ALGEBRA AND GEOMETRY OF DIRAC'S MAGNETIC MONOPOLE

by

Graham Matthew Kemp

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Certificate of originality

This is to certify that I am responsible for the work submitted in this thesis, that the original work is my own except as specified in acknowledgments or in footnotes, and that neither the thesis nor the original work contained therein has been submitted to this or any other institution for a degree.

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Abstract

This thesis is concerned with the quantum Dirac magnetic monopole and two classes of its generalisations.

The first of these are certain analogues of the Dirac magnetic monopole on coadjoint orbits of compact Lie groups, equipped with the normal metric. The original Dirac magnetic monopole on the unit sphere S^2 corresponds to the particular case of the coadjoint orbits of $SU(2)$. The main idea is that the Hilbert space of the problem, which is the space of L^2-sections of a line bundle over the orbit, can be interpreted algebraically as an induced representation. The spectrum of the corresponding Schödinger operator is described explicitly using tools of representation theory, including the Frobenius reciprocity and Kostant's branching formula.

In the second part some discrete versions of Dirac magnetic monopoles on S^2 are introduced and studied. The corresponding quantum Hamiltonian is a magnetic Schödinger operator on a regular polyhedral graph. The construction is based on interpreting the vertices of the graph as points of a discrete homogeneous space G/H, where G is a binary polyhedral subgroup of $SU(2)$. The edges are constructing using a specially selected central element from the group algebra, which is used also in the definition of the magnetic Schrödinger operator together with a character of H. The spectrum is computed explicitly using representation theory by interpreting the Hilbert space as an induced representation.

Keywords: magnetic monopole, induced representation, coadjoint orbit, regular graph.

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Chapter 1

Introduction

The Dirac magnetic monopole is one of the most remarkable and one of the first integrable systems of quantum mechanics. In his pioneering paper [11] Dirac showed that an isolated magnetic charge q should be quantized: $q \in \mathbb{Z}$. The corresponding Schrödinger equation was solved by Tamm [52] while he was visiting Dirac in Cambridge in 1931. The main theoretical implication of the quantization of magnetic charge is should magnetic monopoles exist then this forces the quantization of electric charge and the quantization of electric charge is unexplained.

Thus the game for experimenters was to try and detect magnetic monopoles, but this was played without success. Meanwhile, theoreticians moved on and the theory of magnetic monopoles lay largely dormant. However, a series of papers [57], [58], [59] by Wu and Yang sparked something of a revival in Dirac's original idea by explaining Dirac's monopole 'without strings'. The point being that Dirac and Tamm had described the wavefunction of an electron in the field of a monopole, but found that it was singular (and hence not-defined) along a half-line — this half-line is now known as the Dirac string. This is something of a paradox when compared to the actual physical situation, which is manifestly spherically-symmetric about the monopole.

The global nature of the wavefunctions was understood only in 1976 by Wu and Yang [57], who explained that the corresponding eigenfunctions of the Schrödinger equation (known as *monopole harmonics*) are sections of the complex line bundle L over S^2 whose first Chern class is q. In describing them using overlapping coordinate charts, the eigenfunctions can be defined globally without singularity and recovering the spherical symmetry of the problem. For this reason, Dirac's Magnetic Monopole is a remarkable case study — since it shows that many of the concepts learned in differential geometry are completely natural.

It is worth mentioning that different monopoles have been investigated, which approximate Dirac's at long range but have a different structure nearby: these are known as non-abelian monopoles and a survey is given in [9].

We now describe the geometry of Dirac's monopole in more detail: consider such a magnetic charge of strength q situated at the origin in $\mathbb{R}^3$. The magnetic field it generates is radially symmetric and given by Coulomb's law as

$$\mathbf{B} = q\frac{\mathbf{r}}{r^3}$$

and it can be represented pictorially as in Figure 1.1, with the field strength being constant at a constant distance from the monopole.

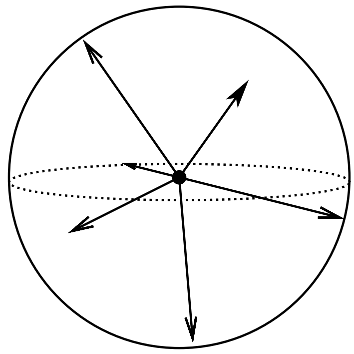

Figure 1.1: Radially symmetric magnetic field lines due to a monopole in $\mathbb{R}^3$.

On the classical level, the Dirac magnetic monopole is also described well in terms of differential geometry — the classical phase space of an electron orbiting a monopole at constant distance is given by the symplectic manifold

$$\left(T^*S^2, dp \wedge dx + \frac{q}{2}\pi^*(d\mathbf{S})\right), \tag{1.1}$$

where $d\mathbf{S}$ is the area form on S^2 and $dp \wedge dx$ is the canonical form on T^*S^2.

If we consider a sphere S^2 centred on the monopole then Gauss' law gives the magnetic flux through the sphere as

$$\Phi = \int_{S^2} \mathbf{B} \cdot d\mathbf{S} = 4\pi q.$$

Alternatively, one could use a magnetic potential $\mathbf{A}$, satisfying $\mathbf{B} = \nabla \times \mathbf{A}$ and use Stokes' Theorem separately on each hemisphere to give the magnetic flux as

$$\Phi = \int_{S^2} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{S^1} \mathbf{A} \cdot d\mathbf{l} = \oint_{S^1} (\mathbf{A}_N - \mathbf{A}_S) \cdot d\mathbf{l} = 4\pi q,$$

where $\mathbf{A}_N$ and $\mathbf{A}_S$ are the magnetic potentials in the Northern and Southern hemispheres respectively and S^1 is the equator. That this integral does not vanish leads to the fact that the magnetic potentials on each hemisphere are not given by the same expression and so $\mathbf{A}$ is not given by the same expression over the whole of S^2.

There is no contradiction here if one divides the space outisde the monopole into two overlapping regions U_a and U_b and defines a vector potential $(A_\mu)_a$ in U_a and $(A_\mu)_b$ in U_b. Using spherical coordinates r, θ, ϕ with the monopole at the origin, we set

$$\begin{aligned} U_a : \qquad & 0 \leq \theta < \frac{\pi}{2} + \delta, \qquad 0 < r, \qquad 0 \leq \phi < 2\pi \\ U_b : \qquad & \frac{\pi}{2} - \delta < \theta < \pi, \qquad 0 < r, \qquad 0 \leq \phi < 2\pi, \end{aligned}$$

with δ such that $0 < \delta \leq \frac{\pi}{2}$. The *Wu–Yang potential* is then

$$\begin{aligned} (A_r)_a =& (A_\theta)_a = 0, \qquad (A_\phi)_a = q\frac{1 - \cos\theta}{r \sin\theta} \\ (A_r)_a =& (A_\theta)_a = 0, \qquad (A_\phi)_a = -q\frac{1 + \cos\theta}{r \sin\theta} \end{aligned}$$

where A_r, A_θ, A_ϕ are the projections of $\mathbf{A}$ in the three local orthogonal directions. The two half-lines of singularity at $\theta = \pi$ and $\theta = 0$ are known in the physics literature as *Dirac strings* and necessarily arise if one tries to represent $\mathbf{A}$ by a single expression. On the overlap of U_a and U_b, the difference between the $\mathbf{A}_a$ and $\mathbf{A}_b$ is the gradient of a function

$$\mathbf{A}_a - \mathbf{A}_b = \frac{2q}{\sin\theta}\hat{e}_\phi = \nabla(2q\phi). \tag{1.2}$$

Dirac considered the interaction of the monopole with an electron of charge e. The wavefunction $\psi(x, t)$ of the electron must satisfy the Schrödinger equation

$$\frac{1}{2}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2\psi = i\hbar\frac{\partial\psi}{\partial t},$$

where $p_j = -i\hbar\partial_j$ are the components of $\mathbf{p}$ in Cartesian coordinates. Corresponding to the two potentials $\mathbf{A}_a$ and $\mathbf{A}_b$ there are two solutions ψ_a and ψ_b in the different coordinate charts. The transformation between these solutions is given by

$$\psi_a = \psi_b \cdot \exp\left(\frac{2iqe}{\hbar c}\phi\right), \tag{1.3}$$

where $\exp\left(\frac{2iqe}{\hbar c}\phi\right)$ is the transition function from Southern to the Northern hemisphere. Requiring this function to be single valued gives that

$$\frac{2eq}{\hbar c} \in \mathbb{Z},$$

i.e. that magnetic charge is quantized. This result was used by Dirac to infer that the existence of a single magnetic monopole in the universe would explain why electric charge is quantized. Whilst, from a logical point of view, the fact that electric charge is quantized does not of course imply the existence of a magnetic monopole it is somehow bewitching because one somehow feels like the theory is correct but with no facts to back it up.

It is clear from equations (1.2) and (1.3) that the wavefunctions ψ are sections of a complex line bundle $L \to S^2$ and the magnetic potential $\mathbf{A}$ is essentially the connection one-form of that bundle. Pursuing this line of thought, one can compute the curvature Ω of $\mathbf{A}$ to find that the magnetic charge q can be identified with the first Chern class of the bundle L, which is necessarily integer-valued. Indeed,

$$\Omega = -\frac{ie}{\hbar c}\sum_{i<j}\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right) dx^i \wedge dx^j$$

and so the first Chern number of L is given by

$$\begin{aligned} c_1 =& \frac{i}{2\pi}\int_{S^2}\Omega = \left(\frac{i}{2\pi}\right)\left(-\frac{ie}{\hbar c}\right)\int_{S^2} d\mathbf{A} \\ =& \frac{e}{2\pi\hbar c}\oint_{S^1}(\mathbf{A}_N - \mathbf{A}_S)\cdot d\mathbf{l} \\ =& \frac{2eq}{\hbar c} \in \mathbb{Z}. \end{aligned}$$

To make the mathematical considerations clearer, from here onwards all non-essential physical constants will be set to be 1, i.e. $e = 1, \hbar = 1$ and $c = 1$.

It is remarkable that the different magnetic charges are essentially classified by homotopy classes of maps $\pi_1(S^1)$ that describe principal $U(1)$-bundles over S^2 with different Chern numbers: bundles of this kind are called Hopf bundles after the seminal work of Hopf [22] in 1931 — the same year as Dirac's paper [11]. However, the relation between the two works was not noticed until much later.

Wu and Yang's description of the wavefunction of the electron as a section of a line bundle is highly illuminating from the geometric viewpoint. However, in solving the corresponding Schrödinger equation and finding the corresponding eigenfunctions, it is best to describe the Dirac magnetic monopole as an algebraic object, namely an induced representaion — it is this idea that forms the basis for this thesis.

More precisely: if G and H are two groups, with H a subgroup of G, then it is clear that any representation of G gives a representation of H by restriction. There is a dual notion of induction, due to Frobenius, which takes a representation ρ of H on a vector space W and forms a representation $\mathrm{ind}_H^G(\rho)$ of G. The different manifestations that this construction takes will be explained at differnt places in the thesis. For the case of Lie groups, the induced representation of G is given on the space of sections of a vector bundle over G/H, with the fibre over a point being isomorphic to the representation space W. If W is one-dimensional then the corresponding vector bundle is a line bundle. First, we explain how this construction gives us the same line bundles that were considered by Wu and Yang.

Chapter Two sets the tone for the whole thesis by explaining how the quantum problem of the Dirac magnetic monopole on a sphere, as considered by Wu and Yang, may be interpreted in terms of representation theory. This chapter is based on work that appeared in [25].

The starting point for the work in this chapter was a calculation by Novikov and Schmeltzer [43] of the coadjoint orbits of the Euclidean group of motions $E(3) \cong SO(3) \ltimes \mathbb{R}^3$. Denote by $e(3)$ the Lie algebra of $E(3)$: it has basis $l_1, l_2, l_3, p_1, p_2, p_3$, where l and p are generators of rotations and translations respectively.

The dual space $e(3)^*$ with the coordinates $\{l_1, l_2, l_3, p_1, p_2, p_3\}$ has the canonical Poisson bracket

$$\{l_i, l_j\} = \epsilon_{ijk} l_k, \quad \{l_i, p_j\} = \epsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0.$$

The symplectic leaves of this Poisson bracket are the coadjoint orbits of $E(3)$, which are the level sets of the Casimir functions

$$C_1 := (p, p) = R^2, \qquad C_2 := (l, p) = \alpha R,$$

where $\alpha \in \mathbb{R}$ and $R \in \mathbb{R}^+$. By introducing the variables

$$\sigma_i = l_i - \frac{\alpha}{R} p_i$$

the authors identify the coadjoint orbits with $T^*S^2 \subset T^*\mathbb{R}^3$

$$(p, p) = R^2, \qquad (\sigma, p) = 0,$$

where TS^2 and T^*S^2 have been identified using the standard Riemannian metric on the radius R sphere. The authors show that the canonical symplectic form on the orbits is given by $dp \wedge dx + \alpha\pi^*(d\mathbf{S})$, where $d\mathbf{S}$ is the area form on S^2. Therefore the coadjoint orbits may be naturally identified with the classical phase space of an electron orbiting a non-quantized Dirac magnetic monopole given in (1.1).

The prequantization condition of geometric quantization gives us exactly Dirac's quantization condition that $q = 2\alpha \in \mathbb{Z}$. We also discover that the quantum version of the new coordinates σ_i have a natural interpretation as covariant derivatives acting on the space of sections of the line bundle over S^2 with Chern class q. Equally important is noticing that the space of sections of this line bundle may be identified as a representation of $SU(2)$. (The appearance of $SU(2)$ stems from $SU(2)$ being a double covering of $SO(3) \subset E(3)$, see Appendix A.1 for details.)

To make this clearer, note that S^2 can be considered as the base space of the principal fibre bundle $SU(2) \to S^2$ with fibre $U(1)$. The induced representation construction then allows us to construct a representation of $SU(2)$ from a representation of $U(1)$. Geometrically, the representation space will be exactly the space of sections of a line bundle over S^2 and it turns out that the magnetic charge q is obtained from nothing but the character of $U(1)$ given by $\exp[i\theta] \mapsto \exp[-iq\theta]$. This then gives another interpretation of the quantization of magnetic charge — for this character to be well-defined we must have that $q \in \mathbb{Z}$.

The tools of representation theory, in particular the Frobenius Reciprocity Theorem, then allow us to decompose the space of sections of this line bundle into irreducible representations of $SU(2)$. Computing the spectrum of the corresponding Schrödinger equation (which reproduces the answer given in [52], [57] and [15]) is essentially a corollary of this.

Chapter Three can be thought of as a broad generalization of Chapter Two. We started in Chapter Two from the calculation of Novikov and Schmeltzer that the regular coadjoint orbits of $E(3)$ may be identified with the classical phase space of an electron orbiting a magnetic monopole at a constant radius. The appearance of the 'magentic term' in the symplectic form was unexpected and remarkable and led to a flurry of work (see the references in [5]) in the investigation of the classical dynamics of what are now called magnetic cotangent bundles — namely, symplectic manifolds of the form $(T^*M, dp \wedge dx + \pi^*(\omega))$, where $\pi^*(\omega)$ is the pullback to T^*M of a closed form on T^*M. In particular, this classical system was investigated for the case that M is a coadjoint orbit $\mathcal{O}(a)$ of a compact Lie group G in [4], [5] and [14].

Coadjoint orbits of compact Lie groups are classes of manifolds with a very rich geometry — on the topological level they are known as generalized flag manifolds — and are an ideal case study for looking for analogues of the Dirac magnetic monopole on S^2, because their second cohomology group (which classifies line bundles) is non-trivial. In Chapter Two we look to apply geometric quantization to the classical system of a free particle on a magnetic cotangent bundle to a coadjoint orbit that was considered in [4], [5] and [14].

By identifying $\mathcal{O}(a) \cong G/G_a$, where G_a is the stabilizer of the point a, we can describe the analogues of magnetic charge in this situation. In Dirac's case the magnetic charge was just a real number q. The situation here is more delicate, if $\mathfrak{g}_a$ is the Lie algebra of G_a then the analogue of magnetic charge is given by a character of $\mathfrak{g}_a$, i.e. a map $f : \mathfrak{g}_a \to \mathbb{R}$ such that $f([X, Y]) = 0$.

In Dirac's situation there was a quantization condition that $q \in \mathbb{Z}$. A similar situation persists here, in that the quantization condition is that f should in fact give a well-defined character χ_f of the Lie group G_a under the rule

$$\chi_f(\exp(X)) = \exp[if(X)].$$

For this to be well-defined, it is necessary that f should belong to the lattice $L \subset \mathfrak{g}_a$, whose elements take values in $2\pi\mathbb{Z}$ when applied to any element of $\exp^{-1}(e)$. If $G = SU(2)$, this exactly corresponds to the case considered by Dirac.

Even more of the method from Chapter Two carries over: it turns out that the analogue of the wavefunction of an electron should again live in the representation of G that is induced from χ_f, which will be on the space of sections of a line bundle over $\mathcal{O}(a) \cong G/G_a$. This space can be decomposed into irreducible representations of G using the Kostant Branching Formula and the Frobenius Reciprocity Formula and the spectrum of the corresponding Schrödinger equation can be written in terms of the Kostant formula.

It is worth mentioning as well that the most natural Schrödinger operator to consider corresponds exactly to the Bochner Laplacian, which is a second-order self-adjoint differential operator acting on sections of vector bundles over a Riemannian manifold, see e.g. [54]. This operator is an extension of the classical Laplace–Beltrami operator that acts on functions on a Riemannian manifold and is of interest to geometers. Specific calculations of the spectrum may be done for individual coadjoint orbits, with the spectrum being easiest to compute when the coadjoint orbit is maximally degenerate. When $G = SU(n)$, such calculations were done to find the spectrum of the Bochner Laplacian acting in line bundles over complex Grassmannians in [21]. We note that the branching rules calculated there may be obtained in a different way using results of [44].

Chapter Four is in a slightly different vein to Chapters Two and Three. In Chapters Two and Three we looked at the quantization of some classical phase spaces that can be identified with that of Dirac monopoles on coadjoint orbits of a compact Lie group G. The procedure of geometric quantization identifies from the classical phase space a Hilbert space and it turns out that this Hilbert space can be identified with an induced representation of G. For each character of G_a we can induce a representation of G and this can be associated with a classical phase space that corresponds to a Dirac magnetic monopole.

The question we address in Chapter Four is, given two finite groups $H \subset G$: 'does it still make sense to describe a representation of G induced from a character of H in terms of a Dirac magnetic monopole?' It turns out that the answer is yes! In some sense this is an inverse problem to that considered in Chapters Two and Three, where we arrived at the induced representation as the algebraic answer to a geometric question — here we start with the algebraic answer and try to discover the geometry.

For finite groups $H \subset G$ the coset space G/H is not a manifold in any meaningful way — it is really just a collection of points, which is in marked contrast to when H and G are Lie groups. To try and impose some geometry on the coset space we look to draw a graph whose vertices are the elements of G/H, this is a natural thing to do because a graph can be thought of as the discrete analogue of a Riemannian manifold. To draw a graph Γ_K we act formally on G/H by certain elements K in the centre of the group algebra of G, which we call Casimir elements and which are formed by taking the formal sum of each element in a conjugacy class. Under certain conditions on the Casimir element (namely that the entries in the character table of G of the corresponding conjugacy class are real, in which case we call it a real Casimir) the graph generated is regular, i.e. the local structure of each vertex looks the same as any other. If we assume that G acts transitively on Γ_K then the adjacency matrix of Γ_K is essentially given by the matrix of K acting in the representation of G induced from the trivial representation of H.

A magnetic field on a graph is given by associating to each oriented edge $[x, y]$ an element $\exp[i\alpha_{xy}]$ such that $\exp[i\alpha_{yx}] = \exp[-i\alpha_{xy}]$, with $\alpha_{xy} \in \mathbb{R}$. The notion of magnetic fields (and indeed arbitrary gauge fields) on lattices has been around for some time — physicists have been studying this since the 1950s (see [38] for a review) and there are two interpretations that can be placed on the lattice. Firstly, the lattice sites can be viewed as atoms in a crystalline solid, with the edges corresponding to electron bonds between the atoms. Alternatively, the lattice points can be viewed as a discretization of space, with the continuous Laplacian being replaced by a finite difference operator. However, the extension of these ideas to arbitrary graphs seems to be a relatively new development, with [38] being one of the first and more recently [10] and [45] contain interesting results. All of the papers cited here use analysis to derive results about general classes of graphs with magnetic fields. However, we take the opposite view: we are concerned with graphs of a special type equipped with special magnetic fields. This can be taken in the same vein as looking at coadjoint orbits rather than arbitrary manifolds. Indeed, the work in this chapter seems closer in spirit to that of Manton [39], who explained the differential geometry of discrete principal fibre bundles, i.e. where the total space is discrete. The objects described in this chapter may be thought of as the associated vector bundle analogue of Manton's construction. He also gives a definition of the Chern number of such a bundle, which we take as the description of magnetic charge.

We define a discrete Dirac magnetic monopole on a regular graph by the following general construction: we act with a real Casimir K of G on a representation of G that is induced from a non-trivial character of a subgroup H. The matrix of K in the induced representation can essentially be taken to be a magnetic adjacency matrix for the graph Γ_K and the magnetic field on the graph has many properties that the magnetic field due to a monopole has. Namely, consider for a moment the magnetic flux through a part of the surface of the sphere centred on a magnetic monopole — we see immediately that the flux contained is proportional to the area of the part of the surface of the sphere. In the discrete case, we have that the magnetic flux through each two cycles that are related by an element $g \in G$ is constant. The spectrum of the corresponding magnetic Laplacian can be obtained using the tools of representation theory.

We demonstrate this construction by trying to find magnetic monopoles on the graphs of the Platonic solids. This can be thought of as a discrete version of Dirac's monopole, since Platonic solids can be thought of as discrete approximations of S^2. We do this by taking $G \subset SO(3)$ to be the orientation preserving symmetry group of the solid in $\mathbb{R}^3$ and H to be the stabilizer of a vertex. This is non-trivial, since a priori there is no way of knowing if a Casimir element of G will generate the desired graph. It turns out that this is possible for the tetrahedron, the octahedron, the cube and the icosahedron, i.e. it is not possible for the dodecahedral graph.

By embedding the polyhedron in $\mathbb{R}^3$, we see that we should expect to find as many different magnetic charges as there are faces of the polyhedron, which we denote by n. This is because, since the total flux is an integer we should have that the flux through each face is the argument of an n^{th} root of unity.

For technical reasons, instead of the symmetry group G of the polyhedron we consider G^* the binary symmetry group of the polyhedron, which is a double cover of G. There are various reasons why G^* should be thought of as more fundamental than G, but here we use G^* instead of G because otherwise we miss half of the magnetic charges — namely those with odd number. This is analagous to the situation in quantum mechanics where instead of looking at the representation theory of $SO(3)$, one instead studies the representation theory of $SU(2)$ — leading to the notion of half-integer spin.

Chapter 2

Geometric quantization of the Dirac magnetic monopole

The considerations made in this chapter lay the foundation for the rest of the thesis. In addition to explaining the geometry of Dirac's magnetic monopole with wavefunctions being sections of a complex line bundle $L \to S^2$, Wu and Yang explicitly solved the Schrödinger equation and computed the spectrum of the corresponding wavefunctions to be

$$\lambda = \left[l(l+1) + |q| \left(l + \frac{1}{2} \right) \right], l = 0, 1, 2, \dots \text{ with degeneracy } \ 2l + |q| + 1. \quad (2.1)$$

A different derivation of this result in terms of integrable systems was given by Ferapontov and Veselov [15], who extended the classical factorisation method going back to Darboux and Schrödinger [51] to curved surfaces. This provides an explicit description of the monopole harmonics, facillitated by recursive application of the lowering operators to the ground states: under the isomorphism $S^2 \cong \mathbb{C}P^1 \cong \mathbb{C} \cup \infty$, the ground states for positive q are given by polynomials of degree $\leq q$.

The starting point of the work in this chapter was the calculation by Novikov and Schmelzer [43] of the canonical symplectic structure on the coadjoint orbits of the Euclidean group $E(3)$ of motions of $\mathbb{E}^3$, which showed the relation with the classical Dirac monopole. A similar calculation for Poincare and Galilean groups was done by Reiman [48], who also seems to have the idea of geometric quantization in mind, but did not pursue it.

The variables introduced by Novikov and Schmelzer have a natural quantum version as covariant derivatives acting on the space of sections $\Gamma(L)$ of the corresponding line bundle L. With this interpretation, the modification of the angular momentum in the presence of the Dirac magnetic monopole by Fierz [16] appears naturally.

Here a simple derivation of the spectrum of the Dirac monopole on a unit sphere is using geometric quantization is presented. It should be mentioned that geometric quantization of the Dirac magnetic monopole and related problems were already discussed in [41, 53], but the approach taken here is perhaps simpler and clearer.

For magnetic charge q, the space $\Gamma(L)$ is the representation space of the representation of $SU(2)$ induced from the character of $U(1) \subset SU(2)$ given by $z \to z^{-q}, z \in U(1)$. This space can be decomposed into irreducible representations of $SU(2)$ using the classical Frobenius Reciprocity Theorem [17] and the formula for the Dirac monopole spectrum (2.1) is a simple corollary of this.

2.1 Coadjoint orbits of the Euclidean group $E(3)$

Let $e(3)$ be the Lie algebra of the Euclidean group $E(3)$ of motions of $\mathbb{E}^3$. It has the basis $l_1, l_2, l_3, p_1, p_2, p_3$, where p and l are generators of translations and rotations (momentum and angular momentum) respectively.

The dual space $e(3)^*$ with the coordinates $\{l_1, l_2, l_3, p_1, p_2, p_3\}$ has the canonical Poisson bracket

$$\{l_i, l_j\} = \epsilon_{ijk} l_k, \quad \{l_i, p_j\} = \epsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0.$$

The regular symplectic leaves of this Poisson bracket are the coadjoint orbits of $E(3)$, which are the level sets of the Casimir functions

$$C_1 := (p, p) = R^2, \qquad C_2 := (l, p) = \alpha R.$$

Following Novikov and Schmelzer [43], introduce the variables

$$\sigma_i = l_i - \frac{\alpha}{R} p_i \tag{2.2}$$

to identify the coadjoint orbits with $T^*S^2 \subset T^*\mathbb{R}^3$

$$(p, p) = R^2, \qquad (\sigma, p) = 0,$$

where TS^2 and T^*S^2 have been identified using the standard Riemannian metric on the radius R sphere.

