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Infinitesimal generators and quantum operators in de Sitter space

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Abstract The 1 + 3-de Sitter space, as a solution of Einstein's equations, provides a framework for investigating various physical concepts, including quantum mechanics. In this context, the group Sp(2,2) plays a crucial role as the universal covering of the de Sitter space, contributing significantly to the understanding of associated quantum mechanics. In this study, the presentation of infinitesimal generators of the group Sp(2,2) and quantum operators on the de Sitter space is conducted using the coherent states quantization method, also known as Berezin's quantization method. This approach allows for a systematic examination of the generators and operators within the quantum framework. Furthermore, the exploration of these generators and operators is facilitated by the unitary representation of the principal series of the group Sp(2,2), offering insights into the underlying quantum mechanics principles governing the de Sitter space.

1 Introduction

The general theory of the relativity, proposed by Einstein, is a great advance in the understanding the Nature, dealing with the structure of the space-time. The de Sitter metric $g_{\mu\nu}$ is one of the solutions of Einstein's equation [1,2]:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -T_{\mu\nu}, \tag{1}$$

with a positive cosmological constant Λ and null energy-impulsion tensor, $T_{\mu\nu} = 0$. This space is a concept in theoretical physics that visualizes a particular metric as a submanifold of a higher-dimensional generalized Minkowski space, specifically a 1 + 3-de Sitter space(-time). In this model, the universe is represented as spatially flat and does not consider the arrangement of matter, focusing instead on the effects of the cosmological constant or dark energy. This concept is commonly used in inflationary cosmological theories and can also be applied to understand the current structure of our universe, where dark energy makes up around 69% of the universe's energy [3,4]. This makes de Sitter space a useful approximation for studying both early and late stages of our universe's evolution.

Quantum field theory in curved space, especially on de Sitter space, opens up a whole new realm of exploration for physicists. The connection between fundamental particles and the geometry of space-time is truly intriguing. By introducing infinitesimal generators and quantum operators on de Sitter space, researchers can delve into the mathematical structures that underlie quantum mechanics in this unique setting. The choice of using the symmetry group $SO_0(1,4)$ or its universal covering group $Sp(2,2)$ is crucial for understanding the quantum behavior on de Sitter space. By decomposing these groups and examining their Lie algebra, physicists can uncover the fundamental building blocks that govern the dynamics of particles in this curved space.

The de Sitter group has 10 parameters, which correspond to the degrees of freedom associated with space-time symmetries. In the associated Hilbert space, one can consider 10 infinitesimal generators, which are crucial for defining the symmetries and transformations of the de Sitter group. These generators represent the transformations that preserve the structure of the group and play a key role in understanding the symmetries of space-time within the framework of the de Sitter group. In this work, we introduce these infinitesimal generators.

The quantum operators for a free massive particle in de Sitter space are often defined in terms of representation theory and symplectic geometry. The phase space associated with this system can be constructed using the co-adjoint orbit of the group Sp(2, 2). Coherent states, which are vectors in the Hilbert space, play an important role in quantum mechanics and quantum field theory. These states are usually defined as eigenstates of the annihilation operator and represent the

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most classical-like quantum states. The quantum operator, in this context, can be expressed as an integral mapping from the phase space of the free massive particle to the specific Hilbert space associated with coherent states. This operator acts on the quantum states and provides information about the observables of the system.

We organize our paper as follows: in Sect. 2 we introduce the 1 + 3-de Sitter space and its associated symmetry group and Lie algebra. The infinitesimal generators on the de Sitter hyperboloid are presented in Sect. 3. Section 4 is devoted to constructing the Hilbert subspace (and coherent states) of a free massive particles on this hyperboloid. In Sect. 5 we introduce the Berezin integral mapping to construct the quantum operators. Finally, in Sect. 6 we present the conclusions of this work.

2 The de Sitter group and its Lie algebra

The de Sitterian metric $g_{\mu\nu}$ is a solution of Einstein's equations with a positive cosmological constant Λ and a null energy-momentum tensor $T_{\mu\nu} = 0$, which means

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2}$$

The de Sitter space can be visualized as a one-sheeted hyperboloid embedded in five-dimensional Minkowski space (ambient space) as follows:

$$\mathcal{X} \equiv \left\{ x \in \mathcal{R}^5 | x^2 = \breve{\eta}_{\gamma\beta} x^\gamma x^\beta = -H_0^{-2} \right\}, \tag{3}$$

where $\gamma, \beta = 0, 1, 2, 3, 4,\ \breve{\eta}_{\gamma\beta} = \mathrm{diag}(1, -1, -1, -1, -1)$. The symmetry group associated to this space is the group $SO_0(1, 4)$ or its universal covering i.e. the group $Sp(2, 2)$. The group $SO_0(1, 4)$ is defined as:

$$SO_0(1,4) = \left\{ \Lambda \in \mathbf{M}_5(\mathbf{R}) :\ \Lambda^{00} \geq 1,\ \Lambda^t\, \breve{\eta}\, \Lambda = \breve{\eta} \right\}, \tag{4}$$

where this applies over vector x in ambient space as:

$$x' = \Lambda x. \tag{5}$$

However, the symplectic group $Sp(2, 2)$ is described by 2×2 matrices with quaternionic coefficients as follows [5,6]:

$$Sp(2,2) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | detg = 1,\ g^\dagger \gamma^0 g = \gamma^0 \right\}, \tag{6}$$

where a, b, c, d belong to the quaternion field $\mathbf{Q}$ ($\simeq R_+ \times SU(2)$), $\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$ that $\mathbf{1}$ and $\mathbf{0}$ are respectively 2×2 unit and zero matrices. This group acts on the matrix $\mathbf{X}$ as follows:

$$\mathbf{X}' = g\mathbf{X}g^{-1}, \tag{7}$$

where

$$g^{-1} = \gamma^0 g^\dagger \gamma^0 = \begin{pmatrix} \overline{a} & -\overline{c} \\ -\overline{b} & \overline{d} \end{pmatrix}, \tag{8}$$

$$\mathbf{X} = \begin{pmatrix} x^0\mathbf{1} & -P \\ \overline{P} & -x^0\mathbf{1} \end{pmatrix}, \tag{9}$$

$$P = \begin{pmatrix} x^4 + ix^3 & ix^1 - x^2 \\ ix^1 + x^2 & x^4 - ix^3 \end{pmatrix} \tag{10}$$

and $\overline{P}$ is conjugate of P. There is a homomorphism between these two groups as [5]:

$$\Lambda^\alpha{}_\beta = \frac{1}{4}\mathrm{tr}(\gamma^\alpha\ g\ \gamma_\beta\ g^{-1}). \tag{11}$$

In the following we use the group $Sp(2, 2)$ for our calculations.

