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# Initial State Factorization and the Kinoshita-Lee-Nauenberg Theorem

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**Abstract.** We carry out for the first time a fully correct, self-consistent application of the Kinoshita-Lee-Nauenberg theorem in a next-to-leading order scattering calculation. We improve on previous works by including all initial and final state soft radiative processes, including absorption and an infinite sum of partially disconnected amplitudes. We show in general that the sum over the initial state radiation factorizes. We apply our approach to the Rutherford scattering cross section at next-to-leading order correction.

## 1. Introduction

The infrared (IR) divergences in purely massless theories come in two flavors: *soft*, due to the massless nature of the radiation, and *collinear*, which comes from treating the radiating particle as massless. The Bloch-Nordsieck (BN) theorem [1, 2] is known as the first theoretical response towards solving the IR problem by summing over the final degenerate states including the emission of soft photons within the energy window  $\Delta$ . The BN theorem gave an exact cancellation of the soft IR divergences leaving behind some collinear singularities. The non-cancelling collinear singularities have been studied later separately by Kinoshita [3], Lee and Nauenberg [4] (KLN) where they came up with a more general theorem which states that one should sum over both initial and final indistinguishable degenerate states in order to get rid of all the collinear divergences. Lee and Nauenberg (LN) included an example demonstrating the IR finite cross section for elastic Rutherford scattering computed at next-to-leading order (NLO) when one integrates over a degenerate final-state energy window and degenerate initial and final state angular windows. However, for consistency, one should also sum over the initial degenerate soft photon states which spoil the IR cancellation.

Many attempts has been made to lead the non-cancelling IR divergences issue from the initial state soft absorption by invoking the role of the disconnected diagrams, where the disconnected amplitudes can interfere with fully connected amplitudes in which there are both emission and absorption of a soft photon, forming a connected cut diagram [5–9]. However, the former subsequent works summed over some, but not all, possible degenerate initial and final states, a full discussion on the incompleteness of their calculations can be found in Ref. [7]. Ito [10] and Akhoury, Sotiropoulos, and Zakharov [11] (IASZ) included all possible initial and final degenerate states with an infinite number of disconnected soft photons forming a divergent infinite series at NLO. They proved that by rearranging the series through formal manipulations, it will be sufficient to sum over a finite number of diagrams, but, as was pointed out in [7], the IASZ rearrangement yields a NLO result that is exactly 0 at tree level.



We will give an example where an electron is scattered off of a point charge by including the NLO diagrams excluding the vacuum polarization and the box correction. We show that the resolution is to use the IASZ approach. In contrary to the IASZ rearrangement, we provide a different rearrangement under a control of a convergent factor to prove the uniqueness of our result, we note that the disconnected part remains factorized.

## 2. NLO Rutherford

Consider an electron with four-momentum  $p = (E, \vec{p})$  scatters off a static point charge into a state with final four-momentum  $p' = (E, \vec{p}')$ , the momentum transfer is then  $q \equiv p' - p$ . We use the photon mass  $m_\gamma$  and keep the electron mass  $m_e$  non-zero to regulate the soft and collinear IR divergences respectively. The application of the BN theorem requires including the tree level Fig. 1 (a) and NLO vertex correction (b) as well as the final state soft bremsstrahlung process (c) and (d). Summing these processes yields