The new coordinates $\{\sigma_1, \sigma_2, \sigma_3, p_1, p_2, p_3\}$ have Poisson brackets

$$\{\sigma_i, \sigma_j\} = \epsilon_{ijk} \left(\sigma_k - \frac{\alpha}{R} p_k \right), \quad \{\sigma_i, p_j\} = \epsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0. \tag{2.3}$$

Novikov and Schmelzer computed the canonical symplectic form on the coadjoint orbits and showed that it is given by

$$\omega = \mathrm{d}p \wedge \mathrm{d}x + \frac{\alpha}{R^2} \pi^*(\mathrm{d}\mathbf{S}) \tag{2.4}$$

where $\mathrm{d}p \wedge \mathrm{d}x$ is the standard symplectic form on T^*S^2 and $\mathrm{d}\mathbf{S}$ is the area form on S^2 (see also [48]). As was pointed out in [43], the second term corresponds to the magnetic field of the (non-quantized) Dirac monopole:

$$\mathbf{B} = \frac{\alpha}{R^2}\,\mathrm{d}\mathbf{S}.$$

The value of the magnetic flux through a sphere around the monopole is defined by

$$q := \frac{1}{2\pi}\int_{S^2} \mathbf{B}$$

and is called the *charge* of the Dirac monopole. Dirac's *quantization condition* [11] is

$$q = \frac{1}{2\pi}\int_{S^2} \mathbf{B} = \frac{1}{2\pi}\int_{S^2} \frac{\alpha}{R^2}\mathrm{d}\mathbf{S} = 2\alpha \in \mathbf{Z}.$$

Comparing this with (2.4), we see that this is identical to the *geometric quantization* condition [26], (i.e. that the symplectic form should give 2π times an integer when integrated over any 2-cycle) which here yields

$$\frac{1}{2\pi}\int_{[S^2]} \omega = \frac{1}{2\pi}\int_{S^2} \frac{\alpha}{R^2}\mathrm{d}\mathbf{S} \in \mathbf{Z}.$$

2.2 Line bundles over S^2

It is convenient to use the scaled variables

$$x_i = p_i/R, \quad x^2 = x_1^2 + x_2^2 + x_3^2 = 1 \tag{2.5}$$

to work with the unit sphere S^2.

The quantum version of the Poisson brackets (2.3) are the following commutation relations (we are using the units in which Planck's constant $\hbar = 1$)

$$[\hat\sigma_k, \hat\sigma_l] = i\epsilon_{klm}(\hat\sigma_m - \alpha\hat x_m), \quad [\hat\sigma_k, \hat x_l] = i\epsilon_{klm}\hat x_m, \quad [\hat x_k, \hat x_l] = 0. \tag{2.6}$$

We are going to show now that the algebra generated by these elements has a natural representation on the space of sections of a certain line bundle over S^2.

Recall that a connection on a vector bundle E over a manifold M^n associates to every vector field X on M^n the operator of covariant derivative ∇_X acting on sections of E. The corresponding curvature tensor $\mathcal{R}$ is defined for each pair of vector fields X, Y as

$$\mathcal{R}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]},$$

where $[X, Y]$ is the standard Lie bracket of vector fields (see e.g. [30]).

Consider a complex line bundle over S^2 with a $U(1)$-connection having the curvature form

$$\mathcal{R} = i\mathbf{B} = i\alpha\, \mathrm{d}\mathbf{S},$$

which is motivated by geometric quantization. Since the first Chern class of the bundle must be an integer we have that

$$q = \frac{1}{2\pi i}\int_{S^2} \mathcal{R} = \frac{1}{2\pi}\int_{S^2} \alpha \mathrm{d}\mathbf{S} = 2\alpha \in \mathbf{Z},$$

which is precisely Dirac's quantization condition.

Let

$$X_1 = x_3\partial_2 - x_2\partial_3,\ X_2 = x_1\partial_3 - x_3\partial_1,\ X_3 = x_2\partial_1 - x_1\partial_2$$

be the vector fields generating rotations of $S^2 \subset \mathbb{R}^3$ and let ∇_{X_j} be the corresponding covariant derivatives. We claim that

$$\hat{\nabla}_j := i\nabla_{X_j}$$

and the operators $\hat{x}_j$ of multiplication by x_j satisfy the commutation relations (2.6).

Indeed, by definition of the curvature form, we have

$$\mathcal{R}(X_1, X_2) = \nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]} = i\alpha x_3$$

since

$$\alpha\, \mathrm{d}S(X_1, X_2) = \alpha \begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \end{vmatrix} = \alpha x_3(x_1^2 + x_2^2 + x_3^2) = \alpha x_3.$$

This implies

$$[\nabla_{X_1}, \nabla_{X_2}] = \nabla_{X_3} + i\alpha x_3$$

since $[X_1, X_2] = X_3$. Consequently, we have

$$[\hat{\nabla}_k, \hat{\nabla}_l] = i\epsilon_{klm}(\hat{\nabla}_m - \alpha\hat{x}_m)$$

for all $k, l, m = 1, 2, 3$; with the rest of the relations (2.6) being obvious.

Alternatively, we can look for the quantization of Novikov-Schmelzer variables as covariant derivatives:

$$\hat{\sigma}_j = i\nabla_{X_j}.$$

Then the same calculation shows that the curvature form of the corresponding connection must be $i\alpha\, \mathrm{d}\mathbf{S}$.

Finally, returning to the original variables we have the operators

$$\hat{l}_j = \hat{\nabla}_j + \alpha\hat{x}_j, \tag{2.7}$$

which coincides with the famous modification of the angular momentum in the presence of the Dirac magnetic monopole [16]. This provides us with one more explanation of this well-known, but a bit mysterious [1] physical notion.

2.3 Induced representations and Frobenius reciprocity

Let L_q be the complex line bundle over S^2 with first Chern class q. We are interested in the space $\Gamma(L_q)$ of L^2-sections of L_q. Viewing S^2 as $SU(2)/U(1)$ (with $U(1)$ as the diagonal subgroup) we have a natural interpretation of $\Gamma(L_q)$ as a representation of $SU(2)$. This is unsurprising since Wu and Yang showed that the Lie algebra $\mathfrak{so}_3$ acts naturally in this space, just as it does on functions in $L^2(S^2)$.

What is perhaps more surprising is that ths space can be described directly in terms of representation theory, where it is known as an induced representation (see e.g. [17]). One can use the classical Frobenius Reciprocity Formula from this theory to decompose $\Gamma(L_q)$ into irreducible representations of $SU(2)$.

We recall the details of the construction of induced representations for Lie groups first, before demonstrating exactly how it works for the case at hand of $U(1) \subset SU(2)$. Induced representations were first described by Frobenius who was looking at representations of finite groups. The construction is perhaps easier to understand for Lie groups, where the result may be interpreted in terms of differential geometry. For more details, one can see [1], [7], [17], [23] and [26] amongst others. The construction for finite groups is described in Chapter 4.

Given a group G and a subgroup H there are two natural functors between the category of representations of each. Given a representation V of G, it is clear that one can get a representation of H by restricting the action of G on V to H

$$V|_H := \mathrm{res}_G^H(V).$$

The functor going in the other direction can be described in terms of differential geometry. If G is a Lie group, H a closed subgroup and $\sigma : H \to Aut(W)$ a unitary representation of H then the representation of G induced from (σ, W) can be explained geometrically as acting on the space of L^2-sections of the associated vector bundle to the principal fibre bundle $H \to G \to G/H$ and the representation (σ, W) of H. This construction can be explained as follows.

[1] Sidney Coleman, in his famous lectures on Dirac monopoles [9], wrote about this modification of angular momentum: "The second term looks very strange indeed; in Rabi's immortal words about something else altogether, "Who ordered that ?""

We define a vector bundle $E_W := G \times_H W$ with fibre W over the homogeneous space G/H, whose L^2-sections are W-valued L^2 functions on G satisfying the following equivariance condition with respect to the H-action

$$L^2(G, W, \sigma) := \left\{\psi : G \to W \mid \psi(gh) = \sigma(h^{-1})\psi(g) \text{ for almost every } g \in G\right\}.$$

It is traditional [29] to abuse notation and work with the subspace of continuous elements of $L^2(G, W, \sigma)$ — this is legitimate, since it can be shown [46] that this is a dense subspace. If this is done then one does not have to worry about the condition 'for almost every g'.

The bundle E_W itself is formed by taking the quotient of the trivial bundle $G \times W \to G$ with respect to the equivalence relation for the action of H

$$E_W = G \times W/\sim, \quad \text{where} \quad (gh, w) \sim (g, \sigma(h)w).$$

The projection on E_W is induced from the natural projection on the trivial vector bundle $G \times W \to G$

$$\pi : E_W \to G/H, \qquad \pi(g, w) \mapsto (gH).$$

The bundle E_W is more special than a typical associated vector bundle and is known in the literature as a homogeneous vector bundle. This means that, unlike the associated vector bundle construction in general, there is a transitive action of G on the base space.

The induced representation can be defined using this bundle as being on the space of L^2-sections of the bundle.

Definition. Given a unitary representation (σ, W) of $H \subset G$, the representation of G induced from (σ, W) is on the space $L^2(G, V, \sigma)$ with G acting on an element ψ by

$$g \cdot \psi(x) = \psi(g^{-1}x).$$

This representation is unitary and is called *the representation of G induced from the representation (σ, W) of H* and is written $\mathrm{ind}_H^G(W)$.

Notice that if $W \cong \mathbb{C}$ then E_W is a line bundle over G/H and $\lambda := \rho$ is a character of H — in this case we refer to the line bundle by the weight as L_λ. Furthermore, if the representation σ is trivial then the line bundle is trivial $E_W \cong G/H \times \mathbb{C}$ and the induced representation space $\mathrm{ind}_H^G(W)$ is just the space of functions $L^2(G/H)$.

Remark. In general the bundle L_λ is not trivial and the degree of twisting is described by the character λ. For compact Lie groups there is the famous Borel–Weil–Bott theorem [6], which is a culminating result in the representation theory of compact Lie groups — giving as it does a uniform geometric construction of all irreducible representations of all compact connected Lie groups. Loosely, it describes irreducible representations of G in terms of sheaf cohomology groups $H^i(G/H, \mathcal{O}(L_\lambda))$. For $i = 0$ the result is due to Borel and Weil, with Bott providing the generalization for the higher cohomology groups when $i > 0$.

The Borel–Weil part of the Theorem is essentially just a reformulation of E. Cartan's theory of highest weights. Let G be a compact Lie group and T its maximal torus. The homogeneous space G/T is a complex manifold (the full flag manifold). Given a weight λ of T, we can induce a representation of G on $\Gamma(L_\lambda)$, where L_λ is a line bundle over G/T constructed as above.

Theorem 2.3.1 *(Borel–Weil Theorem) The space $\Gamma_{hol}(L_{-\lambda})$ of holomorphic sections of $L_{-\lambda}$ is non-zero exactly when λ is a dominant weight for an irreducible representation V_λ of G. If this is the case then $\Gamma_{hol}(L_{-\lambda}) \cong V_\lambda$ as representations of G.*

Remark. For more details see [7] and [17]; with the full statement of the Borel–Weil–Bott Theorem being included in [26] and proved in [6]. The Borel–Weil–Bott Theorem allows the heavy machinery of Algebraic Geometry to be employed to solve problems in Representation Theory. For instance: the Riemann–Roch Theorem can be applied to compute the dimension of the corresponding irreducible representation of G, which gives exactly Weyl's dimension formula from representation theory; a more refined analysis using the Atiyah–Bott fixed point formula [1] can be used to deduce Weyl's character formula for the character of V_λ, see also [7]. The Borel–Weil–Bott theorem was rederived algebraically using Lie algebra cohomology by Kostant [33].

For compact Lie groups, the induced representation is infinite-dimensional and is not, in general, irreducible. It can be decomposed into irreducible representations of G using the Frobenius reciprocity formula [17], which if W is a representation of H and V of G gives the following relation between the Hermitian scalar product of characters

$$\left\langle V, \mathrm{ind}_H^G\left(W\right)\right\rangle_G = \left\langle \mathrm{res}_G^H\left(V\right), W\right\rangle_H .$$

More formally, this can be stated as the following theorem, see [17] for details.

Theorem 2.3.2 *(Frobenius reciprocity theorem) If (ρ, V) is a representation of G and (σ, W) is a representation of H then there is an isomorphism of vector spaces*

$$\mathrm{Hom}_G(V, \mathrm{ind}_H^G(V)) = \mathrm{Hom}_H(\mathrm{res}_G^H(V), W).$$

Remark. Frobenius reciprocity can be interpreted formally as the statement that ind : $Reps(H) \to Reps(G)$ and res : $Reps(G) \to Reps(H)$ are adjoint functors. Loosely speaking, this formula says that the number of times that each irrep V of G appears in $\mathrm{ind}_H^G(W)$ is equal to the number of times that W appears in $\mathrm{res}_G^H(V)$.

The point is that Frobenius reciprocity permits the decomposition of the spaces of L^2-sections of vector bundles over homogeneous spaces into irreducible representations of G. In the specific case that (σ, W) is a one-dimensional representation, then E_W is a line bundle over G/H and the space of sections can be decomposed according by calculating certain branching rules.

We now demonstrate this explicitly for the case at hand with $U(1) \subset SU(2)$. First recall that all finite-dimensional irreducible representations of $SU(2)$ are labelled by a highest weight $k \in \mathbf{Z}_{\geq 0}$. The corresponding spaces V_k have dimension $k+1$ and weights

$$-k, -k+2, \ldots, k-2, k. \tag{2.8}$$

Since $SU(2)$ acts on $\mathbb{C}^2$ we can take as the spaces V_k homogeneous polynomials in two variables of degree k, i.e. $V_k \cong Sym^k(\mathbb{C}^2)$. It is clear that taking $U(1)$ to be the diagonal subgroup of $SU(2)$ gives the weights as claimed in equation (2.8).

Recall also that all finite-dimensional irreducible representations W_q of $U(1)$ have dimension 1 and are given by

$$e^{i\theta} \mapsto e^{iq\theta}, \ q \in \mathbb{Z}.$$

Using W_{-q} to induce a representation of $SU(2)$, we have that $\mathrm{ind}_{U(1)}^{SU(2)}(W_{-q})$ can be described geometrically as the space of L^2-sections of the line bundle L_q over S^2 with the first Chern class q

$$\Gamma(L_q) := \mathrm{ind}_{U(1)}^{SU(2)}(W_{-q}).$$

This can be seen using the Borel–Weil Theorem 2.3.1, since for $q \geq 0$ we have that $\Gamma_{hol}(L_q) \cong V_q$; consequently

$$\dim(\Gamma_{hol}(L_q)) = \dim(V_q) = q+1,$$

whilst using the Index Theorem gives that

$$\dim(\Gamma_{hol}(L_q)) = c_1(L_q) + \frac{c_1(TS^2)}{2} = c_1(L_q) + 1.$$

The induced representation is not irreducible: to decompose it we will use the Frobenius reciprocity formula, which in our concrete case reads

$$\left\langle V_k, \mathrm{ind}_{U(1)}^{SU(2)}\left(W_q\right)\right\rangle_{SU(2)} = \left\langle W_q, \mathrm{res}_{SU(2)}^{U(1)}(V_k)\right\rangle_{U(1)}. \tag{2.9}$$

with the brackets denoting the multiplicity of the first representation entering into the second one (see e.g. [17]).

Since the restriction of V_k to $U(1)$ is the sum of the weight spaces

$$\mathrm{res}_{SU(2)}^{U(1)}(V_k) = \bigoplus_{j \in S_k} W_j,$$

where $S_k = \{-k, -k+2, \ldots, k-2, k\}$ we see that each V_k, which (after restriction) contains W_q will appear once in the decomposition of $\Gamma(L_q)$. Clearly this can only happen if $k \geq |q|$ and $k - |q|$ is even. Therefore $\Gamma(L_q)$ decomposes into $SU(2)$-modules according to the following rule

$$\mathrm{ind}_{U(1)}^{SU(2)}\left(W_q\right) = \Gamma(L_q) = \widehat{\bigoplus}_{l \in \mathbb{Z}_{\geq 0}} V_{2l+|q|}, \tag{2.10}$$

where the hat over the direct sum indicates that an infinite number of terms may be taken.

2.4 Calculation of the monopole's spectrum

The Hamiltonian of the Dirac monopole can be written in terms of Novikov–Schmelzer operators as

$$H = \hat{\sigma}^2$$

or, equivalently, in terms of magnetic angular momentum $\hat{l}$ as

$$H = \hat{l}^2 - \alpha^2 = \hat{l}^2 - \frac{1}{4}q^2.$$

Since the components of $\hat{l}_m$ satisfy the standard commutation relations

$$[\hat{l}_k, \hat{l}_m] = i\epsilon_{kmn}\hat{l}_n,$$

the operator $\hat{l}^2$ is a Casimir operator for $SU(2)$ and acts on V_k as a scalar. If $v \in V_k$ and $s = k/2$ then $\hat{l}^2$ acts as

$$\hat{l}^2 v = s(s+1)v = \frac{1}{4}k(k+2)v, \tag{2.11}$$

see e.g. [17]. The space $V_{2l+|q|}$ has dimension $2l + |q| + 1$, and for $\psi \in V_{2l+|q|}$, the operator H acts as

$$H\psi = (\hat{l}^2 - \frac{1}{4}q^2)\psi = \left[\frac{1}{4}\left(2l+|q|\right)\left(2l+|q|+2\right) - \frac{1}{4}q^2\right]\psi = \left[l(l+1) + |q|\left(l + \frac{1}{2}\right)\right]\psi.$$

Thus for a Dirac monopole of charge q the spectrum is

$$\left[l(l+1)+|q|\left(l+\frac{1}{2}\right)\right], l \in \mathbb{Z}_{\geq 0} \text{ with degeneracy } \; 2l+|q|+1$$

agreeing exactly with (2.1). This result was also derived in [15], where the corresponding eigenfunctions were computed using Darboux-Schrödinger factorisation method applied to curved surfaces. For non-negative q, the ground eigenstates were identified with the space of polynomials of degree $\leq q$ in one complex variable on $\mathbb{C} \cup \infty \cong S^2$. In our picture this ground eigenspace corresponds to the subspace of holomorphic sections of L_q, which by the Borel–Weil Theorem 2.3.1 can be identified with the corresponding irreducible SU(2)-module V_q.

Chapter 3

Magnetic monopoles on coadjoint orbits

This chapter may be thought of as a continuation of the previous one. There the Dirac magnetic monopole was considered from the point of view of geometric quantization. The starting point there was the observation in [43] that the regular coadjoint orbits of the Euclidean group $E(3) = SO(3) \ltimes \mathbb{R}^3$ coincide with the phase spaces of classical Dirac magnetic monopoles. Geometrically, the phase space of (an electron moving on a sphere surrounding) a Dirac magnetic monopole of charge $q \in \mathbb{R}$ is the symplectic manifold

$$(T^*S^2, dp \wedge dx + \frac{q}{2}\pi^*(dS)),$$

where $\pi^*(dS)$ is the pullback of the area form on S^2 to the manifold T^*S^2 under the natural projection $\pi : T^*S^2 \to S^2$.

This is an example of a magnetic cotangent bundle: a symplectic manifold

$$(T^*M, dp \wedge dx + \pi^*(\omega)),$$

whose symplectic form has been twisted away from the canonical form $dp \wedge dx$ by the addition of $\pi^*(\omega)$, where ω is a closed 2-form on M and $\pi^*(\omega)$ is its pullback to T^*M.

For S^2, the prequantization condition of geometric quantization (requiring the symplectic form to be integral) then gives exactly Dirac's quantization condition for magnetic charge: namely that $q \in \mathbb{Z}$, since the prequantization condtion is that

$$\frac{1}{2\pi}\int_{S^2} \frac{q}{2} dS = q \in \mathbb{Z}.$$

On choosing the vertical polarization on T^*S^2, the quantum Hilbert space is given by the space $L^2(L_q)$ of square-integrable sections of $L_q \to S^2$, the complex line bundle over S^2 with Chern class q. The crucial point is that this Hilbert space can be related to an induced representation as follows. The sphere S^2 is a coadjoint orbit for $SU(2)$, being given by $S^2 \cong SU(2)/U(1)$, where $U(1)$ is a maximal torus in $SU(2)$.

If W_q denotes the weight of $U(1)$ given by $e^{i\theta} \mapsto e^{iq\theta}$ (for $q \in \mathbb{Z}$), then there is the isomorphism

$$L^2(L_q) \cong \mathrm{ind}_{U(1)}^{SU(2)}(W_{-q}).$$

This induced representation can be decomposed into irreducible representations of $SU(2)$ explicitly using the Frobenius reciprocity formula.

The classical observables can be explicitly quantized and so the quantum Hamiltonian can be computed — it is given by the Casimir operator for $SU(2)$ minus a constant q^2. Thus, the spectrum of (an electron orbiting) a Dirac magnetic monopole is calculated explicitly and the answer agrees with the straightforward calculations in [57] and [15].

In this chapter this construction is carried over mutatis mutandis for magnetic cotangent bundles to coadjoint orbits of compact, connected and semisimple Lie groups — which will be called 'Dirac magnetic monopoles on coadjoint orbits'.

It is well-known that coadjoint orbits of Lie groups are symplectic manifolds, possessing the canonical Kostant–Kirillov symplectic form ω_{KK}. This observation is the starting point for 'The Orbit Method' [26], which aims to connect the representation theory of the Lie group to the geometry of its coadjoint orbits.

This is not the end of the story as far as the symplectic geometry of coadjoint orbits is concerned. Suppose that $\mathcal{O}(a) \cong G/G_a$ is a coadjoint orbit of G, with G_a being the stabilizer of the point $a \in \mathfrak{g}^*$.

If $\mathfrak{g}_a$ is the Lie algebra of G_a then any $f \in (Z(\mathfrak{g}_a))^*$ (i.e. f belongs to the dual of the centre of $\mathfrak{g}_a$) can be used to define a G-invariant closed 2-form (a 'pre-symplectic form') ω_f on $\mathcal{O}_a$. The condition $f \in (Z(\mathfrak{g}_a))^*$ is equivalent to saying that f is a character of $\mathfrak{g}_a$, i.e. $f \in \mathrm{Hom}(\mathfrak{g}_a, \mathbb{R})$.

One may then study the magnetic cotangent bundle $(T^*\mathcal{O}(a), dp \wedge dx + \pi^*(\omega_f))$. By describing a natural (although not well-known) group structure on $TG \cong G \ltimes_{Ad} \mathfrak{g}$, the tangent bundle to G, it is shown that this is actually symplectomorphic to the coadjoint orbit $\mathcal{O}(f, a) \subset (\mathfrak{g} \ltimes_{ad} \mathfrak{g})^*$ equipped with the Kostant–Kirillov form. This can be thought of as an extension of the observation in [43] that the coadjoint orbits of $E(3)$ are symplectomorphic to the phase space of Dirac magnetic monopoles.

Taking inspiration from the previous chapter, geometric quantization is then used to quantize the phase space $(T^*\mathcal{O}(a), dp \wedge dx + \pi^*(\omega_f)) \cong \mathcal{O}(f, a) \subset (\mathfrak{g} \ltimes_{ad} \mathfrak{g})^*$.

The analogue of Dirac's quantization condition for magnetic charge is given by the integrality condition of geometric quantization, which is satisfied iff f exponentiates to give a character $\chi_f \in \mathrm{Hom}(G_a, U(1))$. This connects to representation theory, because the Hilbert space $\mathcal{H}$ suggested by geometric quantization is exactly $\mathrm{ind}_{G_a}^G(\chi_f)$ — the representation of G induced from the character χ_f.

This induced representation is not irreducible and it can be decomposed into irreducible representations of G according to Frobenius reciprocity, which requires computing how each irreducible representation decomposes when restricted to G_a.

Because G_a contains a subgroup of G that is a maximal torus T, the answer to this branching problem is given by Kostant's branching formula (see e.g. [29]) — which seems to have been purpose-built for this occasion.

Lastly, the action of the quantum Hamiltonian (which is the quantization of the free particle Hamiltonian $H = \frac{1}{2}g^{ij}p_i p_j$ on $T^*\mathcal{O}(a)$)

$$\mathcal{H} = \mathrm{ind}_{G_a}^G(\chi_f) = \left\{ f : G \to \mathbb{C}, f \in L^2(G) \mid f(gh) = \chi_f(h^{-1})f(g), g \in G, h \in G_a \right\}$$

is computed. This is shown to be the same as the action of the Bochner Laplacian and is given by

$$\hat{H}_f = \Omega_G - \langle f, f \rangle \,,$$

where Ω_G is the second order Casimir element of G and this is acting on $\mathcal{H}$.

In conjunction with Kostant's branching formula this gives the spectrum of a 'Dirac magnetic monopole on a coadjoint orbit.'

3.1 Magnetic geodesic flow

The topic for this chapter will be the geometric quantization of the magnetic geodesic flow on a coadjoint orbit with respect to the normal metric. The magnetic geodesic flow describes the motion of a free particle on a manifold in the presence of a magnetic field and may be described as a distortion of the usual geodesic flow.

On a Riemannian manifold (M^n, g_{ij}) one may study the geodesic flow. In this subsection this system will be described — together with the 'magnetic geodesic flow'. If $\{x^i : i = 1, \ldots, n\}$ are local coordinates on M and $p_i = g_{ij}\dot{x}^j$ are local momenta then local coordinates on T^*M are given by $\{x^i, p_i, i = 1, \ldots, n\}$ and the canonical symplectic form is defined globally by $dp \wedge dx = \sum_{i=1}^n dp_i \wedge dx^i$. This defines a Poisson bracket on $C^\infty(M)$, which for $f_1, f_2 \in C^\infty(M)$ is given in local coordinates by

$$\{f_1, f_2\} = \sum_{i=1}^n \left(\frac{\partial f_1}{\partial x^i}\frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial x^i} \right). \tag{3.1}$$

The Hamiltonian function

$$H = \frac{1}{2} g^{ij} p_i p_j$$

describes the kinetic energy of a particle of unit mass on M. Hamilton's equations are then

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i},$$

which hold iff

$$\frac{df}{dt} = \{f, H\},$$

where $f \in C^\infty(M)$; this system is called the geodesic flow on (M, g).

Definition. A *magnetic cotangent bundle* is a symplectic manifold of the form

$$(T^*M, dp \wedge dx + \pi^*(\omega))$$

where ω is a closed 2-form on M being represented in local coordinates by $\omega = F_{ij}(x) dx^i \wedge dx^j$ and describing a magnetic field on M and $\pi^*(\omega)$ denotes its pullback to T^*M.

The Hamiltonian function $H = \frac{1}{2} g^{ij} p_i p_j$ now represents the kinetic energy of a particle of unit mass and electrical charge moving on M with the magnetic field ω. This changes the Poisson bracket from (3.1) to

$$\{f_1, f_2\}_\omega = \sum_{i=1}^{n} \left(\frac{\partial f_1}{\partial x^i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial x^i} \right) + \sum_{i,\, j=1}^{n} F_{ij}(x) \frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial p_j}.$$

Consequently Hamilton's equations change to

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} + \sum_{j=1}^{n} F_{ij}(x) \frac{\partial H}{\partial p_j}.$$

Of particular interest is the case when M is a coadjoint orbit of a compact, connected and simply-connected Lie group G — the geometry of which will be described in the next section. The classical dynamics in this case studied in [4], [5] and [14], with the result being that the classical system is integrable under certain conditions on the magnetic form — namely that it is a scalar multiple of the Kostant–Kirillov form on the orbit (see the next section for the definition of this form).