The *sp(2,2)* Lie algebra

The study of Lie algebra, and consequently the infinitesimal generators of a group is accomplished by decomposing every element of that group. There are three decompositions for the presentation of the group $Sp(2, 2)$: the Lorentz space-time, Kartan, and Ivasava decompositions [7–9]. In this case we will be focusing on the first decomposition. According to the Lorentz space-time decomposition, each element of the group is represented as [6]:

$$Sp(2,2) \ni g = jl, \tag{12}$$

$$j = j_1 j_2 = \begin{pmatrix} \eta & 0 \\ 0 & \overline{\eta} \end{pmatrix} \begin{pmatrix} \cosh(\frac{\psi}{2}) & \sinh(\frac{\psi}{2}) \\ \sinh(\frac{\psi}{2}) & \cosh(\frac{\psi}{2}) \end{pmatrix}, \tag{13}$$

$$l = l_1 l_2 = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \begin{pmatrix} \cosh(\frac{\varphi}{2}) & \widehat{u}\sinh(\frac{\varphi}{2}) \\ -\widehat{u}\sinh(\frac{\varphi}{2}) & \cosh(\frac{\varphi}{2}) \end{pmatrix}, \tag{14}$$

where $\psi, \varphi \in R,\ \eta, \xi \in \mathrm{SU}(2)$ and $\widehat{u}$ is a pure quaternion i.e. $\overline{\widehat{u}} = -\widehat{u} \in SU(2)$. The factors l and j correspond to the Lorentz subgroup (boosts and rotations of space) and the set of time and space translations, respectively. This shows that the group has ten parameters (three for space translation, one for time translation, three for spatial rotation and three for Lorentz boost). Therefore we have ten one-parameter subgroups as follows:

$$S_i = \begin{pmatrix} e^{e_i t} & \mathbf{0} \\ \mathbf{0} & e^{-e_i t} \end{pmatrix}, \tag{15}$$

$$\Theta = \begin{pmatrix} \cosh(\frac{t}{2}) & \sinh(\frac{t}{2}) \\ \sinh(\frac{t}{2}) & \cosh(\frac{t}{2}) \end{pmatrix}, \tag{16}$$

$$R_i = \begin{pmatrix} e^{e_i t} & \mathbf{0} \\ \mathbf{0} & e^{e_i t} \end{pmatrix}, \tag{17}$$

$$B_i = \begin{pmatrix} \cosh(\frac{t}{2}) & e_i\sinh(\frac{t}{2}) \\ -e_i\sinh(\frac{t}{2}) & \cosh(\frac{t}{2}) \end{pmatrix}, \tag{18}$$

and ten corresponding algebraic bases:

$$X_{i4} = \frac{1}{2}\begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix}, \tag{19}$$

$$X_{04} = \frac{1}{2}\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{20}$$

$$X_{ij} = \frac{1}{2}\varepsilon_{ijk}\begin{pmatrix} e_k & \mathbf{0} \\ \mathbf{0} & e_k \end{pmatrix}, \tag{21}$$

$$X_{0i} = \frac{1}{2}\begin{pmatrix} \mathbf{0} & e_i \\ -e_i & \mathbf{0} \end{pmatrix}, \tag{22}$$

where i, j, k=1, 2, 3 and $e_k = (-1)^{k+1} i\sigma_k$ that σ_k is Pauli matrix ("t" is a parameter). These bases satisfy the following commutation relations:

$$[X_{\alpha\beta}, X_{\gamma\delta}] = \eta_{\alpha\delta}X_{\beta\gamma} + \eta_{\beta\gamma}X_{\alpha\delta} - \eta_{\alpha\gamma}X_{\beta\delta} - \eta_{\beta\delta}X_{\alpha\gamma}. \tag{23}$$

Equations (19)–(22) show that an element of this algebra is given by:

$$\textbf{\textit{sp(2,2)}} \ni X = \begin{pmatrix} \overrightarrow{y_1} & q \\ \bar{q} & \overrightarrow{y_2} \end{pmatrix}, \tag{24}$$

where $q = (q^0, \overrightarrow{q}) = q^0\mathbf{1} + q^k e_k$ is quaternion and $\overrightarrow{y_1}, \overrightarrow{y_2}$ are pure quaternions.

3 Infinitesimal generators of group $Sp(2, 2)$

The representation $\mathcal{L}^\tau$ of $Sp(2, 2)$ in Hilbert space $\mathcal{H}$ $=L^2_{\mathcal{C}}(SU(2))$ is of the form [10]:

$$\mathcal{L}^\tau(g)\mathcal{F}(\zeta) = \big(\chi(\zeta, g)\big)^{-2\tau}\mathcal{F}(g^{-1}\cdot\zeta) = \big(\chi(\zeta, g)\big)^{-2\tau}\mathcal{F}\big((a'\zeta + b')(c'\zeta + d')^{-1}\big) \tag{25}$$

where

$\chi(\zeta, g) = det(c'\zeta + d')$ if $g \in Sp(2, 2)$, $\zeta \in SU(2)$, $g^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ and $\mathcal{F} \in \mathcal{H}$. Indeed the action $\zeta \mapsto g^{-1}\cdot\zeta = (a'\zeta + b')(c'\zeta + d')^{-1}$ is a homomorphism of $SU(2)$. This representation is linear. It is equivalent to a unitary representation of the principal series if and only if (see Appendix A.1):

$$\tau = \frac{3}{2} + i\nu, \quad \nu \in \mathcal{R}. \tag{26}$$

This representation in one dimension specified by a parameter is called the "generator" of the group action. The infinitesimal generator is obtained by taking the derivative of this generator with respect to that parameter.

Now we will construct the representation of each one-parameter subgroup of the group Sp(2,2). To do this, we choose $\zeta \in$ SU(2)$\sim$ S^3 where

$$\begin{cases} \zeta_1 = \sin\alpha\sin\theta\cos\phi \\ \zeta_2 = \sin\alpha\sin\theta\sin\phi \\ \zeta_3 = \sin\alpha\cos\theta \\ \zeta_4 = \cos\alpha \end{cases}$$

with $0 \le \alpha, \theta \le \pi$ et $0 \le \phi < 2\pi$.

