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Special Issue

New Prospectives in Quantum Control of Atomic and Molecular Systems

Edited by

Dr. Teodora Kirova and Prof. Ennio Arimondo

MDPI

https://doi.org/10.3390/photonics11100918

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Selection Rules of Linear and Nonlinear Polarization-Selective Absorption in Optically Dressed Matter

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Abstract: Dynamical symmetries of laser-dressed matter determine the selection rules that influence its absorption spectrum. We explore selection rules for polarization-sensitive absorption in Floquet matter, using Floquet group theory in synthetic dimensions. We present comprehensive tables of selection rules that polarization-structured light impose on Floquet dark states and Floquet dark bands. Notably, our tables encompass nonlinear absorption for all nonlinear orders, revealing that different nonlinear orders follow distinct polarization selection rules, potentially leading to polarization-tunable optical filters.

Keywords: Floquet; dynamical symmetries; selection rules; tailored light

Citation: Feldman, M.; Even Tzur, M.; Cohen, O. Selection Rules of Linear and Nonlinear Polarization-Selective Absorption in Optically Dressed Matter. *Photonics* **2024**, *11*, 918. https://doi.org/10.3390/photonics11100918

Received: 25 July 2024
Revised: 9 September 2024
Accepted: 24 September 2024
Published: 28 September 2024

1. Introduction

Recently, *Floquet engineering* techniques have been developed for controlling the spectral absorption of materials [1]. For example, the electronic structure of an optically dressed material (described by a time-periodic Hamiltonian [2]) comprises Floquet states and Floquet bands, and proper engineering of these unique properties can imbue materials with intriguing features. Such Floquet systems are ubiquitous in various areas of physics such as lasers [3] condensed matter [4] and more [5–9].

A Floquet system exhibits spatio-temporal symmetries, also known as dynamical symmetries [10], that determine selection rules, e.g., the generation of odd-only [10] or circularly polarized [11,12] harmonics in high harmonic generation from isotropic media. Floquet group theory was introduced as a general framework to describe dynamical symmetries, and their associated selection rules on physical observables such as the nonlinear emission spectrum [13]. Notably, it was recently shown (theoretically and experimentally) that DSs also impose selection rules on linear absorption in the form of symmetry-protected dark states, symmetry-protected dark bands, and symmetry-induced transparency [14]. However, nonlinear absorption was not explored. Also, only absorbed light with DS that matches the DS of the absorbing Floquet system was considered. That is, the DSs of the dressing and absorbed fields were matched, so that the full light–matter system exhibited DS. Notably, it was recently shown that even in the presence of broken dynamical symmetries, selection rules can still be systematically manifested [15,16].

Here, we present a generalized framework for absorption and transparency of an arbitrarily polarized light by dressed matter with DS, even when this probe light does not uphold the DS of the dressed matter. That is, the light–matter system exhibits broken DS. Nevertheless, the system exhibits dynamical symmetries in space–time and synthetic dimensions (SDSs), where the synthetic dimensions represent the polarization of the probe light. We derive the selection rules for linear and nonlinear absorption associated with these SDSs. We find that polarization-structured dressing light leads to polarization selective Floquet dark states and Floquet dark bands (i.e., polarization-selective absorption). Notably, the derived selection rules encompass nonlinear absorption for all nonlinear orders. We found that different nonlinear orders adhere to different selection rules.

2. Results

We begin our treatment by considering a specific example that was previously considered absorption of light by a molecule with planar hexagonal symmetry, e.g., benzene driven (dressed) by a circularly polarized laser field with frequency Ω [14]. The molecule and driving laser polarization are aligned in the XY plane (Figure 1). The k-vector of the driving field is in the z direction $\left(\hat{k}_{drive} = \hat{z}\right)$, while the probe beam k-vector is in the y-direction $\left(\hat{k}_{prob} = \hat{y}\right)$ and it is linearly polarized in the z-direction. The time-dependent Hamiltonian of the system is given by [14].

$$\widehat{H}(t) = \widehat{H}_0 + \lambda\widehat{\boldsymbol{d}} \cdot \left(\boldsymbol{Q}_{\rm d} e^{i\Omega t} + H.c.\right) \tag{1}$$

in which $\widehat{H}_0$ is the time-independent Hamiltonian of the field-free molecule, $\widehat{\boldsymbol{d}}$ is the dipole moment operator, $\boldsymbol{Q}_{\rm d}$ is the complex Jones vector of the pump, λ is proportional to the electric field amplitude of the dressing laser and Ω is the frequency of the driven light. We note that employing a quantum-optical Hamiltonian [14] is not required, as the semi-classical and quantum-optical Hamiltonians share the same DSs and selection rules. We consider the absorption of a $\hat{z}$ polarized (out of the molecule plane) probe beam of frequency ω_p that interacts with the system (Figure 1a). The interaction term in Hamiltonian (1), including the probe, is shown below:

$$\widehat{W} = \lambda\boldsymbol{Q}_{\rm d} \cdot \widehat{\boldsymbol{d}} e^{i\Omega t} + \lambda\boldsymbol{Q}_{\rm p} \cdot \widehat{\boldsymbol{d}}\, e^{i\omega_p t} + H.c. \tag{2}$$

in which $\boldsymbol{Q}_{\rm d,p}$ are the complex Jones vectors of the complex pump and the probe beams, respectively, and ω_p is the frequency of the probe. Within either linear response theory [14] or Floquet perturbation theory [16], the linear absorption at the frequency $\omega = \omega_p + n\Omega$ is proportional to the susceptibility $I(\omega) \propto -i\widetilde{\chi}_n(\omega_p)$, which is given by the following equation:

$$\widetilde{\chi}_n(\omega_p) = i\lambda^2 \sum_{\nu,\mu,m} \frac{V_{\nu,\mu}^{(-n-m)} V_{\mu,\nu}^{(m)} (p_\nu - p_\mu)}{\epsilon_\mu - \epsilon_\nu + m\Omega - \omega_p - i\gamma_{\nu,\mu}^{(m)}} \tag{3}$$

In the above equation, m and n are integers, μ, ν are Floquet state indices and $\gamma_{\nu,\mu}^{(m)}$ are dephasing rates. $V_{\mu,\nu}^{(m)} \equiv \frac{1}{\tau}\int_0^\tau \langle u_\mu^{(m)}(t)|\widehat{\boldsymbol{d}}|u_\nu^{(m)}(t)\rangle\, dt$ are transition dipole moments between Floquet states $\left|u_{\mu,\nu}(t)\right\rangle$, and $\varepsilon_{\mu,\nu}$ are their corresponding quasi-energies, defined by the following eigenvalue equation:

$$\left[\widehat{H}(t) - \frac{id}{dt}\right] |u_\eta^{(m)}(t)\rangle = \varepsilon_\eta\, |u_\eta^{(m)}(t)\rangle \tag{4}$$