3.2 Coadjoint orbits

In this section the geometry of coadjoint orbits of compact Lie groups is reviewed and also some topological information is derived. A class of G-invariant closed 2-forms on the orbit is defined, which may be taken as magnetic fields on the orbits.

To aid digestion, it is perhaps worth giving some examples of coadjoint orbits of compact Lie groups. Topologically they are known as generalized flag manifolds and have very nice properties, which derive from the fact that the stabilizer of a point contains a subgroup that is a maximal torus. A first non-obvious result is that they are complex manifolds, something that is necessary for the Borel–Weil–Bott Theorem (stated for generalized flag manifolds) to even be plausible.

The generic case is when this stabilizer is as small as possible, i.e. when $G_a \cong T$. In this case, the coadjoint orbit is topologically a flag manifold, i.e. the quotient of $G_{\mathbb{C}}$ by its Borel subgroup B. This is a class that contains the standard flag manifold, which is given by

$$\mathcal{F}_n = \{F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset \mathbb{C}^n\},$$

which is easily seen that $\mathcal{F}_n \cong SL(n,\mathbb{C})/B$. It is a remarkable fact that coadjoint orbits of compact Lie groups are hyperkähler manifolds — this result was proved for generic orbits in [36] and for non-generic orbits in [35].

At the other end of the spectrum, the most degenerate coadjoint orbits are given when the stabilizing subgroup is as large as possible — topologically, these spaces are Grassmannians and projective spaces. For example, the Grassmannian of k-planes in $\mathbb{C}^n$ is a coadjoint orbit of $SU(n)$ with $G_a \cong S(U(k) \times U(n-k))$ — in particular, $k=1$ and $k=n-1$ correspond to complex projective space $\mathbb{C}P^{n-1}$.

3.2.1 Geometry of coadjoint orbits

A matrix Lie group G acts on itself by conjugation, given $g \in G$ define C_g by

$$C_g : G \to G, \qquad C_g : h \mapsto ghg^{-1}.$$

The identity e is a fixed point of this map and so one may look at the derived map of tangent spaces $(C_g)_* := Ad_g : \mathfrak{g} \to \mathfrak{g}$, which is called the adjoint map of G on $\mathfrak{g}$. Given a curve $h(s) = \exp(sY)$ through e in G, by differentiating one sees that the adjoint map is nothing but conjugation

$$Ad_g(Y) := \left.\frac{d}{ds}\right|_{s=0} g\exp(sY)g^{-1} = gYg^{-1}.$$

This map is a representation of G, since $Ad_g \cdot Ad'_g = Ad_{gg'}$. One can again look at the derived version of this map: if $g = \exp(tX)$ then the adjoint map of $\mathfrak{g}$ on itself is defined by

$$ad_X(Y) := \left.\frac{d}{dt}\right|_{t=0} \exp(tX)Y\exp(-tX) = [X,Y].$$

One can define dual maps of G and $\mathfrak{g}$ (written Ad^* and ad^*) on $\mathfrak{g}^*$, which is the dual of the Lie algebra $\mathfrak{g}$. If the pairing between $f \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$ is written $\langle f, X\rangle$ then the coadjoint action of G on $\mathfrak{g}^*$ is defined by

$$\left\langle Ad^*_g(f), X\right\rangle := \langle f, Ad_{g^{-1}}X\rangle$$

and the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is defined by

$$\langle ad^*_Y(f), X\rangle := -\langle f, ad_Y X\rangle = \langle f, [X,Y]\rangle\,.$$

Given a in $\mathfrak{g}^*$, the coadjoint orbit $\mathcal{O}(a)$ is defined by

$$\mathcal{O}(a) := \left\{x \in \mathfrak{g}^* \mid x = Ad^*_g(a), g \in G\right\} = Ad^*_G(a).$$

Define G_a to be the stabilizer of the point a, i.e. $G_a := \left\{g \in G : Ad^*_g(a) = a\right\}$, clearly if $x = Ad^*_g(a)$ then $G_x = g(G_a)g^{-1}$. Thus, the stabilizers of each point of $\mathcal{O}(a)$ are conjugate and so we may identify $\mathcal{O}(a)$ with the homogeneous space $\mathcal{O}(a) \cong G/G_a$.

In this chapter we are considering simple Lie groups G that are compact, connected and simply-connected. A simple compact group has a positive-definite and Ad-invariant inner product defined on its Lie algebra $\mathfrak{g}$, which is given by

$$(X,Y) := -\mathrm{tr}(XY),$$

for $X, Y \in \mathfrak{g}$. Up to a constant this coincides with the Cartan–Killing form (or sometimes it is just called the Killing form) on $\mathfrak{g}$, which is defined by

$$(X,Y)_{CK} := -\mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y : \mathfrak{g} \to \mathfrak{g}).$$

This is a symmetric billinear form taking values in $\mathfrak{g}$, which is non-degenerate if and only if $\mathfrak{g}$ is semisimple. For semisimple $\mathfrak{g}$ this permits the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$. It also allows the identification of coadjoint orbits of G with the adjoint orbits in a natural way. Thus, for the class of groups we are considering here we could just as well look at adjoint orbits instead of coadjoint orbits — this is indeed the approach taken in [4], [5] and [14], where a related classical system is considered.

If G is a compact Lie group then, since any element in $\mathfrak{g}$ can be diagonalized by conjugating within the group G, we have the following lemma.

Lemma 3.2.1 *The stabilizer $G_a \subset G$ of a point $a \in \mathfrak{g}^*$ contains as a subgroup a maximal torus T of G.*

Proof. Any element of $\mathfrak{g}^*$ can be brought into diagonal form by the coadjoint action, i.e. for any $a \in \mathfrak{g}^*$ there is a $g \in G$ such that $Ad^*_g(a) \in \mathfrak{t}^*$, where $\mathfrak{t}$ is the Lie algebra of T. Such an element is clearly fixed by all elements of T, proving the lemma. □

Remark. Thus whilst coadjoint orbits of a compact Lie group are homogeneous spaces, they are homogeneous spaces of a special type — where the stabilizing subgroup contains a maximal torus. This excludes many homogeneous spaces where the spectral problem for the Laplace–Beltrami operator has been traditionally studied, e.g. any sphere is a homogeneous space since $S^n \cong SO(n+1)/SO(n)$, but only for $n = 2$ can this also be realized as a coadjoint orbit. A good survey of the spectral problem for homogeneous spaces is given in [29].

We move now to describe some of the geometry of coadjoint orbits: a first question might be to ask how tangent vectors to coadjoint orbits can be described. Suppose $g(t) = \exp(t\xi)$ is a curve in G, then $Ad^*_{g(t)}(x)$ is a curve in $\mathcal{O}(a)$, passing through x at $t = 0$. Differentiating gives

$$\left.\frac{d}{dt}\right|_{t=0} \left\langle Ad^*_{g(t)}x, \eta \right\rangle = \left.\frac{d}{dt}\right|_{t=0} \left\langle x, Ad_{g(-t)}\eta \right\rangle = \langle x, [\eta, \xi] \rangle = \left\langle ad^*_{\xi}(x), a \right\rangle .$$

Therefore, as might be expected, the tangent space at a point of $\mathcal{O}(a)$ is generated by the infinitessimal version of the coadjoint action of G

$$T_x\mathcal{O}(a) = \left\{ f \in \mathfrak{g}^* \mid f = ad^*_{\xi}(x) \ \text{ for } \xi \in \mathfrak{g} \right\} = ad^*_{\mathfrak{g}}(x).$$

One sees from the above discussion that tangent vectors to coadjoint orbits are naturally coadjoint vectors (i.e. they live in $\mathfrak{g}^*$). By using the identificiation $((\mathfrak{g})^*)^* \cong \mathfrak{g}$ one also sees that cotangent vectors are naturally adjoint vectors (i.e. they live in $\mathfrak{g}$).

As remarked above, one could also study adjoint orbits, which for the class of groups we are considering are isomorphic to coadjoint orbits. For general Lie groups though, it turns out that coadjoint orbits have defined on them a natural symplectic form — immediately exalting them above adjoint orbits. This symplectic form is 'inherited functorially and so the coadjoint orbit is the correct object', as explained in the highly readable [7]. This is not just a technicality however: for nilpotent Lie groups, which are the class of Lie groups where the orbit method of Kirillov works best (see Kirillov's monograph [26]) if one investigates the adjoint orbits one sees that they need not even be even-dimensional.

Definition. Each coadjoint orbit $\mathcal{O}(a)$ of a Lie group G possesses a canonical, G-invariant, symplectic form ω_{KK}; whose value on two tangent vectors $\xi_a = ad^*_{\xi}(a)$ and $\eta_a = ad^*_{\eta}(a)$ at a is given by

$$\omega_{KK}(\xi_a, \eta_a) = \langle a, [\xi, \eta] \rangle .$$

This symplectic form is known generally as the *Kostant–Kirillov form.* The proof that it gives a well-defined, G-invariant, symplectic form can be found in many places, for example: [23], [26] and [56].

There is a natural metric on the coadjoint orbit $\mathcal{O}(a)$, called the normal metric.

Definition. The projection map $pr : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_x \cong \mathfrak{g}_x^\perp$ is used to define the *normal metric* by the formula

$$(\xi_x, \eta_x) := (pr(\xi), pr(\eta))_{CK}.$$

Notice that the normal metric is well-defined: a tangent vector ξ_x defines a $\xi \in \mathfrak{g}$ up to an element in $\mathfrak{g}_x$, but the projection renders all such elements null. Namely, if $\tilde{\xi} = \xi + \zeta$ where $\zeta \in \mathfrak{g}_x$ then $pr(\tilde{\xi}) = pr(\xi)$ and so the value of (ξ_x, η_x) does not depend on this freedom.

If $\xi \in \mathfrak{g}$ generates a tangent vector $ad^*_\xi(x)$ at x then the cotangent vector that is generated by ξ is exactly $p_\xi = pr_{\mathfrak{g}_x^\perp}(\xi)$.

We show how this works specifically for the case when $\mathcal{O}(a) \cong \mathbb{C}P^n$, which is a coadjoint orbit of $SU(n+1)$. If we take as the point a an element in $\mathfrak{su}^*_{n+1}$ of the form $a = \mathrm{diag}(nx, -x, \dots, -x)$ then we see that the stabilizing subgroup G_a is given by $S(U(1) \times U(n))$ as follows

$$G_a \cong S(U(1) \times U(n)) = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & A \end{pmatrix} \in SU(n+1); \epsilon \in U(1), A \in U(n), \right\}$$

with the corresponding Lie algebra being given by

$$\mathfrak{g}_a = \left\{ \begin{pmatrix} i\alpha & 0 \\ 0 & W \end{pmatrix} \in \mathfrak{su}_{n+1}; ia + tr(W) = 0 \right\}.$$

The Cartan–Killing form $(\ ,\ )$ for $X, Y \in \mathfrak{su}_{n+1}$ is given by the well-known formula (see [17])

$$(X, Y) = 2(n+1) tr(XY).$$

The tangent space at a may be identified with $\mathfrak{g}/\mathfrak{g}_a$ as

$$\mathfrak{m} = \left\{ \xi = \begin{pmatrix} 0 & -\overline{\xi}_1 & \dots & -\overline{\xi}_n \\ \xi_1 & & & \\ \vdots & & 0 & \\ \xi_n & & & \end{pmatrix} \in \mathfrak{su}_{n+1}; \xi_1, \dots, \xi_n \in \mathbb{C} \right\} \cong \mathbb{C}^n$$

Thus the normal metric applied to $\xi, \eta \in \mathfrak{m}$ gives

$$(\xi, \eta) = 2(n+1) \sum_{i=1}^{n} \xi_i \overline{\eta_i}$$

3.2.2 Topology of coadjoint orbits

In this section some topological information pertaining to coadjoint orbits is deduced from topological information of G and G_a.

Firstly some of the homotopy groups of $\mathcal{O}(a)$ are computed by means of the identification of the orbit $\mathcal{O}(a)$ through a with the homogeneous space $\mathcal{O}(a) \cong G/G_a$. Therefore G is the total space for a principal fibre bundle over $\mathcal{O}(a)$ with fibre G_a, which gives the fibration

$$G_a \to G \to \mathcal{O}(a). \tag{3.2}$$

Associated to the fibration (3.2) is a long exact sequence of homotopy groups

$$\begin{aligned}
&\ldots \to \pi_k(G_a) \to \pi_k(G) \to \pi_k(\mathcal{O}(a)) \to \pi_{k-1}(G_a) \to \ldots \\
&\ldots \to \pi_2(G_a) \to \pi_2(G) \to \pi_2(\mathcal{O}(a)) \to \pi_1(G_a) \ldots \\
&\ldots \to \pi_1(G) \to \pi_1(\mathcal{O}(a)) \to \pi_0(G_a) \to \pi_0(G) \to \pi_0(\mathcal{O}(a)) \to 1,
\end{aligned} \tag{3.3}$$

the details of which may be found in [12].

The information obtained from the long exact sequence (3.3) is summarized in a series of simple lemmas, given without proof.

Lemma 3.2.2 *If G is connected, i.e. $\pi_0(G) = 1$, then $\mathcal{O}(a)$ is connected.*

Lemma 3.2.3 *If G is connected and simply-connected, i.e. $\pi_1(G) = 1$, then we have $\pi_1(\mathcal{O}(a)) \cong \pi_0(G_a)$ — in particular, $\mathcal{O}(a)$ is simply-connected iff G_a is connected.*

If G is connected and simply-connected, then one can also show that $\pi_2(G) = 0$ (see [26] for details): this fact spawns the following lemma.

Lemma 3.2.4 *If G is connected and simply-connected, then $\pi_2(\mathcal{O}(a)) \cong \pi_1(G_a)$.*

Remark. In particular, this gives that $\pi_1(G_a)$ is commutative, since $\pi_2(M)$ is commutative for any manifold M. Indeed, it is a general fact that the fundamental group of a Lie group is commutative — this derives from the group operation.

Whilst the homotopy groups of a principal fibre bundle may be computed in a reasonably straightforward way, in principle it is difficult to extract information about the homology of a fibre bundle. For instance, one could study the spectral sequence associated to the fibre bundle (see e.g. [13]).

However, in low degrees the homology groups of a manifold M are related to its homotopy groups in a relatively simple way. In the first degree we have that the first homology group is isomorphic to the abelianization of the fundamental group $H_1(M,\mathbb{Z}) \cong \pi_1(M)/\left[\pi_1(M),\pi_1(M)\right]$ — indeed, this may be taken as a definition [12]. In the second degree we have that $H_2(M,\mathbb{Z}) \cong \pi_2(M)$ if $\pi_1(M) = 0$ (this is the Hurewitz isomorphism, see e.g. [13]).

Combining the above paragraph (which relates homotopy groups in low-degree with homology groups) with Lemmas 3.2.2, 3.2.3 and 3.2.4 gives the following Lemma.

Lemma 3.2.5 *For G connected and simply-connected, then*

$$H_2(G,\mathbb{Z}) = 0,$$

$$H_2(\mathcal{O}(a),\mathbb{Z}) \cong H_1(G_a,\mathbb{Z}) \cong \pi_1(G_a),$$

$$H_1(G,\mathbb{Z}) \cong H_1(\mathcal{O}(a),\mathbb{Z}) = 0.$$

De Rham cohomology can be defined as dual to homology in the following sense:

$$H^k(M,\mathbb{R}) := \mathrm{Hom}(H_k(M,\mathbb{Z}),\mathbb{R}).$$

Definition. Define the integral cohomology group as

$$H^k(M,\mathbb{Z}) := \left\{ \omega \in H^k(M,\mathbb{R}) \,\middle|\, \int_A \omega \in \mathbb{Z} \text{ for any } A \in H_k(M,\mathbb{Z}) \right\}. \tag{3.4}$$

One can show that this definition is equivalent to the definition of integer cohomology using using the simplicial or Cech theories [26].

Proposition 3.2.6 *If G is connected and simply-connected then*

$$H^2(G,\mathbb{R}) = 0,$$

$$H^2(\mathcal{O}(a),\mathbb{R}) \cong H^1(G_a,\mathbb{R}) \cong \mathrm{Hom}(\pi_1(G_a),\mathbb{R})$$

and

$$H^1(G,\mathbb{R}) \cong H^1(\mathcal{O}(a),\mathbb{R}) = 0.$$

This extends to integral cohomology, i.e. we have that

$$H^2(\mathcal{O}(a),\mathbb{Z}) \cong H^1(G_a,\mathbb{Z}).$$

3.2.3 Characters and cohomology

We begin this section with the definitions of characters of Lie groups and Lie algebras, before explaining the connection we seek to understand.

Definition. A *character* of a compact Lie group H is a homomorphism from H into $U(1)$. The *character group* $\widehat{H}$ is the abelian group of all characters of H

$$\widehat{H} := \mathrm{Hom}(H, U(1)),$$

with multiplication given by

$$(\Psi \cdot \Psi')(h) = \Psi(h) \cdot \Psi'(h), \quad \text{for } \Psi, \Psi' \in \widehat{H} \text{ and } h \in H.$$

Since Ψ is a homomorphism into an abelian group we have that

$$\Psi(h \cdot h') = \Psi(h) \cdot \Psi(h') = \Psi(h') \cdot \Psi(h) \Rightarrow \Psi(h'hh'^{-1}) = \Psi(h),$$

i.e. Ψ is invariant under conjugation, explaining why Ψ is called a character.

Definition. A *character* of a Lie algebra $\mathfrak{h}$ is a Lie algebra homomorphism from $\mathfrak{h}$ into $\mathfrak{u}_1 = i\mathbb{R}$. The *character algebra* $\widehat{\mathfrak{h}}$ is the abelian group of all characters of $\mathfrak{h}$

$$\widehat{\mathfrak{h}} := \mathrm{Hom}(\mathfrak{h}, i\mathbb{R}),$$

with addition given by

$$(\psi + \psi')(Y) = \psi(Y) + \psi'(Y), \quad \text{for } \psi, \psi' \in \widehat{\mathfrak{h}} \text{ and } Y \in \mathfrak{h}.$$

Since ψ is a Lie algebra homomorphism into the abelian group $i\mathbb{R}$ we have that

$$\psi([Y, Y']) = [\psi(Y), \psi(Y')] = \psi(Y)\psi(Y') - \psi(Y')\psi(Y) = 0. \tag{3.5}$$

In this section we look to make a connection between characters of a Lie group and its first de Rham cohomology. The purpose of this is to see whether we can describe the integer cohomology classes $H^1(G_a, \mathbb{Z})$ and so (by Proposition 3.2.6) $H^2(\mathcal{O}(a), \mathbb{Z})$ in terms of characters of G_a. In other words, we know that given a character of G_a we can induce a representation of G on the space of L^2-sections of a line bundle over $\mathcal{O}(a)$ and that the Chern class of this bundle is an element of $H^2(\mathcal{O}(a), \mathbb{Z})$ — the question is, can every element of $H^2(\mathcal{O}(a), \mathbb{Z})$ be described in this way?

The motivation behind this question lies in the case considered in Chapter 2, when the answer is easily seen to be yes. Specifically, we had that

$$\widehat{\mathfrak{u}_1} \cong \{\lambda : i\theta \to i\lambda\theta, \lambda \in \mathbb{R}\} \cong \mathbb{R} \text{ and } \widehat{U(1)} \cong \left\{n : e^{i\theta} \to e^{in\theta}, n \in \mathbb{Z}\right\} \cong \mathbb{Z}$$

and that

$$H^1(U(1), \mathbb{R}) \cong \left\{ \frac{\lambda}{2\pi} d\theta, \lambda \in \mathbb{R} \right\} \cong \mathbb{R} \text{ and } H^1(U(1), \mathbb{Z}) \cong \left\{ \frac{n}{2\pi} d\theta, n \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

Moreover, inducing a representation of $SU(2)$ from the character $-n$ gives a line bundle over S^2 with Chern class n.

We suppose that the Lie group H is compact and connected and we investigate its first-degree cohomology and characters of it and its Lie algebra. We make the following observation, that must be well-known to experts.

Lemma 3.2.7 *Elements of $\widehat{\mathfrak{h}}$ correspond to elements of $H^1(H, \mathbb{R})$ and vice-versa.*

Proof. The Lie algebra $\mathfrak{h}$ of H may be viewed as both $T_e H$ and also the space of left-invariant vector fields on H. Thus, the dual space $\mathfrak{h}^*$ may be viewed as both $T_e^* H$ and the space of left-invariant 1-forms on H. A left-invariant 1-form ω is defined by its value on $T_e H$ — if $X \in T_h H$ then the value of ω applied to X is given by left-translating X back to $T_e H$ and then computing its value there, i.e. $\omega(X) := \omega((L_{h^{-1}})_* X)$. The algebra of left-invariant forms on H is thus isomorphic to the exterior algebra $\Lambda[\mathfrak{h}^*]$ in a natural way. The cohomology groups of Lie groups were investigated by Weyl [55] who showed that a k-form ω is closed if and only if it is bi-invariant, i.e. invariant under both left and right translation, i.e. if

$$\omega(hX_1h^{-1}, \ldots, hX_kh^{-1}) = \omega(X_1, \ldots, X_k).$$

This is most easily demonstrated for 1-forms, where a form ω is closed if and only if

$$\omega(hXh^{-1}) = \omega(X). \tag{3.6}$$

This can be seen by computing the action of $d\omega$ on two tangent vectors X and Y,

$$d\omega(X, Y) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X, Y]) = -\omega([X, Y]),$$

since $\omega(Y)$ and $\omega(X)$ are left-invariant functions, i.e. constants on H. Thus, $d\omega$ is closed iff $\omega([X, Y]) = 0$, which can be seen by differentiating (3.6). However, this is exactly the condition that characterises characters of $\mathfrak{h}$. Algebraically, an element $\omega \in h^*$ corresponds to a closed 1-form on H if it is an element of the dual of the centralizer of $\mathfrak{h}$, i.e. $\omega \in (Z(\mathfrak{h}))^*$, since

$$\omega([X, Y]) = 0 \Leftrightarrow ad_X^*(\omega)(Y) = 0.$$

□

Remark. Weyl's investigation into the cohomology groups of Lie groups is by no means the end of the story. Hopf looked at them and showed that the group operation induces on $H^*(H, \mathbb{R})$ the structure of what is now called a Hopf algebra. This allows the deduction of many non-trivial facts about the cohomology groups of Lie groups (or more generally H-spaces) — see, e.g., [13] for details.

Lemma 3.2.8 *The character group $\widehat{H}$ can be identified with the lattice $L \subset \widehat{\mathfrak{h}}$ that is defined by*

$$f \in L \Leftrightarrow f(Z) \in 2\pi i\mathbb{Z}, \quad \text{if} \;\; Z \in \exp^{-1}(e).$$

Proof. Given $\Psi \in \widehat{H}$ we can differentiate Ψ to get a character ψ of the Lie algebra, whose action on $X \in \mathfrak{h}$ is given by the rule

$$\psi(X) := \left.\frac{d}{dt}\right|_{t=0} \Psi(\exp(tX)).$$

To show that this is indeed a character of $\mathfrak{h}$, consider (3.2.3) for $g = \exp(sX)$ and $h = \exp(tY)$, differentiating with respect to t gives

$$\left.\frac{d}{dt}\right|_{t=0} \Psi(g\exp(tY)g^{-1}) = \left.\frac{d}{dt}\right|_{t=0} \Psi(\exp(tY)) \Rightarrow \psi(gYg^{-1}) = \psi(Y)$$

and now differentiating with respect to s at $s = 0$ gives

$$\psi([X, Y]) = 0$$

agreeing with equation (3.5). This map is an injective homomorphism of $\widehat{H}$ into L. First we show that $\psi \in L$, i.e. that $\psi(Z) \in 2\pi i\mathbb{Z}$ for $Z \in \exp^{-1}(e)$. Since Ψ is a homomorphism we must have that $\Psi(e) = 1$, this implies that for $Z \in \exp^{-1}(e)$

$$\Psi(\exp(Z)) = \exp(2\pi in), \quad \text{for} \;\; n \in \mathbb{Z} \Rightarrow \Psi(\exp(tZ)) = \exp(2\pi int).$$

Now differentiating at $t = 0$ gives that $\psi(Z) = 2\pi in$. That this is a homomorphism $\Psi \cdot \Psi' \mapsto \psi + \psi'$ follows from the product rule, and that it is injective is clear.

Conversely, suppose that $\psi \in \widehat{h}$: one can try and form a character of H by exponentiation

$$\Psi(\exp(X)) := \exp(\psi(X)).$$

This map is only well-defined if $\psi \in L$, since we must have that $\Psi(e) = 1$. If this is the case then this does indeed give a character of H, with the mapping being an injective homomorphism, with $\psi + \psi' \mapsto \Psi \cdot \Psi'$. □

We see that characters of H form a lattice in characters of $\mathfrak{h}$. Similarly, we know that integer cohomology classes of H form a lattice in real cohomology classes of H. Lemma 3.2.7 says that characters of $\mathfrak{h}$ correspond to real cohomology classes.

Proposition 3.2.9 *The character group $\widehat{H}$ can be identified with $H^1(H,\mathbb{Z})$.*

Proof. Given $\omega \in H^1(H,\mathbb{R})$, one can try and form a character of H by setting

$$\Psi_\omega(h) := \exp\left(2\pi i \int_e^h \omega\right).$$

Clearly this defines an element of $U(1)$. For it to be a character it is necessary that we have $\Psi_\omega(hh') = \Psi_\omega(h)\Psi_\omega(h')$, i.e. that

$$\exp\left(2\pi i \int_e^{hh'} \omega\right) = \exp\left(2\pi i \int_e^h \omega\right) \cdot \exp\left(2\pi i \int_e^{h'} \omega\right). \tag{3.7}$$

Rewriting the left hand side of (3.7) gives

$$\exp\left(2\pi i \int_e^{hh'} \omega\right) = \exp\left[2\pi i \left(\int_e^h \omega + \int_h^{hh'} \omega\right)\right] = \exp\left(2\pi i \int_e^h \omega\right) \cdot \exp\left(2\pi i \int_h^{hh'} \omega\right)$$

and this equals the right hand side of (3.7) since ω is left-invariant. For Ψ_ω to be well-defined, we must have that $\Psi_\omega(e) = 1$, thus it is necessary that $\int_e^e \omega \in \mathbb{Z}$. Since all 1-cycles are homologous to such a loop through e we have that this happens only if $\omega \in H^1(H,\mathbb{Z})$. Thus we have an injective group homomorphism from $H^1(H,\mathbb{Z})$ into $\widehat{H}$, with $\omega + \omega' \mapsto \Psi_\omega \cdot \Psi_{\omega'}$.