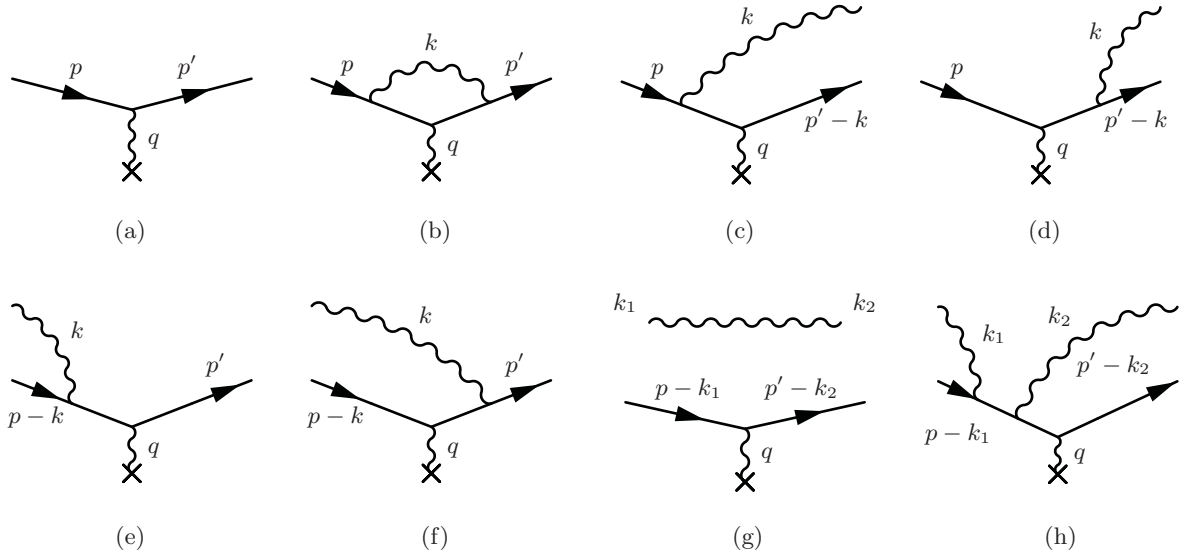
$$d\sigma_{BN} = d\sigma_0 \left\{ 1 + \frac{\alpha_e}{\pi^2} \left[ \log \left( \frac{E^2}{\Delta^2} \right) \left( 1 - \log \left( \frac{-q^2}{m_e^2} \right) \right) + \frac{3}{2} \log \left( \frac{-q^2}{m_e^2} \right) + \mathcal{O}(1) \right] \right\}, \quad (1)$$

which is free of soft IR divergences. We now use the KLN theorem to remove the potentially large logs,  $\log(-q^2/m_e^2)$ , in the high energy limit ( $-q^2 \gg m^2$ ) by summing over the indistinguishable hard collinear final emission Fig. 1 (d) and initial absorption (e) processes. We note that, the indistinguishably fro such a process is determined by the angular resolution ( $\delta$ ). One can easily see that the LN result can be obtained by replacing the  $m_e^2$  by  $\delta^2 E^2$  [4], and the result is free of any IR divergences. Here, we point out to the fact that we used a combination between the BN and KLN theorems for such a cancellation, which is clearly inconsistent [7], and to stay in the spirit of the KLN theorem one should also add the absorption of a soft photon as a degenerate initial state which devastate the finiteness of the cross section at NLO.

It is also shown that including the contributions from the interference of the disconnected diagram Fig. 1 (g) with the absorption-emission diagrams Fig. 1 (h) is not sufficient for the cancellation, even adding the contributions from the absorption or emission with one disconnected photon keeps the cross section IR divergent and raises the question of adding even more disconnected photons to the former processes, while the most important question now: is there a full systematic method to apply the KLN theorem properly? We show here that the resolution is to include all possible contributions from degenerate initial and final states, including the contributions from the disconnected diagrams with an arbitrary number of disconnected soft photons. Following [10, 11], we consider a general form of our process with  $m$  incoming soft photons and  $n$  outgoing soft photons, each process is described by the transition probability  $\mathcal{P}_{mn}$  corresponding to an amplitude  $\mathcal{M}_{mn}$ . The total Lee-Nauenberg probability will be

$$\mathcal{P} = \sum_{m,n=0}^{\infty} \mathcal{P}_{mn} = \sum_{m,n=0}^{\infty} \frac{1}{m!} \frac{1}{n!} \sum_{i,f} |\mathcal{M}_{mn}|^2, \quad (2)$$

It is shown in [10, 11] that any cut diagram from  $\mathcal{P}_{mn}$  at NLO correction can be constructed from a four essential probabilities:  $\mathcal{P}_{00}$  which is the cut diagram with no real emission or absorption of soft photons, this may include the leading term, the vertex correction, the vacuum polarization, etc.;  $\mathcal{P}_{10}$  and  $\mathcal{P}_{01}$  which includes all cut diagrams with one soft photon in the initial or the final states respectively;  $\mathcal{P}_{11}$  which includes all cut diagrams from the interference between the disconnected diagram Fig. 1 (g) and the absorption-emission diagrams like Fig. 1 (h). The fully connected cut diagrams are given by any of the previous basic probabilities while the partially connected ones are these probabilities multiplied by a number of  $\delta$  functions