For each one-parameter subgroup "g(t)" of the group $Sp(2, 2)$ and $(\alpha', \theta', \phi') = g(t)\cdot(\alpha, \theta, \phi)$, the infinitesimal generator (Υ) is denoted by [5]:

$$\begin{aligned}
\frac{\partial[\mathcal{L}^\tau(g)\mathcal{F}(\alpha,\theta,\phi)]}{\partial t}\Big|_{t=0} &= \frac{\partial[(\chi(\zeta,g))^{-2\tau}]}{\partial t}\Big|_{t=0}\mathcal{F}(\alpha,\theta,\phi) \\
&\quad + (\chi(\zeta,g))^{-2\tau}\Big|_{t=0}\frac{\partial\mathcal{F}(\alpha',\theta',\phi')}{\partial t}\Big|_{t=0} \\
&= \Big[\frac{\partial[(\chi(\zeta,g))^{-2\tau}]}{\partial t}\Big|_{t=0} + \frac{\partial\alpha'}{\partial t}\Big|_{t=0}\frac{\partial}{\partial\alpha} + \frac{\partial\theta'}{\partial t}\Big|_{t=0}\frac{\partial}{\partial\theta} \\
&\quad + \frac{\partial\phi'}{\partial t}\Big|_{t=0}\frac{\partial}{\partial\phi}\Big]\mathcal{F}(\alpha,\theta,\phi) = -i\ \Upsilon\mathcal{F}(\alpha,\theta,\phi) \\
\Rightarrow \Upsilon &= i\Big(\frac{\partial[(\chi(\zeta,g))^{-2\tau}]}{\partial t}\Big|_{t=0} + \frac{\partial\alpha'}{\partial t}\Big|_{t=0}\frac{\partial}{\partial\alpha} \\
&\quad + \frac{\partial\theta'}{\partial t}\Big|_{t=0}\frac{\partial}{\partial\theta} + \frac{\partial\phi'}{\partial t}\Big|_{t=0}\frac{\partial}{\partial\phi}\Big).
\end{aligned} \tag{27}$$

From the latter equation, along with Eqs. (13) and (14), we can construct ten associated infinitesimal generators of the group $Sp(2, 2)$ as follows:

The calculation for one of the generators (with calculations for the other generators being similar) is provided in Appendix A.2.

1. **The infinitesimal generators of "space translations":** From Eqs. (15) and (27), we can derive the infinitesimal generators of space translation. These generators which are the components of the momentum generator, are obtained as:

$$\mathbf{P}_1 = -i\Big(\sin\theta\cos\phi\frac{\partial}{\partial\alpha} + \cot\alpha\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\cot\alpha\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\Big), \tag{28}$$

$$\mathbf{P}_2 = -i\Big(\sin\theta\sin\phi\frac{\partial}{\partial\alpha} + \cot\alpha\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cot\alpha\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\Big), \tag{29}$$

$$\mathbf{P}_3 = -i\Big(\cos\theta\frac{\partial}{\partial\alpha} - \cot\alpha\sin\theta\frac{\partial}{\partial\theta}\Big). \tag{30}$$

2. **The infinitesimal generator of "time translation":** This generator is derived as follows (see Appendix A.2:

$$\mathbf{H}_0 = i\Big(\tau\cos\alpha + \sin\alpha\frac{\partial}{\partial\alpha}\Big). \tag{31}$$

3. **The infinitesimal generators of space rotations:** From Eqs. (17) and (27), we can derive the infinitesimal generators of space rotations. These generators which are

the components of the angular momentum generator, are obtained as:

$$\mathbf{J}_1 = i\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right), \tag{32}$$

$$\mathbf{J}_2 = i\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right), \tag{33}$$

$$\mathbf{J}_3 = i\left(-\frac{\partial}{\partial\phi}\right). \tag{34}$$

4. **The infinitesimal generators of the lorentz boosts:** From Eqs. (18) and (27), we can derive the components of the infinitesimal generator of the Lorentz boosts as follows:

$$\mathbf{K}_1 = i\Big(\tau\sin\alpha\sin\theta\cos\phi - \cos\alpha\sin\theta\cos\phi\frac{\partial}{\partial\alpha} - \frac{\cos\theta\cos\phi}{\sin\alpha}\frac{\partial}{\partial\theta} + \frac{\sin\phi}{\sin\alpha\sin\theta}\frac{\partial}{\partial\phi}\Big),$$

$$\mathbf{K}_2 = i\Big(\tau\sin\alpha\sin\theta\sin\phi - \cos\alpha\sin\theta\sin\phi\frac{\partial}{\partial\alpha} - \frac{\cos\theta\sin\phi}{\sin\alpha}\frac{\partial}{\partial\theta} - \frac{\cos\phi}{\sin\alpha\sin\theta}\frac{\partial}{\partial\phi}\Big), \tag{35}$$

$$\mathbf{K}_3 = i\Big(\tau\sin\alpha\cos\theta - \cos\alpha\cos\theta\frac{\partial}{\partial\alpha} \tag{36}$$

$$+\frac{\sin\theta}{\sin\alpha}\frac{\partial}{\partial\theta}\Big). \tag{37}$$

These ten generators satisfy the following commutation relations:

$$[\mathbf{J}_i, \mathbf{H}_0] = 0, \tag{38}$$
$$[\mathbf{J}_i, \mathbf{J}_j] = i\ \varepsilon_{ijk}\mathbf{J}_k, \tag{39}$$
$$[\mathbf{J}_i, \mathbf{P}_j] = i\ \varepsilon_{ijk}\mathbf{P}_k, \tag{40}$$
$$[\mathbf{J}_i, \mathbf{K}_j] = i\ \varepsilon_{ijk}\mathbf{K}_k, \tag{41}$$
$$[\mathbf{P}_i, \mathbf{P}_j] = i\ \varepsilon_{ijk}\mathbf{J}_k, \tag{42}$$
$$[\mathbf{P}_i, \mathbf{K}_j] = -i\ \delta_{ij}\mathbf{H}_0, \tag{43}$$
$$[\mathbf{K}_i, \mathbf{K}_j] = -i\ \varepsilon_{ijk}\mathbf{J}_k, \tag{44}$$
$$[\mathbf{H}_0, \mathbf{P}_i] = -i\ \mathbf{K}_i, \tag{45}$$
$$[\mathbf{H}_0, \mathbf{K}_i] = -i\ \mathbf{P}_i. \tag{46}$$

Effect of infinitesimal generators on normalized hyperspherical harmonics

The normalized hyperspherical harmonics on S^3 are given by [5]

$$Y_{n,l,m}(\alpha,\theta,\phi) = M_{nl}C^{l+1}_{n-l-1}\sin^l\alpha\, Y_{lm}(\theta,\phi), \tag{47}$$

where

$$M_{nl} = l!2^{l+1}\left(\frac{(n-l-1)!n}{2\pi(n+l)!}\right)^{\frac{1}{2}},$$

$$C^{l+1}_{n-l-1} = \sum_{p=0}^{[\frac{n-l-1}{2}]}\frac{(-1)^p(n-p-1)!(2\cos\alpha)^{n-l-1-2p}}{l!p!(n-l-1-2p)!},$$

$$Y_{lm}(\theta,\phi) = \left(\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right)^{\frac{1}{2}} P_l^m(\cos\theta)e^{im\phi},$$

$$P_l^m(\cos\theta) = \frac{(-1)^{l+m}}{2^l l!}\left(\sin\theta\right)^m\left(\frac{\partial}{\partial\cos\theta}\right)^{l+m}\left(\sin\theta\right)^{2l}.$$

Here we apply ten infinitesimal generators to $Y_{n,l,m} \equiv Y_{n,l,m}(\alpha,\theta,\phi)$.