Conversely, given $\Psi \in \widehat{H}$, we will show that there exists a corresponding form $\omega_\psi \in H^1(H,\mathbb{Z})$, with the mapping $\Psi \mapsto \omega_\Psi$ being an injective group homomorphism. Now, we know that Ψ differentiates to give $\psi \in L$ (see Lemma 3.2.8). This defines an element $\omega_\psi \in H^1(H,\mathbb{R})$, by

$$\omega_\psi = \frac{-i}{2\pi}\psi$$

It will be shown that $\omega_\psi \in H^1(H,\mathbb{Z})$, i.e. that $\oint \omega_\psi \in \mathbb{Z}$. This follows immediately, once we realise that any 1-cycle in H is homologous to the image under exp of the straight line tZ, with $t \in [0,1]$ and $Z \in \exp^{-1}(e)$. Call this image $\gamma(t) = \exp(tZ)$. Now, since ω_ψ is a left-invariant form, its value on a tangent vector at the point $h \in H$ is given by translating the tangent vector back to e and computing its value there. Integrating ω_ψ over γ gives

$$\int_\gamma \omega_\psi = \int_{t=0}^1 \omega_\psi(Z)dt = \omega_\psi(Z) \in \mathbb{Z}$$

by virtue of Lemma 3.2.8. Again, this is an injective group homomorphism from $\widehat{H}$ into $H^1(H,\mathbb{Z})$, with $\Psi \cdot \Psi' \mapsto \omega_\psi + \omega_{\psi'}$. □

In Proposition 3.2.6 we gave an isomorphism of the cohomology groups $H^1(G_a, \mathbb{R})$ and $H^2(\mathcal{O}(a), \mathbb{R})$, whilst from Lemma 3.2.7 we know that $H^1(G_a, \mathbb{R}) \cong \widehat{\mathfrak{g}_a}$.

We now put these two isomorphisms together by defining a class of invariant closed 2-forms on $\mathcal{O}(a)$ using an element of $\widehat{\mathfrak{g}_a}$. We will also show that the forms defined are integer cohomology classes if and only if the character used to describe them lives in the lattice L.

Definition. For $x \in \mathcal{O}(a)$ let ξ_x, η_x denote the vectors at x generated by ξ and $\eta \in \mathfrak{g}$, i.e. $\xi_x = ad^*_\xi(x)$ and $\eta_x = ad^*_\eta(x)$. If $y = Ad^*_g(x)$, then given $f_x \in (Z(\mathfrak{g}_x))^* \cong \widehat{\mathfrak{g}_x}$ (where $Z(\mathfrak{g}_x)$ is the centre of the lie algebra of the stabilizer of x) set $f_y = Ad^*_g(f_x) \in (Z(\mathfrak{g}_y))^*$. Now define a 2-form σ_f on $T_x\mathcal{O}(a)$ by

$$\sigma_{f_x}(\xi_x, \eta_x) := \langle f_x, [\xi, \eta] \rangle . \tag{3.8}$$

Remark. The space $(Z(\mathfrak{g}_a))^*$ is never empty, since a always belongs to it. When we denote an element f of this space without a subscript it is understood that this means $f = f_a$.

Proposition 3.2.10 *The 2-form σ_f is invariant and closed.*

Proof. The form σ_f will be shown to be well-defined, closed and invariant.

It is clear that σ_f is skew-symmetric. To show that it is well-defined, recall that a tangent vector only specifies an element of $\mathfrak{g}$ up to the addition of an element in $\mathfrak{g}_a$. Namely, the vector fields ξ_x, η_x generated by ξ and η are only defined up to elements of the centralizer of x, i.e. for $\lambda \in \mathfrak{g}_x$ consider $\tilde{\xi} = \xi + \lambda$. Then $\tilde{\xi}$ and ξ generate the same tangent vector at x since

$$\tilde{\xi}_x = ad^*_{\tilde{\xi}}(x) = ad^*_{\xi+\lambda}(x) = ad^*_\xi(x) + ad^*_\lambda(x) = ad^*_\xi(x) = \xi_x,$$

since $\lambda \in \mathfrak{g}_x$. To show that σ_f is well-defined, it needs to be shown that this does not affect the value of the form, i.e.

$$\sigma_{f_x}(\tilde{\xi}_x, \eta) = \sigma_{f_x}(\xi_x, \eta).$$

This is indeed true, since by specification f_x must belong to the centre of the centralizer of x and so

$$\begin{aligned}
\sigma_{f_x}(\tilde{\xi}_x, \eta) = \left\langle f_x, [\tilde{\xi}, \eta] \right\rangle &= \langle f_x, [\xi + \lambda, \eta] \rangle \\
&= \langle f_x, [\xi, \eta] \rangle + \langle f_x, [\lambda, \eta] \rangle \\
&= \langle f_x, [\xi, \eta] \rangle + \langle ad^*_\lambda(f_x), \eta \rangle \\
&= \langle f_x, [\xi, \eta] \rangle = \sigma_{f_x}(\xi_x, \eta),
\end{aligned}$$

since $\lambda \in \mathfrak{g}_x$ and $f_x \in (Z(\mathfrak{g}_x))^*$ and so $ad^*_\xi(f_x) = 0$. It is well-defined in the second argument since σ_f is skew-symmetric, hence σ_f is well-defined.

By construction, σ_{f_x} is invariant since f_y is given by $f_y = Ad^*_g(f_x)$ if $y = Ad^*_g(x)$.

It remains to check that σ_{f_x} is closed, i.e. that $d\sigma_{f_x} = 0$ — this is due to the Jacobi identity. Namely, given $\xi, \eta, \zeta \in \mathfrak{g}$ there are associated vector fields at x given by $\xi_x = ad^*_\xi(x)$ etc. The action of $d\sigma_{f_x}$ on ξ_x, η_x, ζ_x is given by

$$d\sigma_{f_x}(\xi_x, \eta_x, \zeta_x) = \circlearrowleft \xi_x \cdot \sigma_{f_x}(\eta_x, \zeta_x) - \circlearrowleft \sigma_{f_x}([\xi_x, \eta_x], \zeta_x),$$

where the symbol $\circlearrowleft$ means sum over all cyclic permuations of the arguments. The first three terms are given by

$$\xi_x \cdot \sigma_{f_x}(\eta, \zeta) = \left\langle ad^*_\xi(f_x), [\eta, \zeta] \right\rangle = - \left\langle f_x, [\xi, [\eta, \zeta]] \right\rangle .$$

Taking the sum over all cyclic permutations means that the Jacobi identity kills this group of three terms. The second three terms vanish for exactly the same reason, since

$$\sigma_{f_x}([\xi_x, \eta_x], \zeta_x) = \left\langle f_x, [\xi, [\eta, \zeta]] \right\rangle .$$

Therefore, evaluated on any three vector fields the form $d\sigma_{f_x}$ vanishes — hence it is closed. □

Remark. In general σ_f does not define a symplectic form, since there is no guarantee that it is non-degenerate. Indeed, $0 \in (Z(\mathfrak{g}))^*$ defines a form σ_0 that is degenerate everywhere. Therefore σ_f defines what is sometimes called a pre-symplectic form, i.e. a closed 2-form. However, if $f = \epsilon a$ for $\epsilon \neq 0$ then it is clear that σ_f is indeed a symplectic form, being given by $\epsilon \omega_{KK}$. (Note that this is not a necessary condition for an element f to define a symplectic form.) This is the class of forms that were considered in [4], [5] and [14], where it was shown that the classical dynamics are integrable.

The upshot is that given a character $f \in \widehat{\mathfrak{g}_a}$ we have explicitly constructed a real 2-form ω_f on $\mathcal{O}(a)$ from it — the form ω_f may be explicitly described in terms of the geometry of the principal fibre bundle $\rho : G \overset{G_a}{\to} \mathcal{O}(a)$.

Write Θ for the Maurer-Cartan form on G, which is a $\mathfrak{g}$-valued left-invariant 1-form on G such that $\Theta(e)(X) = X$. In matrix notation, the explicit formula for Θ is given by

$$\Theta(g)(X) = g^{-1} \cdot X \quad \text{for} \quad X \in T_g G$$

and it is sometimes denoted by $g^{-1}dg$ to highlight its left-invariance.

Now define the 1-form $\theta_f = - \left\langle f, \Theta \right\rangle$ on G.

Proposition 3.2.11 *The form ω_f on $\mathcal{O}(a)$ is such that $\rho^*(\omega_f) = d\theta_f$.*

Proof. This result appears in [26]. Write $\widetilde{X}, \widetilde{Y}$ for left-invariant vector fields on G and X, Y for the corresponding vector fields on $\mathcal{O}(a)$. Then, by definition:

$$\begin{aligned} d\theta_f(\widetilde{X}, \widetilde{Y}) =& \widetilde{X} \cdot \theta_f(\widetilde{Y}) - \widetilde{Y} \cdot \theta_f(\widetilde{X}) - \theta_f([\widetilde{X}, \widetilde{Y}]) \\ =& - \theta_f([\widetilde{X}, \widetilde{Y}]), \end{aligned}$$

since $\theta_f(\widetilde{X})$ and $\theta_f(\widetilde{Y})$ are left-invariant functions on G (i.e. constants) and so the Lie derivative of them vanishes. This can be rewritten as

$$-\theta_f([\widetilde{X}, \widetilde{Y}]) = -\theta_f(\widetilde{[X, Y]}) = \langle f, [X, Y] \rangle = \rho^*(\omega_f)(\widetilde{X}, \widetilde{Y}).$$

□

Remark. This description of ω_f can be used to give another proof of the fact it is closed. Since $\rho : G \to \mathcal{O}(a)$ is a submersion, the map $\rho_* : T_g G \to T_{\rho(g)}\mathcal{O}(a)$ is surjective. Therefore the map ρ^* is injective, hence

$$\rho^*(d\omega_f) = d\rho^*(\omega_f) = d^2\theta_f = 0.$$

Proposition 3.2.12 *The form ω_f on $\mathcal{O}(a)$ defines an integer cohomology class if and only if $2\pi i f \in L$, in which case f defines a character of G_a and so $H^2(\mathcal{O}(a), \mathbb{Z}) \cong \widehat{G_a}$.*

Proof. We need to show that if $2\pi i f \in L$ then the integral of ω_f over a 2-cycle lies in $\mathbb{Z}$. This extends the argument from Proposition 3.2.9 Given $Z \in \exp^{-1}(e)$, denote by γ the loop in G_a that is the image of the segment $[0, Z]$ under exp. Since G is simply-connected by assumption, γ is the boundary of a 2-dimensional surface S in G, which projects to give a 2-cycle $\rho(S)$ on $\mathcal{O}(a)$. Therefore, we have that

$$\int_{\rho(S)} \omega_f = \int_S d\theta_f = \int_\gamma \theta_f = \langle f, Z \rangle$$

and this lies in $\mathbb{Z}$ if and only if $2\pi i f$ defines a character of G_a, by Lemma 3.2.8. The correspondence of $[\gamma]$ and $\rho(S)$ is precisely the isomorphism $\pi_1(G_a) \cong \pi_2(\mathcal{O}(a))$ given in Lemma 3.2.4. □

3.2.4 Magnetic cotangent bundles to coadjoint orbits

As described at the start of this chapter, the classical phase space that we are quantizing is a magnetic cotangent bundle to a coadjoint orbit. Here we explain some properties of this object as well as giving an analogue of the result of Novikov and

Schmeltzer [43] by embedding them as coadjoint orbits themselves. Although technically unnecessary, this result is somehow pleasing because it takes us back to the starting point for the work of the previous chapter. Most of the results in this section are based on corresponding results in [5] for adjoint orbits.

Definition. A *magnetic cotangent bundle to a coadjoint orbit* is a symplectic manifold of the form

$$(T^*\mathcal{O}(a), dp \wedge dx + \pi^*\omega_f), \tag{3.9}$$

where ω_f is defined in (3.8).

This object was essentially considered in [4], [5] and [14], and most of the results in this section were essentially given there, although perhaps phrased differently.

If $\mathcal{O}(a)$ is a coadjoint orbit of a compact Lie group G then a magnetic cotangent bundles to $\mathcal{O}(a)$ can be obtained by symplectic reduction from T^*G, as explained in [5] and also [49].

First note that the action of $G \times G$ on G

$$(g_1, g_2) \cdot (g) = (g_1 g g_2^{-1})$$

extends to a Hamiltonian action on T^*G by

$$(g_1, g_2) \cdot (g, F) = (g_1 g g_2^{-1}, Ad^*_{g_2}(F)).$$

The tangent space to T^*G at the point (g, F) is given by $\mathfrak{g} \oplus \mathfrak{g}^*$, so an element of it can be described as $v = (g, F; X, F')$, where $X \in \mathfrak{g}$ and $F, F' \in \mathfrak{g}^*$. The canonical 1-form θ on T^*G can be described in terms of its action on v as

$$\theta(v) = \langle F, X \rangle .$$

Therefore the moment maps associated with the left and right actions are given by

$$\mu_l : (g, F) \mapsto Ad^*_g(F) \qquad \mu_r : (g, F) \mapsto F.$$

Now consider the right action of $G_a \subset G$ on T^*G. The moment map $\Psi : T^*G \to \mathfrak{g}^*_a$ for this action is given by

$$\Psi(g, F) = pr_{\mathfrak{g}^*_a}(F),$$

where pr denotes the orthogonal projection with respect to the Cartan–Killing form.

Symplectic reduction is accomplished by taking the quotient of the fibre of the moment map

$$\Psi^{-1}(f) = \big\{(g, F) \,|\, pr_{\mathfrak{g}^*_a}(F) = f\big\}$$

with respect to the G_a action. It is well-known (see [5]) that the reduced phase space $\Psi^{-1}(0)/G_a$ is symplectomorphic to the cotangent bundle $(T^*(G/G_a), dp \wedge dx)$. This can be seen by noting that $\Psi^{-1}(0) \cong G \times (\mathfrak{g}/\mathfrak{g}_a)^*$ consists of those covectors that are killed on the orbits of the G_a-action on G and checking that the restriction of the canonical 1-form on T^*G to $\mu^{-1}(0)$ coincides with the pullback to $\mu^{-1}(0)$ of the canonical 1-form on $T^*(G/G_a)$.

If we consider the symplectic reduction over non-trivial points then we can also describe magnetic cotangent bundles. Suppose $f \in \mathfrak{g}_a^*$ defines a magnetic form ω_f on $\mathcal{O}(a)$ as in (3.8). In that case the reduced phase space $\Psi^{-1}(f)/G_a$ is again diffeomorphic to $T^*(G/G_a)$, since $\Psi^{-1}(f) \cong G \times (\mathfrak{g}/\mathfrak{g}_a)^*$ and f is invariant under G_a. However, the symplectic form on the reduced space is given by $dp \wedge dx + \pi^*(\omega_f)$. A proof of this result is given in [5], and is essentially accomplished by showing that the 1-form on G_a transgresses to give the 2-form ω_f on G/G_a.

Going back to the left action of G on T^*G, we have that this commutes with the right action of G_a and leaves $\Psi^{-1}(f)$ invariant. Representing a point in the reduced space by (g, F) we have that the moment map for the left action of G on $T^*(G/G_a)$, with respect to the symplectic form $dp \wedge dx + \pi^*(\omega_f)$ is given by

$$\Phi_f(g, F) = Ad_g^*(F + f).$$

Magnetic cotangent bundles to coadjoint orbits can also be realised directly as coadjoint orbits themselves of a group S whose dimension is twice that of G. This procedure was essentially explained in [5] (and see also the references therein). This extends the construction given in [43], where the phase space of the classical Dirac monopole on S^2 was given as a coadjoint orbit of $E(3)$. Here we show that S has a natural interpretation as the tangent bundle of G when considered as a Lie group in its own right. Although this is not essential for the scheme of geometric quantization it is satisfying to have such a natural interpretation for the result of Novikov and Schmeltzer, which was the starting point for the investigation conducted in the previous chapter.

As is well-known, the tangent bundle to a Lie group G is, as a manifold, a trivial vector bundle $TG \cong G \times \mathfrak{g}$ over G. Simply choosing a basis of left-invariant vector fields at e and translating around G gives the required identification.

Similarly, one can also consider TG to be a Lie group — it is the direct product of a Lie group and a vector space. The 'obvious' group structure on TG, namely the direct product structure $(g_1, X_1) \cdot (g_2, X_2) = (g_1 g_2, X_1 + X_2)$, can be obtained by first translating the tangent vectors $g_1 X_1 \in T_{g_1}G$ and $g_2 X_2 \in T_{g_2}G$ back to e, adding them there and then translating to $g_1 g_2$.

However there is a more natural group structure on TG is given as follows: if $\gamma_1(t) = g_1 + tg_1X_1 + \mathcal{O}(t^2)$ and $\gamma_2(t) = g_2 + tg_2X_2 + \mathcal{O}(t^2)$ represent (g_1, X_1) and (g_2, X_2) in TG then their product is defined to be

$$(g_1, X_1) \cdot (g_2, X_2) = (g_1g_2, Ad_{g_2^{-1}}X_1 + X_2),$$

since naively multiplying γ_1 and γ_2 gives

$$\begin{aligned}\gamma_1(t) \cdot \gamma_2(t) &= g_1g_2 + t(g_1X_1g_2 + g_1g_2X_2) + \mathcal{O}(t^2)\\ &= g_1g_2 + tg_1g_2(Ad_{g_2^{-1}}X_1 + X_2) + \mathcal{O}(t^2).\end{aligned}$$

Identifying $(G, 0)$ with G and $(e, \mathfrak{g})$ with $\mathfrak{g}$, gives the tangent bundle TG the structure of the semidirect product $S := TG = G \ltimes_{Ad} \mathfrak{g}$.

If the group elements of S are denoted by (g, X) and the Lie algebra elements of $\mathfrak{s}$ denoted by (u, v), then one may calculate various important operations on S, $\mathfrak{s} \cong \mathfrak{g} \ltimes \mathfrak{g}$ and $\mathfrak{s}^* \cong (\mathfrak{g} \ltimes \mathfrak{g})^*$. These are summarized below, the first few without proof. A similar object was considered in [5] and one can see that for more details, also a more general set of results are given in [49].

Lemma 3.2.13 *The Lie group structure on S is given by:*

$$(g_1, X_1) \cdot (g_2, X_2) = (g_1g_2, Ad_{g_2^{-1}}X_1 + X_2) \tag{3.10}$$

and so the inverse element to $(g, X) \in S$ is given by

$$(g, X)^{-1} = (g^{-1}, -Ad_g(X)). \tag{3.11}$$

Lemma 3.2.14 *The adjoint action of S on $\mathfrak{s}$ is given by*

$$Ad_{(g,x)}(u, v) = (Ad_g(u), Ad_g([x, u] + v)). \tag{3.12}$$

Lemma 3.2.15 *The Lie algebra structure on $\mathfrak{s}$ is given by*

$$[(u_1, v_1), (u_2, v_2)] = ([u_1, u_2], [u_1, v_2] - [u_2, v_1]). \tag{3.13}$$

Lemma 3.2.16 *The coadjoint action of $(g, x) \in S$ on $(f, a) \in \mathfrak{s}^*$ is given by*

$$Ad^*_{(g,x)}(f, a) = \left(Ad^*_g(f + ad^*_x(a)), Ad^*_g(a)\right). \tag{3.14}$$

Proof. The proof is by direct calculation, recall that the coadjoint action of $(g, x) \in S$ on $(f, a) \in \mathfrak{s}^*$ is defined by means of the pairing with any $(u, v) \in \mathfrak{s}$ as

$$\begin{aligned}\left\langle Ad^*_{(g,x)}(f, a), (u, v)\right\rangle &= \left\langle (f, a), Ad^{-1}_{(g,x)}(u, v)\right\rangle\\ &= \left\langle (f, a), Ad_{(g^{-1}, -Ad_g(x))}(u, v)\right\rangle.\end{aligned}$$

This calculation is somewhat easier to read if notation is abused and the pairing is split on each factor. Then, using (3.11) and (3.12), the above can be rewritten as

$$\begin{aligned}\left\langle (f,a), Ad_{(g^{-1},-Ad_g(x))}(u,v)\right\rangle &= \langle f, Ad_{g^{-1}}(u)\rangle + \langle a, Ad_{g^{-1}}\left([-Ad_g(x),u]+v\right)\rangle \\ &= \left\langle Ad_g^*(f),u\right\rangle + \langle a, -\left[x, Ad_{g^{-1}}(u)\right]\rangle + \left\langle Ad_g^*(a),v\right\rangle \\ &= \left\langle Ad_g^*(f),u\right\rangle + \left\langle Ad_g^*\left(ad_X^*(a)\right),u\right\rangle + \left\langle Ad_g^*(a),v\right\rangle \\ &= \left\langle Ad_g^*\left(f + ad_X^*(a)\right),u\right\rangle + \left\langle Ad_g^*(a),v\right\rangle\end{aligned}$$

and so the coadjoint action is indeed given by (3.14). □

One may also ask for the coadjoint action of $\mathfrak{s}$ on $\mathfrak{s}^*$.

Lemma 3.2.17 *The coadjoint action of* $(u,v) \in \mathfrak{s}$ *on* $(f,a) \in \mathfrak{s}^*$ *is given by*

$$ad^*_{(u,v)}(f,a) = \left(ad_u^*(f) + ad_v^*(a), ad_u^*(a)\right). \tag{3.15}$$

Proof. Again, the proof is by direct calculation. For $(s,t) \in \mathfrak{s}$, the coadjoint action of $(u,v) \in \mathfrak{s}$ on $(f,a) \in \mathfrak{s}^*$ is defined by

$$\begin{aligned}\left\langle ad^*_{(u,v)}(f,a),(s,t)\right\rangle &= \left\langle (f,a), -ad_{(u,v)}(s,t)\right\rangle \\ &= \langle (f,a), ([s,u],[s,v]+[t,u])\rangle \\ &= \langle ad_u^*(f),s\rangle + \langle ad_v^*(a),s\rangle + \langle ad_u^*(a),t\rangle \\ &= \langle \left(ad_u^*(f)+ad_v^*(a), ad_u^*(a)\right),(s,t)\rangle\end{aligned}$$

where the right hand side has been rewritten using (3.12). □

One might then ask, what is the stabilizer of (f,a) under the coadjoint action.

Lemma 3.2.18 *The stabilizer of* (f,a) *in* S *under the coadjoint action is given by* $S_{(f,a)} = G_a \ltimes \mathfrak{g}_a$ *and the stabilizer of* (f,a) *in* $\mathfrak{s}$ *under the coadjoint action is given by* $\mathfrak{s}_{(f,a)} = \mathfrak{g}_a \ltimes \mathfrak{g}_a$.

Proof. The proof is simple. For the first statement, (3.14) shows that if $Ad^*_{(g,x)}(f,a) = (f,a)$ then $g \in G_a$ and $x \in \mathfrak{g}_a$. For the second statement, (3.15) shows that if $ad^*_{(u,v)}(f,a) = 0$ then both u and $v \in \mathfrak{g}_a$. These two answers agree in the sense that $\mathfrak{s}_{(f,a)}$ is the Lie algebra of $S_{(f,a)}$. □

Lemma 3.2.19 *For* $g \in G$, $\xi \in \mathfrak{g}$ *and* $a \in \mathfrak{g}^*$, *the following 'commutativity relation' between* ad^* *and* Ad^* *holds:*

$$ad^*_{Ad_g(\xi)}Ad_g^*(a) = Ad_g^* ad_\xi^*(a). \tag{3.16}$$

Proof. Again, the proof is by direct calculation using the definitions of Ad^* and ad^*, for any $\eta \in \mathfrak{g}$ we have that

$$\begin{aligned} \left\langle ad^*_{Ad_g(\xi)} Ad^*_g(a), \eta \right\rangle &= -\left\langle Ad^*_g(a), ad_{Ad_g(\xi)}(\eta) \right\rangle \\ &= -\left\langle a, Ad_{g^{-1}}(ad_{Ad_g(\xi)}(\eta)) \right\rangle \\ &= -\left\langle a, ad_\xi(Ad_{g^{-1}}(\eta)) \right\rangle \\ &= \left\langle ad^*_\xi(a), Ad_{g^{-1}}(\eta) \right\rangle \\ &= \left\langle Ad^*_g ad^*_\xi(a), \eta \right\rangle . \end{aligned}$$

□

Corollary 3.2.20 *Suppose that $\xi \in \mathfrak{g}_a$, then $Ad_g(\xi) \in \mathfrak{g}_{Ad^*_g(a)}$. Likewise, if $\xi \in \mathfrak{g}_a^\perp$ then $Ad_g(\xi) \in \mathfrak{g}^\perp_{Ad^*_g(a)}$.*

The Cartan–Killing form allows us to embed $T\mathcal{O}(a)$ in $\mathfrak{g}^* \oplus \mathfrak{g}^*$ by

$$T\mathcal{O}(a) \cong \left\{ (x, v) \in \mathfrak{g}^* \oplus \mathfrak{g}^* \mid x = Ad^*_g(a), v \in \mathfrak{g}_x^{\perp *} \right\}$$

and similarly $T^*\mathcal{O}(a)$ can be embedded in $\mathfrak{g}^* \oplus \mathfrak{g}$ by

$$T^*\mathcal{O}(a) \cong \left\{ (x, p) \in \mathfrak{g}^* \oplus \mathfrak{g} \mid x = Ad^*_g(a), p \in \mathfrak{g}_x^\perp \right\}$$

Now, G acts naturally on $T^*\mathcal{O}(a)$ by

$$g : (x, p) \mapsto (Ad^*_g(x), Ad_g(p)) \tag{3.17}$$

since Corollary 3.2.20 says that if $p \in \mathfrak{g}_x^\perp$ then $Ad_g(p) \in \mathfrak{g}^\perp_{Ad^*_g(x)}$.

This can be extended to an action of TG on $T^*\mathcal{O}(a)$ by adding in the action of $\mathfrak{g}$

$$(g, X) : (x, p) \mapsto (Ad^*_g(x), Ad_g(p + pr_{\mathfrak{g}_x^\perp}[X])). \tag{3.18}$$

This genuinely gives an action of TG, since acting with (g_1, X_1) on $(g_2, X_2) \cdot (x, p)$ gives

$$\begin{aligned} (g_1, X_1) \cdot [(g_2, X_2) \cdot (x, p)] &= \left(Ad^*_{g_1 g_2}(x), Ad_{g_1 g_2}\left(p + pr_{\mathfrak{g}_x^\perp}[X_2]\right) + Ad_{g_1}\left(pr_{\mathfrak{g}_{x'}^\perp}[X_1] \right) \right) \\ &= \left(Ad^*_{g_1 g_2}(x), Ad_{g_1 g_2}\left(p + pr_{\mathfrak{g}_x^\perp}[X_2] + pr^\perp_{\mathfrak{g}_x}[Ad^{-1}_{g_2}(X_1)]\right)\right), \end{aligned}$$

where $x' = Ad^*_{g_2}(x)$ and the commutativity relation (3.16) has been used. This is the same as acting on (x, p) with $(g_1, X_1) \cdot (g_2, X_2) = (g_1 g_2, Ad_{g_2^{-1}}(X_1) + X_2)$. The results in this section have been leading up to the following, which is essentially the same as a result from [5].