**Figure 1.** The (a) leading order (LO) and (b) next-to-leading order (NLO) vertex contributions to the  $1 \rightarrow 1$  process in Rutherford scattering. The (c) and (d) potentially degenerate final state radiative processes. The (e) and (f) potentially degenerate initial state radiative processes. A disconnected photon amplitude such as (g) might interfere with an amplitude with a photon absorption and emission process such as (h).

according to the number of disconnected photons, so we can construct  $\mathcal{P}_{mn}$  by splitting each cut diagram up into connected and disconnected parts. We define the disconnected part by the function  $\mathcal{D}(m-i, n-j)$  that describes the number of  $m-i$  incoming and  $n-j$  outgoing soft photons that can be joined together and become disconnected from the electron line. By definition  $\mathcal{D}(0,0) = 1$  and  $\mathcal{D}(a,b) = 0$  for  $a \neq b$ . One may show that [10, 11] the transition probability for the general process at NLO is given by

$$\mathcal{P}_{mn} = \frac{\mathcal{D}(m,n)}{m!n!} \mathcal{P}_{00} + \sum_{i=0} \frac{\mathcal{D}(m-i, n-i-1)}{(m-i)!(n-i-1)!} \mathcal{P}_{01} + \sum_{i=0} \frac{\mathcal{D}(m-i-1, n-i)}{(m-i-1)!(n-i)!} \mathcal{P}_{10} + \sum_{i=0} \frac{\mathcal{D}(m-i-1, n-i-1)}{(m-i-1)!(n-i-1)!} \tilde{\mathcal{P}}_{11}. \quad (3)$$

IASZ were able to rearrange the full series to give an IR finite value for  $\mathcal{P}$ . However, as it is shown in [7] that their argument is not safe because the probability  $\mathcal{P}$  tends to zero once we send it back to the tree level. One may show that the series can be rearranged in different ways. However, we are looking for a rearrangement with special features: to be IR safe and to keep the tree level contribution finite. We give here the rearrangement that satisfies these criteria and rigorously prove that our result is unique. In order to give such a proof, we manipulate Eq. (3) in a controlled way by introducing a convergence factor that becomes small for large  $i$ : we take

$$\mathcal{D}(m-i, n-j) \rightarrow \mathcal{D}(m-i, n-j) e^{-(i+j)/\Lambda} \quad (4)$$

with  $\Lambda \gg 1$ . The convergent factor  $\Lambda$  allows us to sum up to finite number of disconnected soft photons  $N$ . Since  $\mathcal{D}(m,n) = 0$  for  $m \neq n$ , we may simplify our manipulations by replacing the double sum over  $m$  and  $n$  with a single sum over  $n$ . With the above convergence factor, we

are guaranteed that  $\mathcal{P} = \lim_{\Lambda \rightarrow \infty} \lim_{N \rightarrow \infty} \mathcal{P}_N(\Lambda)$  converges. Now, we rearrange the partial sum  $\mathcal{P}_N(\Lambda)$  up to  $N$  soft photons to find

$$\mathcal{P}_N(\Lambda) = \sum_{n=0}^N \frac{\mathcal{D}(n, n)}{(n!)^2} \left[ \mathcal{P}_{00} + e^{-\frac{1}{\Lambda}} \mathcal{P}_{01} \right] + \sum_{n=1}^N \sum_{i=1}^n \frac{\mathcal{D}(n-i, n-i)}{[(n-i)!]^2} \left[ e^{-\frac{2i+1}{\Lambda}} \mathcal{P}_{01} + e^{-\frac{2i-1}{\Lambda}} \mathcal{P}_{10} + e^{-\frac{2i}{\Lambda}} \tilde{\mathcal{P}}_{11} \right], \quad (5)$$

