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A $U(6)$ Boson Model for Deformed Nuclei

Andriana Martinou

Institute of Nuclear and Particle Physics at the NCSR "Demokritos", 15341 Agia Paraskevi, Greece; martinou@inp.demokritos.gr

Abstract: The Interacting Boson Model is one of the most famous group-theoretical nuclear models, which established the use of the $U(6)$ symmetry in nuclei, built upon the s, d bosons, which derive by nucleon pairs. In this article, it is suggested that the symmetric pairs of the valence harmonic oscillator quanta can be used approximately as the s and d bosons of a new $U(6)$ Boson Model, applicable in medium mass and heavy nuclei. The main consequence of this interpretation is that the number of bosons is the number of the pairs of the valence harmonic oscillator quanta, which occur from the occupation of the Shell Model orbitals by nucleons.

Keywords: $U(3)$ boson model; interacting boson model; Elliott $SU(3)$ symmetry; proxy-$SU(3)$ symmetry

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1. Introduction

Elliott was the first to introduce a group-theoretical nuclear model in 1958, when he published the Shell Model $SU(3)$ symmetry (or nowadays called the Elliott $SU(3)$ symmetry) [1–4]. This happened 3 years earlier than the first use of symmetries in high energies physics in 1961 [5]. With his work, Elliott explained how the nucleons in a valence shell, which consists of orbitals with a common number of harmonic oscillator quanta, generate the rotational spectrum. Thus, he bridged the microscopic picture given by the nuclear Shell Model of Mayer, Haxel, Jensen and Suess [6,7] with the collective, and especially, with the rotational nuclear properties. Elliott, along with Harvey and Wilsdon, had applied the Shell Model $SU(3)$ symmetry in the s, d nuclear shell among the harmonic oscillator magic numbers 8–20. This work begun in 1958 [1] and lasted till 1968 [4].

Afterwards, in 1975, the idea that the nuclear spectrum can be produced using spherical tensors of degree 0 and 2 (the s, d bosons) was proposed by Arima and Iachello [8]. This gave rise to the Interacting Boson Model (IBM) [9–12], which supposes that the valence nuclear shell can be described by the $U(6)$ symmetry. The $U(6)$ symmetry accommodates three limiting symmetries: the $SU(3)$ symmetry for rotational nuclei, the $U(5)$ symmetry for vibrational nuclei and the $O(6)$ for the γ- unstable. The connection of the Collective Model of Bohr, Mottelson [13] and Rainwater [14] with the IBM has also been studied in Refs. [15–19].

Another $U(6)$ Boson Model is introduced in this article, which is much more similar to the Elliott $SU(3)$ symmetry. To this purpose, we have to revisit the articles of Rosensteel and Rowe, where they introduced the so-called Symplectic Model in 1979 [20–22]. In this model, an $Sp(3,\mathbb{R})$ symmetry is assumed for nuclei, which encloses the Elliott $SU(3)$ symmetry. A very interesting truncation of the Symplectic Model is the $U(3)$ Boson Model [23–28], introduced by the same authors, where they elaborate $\mathcal{S}$ and $\mathcal{D}$ operators, which are approximately boson operators for medium mass and heavy nuclei.

In this article, we use similar s, d operators as those of the $U(3)$ Boson Model to introduce a $U(6)$ Boson Model which (a) shall be applicable in medium mass and heavy nuclei, (b) shall have the same $SU(3)$ irreducible representation (irreps) as those of the Elliott or the proxy-SU(3) (approximate-SU(3)) symmetry [29–31] and (c) its wave functions will be the coherent states of Ginocchio and Kirson [16,17] with the correct values of the deformation variables of the Bohr–Mottelson Model (β, γ).

In this $U(6)$ Boson Model, the s, d bosons are the symmetric pairs of the valence harmonic oscillator quanta, in contradiction with the IBM and the Otsuka–Arima–Iachello (OAI) mapping [32,33], where the s, d bosons come from the valence nucleon pairs. Yet the algebraic mathematical structure of this $U(6)$ Boson Model is identical with that of the IBM, in the sense that both models have a $U(6)$ group, which is accompanied by three limits: the $SU(3)$, the $O(6)$ and the $U(5)$ limit; the only difference among the two models is the physical interpretation of the s, d bosons and the way to produce the irreps. Specifically, the irreps of the $SU(3)$ limit of this $U(6)$ Boson Model are identical to those of the Elliott $SU(3)$ symmetry [1] for valence shells among the 3D harmonic oscillator magic numbers.

The main difference in the derivation of the s, d bosons in this $U(6)$ Boson Model with the one used in the IBM is that here we mapped a pair of bosons (the harmonic oscillator quanta) into a new approximate boson (the s, d), while in the IBM, a pair of fermions (the nucleons) was mapped into an approximate boson. Therefore, in this $U(6)$ Boson Model, we did not perform a "boson mapping", in the sense that we did not map a pair of fermions into a boson. Consequently, in this procedure, no "spurious states" [34] emerge due to a boson mapping.

2. The Nuclear Shell Model

The Nuclear Shell Model [6,7] is the state-of-the-art theoretical model which describes the microscopic structure of atomic nuclei. The first assumption of the model is that the protons and neutrons move inside a mean field potential, which may be represented by the three-dimensional isotropic harmonic oscillator (3D-HO). Harvey, in Section 4.2 of Ref. [35], explains in simple words that *any* effective nucleon–nucleon interaction can be expanded into terms, out of which the leading term is the harmonic oscillator potential. The second assumption of the Nuclear Shell Model is the existence of a spin–orbit interaction [6,7], which leads to the prediction of the so-called nuclear magic proton or neutron numbers 2, 8, 20, 28, 50, 82 and 126, above which large single-particle energy gaps appear. This prediction was the major success of the Shell Model.

The Hamiltonian of a single particle with mass m, momentum p_x, p_y, p_z and position x, y, z in a 3D-HO potential with frequency ω, in the Cartesian coordinate system, reads

$$h_0 = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) \quad (1)$$

The eigenstates of the above Hamiltonian can be expressed either in the Cartesian coordinate system (x, y, z) as $|n_z, n_x, n_y\rangle$, or in the spherical coordinate system (r, θ, ϕ) as $|\mathbf{n}, \mathbf{l}, \mathbf{m_l}\rangle$ [31]. The labels n_z, n_x, n_y represent the harmonic oscillator quanta in each Cartesian axis, obtaining values $0, 1, 2, \ldots$, while the $n = 0, 1, 2, \ldots$ represents the radial quantum number and the l, m_l stand for the orbital angular momentum and its projection, respectively. Notice that the bold figure kets $|\mathbf{n}, \mathbf{l}, \mathbf{m_l}\rangle$ are used to distinguish the spherical eigenstates from the Cartesian ones $|n_z, n_x, n_y\rangle$ in this article. The total number of the harmonic oscillator quanta for each eigenstate is [36]

$$\mathcal{N} = n_z + n_x + n_y = 2n + l. \quad (2)$$

A unitary transformation among the $|n_z, n_x, n_y\rangle$ and the $|\mathbf{n}, \mathbf{l}, \mathbf{m_l}\rangle$ eigenstates is presented in Ref. [31]. Specifically, one may use Equation (5) of Ref. [31] to transform the eigenstates of the 3D-HO Hamiltonian from the Cartesian to the spherical basis and vice versa.

For instance, for the p shell with $\mathcal{N} = 1$ number of quanta, the following transformations can be deduced from the conjugate of Equation (5) of Ref. [31]:

$$|\mathbf{n}, \mathbf{l}, \mathbf{m_l}\rangle \to |n_z, n_x, n_y\rangle :$$
$$|\mathbf{0}, \mathbf{1}, -\mathbf{1}\rangle = \frac{|0,1,0\rangle - \mathbf{i}\,|0,0,1\rangle}{\sqrt{2}}, \quad (3)$$
$$|\mathbf{0}, \mathbf{1}, \mathbf{0}\rangle = |1,0,0\rangle\,, \quad (4)$$
$$|\mathbf{0}, \mathbf{1}, \mathbf{1}\rangle = -\frac{|0,1,0\rangle + \mathbf{i}\,|0,0,1\rangle}{\sqrt{2}}, \quad (5)$$

where $\mathbf{i}$ stands for the imaginary unit.

The operators, which annihilate or create a harmonic oscillator quantum in each Cartesian direction, are the [37]:

$$a_k = \sqrt{\frac{m\omega}{2\hbar}} k + \frac{\mathbf{i}}{\sqrt{2m\omega\hbar}} p_k, \qquad a_k^\dagger = \sqrt{\frac{m\omega}{2\hbar}} k - \frac{\mathbf{i}}{\sqrt{2m\omega\hbar}} p_k, \quad (6)$$

with $k = x, y, z$. The operators of Equation (6) satisfy the boson commutation relations [37]:

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}, \qquad [a_k^\dagger, a_{k'}^\dagger] = [a_k, a_{k'}] = 0 \quad (7)$$

with $k, k' = x, y, z$. The action of the annihilation and creation operators of Equation (6) on the Cartesian eigenstates of the 1D-HO is [36]

$$a_k^\dagger\,|n_k\rangle = \sqrt{n_k + 1}\,|n_k + 1\rangle\,, \qquad a_k\,|n_k\rangle = \sqrt{n_k}\,|n_k - 1\rangle \quad (8)$$

for $n_k = 0, 1, 2, \ldots$ and $k = x, y, z$.