(I) *Effect of the infinitesimal generators of space translations on $Y_{n,l,m}$:*

In the extensive calculation, we obtain:

$$\mathbf{P}_1Y_{n,l,m} = \frac{-i}{2}\Big(p_{11}Y_{n,l+1,m-1} + p_{12}Y_{n,l-1,m-1} - p_{13}Y_{n,l+1,m+1} - p_{14}Y_{n,l-1,m+1}\Big), \tag{48}$$

$$\mathbf{P}_2Y_{n,l,m} = \frac{1}{2}\Big(p_{11}Y_{n,l+1,m-1} + p_{12}Y_{n,l-1,m-1} + p_{13}Y_{n,l+1,m+1} + p_{14}Y_{n,l-1,m+1}\Big), \tag{49}$$

$$\mathbf{P}_3Y_{n,l,m} = ip_{31}Y_{n,l+1,m} - ip_{32}Y_{n,l-1,m}, \tag{50}$$

where

$$p_{11} = \sqrt{\frac{(l-m+1)(l-m+2)(n+l+1)(n-l-1)}{(2l+1)(2l+3)}},$$

$$p_{12} = \sqrt{\frac{(l+m-1)(l+m)(n+l)(n-l)}{(2l+1)(2l-1)}},$$

$$p_{13} = \sqrt{\frac{(l+m+1)(l+m+2)(n+l+1)(n-l-1)}{(2l+1)(2l+3)}},$$

$$p_{14} = \sqrt{\frac{(l-m)(l-m-1)(n+l)(n-l)}{(2l+1)(2l-1)}},$$

$$p_{31} = \sqrt{\frac{(l-m+1)(n+l+1)(n-l-1)(l+m+1)}{(2l+1)(2l+3)}},$$

$$p_{32} = \sqrt{\frac{(l+m)(n-l)(n+l)(l-m)}{(2l+1)(2l-1)}}.$$

textbi(II) *Effect of the infinitesimal generator of time translation on $Y_{n,l,m}$:*

$$\mathbf{H}_0Y_{n,l,m} = \frac{i}{2}\Big((\tau-n-1)d_{11}Y_{n-1,l,m} + (\tau+n-1)d_{22}Y_{n+1,l,m}\Big), \tag{51}$$

where

$$d_{11}=\sqrt{\frac{(n+l)(n-l-1)}{n(n-1)}},\quad d_{22}=\sqrt{\frac{(n-l)(n+l+1)}{n(n+1)}}.$$

The effect of this infinitesimal generator on $Y_{n,l,m}$ is to induce a shift in the time variable of the function $Y_{n,l,m}$ by an infinitesimal amount. This transformation allows us to study how the function $Y_{n,l,m}$ changes under small variation in time, providing insight into the dynamics of the system being described.(We can explain the other nine generators in a similar manner.)

(III) *Effect of the infinitesimal generators of space rotations on $Y_{n,l,m}$*:

$$\mathbf{J}_1 Y_{n,l,m}=-\frac{1}{2}\Big(\sqrt{(l+m)(l-m+1)}Y_{n,l,m-1}+\sqrt{(l-m)(l+m+1)}Y_{n,l,m+1}\Big), \quad (52)$$

$$\mathbf{J}_2 Y_{n,l,m}=-\frac{i}{2}\Big(\sqrt{(l+m)(l-m+1)}\;Y_{n,l,m-1}-\sqrt{(l-m)(l+m+1)}Y_{n,l,m+1}\Big), \quad (53)$$

$$\mathbf{J}_3 Y_{n,l,m}=m\;Y_{n,l,m}. \quad (54)$$

(IV) *Effect of the infinitesimal generators of the lorentz boosts on $Y_{n,l,m}$*:

$$\mathbf{K}_1 Y_{n,l,m}=-\frac{i(\tau-n-1)}{4}\Big(b_{11}Y_{n-1,l-1,m+1}+b_{12}Y_{n-1,l+1,m+1}+b_{13}Y_{n-1,l+1,m-1}+b_{14}Y_{n-1,l-1,m-1}\Big)-\frac{i(\tau+n-1)}{4}\Big(b_{21}Y_{n+1,l+1,m+1}+b_{22}Y_{n+1,l-1,m+1}+b_{23}Y_{n+1,l+1,m-1}+b_{24}Y_{n+1,l-1,m-1}\Big), \quad (55)$$

$$\mathbf{K}_2 Y_{n,l,m}=-\frac{(\tau-n-1)}{4}\Big(b_{11}Y_{n-1,l-1,m+1}+b_{12}Y_{n-1,l+1,m+1}-b_{13}Y_{n-1,l+1,m-1}-b_{14}Y_{n-1,l-1,m-1}\Big)-\frac{(\tau+n-1)}{4}\Big(b_{21}Y_{n+1,l+1,m+1}+b_{22}Y_{n+1,l-1,m+1}-b_{23}Y_{n+1,l+1,m-1}-b_{24}Y_{n+1,l-1,m-1}\Big), \quad (56)$$

$$\mathbf{K}_3 Y_{n,l,m}=\frac{i(\tau-n-1)}{2}\Big(k_{11}Y_{n-1,l-1,m}+k_{12}Y_{n-1,l+1,m}\Big)+\frac{i(\tau+n-1)}{2}\Big(k_{21}Y_{n+1,l-1,m}+k_{22}Y_{n+1,l+1,m}\Big), \quad (57)$$