Theorem 3.2.21 *Given $a \in \mathfrak{g}^*$, let $f \in \mathfrak{g}^*$ be such that $ad^*_X(f) = 0$ for $X \in \mathfrak{g}_a$, i.e. f belongs to the centre of the centralizer of a. The coadjoint orbit $\mathcal{O}(f,a)$ of the point $(f,a) \in \mathfrak{s}^*$, equipped with the Kostant–Kirillov canonical symplectic form is symplectomorphic to the magnetic cotangent bundle $(T^*\mathcal{O}(a), dp \wedge dx + \pi^*(\omega_f))$.*

Proof. First note that the two manifolds have the same dimension, as demonstrated by Lemma 3.2.18.

Recall from (3.14) that $\mathcal{O}(f,a)$ may be parametrized as

$$\mathcal{O}(f,a) = \left(Ad^*_g(f_a + ad^*_\xi(a)), Ad^*_g(a)\right) \subset (\mathfrak{g} \times \mathfrak{g})^*$$

and that by identifying $T^*\mathcal{O}(a)$ with $T\mathcal{O}(a)$ using the normal metric, $T^*\mathcal{O}(a)$ can be parametrized as

$$T^*\mathcal{O}(a) = \{(x,p) : x = Ad^*_g(a), p \in \mathfrak{g}_x^\perp\}.$$

Since $f_a \in \mathfrak{g}_a$ and $ad^*_\xi(a) \in \mathfrak{g}_a^\perp$, it is clear that $\mathcal{O}(f,a)$ is a one-point orbit over $T^*\mathcal{O}(a)$, with the mapping between the two given by

$$\phi : \left(Ad^*_g(f + ad^*_\xi(a)), Ad^*_g(a)\right) \mapsto \left(Ad^*_g(a), ad^*_{Ad_g(\xi)} Ad^*_g(a)\right).$$

This map is well-defined because of Corollary 3.2.20.

The second part of the proof is showing that the symplectic forms coincide. The Kostant–Kirillov form on $\mathcal{O}(f,a)$ at (f,a) is defined by

$$\begin{aligned}\sigma_{KK}((u_1,v_1),(u_2,v_2)) &= \langle (f,a), [(u_1,v_1),(u_2,v_2)] \rangle \\ &= \langle (f,a), ([u_1,u_2],[u_1,v_2]-[u_2,v_1]) \rangle \\ &= \langle f, [u_1,u_2] \rangle + \langle a, [u_1,v_2]-[u_2,v_1] \rangle,\end{aligned}$$

where second equality is given by using (3.13). But this is exactly the symplectic form $dp \wedge dx + \pi^*(\omega_f)$ on $T^*\mathcal{O}(a)$ and so we are done. The inverse of ϕ is actually the moment map for the action of TG on $T^*\mathcal{O}(a)$ with respect to the form $dp \wedge dx + \pi^*(\omega_f)$ on $T^*\mathcal{O}(a)$, i.e. $\phi^{-1} : T^*\mathcal{O}(a) \to \mathfrak{s}^*$. This can be seen by noting that the moment map for the action of G on $(T^*\mathcal{O}(a), dp \wedge dx + \pi^*(\omega_f))$ (3.17) is given by

$$\Phi : T^*\mathcal{O}(a) \to \mathfrak{g}^* \qquad \Phi_f(x,p) = ad^*_p(x) + f_x,$$

since the Hamiltonian function associated to the element $\xi \in \mathfrak{g}$ is given by

$$H_f(\xi) = \left\langle ad^*_\xi(x), p \right\rangle + \langle f_x, \xi \rangle \tag{3.19}$$

and the Hamiltonian vector field of ξ at (x,p) is

$$\xi_x = ad^*_\xi(x) + \langle f_x, \xi \rangle. \tag{3.20}$$

Adding in the action of $\mathfrak{g}$ given in (3.18), we find that the moment map is given by

$$\Theta_f : T^*\mathcal{O}(a) \to \mathfrak{s}^* \qquad \Theta_f(x,p) = (ad_p^*(x) + f_x, x),$$

which is exactly ϕ^{-1}. □

Remark. It is worth remarking that this is a very special orbit of TG: if f is not fixed by all $X \in \mathfrak{g}_a$, then the coadjoint orbit through (f, a) will be a fibre bundle over $T^*\mathcal{O}(a)$ with fibre the G_a orbit of f (see e.g. [49]). The case considered here is a 'one-point' orbit over $T^*\mathcal{O}(a)$, exactly as considered by Novikov and Schmelzer [43] and described in Chapter Two.

Going back to the description of the classical mechanics given at the start of the Chapter we can describe the Hamiltonian function for the geodesic flow. The Hamiltonian function for the classical geodesic flow on $\mathcal{O}(a)$ with respect to the normal metric is given by

$$H_0 = \frac{1}{2}\langle \Phi_0, \Phi_0 \rangle = \frac{1}{2}\left\langle ad_p^*(x), ad_p^*(x) \right\rangle,$$

whilst the Hamiltonian function for the magnetic geodesic flow with magnetic term ω_f is given by

$$\begin{aligned} H_f = \frac{1}{2}\langle \Phi_f, \Phi_f \rangle &= \frac{1}{2}\left\langle ad_p^*(x), ad_p^*(x) \right\rangle + \left\langle ad_p^*(x), f_x \right\rangle + \frac{1}{2}\langle f_x, f_x \rangle \\ &= \frac{1}{2}\left\langle ad_p^*(x), ad_p^*(x) \right\rangle + \frac{1}{2}\langle f_x, f_x \rangle = H_0 + const, \end{aligned}$$

since $f_x \in \mathfrak{g}_x^*$ and $ad_p^*(x) \in \mathfrak{g}_x^\perp$.

3.3 Geometric quantization

We now look to apply Geometric Quantization to the classical mechanical system of magnetic geodesic flow on a coadjoint orbit that was described in Section 3.1: this is an extension of the problem considered in the previous chapter. Recall from Section 3.2.4 that the classical phase space is described by the symplectic manifold

$$(T^*\mathcal{O}(a), dp \wedge dx + \pi^*(\omega_f)).$$

We use the formalism of Geometric Quantization to a greater extent to achieve this. We describe the essence of the procedure, before explaining how it works for the case at hand. The goal is to produce from a classical phase space (i.e. a pre-symplectic manifold (M, ω), where ω is a closed 2-form on M) a quantum phase space (i.e. a

Hilbert space $\mathcal{H}$ and an algebra $\mathcal{A}$ of operators on it), in such a way that classical observables (i.e. functions on M) are sent to operators in $\mathcal{A}$ in a sensible way. (In our setup we have that $(M, \omega) = (T^*\mathcal{O}(a), dp \wedge dx + \pi^*(\omega_f))$, as described in the previous chapter.)

The starting point is a natural geometric idea: to ask that ω is the curvature form of a line bundle $L \to M$. This imposes an integrality condition on ω that is reminiscent of a quantization condition, namely that $\omega/2\pi \in H^2(M, \mathbb{Z})$. This is natural from the point of view of physics too: being essentially a rephrasing of the Bohr–Sommerfeld condition and is known as the prequantization condition.

Definition. A symplectic manifold (M, ω) is said to be *prequantizable* if $\omega/2\pi$ lies in the image of $H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})$. This means that the integral of ω around any 2-cycle should lie in $2\pi\mathbb{Z}$.

A first guess at a Hilbert space might be the space $L^2(L)$ of L^2-sections of this bundle — however, it turns out that this space is too large. A colloquial explanation of this is given by taking $M = \mathbb{R}^{2n}$ being the cotangent bundle to $\mathbb{R}^n$. The space $L^2(L)$ then consists of functions in both coordinates and momenta, but the equations of quantum mechanics can be formulated with respect to just coordinates or just momenta (the Schrödinger formulation or the Heisenberg formulation). The upshot being that there are twice as many variables as needed: the established procedure to strip them away is to choose a Lagrangian polarization on M and choose for $\mathcal{H}$ those sections in $L^2(L)$ that are constant along the leaves of the polarization.

One then has to quantize the classical observables, i.e. to each element $f \in C^\infty(M)$ (a classical observable), one wants to associate an operator $\hat{f} : \mathcal{H} \to \mathcal{H}$. This is to be done in such a way that Poisson brackets of functions go into commutators of operators:

$$\{f, g\} = h \to [\hat{f}, \hat{g}] = -i\hat{h}. \tag{3.21}$$

The theory is very deep-rooted and has moved in several different directions: good references are [23], [27], [31] and [56].

We start by explaining the general theory of geometric quantization, before applying it to the case of magnetic cotangent bundles to coadjoint orbits.

3.3.1 Broad scheme of geometric quantization

In giving an outline of geometric quantization, we follow here mainly [56]. It is necessary to begin with a few definitions.

Definition. A *Hermitian structure* on a vector bundle $E \xrightarrow{V} M$ is a Hermitian inner product $(\cdot,\cdot)$ on each fibre V_x. The inner product should be smooth, in the sense that the function $v \mapsto (v,v)$ is smooth for $v \in V$.

Definition. If $E \to M$ has a connection ∇ then the Hermitian structure is *compatible with the connection* ∇ if for all smooth sections $s, s' \in \Gamma(E)$ and smooth vector fields $X \in \mathfrak{X}(M)$:

$$i_X d(s,s') = (\nabla_X s, s') + (s, \nabla_X s').$$

Definition. Two vector bundles (E,π) and (E',π') over M are *equivalent* if there is a morphism $\phi : E \to E'$ respecting any structures on the bundles. For a Hermitian line bundle with connection, this requires that

$$\pi' \circ \phi(e) = \pi(e), \qquad \phi(\nabla_X s) = \nabla'_X(\phi(s)), \qquad (\phi(s),\phi(s))' = (s,s).$$

Definition. Given a connection ∇ on $E \to M$, its *curvature* is the 2-form B defined by

$$B(X,Y)s = i(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})s,$$

for any two smooth vector fields X, Y and a smooth section s.

Lemma 3.3.1 *The set (group) of equivalence classes of topological (i.e. with no additional structures imposed) smooth line bundles over M is isomorphic to $H^2(M,\mathbb{Z})$.*

Proof. The sheaf of smooth functions on M is denoted by ϵ and the sheaf of non-vanishing smooth functions on M is denoted ϵ^*. An equivalence class of smooth line bundles is defined by an element of the Cech cohomology group $H^1(M,\epsilon^*)$ and it turns out that $H^1(M,\epsilon^*) \cong H^2(M,\mathbb{Z})$. The isomorphism is given by sending a line bundle to its Chern class, which is the connecting homomorphism in the long exact sequence in sheaf cohomology corresponding to the exponential sheaf sequence

$$0 \to \mathbb{Z} \to \epsilon \xrightarrow{\exp} \epsilon^* \to 1.$$

In particular, since ϵ is a fine sheaf, all of the higher sheaf cohomology groups $H^i(M,\epsilon)$ vanish for $i \geq 1$. See any of [18], [26], [31] or [56] (amongst others) for details. $\square$

The first fundamental result of geometric quantization is the following, see [56] for more details.

Proposition 3.3.2 *Given a manifold M and a closed 2-form ω, there exists a Hermitian line bundle $L \to M$ and a connection ∇ on L with curvature ω iff $\omega/2\pi$ is an integral 2-form, i.e. iff (M,ω) is prequantizable.*

Proof. Suppose that ω is an integral 2-form. Then, there exists a contractible open cover $\{U_i\}$ of M and a collection of 'symplectic potentials' $\theta_j \in \Omega^1(U_j)$, i.e. forms such that $d\theta_j = \omega$. There also exist functions $u_{jk} \in C^\infty(U_j \cap U_k)$ such that

$$du_{jk} = \theta_j - \theta_k \quad \text{if} \quad U_j \cap U_k \neq \emptyset$$

and

$$\frac{1}{2\pi}\left(u_{jk} + u_{kl} + u_{lj}\right) \in \mathbb{Z} \quad \text{if } U_j \cap U_k \cap U_l \neq \emptyset.$$

Now, setting $c_{jk} = \exp(iu_{jk})$, we have that on non-empty intersections

$$\frac{dc_{jk}}{c_{jk}} = i(\theta_j - \theta_k)$$

and

$$c_{jk}c_{kl}c_{lj} = \exp\left[2\pi i(u_{jk} + u_{kl} + u_{lj})\right] = 1$$

This means that the c's are the transition functions of a line bundle $L \to M$ with curvature ω. Since the potentials are real and the transition functions are of unit modulus there exists a compatible Hermitian structure.

Conversely, suppose that we are given a line bundle L with connection ∇ and curvature ω — it will be shown that $\omega \in H^2(M, \mathbb{Z})$. Let c_{jk} be the transition functions of L relative to some open cover. On each non-empty triple intersection, set

$$z_{jkl} := \frac{1}{2\pi i}\left[\log(c_{jk}) + \log(c_{kl}) + \log(c_{lj})\right].$$

Since the c's are smooth functions satisfying the cocycle condition, we have that z_{jkl} is an integer, and hence a constant: moreover, the z's cocycles. Note that there is an ambiguity in the definition of the logarithms since log is only defined up to an integer. However, the cohomology class $[z]$ of z does not depend on the choice of branches — it is called the Chern class of L. Therefore, $2\pi z$ is a representative cocycle in the class of $H^2(M, \mathbb{R})$ determined by ω. □

Proposition 3.3.3 *[56] The inequivalent choices of L and ∇ are parametrized by $H^1(M, U(1))$. If M is simply-connected then $H^1(M, U(1)) = 0$ and so any connection on a line bundle is uniquely determined by its curvature and vice-versa.*

Proof. Again, the proof of this proposition is taken from [56].

In the above construction of L and ∇ from ω there is a freedom of choice, since sending $u_{jk} \mapsto u_{jk} + y_{jk}$, where the y's are real constants satisfying

$$y_{jk} = -y_{kj} \quad \text{and} \quad \frac{1}{2\pi}(y_{jk} + y_{kl} + y_{lj})$$

on non-empty intersections. This sends $L \mapsto L \otimes F$, where F is the Hermitian line bundle with transition functions $t_{jk} = \exp(iy_{jk})$. Since t_{jk} is constant, F has a connection with curvature 0 and so $L \otimes F$ has the same curvature as L.

Conversely, if (L, ∇) and (L', ∇') are Hermitian line bundles both having curvature ω then $F = L^{-1} \otimes L$ is a Hermitian line bundle with flat connection labelled by elements of $H^1(M, U(1))$.

If $\pi_1(M) = 0$ then $H^1(M, U(1)) = 0$ and so there is a unique up to equivalence Hermitian line bundle L with connection ∇ and curvature ω. □

3.3.2 Magnetic cotangent bundles

Some of the first steps in the geometric quantization of magnetic cotangent bundles over an arbitrary manifold M are outlined in this subsection. This will be specialized later on to the specific case when M is a coadjoint orbit — but in the first part the reasoning applies to a general magnetic cotangent bundle.

Lemma 3.3.4 *A magnetic cotangent bundle $(T^*M, dp \wedge dx + \pi^*(\omega))$ is prequantizable (i.e. the form $dp \wedge dx + \pi^*(\omega)$ is integral on T^*M) iff (M, ω) is prequantizable.*

Proof. Given any $[\alpha] \in H_2(T^*M, \mathbb{Z})$, we have that $[\alpha] \sim [\beta] \in H_2(M, \mathbb{Z})$ and that

$$\int_{[\alpha]} (dp \wedge dx + \pi^*(\omega)) = \int_{[\beta]} \omega.$$

Therefore $dp \wedge dx + \pi^*(\omega)$ is an integral 2-form iff ω is an integral 2-form. □

Therefore, if ω is an integral 2-form then Proposition 3.3.2 guarantees that there is a Hermitian line bundle $L' \to T^*M$ with a connection over T^*M whose curvature is $dp \wedge dx + \pi^*(\omega)$.

The next step is to choose a polarization on T^*M. There is one very natural polarization, namely the one given by the vertical vectors. This means that only those sections that are constant along the fibres of T^*M are picked out.

Using the standard local coordinates (x^i, p_i) on T^*M about x and choosing as a distribution $F_x = \left\{ \frac{\partial}{\partial p_i}\Big|_x \right\}$ defines a Lagrangian polarization since $\Omega|_{F_x} = 0$.

If M is simply-connected then taking the vertical polarization on $(T^*M, dp \wedge dx + \pi^*\omega)$ means that instead of looking at sections of (the unique up to equivalence) Hermitian line bundle with connection $L' \to T^*\mathcal{O}(a)$, whose curvature is $dp \wedge dx + \pi^*\omega$; we should instead look at sections of the (again, unique up to equivalence) Hermitian line bundle with connection $L \to M$ whose curvature is ω.

For the case at hand, we summarise this reasoning in a Theorem.

Theorem 3.3.5 *Suppose that the magnetic cotangent bundle* $(T^*\mathcal{O}(a), dp \wedge dx + \pi^*(\omega_f))$ *is prequantizable: the Hilbert space* $\mathcal{H}$ *that is associated to it by geometric quantization is*

$$\mathcal{H} = L^2(L_f),$$

where L_f *is the line bundle over* $\mathcal{O}(a)$ *with curvature* ω_f*.*

We also want to quantize the classical observables, which are functions on $\mathcal{O}(a)$, in such a way that (3.21) holds. There is a very natural way to do this, using the formalism of symplectic geometry. Recall that there is a Hamiltonian action of G on $(T^*\mathcal{O}(a), dp \wedge dx + \pi^*(\omega_f))$ that was described in (3.17). This means that to each $\xi \in \mathfrak{g}$ there is a Hamiltonian function $H_\xi(x,p)$ and a Hamiltonian vector field ξ_x and these satisfy

$$\{H_\xi, H_\eta\} = H_{[\xi,\eta]}.$$

Therefore, if we associate to the Hamiltonian function H_ξ the operator

$$\widehat{H}_\xi = -i\xi_x$$

we have that

$$[\widehat{H}_\xi, \widehat{H}_\eta] = -i\widehat{H}_{[\xi,\eta]}$$

as desired — we will return to this in Section 3.4.

3.3.3 Homogeneous line bundles

The purpose of this section is to show that the quantum Hilbert space described in Theorem 3.3.5 may indeed be identified with the representation of G induced from the character χ_f of G_a.

Recall from Section 2.3 the construction of the induced representation of a Lie group G from a representation of a subgroup H. If the representation χ of H is 1-dimensional, then the representation of G is on the space of L^2-sections of a line bundle $L_\chi \to X$, where $X := G/H$. In this section a connection will be defined on the line bundle and the curvature form of it will be computed and shown to be exactly given by the differential of the representation of H. In light of Theorem 3.3.5, this enables the identification of the quantum Hilbert space suggested by geometric quantization with an induced representation.

We first define a connection on the principal H-bundle $G \xrightarrow{H} X$. Recall that this is an $\mathfrak{h}$-valued 1-form θ with the property that the horizontal distribution $D = \ker(\theta) \subset TG$ is H-invariant and transversal to the orbits of the H-action. At each point $g \in G$ the connection θ defines a splitting of the tangent space: $T_gG \cong \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{p} \cong \mathfrak{h}^\perp$ is the orthogonal complement to $\mathfrak{h}$ with respect to the Cartan–Killing form. This means that each $u \in T_gG$ can be decomposed into horizontal and vertical parts $u = u_h + u_v$ where $u_h \in g\mathfrak{p}$ and $u_v = \theta(u) \in g\mathfrak{h}$.

The curvature Θ of the connection θ is the $\mathfrak{h}$-valued 2-form that is equal to $d\theta$ on the horizontal distribution

$$\Theta(u_h, v_h) = d\theta(u_h, v_h) = -\theta([u_h, v_h]).$$

Recall from Section 2.3 that sections of the line bundle $L_\chi \to X$ satisfy the following equivariance condition with respect to the action of H

$$\Gamma(L_\chi) = \left\{\psi : G \to \mathbb{C} \mid \psi(gh) = \chi(h^{-1})\psi(g) : g \in G, h \in H\right\} \tag{3.22}$$

and that L_χ is formed by taking the quotient of $G \times \mathbb{C}$ by the action of H

$$L_\chi := G \times \mathbb{C}/\sim \quad \text{where} \quad (gh, z) \sim (g, \chi(h)z).$$

The curvature of the line bundle $L_\chi \to X$ (which is a 2-form on X) may be computed in terms of Θ (which is an $\mathfrak{h}$-valued 2-form on X). In order that this may be done, a connection needs to be defined on L_χ.

Definition. Let $u \in \mathfrak{X}(X)$ be a vector field on X and $s \in \Gamma(L_\chi)$ be a section of L_χ. Then define the covariant derivative of s with respect to u at x by the formula

$$\nabla_u s(x) := \mathcal{L}_{u_h} s(x),$$

where $\mathcal{L}_{u_h} s(x)$ is the Lie derivative of the section s in the direction u_h at x.

Lemma 3.3.6 *This does indeed define a connection on L_χ.*

Proof. For $v \in \mathfrak{X}(X)$, $f \in C^\infty(X)$ and $s \in \Gamma(L_\chi)$, three things must be shown:

1. the mapping $\nabla_v : \Gamma(L_\chi) \to \Gamma(L_\chi)$, given by $s \mapsto \nabla_v s$ is linear,
2. that it satisfies the Leibniz rule
$$\nabla_v(fs) = (\mathcal{L}_v f)s + f\nabla_v s$$
3. and that
$$\nabla_{fv} s = f\nabla_v s.$$

All of these can be shown using properties of the Lie derivative, indeed the first is clear since the Lie derivative is linear. For the second, consider the homotopy identity for the Lie derivative of a differential form ω with respect to a vector field X

$$\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega),$$

where $i_X\omega$ is the interior product of X with ω. If $\omega = s$ is a 0-form (i.e. a section), then this takes the simpler form

$$\mathcal{L}_X s = i_X ds = ds(X).$$

Then one sees that

$$\begin{aligned}\mathcal{L}_{v_h}(fs) = i_{v_h}d(fs) =& i_{v_h}(df)s + f i_{v_h} ds\\ =& \mathcal{L}_{v_h}(f)s + f\nabla_v s,\end{aligned}$$

as required. The last property is true, since for any form ω

$$\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge i_X\omega,$$

which if $\omega = s$ is a 0-form simplifies to

$$\mathcal{L}_{fv_h} s = f\mathcal{L}_{v_h} s = f\nabla_v s.$$

□

Definition. The curvature of this connection ∇ is a linear operator on $\Gamma(L_\chi)$ defined by

$$R(u,v)s := [\nabla_u, \nabla_v]\, s - \nabla_{[u,v]} s,$$

where $u, v \in \mathfrak{X}(X)$.

Proposition 3.3.7 *The curvature of this connection on L_χ is given by*

$$R(u,v)s = d\chi(\Theta(u_h, v_h)),$$

where $d\chi$ is the representation of $\mathfrak{h}$ obtained by differentiating χ

$$d\chi(Y) := \left.\frac{d}{dt}\right|_{t=0} \chi(\exp(tY))$$

for $Y \in \mathfrak{h}$.

Proof. First note that this makes sense: Θ is an $\mathfrak{h}$-valued 2-form on X and so applying $d\chi(X) \in \mathfrak{h}^*$ to Θ gives a 2-form on X. Define the vector field $w := [u_h, v_h] - [u, v]_h$ on G. Then

$$R(u, v)s = \mathcal{L}_w s,$$

since

$$\begin{aligned} R(u, v)s :&= [\nabla_u, \nabla_v]\, s - \nabla_{[u,v]} s \\ &= \mathcal{L}_{[u_h, v_h]} s - \mathcal{L}_{[u,v]_h} s \\ &= \mathcal{L}_{([u_h, v_h] - [u,v]_h)} s = \mathcal{L}_w(s). \end{aligned}$$

Now the vector field w on G is vertical; this means that applying the connection 1-form θ to w will give an element Y of $\mathfrak{h}$

$$\begin{aligned} Y := \theta(w) &= \theta([u_h, v_h] - [u, v]_h) \\ &= \theta([u_h, v_h]) \qquad\qquad (\text{since } [u, v]_h \in \ker\theta) \\ &= -d\theta(u_h, v_h) \\ &= -\Theta(u_h, v_h). \end{aligned}$$

Recall that any section s of L_χ is a function on G satisfying the following equivariance condition with respect to the action of H

$$s(gh) = \chi(h^{-1})s(g).$$

By letting $h = \exp(tY)$ one sees that the infinitesimal version of this condition is that

$$\begin{aligned} R(u, v)s = \mathcal{L}_Y s(g) &= \left.\frac{d}{dt}\right|_{t=0} \chi(\exp(-tY))s(g) \\ &= -d\chi(Y)s \\ &= d\chi(\Theta(u_h, v_h)), \end{aligned}$$

as required. □

As a corollary of this result and Theorem 3.3.5, we see that if the magnetic cotangent bundle $(T^*\mathcal{O}(a), dp \wedge dx + \pi^*(\omega_f))$ is prequantizable (which means that the form ω_f is integral and so by Proposition 3.2.9 can be used to define a character χ_f of G_a), then the Hilbert space suggested by geometric quantization (i.e. the space of L^2-sections of the line bundle over $\mathcal{O}(a)$ with curvature ω_f) can be identified with the representation space of the representation of G induced from the character χ_f of G_a.

3.3.4 Branching rules

The Frobenius reciprocity theorem (Theorem 2.3.2) says that to calculate how the induced representation of W_μ splits up as irreducible representations V_λ of G, one only needs to calculate whether $V_\lambda|_H$ contains W_μ — and with what multiplicity.

It turns out that when $H \subset G$ are compact Lie groups and H contains a maximal torus T, the problem was solved by Kostant — this section is expository and summarizes the discussion given in [29]. It employs a simple argument due to Cartier [8] to find a formula for the restriction of a representation of G to H to calculate the branching multiplicities — Kostant originally proved the multiplicity formula using arguments rooted in Lie algebra cohomology.

The generic case — Kostant's multiplicity formula

The generic case is that the stabilizing subgroup G_a is conjugate to the maximal torus T. Thus, to work out the branching rules in this case, one needs to know how irreducible representations of G split when restricted to the maximal torus.

A fundamental result in the representation theory of compact Lie groups is Weyl's character formula, which gives the character of an irreducible representation as the ratio of two alternating trigonometric polynomials. The material in this section is standard and can be found in several textbooks on representation theory, e.g. [17], [19] or [29].

For a compact Lie group G with maximal torus T, the irreducible representations V_λ of G are classified by a highest weight λ. The Weyl group of G is defined to be the group $W_G = N_G(T)/T$, where $N_G(T) = \{g \in G : gTg^{-1} = T\}$ is the normalizer of T in G. The Weyl group, which turns out to be finite, acts on the weights of T as a permutation group. We also define the Weyl vector ρ_G to be half the sum of the positive roots of $G_\mathbb{C} = G \otimes \mathbb{C}$, or more precisely its Lie algebra $\mathfrak{g}_\mathbb{C}$: $\rho_G = \frac{1}{2}\sum_{\alpha \in R_G^+} \alpha$.