The dressed system exhibits the DS $\hat{C}_6 = \hat{R}_6 \cdot \hat{\tau}_6$ where $\hat{R}_6$ is the spatial rotation of $\frac{2\pi}{6}$ around the $\hat{z}$ axis, and $\hat{\tau}_6$ is $\frac{T}{6} = \frac{2\pi}{6\Omega}$ [13,14] time translation. The DS $\hat{C}_6$ results in symmetry-protected dark states (spDS) [14], as shown below:

$$\hat{V}_{\nu,\mu}^{(m)} = \begin{cases} 1 \text{ if } e^{i\frac{2\pi}{N}(m_\mu - m_\nu + n)} = 1 \\ 0 \quad else \end{cases} \quad m_\mu, m_\nu \in \{0, 1 \ldots N-1\} \tag{5}$$

Symmetry-protected Floquet bands are also shown below:

$$\widetilde{\chi}_n(\omega_p) = \begin{cases} 1 \text{ if } \mathrm{e}^{i\frac{2\pi}{N}n} = 1 \\ 0 \quad else \end{cases} \tag{6}$$

In the above equation, $N = 6$. We emphasize that within the linear response approach, Equations (5) and (6) are only correct for probes that do not break the $\hat{C}_6$ symmetry, essentially limiting the discussion to $\hat{z}$ polarized probes.

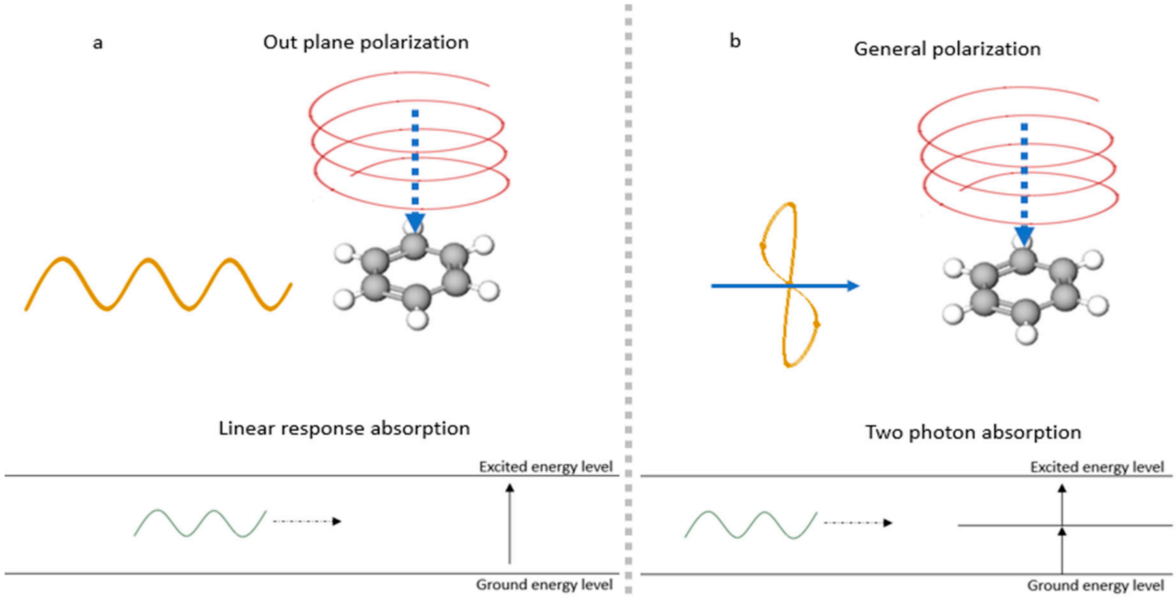

Figure 1. The differences between perturbation theory (**a**) and dynamical symmetry (**b**) for the polarization of the probe (1) and the absorption order (2).

To generalize this result beyond the regime of linear response, and for arbitrarily polarized probe fields, we look for synthetic dynamical symmetries [16] of the perturbed (probed) Hamiltonian $\widehat{H} = \widehat{H}_0 + \widehat{W}$. While the probe may break the $\hat{C}_6$ symmetry imposed by the circularly polarized Ω field, for any choice of $\boldsymbol{Q}_p$, reduced dynamical symmetry in synthetic dimensions (synthetic dynamical symmetry—SDS) remains. The SDS is constructed as the composition $\hat{X} = \hat{C}_6 \circ \hat{\zeta}$, where $\hat{\zeta}$ operates in the synthetic space spanned by the polarization vector of the probe $\boldsymbol{Q}_{\mathrm{p}}$.

Here, $\hat{\zeta}_{\hat{C}_6}$ operates in the synthetic space spanned by the components of the Jones vector $\boldsymbol{Q}_p$, and is given by $\hat{\zeta}_{\hat{C}_6}\left[\boldsymbol{Q}_p\right] = e^{\frac{\mathrm{i}\omega 2\pi}{6\Omega}}\boldsymbol{Q}_p$. By employing the invariance of the emitted field $\boldsymbol{E}(t)$ (equivalent to the dipole moment expectation value in our semi-classical description) under the SDS operation, we obtain the generalized selection rule for symmetry-protected dark bands (generalizing Equation (6)), as shown below:

$$\boldsymbol{E}\left(t, \boldsymbol{Q_p}\right) = \hat{C}_6 \boldsymbol{E}\left(t, \hat{\zeta}_{\hat{C}_6}\left[\boldsymbol{Q_p}\right]\right) \tag{7}$$

where $\boldsymbol{E}$ is the emitted electric field and may also be replaced by the nonlinear polarization $P_{NL}\left(t, Q_p\right)$. This represents the most general selection rule for the absorption of the probe. For example, assuming $\boldsymbol{Q}= q_z\hat{z}$ and that $\boldsymbol{E}$ is linear with q_z, Equation (7) reduces to Equation (6), reproducing the result of ref. [14]. However, Equation (7) contains a wealth of additional information, as it provides selection rules on the nonlinear response to all nonlinear orders, and for arbitrary probe polarization. We denote $\omega \equiv \mathrm{n}\Omega + \omega_{\mathrm{p}}$ to form Equation (8), as shown below:

$$\sum_n \widetilde{\boldsymbol{E}}_\omega\left(\boldsymbol{Q_p}\right)e^{i\omega t} = \sum_n \hat{C}_6^{(XY)}\widetilde{\boldsymbol{E}}_\omega\left(\hat{\zeta}_{\hat{C}_6}\left(\boldsymbol{Q_p}\right)\right)e^{i\omega t} \tag{8}$$

in which $\overset{\sim}{\boldsymbol{E}}_{\omega}\left(\boldsymbol{Q_p}\right)$ is the Fourier element of $\boldsymbol{E}$. For the first order in $\boldsymbol{Q}$, we obtain the following equation:

$$1 = e^{\frac{i2\pi}{6}\left(\frac{\omega-\omega p}{\Omega}\right)} = e^{\frac{i2\pi}{6}n} \tag{9}$$