where

$$b_{11}=\sqrt{\frac{(n+l)(n+l-1)(l-m)(l-m-1)}{n(n-1)(2l+1)(2l-1)}},$$

$$b_{12}=\sqrt{\frac{(n-l-1)(n-l-2)(l+m+1)(l+m+2)}{n(n-1)(2l+1)(2l+3)}},$$

$$b_{13}=-\sqrt{\frac{(n-l-1)(n-l-2)(l-m+1)(l-m+2)}{n(n-1)(2l+1)(2l+3)}},$$

$$b_{14}=-\sqrt{\frac{(n+l)(n+l-1)(l+m)(l+m-1)}{n(n-1)(2l+1)(2l-1)}},$$

$$b_{21}=-\sqrt{\frac{(n+l+1)(n+l+2)(l+m+1)(l+m+2)}{n(n+1)(2l+1)(2l+3)}},$$

$$b_{22}=-\sqrt{\frac{(n-l)(n-l+1)(l-m)(l-m-1)}{n(n+1)(2l+1)(2l-1)}},$$

$$b_{23}=\sqrt{\frac{(n+l+1)(n+l+2)(l-m+1)(l-m+2)}{n(n+1)(2l+1)(2l+3)}},$$

$$b_{24}=\sqrt{\frac{(n-l)(n-l+1)(l+m)(l+m-1)}{n(n+1)(2l+1)(2l-1)}},$$

$$k_{11}=\sqrt{\frac{(n+l)(n+l-1)(l-m)(l+m)}{n(n-1)(2l+1)(2l-1)}},$$

$$k_{12}=-\sqrt{\frac{(n-l-1)(n-l-2)(l-m+1)(l+m+1)}{n(n-1)(2l+1)(2l+3)}},$$

$$k_{21}=-\sqrt{\frac{(n-l)(n-l+1)(l-m)(l+m)}{n(n+1)(2l+1)(2l-1)}},$$

$$k_{22}=\sqrt{\frac{(n+l+1)(n+l+2)(l-m+1)(l+m+1)}{n(n+1)(2l+1)(2l+3)}}.$$

As an interesting result, we obtained the effect of generators " $\mathbf{P}^2=\mathbf{P}_1^2+\mathbf{P}_2^2+\mathbf{P}_3^2$ " and " $\mathbf{J}^2=\mathbf{J}_1^2+\mathbf{J}_2^2+\mathbf{J}_3^2$ " on $Y_{n,l,m}$ as follows:

$$\mathbf{P}^2 Y_{n,l,m}=\mathbf{P}_1^2 Y_{n,l,m}+\mathbf{P}_2^2 Y_{n,l,m}+\mathbf{P}_3^2 Y_{n,l,m}=\Big((n^2-1)-l(l+1)\Big)Y_{n,l,m}, \quad (58)$$

$$\mathbf{J}^2 Y_{n,l,m}=\mathbf{J}_1^2 Y_{n,l,m}+\mathbf{J}_2^2 Y_{n,l,m}+\mathbf{J}_3^2 Y_{n,l,m}=l(l+1)Y_{n,l,m}. \quad (59)$$

These equations and Eq. (54) show that "$Y_{n,l,m}(\alpha,\theta,\phi)$" is an eigenfunction of the generators $\mathbf{P}^2$, $\mathbf{J}^2$ and $\mathbf{J}_3$ with eigenvalues "$(n^2-1)-l(l+1)$", "$l(l+1)$" and "m" respectively. It is interesting that the effect of $\mathbf{J}^2$ on "$Y_{n,l,m}(\alpha,\theta,\phi)$" is similar to the effect of the square of the angular momentum generator on normalized spherical harmonics "$Y_{l,m}(\theta,\phi)$" in three-dimensional space.

4 A Hilbert subspace characterized by coherent states

In order to study quantum mechanics on 1 + 3-de Sitter space, we must introduce the associated Hilbert subspace. This space is represented by vectors known as coherent states. To do this, we begin by constructing the phase space associated with a movement massive particle (on the basis of irreducible representation of de Sitter group) on 1 + 3-de Sitter space. Subsequently, we introduce the coherent states corresponding to each point in this phase space.

4.1 The complex sphere "$S^3_{\mathcal{C}}$" as the phase space

To construct this phase space, we use the orbit method [11–13]. This method expresses that for a Hamiltonian system with a symmetric group, the phase space can be constructed by a co-adjoint orbit. On the other hand, for a simple group such as the de Sitter group, the adjoint orbit is equivalent to the co-adjoint orbit. In other words, we can define our phase space by constructing the adjoint orbit of a movement massive particle on de Sitter space.

To do this, we choose the point $X_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in$ ***sp(2,2)*** (the case $\overrightarrow{y_1} = \overrightarrow{y_2} = \overrightarrow{0}$ and q=(1,0) of Eq. (24)). This point is invariant under adjoint action of "space" rotation subgroup and "time" translation matrix i.e.

$$l_1.X_0 = l_1 X_0 l_1^{-1} = X_0, \tag{60}$$

$$j_2.X_0 = j_2 X_0 j_2^{-1} = X_0. \tag{61}$$

Therefore, a point of our adjoint orbit is obtained by :

$$g \cdot X_0 = l_2 j_1 \, X_0 j_1^{-1} l_2^{-1} = \begin{pmatrix} \sinh(\varphi) \; \widehat{\upsilon} & \cosh(\varphi) \; \zeta \\ \cosh(\varphi) \; \overline{\zeta} & -\sinh(\varphi) \; (\overline{\zeta} \widehat{\upsilon} \zeta) \end{pmatrix}, \tag{62}$$

where $\zeta = \eta^2 \in SU(2) \simeq S^3$ and $\widehat{\upsilon} = \eta \widehat{u} \overline{\eta}$. The equations (60)–(62) show that the X_0 is invariant under adjoint action of $SO(1,1) \times SU(2)$ and the adjoint orbit is identified by:

$$\mathcal{M}_H = Sp(2,2) \Big/ SO(1,1) \times SU(2). \tag{63}$$

Also, we know that the homogeneous space for a movement massive particle on de Sitter space (on the basis of irreducible representation of de Sitter group) is given by Eq. (63) (see [5]). Therefore, Eq. (62) expresses a point of adjoint orbit for a movement massive particle on de Sitter space that corresponds to a point of phase space. By choosing "$\cosh(\varphi) = \frac{p_0}{m}$" and "$\widehat{\upsilon} \sinh(\varphi) = \frac{\overrightarrow{p}}{m}$" we parameterize the Eq. (62):

$$g \cdot X_0 = \begin{pmatrix} \dfrac{\overrightarrow{p}}{m} & \dfrac{p_0}{m}\zeta \\ \dfrac{p_0}{m}\overline{\zeta} & -\overline{\zeta}\dfrac{\overrightarrow{p}}{m}\zeta \end{pmatrix} \equiv X(\overrightarrow{p}, \zeta), \tag{64}$$

where m is a real parameter (like the mass), $\overrightarrow{p} \in \mathcal{R}^4$ and $p_0 = \pm\sqrt{m^2 + p^2}$. The matrix (64) expresses that our adjoint orbit (or phase space) is identified with cotangent space $T^*(S^3)$ that the $(\zeta, \overrightarrow{p})$ play the role of pair varieties of $T^*(S^3)$.