Weyl's character formula gives the character of the representation with highest weight V_λ as

$$\chi(V_\lambda) = \frac{\sum_{\sigma \in W_G} (-1)^\sigma e^{\sigma(\lambda+\rho_G)}}{\sum_{\sigma \in W_G} (-1)^\sigma e^{\sigma(\rho_G)}}.$$

Since representations are determined by their character, this is the best possible result — a (seemingly) simple expression for the character of an irreducible representation in terms of its highest weight.

Recall that all irreps of $T \cong U(1)^n$ are given by weights $\mu : T \to U(1)$, with $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Specifically, for $U(n)$, if $T \ni t = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ then $\mu(t) = e^{i(m_1\theta_1 + \ldots + m_n\theta_n)}$.

One might ask how V_λ breaks up when restricted to T. The answer is given by Kostant's multiplicity formula, which can be obtained by rewriting Weyl's character formula as an infinite sum over weight spaces. The first step is to utilise Weyl's denominator formula (see [17])

$$\frac{1}{\sum_{\sigma \in W_G} (-1)^\sigma e^{\sigma(\rho_G)}} = \frac{e^{-\rho_G}}{\prod_{\alpha \in R_G} (1 - e^{-\alpha})},$$

where R_G is the set of roots of G. Expanding each term in the denominator on the right as a geometric series gives

$$\frac{e^{-\rho_G}}{\prod_{\alpha \in R_G} (1 - e^{-\alpha})} = e^{-\rho_G} \sum_\nu \mathcal{P}(\nu) e^{-\nu}, \tag{3.23}$$

where the (infinite) sum on the right is over all positive weights and the function $\mathcal{P}(\nu)$ is the *Kostant partition function*, which is defined as the number of distinct ways to write ν as a sum of positive roots.

Substituting this into Weyl's formula yields:

$$\begin{aligned} \chi(V_\lambda) &= \sum_{\sigma \in W_G} (-1)^\sigma e^{\sigma(\lambda+\rho_G)} \sum_\nu \mathcal{P}(\nu) e^{-\nu-\rho_G} \\ &= \sum_{\sigma,\nu} (-1)^\sigma e^{\sigma(\lambda+\rho_G)-\nu-\rho_G} \mathcal{P}(\nu). \end{aligned} \tag{3.24}$$

Theorem 3.3.8 *(Kostant Multiplicity Formula) [32] The multiplicity of the weight μ in the irreducible representation V_λ with highest weight λ is given by*

$$n_\mu(V_\lambda) = \sum_\sigma (-1)^\sigma \mathcal{P}(\sigma(\lambda + \rho_G) - \mu - \rho_G).$$

where $\mu = \sigma(\lambda + \rho_G) - \nu - \rho_G$ in (3.24) and $\mathcal{P}(\nu)$ is Kostant's partition function.

The point is that we have obtained a formula for the multiplicity of a weight μ in the irreducible representation V_λ, but at the cost of summing over the Weyl group. In principle though this gives an explicit answer to the branching problem for a magnetic cotangent bundle over a generic coadjoint orbit, where the stabilizer of a point is a maximal torus.

Whilst we have a formal answer to the branching problem that can be written on one line, the downside is that this formula is extremely difficult to evaluate — some asymptotic properties of Kostant's formula are given in [20], see also [26] for a discussion.

It is interesting to note the opinion of Gelfand on Kostant's formula, as related by Kostant himself, who ends his recollections of I.M. Gelfand in [34] by "citing a mathematically philosophical statement of Gelfand" that he thinks deserves considerable attention. "It also opens a little window, presenting us with a view of the way Gelfand's mind sometimes works." He says:

"One of my first papers gave a formula for the multiplicity of a weight in finite-dimensional (Cartan–Weyl) representation theory. A key ingredient of the formula was the introduction of a partition function on the positive part of the root lattice. The partition function was very easy to define combinatorially, but giving an expression at a particular lattice point was altogether a different matter. Gelfand was very interested in this partition function and mentioned it on many occasions. He finally convinced himself that no algebraic formula existed that would give its values everywhere. He dealt with this realization as follows. One day he said to me that in any good mathematical theory there should be at least one "transcendental" element and this transcendental element should account for many of the subtleties of the theory. In the Cartan–Weyl theory, he said that my partition function was the transcendental element."

Non-generic cases — Kostant's branching formula

The situation in the non-generic case is the following: $O(a) = G/H$, with $T \subset H \subset G$. When $H \not\cong T$, its irreducible representations are no longer one-dimensional, as they are for T — however, a similar argument can be employed.

Denote the irreps of H and G by W_μ and V_λ respectively. V_λ breaks up into a finite sum of irreducibles on restriction to H, which may be written as a formal sum

$$V_\lambda|_H = \sum_\mu n_\mu(V_\lambda) W_\mu,$$

where each μ is a dominant weight for H. Since H is a compact Lie group, Weyl's character formula can be used to find that

$$\chi(W_\mu) = \frac{\sum_{\tau \in W_H} (-1)^\tau e^{\tau(\mu+\rho_H)}}{\sum_{\tau \in W_H} (-1)^\tau e^{\tau(\rho_H)}}.$$

The trace of an element depends only on its conjugacy class in T, therefore

$$\chi(V_\lambda) = \chi(V_\lambda|_H) = \sum_\mu n_\mu(V_\lambda)\chi(W_\mu).$$

Applying the two Weyl formulas for G and H gives

$$\frac{\sum_{\sigma \in W_G} (-1)^\sigma e^{\sigma(\lambda+\rho_G)}}{\sum_{\sigma \in W_G} (-1)^\sigma e^{\sigma(\rho_G)}} = \sum_\mu n_\mu(V_\lambda) \frac{\sum_{\tau \in W_H} (-1)^\tau e^{\tau(\mu+\rho_H)}}{\sum_{\tau \in W_H} (-1)^\tau e^{\tau(\rho_H)}}.$$

Expanding the two denominators, as in (3.23) gives

$$\frac{\sum_{\sigma\in W_G}(-1)^\sigma e^{\sigma(\lambda+\rho_G)-\rho_G}}{\prod_{\alpha\in R_G^+}(1-e^{-\alpha})} = \sum_\mu n_\mu(V_\lambda)\frac{\sum_{\tau\in W_H}(-1)^\tau e^{\tau(\mu+\rho_H)-\rho_H}}{\prod_{\beta\in R_H^+}(1-e^{-\beta})}. \tag{3.25}$$

Note that each term in the product in the denominator on the right appears on the left and so can be cancelled. To this end, a modified Kostant partition function $\tilde{\mathcal{P}}_{\mathfrak{g}/\mathfrak{h}}(\nu)$ is defined to be the number of ways that the weight ν can be constructed using positive roots of $\mathfrak{g}$ that do not appear in $\mathfrak{h}$. Then (3.25) can be rewritten as

$$\sum_{\sigma\in W_G}(-1)^\sigma e^{\sigma(\lambda+\rho_G)-\rho_G-\nu}\tilde{\mathcal{P}}_{\mathfrak{g}/\mathfrak{h}}(\nu) = \sum_\mu \tilde{n}_\mu(V_\lambda)\sum_{\tau\in W_H}(-1)^\tau e^{\tau(\mu+\rho_H)-\rho_H}.$$

To calculate the multiplicity of the irreducible representation W_μ in the restricted representation, compare coefficients of $e(\mu)$ on the left and the right.

On the right, $\mu = \tau(\mu+\rho_H)-\rho_H$ means that $\tau = 1$. On the left, setting $\mu = \sigma(\lambda+\rho_G)-\rho_G-\nu$ gives that:

$$\sum_{\sigma\in W_G}(-1)^\sigma\mathcal{P}_{\mathfrak{g}/\mathfrak{h}}(\sigma(\lambda+\rho_G)-\rho_G-\mu)e^\mu = \sum_\mu n_\mu(V_\lambda)e^\mu.$$

The above discussion gives the following theorem, see [19] or [29].

Theorem 3.3.9 *(Kostant's Branching Formula) The multiplicity $n_\mu(V_\lambda)$ of an irreducible representation W_μ of H in the restriction of V_λ to H is given by*

$$n_\mu(V_\lambda) = \sum_{\sigma\in W_G}(-1)^\sigma\mathcal{P}_{\mathfrak{g}/\mathfrak{h}}(\sigma(\lambda+\rho_G)-\rho_G-\mu).$$

This subsumes the Kostant multiplicity formula (Theorem 3.3.8).

Corollary 3.3.10 *Frobenius reciprocity gives that the representation of G induced from the weight χ_f is decomposed into irreducible representations of G by*

$$\mathrm{ind}_{G_a}^G(\chi_f) \cong \bigoplus_{V_\lambda} n_f(V_\lambda)\cdot V_\lambda,$$

where this is considered as a virtual sum since most of the coefficients $n_f(V_\lambda)$ given by the Kostant Branching Formula will be zero.

3.4 The magnetic Schrödinger operator

In this section we discuss the quantization of the magnetic geodesic flow on a coadjoint orbit of a compact Lie group. The most natural definition of the quantum

Hamiltonian is to replace ordinary derivatives with covariant derivatives in the presence of a magnetic field. A related geometric approach is to consider the Bochner Laplacian, which is a self-adjoint second-order differential operator acting on sections of vector bundles, see [54] for details. We show that in the cases that we are considering these two approaches give the same result. It is then shown that the spectrum of the quantum Hamiltonian can be computed in terms of the Kostant branching formula.

3.4.1 The quantum Hamiltonian

Some of the early work in quantum mechanics proposed that the quantization of the geodesic flow on a Riemannian manifold (M, g) is given by the Laplace–Beltrami operator acting on $C^\infty(M)$ (a discussion on this is in [23], see also [56]), i.e.

$$H_0 = g^{ij} p_i p_j \mapsto \qquad \hat{H}_0 \psi = \Delta\psi := -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial \psi}{\partial x^i} \right).$$

If $\{X_1, \dots, X_n\}$ is an orthonormal basis of vector fields at the point x

$$\hat{H}_0 \psi(x) := \sum_{i=1}^{n} \mathcal{L}^2_{(X_i)} \psi(x),$$

where $\psi \in C^\infty(M)$. It is worth mentioning that there exist other schemes for quantizing the geodesic flow on a manifold, notably the BKS construction (due to Blattner Kostant and Sternberg) (see [56] and references therein). The resulting operator differs from the one considered here by the addition of a correction term related to the scalar curvature of M.

The most intuitive quantization of the magnetic geodesic flow is obtained by replacing ordinary derivatives by covariant derivatives, so that at the point x

$$\hat{H} \psi(x) := \sum_{i=1}^{n} \nabla^2_{X_i} \psi(x),$$

where now $\psi \in \Gamma^\infty(L_\omega)$, i.e. ψ is a smooth section of the line bundle $L_\omega \to M$ with curvature ω. We will describe this in detail for our case.

Let $\{\xi_i\}$ with $i = 1, \dots, m$ be an orthonormal basis of $\mathfrak{g}$, with respect to the Cartan–Killing form, such that $\{\xi_1, \dots, \xi_r\}$ is a basis of $\mathfrak{g}_a$ and $\{\xi_{r+1}, \dots, \xi_m\}$ is a basis of $\mathfrak{g}_a^\perp \cong T_a\mathcal{O}(a)$. Recall that the Cartan–Killing form is defined for $\xi, \eta \in \mathfrak{g}$ by

$$(\xi, \eta) := -tr(ad_\xi \cdot ad_\eta).$$

Since $\mathfrak{g}$ is semisimple, the form $(\ ,\ )$ is non-degenerate and so provides an isomorphism of $\mathfrak{g}$ with $\mathfrak{g}^*$ and an induced form on $\mathfrak{g}^*$, which is again denoted by $(\ ,\ )$.

Since we are using the normal metric on $\mathcal{O}(a)$, then the set $\{\xi_{r+1}, \ldots, \xi_m\}$ provides an orthonormal basis for $T_a\mathcal{O}(a)$.

For $(\mathcal{O}(a), ds_0^2)$, the coadjoint orbit through a equipped with the normal metric, the quantization of the ordinary geodesic flow is given by

$$H_0 = \langle \Phi_0, \Phi_0 \rangle \mapsto \qquad \widehat{H_0} = \Omega_G = \sum_{i=1}^{m} \mathcal{L}_{\xi_i}^2$$

acting on $C^\infty(\mathcal{O}(a)) \cong \{f \in C^\infty(G) \mid f(gh) = f(g), \text{ for } h \in G_a\}$.

Definition. The quantum Hamiltonian of the magnetic geodesic flow is given by

$$H_f = \langle \Phi_f, \Phi_f \rangle \mapsto \qquad \widehat{H_f} = \sum_{i=1}^{m} \nabla_{\xi_i}^2$$

acting by its self-adjoint extension [46] on

$$L^2(L_{\chi_f}) \supset \left\{\psi \in C^\infty(G) \mid \psi(gh) = \chi_f(h^{-1})\psi(g), \text{ for } h \in G_a\right\}.$$

Remark. This definition of the quantum Hamiltonian is motivated by the quantum Hamiltonian for an 'ordinary' Dirac magnetic monopole, in the following sense. The generators of $\mathfrak{so}(3)$ l_1, l_2, l_3 generate vector fields X_1, X_2, X_3 on S^2. The quantum Hamiltonian used in [57] is essentially given by

$$\hat{H} = -(\nabla_{X_1}^2 + \nabla_{X_2}^2 + \nabla_{X_3}^2),$$

where $\nabla_{X_j} = X_j - iA_j$ is the covariant derivative with respect to the vector field X_j and the vector potential $\mathbf{A}$ satisfies

$$\nabla \times \mathbf{A} = q.$$

Given $\xi \in \mathfrak{g}$, generating a vector field X_ξ on $\mathcal{O}(a)$ the covariant derivative with respect to X_ξ, acting on $s \in \Gamma^\infty(L)$ at $x \in \mathcal{O}(a)$ is given by

$$\nabla_{X_\xi}(s(x)) = \mathcal{L}_{pr_{\mathfrak{g}_x^\perp}(\xi)} s(x).$$

Lemma 3.4.1 *The quantum Hamiltonian acts on smooth sections by*

$$\widehat{H}_f s = (\Omega_G - \Omega_{G_a})s$$

where $\Omega_G = \sum_{i=1}^m \mathcal{L}_{\xi_i}^2$ *and* $\Omega_{G_a} = \sum_{i=1}^r \mathcal{L}_{\xi_i}^2$ *are the second order Casimir elements of* G *and* G_a *and* $s \in \{\psi \in C^\infty(G) \mid \psi(gh) = \chi_f(h^{-1})\psi(g), \text{ for } h \in G_a\}$.

Proof. At the point a

$$\widehat{H}_f = \sum_{i=1}^{m} \nabla_{\xi_i}^2 = \sum_{i=1}^{m} \mathcal{L}^2_{pr_{\mathfrak{g}_a^\perp}(\xi_i)} = \sum_{i=r+1}^{m} \mathcal{L}^2_{\xi_i} = \Omega_G - \Omega_{G_a}$$

and this is G-invariant. □

Lemma 3.4.2 *The Casimir Ω_{G_a} acts on $s(g)$ by multiplication by (f, f).*

Proof. For $j = p+1, \ldots, N$ then X_j acts on $\tilde{s}$ by

$$\begin{aligned}(X_j \cdot s)(g) &= \left.\frac{d}{dt}\right|_{t=0} s(g \exp(tX_j)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \chi_f(\exp(-tX_j))s(g) \\ &= -\langle f, X_j \rangle\, s(g).\end{aligned}$$

Since $\Omega_{G_a} = \sum_{j=p+1}^{N} X_j^2$, then

$$\Omega_{G_a} s = \sum_{j=p+1}^{N} \langle f, X_j \rangle^2 s = (f, f)s$$

□

Remark. Alternatively, for $\xi \in \mathfrak{g}$ consider the Hamiltonian vector field $\xi(x)$

$$\xi(x) = ad^*_\xi(x) + \langle f_x, \xi \rangle .$$

Since the mapping $\xi \mapsto \xi(x)$ is a Lie algebra homomorphism, at the point a we have

$$\begin{aligned}\Omega_G = \sum_{i=1}^{m} (\xi_i(a))^2 &= \sum_{i=1}^{m} \left(ad^*_{\xi_i}(a) + \langle f_a, \xi_i \rangle\right)^2 \\ &= \sum_{i=1}^{m} \nabla_{\xi_i}^2 + \mathcal{L}_{pr_{\mathfrak{g}_a^\perp}(\xi_i)} \langle f_a, \xi_i \rangle + \langle f_a, \xi_i \rangle \mathcal{L}_{pr_{\mathfrak{g}_a^\perp}(\xi_i)} + \langle f_a, \xi_i \rangle^2 \\ &= \sum_{i=1}^{m} \nabla_{\xi_i}^2 + (f, f)\end{aligned}$$

since $\langle f_a, \xi \rangle$ vanishes if $\xi_i \notin \mathfrak{g}_a$ (due to the orthogonal decomposition) and $pr_{\mathfrak{g}_a^\perp}(\xi_i) = 0$ if $\xi_i \in \mathfrak{g}_a$. Putting this together gives

$$\sum_{i=1}^{m} \nabla_{\xi_i}^2 = \Omega_G - (f, f),$$

as before. We also have that the curvature form is given by

$$\begin{aligned}R(\xi, \eta) =& \xi(a)\eta(a) - \eta(a)\xi(a) - [\xi, \eta](a) \\ =& ad^*_\xi ad^*_\eta(a) - ad^*_\eta ad^*_\xi(a) + \langle f_a, [\xi, \eta] \rangle - \langle f_a, [\eta, \xi] \rangle - ad^*_{[\xi,\eta]}(a) - \langle f_a, [\xi, \eta] \rangle \\ =& \langle f_a, [\xi, \eta] \rangle .\end{aligned}$$

The reasoning in this section is summarised in the following theorem.

Theorem 3.4.3 *The quantum Hamiltonian acts on smooth sections as*

$$\widehat{H}_f s = (\Omega_G - (f, f))s$$

on each irreducible representation of G *occuring in the decomposition of* $\mathrm{ind}_H^G(\chi_f)$.

The quantum Hamiltonian that we are considering has a natural quantum interpretation in terms of the Bochner Laplacian, see e.g. [54] for more details.

If $E \to M$ is a vector bundle with connection ∇ over a Riemannian manifold (M, g), with a metric on each fibre then the Bochner Laplacian Δ on E is a second order differential operator $\Delta : \Gamma(E) \to \Gamma(E)$ defined using the metric structures.

The covariant derivative ∇ is a map $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$; using the metric structures on E the L^2 adjoint to ∇ may be defined as

$$\nabla^* : \Gamma(E \otimes T^*M) \to \Gamma(E), \qquad (\nabla^* s, s') = (s, \nabla s').$$

The Bochner Laplacian is then defined by

$$\Delta := \nabla^*\nabla : \Gamma(E) \to \Gamma(E).$$

Given a homogeneous space $G/H = M$ and a representation of H on some vector space V; how does the Bochner Laplacian act on the homogeneous vector bundle $E = G \times_H V_\rho$, whose space of smooth sections is linearly isomorphic to the space $C^\infty(G, V)^{H_\rho}$ of smooth functions $f : G \to V$ satisfying

$$f(gh) = \rho(h^{-1})f(g) \qquad g \in G, h \in H.$$

An element of $\Gamma(E)$ is denoted by s and the corresponding element of $C^\infty(G, V)^{H_\rho}$ is denoted by $\tilde{s}$.

Lemma 3.4.4 *In the basis above, define* $\Omega_G = \sum_{j=1}^N \xi_j^2$ *and* $\Omega_{G_a} = \sum_{j=p+1}^N \xi_j^2$ *to be the second order Casimir elements of* G *and* H *respectively. Then the Bochner Laplacian acts on sections of the line bundle* L_f *at* a *by*

$$\Delta\tilde{s}(a) = \nabla^*\nabla\tilde{s}(a) = (\Omega_G - \Omega_{G_a}) \cdot \tilde{s}(a).$$

Theorem 3.4.5 *Since this is* G *invariant, we have that the Bochner Laplacian acts on smooth sections relative to the* L^2 *structure as*

$$\Delta\tilde{s} = (\Omega_G - (f, f))\tilde{s}.$$

Hence, the Bochner Laplacian acts on sections of line bundles in the same way that the quantum Hamiltonian in the presence of magnetic field does.

So we see that a natural question from the physical point of view is essentially the same as a natural geometric question. This question is still of current interest: in 2007 there was a paper [21] that attempted to solve the spectral problem for the Bochner Laplacian acting on sections of line bundles over complex Grassmannians — we note that this is the same as considering maximally degenerate (non-trivial) coadjoint orbits for $SU(n)$. We will return to this in Section 3.4.3.

3.4.2 Calculation of the spectrum

We give here a formal answer to the spectral problem for quantization of the magnetic geodesic flow. We say that it is a formal answer, because it is given in terms of Kostant's branching formula. Specific examples will be computed in Section 3.4.3.

Recall that Kostant's branching formula and the Frobenius reciprocity theorem (Theorems 3.3.9 and 2.3.2) give the decomposition of the induced representation in terms. Kostant's branching formula gives coefficients $n_\mu(V_\lambda)$ for each V_λ which are equal to the number of times that the representation V_λ occurs in the induced representation $\mathrm{ind}_H^G(W_\mu)$. Since the quantum Hamiltonian acts on each representation as a scalar (Theorem 3.4.3) then the degeneracy of any particular eigenvalue E_μ is given by the formula

$$\mathrm{degen}(E_\mu) = n_\mu(V_\lambda) \cdot \dim V_\mu.$$

The dimension of an irreducible representation is given by the Weyl dimension formula. This can be obtained from the Weyl character formula by evaluating $\chi_\mu(e)$ using an appropriate limit. The answer is

$$\dim(V_\mu) = \frac{\prod_{\alpha\in\Phi^+} (\lambda+\rho, \alpha)}{\prod_{\alpha\in\Phi^+} (\rho, \alpha)}. \tag{3.26}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ and $(\ , \ )$ is the Cartan–Killing form — see e.g. [17] for details.

The value of the second-order Casimir element $\Omega_G = \sum_{i=1}^m \xi_i^2$ of G acting on an irreducible representation V_μ is given by the well-known formula

$$\Omega_G(V_\mu) = (\mu, \mu + 2\rho)\,.$$

where $(\ , \ )$ is the induced Cartan–Killing form on $\mathfrak{g}^*$. Derivations of this can be found in e.g. [17], [29] or [19].

Summarizing everything gives the following theorem:

Theorem 3.4.6 *The spectrum of the quantum Hamiltonian $\widehat{H}_f$ is given by*

$$(\lambda + 2\rho, \lambda) - (f, f)\,,$$

with multiplicity

$$n_f(V_\lambda) \cdot \dim V_\lambda,$$

where λ ranges over the highest weights of irreducible representations of the group G, $\dim V_\lambda$ is given in (3.26) and $n_f(V_\lambda)$ is given in Theorem 3.3.9.

Remark. This should be thought of as a virtual sum, since most of the V_λ's the multiplicity with which they appear in the decomposition of $G \times_H \mathbb{C}_f$ is zero. Also, notice that whilst the element $f = f_x$ is defined relative to some $x \in \mathcal{O}(a)$, the quantum operator $\widehat{H}_f$ does not care about which x this is, since if $y = Ad^*_g(x)$ then $f_y = Ad^*_g(f_x)$ and so

$$(f_y, f_y) = (Ad^*_g(f_x), Ad^*_g(f_x)) = (f_x, f_x) = (f, f).$$

3.4.3 Examples

The answer given in Theorem 3.4.6 is neat and concise. However, it is worthwhile spending some time to compute some specific examples, so that one can see how it really works. Calculating specific examples of this construction is a task that is limited by one's patience and ingenuity. In general it is a very difficult task, since we have to calculate $n_\mu(V_\lambda)$ for every irrep V_λ of G. The most difficult examples to compute are the generic cases, when G_a contains a maximal torus T — this is because this is when there are the most relations between the different positive roots of $\mathfrak{g}$. It is clear from looking at Kostant's Branching formula that the problem gets easier as the stabilizing subgroup G_a gets bigger, since there become fewer relations between the available roots.

It turns out that specific examples for $SU(n)$ with G_a as large as possible have already been computed by Halima [21] and indeed used to compute the spectrum of the Bochner Laplacian acting in various line bundles over G/G_a — exactly the problem that we are looking at. This being the case, we can give the spectrum of the corresponding Schrödinger equation for coadjoint orbits that are topologically complex Grassmannians. The branching rules computed in [21] can in fact be derived directly using Kostant's branching formula. Alternatively, after some calculations they can be seen to be a consequence of earlier work [44] where some remarkable examples of multiplicity free branching are given for rectangular partitions for the classical groups.

Halima's work extends earlier work of Kuwabara [37], who calculated the spectrum of the Bochner Laplacian acting on sections of line bundles over $\mathbb{C}P^n$.

First however, we recall some facts about the representation theory of $SU(n)$ — see [17], [19] or [29] for more information. A weight is a collection of integers $\alpha = (\alpha_1, \dots, \alpha_n)$ that acts on a diagonal matrix $g = \mathrm{diag}(x_1, \dots, x_n)$ as

$$g \mapsto g^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}.$$

Note that because for $g \in SU(n)$ we have that $\det g = 1$ and consequently we have that $\prod_{i=1}^n x_i = 1$. This means that the weight $\alpha = (\alpha_1, \dots, \alpha_n)$ acts identically to the weight $\alpha' = (\alpha_1 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n, 0)$ and so we take $\alpha_n = 0$.
Every irreducible representation of $SU(n)$ is labelled by a highest weight vector, which is an integer partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0)$, with each such partition giving an irreducible representation of $SU(n)$.

For a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0)$ we have that the character of the representation V_λ is computed using Weyl's character formula: if $g \in SU(n)$ is given by a diagonal matrix $g = \mathrm{diag}(x_1, \dots, x_n)$, then the character of g acting in V_λ is given by the Schur polynomial in the x_i corresponding to λ

$$S_\lambda(x_1, \dots, x_n) = \frac{\left|x_j^{\lambda_i + n - i}\right|}{\left|x_j^{n-i}\right|}.$$

The Weyl dimension formula is obtained by evaluating the character formula on the identity element in G: the dimension of V_λ is given by

$$\dim V_\lambda = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{3.27}$$

Let Ω to be the second order Casimir for $SU(n)$. The value of Ω on V_λ is given by

$$\Omega = ||\lambda + \rho||^2 - ||\rho||^2$$

where the norm is taken with respect to the Cartan–Killing form and the weight vector $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{-n+1}{2}) = (n-1, n-2, \dots, 1, 0) = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$ is half the sum of the positive roots of $\mathfrak{sl}_n \cong \mathfrak{su}_n \otimes \mathbb{C}$. Explicitly, we have that the value of Ω on V_λ is given by the famous formula

$$\Omega(\lambda) = (\lambda + 2\rho, \lambda)$$

where $(\ ,\ )$ is the Cartan–Killing form. For $SU(n)$ we have that this explicitly evaluates to

$$\Omega(\lambda) = \frac{1}{2n} \sum_{j=1}^n \left(\lambda_j \left(\lambda_j + 2\rho_j\right) - \frac{1}{n} \left(\sum_{k=1}^n \lambda_j \right)^2 \right) \tag{3.28}$$

by applying the specific form of the Cartan–Killing form for $\mathfrak{sl}_n$ to the case at hand.