Inspired from the spherical harmonics $Y_m^{l=1}$ (Appendix A.1 of [38]):

$$Y_{-1}^1 \propto \frac{x - \mathbf{i}y}{\sqrt{2}}, \qquad Y_0^1 \propto z, \qquad Y_1^1 \propto -\frac{x + \mathbf{i}y}{\sqrt{2}}, \quad (9)$$

we may define a slightly different tensor operator η_m and its conjugate $\xi_m = \eta_m^\dagger$ with components $m = -1, 0, 1$ as (see Equation (3.17) of Ref. [39])

$$\eta_{-1} = \frac{a_x^\dagger - \mathbf{i}a_y^\dagger}{\sqrt{2}}, \qquad \xi_{-1} = \frac{a_x + \mathbf{i}a_y}{\sqrt{2}}, \quad (10)$$
$$\eta_0 = a_z^\dagger, \qquad \xi_0 = a_z, \quad (11)$$
$$\eta_1 = -\frac{a_x^\dagger + \mathbf{i}a_y^\dagger}{\sqrt{2}}, \qquad \xi_1 = -\frac{a_x - \mathbf{i}a_y}{\sqrt{2}}. \quad (12)$$

Alternatively,

$$a_x^\dagger = \frac{\eta_{-1} - \eta_1}{\sqrt{2}}, \qquad a_y^\dagger = \mathbf{i}\frac{\eta_{-1} + \eta_1}{\sqrt{2}}, \qquad a_z^\dagger = \eta_0. \quad (13)$$

In order to keep up with the commutation relations of the spherical tensors of degree 1, we also define (Equation (3.19) of Ref. [39])

$$\tilde{\xi}_m = (-1)^m \xi_{-m}. \quad (14)$$

If we express the angular momentum operators in terms of the $a_k^\dagger, a_k$, we obtain [37]

$$L_x = yp_z - p_y z = \mathbf{i}(a_y a_z^\dagger - a_z a_y^\dagger), \tag{15}$$
$$L_y = zp_x - p_z x = \mathbf{i}(a_z a_x^\dagger - a_x a_z^\dagger), \tag{16}$$
$$L_z = xp_y - p_x y = \mathbf{i}(a_x a_y^\dagger - a_y a_x^\dagger). \tag{17}$$

The ladder operators of the angular momentum are

$$L_+ = L_x + \mathbf{i}L_y = a_x a_z^\dagger - a_z a_x^\dagger + \mathbf{i}(a_y a_z^\dagger - a_z a_y^\dagger), \tag{18}$$
$$L_- = L_x - \mathbf{i}L_y = a_z a_x^\dagger - a_x a_z^\dagger - \mathbf{i}(a_z a_y^\dagger - a_y a_z^\dagger). \tag{19}$$

The $\eta_m, \tilde{\xi}_m$ operators satisfy the commutation relations with the angular momentum operator (Equation (3.21) of Ref. [39]):

$$[L_0, \eta_m] = m\eta_m, \tag{20}$$
$$[L_\pm, \eta_m] = \mp\frac{1}{\sqrt{2}}\sqrt{2 - m(m \pm 1)}\eta_{m\pm1}, \tag{21}$$
$$[L_0, \tilde{\xi}_m] = m\tilde{\xi}_m, \tag{22}$$
$$[L_\pm, \tilde{\xi}_m] = \mp\frac{1}{\sqrt{2}}\sqrt{2 - m(m \pm 1)}\tilde{\xi}_{m\pm1}. \tag{23}$$

Therefore, the $\eta_m, \tilde{\xi}_m$ are spherical tensor operators of degree $l = 1$.

The physical meaning of the η_m is revealed, when acting on the vacuum eigenstate of the Hamiltonian h_0, namely on the $|n_z, n_x, n_y\rangle = |0,0,0\rangle$ orbital:

$$\eta_{-1}|0,0,0\rangle = \frac{|0,1,0\rangle - \mathbf{i}\,|0,0,1\rangle}{\sqrt{2}}, \tag{24}$$
$$\eta_0|0,0,0\rangle = |1,0,0\rangle\,, \tag{25}$$
$$\eta_1|0,0,0\rangle = -\frac{|0,1,0\rangle + \mathbf{i}\,|0,0,1\rangle}{\sqrt{2}}, \tag{26}$$

where Equations (10)–(12) and (8) were used. Interestingly the right-hand sides of Equations (24)–(26) are equal to the spherical eigenstates $|\mathbf{n}, \mathbf{l}, \mathbf{m_l}\rangle$ of Equations (3)–(5), respectively. Therefore, the operators η_m create a harmonic oscillator quantum with angular momentum $l = 1$ and projection of the angular momentum $m_l = m = \pm1, 0$, when acting on the vacuum state.

Since the quanta are bosons, the η_m operators must obey the boson commutators. Indeed, with the definitions (10)–(12) and the commutators of Equation (7), one may prove that

$$[\xi_m, \eta_{m'}] = \delta_{mm'}, \qquad [\eta_m, \eta_{m'}] = [\xi_m, \xi_{m'}] = 0. \tag{27}$$

The spin–orbit interaction $\mathbf{l}\cdot\mathbf{s}$ has to be added in the nuclear Hamiltonian:

$$h = h_0 + v_{ls}\hbar\omega\mathbf{l}\cdot\mathbf{s}, \tag{28}$$

where $\mathbf{s}$ is the spin of the particle, and v_{ls} is the strength parameter of the spin–orbit interaction (see Table I of [40,41]). The spin–orbit interaction leads to the derivation of the total angular momentum:

$$\mathbf{j} = \mathbf{l} + \mathbf{s}. \tag{29}$$

Thus, the spinor $|s, m_s\rangle$ with $s = \frac{1}{2}$ and $m_s = \pm\frac{1}{2}$ must also be considered. Consequently, the single-particle states may be written as

$$|\mathbf{n}, \mathbf{l}, \mathbf{m_l}\rangle\, |s, m_s\rangle = |\mathbf{n}, \mathbf{l}, \mathbf{m_l}, \mathbf{m_s}\rangle, \quad (30)$$

$$|n_z, n_x, n_y\rangle\, |s, m_s\rangle = |n_z, n_x, n_y, m_s\rangle\,, \quad (31)$$

having in mind that a unitary transformation among the two bases of Equations (30) and (31) exists [31].

The coupling of the spatial part of the wave function $|\mathbf{n}, \mathbf{l}, \mathbf{m_l}\rangle$ with the spinor leads to the Shell Model states:

$$|n, l, j, m_j\rangle = \sum_{m_l, m_s} (l m_l s m_s | j m_j) |\mathbf{n}, \mathbf{l}, \mathbf{m_l}, \mathbf{m_s}\rangle, \quad (32)$$

with m_j being the projection of the total angular momentum and $(l m_l s m_s | j m_j)$ being the Clebsch–Gordan coefficients [42]. The $|n, l, j, m_j\rangle$ denote the usual Shell Model orbitals, if one adds 1 unit in the radial quantum number n and represents the angular momentum $l = 0, 1, 2, \ldots$ by the small Latin characters s, p, d, etc. For instance, the orbital $|n, l, j, m_j\rangle$= $|0, 1, \frac{3}{2}, \frac{1}{2}\rangle$ is labeled $1\mathrm{p}^{j=3/2}_{m_j=1/2}$.

The spherical states $|n, l, j, m_j\rangle$ can be transformed to the Cartesian states $|n_z, n_x, n_y, m_s\rangle$, as in Ref. [31]:

$$|n, l, j, m_j\rangle = \sum_{n_z + n_x + n_y = 2n + l} \langle n_z, n_x, n_y, m_s | n, l, j, m_j\rangle\, |n_z, n_x, n_y, m_s\rangle \quad (33)$$

Consequently, one may consider the $|n_z, n_x, n_y, m_s\rangle$ states as an *alternative Shell Model basis*, expressed in the Cartesian coordinate system. The necessity for this Cartesian basis is demonstrated by Elliott and Harvey in Refs. [3,35].

3. The Shell Model SU(3) Symmetry

A very instructive illustration of the algebraic chains, which lead from the valence Shell Model space to the Shell Model $SU(3)$ symmetry lies in the Figure 7.1 of Ref. [43]. We discuss the algebraic chains and their physical meaning in this article for completeness.

The 3D-HO Hamiltonian of Equation (1) has eigenstates, which constitute the harmonic oscillator shells. The eigenstates of the h_0 of the harmonic oscillator shell with $\mathcal{N}$ =0, 1, 2, 3, 4, 5, 6 quanta lie among the proton or neutron magic numbers 0–2, 2–8, 8–20, 20–40, 40–70, 70–112 and 112–168, respectively.

Such harmonic oscillator shells, which consist of orbitals with common number of quanta $\mathcal{N}$, posses the

$$U(4\Omega) = U(\Omega) \times U(4) \quad (34)$$

symmetry [1,2], where $\Omega = \frac{(\mathcal{N}+1)(\mathcal{N}+2)}{2}$ is the number of the spatial harmonic oscillator eigenstates (for instance the $|n_z, n_x, n_y\rangle$ or the $|\mathbf{n}, \mathbf{l}, \mathbf{m_l}\rangle$), and 4 stands for the four possible projections of spin and isospin $m_s = \pm\frac{1}{2}, m_t = \pm\frac{1}{2}$ a nucleon may adopt. As an example, the shell with $\mathcal{N} = 0$ lies among the magic numbers 0–2, contains 1 orbital $|0, 0, 0\rangle$, accommodates up to 2 protons and 2 neutrons and possesses a $U(4) = U(1) \times U(4)$ symmetry. This $U(4\Omega)$ algebra has totally antisymmetric irreps. The $U(4)$ symmetry of the isospin leads to the Wigner $SU(4)$ symmetry [44].