On the other hand , the cotangente space $T^*(S^d)$ is isomorphic with the complex sphere "$S^d_{\mathcal{C}}$" [14,15] where

$$\mathrm{T^*(S^d)} = \left\{(\overrightarrow{x}, \overrightarrow{p}) \in \mathcal{R}^{d+1} \times \mathcal{R}^{d+1} | x^2 = r^2, \overrightarrow{x}.\overrightarrow{p} = 0\right\},$$

$$S^d_{\mathcal{C}} = \left\{\overrightarrow{a} \in \mathcal{C}^{d+1} \mid a_1^2 + a_2^2 + \cdots + a_{d+1}^2 = r^2 = 1\right\}.$$

Note that "r" is radius of d-dimensional sphere and complex vector $\overrightarrow{a}$ is given by:

$$\overrightarrow{a}(\overrightarrow{x}, \overrightarrow{p}) = \cosh(p) \, \overrightarrow{x} + i \frac{\sinh(p)}{p} \, \overrightarrow{p}, \tag{65}$$

where $p = (\overrightarrow{p}.\overrightarrow{p})^{\frac{1}{2}}$. Therefore by choosing $d = 3$, we can say that our necessary phase space is equivalent to the complex sphere "$S^3_{\mathcal{C}}$" and we can obtain the coherent states on this complex sphere.

4.2 Our coherent states

As mentioned in the previous subsection, the phase space of a movement massive particle on 1 + 3-de Sitter space is isomorphic to "$S^3_{\mathcal{C}}$". This complex 3-sphere and its configuration space "S^3" were introduced by [6]:

$$\begin{aligned} S^3_{\mathcal{C}} &= \{\overrightarrow{a} \in \mathcal{C}^4 | \overrightarrow{a} = \cosh(p) \, \overrightarrow{x} + i \frac{\sinh(p)}{p} \, \overrightarrow{p}\} \\ &= \left\{(a_1, a_2, a_3, a_4)\right\} \\ &= \{(\sin\alpha_a \sin\theta_a \cos\phi_a, \sin\alpha_a \sin\theta_a \sin\phi_a, \\ &\quad \sin\alpha_a \cos\theta_a, \cos\alpha_a)\}, \end{aligned} \tag{66}$$

$$\begin{aligned} S^3 &= \left\{\mathcal{R}^4 \ni \overrightarrow{x} = (x_1, x_2, x_3, x_4)\right\} \\ &= \{(\sin\alpha \sin\theta \cos\phi, \sin\alpha \sin\theta \sin\phi, \\ &\quad \sin\alpha \cos\theta, \cos\alpha)\}, \end{aligned} \tag{67}$$

where $0 \leq \alpha, \theta, Re(\alpha_a), Re(\theta_a) \leq \pi, \quad 0 \leq \phi, Re(\phi_a) \leq 2\pi$. According to $\overrightarrow{x}.\overrightarrow{p} = 0$, $\overrightarrow{p} \in \mathcal{R}^4$ and Eq. (67) we can write:

$$\overrightarrow{p} \equiv \begin{cases} p_1 = \cos\alpha \sin\theta \cos\phi p_\alpha + \cos\theta \cos\phi p_\theta - \sin\phi p_\phi \\ p_2 = \cos\alpha \sin\theta \sin\phi p_\alpha + \cos\theta \sin\phi p_\theta + \cos\phi p_\phi \\ p_3 = \cos\alpha \cos\theta p_\alpha - \sin\theta p_\theta \\ p_2 = -\sin\alpha p_\alpha, \end{cases}$$

where $p_\alpha, p_\theta, p_\phi$ are three variables on tangent space of S^3 that satisfy the following relation:

$$p^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2 = p_\alpha^2 + p_\theta^2 + p_\phi^2. \tag{68}$$

For the presentation of coherent states, we use the heat kernel on the complex 3-sphere ($S^3_{\mathcal{C}}$) as introduced by Hall–Mitchell

[15]. This kernel is given by:

$$\rho^3_\epsilon(\overrightarrow{a}, \overrightarrow{x}) = \frac{(2\pi\epsilon)^{-\frac{3}{2}} e^{\frac{\epsilon}{2}}}{\sin\tilde{\theta}} \sum_{n=-\infty}^{\infty} (\tilde{\theta} - 2\pi n) e^{\frac{-(\tilde{\theta}-2\pi n)^2}{2\epsilon}}, \quad (69)$$

where $\tilde{\theta}$ is a complex angle with $0 \leq Re\,\tilde{\theta} \leq \pi$. This kernel is related to coherent states "$|\Psi^\epsilon_3(\overrightarrow{a}) >$" on $S^3_{\mathcal{C}}$ as follows:

$$\rho^3_\epsilon(\overrightarrow{a}, \overrightarrow{x}) =< x|\Psi^\epsilon_3(\overrightarrow{a}) > . \quad (70)$$

By using the method mentioned in Ref. [6], we can construct our coherent states as:

$$|\Psi^\epsilon_3(\overrightarrow{a}) >= \sum_{n,l,m} \frac{e^{-\frac{\epsilon(n^2-1)}{2}}}{\sqrt{\mathcal{N}_3}} Y_{n,l,m}(\alpha_a, \theta_a, \phi_a)|n, l, m >, \quad (71)$$

where

$$\mathcal{N}_3 = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\sum_{m=-l}^{l} e^{-\epsilon(n^2-1)} \mid Y_{n,l,m} \mid^2 < \infty,$$

$$\mid x > = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\sum_{m=-l}^{l} Y_{n,l,m}(\alpha, \theta, \phi) \mid n, l, m >,$$

and $Y_{n,l,m}(\alpha_a, \theta_a, \phi_a)$ is the normalized hyperspherical harmonics on $S^3_{\mathcal{C}}$ (see Eq. (47) for complex angles). These states satisfy the resolution of the unity as:

$$\mathbf{I}_d = \int_{\overrightarrow{x}\in S^3} \int_{\overrightarrow{x}.\overrightarrow{p}=0} \mid \Psi^\epsilon_3(\overrightarrow{a}) >< \Psi^\epsilon_3(\overrightarrow{a}) \mid \, d\mu_3(\overrightarrow{a})$$
$$= \sum_{n,l,m} \mid n, l, m >< n, l, m \mid, \quad (72)$$

where

$$Q^{n',l',m'}_{n,l,m} = \frac{\pi^2\sqrt{(4\pi\epsilon)^3}}{4} \delta_{n,n'}\delta_{l,l'}\delta_{m,m'}, \quad (73)$$

and the relevant measure on 1 + 3-de Sitter space is given by:

$$d\mu_3(\overrightarrow{a}) = \frac{4\mathcal{N}_3}{\pi^2} \frac{e^{-\epsilon}}{(4\pi\epsilon)^{\frac{3}{2}}} \left(\frac{\sinh(2p)}{2p}\right) e^{-\frac{p^2}{\epsilon}} d\overrightarrow{x}\, d\overrightarrow{p},$$
$$1 = \int_{\overrightarrow{x}\in S^3} \int_{\overrightarrow{x}.\overrightarrow{p}=0} d\mu_3(\overrightarrow{a}).$$

The set of these coherent states with bases $\{| \, n, l, m \, >\}$ construct our Hilbert subspace"$\mathcal{H}(S^3_{\mathcal{C}}, d\mu_3(\overrightarrow{a}))$".