We consider the case when $G = SU(n)$ and $G_a \cong S(U(n-1) \times U(1))$, when we have that $\mathcal{O}(a) \cong G/G_a \cong \mathbb{C}P^{n-1}$. Weights of G_a are given by those weights of $SU(n)$ of the form $\mu = (q, \ldots, q, 0)$.

Lemma 3.4.7 *For $q \geq 0$, the only λ that branch to μ are of the form*

$$\lambda_l^+ = (q + 2l, 0) \qquad l \geq 0 \qquad \text{for } n = 2 \tag{3.29}$$

$$\lambda_l^+ = (q + 2l, q + l, \ldots, q + l, 0) \qquad l \geq 0 \qquad \text{for } n \geq 3. \tag{3.30}$$

and these have branching multiplicity 1. For $q \leq 0$ the partition λ is of the form

$$\lambda_l^- = (|q| + 2l, 0) \qquad l \geq 0 \qquad \text{for } n = 2$$

$$\lambda_l^- = (|q| + 2l, l, \ldots, l, 0) \qquad l \geq 0 \qquad \text{for } k \geq 3,$$

again the branching multiplicity is 1.

Proof. This Lemma appeared in [21], but the result must be well-known to specialists in the area. We give an alternative proof to that given in [21], by making use of Kostant's Branching Theorem 3.3.9.

It is apparent that (3.29) is true since this is exactly the result we had in Chapter Two. In this case Kostant's Branching Theorem is easy to apply since there are no relations among the roots. Indeed, it is clear that the roots that make up $(\mathfrak{g}/\mathfrak{g}_a) \otimes \mathbb{C}$ are given by $L_k - L_n$, for $k = 2, \ldots n$, with $L_1 - L_2$ corresponding to the weight $(1, -1, 0, \ldots, 0)$ etc. We demostrate how to get (3.30) using Kostant's branching formula.

Since λ is supposed to be a dominant weight we have that $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{n-1}, 0)$. For μ dominant as well, i.e. for $q \geq 0$, it is clear that the only summand in Kostant's formula that contributes is when the corresponding element of the Weyl group is the identity, i.e. $w = e$. The only $L_k - L_n$ that gives a dominant weight λ on repeated application to $\mu = (q, \ldots, q, 0)$ is $L_1 - L_n$ — indeed we have that

$$\mu + l(L_1 - L_n) = (q + l, q, \ldots, q, -l) = (q + 2l, q + l, \ldots, q + l, 0) = \lambda_l.$$

The result for $q \leq 0$ can be proved similarly by taking account of the shift by ρ. $\square$

For $q \geq 0$ the Weyl dimension formula gives its dimension as

$$\dim V_{\lambda_l} = \left(\prod_{j=2}^{n-1} \frac{l + j - 1}{j - 1} \right) \frac{q + 2l + n - 1}{n - 1} \left(\prod_{j=2}^{n-1} \frac{q + l + j - 1}{j - 1} \right) \tag{3.31}$$

For $q \leq 0$ the Weyl dimension formula gives the dimension of V_{λ_l} as

$$\dim V_{\lambda_l} = \left(\prod_{j=2}^{n-1} \frac{|q| + l + j - 1}{j-1}\right) \frac{|q| + 2l + n - 1}{n-1} \left(\prod_{j=2}^{n-1} \frac{l + j - 1}{j-1}\right) \tag{3.32}$$

Theorem 3.4.8 *If $\mathcal{O}(a) \cong \mathbb{C}P^{n-1}$, (with $n \geq 3$) with the weight corresponding to the magnetic form being given by $(q, \ldots, q, 0)$, we have that the spectrum of the quantum Hamilitonian is given for $q \geq 0$ by*

$$E_l^+ = \frac{1}{n}\left(l(l+n-1) + q\left(l + \frac{n-1}{2}\right)\right)$$

with the multiplicity of the l^{th} eigenvalue being

$$\dim(E_l^+) = \dim V_{\lambda_l^+} = \left(\prod_{j=2}^{n-1} \frac{l + j - 1}{j-1}\right) \frac{q + 2l + n - 1}{n-1} \left(\prod_{j=2}^{n-1} \frac{q + l + j - 1}{j-1}\right).$$

For $q \leq 0$ we have that the spectrum is given by

$$E_l^- = \frac{1}{n}\left(l(l+n-1) + |q|\left(l + \frac{n-1}{2}\right)\right)$$

with the multiplicity of the l^{th} eigenvalue being

$$\dim(E_l^-) = \dim V_{\lambda_l^-} = \left(\prod_{j=2}^{n-1} \frac{l + j - 1}{j-1}\right) \frac{q + 2l + n - 1}{n-1} \left(\prod_{j=2}^{n-1} \frac{q + l + j - 1}{j-1}\right).$$

Proof. The spectrum is given by applying to (3.28) to the result of Lemma 3.4.7 and subtracting off (μ, μ), where $\mu = (q, \ldots, q, 0)$, which is given by $\frac{1}{2n}\frac{(n-1)}{n}q^2$. The multiplicity of each eigenvalue is just given by using the Weyl Dimension Formula (3.27) for each of the partitions $\lambda_l^{\pm}$. □

Remark. We can relate this to the spectral problem considered in [24], where the spectrum of the Laplace–Beltrami operator and the Hodge Laplacian acting on differential forms of $\mathbb{C}P^n$ is computed. Our result for $q = 0$ agrees with the result for degree 0 forms that is given there. The corresponding calculation for the spectrum of the Bochner Laplacian acting on sections of line bundles over $\mathbb{C}P^n$ was performed in [37], with the spectrum here differing from there by multiplication by a constant factor that arises from choosing a metric that is a scalar multiple of their metric.

Similarly we can give the spectrum of the corresponding quantum Hamiltonians for $\mathcal{O}(a) \cong SU(n)/S(U(k) \times U(n-k)) \cong G(k, n)$. These results are given in [21] — the calculation of which λ branch to give the corresponding weights μ (which are of the form $(q, \ldots, q, 0, \ldots, 0)$ with k $q's$) can again be done using Kostant's branching formula. Alternatively, the branching calculations can be done in yet another different way, by using a result in [44] in conjunction with the Schur functor (which is one way of describing all the irreducible representations of $SU(n)$).

Chapter 4

Magnetic fields on regular graphs

This chapter provides a discrete analogue of the previous two — we give a general construction of special magnetic fields on regular graphs using induced representations for finite groups.

Specifically, given a finite group G and a subgroup H of it, we draw a graph Γ_K, whose vertices are the points of G/H by acting on G/H with a special element K that lives in the centre of the group ring $\mathbb{Z}[G]$. Under certain conditions on K, the graph Γ_K has nice properties and its adjacency matrix can be described algebraically by computing the matrix of K acting in $\mathrm{ind}_H^G(\mathbf{1})$, the representation of G induced from the trivial representation of H.

A magnetic field is defined on Γ_K using a non-trivial character $\rho : H \to U(1)$. Specifically, a magnetic adjacency matrix for the graph Γ_K is given by acting with K in the representation $\mathrm{ind}_H^G(\rho)$. This magnetic field on Γ_K has properties reminiscent of those of the magnetic field due to a magnetic monopole; namely, the flux through any two cycles of the graph that are related by an element $g \in G$ is the same.

The corresponding magnetic Schrödinger operator is the magnetic Laplacian on the graph Γ_K. The spectrum of the magnetic Laplacian, for the magnetic field given by our general construction, can be found using the tools of representation theory.

Having given a general construction of a magnetic field on a regular graph, we then consider what is, in some sense, the inverse problem — namely, given a regular graph equipped with a transitive action of a group G on its vertices and edges and a G-invariant magnetic field can we describe this by our construction? Specifically, we look at the graphs of the Platonic solids and ask whether we can realise magnetic fields on them by our construction. We attempt to do this by taking $G \subset SO(3)$ to be the symmetry group of the polyhedron and H to be the stabilizer of a vertex. The answer is: sometimes — the biggest omission being a complete failure for the dodecahedron. Since the Platonic solids can be thought of as discete approximations to S^2, it is illuminating to view invariant magnetic fields on graphs of the Platonic solids as discrete approximations of Dirac's original construction.

4.1 Magnetic fields on graphs

Since the 1950's physicists have been interested in defining gauge field theories on a lattice. There are two main interpretations that are placed upon the lattice. One can imagine that the lattice points are the locations of atoms in a solid, with the edges of the graph being drawn in the obvious way and corresponding to electron bonds between the atoms. This is known as the *tight-bonding model* or the *Hückel model* — see [38] and references therein. Alternatively, the vertices of the lattice may be thought of as a discretization of space, with the continuous Laplacian being replaced by a finite difference operator. This has proved to be quite fertile ground and the theory of gauge fields on a lattice has grown healthily.

Graph theory has been an active area of study in Mathematics for some time, a good reference for the algebraic side is [3]. However, it seems to be a relatively recent development for mathematicians to look at gauge fields — and in particular magnetic fields — on an abstract graph (as opposed to a lattice). Also, it has to be said that most results obtained by looking at magnetic fields on a graph come from the point of view of analysis — looking for results about the most general graphs. The point of view taken here is diametrically opposed to this, here we study very special graphs giving an exact solution to the eigenvalue problem for graphs with a high degree of symmetry.

This section briefly summarises the relevant definitions, before giving a summary of some notable works in this direction.

A graph $\Lambda = (V, E)$ is a collection of vertices V joined by a set of edges E. An unoriented edge is denoted between x and y is denoted by $\{x, y\} \in E$ and $[x, y]$ and $[y, x]$ are its two orientations. In this chapter graphs are assumed to be finite — this is not necessary in general, but requires more analysis than is needed here.

The most basic object that one can associate to a graph Λ is its adjacency matrix T. The adjacency matrix records which vertices are connected by edges: for example, for the triangular graph K_3, which is shown in Figure 4.1 the adjacency matrix is

$$T_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \tag{4.1}$$

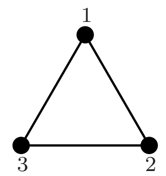

Figure 4.1: The graph of K_3, the complete graph with three vertices.

acting on the free vector space generated by the vertices 1,2 and 3.

In general, the adjacency matrix T is defined as follows: if x and y are two vertices of a graph Λ then T_{xy} is equal to the number of edges joining x to y. The graph is undirected if $T_{xy} = T_{yx}$ for all x and y. The xy^{th} entry of the n^{th} power of the adjacency matrix gives the number of paths of length n between x and y.

One may also associate to a graph its Laplacian matrix, which records which vertices are linked by an edge and also records how many edges are attached to each vertex (the valency or degree of the vertex): the Laplacian matrix for the triangular graph is given by

$$L = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \tag{4.2}$$

In general, if x and y are two different vertices of a graph Λ then L_{xy} is equal to the number of edges joining x to y and $-L_{xx}$ is equal to the number of edges attached to x. Again, the graph is undirected if $L_{xy} = L_{yx}$. One sees that T may be obtained from L by forgetting the diagonal terms.

Attaching the name 'Laplacian' to this operator was not done tritely — as can be seen when one considers functions on Λ. The space of complex valued functions on Λ is denoted by

$$C(V) = \{f : V \to \mathbb{C}\}.$$

At the point x, the Laplacian acts on functions by the rule

$$L(f(x)) = \sum_{y \sim x} [f(y) - f(x)]$$

where the summation is over all vertices y that are at the other end of an edge attached to x, taken with multiplicity. The Laplacian on a graph may be thought of as playing the role of the Laplace–Beltrami operator on a manifold. An easy demonstration of this is afforded by taking the discrete limit of the Laplacian on $\mathbb{R}^2$, which gives (essentially) the Laplacian on $\mathbb{Z}^2$.

More than this though, as is well-known in Riemannian geometry a lot of geometric information is contained in the eigenvalues of the Laplace–Beltrami operator (see e.g. [2] for a review). A similar situation exists in the realm of graphs: the eigenvalues of the Laplacian of a graph encode certain geometric information from the graph. For example: the multiplicity of 0 is equal to the number of connected components of the graph, and the smallest non-zero eigenvalue of L gives the algebraic connectivity of Λ.

Definition. [45] A graph Λ is said to be d-regular if the number of neighbours of each vertex is d and there are no multiple edges or self-connections allowed.

In general, the eigenvalues of T and L are essentially different. However, for d-regular graphs, the Laplacian can be obtained from the adjacency matrix by subtracting d times the identity matrix. Therefore the eigenvalues of T and of L only differ by a shift by a constant d. The graphs considered in this chapter will be (essentially) d-regular.

Definition. [10] A magnetic potential A on a graph Λ with no multiple edges or cycles is given by associating to each edge $[x, y]$ an element $\exp[i\alpha_{xy}] \in U(1)$ such that $\alpha_{xy} = -\alpha_{yx} \in \mathbb{R}$.

This has the effect of replacing the off diagonal elements of the adjacency matrix and the Laplacian matrix by the corresponding elements of $U(1)$, i.e. the elements of the adjacency matrix change by $T_{xy} \mapsto T_{xy} \cdot \exp[i\alpha_{xy}]$ and the elements of the Laplacian matrix change by $L_{xy} \mapsto L_{xy} \cdot \exp[i\alpha_{xy}]$. For example, for a general magnetic field on the triangular graph the adjacency matrix (4.1) changes to

$$T_A = \begin{pmatrix} 0 & \exp[i\alpha_{12}] & \exp[i\alpha_{13}] \\ \exp[-i\alpha_{12}] & 0 & \exp[i\alpha_{23}] \\ \exp[-i\alpha_{13}] & \exp[-i\alpha_{23}] & 0 \end{pmatrix}.$$

and the Laplacian matrix (4.2) changes to

$$-L_A = \begin{pmatrix} -2 & \exp[i\alpha_{12}] & \exp[i\alpha_{13}] \\ \exp[-i\alpha_{12}] & -2 & \exp[i\alpha_{23}] \\ \exp[-i\alpha_{13}] & \exp[-i\alpha_{23}] & -2 \end{pmatrix},$$

for some $\alpha_{ij} \in \mathbb{R}$. One sees that the condition $\alpha_{xy} = -\alpha_{yx}$ guarantees that the operators T_A and L_A are Hermitian (and consequently have real eigenvalues).

The combinatorial magnetic Laplacian acting on functions may be introduced formally as follows. The space of functions $C(V)$ on Λ may be made into a Hilbert space $l^2(V)$ by defining the Hermitian inner product $\langle f, g\rangle_{l^2} = \sum_{x\in V} f(x)\overline{g(x)}$.

It is convenient to introduce the Hermitian form Q by

$$Q_A(f) = \sum_{\{x,y\}\in E} |f(x) - \exp[i\alpha_{xy}f(y)]|^2 \tag{4.3}$$

where each edge is only taken once and the choice of orientation turns out not to matter. The combinatorial magnetic Laplacian L_A is then defined formally by the relation

$$\langle L_A(f), f\rangle = Q_A(f),$$

or more explicitly by

$$L_A(f) = \sum_{y\sim x} [f(x) - \exp[i\alpha_{xy}]f(y)], \tag{4.4}$$

where $y \sim x$ means the summation is taken over all vertices y that are joined to x by an edge. Obviously, if $\alpha_{xy} = 0$ for all edges $\{x, y\}$ then the magnetic Laplacian reduces to the ordinary Laplacian on Λ.

Again, for d-regular graphs, the eigenvalues of L_A and T_A differ only by a shift by d. In this chapter every graph considered will be essentially d-regular.

A natural first question to ask is what effect the introduction of a magnetic field A has on the eigenvalues of the operators T and L.

One of the first works in this direction is [38], which blends mathematics with physics and is very readable. It takes the point of view that the Hamiltonian of a single electron on the graph is given by T_A. Denote the eigenvalues of this operator by λ_i. (One could also take the operator L_A, but for various reasons the authors prefer T_A.) They then move to answer the question of what happens to the eigenvalues if there are more than one free electrons on Λ, and in particular, which choices of θ minimise the ground state of this system. The answer is quite surprising, in that if there is only one electron then the introduction of a magnetic field raises the energy of the system — this result is known as the *diamagnetic inequality*. However, if the number of electrons approaches the number of vertices of the lattice the magnetic field actually lowers the ground state energy. They also give an alternative proof of Kasteleyn's Theorem, which is one of the main tools for counting 'dimer configurations' on a graph. (A dimer configuration on a graph is a subset $\{e_1, \dots, e_n\}$ of E such that each vertex is the end point of exactly one of the e_i's.)

More mathematically-minded is the paper [10], which considers an extension of the magnetic Laplacian defined above for locally finite connected graphs. They define the more general magnetic Schrödinger operator on a weighted graph by the data of a magnetic field A, and some weights $\omega_x \in \mathbb{R}^+$ on the vertices and $c_{xy} \in \mathbb{R}^+$ on the edges to be the operator

$$H_{\omega,c,A}(f)(x) = \frac{1}{\omega_x^2} \sum_{y \sim x} c_{xy} \left[f(x) - \exp[i\alpha_{xy}] f(y) \right].$$

Taking $\omega_x = 1$ and $c_{xy} = 1$ gives the combinatorial magnetic Laplacian defined in (4.4). This operator is Hermitian symmetric on the Hilbert space

$$l^2_\omega(V) = \left\{ f \in C(V) \mid \sum_{x \in V} \omega_x^2 \left| f(x) \right|^2 < \infty \right\}$$

with Hermitian inner product

$$\langle f, g \rangle_{l^2_\omega} = \sum_{x \in V} \omega_x^2 f(x) \overline{g(x)}.$$

Defining the norm $|B|$ of the magnetic field A to be the smallest eigenvalue of $H_{\omega,c,A}$, they prove that under certain growth conditions on c and $|B|$ the operator $H_{\omega,c,A}$ is essentially self-adjoint. This extends previous results of the authors.

An interesting recent paper [45] establishes a trace formula for certain discrete Laplacians on d-regular graphs that depends upon a continuous parameter. (The graphs considered in this chapter will all be d-regular.) For a special value of the parameter this gives exactly the magnetic Laplacian. The trace formula is then used in a following paper to show a connection between the spectral properties of d-regular graphs and random matrices.

Perhaps the most interesting, and relevant to the problem considered here is the highly illuminating paper [39] of Manton. This is along different lines to all of the other works referenced here because it is written from the point of view of Differential Geometry and not Analysis. He starts by recalling that the most efficient and natural way to describe topologically non-trivial gauge fields in the continuous case is to use the language of connections on a principal fibre bundle. He then notes that in standard gauge theory on a lattice, the total space of the bundles considered is not usually discrete, but a Lie group bundle over a finite set of points and is topologically trivial. With this in mind he looks to develop the notion of a connection on a discrete fibre bundle (one whose total space is discrete).

He considers two examples in particular, which are discretizations of the Hopf fibrations $S^3 \overset{S^1}{\to} S^2$ and $S^7 \overset{S^3}{\to} S^4$. The total space for the first bundle is a set of 24 points in $\mathbb{R}^4 \cong \mathbb{C}^2$ (which may be identified with the binary tetrahedral group). The gauge group is the group $\mathbb{Z}_4$ and it acts on the total space, with the base space being the 6 points that may be identified with the vertices of an octahedron. For the second bundle he uses the 240 roots of E_8 as the total space, with gauge group the non-abelian subgroup of $SU(2)$ with order 24 (again, the binary tetrahedral group). The base space is then 10 points and can be thought of as the vertices of a '5-dimensional octahedron'. He supposes that points in the total spaces of these bundles then come equipped with a notion of neighbouring points. (This is not unreasonable, if one considers a standard metric in the ambient space.) This can be used to draw an edge between neighbouring points in the bundle and explain how to use the notion of holonomy to conduct parallel transport on the bundle and measure the curvature of the bundle. Having done this he then defines a notion of the first Chern number of the bundle, which will be used later in this chapter.

His ideas were taken slightly further in [42], where the author defines a discrete Yang–Mills action and shows that the connection on the octahedral bundle described by Manton is a minimal connection for this action. He also resolves a certain troubling asymmetry in Manton's bundle by using the binary octahedral group O^* as the total space for the bundle and identifying the vertices of the octahedron with $O^*/\mathbb{Z}_4^*$. By definition the group O^* is the preimage of the symmetry group of the octahedron under the double-covering of $SO(3)$ by $SU(2)$. To distinguish symmetry groups G from their binary versions G^*, the binary versions are affixed with a $*$.

4.2 Dirac monopoles on homogeneous graphs

This section describes in detail the construction, outlined at the start of the chapter, of what may reasonably be called *Dirac magnetic monopoles on homogeneous graphs*. Firstly, some basic lemmas concerning magnetic fields on graphs are given — these are mostly known results and can be found in [10] and [38]. Next, it is explained how to construct certain d-regular graphs using the representation theoretic notion of an induced representation. Finally, it is explained how to define a Dirac magnetic monopole on a homogeneous graph using this language.

4.2.1 Magnetic fields on graphs

Recall from the Section 4.1 that a magnetic field is defined on a graph Λ by specifying a magnetic potential A, which associates to each edge $[x, y]$ an element $\exp[i\alpha_{xy}] \in U(1)$ with $\alpha_{xy} = -\alpha_{yx} \in \mathbb{R}$.

As might be expected, there is a notion of a gauge transformation, which renders some potentials equivalent.

Definition. A *gauge transformation* U is given by a sequence of complex numbers

$\exp[i\sigma_x]$, where $\sigma_x \in \mathbb{R}$. It acts on a function $f \in l^2(V)$ by

$$(Uf)(x) = \exp[i\sigma_x]f(x),$$

on the quadratic form Q_A defined in (4.3) by $Q_A(f) \mapsto Q_A(Uf)$ and on the magnetic Laplacian L_A by $L_A \mapsto L_{U^*(A)}$, where $U^*(A)_{xy} = \alpha_{xy} + \sigma_y - \sigma_x$.

For finite graphs a gauge transformation just acts on a magnetic Laplacian by

$$L_A \mapsto \overline{U}^t L_A U,$$

where $U = \exp[i\sigma_x]\delta_{xy}$.
It is clear that a gauge transformation leaves the spectrum of L_A unchanged.

It is convenient to introduce the formalism of homology to describe magnetic fields on graphs. Define the space of 1-chains $C_1(\Lambda)$ on the graph Λ to be the $\mathbb{Z}$-module generated by oriented edges subject to the relation $[x, y] = -[y, x]$. A boundary operator can be defined by

$$\partial : C_1(\Lambda) \to C(V) \qquad \partial([x, y]) = \delta_y - \delta_x,$$

where $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for $y \neq x$.

The space of 1-cycles $Z_1(\Lambda)$ is defined as the kernel of the boundary operator.

Definition. Let $\gamma = [x_0, x_1]+[x_1, x_2]+\ldots+[x_{n-1}, x_0]$ be a cycle on Λ. The *holonomy map* is defined by $\Phi_A : Z_1(\Lambda) \to \mathbb{R}/2\pi\mathbb{Z}$ by

$$\Phi_A(\gamma) = \alpha_{x_0x_1} + \ldots \alpha_{x_{n-1}x_0} \pmod{2\pi}.$$

Physically this may be interpreted as the magnetic flux through the cycle, as can be seen by writing

$$\Phi_A(\gamma) = \arg\left(\prod_{i=0}^{n-1} \exp[i\alpha_{xy}]\right)$$

and applying Stokes' theorem.

Lemma 4.2.1 *A gauge transformation leaves the flux through each cycle unchanged.*

Proof. If U is a gauge transformation of A and $\gamma = [x_0, x_1]+[x_1, x_2]+\ldots+[x_{n-1}, x_0]$ is a cycle of Λ then writing out the total flux through γ for $U^*(A)$ and A gives

$$\begin{aligned}\Phi_{U^*(A)}(\gamma) &= \alpha_{x_0x_1} + \sigma_1 - \sigma_0 + \alpha_{x_1x_2} + \sigma_2 - \sigma_1 + \ldots + \alpha_{x_{n-1}x_0} + \sigma_0 - \sigma_{n-1} \\ &= \alpha_{x_0x_1} + \alpha_{x_1x_2} + + \ldots + \alpha_{x_{n-1}x_0} = \Phi_A(\gamma).\end{aligned}$$

□

Definition. Let F be the set of faces of a planar graph. The *Chern number* of the magnetic field (in the sense of [39]) is defined to be

$$c_1(A) = \frac{1}{2\pi} \sum_{f \in F} \Phi(f). \tag{4.5}$$

Remark. This definition approximates the definition of the first Chern number in the case of smooth bundles. However, we have lost something: for the graphs considered here this quantity is not very well-defined. This is because for planar graphs there is no natural way to orient the graph, since the mirror image of the graph is still the graph and so anti-clockwise and clockwise orientations are indistinguishable. However, provided for each graph we are consistent in the orientation used this definition can be used for comparison.

One might wonder what influence the fluxes have on the spectrum of the adjacency matrix — an answer is provided by the following Lemma from [38].

Lemma 4.2.2 *Let T and T' be two magnetic adjacency matrices of a finite graph Λ, with the property that $|T_{xy}| = |T'_{xy}|$ and also such that the flux through each face is equal, i.e. if $\gamma \in Z_1(\Lambda)$ then $\Phi_T(\gamma) = \Phi_{T'}(\gamma)$. Then there exists a gauge transformation U such that $T' = \overline{U}^t TU$ and so T and T' are isospectral.*

Proof. It is sufficient to prove that if $\Phi_T(\gamma) = 0$ for every γ then T is gauge-equivalent to 0. Fix a point $x_0 \in V$. For any $x \in V$ that is linked to x_0 by a path γ_x, consider the function describing the phase acquired in moving from x_0 to x $\phi_x := \arg \prod_\gamma T$. This turns out not to depend on γ_x, for if γ'_x is any other path from x_0 to x then $\gamma - \gamma'$ is a cycle on Λ and the flux through every cycle is 0 by assumption. For any y that is linked to x by an edge, consider a path γ_y from x_0 to y. Then since $\gamma_x + [x, y] - \gamma_y$ is a cycle, it must hold that

$$1 = \prod_{\gamma_x} T \cdot T_{xy} \cdot \prod_{\gamma_y} \bar{T} = \exp[i(\phi_x - \phi_y)]T_{xy}$$

and so $T_{xy} = \exp[i(\phi_y - \phi_x)]$ — therefore T is gauge equivalent to 0. □

A similar Lemma is proved in [10], when the graph is not assumed to be finite.