This equation is equivalent to Equation (6). In a similar way, $\hat{X} = \hat{C}_6 \circ \hat{\zeta}$ imposes a selection rule on the transition between Floquet bands. To derive this selection rule, we expand $\left\langle \nu \middle| \hat{\boldsymbol{d}} \middle| \mu \right\rangle = \left\langle \nu \middle| \hat{\boldsymbol{d}} \middle| \mu \right\rangle \equiv \sum_n \overset{\sim}{\boldsymbol{d}}_n^{(\nu,\mu)}\left(\boldsymbol{Q_p}\right) e^{in\omega t}$ in a Fourier series, and employ the relation $\langle \nu | \hat{X}^{+}\hat{X}\hat{\boldsymbol{d}}\ \hat{X}^{+}\hat{X} | \mu \rangle$. The corresponding generalized selection rule is as follows:

$$\overset{\sim}{\boldsymbol{d}}_n^{(\nu,\mu)}\left(\boldsymbol{Q_p}\right) = e^{\frac{i2\pi}{6\Omega}(m_\nu - m_\mu + n)} \hat{R}_6 \overset{\sim}{\boldsymbol{d}}_n^{(\nu,\mu)}\left(\hat{\zeta}_{\hat{C}_6}\left[\boldsymbol{Q_p}\right]\right) \tag{10}$$

Equation (10) is a generalized selection rule for symmetry-protected dark states, which reduces to Equation (5) for $\boldsymbol{Q}_p \parallel \hat{z}$ and the linear term of $\overset{\sim}{\boldsymbol{d}}_n^{(\nu,\mu)}\left(\boldsymbol{Q}_p\right)$. As a generalization, the polarization of the probe can also account for the absorption of in-plane (the molecule XY plane) probes, e.g., right- (RCP) or left (LCP)-handed circularly polarized probes with Jones vectors $\boldsymbol{Q}_p \equiv Q(\hat{x} \pm i\hat{y})$ where a plus (minus) corresponds to an RCP (LCP)-polarized probe. Expanding $\overset{\sim}{\boldsymbol{d}}_n^{(\nu,\mu)}$ into orders of Q^k, Equation (10) becomes the following:

$$1 = e^{\frac{i2\pi}{6}\left(n - \left(\frac{\omega p}{\Omega} \pm 1\right)(k-1)\right)} \tag{11}$$

in which k is the nonlinearity order of the absorption, and $\pm$ corresponds to RCP/LCP, respectively.

Through a similar procedure (see Appendix A), we tabulate the generalized selection rules for symmetry-protected bands (Table 1) and symmetry-protected dark states (Table 2). The complete derivations are outlined in the SI. For a Hamiltonian with time-reversal $(\hat{T})$ symmetry, we find that transitions between Floquet bands (absorption/emission) of linearly polarized probes are allowed only when the electric field is in phase with the dressing field, regardless of the interaction's nonlinearity order. For $\hat{Z}_y$ symmetry, transition between Floquet bands is allowed only for an $\hat{x}$ polarized probe that satisfies $n - \frac{\omega_p}{\Omega}(k-1) + k + 1 = 2l$ and $\hat{y}$ polarized probes satisfying $\frac{\omega_p}{\Omega}(k-1) = 2l$, where k is the nonlinearity order, l is an integer, and n is the Floquet band index. For the circular DS, $\hat{C}_{N,M}$, transition between Floquet bands is allowed by an RCP/LCP probe that only satisfies $n - \left(\frac{\omega_p}{\Omega} \pm M\right)(k-1) = Nl$, where a plus (minus) corresponds to RCP (LCP) and M and N are integers. Similar rules are outlined in Table 2 for symmetry-protected dark states. It is important to note that our work is based solely on symmetry arguments, independent of the specific molecular structure or the various physical mechanisms of optical absorption [17]. Additionally, the presented formalism is general, provided that the light includes enough optical cycles to account for the Flouqet formalism. Notably, on-resonance interaction requires significantly larger numbers of optical cycles [16].

Information on the general SR with other symmetries is shown below:

Table 1. Selection rule of dark bend 'n' with various symmetries in general order and general polarized probes. When k is the order of the absorption, λ is the magnitude of the light. ω_p is the frequency of the probe and Ω is the frequency of the driver $n, l \in \mathbb{Z}$.

Symmetry	Selection Rule of $\overset{\sim}{\chi}_n^{(k,\hat{x})}, l \in \mathbb{Z}, k(\textit{nonlinearity order}) \in \mathbb{N}$
$\hat{T}$	Transition (absorption/emission): any linearly polarized probe in the same phase with the dressing field for any nonlinearity order
$\hat{Q}$	Transition (absorption/emission): linearly polarized probe with the phase πk/2
$\hat{G}$	Transition (absorption/emission): linearly polarized probe with the phase $\pi\left(\mathrm{n} - \frac{\omega_\mathrm{p}}{\Omega}(\mathrm{k}-1) + \mathrm{k} + 1\right)/2$

Table 1. *Cont.*

Symmetry	Selection Rule of $\overset{\sim}{\chi}_n^{(k,\hat{x})}, l \in \mathbb{Z}, k(\textit{nonlinearity order}) \in \mathbb{N}$
$\hat{Z}_y$	$\hat{x}$ polarized probe: $n - \frac{\omega_p}{\Omega}(k-1) + k + 1 = 2l$ $\hat{y}$ polarized probe: $n - \frac{\omega_p}{\Omega}(k-1) = 2l$
$\hat{D}_y$	$\hat{x}$ polarized probe with the phase $\pi(1+k)/2$ $\hat{y}$ polarized probe in phase with the dressing field
$\hat{H}_y$	$\hat{x}$ polarized probe with the phase $\frac{\pi}{2}\left(n - (k-1)\left(\frac{\omega_p}{\Omega}\right) + 1 + k\right)$ $\hat{y}$ polarized probe with the phase $\frac{\pi}{2}\left(n - \frac{\omega_p}{\Omega}(k-1)\right)$
$\hat{C}_{N,M}$	RCP-polarized probe: $n - \left(\frac{\omega_p}{\Omega} + 1\right)(k-1) = Nl$ LCP-polarized probe: $n - \left(\frac{\omega_p}{\Omega} - 1\right)(k-1) = Nl$
$\hat{e}_{N,M}$	RCP-polarized probe: $n - \left(\frac{\omega_p}{\Omega} + M\right)(k-1) = Nl$ LCP-polarized probe: $n - \left(\frac{\omega_p}{\Omega} - M\right)(k-1) = Nl$

Table 2. Selection rule between two states $(\mu.\nu)$ at bend 'n' with various symmetries in general order and general polarized probes.