In order to make the concept of the Shell Model $SU(3)$ symmetry clear, we work out an example from basic Quantum Mechanics throughout the text. In our example, we suppose that a nucleus has 2 valence protons in the s, d nuclear shell, which lies among the proton magic numbers 8–20. This valence shell consists of orbitals with $\mathcal{N} = 2$ number of quanta. Thus, the 2 protons shall occupy the Cartesian orbital $|n_z, n_x, n_y, m_s, m_t\rangle$ = $|2, 0, 0, \pm\frac{1}{2}, +\frac{1}{2}\rangle$, according to the highest weight irrep (see Refs. [2,30,45–47] for the explanation). In this article, $m_t = +\frac{1}{2}$ for protons, while $m_t = -\frac{1}{2}$ for neutrons. The wave function of the 2 protons has to be totally antisymmetric Slater determinant [48], according to the Pauli Principle [49,50]. If the state:

$$\phi^{m_s,m_t}(i_1) = |n_z, n_x, n_y, m_s, m_t\rangle_{i_1} = \\ \phi(i_1)\,|m_s, m_t\rangle_{i_1} \tag{35}$$

represents the orbital of the i_1^{th} nucleon, with the spin–isospin part being

$$|s, m_s\rangle_{i_1}\,|t, m_t\rangle_{i_1} = |m_s, m_t\rangle_{i_1} = |\pm, \pm\rangle_{i_1} \tag{36}$$

while the spatial part is

$$\phi(i_1) = |n_z, n_x, n_y\rangle_{i_1}, \tag{37}$$

then the wave function of the two particles is the Slater determinant:

$$\Phi = \frac{1}{\sqrt{2!}}\begin{vmatrix} \phi^{++}(1) & \phi^{-+}(1) \\ \phi^{++}(2) & \phi^{-+}(2) \end{vmatrix}. \tag{38}$$

The ϕ^{m_s,m_t} are the states of the 4Ω space, with $\Omega = 6$ for the s, d nuclear shell. The irreps of the $U(4\Omega)$ symmetry show the ways one may place the A_{val} objects (valence protons and neutrons) in the 4Ω states.

Then, the $U(4)$ symmetry is decomposed into the nuclear spin (S) and the nuclear isospin (T) symmetries:

$$U(\Omega) \times U(4) \to U(\Omega) \times [SU_S(2) \times SU_T(2)] \tag{39}$$

Emphasis has to be given to the fact that the spatial part of the wave function, which is represented by the $U(\Omega)$ algebra, is treated separately by the spin and the isospin part, which are represented by the $SU_S(2)$ and the $SU_T(2)$ algebras, respectively.

To make this statement clear, we go on with our example. If the LS coupling scheme is to be followed, i.e.,

$$\mathbf{L} = \sum_i \mathbf{l}_i, \qquad \mathbf{S} = \sum_i \mathbf{s}_i, \qquad \mathbf{J} = \mathbf{L} + \mathbf{S} \tag{40}$$

(with $\mathbf{l}_i, \mathbf{s}_i$ being the angular momentum and the spin, respectively, of the ith particle), then the Slater determinant of Equation (38) can be decomposed into a spatial part and a spin–isospin part:

$$\Phi = (\phi(1)\phi(2)) \\ \left(\frac{1}{\sqrt{2!}}(|{++}\rangle_1\,|{-+}\rangle_2 - |{++}\rangle_2\,|{-+}\rangle_1)\right). \tag{41}$$

Obviously, the spatial part of the wave function is

$$\Phi_{space} = \phi(1)\phi(2) \tag{42}$$

and it is totally symmetric in the transposition of the two particles, while the spin–isospin part is

$$\Phi_{spin-isospin} = \frac{1}{\sqrt{2!}}(|{++}\rangle_1\,|{-+}\rangle_2 - |{++}\rangle_2\,|{-+}\rangle_1) \tag{43}$$

and it is totally antisymmetric in the transposition of the particles. The overall product of the wave functions

$$\Phi = \Phi_{space} \cdot \Phi_{spin-isospin} \tag{44}$$

is thus antisymmetric, as it should be according to the Pauli Principle. The spatial part of the wave function possesses the $U(\Omega)$ symmetry, while the spin–isospin the $U(4)$.

This is the very essence of the LS coupling scheme: that the spatial part of the nuclear wave function generates the nuclear angular momentum L, the spinor part generates the nuclear spin S and that one may treat these two parts separately, as long as the product of the two of them respects the Pauli Principle for the multifermion system. The antisymmetry of the overall multinucleon wave function is guaranteed if the Young diagram of spin–isospin part ($U(4)$) is the conjugate of the Young diagram of the spatial part ($U(\Omega)$). The interested reader can find more details about this conjugation in Section 7.1.2 of Ref. [43] of Draayer's chapter or in Chapter 29 of Talmi's book [51]. This separation of the spatial wave function from the spin–isospin part is achieved in the LS coupling scheme and leads to the Shell Model $SU(3)$ symmetry.

A significant spin–orbit splitting of the single-nucleon energies may cause the rise of the spin–orbit-like shells, among proton or neutron numbers 6–14, 14–28, 28–50, 50–82, 82–126 and 126–182 [7]. These shells consist of some harmonic oscillator eigenstates with $\mathcal{N}$ quanta and some others with $\mathcal{N}+1$ quanta (see Table 7 of Ref. [31]), and so the $U(4\Omega) = U(\Omega) \times U(4)$ symmetry no longer has a straightforward application.

One of the possible ways [52,53] to overpass this problem is the use of the proxy-$SU(3)$ symmetry [29,54,55]. In this type of approximate symmetry, one may apply a unitary transformation in the intruder orbitals with $\mathcal{N}+1$ quanta [31], so as to transform them to their de Shalit–Goldhaber counterparts [56]. This unitary transformation [57] reduces the total number of quanta of the intruder orbitals by 1 unit ($\mathcal{N}+1 \rightarrow \mathcal{N}$), and it is similar in spirit with the unitary transformation introduced in the pseudo-$SU(3)$ symmetry [58–60].

The advantages of the proxy-$SU(3)$ symmetry are the following:

(a) The relation of the intruder orbitals to their proxies is based on the experimental observations of de Shalit and Goldhaber [56] and of Cakirli, Blaum and Casten [61];
(b) The unitary transformation used in the proxy-$SU(3)$ symmetry leaves the normal parity orbitals (those with $\mathcal{N}$ quanta) intact and affects only the intruder orbitals (those with $\mathcal{N}+1$ quanta);
(c) The proxy transformation affects only the z-axis of the intruder orbitals, and so the the number of quanta in the x, y plane is conserved. This means that the projection of the total and the orbital single-particle angular momenta, which are good quantum numbers in the deformed nuclei [62,63], are not affected by the transformation. We have zero error in the prediction of the band label K and minimum error in the cutoff of the nuclear angular momentum (L_{max}) for each band [31].

Furthermore, the irreps of the proxy-$SU(3)$ symmetry gave parameter-free predictions for the prolate–oblate transition [30] and for the islands of inversion and shape coexistence [45,64], while within a single parameter, they gave promising early stage results for the binding and the two-neutron separation energies [65].

As a result, in a harmonic oscillator shell, one may use the $U(4\Omega) = U(\Omega) \times U(4)$ symmetry in a straightforward way, as in Refs. [1–4], while in a spin–orbit-like shell, the $U(4\Omega) = U(\Omega) \times U(4)$ can be approximately applied within the proxy-$SU(3)$ scheme [30,31,66]. The gain is that in any of the two types of shells, the spatial $U(\Omega)$ symmetry exists and is decomposed as [1,2]

$$U(\Omega) \supset U(3) \supset SU(3) \supset SO(3) \supset SO(2), \tag{45}$$

Clearly, the Shell Model $SU(3)$ symmetry derives from the spatial $U(\Omega)$ symmetry. The labels of each of the above symmetries are [38]

$$\begin{aligned}
[f] = [f_1, f_2, ..., f_\Omega] &: \text{ for the } U(\Omega), \\
[f_1, f_2, f_3] &: \text{ for the } U(3), \\
(\lambda, \mu) &: \text{ for the } SU(3), \\
L &: \text{ for the } SO(3), \\
M &: \text{ for the } SO(2),
\end{aligned} \tag{46}$$

where M is the projection of the nuclear orbital angular momentum.

In our example (the one of the two protons in the s, d shell), the irrep of the $U(\Omega = 6)$ is $[2,0,0,0,0,0]$; since the two protons occupy the same $|n_z, n_x, n_y\rangle$ orbital, the irrep of the $U(3)$ is $[4,0,0]$, since in the highest weight irrep $f_1 = \sum_i n_{iz}$, $f_2 = \sum_i n_{ix}$ and $f_3 = \sum_i n_{iy}$ [2,46], and $(\lambda, \mu) = (4,0)$, since $\lambda = f_1 - f_2$ and $\mu = f_2 - f_3$. The subscript i is for every valence nucleon. Therefore, the spatial part of the Shell Model $SU(3)$ wave function is labeled as

$$\Phi_{spatial}([f](\lambda, \mu)). \tag{47}$$

The spin–isospin part, which is the conjugate of the spatial, is labeled by the nuclear spin $\mathbf{S}$, the nuclear isospin $\mathbf{T} = \sum_i \mathbf{t}_i$ and their projections M_S, M_T:

$$\Phi_{spin-isospin}(T, M_T, S, M_S) \tag{48}$$

In our example, $S = 0$, $M_S = 0$, $T = 1$ and $M_T = 1$. Thus, the overall nuclear wave function is labeled by the [3]:

$$\Phi(TS[f](\lambda, \mu)M_T M_S). \tag{49}$$

Despite the fact that the overall Shell Model $SU(3)$ wave function is labeled by both the spatial and the spin–isospin irreps, one has to remember that the $U(3)$ and $SU(3)$ lie solely in the spatial part of the state, and this is the privilege of the LS coupling scheme.

The Shell Model $U(3)$ algebra is generated by the nine Cartesian generators of the form [3,35]:

$$G_{k,k'} = a_k^\dagger a_{k'}, \text{ with } k, k' = x, y, z. \tag{50}$$

The three components of the angular momentum $L_z, L_\pm$, the five components of the quadrupole operator $Q_m, m = \pm 2, \pm 1, 0$ and the number (of quanta) operator can be expressed as linear combinations of the generators of Equation (50), and their commutators close the $U(3)$ algebra [1,2].