One can easily show, using the same convergent factor, that the IASZ result is very different from Eq. (5) after swapping the limit in the full sum. So we check the possibility of switching the limits in order to prove the uniqueness of our result. We exploit the Monotone Convergence Theorem to justify that our result converges to the same value under the interchanging limits procedure by proving that the partial sum  $\mathcal{P}_N(\Lambda)$  1) monotonically increase in  $N$  for each  $\Lambda$  and 2) monotonically increase in  $\Lambda$  for each  $N$  [12]. We use the fact that  $\mathcal{P}_{01} + \mathcal{P}_{10} = -\tilde{\mathcal{P}}_{11}$  [6] to simplify Eq. (5). Then we have

$$\mathcal{P}_N(\Lambda) = \sum_{n=0}^N \frac{\mathcal{D}(n, n)}{(n!)^2} \left[ \mathcal{P}_{00} + e^{-\frac{1}{\Lambda}} \mathcal{P}_{01} \right] + \sum_{n=1}^N \sum_{i=1}^n \frac{\mathcal{D}(n-i, n-i)}{[(n-i)!]^2} 2\mathcal{P}_{01} e^{-\frac{2i}{\Lambda}} \left[ \cosh\left(\frac{1}{\Lambda}\right) - 1 \right]. \quad (6)$$

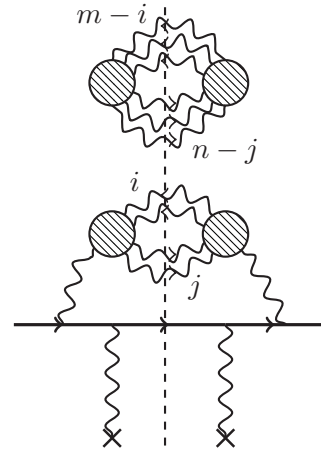
Since  $\mathcal{P}_{00}$ ,  $\mathcal{P}_{01}$ ,  $\Lambda$ , and  $\mathcal{D}(n, n)$  are all strictly positive, Eq. (6) clearly increases monotonically in  $N$  for fixed  $\Lambda$ . To show that  $\mathcal{P}_N(\Lambda)$  increases monotonically in  $\Lambda$  for fixed  $N$ , we take the derivative with respect to  $\Lambda$ :

$$\frac{d\mathcal{P}_N(\Lambda)}{d\Lambda} = \sum_{n=0}^N \frac{\mathcal{D}(n, n)}{(n!)^2} \left[ \frac{1}{\Lambda^2} e^{-\frac{1}{\Lambda}} \mathcal{P}_{01} \right] + \mathcal{O}\left(\frac{1}{\Lambda^3}\right). \quad (7)$$

Although one finds that the higher order in  $1/\Lambda$  correction term is negative, for any  $N$  we can find a  $\Lambda$  large enough such that the first term, which is strictly positive, dominates. We have thus proved that we may exchange limits for our rearranged formula Eq. (6), and we may evaluate the  $\Lambda \rightarrow \infty$  limit first, yielding our main result:

$$\mathcal{P} = (\mathcal{P}_{00} + \mathcal{P}_{01}) \sum_{n=0}^{\infty} \frac{\mathcal{D}(n, n)}{(n!)^2}. \quad (8)$$

We find that all the soft initial state physics of the infinite number of undetectably soft photons completely factorizes. When the cross section is computed, one simply divides out by this unobserved infinity. We have thus rendered all soft and collinear IR divergences harmless.



**Figure 2.** A generic cut diagram with  $m$  incoming and  $n$  outgoing soft photons that is partially disconnected. (Note that the blobs include the possibility of connecting incoming photon lines from the left with conjugate incoming photon lines from the right.)

### 3. Conclusions

A self-consistent application of the KLN theorem requires a sum over all degenerate initial and final states to arrive at an IR safe cross section. We checked the IR cancellation to the Rutherford scattering cross section at NLO correction by including all possible diagrams with an arbitrary number of disconnected soft photons which forms a divergent infinite series. Introducing a convergent factor helped us to manipulate the formally divergent series under control, which helps us to apply the Monotone Convergence Theorem to prove that the given rearrangement is unique through interchanging the limits in the full sum. Our rearrangement shows that we only need to sum up to a finite number of diagrams where the disconnected part remains factorized, and more important that the sum of these diagrams is IR safe and leads to the tree level contribution when we take it back to the LO. We, therefore, arrived at the extremely nontrivial result that for NLO Rutherford scattering the summation over *all* indistinguishable initial and final states is equivalent to the summation over *only* the initial hard collinear and final soft, hard collinear, and soft and collinear degenerate states.

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