Symmetry	Selection Rule of $\hat{V}_{\nu,\mu}^{(n,k)} l \in \mathbb{Z}, k(\textit{nonlinearity order}) \in \mathbb{N}$
$\hat{T}$	Linearly polarized probe with a phase: $\frac{\pi}{2}(m_\mu - m_\nu)$
$\hat{Q}$	Linearly polarized probe with a phase: $\frac{\pi}{2}(m_\mu - m_\nu + k + 1)$
$\hat{G}$	Linearly polarized probe with a phase: $\frac{\pi}{2}\left(n - \frac{\omega_p}{\Omega}(k-1) + k + 1 + m_\mu - m_\nu\right)$
$\hat{Z}_y$	$\hat{x}$ polarized probe: $n - \frac{\omega_p}{\Omega}(k-1) + k + 1 + m_\mu - m_\nu = 2l$ $\hat{y}$ polarized probe: $n - \frac{\omega_p}{\Omega}(k-1) + m_\mu - m_\nu = 2l$
$\hat{D}_y$	$\hat{x}$ polarized probe with a phase: $\frac{\pi}{2}(1 + k1 + k + m_\mu - m_\nu)$ $\hat{y}$ polarized probe with a phase: $\frac{\pi}{2}(m_\mu - m_\nu)$
$\hat{H}_y$	$\hat{x}$ polarized probe with a phase: $\frac{\pi}{2}\left(n - (k-1)\left(\frac{\omega_p}{\Omega}\right) + 1 + k + m_\mu - m_\nu\right)$ $\hat{y}$ polarized probe with a phase: $\frac{\pi}{2}\left(n - \frac{\omega_p}{\Omega}(k-1) + m_\mu - m_\nu\right)$
$\hat{C}_N$	RCP-polarized probe: $n - \left(\frac{\omega_p}{\Omega} + M\right)(k-1) + m_\mu - m_\nu = Nl$ LCP-polarized probe: $n - \left(\frac{\omega_p}{\Omega} - M\right)(k-1) + m_\mu - m_\nu = Nl$
$\hat{e}_N$	RCP-polarized probe: $n - \left(\frac{\omega_p}{\Omega} + M\right)(k-1) + m_\mu - m_\nu = Nl$ LCP-polarized probe: $n - \left(\frac{\omega_p}{\Omega} - M\right)(k-1) + m_\mu - m_\nu = Nl$

3. Discussion

In summary, in this study, we derived selection rules for symmetry-protected dark bands and states in Floquet systems by applying the concept of synthetic dynamical symmetries [16]. By employing synthetic dynamical symmetry, we extended the applicability of previous linear response theory treatments to nonlinear absorption by Floquet systems. Notably, our approach enables the identification of selection rules even when one of the fields disrupts the system's DS, accommodating complex field geometry. Our selection rules derived from dynamical symmetry generalized previously reported [14,18] selection rules for absorption in dressed matter. This work holds promise for the development of tunable polarization sensitive optical elements using Floquet dressing. Additionally, it connects the concepts of Floquet engineering and tailored light. Our selection rules can serve as a guide for further research on nonlinear absorption in complex geometries [19].

Author Contributions: All authors made substantial contributions to all aspects of the work. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Israel Science Foundation (ISF) grant number 2626/23.

Data Availability Statement: The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Acknowledgments: We thank the Helen Diller Quantum Center for their support. This work was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (819440), and by the Israel Science Foundation (Grant No. 2626/23). M.E.T. gratefully acknowledges the support of the Council for Higher Education scholarship for excellence in quantum science and technology.

Conflicts of Interest: The authors declare no competing interests.

Appendix A. Derivation

$$\hat{W} = \lambda \boldsymbol{Q}_{\mathrm{p}} \cdot \hat{\boldsymbol{d}}\, e^{i\omega_p t} + H.c.$$

$$\omega = \Omega n + \omega_p$$

$$\boldsymbol{E}_{emit}(\boldsymbol{Q}, t) = \lambda \boldsymbol{Q} \cdot \boldsymbol{r}\, e^{i\omega t}$$

In order to calculate $\overset{\sim}{\chi}_n^{(k,\hat{x})}$ for every symmetry $\hat{x}$, we first find $\hat{\zeta}_{\hat{x}}$ using the following equation:

$$\hat{X} = \hat{x} \cdot \hat{\zeta}_{\hat{x}}$$

$$\hat{x} \cdot \hat{\zeta}_{\hat{x}}(\hat{W}) = \hat{W}$$

We then find the SR of the emitter field using the following equation:

$$\hat{x} \cdot \hat{\zeta}_{\hat{T}}\, \boldsymbol{E}(t, \mathbf{Q}) = \boldsymbol{E}(t, \mathbf{Q})$$

$\boldsymbol{E}(t, \mathbf{Q})$ is the expectation value of the field. Additionally, we use Fourier series to separate the frequencies, and Tailor series to separate the order of the absorption.

