\rangle$, require that the series

$$\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w\rangle, \tag{9.6}$$
$$\langle w', Y_W(u_2, z_2)Y_W(u_1, z_1)w\rangle, \tag{9.7}$$
$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w\rangle, \tag{9.8}$$

are absolutely convergent in the regions

$$|z_1| > |z_2| > 0,$$
$$|z_2| > |z_1| > 0,$$
$$|z_2| > |z_1 - z_2| > 0,$$

respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1 = 0 = z_2$ and $z_1 = z_2$.

(5) $L_W(0)$-bracket formula: For $v \in V$,

$$[L_W(0), Y_W(v,z)] = Y_W(L_V(0)v, z) + z\frac{d}{dz}Y_W(v,z); \tag{9.9}$$

(6) $L_W(0)$-grading property: For $w \in W_{(\alpha)}$, there exists $N \in \mathbb{Z}_+$ such that

$$(L_W(0) - \alpha)^N w = 0; \tag{9.10}$$

(7) $L_W(-1)$-derivative property: For $v \in V$,

$$\frac{d}{dz}Y_W(u,z) = Y_W(L_V(-1)u, z) = [L_W(-1), Y_W(u,z)]. \tag{9.11}$$

The translation property of vertex operators

$$Y_W(u,z) = e^{-z' L_W(-1)} Y_W(u, z+z') e^{z' L_W(-1)}, \tag{9.12}$$

for $z' \in \mathbb{C}$, follows from from (9.11). For $v \in V$, and $w \in W$, the intertwining operator

$$\begin{aligned} Y^W_{WV} &: V \to W, \\ v &\mapsto Y^W_{WV}(w,z)v, \end{aligned} \tag{9.13}$$

is defined by

$$Y^W_{WV}(w,z)v = e^{zL_W(-1)} Y_W(v,-z)w. \tag{9.14}$$

For $a \in \mathbb{C}$, the conjugation property with respect to the grading operator $L_W(0)$ is given by

$$a^{L_W(0)}\, Y_W(v,z)\, a^{-L_W(0)} = Y_W(a^{L_W(0)}v, az). \tag{9.15}$$

.2. Generators of Virasoro algebra and the group of automorphisms. Let us recall some further facts from [7] relating generators of Virasoro algebra with the group of automorphisms in complex dimension one. Let us represent an element of $\mathrm{Aut}_z\, \mathcal{O}^{(1)}$ by the map

$$z \mapsto \rho = \rho(z), \tag{9.16}$$

given by the power series

$$\rho(z) = \sum_{k\geq 1} a_k z^k, \tag{9.17}$$

$\rho(z)$ can be represented in an exponential form

$$\rho(z) = \exp\left(\sum_{k>-1} \beta_k\, z^{k+1}\partial_z\right)(\beta_0)^{z\partial_z}\, .z, \tag{9.18}$$

where we express $\beta_k \in \mathbb{C}$, $k \geq 0$, through combinations [37] of a_k, $k \geq 1$. A representation of Virasoro algebra modes in terms of differential operators is given by [47]

$$L_W(m) \mapsto -\zeta^{m+1}\partial_\zeta, \tag{9.19}$$

for $m \in \mathbb{Z}$. By expanding (9.18) and comparing to (9.17) we obtain a system of equations which, can be solved recursively for all β_k. In [7], $v \in V$, they derive the formula

$$[L_W(n), Y_W(v, z)] = \sum_{m \geq -1} \frac{1}{(m+1)!} \left(\partial_z^{m+1} z^{m+1} \right) Y_W(L_V(m)v, z), \quad (9.20)$$

of a Virasoro generator commutation with a vertex operator. Given a vector field

$$\beta(z)\partial_z = \sum_{n \geq -1} \beta_n z^{n+1} \partial_z, \quad (9.21)$$

which belongs to local Lie algebra of $\mathrm{Aut}_z\ \mathcal{O}^{(1)}$, one introduces the operator

$$\beta = -\sum_{n \geq -1} \beta_n L_W(n).$$

We conclude from (9.21) with the following

Lemma 13.

$$[\beta, Y_W(v, z)] = \sum_{m \geq -1} \frac{1}{(m+1)!} \left(\partial_z^{m+1} \beta(z) \right) Y_W(L_V(m)v, z). \quad (9.22)$$

The formula (9.22) is used in [7] (Chapter 6) in order to prove invariance of vertex operators multiplied by conformal weight differentials in the case of primary states, and in generic case.