Taking advantage of the equivalence of the η_m, ξ_m with the $a_k^\dagger, a_k$ operators, which derives from the Equations (10)–(12), one may construct the spatial $U(3)$ algebra of a valence shell from the *spherical quanta states*. In this scenario, the quanta are created by spherical tensors of degree $l = 1$, and thus they may be arranged according to the three components $m = \pm 1, 0$, instead of being arranged according to the three Cartesian directions of the Elliott–Harvey point of view [3,35]. The $U(3)$ algebra of the spherical quanta is generated by the nine operators of the form:

$$\mathcal{A}_{m,m'} = \eta_m \xi_{m'}, \text{ with } m, m' = \pm 1, 0. \tag{51}$$

Using the boson commutators, (27) along with the identity

$$\begin{aligned}
[AB, CD] = A[B, C]D + C[A, D]B \\
+[A, C]BD + CA[B, D],
\end{aligned} \tag{52}$$

one may calculate all the commutators of the type $[\mathcal{A}_{m,m'}, \mathcal{A}_{m'',m'''}]$ with $m, m', m'', m''' = \pm 1, 0$ and produce the Multiplication Table (see Table 1). Since the set of generators of an algebra is not unique, one may consider the $G_{k,k'}$ of the expression (50) and the $\mathcal{A}_{m,m'}$ of (51) as two generator sets of the Shell Model $U(3)$ algebra.

Table 1. Multiplication table of the $U(3)$ algebra, which is generated by the operators $\mathcal{A}_{m,m'}$. For instance, $[\mathcal{A}_{1,0}, \mathcal{A}_{1,1}] = -\mathcal{A}_{1,0}$.

$\mathcal{A}_{m,m'}$	$\mathcal{A}_{1,1}$	$\mathcal{A}_{1,0}$	$\mathcal{A}_{1,-1}$	$\mathcal{A}_{0,1}$	$\mathcal{A}_{0,0}$	$\mathcal{A}_{0,-1}$	$\mathcal{A}_{-1,1}$	$\mathcal{A}_{-1,0}$	$\mathcal{A}_{-1,-1}$
$\mathcal{A}_{1,1}$	0	$\mathcal{A}_{1,0}$	$\mathcal{A}_{1,-1}$	$-\mathcal{A}_{0,1}$	0	0	$-\mathcal{A}_{-1,1}$	0	0
$\mathcal{A}_{1,0}$	$-\mathcal{A}_{1,0}$	0	0	$\mathcal{A}_{1,1}-\mathcal{A}_{0,0}$	$\mathcal{A}_{1,0}$	$\mathcal{A}_{1,-1}$	$-\mathcal{A}_{-1,0}$	0	0
$\mathcal{A}_{1,-1}$	$-\mathcal{A}_{1,-1}$	0	0	$-\mathcal{A}_{0,-1}$	0	0	$\mathcal{A}_{1,1}-\mathcal{A}_{-1,-1}$	$\mathcal{A}_{1,0}$	$\mathcal{A}_{1,-1}$
$\mathcal{A}_{0,1}$	$\mathcal{A}_{0,1}$	$\mathcal{A}_{0,0}-\mathcal{A}_{1,1}$	$\mathcal{A}_{0,-1}$	0	$-\mathcal{A}_{0,1}$	0	0	$-\mathcal{A}_{-1,1}$	0
$\mathcal{A}_{0,0}$	0	$-\mathcal{A}_{1,0}$	0	$\mathcal{A}_{0,1}$	0	$\mathcal{A}_{0,-1}$	0	$-\mathcal{A}_{-1,0}$	0
$\mathcal{A}_{0,-1}$	0	$-\mathcal{A}_{1,-1}$	0	0	$-\mathcal{A}_{0,-1}$	0	$\mathcal{A}_{0,1}$	$\mathcal{A}_{0,0}-\mathcal{A}_{-1,-1}$	$\mathcal{A}_{0,-1}$
$\mathcal{A}_{-1,1}$	$\mathcal{A}_{-1,1}$	$\mathcal{A}_{-1,0}$	$\mathcal{A}_{-1,-1}$ $-\mathcal{A}_{1,1}$	0	0	$-\mathcal{A}_{0,1}$	0	0	$-\mathcal{A}_{-1,1}$
$\mathcal{A}_{-1,0}$	0	0	$-\mathcal{A}_{1,0}$	$\mathcal{A}_{-1,1}$	$\mathcal{A}_{-1,0}$	$\mathcal{A}_{-1,-1}-\mathcal{A}_{0,0}$	0	0	$-\mathcal{A}_{-1,0}$
$\mathcal{A}_{-1,-1}$	0	0	$-\mathcal{A}_{1,-1}$	0	0	$-\mathcal{A}_{0,-1}$	$\mathcal{A}_{-1,1}$	$\mathcal{A}_{-1,0}$	0

4. The Shell Model SU(3) States

When one is working in the level of the $U(\Omega)$ symmetry, the irreps $[f_1, f_2, \dots, f_\Omega]$ represent with how many and with which ways the "objects" can be placed in the Ω "states". In this level, the "objects" are the indistinguishable valence nucleons, and the "states" are the spatial orbitals $|n_z, n_x, n_y\rangle$. Draayer, Leschber, Park and Lopez in Ref. [67] accomplished the $U(\Omega) \supset U(3)$ decomposition. This is a pure mathematical procedure, but what is the physical meaning of this decomposition?

The fact is that when one is working on the level of the $U(3)$ symmetry, the irreps $[f_1, f_2, f_3]$ represent in how many many ways one may place the "objects" in a three-dimensional space. Now, the three dimensions are the Hermite polynomials $|n_z = 1\rangle$, $|n_x = 1\rangle$ and $|n_y = 1\rangle$, which are eigenstates of the harmonic oscillator, while the "objects" are the indistinguishable harmonic oscillator quanta, which derive from the placement of the nucleons in the $|n_z, n_x, n_y\rangle$ states. For instance, the Shell Model $U(3)$ irrep $[f_1, f_2, f_3] = [2, 1, 0]$ is about two quanta, which have occupied the state $|n_z = 1\rangle$ and about one quantum in the state $|n_x = 1\rangle$. Therefore, the objects of the $U(3)$ wave functions are not the nucleons anymore but the quanta. Thus, the many nucleon wave functions of the $U(\Omega)$ symmetry are being decomposed to the many quanta wave functions of the $U(3)$ symmetry. This is the very meaning of the decomposition Draayer et al. accomplished in Ref. [67]. The $U(3)$ irreps $[f_1, f_2.f_3]$ show with how many and with which ways one may transpose the harmonic oscillator quanta in the three Cartesian axes. Each transposition of the harmonic oscillator quanta is equivalent with a spatial rotation [68].

The third article of the Shell Model $SU(3)$ symmetry was written by Elliott and Harvey. Harvey wrote another article (see Ref. [35]) where he explained the details of the model. We now focus on Equation (3.15) of Section 3.3 of Harvey's article in Ref. [35]. There, he presented that the $U(3)$ wave function is made of states:

$$|pqr\rangle_{i_1,\dots,i_{p+q+r}} = a^\dagger_{i_1 z} a^\dagger_{i_2 z} \dots a^\dagger_{i_p z} a^\dagger_{i_{p+1} x} \dots a^\dagger_{i_{p+q} x} \\ a^\dagger_{i_{p+q+1} y} \dots a^\dagger_{i_{p+q+r} y} |0\rangle . \quad (53)$$

The dagger operators are those introduced in Equation (6). The labels $i_1, \dots, i_{p+q+r}$ take the values $1, 2, 3, \dots, A_{val}$, where A_{val} is the valence number of nucleons. It is possible that a particle number may appear more than once, or not at all; so, it is possible that $i_1 = i_2 = 1$. The $a^\dagger_{i_1 z} |0\rangle$ represents a quantum on the z-axis from the i_1^{th} particle. Clearly, in the state of Equation (53), there are p quanta on the z-axis, q quanta in the x-axis and r quanta in the y-axis. So, the numbers $1, 2, \dots, p, p+1, \dots, p+q, p+q+1, \dots, p+q+r$

enumerate the quanta, which are the "objects" of the $U(3)$ symmetry. Indeed, the quanta are being enumerated, and this is necessary for the construction the particle-number Young tableau of the $U(3)$ states, which is discussed afterwards. The vacuum $|0\rangle$ is the state of no quanta:

$$|0\rangle = [f_1 = 0, f_2 = 0, f_3 = 0]. \tag{54}$$

The many-quanta $U(3)$ wave function of Equation (53) can be represented by a Young tableau. Each box in a Young tableau is an "object", which in the case of the Shell Model $U(3)$ symmetry is a harmonic oscillator quantum in one Cartesian axis. A general quantum-number (left) and particle-number (right) Young tableau [38] of the Shell Model $U(3)$ symmetry looks like:

$$\begin{array}{|c|c|c|c|c|}\hline \mathbf{z} & \mathbf{z} & \dots & \dots & \mathbf{z} \\ \hline \mathbf{x} & \dots & \dots & \mathbf{x} \\ \cline{1-4} \mathbf{y} & \dots & \mathbf{y} \\ \cline{1-3} \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \dots & \dots \\ \hline \dots & \dots & \dots & \dots \\ \cline{1-4} \dots & \dots & \dots \\ \cline{1-3} \end{array} \tag{55}$$

The labels $\mathbf{z}, \mathbf{x}, \mathbf{y}$ on the left signify a quantum in the z, x, y Cartesian axis, respectively. The numbers on the right enumerate the quanta, take values $1, 2, \dots, p, p+1, \dots, p+q, p+q+1, \dots, p+q+r$ using Harvey's notation (Equation (3.15) of Ref. [35]) and can be placed in the boxes so as to increase from left to the right and from up to down [38]. The position of the numbers indicates the permutation symmetry of the quanta. The permutation of the quanta is discussed extensively in Ref. [46].