Derivation of T-symmetry:

The symmetry operator is as follows:

$$\hat{X} = \hat{T} \cdot \hat{\zeta}_{\hat{T}}$$

$$\hat{T} \cdot \hat{\zeta}_{\hat{T}} \lambda \mathbf{Q}_{\mathrm{p}} \cdot \boldsymbol{r}\, e^{i\omega_p t} + h.c = \hat{\zeta}_{\hat{T}} \lambda \mathbf{Q}_{\mathrm{p}} \cdot \boldsymbol{r}\, e^{-i\omega_p t} + h.c$$

Therefore, the synthetic operator is as follows:

$$\hat{\zeta}_{\hat{T}}\left(\lambda \mathbf{Q}_{\mathrm{p}}\right) = \lambda \overline{\mathbf{Q}_{\mathrm{p}}}$$

The expectation value is as follows:

$$\hat{T} \cdot \hat{\zeta}_{\hat{T}}\, \boldsymbol{E}(t, \mathbf{Q}) = \boldsymbol{E}(t, \mathbf{Q})$$

$$\hat{T} \cdot \hat{\zeta}_{\hat{T}}\, \boldsymbol{E}(t, \mathbf{Q}) = \sum_n \boldsymbol{E}_n\left(\overline{\mathbf{Q}}\right) e^{-i\omega t} = \sum_n \boldsymbol{E}_n(\boldsymbol{Q}) e^{i\omega t}$$

$$\boldsymbol{E}_n\left(\overline{\boldsymbol{Q}}\right) = \boldsymbol{E}_{-n}(\boldsymbol{Q}) = \overline{\boldsymbol{E}}_n(\boldsymbol{Q})$$

Therefore, regardless of the polarization, the SR is as follows:

$$\boldsymbol{E}_{k,n} \in \mathbb{R}$$

Derivation of Q-symmetry:

$$\hat{X} = \hat{Q} \cdot \hat{\zeta}_{\hat{Q}}$$

For $\hat{x}$ or $\hat{y}$ polarized probes,

$$\hat{Q}\cdot\hat{\zeta}_{\hat{Q}}\left(\lambda\hat{x}\,e^{i\omega_p t}+h.c\right)=-\hat{\zeta}_{\hat{Q}}\left(\hat{x}\,e^{-i\omega_p t}+h.c\right)$$

Therefore, the synthetic operator is as follows:

$$\hat{\zeta}_{\hat{Q}}\left(\lambda\mathbf{Q}_{\mathrm{p}}\right)=-\lambda\overline{\mathbf{Q}_{\mathrm{p}}}$$

Regarding the expectation value, the following expressions can be used for $\hat{x}$ polarized probes, which are the same for $\hat{y}$ polarized probes:

$$\hat{Q}\cdot\hat{\zeta}_{\hat{Q}}\,\boldsymbol{E}(t,\mathbf{Q})=\boldsymbol{E}(t,\mathbf{Q})$$

$$\hat{Q}\cdot\hat{\zeta}_{\hat{Q}}\,\boldsymbol{E}(t,\mathbf{Q})=\sum_n\boldsymbol{E}_n\left(\hat{\zeta}_{\hat{Q}}(\boldsymbol{Q})\right)e^{-i\omega t}=\sum_n-\boldsymbol{E}_n(-\overline{\mathbf{Q}})e^{-i\omega t}=\sum_n\boldsymbol{E}_n(\boldsymbol{Q})e^{i\omega t}$$

$$\overline{E}_n(Q)=-E_n(-\overline{Q})$$

$$\sum_k\overline{\boldsymbol{E}}_{n,k}\lambda^k=\sum_k-\boldsymbol{E}_{n,k}\left(-\lambda\right)^k$$

$$\overline{\boldsymbol{E}}_{n,k}=(-1)^{k+1}\boldsymbol{E}_{n,k}$$

The SR is as follows:

$$i^{1+k}\cdot\ \boldsymbol{E}_{k,n}\in\mathbb{R}$$

Derivation of G-symmetry:

$$\hat{X}=\hat{G}\cdot\hat{\zeta}_{\hat{G}}$$

For $\hat{x}$ or $\hat{y}$ polarized probes,

$$\hat{G}\cdot\hat{\zeta}_{\hat{G}}\left(\lambda\hat{x}\,e^{i\omega_p t}+h.c\right)=-\hat{\zeta}_{\hat{G}}\left(\hat{x}\,e^{-i\omega_p t}(-1)^{\frac{\omega_p}{\Omega}}+h.c\right)$$

Therefore, the synthetic operator is as follows:

$$\hat{\zeta}_{\hat{G}}\left(\lambda\mathbf{Q}_{\mathrm{p}}\right)=(-1)^{-\frac{\omega_p}{\Omega}-1}\lambda\overline{\mathbf{Q}_{\mathrm{p}}}$$

For the expectation value, the following expressions can be used for $\hat{x}$ polarized probes, which are the same for $\hat{y}$ polarized probes:

$$\hat{G}\cdot\hat{\zeta}_{\hat{G}}\,\boldsymbol{E}(t,\mathbf{Q})=\boldsymbol{E}(t,\mathbf{Q})$$

$$\hat{G}\cdot\hat{\zeta}_{\hat{G}}\,\boldsymbol{E}(t,\mathbf{Q})=\sum_n(-1)^{\frac{\omega}{\Omega}+1}\boldsymbol{E}_n(\hat{\zeta}_{\hat{G}}(\boldsymbol{Q}))e^{-i\omega t}=\sum_n\boldsymbol{E}_n\left(\overline{\mathbf{Q}}(-1)^{-\frac{\omega_p}{\Omega}-1}\right)e^{-i\omega t}=\sum_n\boldsymbol{E}_n(\boldsymbol{Q})e^{i\omega t}$$

$$\overline{\boldsymbol{E}}_n(\boldsymbol{Q})=(-1)^{\frac{\omega}{\Omega}+1}\boldsymbol{E}_n\left(\overline{\mathbf{Q}}(-1)^{-\frac{\omega_p}{\Omega}-1}\right)$$

$$\sum_k\overline{\boldsymbol{E}}_{n,k}\lambda^k=\sum_k(-1)^{\frac{\omega}{\Omega}+1}\boldsymbol{E}_{n,k}((-1)^{-\frac{\omega_p}{\Omega}-1}\lambda)^k$$

$$\overline{\boldsymbol{E}}_{n,k}=(-1)^{\frac{\omega}{\Omega}+1-(\frac{\omega_p}{\Omega}+1)k}\boldsymbol{E}_{n,k}=(-1)^{\frac{\omega+\omega_p}{\Omega}+1-1-(\frac{\omega_p}{\Omega}+1)(k-1)}\boldsymbol{E}_{n,k}$$

The SR is as follows:

$$i^{n-(\frac{\omega_p}{\Omega}+1)(k-1)}\cdot\ \boldsymbol{E}_{n,k}\in\mathbb{R}$$

Derivation of $\boldsymbol{Z_y}$-symmetry:

$$\hat{X}=\hat{Z}\cdot\hat{\zeta}_{\hat{Z}}$$

For $\hat{x}$ polarized probes,

$$\hat{Z}\cdot\hat{\zeta}_{\hat{Z}}\left(\lambda\hat{x}\,e^{i\omega_p t}+h.c\right)=-\hat{\zeta}_{\hat{Z}}\left(\hat{x}\,e^{i\omega_p t}(-1)^{\frac{\omega_p}{\Omega}}+h.c\right)$$