Let us give some further definition:

Definition 35. A grading-restricted vertex algebra V-module W is called quasi-conformal if it carries an action of local Lie algebra of $\mathrm{Aut}_z\ \mathcal{O}$ such that commutation formula (9.22) holds for any $v \in V$, the element

$$L_W(-1) = -\partial_z,$$

as the translation operator T,

$$L_W(0) = -z\partial_z,$$

acts semi-simply with integral eigenvalues, and the Lie subalgebra of the positive part of local Lie algebra of $\mathrm{Aut}_z\ \mathcal{O}^{(n)}$ acts locally nilpotently.

Recall [7] the exponential form $\rho(\zeta)$ (9.18) of the coordinate transformation (9.16) $\rho(z) \in \mathrm{Aut}_z\ \mathcal{O}^{(1)}$. A quasi-conformal vertex algebra posseses the formula (9.22), thus it is possible by using the identification (9.19), to introduce the linear operator representing $\rho(\zeta)$ (9.18) on $\mathcal{W}_{z_1,\ldots,z_n}$,

$$P\left(\rho(\zeta)\right) = \exp\left(\sum_{m>0} (m+1)\ \beta_m\ L_V(m) \right) \beta_0^{L_W(0)}, \quad (9.23)$$

(note that we have a different normalization in it). In [7] (Chapter 6) it was shown that the action of an operator similar to (9.23) on a vertex algebra element $v \in V_n$ contains finitely many terms, and subspaces

$$V_{\leq m} = \bigoplus_{n \geq K}^{m} V_n,$$

are stable under all operators $P(\rho)$, $\rho \in \mathrm{Aut}_z\ \mathcal{O}^{(1)}$. In [7] they proved the following

Lemma 14. *The assignment*

$$\rho \mapsto P(\rho),$$

defines a representation of $\mathrm{Aut}_z\ \mathcal{O}^{(1)}$ *on* V,

$$P(\rho_1 * \rho_2) = P(\rho_1)\ P(\rho_2),$$

which is the inductive limit of the representations $V_{\leq m}$, $m \geq K$.

Similarly, (9.23) provides a representation operator on $\mathcal{W}_{z_1,\ldots,z_n}$.

.3. Non-degenerate invariant bilinear pairing on V. The subalgebra

$$\{L_V(-1), L_V(0), L_V(1)\} \cong SL(2, \mathbb{C}),$$

associated with Möbius transformations on z naturally acts on V, (cf., e.g., [47]). In particular,

$$\gamma_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} : z \mapsto w = -\frac{\lambda^2}{z}, \tag{9.24}$$

is generated by

$$T_\lambda = \exp\left(\lambda L_V(-1)\right)\ \exp\left(\lambda^{-1} L_V(1)\right)\ \exp\left(\lambda L_V(-1)\right),$$

where

$$T_\lambda Y(u, z) T_\lambda^{-1} = Y\left(\exp\left(-\frac{z}{\lambda^2} L_V(1)\right)\left(-\frac{z}{\lambda}\right)^{-2L_V(0)} u, -\frac{\lambda^2}{z}\right). \tag{9.25}$$

In our considerations of Riemann sphere sewing, we use in particular, the Möbius map

$$z \mapsto z' = \epsilon/z,$$

associated with the sewing condition (3.5) with

$$\lambda = -\xi \epsilon^{\frac{1}{2}}, \tag{9.26}$$

with $\xi \in \{\pm\sqrt{-1}\}$. The adjoint vertex operator [24,47] is defined by

$$Y^\dagger(u, z) = \sum_{n \in \mathbb{Z}} u^\dagger(n) z^{-n-1} = T_\lambda Y(u, z) T_\lambda^{-1}. \tag{9.27}$$

A bilinear pairing $\langle ., . \rangle_\lambda$ (see, e.g., [56,66]) on V is invariant if for all $a, b, u \in V$, if

$$\langle Y(u, z) a, b \rangle_\lambda = \langle a, Y^\dagger(u, z) b \rangle_\lambda, \tag{9.28}$$

i.e.,

$$\langle u(n) a, b \rangle_\lambda = \langle a, u^\dagger(n) b \rangle_\lambda.$$