It is common practice that, in a $U(3)$ Young tableau, a column with three boxes is erased. The equivalent $SU(3)$ Young tableaux is

$$\begin{array}{|c|c|c|c|c|}\hline \mathbf{z} & \mathbf{z} & \dots & \dots & \mathbf{z} \\ \hline \mathbf{x} & \dots & \dots & \mathbf{x} \\ \cline{1-4} \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & \dots \\ \hline \dots & \dots & \dots & \dots \\ \cline{1-4} \end{array} \tag{56}$$

In general, two boxes in a row of a Young tableau

$$\begin{array}{|c|c|}\hline \mathbf{z} & \mathbf{z} \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \tag{57}$$

represent a symmetric pair of quanta, while two boxes in a column

$$\begin{array}{|c|}\hline \mathbf{z} \\ \hline \mathbf{x} \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \tag{58}$$

correspond to an antisymmetric pair of quanta. At this point, recall that, since the quanta are bosons, they can form symmetric and antisymmetric pairs, in contradiction with the fermions, which can form only antisymmetric pairs. From the above, it becomes clear that the Shell Model $U(3)$ symmetry has to do with harmonic oscillator quanta, which are bosons and are coupled into symmetric or into antisymmetric pairs.

The Shell Model $SU(3)$ labels for the highest weight irrep are [2,46]

$$\lambda = f_1 - f_2 = \sum_i n_{iz} - \sum_i n_{ix}, \tag{59}$$

$$\mu = f_2 - f_3 = \sum_i n_{ix} - \sum_i n_{iy}. \tag{60}$$

A general irrep (λ, μ) with $\mu \neq 0$ represents an $SU(3)$ state of mixed symmetry, i.e., it is not totally symmetric [46].

Now, we may return to our example, the one of the two protons in the s, d shell. The quantum-number and particle-number Young tableaux of this example are

$$\begin{array}{|c|c|c|c|}\hline \mathbf{z} & \mathbf{z} & \mathbf{z} & \mathbf{z} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array}. \tag{61}$$

This state has a $U(3)$ irrep $[f_1, f_2, f_3] = [4, 0, 0]$, or an $SU(3)$ irrep $(\lambda, \mu) = (4, 0)$ [2].

The question now is if we can construct the above state by symmetric pairs of quanta. Two symmetric pairs of quanta in the z-axis are represented by the Young tableaux:

$$\begin{array}{|c|c|}\hline \mathbf{z} & \mathbf{z} \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|}\hline \mathbf{z} & \mathbf{z} \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} . \tag{62}$$

Each of the two Young tableaux have a $U(3)$ irrep $[f_1, f_2, f_3]$ = $[2, 0, 0]$, or $SU(3)$ irrep $(\lambda, \mu) = (2, 0)$ [2]. The above two Young tableaux can be coupled (outer product) into a new Young tableau. The rules for the coupling are described in Refs. [68,69] and can be accomplished by the online code of Ref. [70], or even by the code of Ref. [71], which has far more reaching capabilities than this task. The results of this outer product are

$$\begin{array}{|c|c|}\hline \mathbf{z} & \mathbf{z} \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \quad \otimes \quad \begin{array}{|c|c|}\hline \mathbf{z} & \mathbf{z} \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \quad =$$
$$\begin{array}{|c|c|c|c|}\hline \mathbf{z} & \mathbf{z} & \mathbf{z} & \mathbf{z} \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array} \quad \oplus$$
$$\begin{array}{|c|c|}\hline \mathbf{z} & \mathbf{z} \\ \hline \mathbf{x} & \mathbf{x} \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \oplus$$
$$\begin{array}{|c|c|c|}\hline \mathbf{z} & \mathbf{z} & \mathbf{z} \\ \hline \mathbf{x} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \quad . \tag{63}$$

The first quantum-number and particle-number Young tableau in the r.h.s. of the above equation is the fully symmetric state of the four quanta, while the remaining two occurrences correspond to spatial rotations (Figure 3 of Ref. [68]). Consequently, the fully symmetric state of (61) may result from the symmetric coupling of two pairs of symmetric quanta.

5. The U(3) Boson Model

The Symplectic Model [20–22] uses the $Sp(3, \mathbb{R})$ algebra. This algebra is spanned by the 21 generators of the type:

$$\mathcal{B}^{\dagger}_{kk'} = \mathcal{B}^{\dagger}_{k'k} = \frac{1}{2} \sum_{i=1}^{A-1} a^{\dagger}_{ik} a^{\dagger}_{ik'}, \tag{64}$$

$$\mathcal{B}_{kk'} = \mathcal{B}_{k'k} = \frac{1}{2} \sum_{i=1}^{A-1} a_{ik} a_{ik'}, \tag{65}$$

$$C_{kk'} = \frac{1}{2} \sum_{i=1}^{A-1} (a^{\dagger}_{ik} a_{ik'} + a_{ik'} a^{\dagger}_{ik}), \tag{66}$$

where A is the mass number and the i enumerates the nucleons (protons or neutrons). Consider that the eigenvalue of the $C_{kk'}$ operator is labeled as $N_{kk'}$. The commutators among the generators of the Symplectic Model are given by Equations (2.2) of Ref. [27] (just be aware of the factor $\frac{1}{2}$ in the definitions). Of special interest is the commutator:

$$[\mathcal{B}_{kk'}, \mathcal{B}^{\dagger}_{k''k'''}] = \frac{1}{4}(C_{k'''k'}\delta_{kk''} + C_{k'''k}\delta_{k''k'} + C_{k''k'}\delta_{kk'''} + C_{k''k'}\delta_{k'k'''}). \tag{67}$$

The spherical tensors of zero degree were to be used in the $U(3)$ Boson Model [23]:

$$\mathcal{B}^{\dagger}_0 = \sqrt{\frac{2}{3}}(\mathcal{B}^{\dagger}_{xx} + \mathcal{B}^{\dagger}_{yy} + \mathcal{B}^{\dagger}_{zz}), \qquad \mathcal{B}_0 = (\mathcal{B}^{\dagger}_0)^{\dagger}, \tag{68}$$

$$C_0 = C_{xx} + C_{yy} + C_{zz}. \tag{69}$$

The commutator among them is

$$[\mathcal{B}_0, \mathcal{B}_0^\dagger] = \frac{2}{3} C_0. \tag{70}$$

The eigenvalue of C_0 is

$$N_0 = \sum_{i=1}^{A-1} (n_{ix} + n_{iy} + n_{iz} + \frac{3}{2}) \tag{71}$$

Rosensteel and Rowe defined the following operators [23]:

$$\mathcal{S}^\dagger = \sqrt{\frac{3}{2N_0}} \mathcal{B}_0^\dagger, \qquad \mathcal{S} = \sqrt{\frac{3}{2N_0}} \mathcal{B}_0, \tag{72}$$

whose commutator is

$$[\mathcal{S}, \mathcal{S}^\dagger] = \frac{C_0}{N_0} = I + \frac{C_0 - N_0 I}{N_0}, \tag{73}$$

where I is the identity operator. For medium mass or heavy nuclei, the number of quanta N_0 is indeed a large number $N_0 \gg 1$, and so

$$[\mathcal{S}, \mathcal{S}^\dagger] \approx I \tag{74}$$

Similarly, the spherical tensors of degree 2 were introduced [23]:

$$\mathcal{D}_M^\dagger = \sqrt{\frac{3}{2N_0}} \mathcal{B}_M^{2\dagger}, \qquad \mathcal{D}_M = \sqrt{\frac{3}{2N_0}} \mathcal{B}_M^2, \tag{75}$$

where the $\mathcal{B}_M^{2\dagger}$ and the $\mathcal{B}_M^2$ are given by Equations (22) and (23) of Ref. [28] (just be aware of the factor $\frac{1}{2}$ in the definitions). The commutator among them for medium mass and heavy nuclei, where $N_0 \gg 1$, is

$$[\mathcal{D}_M, \mathcal{D}_M^\dagger] \approx I. \tag{76}$$

As a result, one may consider that the above S and D operators are approximately bosons. Certain combinations of them form the generators of the $U(3)$ boson algebra of the $U(3)$ Boson Model.

6. A U(6) Boson Model

Now, we may revise the procedure used for the introduction of the $U(3)$ Boson Model to create a $U(6)$ Boson Model, which is appropriate to describe the collective features of the valence nucleons. At this point, we have to be aware that the operators of Equation (64) cause *particle excitations* to two shells above. However, here, we intend to introduce a $U(6)$ Boson Model to describe a nucleus with nucleons below the Fermi level, without particle excitations to two shells above. To this purpose, suppose that a *pair* of protons or neutrons occupies the Shell Model orbital $|n_z, n_x, n_y, m_s\rangle$ of the valence shell. In this case, the numbers $\sum_i n_{iz}$, $\sum_i n_{ix}$, $\sum_i n_{iy}$ are always even numbers, because two particles occupy the same orbital. The nucleon pair consists of the ith nucleon (where i is an odd number 1, 3, 5, etc.) and by the i'th nucleon (with i' being the next even number $i' = i + 1$= 2, 4, 6, etc.).

We may define the operators:

$$B_{kk'}^\dagger = \frac{1}{2} \sum_{i=1,3,5,\ldots}^{A-1} (a_{ik}^\dagger a_{i'k'}^\dagger + a_{ik'}^\dagger a_{i'k}^\dagger), \text{ with } i' = i + 1 \text{ and } k, k' = x, y, z. \tag{77}$$

The $B^{\dagger}_{kk'}$ creates a symmetric pair of quanta in the (k, k') axes deriving from the pair of the (i^{th}, i'^{th}) nucleons. The $B_{kk'}$ is simply its conjugate $B_{kk'} = (B^{\dagger}_{kk'})^{\dagger}$.