Therefore, the synthetic operator is as follows:

$$\hat{\zeta}_{\hat{Z}}\left(\lambda\boldsymbol{Q}_{\mathrm{p}}\right)=(-1)^{-\frac{\omega_p}{\Omega}-1}\lambda\boldsymbol{Q}_{\mathrm{p}}$$

The expectation value of the emitter field is shown below:

$$\hat{Z}\cdot\hat{\zeta}_{\hat{Z}}\,\boldsymbol{E}(t,\mathbf{Q})=\sum_n(-1)^{\frac{\omega}{\Omega}+1}\boldsymbol{E}_n\left(\hat{\zeta}_{\hat{Z}}(\mathbf{Q})\right)e^{i\omega t}=\sum_n\boldsymbol{E}_n\left((-1)^{-\frac{\omega_p}{\Omega}-1}\boldsymbol{Q}\right)e^{i\omega t}=\sum_n\boldsymbol{E}_n(\boldsymbol{Q})e^{i\omega t}$$

$$\boldsymbol{E}_n(Q)=(-1)^{\frac{\omega}{\Omega}+1}\boldsymbol{E}_n\left(\boldsymbol{Q}(-1)^{-\frac{\omega_p}{\Omega}-1}\right)$$

$$\sum_k\boldsymbol{E}_{n,k}\lambda^k=\sum_k(-1)^{\frac{\omega}{\Omega}+1}\boldsymbol{E}_{n,k}((-1)^{-\frac{\omega_p}{\Omega}-1}\lambda)^k$$

$$\boldsymbol{E}_{n,k}=(-1)^{\frac{\omega}{\Omega}+1-(\frac{\omega_p}{\Omega}+1)k}\boldsymbol{E}_{n,k}=(-1)^{\frac{\omega+\omega_p}{\Omega}-(\frac{\omega_p}{\Omega}+1)(k-1)}\boldsymbol{E}_{n,k}$$

The SR is as follows:

$$n-\left(\frac{\omega_p}{\Omega}+1\right)(k-1)=2l\;,\;l\in\mathbb{Z}$$

For $\hat{y}$ polarized probes,

$$\hat{Z}\cdot\hat{\zeta}_{\hat{Z}}\left(\lambda\hat{y}\,e^{i\omega_p t}+h.c\right)=\hat{\zeta}_{\hat{Z}}\left(\hat{x}\,e^{i\omega_p t}(-1)^{\frac{\omega_p}{\Omega}}+h.c\right)$$

Therefore, the synthetic operator is as follows:

$$\hat{\zeta}_{Z}\left(\lambda\boldsymbol{Q}_{\mathrm{p}}\right)=(-1)^{-\frac{\omega_p}{\Omega}}\lambda\boldsymbol{Q}_{\mathrm{p}}$$

The expectation value of the emitter field is shown below:

$$\hat{Z}\cdot\hat{\zeta}_{\hat{Z}}\,\boldsymbol{E}(t,\mathbf{Q})=\sum_n(-1)^{\frac{\omega}{\Omega}}\boldsymbol{E}_n\left(\hat{\zeta}_{\hat{Z}}(\mathbf{Q})\right)e^{i\omega t}=\sum_n(-1)^{\frac{\omega}{\Omega}}\boldsymbol{E}_n\left((-1)^{-\frac{\omega_p}{\Omega}}\boldsymbol{Q}\right)e^{i\omega t}=\sum_n\boldsymbol{E}_n(\boldsymbol{Q})e^{i\omega t}$$

$$\boldsymbol{E}_n(Q)=(-1)^{\frac{\omega}{\Omega}}\boldsymbol{E}_n\left(\boldsymbol{Q}(-1)^{-\frac{\omega_p}{\Omega}}\right)$$

$$\sum_k\boldsymbol{E}_{n,k}\lambda^k=\sum_k(-1)^n\boldsymbol{E}_{n,k}((-1)^{-\frac{\omega_p}{\Omega}}\lambda)^k$$

$$\boldsymbol{E}_{n,k}=(-1)^{\frac{\omega}{\Omega}-(\frac{\omega_p}{\Omega})k}\boldsymbol{E}_{n,k}=(-1)^{\frac{\omega+\omega_p}{\Omega}-(\frac{\omega_p}{\Omega})(k-1)}\,\boldsymbol{E}_{n,k}$$

The SR is as follows:

$$n-\left(\frac{\omega_p}{\Omega}\right)(k-1)=2l\;,\;l\in\mathbb{Z}$$

Derivation of D-symmetry:
The symmetry operator is as follows:

$$\hat{X}=\hat{D}\cdot\hat{\zeta}_{\hat{D}}$$

For $\hat{x}$ polarized probes,

$$\hat{D}\cdot\hat{\zeta}_{\hat{D}}\left(\lambda\hat{x}\,e^{i\omega_p t}+h.c\right)=\hat{\zeta}_{\hat{D}}\left(-\hat{x}\,e^{-i\omega_p t}+h.c\right)$$

Therefore, the synthetic operator is as follows:

$$\hat{\zeta}_{\hat{D}}\left(\lambda\mathbf{Q}_{\mathrm{p}}\right)=-\lambda\overline{\mathbf{Q}}_{\mathrm{p}}$$

In this instance, the $\hat{Q}$ symmetry is the same; therefore, the SR is as follows:

$$i^{1+k}\cdot\ \boldsymbol{E}_{k,n}\in\mathbb{R}$$

For $\hat{y}$ polarized probes,

$$\widehat{D}\cdot\hat{\zeta}_{\hat{D}}\left(\lambda\hat{y}\,e^{i\omega_p t}+h.c\right)=\hat{\zeta}_{\hat{D}}\left(\hat{y}\,e^{-i\omega_p t}+h.c\right)$$

Therefore, the synthetic operator is as follows:

$$\hat{\zeta}_{\hat{D}}\left(\lambda\mathbf{Q}_{\mathrm{p}}\right)=\lambda\overline{\mathbf{Q}}_{\mathrm{p}}$$

In this instance, the operator is similar to $\hat{T}$ symmetry; therefore, the SR is as follows:

$$\boldsymbol{E}_{k,n}\in\mathbb{R}$$

Derivation of H-symmetry:
The symmetry operator is as follows:

$$\hat{X}=\hat{H}_y\cdot\hat{\zeta}_{\hat{H}}$$

For $\hat{x}$ polarized probes,

$$\hat{H}\cdot\hat{\zeta}_{\hat{H}}\left(\lambda\hat{x}\,e^{i\omega_p t}+h.c\right)=\hat{\zeta}_{\hat{H}}\left(-\hat{x}\,e^{-i\omega_p t}+h.c\right)$$