Thus it follows that

$$\langle L_V(0) a, b \rangle_\lambda = \langle a, L_V(0) b \rangle_\lambda, \tag{9.29}$$

so that

$$\langle a, b\rangle_\lambda = 0, \tag{9.30}$$

if $wt(a) \neq wt(b)$ for homogeneous a, b. One also finds

$$\langle a, b\rangle_\lambda = \langle b, a\rangle_\lambda.$$

The form $\langle ., .\rangle_\lambda$ is unique up to normalization if $L_V(1)V_1 = V_0$. Given any V basis $\{u^\alpha\}$ we define the dual V basis $\{\overline{u}^\beta\}$ where

$$\langle u^\alpha, \overline{u}^\beta\rangle_\lambda = \delta^{\alpha\beta}.$$

10. Appendix: Properties of Matrix Elements for a Grading-Restricted Vertex Algebra

Let us recall some definitions and facts about matrix elements for a grading-restricted vertex algebra [43]. Let V be a grading-restricted vertex algebra and W be a grading-restricted generalized V-module. If a meromorphic function $f(z_1, \ldots, z_n)$ on a domain in $\mathbb{C}^n$ is analytically extendable to a rational function in $(z_1, \ldots, z_n)$, we denote this rational function by $R(f(z_1, \ldots, z_n))$.

For $w \in W$, the $\overline{W}$-valued function is given by

$$E^{(n)}_W(v_1, z_1; \cdots ; v_n, z_n; w) = E(\omega_W(v_1, z_1) \ldots \omega_W(v_n, z_n)w),$$

where

$$\omega_W(dz^{\mathrm{wt}(v)} \otimes v, z) = Y_W(dz^{\mathrm{wt}(v)} \otimes v, z),$$

and an element $E(.) \in \overline{W}$ is given by

$$\langle w', E(.)\rangle = R(\langle w', .\rangle),$$

and $R(.)$ denotes the following (cf. [43]). Namely, if a meromorphic function $f(z_1, \ldots, z_n)$ on a region in $\mathbb{C}^n$ can be analytically extended to a rational function in $(z_1, \ldots, z_n)$, then the notation $R(f(z_1, \ldots, z_n))$ is used to denote such rational function. One defines

$$E^{W;(n)}_{WV}(w; v_1, z_1; \ldots ; v_n, z_n) = E^{(n)}_W(v_1, z_1; \ldots ; v_n, z_n; w),$$

where $E^{W;(n)}_{WV}(w; v_1, z_1; \ldots ; v_n, z_n)$ is an element of $\overline{W}_{z_1,\ldots,z_n}$. One defines

$$\left(E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}}\right).\Phi : V^{\otimes m+n} \to \overline{W}_{z_1,\ldots,z_{m+n}},$$

by

$$\begin{aligned}
&\left(E^{(l_1)}_{V;\,\mathbf{1}} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}}\right).\Phi(v_1 \otimes \cdots \otimes v_{m+n-1})\\
&= E(\Phi(E^{(l_1)}_{V;\mathbf{1}}(v_1 \otimes \cdots \otimes v_{l_1}) \otimes \cdots\\
&\otimes E^{(l_n)}_{V;\mathbf{1}}(v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n}))),
\end{aligned}$$

and

$$E^{(m)}_W.\Phi : V^{\otimes m+n} \to \overline{W}_{z_1,\ldots,z_{m+n-1}},$$

is given by

$$E_W^{(m)}.\Phi(v_1 \otimes \cdots \otimes v_{m+n})$$
$$= E(E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))).$$

Finally,

$$E_{WV}^{W;(m)}.\Phi : V^{\otimes m+n} \to \overline{W}_{z_1,\ldots,z_{m+n-1}},$$

is defined by

$$E_{WV}^{W;(m)}.\Phi(v_1 \otimes \cdots \otimes v_{m+n}) = E(E_{WV}^{W;(m)}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m})).$$

In the case $l_1 = \cdots = l_{i-1} = l_{i+1} = 1$ and $l_i = m - n - 1$, for some $1 \leq i \leq n$, we will use $E_{V;\,\mathbf{1}}^{(l_i)}.\Phi$ to denote $(E_{V;\,\mathbf{1}}^{(l_1)} \otimes \cdots \otimes E_{V;\,\mathbf{1}}^{(l_n)}).\Phi$. Note that our notations differ from that of [43].

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