Since $B^{\dagger}_{kk'} = B^{\dagger}_{k'k}$, there exist six types of such operators in the Cartesian form:

$$B^{\dagger}_{xx} \quad B^{\dagger}_{xy} \quad B^{\dagger}_{xz} \quad B^{\dagger}_{yy} \quad B^{\dagger}_{yz} \quad B^{\dagger}_{zz}.$$

Proof. We shall prove that these operators satisfy the same commutation relation as the Equation (67). Consequently, we can mimic the derivation of boson operators in the $U(3)$ Boson Model to derive similar, yet different, s, d boson operators. These new operators, when acting on the vacuum, *will form symmetric pairs of quanta, which derive from the pairs of the nucleons of the valence shell*, while on the contrary, the relevant operators of the Symplectic Model and the $U(3)$ Boson Model excite a nucleon from the valence shell to two shells above (this is called a $2\hbar\omega$ particle excitation).

The boson commutator of Equation (7) for different particles (i, i') becomes

$$[a_{ik'}, a^{\dagger}_{i'k}] = \delta_{ii'}\delta_{kk'}. \tag{78}$$

If $i = i'$:

$$a_{ik'}a^{\dagger}_{ik'} = \delta_{kk'} + a^{\dagger}_{ik'}a_{ik'}. \tag{79}$$

Using the above, the generator of the Elliott $U(3)$ symmetry (see Equation (66)) can be written as

$$\begin{aligned} C_{kk'} &= \sum_{i=1}^{A}\left(a^{\dagger}_{ik}a_{ik'} + \frac{\delta_{kk'}}{2}\right) \Rightarrow \\ \sum_{i=1}^{A}\left(a^{\dagger}_{ik}a_{ik'}\right) &= C_{kk'} - \sum_{i=1}^{A}\left(\frac{\delta_{kk'}}{2}\right) \Rightarrow \\ \sum_{i=1}^{A}\left(a^{\dagger}_{ik}a_{ik'}\right) &= C_{kk'} - \frac{A}{2}\delta_{kk'}. \end{aligned} \tag{80}$$

where the summation is for every particle.

The commutator is

$$[B_{kk'}, B^{\dagger}_{k''k'''}] = \frac{1}{4}\sum_{i=1,3,5,\ldots}^{A}\Big([a_{i'k'}a_{ik}, a^{\dagger}_{ik''}a^{\dagger}_{i'k'''}] + [a_{i'k'}a_{ik}, a^{\dagger}_{ik'''}a^{\dagger}_{i'k''}] + [a_{i'k}a_{ik'}, a^{\dagger}_{ik''}a^{\dagger}_{i'k'''}] + [a_{i'k}a_{ik'}, a^{\dagger}_{ik'''}a^{\dagger}_{i'k''}]\Big). \tag{81}$$

We calculate each of the four commutators from the above relation with the help of the identity (52) and through Equations (78) and (79):

$$[a_{i'k'}a_{ik}, a^{\dagger}_{ik''}a^{\dagger}_{i'k'''}] = a^{\dagger}_{i'k'''}a_{i'k'}\delta_{kk''} + a^{\dagger}_{ik''}a_{ik}\delta_{k'k'''} + \delta_{k'k'''}\delta_{kk''}, \tag{82}$$

$$[a_{i'k'}a_{ik}, a^{\dagger}_{ik'''}a^{\dagger}_{i'k''}] = a^{\dagger}_{i'k''}a_{i'k'}\delta_{kk'''} + a^{\dagger}_{ik'''}a_{ik}\delta_{k'k''} + \delta_{k'k''}\delta_{kk'''}, \tag{83}$$

$$[a_{i'k}a_{ik'}, a^{\dagger}_{ik''}a^{\dagger}_{i'k'''}] = a^{\dagger}_{i'k'''}a_{i'k}\delta_{k'k''} + a^{\dagger}_{ik''}a_{ik'}\delta_{kk'''} + \delta_{kk'''}\delta_{k'k''}, \tag{84}$$

$$[a_{i'k}a_{ik'}, a^{\dagger}_{ik'''}a^{\dagger}_{i'k''}] = a^{\dagger}_{i'k''}a_{i'k}\delta_{k'k'''} + a^{\dagger}_{ik'''}a_{ik'}\delta_{kk''} + \delta_{kk''}\delta_{k'k'''}. \tag{85}$$

If we substitute these four equations into (81), we obtain

$$
\begin{aligned}
[B_{kk'}, B^{\dagger}_{k''k'''}] = \frac{1}{4} \sum_{i=1,3,5,\ldots}^{A-1} [(a^{\dagger}_{i'k'''} a_{i'k'} + a^{\dagger}_{ik'''} a_{ik'})\delta_{kk''} + (a^{\dagger}_{ik''} a_{ik} + a^{\dagger}_{i'k''} a_{i'k})\delta_{k'k'''} \\
+ (a^{\dagger}_{i'k''} a_{i'k'} + a^{\dagger}_{ik''} a_{ik'})\delta_{kk'''} + (a^{\dagger}_{ik'''} a_{ik} + a^{\dagger}_{i'k'''} a_{i'k})\delta_{k'k''}] \\
+ \sum_{i=1,3,5,\ldots}^{A-1} (\delta_{k'k'''}\delta_{kk''} + \delta_{k'k''}\delta_{kk'''} + \delta_{kk'''}\delta_{k'k''} + \delta_{kk''}\delta_{k'k'''}) = \\
\frac{1}{4} \sum_{i=1}^{A} (a^{\dagger}_{ik'''} a_{ik'}\delta_{kk''} + a^{\dagger}_{ik''} a_{ik}\delta_{k'k'''} + a^{\dagger}_{ik''} a_{ik'}\delta_{kk'''} + a^{\dagger}_{ik'''} a_{ik}\delta_{k'k''}) \\
+ \frac{A}{2} (\delta_{k'k'''}\delta_{kk''} + \delta_{k'k''}\delta_{kk'''} + \delta_{kk'''}\delta_{k'k''} + \delta_{kk''}\delta_{k'k'''}).
\end{aligned} \tag{86}
$$

If we make use of Equation (80), the above commutator becomes

$$
[B_{kk'}, B^{\dagger}_{k''k'''}] = \frac{1}{4}(C_{k''k'}\delta_{kk''} + C_{k''k}\delta_{k'k'''} + C_{k''k'}\delta_{kk'''} + C_{k'''k}\delta_{k'k''}). \tag{87}
$$

This result is identical with the commutator of Equation (67). □

Consequently, we can use these $B^{\dagger}_{kk'}$ operators to define boson operators for medium mass and heavy nuclei, where $N_0 \gg 1$, just like Rowe and Rosensteel carried out in the $U(3)$ Boson Model.

Boson Operators in the Spherical Form

In the following, we give the expressions of the spherical tensor operators of degree zero and two in this scheme. The interesting thing which has occurred is that *anything* which is constructed by the Cartesian operators $a_{ik}, a^{\dagger}_{i'k}$ in the Shell Model $SU(3)$ symmetry can be equally constructed by the spherical operators $\tilde{\xi}_{im} = (-1)^m \xi_{im}, \eta_{im}$.

Since the η_{im} are spherical tensors of degree 1, one may couple a pair of them to create a spherical tensor of degree:

(a) $L = 0$;
(b) $L = 1$;
(c) $L = 2$.

We may define the spherical operator $B^{L\dagger}_{M}$, which creates a symmetric pair of quanta with angular momentum L and projection M deriving from the $i, i' = i + 1$ particles:

$$
B^{L\dagger}_{M} = \sum_{i=1,3,5,\ldots}^{A-1} \sum_{m,m'} (1m1m'|LM)\eta_{im}\eta_{i'm'}, \tag{88}
$$

where $(1m1m'|LM)$ is a Clebsch–Gordan coefficient. The tilde operators, in order to ensure that the $\tilde{B}^{L}_{M}$ are spherical tensors, follow the relation:

$$
\tilde{B}^{L}_{M} = (-1)^{L-M} \left(B^{L\dagger}_{-M}\right)^{\dagger}. \tag{89}
$$

Explicitly, through the calculation of the Clebsch–Gordan coefficients of Equation (88), the new creation operators for a pair of spherical quanta result to the expressions:

$$
B^{0\dagger}_{0} = \sum_{i=1,3,5,\ldots}^{A-1} \frac{1}{\sqrt{3}}(\eta_{i1}\eta_{i'-1} + \eta_{i-1}\eta_{i'1} - \eta_{i0}\eta_{i'0}), \tag{90}
$$

$$B^{2\dagger}_{-2} = \sum_{i=1,3,5,\ldots}^{A-1} \eta_{i-1}\eta_{i'-1}, \tag{91}$$

$$B^{2\dagger}_{-1} = \sum_{i=1,3,5,\ldots}^{A-1} \frac{1}{\sqrt{2}}(\eta_{i-1}\eta_{i'0} + \eta_{i0}\eta_{i'-1}) \tag{92}$$

$$B^{2\dagger}_{0} = \sum_{i=1,3,5,\ldots}^{A-1} \frac{1}{\sqrt{6}}(\eta_{i1}\eta_{i'-1} + \eta_{i-1}\eta_{i'1}) + \sqrt{\frac{2}{3}}\eta_{i0}\eta_{i'0}, \tag{93}$$

$$B^{2\dagger}_{1} = \sum_{i=1,3,5,\ldots}^{A-1} \frac{1}{\sqrt{2}}(\eta_{i1}\eta_{i'0} + \eta_{i0}\eta_{i'1}), \tag{94}$$

$$B^{2\dagger}_{2} = \sum_{i=1,3,5,\ldots}^{A-1} \eta_{i1}\eta_{i'1}, \tag{95}$$

while, following the identity $(AB)^\dagger = B^\dagger A^\dagger$ and Equation (89), we obtain the tilde annihilation operators.