Therefore, the synthetic operator is as follows:

$$\hat{\zeta}_{\hat{H}}\left(\lambda\mathbf{Q}_{\mathrm{p}}\right)=-\lambda\overline{\mathbf{Q}}_{\mathrm{p}}(-1)^{-\frac{\omega_p}{\Omega}}$$

In this instance, the $\hat{G}$ symmetry is the same; therefore, the selection rule is as follows:

$$i^{n-(\frac{\omega_p}{\Omega}+1)(k-1)}\cdot\ \boldsymbol{E}_{n,k}\in\mathbb{R}$$

For $\hat{y}$ polarized probes,

$$\hat{H}\cdot\hat{\zeta}_{\hat{H}}\left(\lambda\hat{y}\,e^{i\omega_p t}+h.c\right)=\hat{\zeta}_{\hat{H}}\left(\hat{y}\,e^{-i\omega_p t}+h.c\right)$$

Therefore, the synthetic operator is as follows:

$$\hat{\zeta}_{\hat{H}}\left(\lambda\mathbf{Q}_{\mathrm{p}}\right)=\lambda\overline{\mathbf{Q}}_{\mathrm{p}}(-1)^{-\frac{\omega_p}{\Omega}}$$

The expectation value of the emitter field is shown below:

$$\widehat{H}\cdot\hat{\zeta}_{\hat{H}}\,\boldsymbol{E}(t,\mathbf{Q})=\sum_n(-1)^{\frac{\omega}{\Omega}}\boldsymbol{E}_n\big(\hat{\zeta}_{\hat{H}}(\mathbf{Q})\big)e^{-i\omega t}=\sum_n(-1)^{\frac{\omega}{\Omega}}\boldsymbol{E}_n\left((-1)^{-\frac{\omega_p}{\Omega}}\overline{\mathbf{Q}}\right)e^{-i\omega t}=\sum_n\boldsymbol{E}_n(\boldsymbol{Q})e^{i\omega t}$$

$$(-1)^{\frac{\omega}{\Omega}}\boldsymbol{E}_n\left((-1)^{-\frac{\omega_p}{\Omega}}\overline{\boldsymbol{Q}}\right)=\overline{\boldsymbol{E}}_n(\boldsymbol{Q})e^{i\omega t}$$

$$\sum_k \overline{\boldsymbol{E}}_{n,k}\lambda^k=\sum_k(-1)^{\frac{\omega}{\Omega}}\boldsymbol{E}_{n,k}((-1)^{-\frac{\omega_p}{\Omega}}\lambda)^k$$

$$\overline{\boldsymbol{E}}_{n,k}=(-1)^{\frac{\omega}{\Omega}-\frac{\omega_p}{\Omega}k}\boldsymbol{E}_{n,k}=(-1)^{n-\frac{\omega_p}{\Omega}(k-1)}\boldsymbol{E}_{n,k}$$

The SR is as follows:

$$i^{n-\frac{\omega_p}{\Omega}(k-1)}\boldsymbol{E}_{n,k}\in\mathbb{R}$$

Derivation of C-symmetry:

The symmetry operator is as follows:

$$\hat{X}=\hat{C}_{N,M}\cdot\hat{\zeta}_{\hat{C}}$$

The vectors $\hat{e}_{RH},\hat{e}_{LH}$ are RH and LH polarizations, i.e., $x+iy,\ x-iy$. These vectors are the eigen vectors of the rotation operators, with an eigen value of $e^{\pm\frac{i2\pi}{N}M}$.

For RCP-polarized probes,

$$\hat{C}\cdot\hat{\zeta}_{\hat{C}}\left(\lambda\hat{e}_{RH}\,e^{i\omega_p t}+h.c\right)=\hat{\zeta}_{\hat{C}}\left(e^{\frac{i2\pi}{N}M}\lambda\hat{e}_{RH}e^{-i\omega_p t}e^{2\pi\frac{i\omega_p}{N\Omega}}+h.c\right)$$

Therefore, the synthetic symmetry operator is as follows:

$$\hat{\zeta}_{\hat{C}}\left(\lambda\boldsymbol{Q}_{\mathrm{p}}\right)=\lambda\boldsymbol{Q}_{\mathrm{p}}e^{-2\pi\frac{i\omega_p}{N\Omega}}e^{-\frac{i2\pi}{N}M}$$

The expectation value of the emitter field is shown below:

$$\hat{C}\cdot\hat{\zeta}_{\hat{C}}\,\boldsymbol{E}(t,\mathbf{Q})=\sum_n e^{\frac{i2\pi}{N}M}e^{2\pi i\frac{\omega}{N\Omega}}\boldsymbol{E}_n\left(\hat{\zeta}_{\hat{C}}(\boldsymbol{Q})\right)e^{i\omega t}=\sum_n e^{\frac{i2\pi}{N}M}e^{2\pi\frac{i\omega}{N\Omega}}\boldsymbol{E}_n\left(e^{-\frac{i2\pi}{N}M}e^{-2\pi i\frac{\omega_p}{N\Omega}}\boldsymbol{Q}\right)e^{i\omega t}=\sum_n\boldsymbol{E}_n(\boldsymbol{Q})e^{i\omega t}$$

$$e^{\frac{i2\pi}{N}M}e^{2\pi i\frac{\omega}{N\Omega}}\boldsymbol{E}_n\left(e^{-\frac{i2\pi}{N}M}e^{-2\pi i\frac{\omega_p}{N\Omega}}\boldsymbol{Q}\right)=\boldsymbol{E}_n(\boldsymbol{Q})e^{i\omega t}$$

$$\sum_k\boldsymbol{E}_{n,k}\lambda^k=\sum_k e^{\frac{i2\pi}{N}M}e^{2\pi i\frac{\omega}{N\Omega}}\boldsymbol{E}_{n,k}(e^{-\frac{i2\pi}{N}M}e^{-2\pi i\frac{\omega_p}{N\Omega}}\lambda)^k$$

$$\boldsymbol{E}_{n,k}=\boldsymbol{E}_{n,k}e^{\frac{2\pi i}{N}(\frac{\omega}{\Omega}+M-(\frac{\omega_p}{\Omega}+M)k)}=\boldsymbol{E}_{n,k}e^{\frac{2\pi i}{6}(\frac{\omega-\omega_p}{\Omega}-(\frac{\omega_p}{\Omega}+M)(k-1))}$$

$$n+\left(\frac{\omega_p}{\Omega}+M\right)(k-1)=Nl\ ,\ l\in\mathbb{Z}$$

For LCP-polarized probes,

$$\hat{C}\cdot\hat{\zeta}_{\hat{C}}\left(\lambda\hat{e}_{LH}\,e^{i\omega_p t}+h.c\right)=\hat{\zeta}_{\hat{C}}\left(e^{-\frac{i2\pi}{N}M}\lambda\hat{e}_{RH}e^{-i\omega_p t}e^{2\pi\frac{i\omega_p}{N\Omega}}+h.c\right)$$