5 Quantum operators on 1 + 3-de Sitter space

As mentioned in Sect. 1, 1 + 3-de Sitter space is a curved space. Studying quantum mechanics in this space cannot be done using the standard quantum method. One method that can solve this problem is the "coherent states quantization" or "Berezin quantization" which we will present in the following.

The resolution of the unity (72) leads us to quantize the classical observables (or functions in phase space) on $S^3_{\mathcal{C}}$ using Berezin's method [16]. In this method, for each classical observable, such as $f(\overrightarrow{x}, \overrightarrow{p}) = f(\overrightarrow{a})$, we associate a quantum observable (operator) as follows [6]:

$$f \longmapsto O_f = \int_{S^3_{\mathcal{C}}} f(\overrightarrow{a}) \mid \Psi^\epsilon_3(\overrightarrow{a})\rangle\langle\Psi^\epsilon_3(\overrightarrow{a}) \mid \, d\mu_3(\overrightarrow{a}). \quad (74)$$

In the following we introduce two important operators on $\mathcal{H}(S^3_{\mathcal{C}}, d\mu_3(\overrightarrow{a}))$.

5.1 Kinetic energy operator

By using the Eq. (74) for $f(\overrightarrow{a}) = \frac{p^2}{2}$ (kinetic energy with unit mass) we obtain the associated operator as:

$$O_{\frac{p^2}{2}} = \frac{1}{2}\sum\left((n\epsilon)^2 + \frac{3}{2}\epsilon\right)|n, l, m >< n, l, m|, \quad (75)$$

or equivalently

$$\widehat{O_{\frac{p^2}{2}}} =< n, l, m|O_{\frac{p^2}{2}}|n, l, m >= \frac{1}{2}\epsilon\left(n^2\epsilon + \frac{3}{2}\right). \quad (76)$$

These equations show that the Kinetic energy operator on 1 + 3-De Sitter space is similar to the Hamiltonian operator of a quantum harmonic oscillator in three dimensions [17].

5.2 Angle operator

One of the interesting aspects of Berezin's quantization method is that any function of the phase space, including the angle, can be quantized. Equation (74) presents the Angle operator as follows:

$$O_\phi = \pi\mathbf{I}_d + i\sum_{m\neq m'} \frac{\text{W}^\epsilon}{m - m'} \mid n, l, m >< n', l', m' \mid, \quad (77)$$

where

$$\text{W}^\epsilon = \frac{e^{-\frac{\epsilon}{2}(n^2+n'^2)}}{\pi(\pi\epsilon)^{\frac{3}{2}}} M_{nlm} M_{n'l'm'}\Bigg[\int_\alpha d\alpha \int_\theta d\theta \sin^2\alpha\sin\theta$$
$$\times \int_{p_\alpha}\int_{p_\theta}\int_{p_\phi} dp_\alpha dp_\theta dp_\phi \sqrt{1+p^2}\, e^{-\frac{p^2}{\epsilon}}\left(\frac{\sinh(2p)}{2p}\right)$$
$$\times \mathcal{A}^*_{n,l,m}(\alpha, \theta, p_\alpha, p_\theta, p_\phi)\mathcal{A}_{n',l',m'}(\alpha, \theta, p_\alpha, p_\theta, p_\phi)\Bigg], \quad (78)$$

$$\mathcal{A}_{n,l,m}(\alpha, \theta, p_\alpha, p_\theta, p_\phi) = \Bigg[1 - \Bigg(\cosh p\cos\alpha$$
$$-i\frac{p_\alpha\sinh p\sin\alpha}{p}\Bigg)^2\Bigg]^{\frac{l-m}{2}}\Bigg[\cosh p\sin\alpha\sin\theta$$
$$+i\frac{\sinh p}{p}\left(p_\alpha\cos\alpha\sin\theta + p_\theta\cos\theta + ip_\phi\right)\Bigg]^m$$
$$\times C^{l+1}_{n-l-1}\left(\cosh p\,\cos\alpha - i\frac{p_\alpha\sinh p\sin\alpha}{p}\right)$$

$$\times C_{l-m}^{m+\frac{1}{2}}\left(\frac{\cosh p\sin\alpha\cos\theta+i\frac{\sinh p}{p}\left(\cos\alpha\cos\theta p_\alpha-\sin\theta p_\theta\right)}{\sqrt{1-\left(\cosh p\cos\alpha-i\frac{\sinh p\ \sin\alpha\ p_\alpha}{p}\right)^2}}\right). \tag{79}$$

There is an interesting connection between our Angle operator (77) and the one presented by Aremua et al. for 1 + 1-de Sitter space [18,19].

6 Conclusion

In this work, we have made significant progress in understanding the quantum operators and infinitesimal generators in this space. The eigenfunctions that we have defined and their relation to the generators $\mathbf{P}^2$, $\mathbf{J}^2$, and $\mathbf{J}_3$ show a deep understanding of the mathematical structures at play.

The construction of the phase space for a movement massive particle on 1 + 3-de Sitter space using the (co)adjoint orbit method is a fascinating approach. It's interesting that each point in the phase space corresponds to a coherent state in the associated Hilbert subspace $\mathcal{H}(S^3_{\mathcal{C}}, d\mu_3(\overrightarrow{a}))$, showcasing the connection between geometry and quantum mechanics.

The derivation of the Kinetic Energy operator and Angle operator using the coherent states quantization method opens up new possibilities for studying the dynamics of particles in this space. The resemblance of the Kinetic Energy operator to the Hamiltonian of a quantum harmonic oscillator and the similarity of the Angle operator in 1 + 3-de Sitter space to the 1 + 1-de Sitter space further highlight the intriguing properties of this space. Exploring the potential to derive other operators, such as annihilation and creation operators, using this method in 1 + 3-de Sitter space indicates a promising avenue for further research and possibly uncovering more hidden symmetries and structures in this space.

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Code Availability Statement This manuscript has no associated code/software. [Author's comment: In this article, I have not used any code/software.]

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Appendix A

Appendix A.1: The unitarity of the representation $\mathcal{L}^\tau(g)$

The $\mathcal{L}^\tau(g)$ representations of $Sp(2,2)$ are unitary, as shown by the following result.