The same operators can be written in terms of the Cartesian operators using the correspondence of the Equations (10)–(12):

$$B^{0\dagger}_{0} = \sum_{i=1,3,5,\ldots}^{A-1} -\frac{1}{\sqrt{3}}\left(a^\dagger_{ix}a^\dagger_{i'x} + a^\dagger_{iy}a^\dagger_{i'y} + a^\dagger_{iz}a^\dagger_{i'z}\right), \tag{96}$$

$$B^{2\dagger}_{-2} = \sum_{i=1,3,5,\ldots}^{A-1} \frac{1}{2}\left(a^\dagger_{ix}a^\dagger_{i'x} - a^\dagger_{iy}a^\dagger_{i'y} - \mathbf{i}(a^\dagger_{ix}a^\dagger_{i'y} + a^\dagger_{iy}a^\dagger_{i'x})\right), \tag{97}$$

$$B^{2\dagger}_{-1} = \sum_{i=1,3,5,\ldots}^{A-1} \frac{1}{2}\left(a^\dagger_{ix}a^\dagger_{i'z} + a^\dagger_{iz}a^\dagger_{i'x} - \mathbf{i}(a^\dagger_{iy}a^\dagger_{i'z} + a^\dagger_{iz}a^\dagger_{i'y})\right), \tag{98}$$

$$B^{2\dagger}_{0} = \sum_{i=1,3,5,\ldots}^{A-1} -\frac{1}{\sqrt{6}}\left(a^\dagger_{ix}a^\dagger_{i'x} + a^\dagger_{iy}a^\dagger_{i'y}\right) + \sqrt{\frac{2}{3}}a^\dagger_{iz}a^\dagger_{i'z}, \tag{99}$$

$$B^{2\dagger}_{1} = \sum_{i=1,3,5,\ldots}^{A-1} -\frac{1}{2}\left(a^\dagger_{ix}a^\dagger_{i'z} + a^\dagger_{iz}a^\dagger_{i'x} + \mathbf{i}(a^\dagger_{iy}a^\dagger_{i'z} + a^\dagger_{iz}a^\dagger_{i'y})\right), \tag{100}$$

$$B^{2\dagger}_{2} = \sum_{i=1,3,5,\ldots}^{A-1} \frac{1}{2}\left(a^\dagger_{ix}a^\dagger_{i'x} - a^\dagger_{iy}a^\dagger_{i'y} + \mathbf{i}(a^\dagger_{ix}a^\dagger_{i'y} + a^\dagger_{iy}a^\dagger_{i'x})\right). \tag{101}$$

In other words, if we use the definition (77):

$$B^{0\dagger}_{0} = -\frac{1}{\sqrt{3}}\left(B^\dagger_{xx} + B^\dagger_{yy} + B^\dagger_{zz}\right), \tag{102}$$

$$B^{2\dagger}_{-2} = \frac{1}{2}\left(B^{\dagger}_{xx} - B^{\dagger}_{yy} - 2\mathbf{i}B^{\dagger}_{xy}\right), \tag{103}$$

$$B^{2\dagger}_{-1} = B^{\dagger}_{xz} - \mathbf{i}B^{\dagger}_{yz}, \tag{104}$$

$$B^{2\dagger}_{0} = -\frac{1}{\sqrt{6}}\left(B^{\dagger}_{xx} + B^{\dagger}_{yy}\right) + \sqrt{\frac{2}{3}}B^{\dagger}_{zz}, \tag{105}$$

$$B^{2\dagger}_{1} = -B^{\dagger}_{xz} - \mathbf{i}B^{\dagger}_{yz} \tag{106}$$

$$B^{2\dagger}_{2} = \frac{1}{2}\left(B^{\dagger}_{xx} - B^{\dagger}_{yy}\right) + \mathbf{i}B^{\dagger}_{xy}. \tag{107}$$

If we compare these equations with the Equations (22a–22d) of Ref. [28], we observe that

$$B^{0\dagger}_{0} = -\sqrt{2}\mathcal{B}^{\dagger}_{0}, \qquad B^{2\dagger}_{M} = \sqrt{2}\mathcal{B}^{2\dagger}_{M}. \tag{108}$$

In accordance with the definitions (72) and (75), the new boson operators are

$$s^{\dagger} = -\sqrt{\frac{3}{4N_0}}B^{0\dagger}_{0}, \qquad d^{\dagger}_{M} = \sqrt{\frac{3}{4N_0}}B^{2\dagger}_{M}. \tag{109}$$

In the large N_0 limit, these operators satisfy the boson commutation relations approximately, since the commutator of Equation (87) is identical with the commutator (67).

For brevity, we label the bosons as $b^{L\dagger}_{M}$:

$$b^{0\dagger}_{0} = s^{\dagger}, \qquad b^{2\dagger}_{M} = d^{\dagger}_{M}. \tag{110}$$

The commutators among them in the large N_0 limit are (see Equations (4.4a), (4.4b) of Ref. [27])

$$[b^{L\dagger}_{M}, b^{L'\dagger}_{M}] = [b^{L}_{M}, b^{L'}_{M'}] = 0, \tag{111}$$

$$[b^{L}_{M}, b^{L'\dagger}_{M'}] = \delta_{LL'}\delta_{MM'}. \tag{112}$$

Therefore ,we can use these boson operators to construct a $U(6)$ algebra with generators of the type:

$$b^{L\dagger}_{M}b^{L'}_{M'} \tag{113}$$

where $L, L' = 0, 2$.The algebraic structure of this model is identical with that of the Interacting Boson Model (IBM) of Arima and Iachello [8], in the sense that both models have a $U(6)$ algebra and three limiting symmetries (the $O(6)$, the $U(5)$ and the $SU(3)$ limits). So this $U(6)$ Boson Model has an $SU(3)$ subalgebra, just like there is an $SU(3)$ limit for the IBM. The main difference with the IBM is that the bosons of this $U(6)$ Boson Model come from symmetric pairs of harmonic oscillator quanta, which derive from pairs of nucleons in the same Shell Model orbital.

The number operator in our case is the number of the pairs of quanta:

$$N_b = \sum_{L,M} b^{L\dagger}_{M}b^{L}_{M} = \frac{\sum_{i=1}^{A}(n_{iz} + n_{ix} + n_{iy})}{2}. \tag{114}$$

Let the round ket $|N_{b^L_M})$ represent the state with a certain number $N_{b^L_M}$ of symmetric pairs of quanta with angular momentum L and projection M, deriving from pairs of nucleons in the same orbit for medium mass or heavy nuclei with $N_0 \gg 1$. If the eigenvalue of $s^\dagger s$ is N_s and of $d^\dagger_M d_M$ is N_{d_M}, then:

$$N_b = N_s + \sum_M N_{d_M}. \tag{115}$$

7. The Coherent States

The coherent states were introduced by Ginocchio and Kirson in Refs. [16,17] in order to link the IBM with the Collective Model of Bohr and Mottelson [72]. The authors defined the boson creation operator as [16]:

$$Q^\dagger(\beta,\gamma) = \frac{1}{\sqrt{1+\beta^2}}\left(s^\dagger + \beta\cos\gamma d^\dagger_0 + \frac{1}{\sqrt{2}}\beta\sin\gamma(d^\dagger_2 + d^\dagger_{-2})\right), \tag{116}$$

where β is the quadrupole deformation variable, while γ is an angle, which shows the kind of deformation (prolate, oblate and spherical). Accordingly, the coherent state is defined as [17]

$$|N_b;\beta,\gamma\rangle = \frac{1}{\sqrt{N_b!}}[Q^\dagger(\beta,\gamma)]^N_b|0). \tag{117}$$

In the $SU(3)$ limit of the IBM, the deformation is equal to $\beta = \sqrt{2}$, while the angle γ may adopt any value (see Chapter 13 of Ref. [73]) and N is the number of bosons, which approaches the infinity in the coherent states of the $SU(3)$ limit of the IBM. Luckily, in this $U(6)$ Boson Model, the number of quanta and of the pairs of quanta is $N_b \gg 1$, since $N_0 \gg 1$ for medium mass and heavy nuclei.

The corresponding shape for selected values of the γ is presented in the book by Greiner and Maruhn (see Figure 6.4 of Ref. [74]). Specifically:

$$\begin{aligned}
\gamma &= 0^\circ, \text{ prolate with } x = y, \\
\gamma &= 60^\circ, \text{ oblate with } x = z, \\
\gamma &= 120^\circ, \text{ prolate with } y = z, \\
\gamma &= 180^\circ, \text{ oblate with } x = y, \\
\gamma &= 240^\circ, \text{ prolate with } x = z, \\
\gamma &= 300^\circ, \text{ oblate with } y = z.
\end{aligned}$$

We use the findings of Ginocchio and Kirson to exhibit that the wave functions of the $SU(3)$ limit of this $U(6)$ Boson Model are coherent states with the correct values of β, γ.

8. The SU(3) States

In the following, some basic examples are demonstrated. The $SU(3)$ states consist of symmetric pairs of quanta in the same Cartesian axis [45]. Due to this property, we demonstrate the wave functions of a pair of quanta in the z-, in the x- and in the y-axis.