Therefore, the synthetic symmetry operator is as follows:

$$\hat{\zeta}_{\hat{C}}\left(\lambda\boldsymbol{Q}_{\mathrm{p}}\right)=\lambda\boldsymbol{Q}_{\mathrm{p}}e^{-2\pi\frac{i\omega_p}{N\Omega}}e^{\frac{i2\pi}{N}M}$$

The expectation value of the emitter field is shown below:

$$\hat{C}\cdot\hat{\zeta}_{\hat{C}}\,\boldsymbol{E}(t,\mathbf{Q})=\sum_n e^{-\frac{i2\pi}{N}M}e^{2\pi i\frac{\omega}{N\Omega}}\boldsymbol{E}_n\left(\hat{\zeta}_{\hat{Z}}(\boldsymbol{Q})\right)e^{i\omega t}=\sum_n e^{-\frac{i2\pi}{N}M}e^{2\pi\frac{i\omega}{N\Omega}}\boldsymbol{E}_n\left(e^{\frac{i2\pi}{N}M}e^{-2\pi i\frac{\omega_p}{N\Omega}}\boldsymbol{Q}\right)e^{i\omega t}=\sum_n\boldsymbol{E}_n(\boldsymbol{Q})e^{i\omega t}$$

$$e^{-\frac{i2\pi}{N}M}e^{2\pi i\frac{\omega}{N\Omega}}\boldsymbol{E}_n\left(e^{\frac{i2\pi}{N}M}e^{-2\pi i\frac{\omega_p}{N\Omega}}\boldsymbol{Q}\right)=\boldsymbol{E}_n(\boldsymbol{Q})e^{i\omega t}$$

$$\sum_k \boldsymbol{E}_{n,k}\lambda^k = \sum_k e^{-\frac{i2\pi}{N}M} e^{2\pi i\frac{\omega}{N\Omega}} \boldsymbol{E}_{n,k}(e^{-\frac{i2\pi}{N}M} e^{-2\pi i\frac{\omega_p}{N\Omega}}\lambda)^k$$

$$\boldsymbol{E}_{n,k} = \boldsymbol{E}_{n,k} e^{\frac{2\pi i}{N}(\frac{\omega}{\Omega}-M-(\frac{\omega_p}{\Omega}-M)k)} = \boldsymbol{E}_{n,k} e^{\frac{2\pi i}{N}(\frac{\omega-\omega_p}{\Omega}-(\frac{\omega_p}{\Omega}-M)(k-1))}$$

The selection rule is as follows:

$$n - \left(\frac{\omega_p}{\Omega} - M\right)(k-1) = Nl \ , \ l \in \mathbb{Z}$$

Derivation of e-symmetry:

The symmetry operator is as follows:

$$\hat{X} = \hat{e}_{N,M} \cdot \hat{\zeta}_{\hat{C}}$$

Now, we can define $\hat{e}_{RH/LH}$ as $x \pm iby$, i.e., the right/left elliptical polarized probes. These unit vectors are the eigen vector of the operator $\hat{L}_b \hat{R}_N \hat{L}_{\frac{1}{b}}$ and they have an eigen value of $e^{\pm\frac{i2\pi}{N}M}$.

For RCP-polarized probes,

$$\hat{e} \cdot \hat{\zeta}_{\hat{e}}\left(\lambda \hat{e}_{RH}\, e^{i\omega_p t} + h.c\right) = \hat{\zeta}_{\hat{e}}\left(e^{\frac{i2\pi}{N}M} \lambda \hat{e}_{RH} e^{-i\omega_p t} e^{2\pi\frac{i\omega_p}{N\Omega}} + h.c\right)$$

Therefore, the synthetic symmetry operator is as follows:

$$\hat{\zeta}_{\hat{e}}\left(\lambda \boldsymbol{Q}_{\mathrm{p}}\right) = \lambda \boldsymbol{Q}_{\mathrm{p}} e^{-2\pi\frac{i\omega_p}{N\Omega}} e^{-\frac{i2\pi}{N}M}$$

In this instance, the $\hat{C}$ symmetry is the same; therefore, the selection rule is as follows:

$$n - \left(\frac{\omega_p}{\Omega} + M\right)(k-1) = Nl \ , \ l \in \mathbb{Z}$$

For LCP-polarized probes,

$$\hat{e} \cdot \hat{\zeta}_{\hat{e}}\left(\lambda \hat{e}_{RH}\, e^{i\omega_p t} + h.c\right) = \hat{\zeta}_{\hat{e}}\left(e^{\frac{i2\pi}{N}M} \lambda \hat{e}_{RH} e^{-i\omega_p t} e^{2\pi\frac{i\omega_p}{N\Omega}} + h.c\right)$$

Therefore, the synthetic symmetry operator is as follows:

$$\hat{\zeta}_{\hat{e}}\left(\lambda \boldsymbol{Q}_{\mathrm{p}}\right) = \lambda \boldsymbol{Q}_{\mathrm{p}} e^{-2\pi\frac{i\omega_p}{N\Omega}} e^{-\frac{i2\pi}{N}M}$$

In this instance, the $\hat{C}$ symmetry is the same; therefore, the selection rule is as follows:

$$n - \left(\frac{\omega_p}{\Omega} - M\right)(k-1) = 6l \ , \ l \in \mathbb{Z}$$

The derivation of $\hat{V}^{(n,k)}_{\nu,\mu}$ is very similar to the derivation of $\overset{\sim}{\chi}{}^{(k,\hat{x})}_{n}$, but the expectation value of $\boldsymbol{E}_{emit}$ is added to the eigen value of the state μ, ν in this instance. Therefore, we need to solve the following equation:

$$e^{\frac{i2\pi}{N}(m_\mu - m_\nu)} \hat{x} \cdot \hat{\zeta}_{\hat{T}}\, \boldsymbol{E}(t, \mathbf{Q}) = \boldsymbol{E}(t, \mathbf{Q})$$

In the above equation, N is the number of the different states in every band. Because of this, we obtain the same SR but with the addition of $m_\mu - m_\nu$.

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