We know that unitarity in Hilbert space ($\mathcal{H}$) means the following relation:

$$\left(\mathcal{F}',\mathcal{L}^\tau(g)\mathcal{F}\right)_{\mathcal{H}}=\left(\left(\mathcal{L}^\tau(g)\right)^\dagger\mathcal{F}',\mathcal{F}\right)_{\mathcal{H}}. \tag{A.1}$$

For our choice we have

$$\left(\ \mathcal{F}'\ ,\ \mathcal{L}^\tau(g)\mathcal{F}\ \right)_{\mathcal{H}} = \int_{SU(2)} d\mu(\zeta)\ \left(\chi(\zeta,g)\right)^{-2\tau}\ \mathcal{F}(g^{-1}.\zeta)\ \mathcal{F}'^*(\zeta). \tag{A.2}$$

On one hand if we use the transformation of the Haar measure on SU(2):

$$d\mu(g^{-1}.\zeta)=\left[\chi(\zeta,g)\right]^{-6}\ d\mu(\zeta) \tag{A.3}$$

and on the other hand the following relation:

$$\chi(\zeta,\ g\ g^{-1})=1=\chi(\zeta,g)\ \chi(g^{-1}\cdot\zeta,\ g^{-1}), \tag{A.4}$$

we deduce that

$$\left(\mathcal{F}'\ \mathcal{L}^\tau(g)\mathcal{F}\right)_{\mathcal{H}}=\int_{SU(2)} d\mu(\zeta)\ \left(\chi(g^{-1}\cdot\zeta,g^{-1})\right)^{2\tau-6} \times\mathcal{F}(g^{-1}.\zeta)\mathcal{F}'^*\left(g.(g^{-1}\cdot\zeta)\right)=\left(\left(\mathcal{L}^{3-\tau}(g^{-1})\right)^*\mathcal{F}'\mathcal{F}\right)_{\mathcal{H}}, \tag{A.5}$$

where

$$\left(\mathcal{L}^\tau(g)\right)^\dagger=\left(\mathcal{L}^{3-\tau}(g^{-1})\right)^*. \tag{A.6}$$

From this last equation we conclude that the representation $\mathcal{L}^\tau(g)$ is unitary if and only if:

$$\tau=\frac{3}{2}+i\nu\quad \nu\in\mathcal{R}\,. \tag{A.7}$$

Appendix A.2: Calculation of infinitesimal generator of time translation

The one-parameter subgroup of time translation is defined by Eq. (16). From this equation and Eq. (8), we can derive its inverse as:

$$g^{-1}=\Theta^{-1}=\begin{pmatrix}\cosh(\frac{t}{2}) & -\sinh(\frac{t}{2})\\ -\sinh(\frac{t}{2}) & \cosh(\frac{t}{2})\end{pmatrix}=\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}. \tag{A.8}$$

By choosing ζ as a quaternion i.e.

$$\zeta = (\zeta_4, \overrightarrow{\zeta}) = \zeta_4 + \zeta_1 e_1 + \zeta_2 e_2 + \zeta_3 e_3, \tag{A.9}$$

with

$$\begin{cases} \zeta_1 = \sin\alpha \sin\theta \cos\phi \\ \zeta_2 = \sin\alpha \sin\theta \sin\phi \\ \zeta_3 = \sin\alpha \cos\theta \\ \zeta_4 = \cos\alpha \end{cases}$$

we can write:

$$\begin{aligned} (a'\zeta + b') &= \cosh(\frac{t}{2})\zeta - \sinh(\frac{t}{2}), \\ (c'\zeta + d') &= -\sinh(\frac{t}{2})\zeta + \cosh(\frac{t}{2}), \\ (det(c'\zeta + d'))^2 &= (\chi(\zeta, \Theta))^2 = \cosh t - \sinh t \cos\alpha, \\ (c'\zeta + d')^{-1} &= \frac{\overline{(c'\zeta + d')}}{(det(c'\zeta + d'))^2} \\ &= \frac{-\sinh(\frac{t}{2})\overline{\zeta} + \cosh(\frac{t}{2})}{\cosh t - \sinh t \cos\alpha}, \end{aligned} \tag{A.10}$$

$$\begin{aligned} g^{-1} \cdot \zeta &= \Theta^{-1} \cdot \zeta = (a'\zeta + b')(c'\zeta + d')^{-1} \\ &= \frac{(\cosh t \cos\alpha - \sinh t, \overrightarrow{\zeta})}{\cosh t - \sinh t \cos\alpha} \doteq (\zeta_4', \overrightarrow{\zeta'}), \end{aligned} \tag{A.11}$$

where $\overline{\zeta}$ is conjugate of ζ that is given by

$$\overline{\zeta} = (\zeta_4, -\overrightarrow{\zeta}). \tag{A.12}$$

From the Eq. (A.11), we obtain

$$\zeta_1' = \sin\alpha' \sin\theta' \cos\phi' = \frac{\sin\alpha \sin\theta \cos\phi}{\cosh t - \sinh t \cos\alpha} \tag{A.13}$$

$$\zeta_2' = \sin\alpha' \sin\theta' \sin\phi' = \frac{\sin\alpha \sin\theta \sin\phi}{\cosh t - \sinh t \cos\alpha} \tag{A.14}$$

$$\zeta_3' = \sin\alpha' \cos\theta' = \frac{\sin\alpha \cos\theta}{\cosh t - \sinh t \cos\alpha} \tag{A.15}$$

$$\zeta_4' = \cos\alpha' = \frac{\cosh t \cos\alpha - \sinh t}{\cosh t - \sinh t \cos\alpha}. \tag{A.16}$$

Also, from these four equations we derive following relations:

$$\left.\frac{\partial\alpha'}{\partial t}\right|_{t=0} = \sin\alpha, \tag{A.17}$$

$$\left.\frac{\partial\theta'}{\partial t}\right|_{t=0} = 0, \tag{A.18}$$

$$\left.\frac{\partial\phi'}{\partial t}\right|_{t=0} = 0, \tag{A.19}$$

$$\begin{aligned} \left.\frac{\partial[(\chi(\zeta, \Theta))^{-2\tau}]}{\partial t}\right|_{t=0} &= \left.\frac{\partial[(\cosh t - \sinh t \cos\alpha)^{-\tau}]}{\partial t}\right|_{t=0} \\ &= \tau \cos\alpha. \end{aligned} \tag{A.20}$$

By substituting these values into Eq. (27), we obtain the infinitesimal generator of time translation as follows:

$$\mathbf{H}_0 = i\Big(\tau \cos\alpha + \sin\alpha \frac{\partial}{\partial\alpha}\Big). \tag{A.21}$$

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