8.1. Two Quanta in the z-Axis

Suppose a state with two quanta in the z-axis. The many-quanta $SU(3)$ wave function of this irrep is represented by the Young tableaux:

$$\boxed{\mathbf{z}}\boxed{\mathbf{z}} \quad \boxed{1}\boxed{2}. \tag{118}$$

The quanta 1, 2 are symmetric upon their interchange, so the spatial wave function is

$$\Phi_{zz} = a^{\dagger}_{iz} a^{\dagger}_{i'z} |0\rangle. \tag{119}$$

With the use of the operators of Equation (13), the above wave function is written as

$$\Phi_{zz} = \eta_{i0} \eta_{i'0} |0\rangle. \tag{120}$$

The operators $\eta_{i0}, \eta_{i'0}$ can be written in terms of the $B_0^{0\dagger}$, $B_0^{2\dagger}$ operators of Equations (90), (93) as

$$\eta_{i0}\eta_{i'0} = -\frac{1}{\sqrt{3}} B_0^{0\dagger} + \sqrt{\frac{2}{3}} B_0^{2\dagger}. \tag{121}$$

So, Equation (120) is equal to

$$\Phi_{zz} = \frac{1}{\sqrt{3}} \left(-B_0^{0\dagger} + \sqrt{2} B_0^{2\dagger} \right) |0\rangle. \tag{122}$$

Now, if we substitute Equation (109) and renormalize the wave function, we obtain

$$\Phi_{zz} = \frac{1}{\sqrt{3}} \left(s^{\dagger} + \sqrt{2} d_0^{\dagger} \right) |0\rangle. \tag{123}$$

In the Elliott wave functions, the band label K is equal to the projection of the angular momentum M. Practically, this means that the Cartesian wave function of Equation (119) *projects* into a wave function with good angular momentum L, so as a nuclear state with $L = 0, K = 0$ (the s boson) is included into the Cartesian wave function with probability $\frac{1}{3}$ and an $L = 2, K = 0$ (the d_0 boson) nuclear state lies within the Cartesian wave function with probability $\frac{2}{3}$. This procedure is called *L-projection* and was introduced by J. P. Elliott in 1958 in Ref. [2]. The matrix elements of the projection operator were calculated in 1968 by J. D. Vergados in Ref. [75].

The L-projection of the spatial many-quanta Cartesian wave function using the traditional method of Elliott, Harvey and Vergados [2,3,75] is now presented briefly. The matrix elements of the projection operator within the same K nuclear band are

$$A(K, L, K) = \langle \Phi | P | \Phi \rangle = |a(K, L)|^2. \tag{124}$$

The coefficients $a(K, L)$ are given in Table 2A of Ref. [75]. The $SU(3)$ irrep of our example is the (λ, μ)= $(2, 0)$, and so:

$$a(K = 0, L = 0) = \frac{1}{\sqrt{3}}, \tag{125}$$

$$a(K = 0, L = 2) = \sqrt{\frac{2}{3}}. \tag{126}$$

So, with the use of the symmetric pairs of quanta (the s, d bosons) and with the use of the Elliott, Harvey and Vergados method, we obtained the *same* projection coefficients for the Φ_{zz} state. This successful result indicates that the method we used for the construction of the spatial Shell Model $SU(3)$ state gave consistent results with the relevant bibliography [75].

To further test the interpretation of the s, d bosons as symmetric pairs of quanta, we compare the Φ_{zz} state with the coherent states of Ginocchio and Kirson [16]. The Equation (123) can be written as

$$\Phi_{zz} = \frac{1}{\sqrt{1+\beta^2}} \left(s^{\dagger} + \beta \cos\gamma d_0^{\dagger} + \frac{1}{\sqrt{2}} \beta \sin\gamma \left(d_2^{\dagger} + d_{-2}^{\dagger} \right) \right) |0\rangle \tag{127}$$

with $\beta = \sqrt{2}$ and $\gamma = 0°$. Thus, it is identical with the coherent state of Equation (117). Therefore, the Φ_{zz} state represents a coherent state of the $SU(3)$ limit of the $U(6)$ Boson Model (since $\beta = \sqrt{2}$) with prolate shape (since $\gamma = 0°$).

Consequently, the method we used for the construction of the Φ_{zz} state out of the symmetric pairs of quanta not only gave the correct projection coefficients in comparison with those of Vergados but also proved that the intrinsic Elliott state with $(\lambda, \mu) = (2, 0)$ is a coherent state with the correct value of the deformation variable for the $SU(3)$ limit ($\beta = \sqrt{2}$) and the correct value of the angle $\gamma = 0°$.

8.2. Two Quanta in the x-Axis

Using the same method, we can construct the state of a symmetric pair of quanta in the x-axis:

$$\Phi_{xx} = a^{\dagger}_{ix} a^{\dagger}_{i'x} |0). \tag{128}$$

This state results to be

$$\Phi_{xx} = \frac{1}{\sqrt{3}} \Big(- B_0^{0\dagger} - \frac{\sqrt{2}}{2} B_0^{2\dagger} + \frac{\sqrt{3}}{2} (B_{-2}^{2\dagger} + B_2^{2\dagger}) \Big) |0). \tag{129}$$

If we substitute one more into the Equation (109) and renormalize the wave function, we obtain that

$$\Phi_{xx} = \frac{1}{\sqrt{3}} \Big(s^{\dagger} - \frac{\sqrt{2}}{2} d_0^{\dagger} + \frac{\sqrt{3}}{2} (d_{-2}^{\dagger} + d_2^{\dagger}) \Big) |0). \tag{130}$$

In the Elliott L-projected wave functions $\psi(KLM)$ [2], two opposite K states are equal: $\psi(KLM) = \psi(-KLM)$. Thus, the $d_{-2}^{\dagger}|0)$, $d_2^{\dagger}|0)$ refer to the same nuclear state. All these mean that the wave function (130) reflects to two nuclear bands with $(K, L) = (0, 0), (0, 2)$ and $(K, L) = (2, 2)$. In Equation (130) the state with $K = 0, L = 0$ appears with probability $1/3$, the $K = 0, L = 2$ with probability $1/6$ and the $K = 2, L = 2$ with probability $1/4 + 1/4 = 1/2$. The quanta in the x, y plane are responsible for the μ quantum number (see Equation (15) of Ref. [2]). For two quanta in the x-axis, we obtain that $\mu = 2$ and that $K = 0, 2$ (see Equation (22) of Ref. [2]). So, Equation (130) correctly predicts the existence of two bands with $K = 0, 2$.

Furthermore, Equation (130) can be written as

$$\Phi_{xx} = \frac{1}{\sqrt{1+\beta^2}} \Big(s^{\dagger} + \beta \cos\gamma d_0^{\dagger} + \frac{1}{\sqrt{2}} \beta \sin\gamma (d_2^{\dagger} + d_{-2}^{\dagger}) \Big) |0) \tag{131}$$

with $\beta = \sqrt{2}$ and $\gamma = 120°$. In comparison with Equation (117), the Φ_{xx} represents a coherent state with prolate shape. So, it represents a prolate shape with equal lengths in the z, y axes, as expected.

8.3. Two Quanta in the y-Axis

If the $SU(3)$ wave function has two quanta in the y Cartesian axis, the wave function is

$$\Phi_{yy} = a^{\dagger}_{iy} a^{\dagger}_{i'y} |0), \tag{132}$$

which becomes

$$\Phi_{yy} = \frac{1}{\sqrt{3}} \Big(s^{\dagger} - \frac{\sqrt{2}}{2} d_0^{\dagger} - \frac{\sqrt{3}}{2} (d_{-2}^{\dagger} + d_2^{\dagger}) \Big) |0). \tag{133}$$

Equation (133) represents once more a coherent state of the $SU(3)$ limit, as defined in Equation (117), with $N = 1$, $\beta = \sqrt{2}$, $\gamma = 240°$. Therefore, it represents a prolate shape with equal lengths in the z, x axes.

8.4. The General U(3) Wave Function

We may generalize now and write down an arbitrary $U(3)$ wave function, which consists of symmetric pairs of quanta derived by nucleons in the same orbit of a medium mass or heavy nucleus:

$$\Phi_{space} = \left(a_y^\dagger a_y^\dagger\right)^{\frac{N_y}{2}} \left(a_x^\dagger a_x^\dagger\right)^{\frac{N_x}{2}} \left(a_z^\dagger a_z^\dagger\right)^{\frac{N_z}{2}} |0\rangle, \tag{134}$$

Just like in the IBM, the number of bosons is doubled to give the ground state band $SU(3)$ irreps; similarly, in this scheme, the general $SU(3)$ irrep is

$$(\lambda, \mu) = \left(2\left(\frac{N_z}{2} - \frac{N_x}{2}\right), 2\left(\frac{N_x}{2} - \frac{N_y}{2}\right)\right), \tag{135}$$

where we used that $N_k = \sum_i n_{ik}$, that $\frac{N_k}{2}$ is the number of bosons, i.e., symmetric pairs of quanta in the same Cartesian axis in this $U(6)$ Boson Model, and that $N_z \geq N_x \geq N_y$. The conclusion is that in this $U(6)$ Boson Model, the $SU(3)$ irreps are identical to those of the Elliott $SU(3)$ symmetry:

$$(\lambda, \mu) = (N_z - N_x, N_x - N_y). \tag{136}$$

So, the irreps of the $SU(3)$ limit of this $U(6)$ Boson Model can be found through the proxy-SU(3) symmetry [29] for the spin–orbit-like shells among magic numbers 28, 50, 82, . . . of medium mass and heavy nuclei.

Note that, in this procedure, we mapped a pair of bosons (the quanta) into an approximate boson (the s, d operators). We did not map a pair of fermions (the nucleons) into a boson, as was performed in the OAI mapping of the IBM. Therefore, we did not perform a boson mapping, and so no spurius states emerged in our case.

9. Conclusions

A $U(6)$ Boson Model was introduced, which treats as s, d bosons the symmetric pairs of quanta of the valence protons (neutrons) of the same Shell Model orbital. This model is valid in medium mass and heavy nuclei. The introduction of the s, d operators was based on the commutation relations of the $U(3)$ Boson Model. The algebraic structure of this model is identical with that of the Interacting Boson Model, and so it possesses an $SU(3)$ limit, which has the same $SU(3)$ irreps as those of the Elliott Model. Since the Elliott $SU(3)$ symmetry is broken in medium mass and heavy nuclei, due to the strong spin–orbit interaction, the proxy-SU(3) symmetry [29–31] can be used to identify the $SU(3)$ irreps of the $SU(3)$ limit of this $U(6)$ Boson Model.

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