One might also ask to what extent the fluxes determine the phases. For the case of finite planar graphs this question was answered in [38] as follows.

Lemma 4.2.3 *If a graph Λ is planar (i.e. it can be embedded in $\mathbb{R}^2$ without self-intersections) then the flux through each face of the graph determines the potential A up to a gauge transformation. More specifically, let the graph have faces $F_1, \ldots, F_f$ and let $\Phi_1, \ldots, \Phi_f$ be any numbers in $[0, 2\pi)$. Then there is a function $\theta(x, y) : E(\Lambda) \to [0, 2\pi)$ such that if γ is a cycle then $\Phi(\gamma) = \sum_{interior\ faces\ of\ \gamma} \Phi_j$.*

Proof. For each face F_j inside γ pick an interior point $z_j = (z_j^1, z_j^2) \in \mathbb{R}^2$ and consider the one-form

$$A = \sum_{\text{all faces } F_j \text{ of } \Lambda} \Phi_j \frac{(z_j^2 - y)dx + (x - z_j^1)dy}{2\pi(x^2 + y^2)}.$$

Now, for each edge $[x, y]$ define $\theta(x, y) = \int_x^y A$. Then the flux through any cycle γ is given by $\oint_\gamma A$ and by Stokes' Theorem this equals the $\Phi(\gamma)$ defined above. □

When the graph is no longer assumed to be finite and planar, a Lemma similar in character is given in [10], where it is proved that the map $A \to \Phi_A$ is surjective onto $\mathrm{Hom}(Z_1(\Lambda), \mathbb{R}/2\pi\mathbb{Z})$.

Another interesting question is what influence the magnetic field A has on the lowest eigenvalue of L_A. A partial answer to this question is easily answered using the quadratic form Q_A, as shown in [10].

Lemma 4.2.4 *Let L_A be a magnetic Laplacian on Λ. Then 0 is an eigenvalue of L_A if and only if $\Phi_A(\gamma) = 0$ for every cycle γ.*

Proof. In one direction this is clear, if $\Phi_A(\gamma) = 0$ for every cycle γ then A is gauge equivalent to 0 by Lemma 4.2.2 and so L_A reduces to the combinatorial Laplacian of Γ. The multiplicity of zero as an eigenvalue of the combinatorial Laplacian is equal to the number of connected components of Λ, with eigenfunctions given by constants. In the other direction, suppose that $f \neq 0$ and that $L_A f = 0$. Then, this implies that $Q_A(f) = 0$, which implies that every term in (4.3) vanishes. This means that for any edge $[x, y]$, it must hold that $f(x) = \exp[i\alpha_{xy}]f(y)$. In particular, this means that if $\gamma = [x_0, x_1] + [x_1, x_2] + \ldots + [x_{n-1}, x_0]$ is a cycle then

$$f(x_0) = \exp[-i\alpha_{x_0 x_{n-1}}]f(x_{n-1}) = \ldots = \exp[-i\Phi_A(\gamma)]f(x_0)$$

and so $\Phi_A(\gamma) = 0$. □

4.2.2 Induced representations for finite groups

The main idea of this chapter is to use the definition of the induced representation for finite groups to generate regular graphs with a magnetic field. Recall that if G is any group and H is any subgroup then one can form from any representation V of G a representation of H by just restricting the representation V to H (written $\mathrm{res}_G^H(V)$ or sometimes $V|_H$).

Allied to this is the dual notion of induction: if W is a representation of H then one may form a representation of G called the representation of G induced from W

and written $\mathrm{ind}_H^G(W)$. Recall from Section 2.3 that if G and H are Lie groups then the construction of the induced representation may be explained in terms of the space of sections of a vector bundle with fibre W over G/H; a very readable summary of this is given in [7].

For finite groups the construction, which was first given by Frobenius, is slightly harder to grasp owing to the near absence of geometry. The explicit form of the induced representation will be given in terms of cosets and representatives here and is based on the account in [17], where more details may be found.

Let $H \subset G$ and W be a representation of H. For each coset $x \in G/H$, a representative g_x must be chosen — the choice does not matter. For each coset x, a copy W_x is taken of W. For $w \in W$, denote by $g_x w$ the corresponding element in W_x. The induced representation $\mathrm{ind}_H^G(W)$ is then formed by taking the direct sum of all these copies of W

$$\mathrm{ind}_H^G(W) := \bigoplus_{x \in G/H} W_x.$$

Any element v of $\mathrm{ind}_H^G(W)$ may be written as $v = \sum g_x w_x$. To describe the action of the group G on this space, one needs to write the action of $g \in G$ on any coset representative. An element $g \in G$ acts by the formula

$$g \cdot (g_x w_x) = g_y(h \cdot w_x) \qquad \text{if } \; g \cdot g_x = g_y \cdot h. \tag{4.6}$$

This does indeed give a representation of G, for one can show that

$$g' \cdot (g \cdot (g_x w_x)) = (g' \cdot g) \cdot (g_x w_x)$$

for any other element $g' \in G$, which follows from the associativity of the group.

The induced representation of G is not, in general, irreducible. Indeed, it may be decomposed into irreducible representations of G according to the Frobenius Reciprocity Theorem 2.3.2, which may be stated in terms of the Hermitian scalar product of characters as

$$\left\langle V, \mathrm{ind}_H^G(W) \right\rangle_G = \left\langle \mathrm{res}_G^H(V), W \right\rangle_H.$$

This formula says that the number of times that a given representation V of G appears in the $\mathrm{ind}_H^G(W)$ is equal to the number of times that the representation W of H appears in the restricted representation $\mathrm{res}_G^H(V)$. For finite groups this may be computed very quickly using the character tables of G and H.

4.2.3 Construction of regular graphs

We give here a method of constructing regular graphs using group-theoretic data.

Definition. Given a conjugacy class $[k]$ of a group G, we define the *Casimir element* K corresponding to $[k]$ by taking the formal sum of each element in $[k]$

$$K := \sum_{k \in [k]} k.$$

The elements $k \in [k]$ are called the summands of K.

Lemma 4.2.5 *The Casimir elements form a basis for the centre of the group ring* $\mathbb{Z}[G]$*.*

Proof. To prove the Lemma it needs to be shown that for any $h \in G$ that $hK = Kh$ holds, but this is straightforward:

$$hK = \sum_{g \in G} hgkg^{-1} = \sum_{g \in G} gkg^{-1}h = Kh,$$

where the second equality is a result of the map $g \mapsto h^{-1}g$, which obviously leaves the summation invariant. This proves that the Casimir elements are central; to see that they form a basis see [17]. □

Definition. We define a *Casimir element* of G to be any element in the centre of $\mathbb{Z}[G]$, which is a linear combination $\sum m_i K_i$, by analogy with the elements making up the centre of the universal enveloping algebra of a Lie algebra.

Definition. A *real Casimir element* is a Casimir that acts as multiplication by a real scalar on each irreducible representation of G. By Schur's Lemma, this means that the characters of the elements of the corresponding conjugacy class are real.

Real Casimir elements may be formed by adding together two Casimirs whose corresponding conjugacy classes have characters that are complex conjugate.

Lemma 4.2.6 *For any real Casimir element K and for any summand k of K, we have that k^{-1} is also a summand of K.*

Proof. For any finite group it is true that the character of an element g acting in any irreducible representation of G is related to that of g^{-1} by

$$\chi(g^{-1}) = \overline{\chi(g)}. \tag{4.7}$$

This can be seen by noting that for any g there exists a $l \in \mathbb{Z}$ such that $g^l = e$. If ρ is any representation of G then this means that $\rho(g)^l = 1$ and furthermore that the eigenvalues λ_i of $\rho(g)$ are of modulus 1. Therefore the eigenvalues of $\rho(g^{-1}) = \rho(g)^{-1}$ are given by $1/\lambda_i = \overline{\lambda_i}$ and so (4.7) holds.

Therefore, if all the characters are real then $\chi(g^{-1}) = \chi(g)$. Since conjugacy classes are distinguished by their characters this means that g and g^{-1} belong to the same conjugacy class. □

Definition. Let G be a finite group, H a subgroup of G and K a real Casimir of G. Corresponding to the triple (G, H, K) we define a graph $\Gamma_K(V, E)$ by the following procedure. We let the vertices of Γ_K be the left-cosets of H in G, i.e. $V = G/H$. If x, y are two distinct cosets of G/H, we draw an edge from x to y if there exists a summand k of K such that $k \cdot x = y$. We do not draw loops, i.e. edges starting and ending at the same point and we do not draw multiple edges between different points.

Lemma 4.2.7 *The graph Γ_K defined above is undirected, in the sense that if there is an edge from x to y then there is also an edge from y to x.*

Proof. This follows from specifying that K should be a real Casimir. Suppose that $x, y \in G/H$ are two vertices that are joined by an edge $[x, y]$. This means that there exists a summand k of K such that $k \cdot x = y$, therefore we have that $k^{-1} \cdot y = x$. By Lemma 4.2.6 we have that k^{-1} is also a summand of K and corresponding to the edge $[x, y]$ there is also the edge $[y, x]$. □

Lemma 4.2.8 *The graph Γ_K is d-regular.*

Proof. Recall that a graph Γ is d-regular if it has no loops or multiple edges and if each vertex has d-edges joined to it. By construction the graph Γ_K has no loops or multiple edges, so proving the claim amounts to showing that if there are d edges connected to $x \in G/H$ then there are d edges connected to any other $y \in G/H$. This follows from the transitivity of the G action, since there must exist a g such that $g \cdot x = y$. Therefore, if the edges $[x, x_1], \ldots, [x, x_d]$ are generated by $k_1, \ldots, k_d$, then the edges $[y, g \cdot x_i]$ are generated by gk_ig^{-1} acting on $y = g \cdot x$. □

By analogy with the case of coadjoint orbits, we can say that K defines a discrete analogue of the normal metric on the space $X = G/H$.

Definition. We say that K is a *good Casimir element* if it is real and if G acts transitively on the edges of the graph Γ_K.

We will assume from now on that K is a good Casimir element. This is a strong condition on K, but it is clear that it is not so strong that it is never satisfied — in the next section we will be dealing with graphs of regular polyhedra, and in this case G does act transitively.

On the other hand, it is fairly easy to find cases where G does not act transitively on Γ_K. We give as a specific example the case when $G = S_3$ is the symmetric group on three elements and $H = e$. There are three real Casimirs for S_3: $K = e$, $L = (123) + (132)$ and $M = (12) + (13) + (23)$ and these generate the graphs shown in Figure 4.2. On the left is the graph Γ_K, which has no edges; in the middle is the graph $\Gamma_L = K_3 \oplus K_3$; on the right is the graph $\Gamma_M = K_{3,3}$. In the latter case one can check that there is no $g \in S_3$ that sends $[e, (12)]$ to $[e, (13)]$.

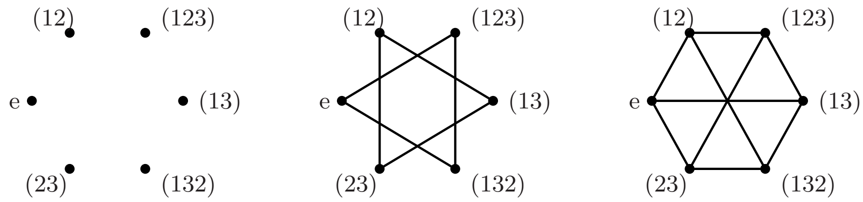

Figure 4.2: The graphs Γ_K, Γ_L and Γ_M for $G = S_3$.

Proposition 4.2.9 *If K is a good Casimir element then the adjacency matrix T_{Γ_K} for Γ_K is given by*

$$T_{\Gamma_K} = \frac{1}{l}\left(P(K) - cI\right), \tag{4.8}$$

where $P(K)$ denotes the matrix of K acting in $\mathrm{ind}_H^G(\mathbf{1})$, with $\mathbf{1}$ being the trivial representation of H and c and l being positive integers.

Proof. We want to show that the matrix $P(K)$ has a constant c along the diagonal and whose non-zero off-diagonal terms are l.

The first part follows from the fact that G acts transitively on G/H. Suppose that $k_1, \ldots, k_c$ are such that $k_i \cdot x = x$, then $k_1' = gk_1g^{-1}, \ldots, k_c' = gk_cg^{-1}$ are such that $k_i' \cdot y = y$ for $y = g \cdot x$.

The second part follows from the assumption of the transitivity of the G-action on the edges of Γ_K. This means that for any edges $[x, y]$ and $[x', y']$, there exists a $g \in G$ such that $g \cdot [x, y] = [x', y']$. Suppose that the xy^{th} entry of P_{xy} equals l, this means that there exist $k_1, \ldots, k_l$ such that $k_i \cdot x = y$. By considering $k_i' = g \cdot k_i \cdot g^{-1}$, we see that there are also l elements mapping $x' = g \cdot x$ to $y' = g \cdot y$. □

Corollary 4.2.10 *The Laplacian matrix L_{Γ_K} for Γ_K is given by*

$$L_{\Gamma_K} = dI - T_{\Gamma_K}, \tag{4.9}$$

where d is the valency of each vertex in Γ_K, which is constant by Lemma 4.2.8.

Theorem 4.2.11 *The spectrum of T_{Γ_K} and L_{Γ_K} can be computed using representation theory and are given by the formulae* (4.10) *and* (4.11) *below.*

Proof. The induced representation $\mathrm{ind}_H^G(\mathbf{1})$ can be decomposed into irreducible representations of G using the Frobenius Reciprocity Theorem 2.3.2, which will lead to a formula of the form

$$\mathrm{ind}_H^G(\mathbf{1}) \cong V_1 \oplus \ldots \oplus V_m$$

where V_i are irreducible representations of G. Since K is an element of the centre of the group algebra of G, by Schur's Lemma it acts as multiplication by a complex-number c_i on each irreducible representation V_i of G. However, since K is a real Casimir, the c_i is in fact real and given by

$$c_i = \frac{n}{d_i}\chi_i(k),$$

where n is the number of elements of conjugacy class corresponding to K, d_i is the dimension of V_i and χ_i is the character of an element k acting in V_i. Therefore the eigenvalues of $P(K)$ are given by

$$Spec(P(K)) = c_1^{d_1}, \ldots, c_m^{d_m}$$

where the notation $c_i^{d_i}$ means that the eigenvalue c_i appears d_i times. By applying (4.8), we see that the spectrum of the adjacency matrix T_{Γ_K} is given by

$$Spec(T_{\Gamma_K}) = \frac{1}{l}(c_1 - c)^{d_1}, \ldots, \frac{1}{l}(c_m - c)^{d_m} \tag{4.10}$$

and by applying (4.9), we see that the spectrum of the Laplacian matrix L_{Γ_K} is

$$Spec(L_{\Gamma_K}) = \left(d + \frac{c}{l} - \frac{c_1}{l}\right)^{d_1}, \ldots, \left(d + \frac{c}{l} - \frac{c_m}{l}\right)^{d_m}. \tag{4.11}$$

□

4.2.4 Construction of regular graphs with magnetic field

We now define a regular graph with magnetic field by replacing the trivial representation of H with any other character $\rho : H \to U(1)$.

Definition. Let G be a finite group, H a subgroup of G, K a good Casimir element of G and ρ a character of H. Suppose that $[x, y]$ is an edge of Γ_K, recall from Proposition 4.2.9 that this means that there exist $k_1, \ldots, k_l$ being summands of K, such that $k_i \cdot x = y$. In terms of coset representatives g_x of x and g_y of y, this means

$$k_i \cdot g_x = g_y \cdot h_i. \tag{4.12}$$

We associate to the edge $[x, y]$ the element of $U(1)$ given by

$$\exp[i\theta_{xy}] = \frac{\rho(h_1) + \ldots + \rho(h_l)}{|\rho(h_1) + \ldots + \rho(h_l)|}. \tag{4.13}$$

assuming that the condition

$$\rho(h_1) + \ldots + \rho(h_l) \neq 0 \tag{4.14}$$

is satisfied.

Remark. If the condition (4.14) holds for one edge then by the transitivity of the G-action on the edges it holds for all edges.

Lemma 4.2.12 *This construction does indeed define a magnetic field $A(\rho)$ on Γ_K, i.e. we have that* $\exp[i\theta_{xy}] = \exp[-i\theta_{yx}]$.

Proof. This follows from the fact that K is a real Casimir. Recall that Lemma 4.2.6 says that if K is a real Casimir then if k is a summand of K, then so is k^{-1}. Rewriting (4.12), we see that

$$k_i^{-1} \cdot g_y = g_x \cdot h_i^{-1}$$

and so

$$\exp[i\theta_{yx}] = \frac{\rho(h_1^{-1}) + \ldots + \rho(h_l^{-1})}{|\rho(h_1^{-1}) + \ldots + \rho(h_l^{-1})|} = \frac{\overline{\rho(h_1)} + \ldots + \overline{\rho(h_l)}}{|\rho(h_1) + \ldots + \rho(h_l)|} = \exp[-i\theta_{xy}]. \tag{4.15}$$

□

One can check that the choice of coset representatives affects the magnetic adjacency matrix by conjugation by a diagonal unitary matrix. Thus, the choice of a coset representative amounts to a choice of gauge.

We have a magnetic analogue of Proposition 4.2.9 and its corollary.

Proposition 4.2.13 *If K is a good Casimir then the magnetic adjacency matrix T_ρ for the magnetic field $A(\rho)$ on Γ_K is given by*

$$T_\rho = \frac{1}{p}\left(P(K) - qI\right), \tag{4.16}$$

where $P(K)$ denotes the matrix of K in $\mathrm{ind}_H^G(\rho)$ and p and q are real constants.

Proof. We want to show that the matrix $P(K)$ has a constant q along the diagonal and that all non-zero off-diagonal terms have constant modulus p.

The first part follows from the fact that G acts transitively on G/H. Suppose that $k_1, \dots, k_c$ are such that $k_i \cdot x = x$, then $k'_1 := gk_1g^{-1}, \dots, k'_c := gk_cg^{-1}$ are such that $k'_i \cdot y = y$ for $y = g \cdot x$. On the level of coset representatives, this means that given $g \cdot g_x = g_y \cdot h$ and $k_i \cdot g_x = g_x \cdot h_i$ we have

$$k'_i g_y = g_y h h_i h^{-1}.$$

We then have that

$$P(K)_{xx} = \rho(h_1) + \ldots + \rho(h_c) \quad \text{and} \quad P(K)_{yy} = \rho(hh_1h^{-1}) + \ldots + \rho(hh_ch^{-1}) = P(K)_{xx}.$$

The reality of the diagonal elements follows from the fact that if $k \cdot x = x$ then $k^{-1} \cdot x = x$.

The second part follows from the assumption of the transitivity of the G-action on the edges of Γ_K. This means that for any edges $[x, y]$ and $[x', y']$, there exists a $g \in G$ such that $g \cdot [x, y] = [x', y']$.

If $[x, y]$ and $[x', y']$ are two edges of Γ_K, then we know from Proposition 4.2.9 that there exist l elements $k_1, \dots, k_l$ such that $k_i \cdot x = y$ and l elements $k'_1, \dots, k'_l$ such that $k'_i \cdot x' = y'$. On the level of coset representatives this means that we have

$$k_i g_x = g_y h_i \quad \text{and} \quad k'_i g_{x'} = g_{y'} h'_i$$

and so we have

$$P_{xy} = \rho(h_1) + \ldots \rho(h_l) \quad \text{and} \quad P_{x'y'} = \rho(h'_1) + \ldots \rho(h'_l).$$

Since G acts transitively on the edges of Γ_K this means that there exists a g such that $gg_x = g_{x'} h_x$ and $gg_y = g_{y'} h_y$. We then see that $gk_ig^{-1} \cdot g_{x'} = g_{y'} h_y h_i h_x^{-1}$ and so

$$P_{x'y'} = \rho(h_y h_1 h_x^{-1}) + \ldots \rho(h_y h_l h_x^{-1}) = \rho(h_y) P_{xy} \rho(h_x^{-1}).$$

Therefore we have that $|P_{x'y'}| = |P_{xy}|$. □

Corollary 4.2.14 *The magnetic Laplacian matrix L_ρ for Γ_K is given by*

$$L_\rho = dI - T_\rho, \tag{4.17}$$

where d is the valency of each vertex and T_ρ is the magnetic adjacency matrix for Γ_K with the magnetic field $A(\rho)$.

Proposition 4.2.15 *The magnetic field constructed by this is G-invariant in the following sense: if γ is a cycle on Γ_K then any other cycle that is the image under G of γ, i.e. $\gamma' = g \cdot \gamma$; then the flux through the cycle γ and the cycle γ' is the same.*

Proof. This follows by iteratively applying the argument from Proposition 4.2.13 to each edge in a cycle. □

Theorem 4.2.16 *The spectrum of the magnetic adjacency matrix and the magnetic Laplacian corresponding to the magnetic field $A(\rho)$ on Γ_K can be found using tools from representation theory and are given by formulae* (4.18) *and* (4.19).

Proof. The proof of this result is essentially the same as the proof of Theorem 4.2.11.

The induced representation $\mathrm{ind}_H^G(\rho)$ can be decomposed into irreducible representations of G using the Frobenius Reciprocity Theorem 2.3.2, which will lead to a formula of the form

$$\mathrm{ind}_H^G(\rho) \cong V_1 \oplus \ldots \oplus V_m$$

where V_i are irreducible representations of G. Since K is an element of the centre of the group algebra of G, by Schur's Lemma it acts as multiplication by a complex-number c_i on each irreducible representation V_i of G. However, since K is a real Casimir, the c_i is in fact real and given by

$$c_i = \frac{n}{d_i}\chi_i(k),$$

where n is the number of elements of conjugacy class corresponding to K, d_i is the dimension of V_i and χ_i is the character of an element k acting in V_i. Therefore the eigenvalues of $P(K)$ are given by

$$Spec(P(K)) = c_1^{d_1}, \ldots, c_m^{d_m}$$

where the notation $c_i^{d_i}$ means that the eigenvalue c_i appears d_i times. By applying (4.8), we see that the spectrum of the adjacency matrix T_{Γ_K} is given by

$$Spec(T_{\Gamma_K}) = \frac{1}{p}(c_1 - q)^{d_1}, \ldots, \frac{1}{p}(c_m - q)^{d_m} \tag{4.18}$$

and by applying (4.9), we see that the spectrum of the Laplacian matrix L_{Γ_K} is

$$Spec(L_{\Gamma_K}) = \left(d + \frac{q}{p} - \frac{c_1}{p}\right)^{d_1}, \ldots, \left(d + \frac{q}{p} - \frac{c_m}{p}\right)^{d_m}. \tag{4.19}$$

□

4.3 Discrete magnetic monopoles on graphs of regular polyhedra

In the previous section we gave a general construction for regular graphs with a magnetic field that is invariant under the action of a symmetry group G. In this section we consider an inverse problem to this, namely we give a definition of a discrete magnetic monopole on a regular polyhedral graph and ask whether it can be obtained by the construction.

Definition. We define a *discrete magnetic monopole* on a polyhedral graph to be given by a magnetic Laplacian on the graph with the magnetic field being G-invariant for some $G \subset SO(3)$ being a symmetry group of the graph.

The question then is, can discrete magnetic monopoles be described by the construction given in the previous section? We show that in many cases the answer is yes; the most important omission is the dodecahedral graph.

As the group we consider $G^* \subset SU(2)$ to be a binary polyhedral subgroup — these are double covers of the corresponding symmetry groups in $G \subset SO(3)$. The binary groups are detailed in Appendix A.1, together with their character tables and Casimir tables. The reason for taking the binary symmetry groups as opposed to the regular symmetry groups is that if we take only the $G \subset SO(3)$ we miss half of the different magnetic charges, picking up only the even Chern numbers. This is entirely analagous to the situation in quantum mechanics of integer and half-integer spin.

It is worth keeping in mind that the Platonic solids may be thought of as discrete approximations to S^2. With this in mind, it is reasonable to think of magnetic monopoles on graphs of the Platonic solids as discrete approximations to a magnetic monopole on a sphere. By pursuing this line of reasoning we can deduce how many distinct magnetic charges, i.e. distinct Chern numbers (in the sense of (4.5)) of the magnetic fields, there should be for each graph.

Theorem 4.3.1 *The number of distinct Chern numbers for each magnetic monopole on a Platonic solid is equal to the number of faces of the Platonic solid.*

Proof. Let a Platonic solid P with f faces be centred around a Dirac monopole of charge $q \in \mathbb{Z}$ in $\mathbb{R}^3$. The flux through each face F of the solid is given by

$$\Phi_F = \frac{2\pi q}{f}.$$

Now consider for a moment the special case when $q = 2$, which corresponds to the tangent bundle to the unit-sphere S^2. Consider also a spherical octahedron — i.e.

one whose edges are geodesics on S^2. If we take a tangent vector at a vertex and parallel transport it along the edges of one of the faces of the octahedron then we find that when it returns to the starting point it has been rotated by $\pi/2 = 4\pi/8$, which is exactly the area of the face. Similarly, for any other P we find that the holonomy of the tangent vector on being parallel transported about the edges of a face is equal to the area enclosed by that face, i.e. $4\pi/f$.

For any other q, we find that the holonomy on parallel transporting along the edges of a face F is given by the flux Φ_F through that F. However, the holonomy is only taken $\mod (2\pi)$ and so we find that $q+f$ and q give the same holonomy. Thus there are f different possible Chern numbers for magnetic monopoles on the graph of the Platonic solid P. □

We can rephrase Theorem 4.3.1 in terms of representation theory, since $G^* \subset SU(2)$ we can use the description of wavefunctions from Chapter Two.

Theorem 4.3.2 *Let P be a Platonic solid, whose binary symmetry group is G^* and with the stabilizer of a vertex given by H^*. This means that $H^* \cong \mathbb{Z}_k^* \cong \langle \xi \rangle$, where $\xi^{2k} = 1$, for some k. Let K be a real Casimir of G^* and suppose that K generates a graph Λ. Let γ be a cycle in Λ and let $q = 0, 1, \ldots, 2k-1$. Denote the flux through γ when acting in the representation $\mathrm{ind}_{H^*}^{G^*}(q)$ by $\phi(\gamma)$. By considering $G^* \subset SU(2)$, we can also consider the flux through γ when K acts in the representation $\mathrm{ind}_{U(1)}^{SU(2)}(W_p)$, denote this by $\Phi_p(\gamma)$. The result is that if $q = p \mod 2k$ then $\Phi_p(\gamma) = \Phi(\gamma)$.*

Proof. If γ is a cycle of length n on the graph Λ that is generated by K acting on G^*/H^* then there exist $g_1, \ldots, g_n$ such that $g_i : x_{i-1} \to x_i$, or equivalently $g_i \cdot x_{i-1} = x_i \cdot h_i$, where $h_i \in H^*$ and a coset x_i has been identified with its representative. For each i we have that $h_i = \xi^{i_m}$, for some m. Acting in the representation $\mathrm{ind}_{H^*}^{G^*}(q)$, we have that the flux through γ is given by

$$\Phi(\gamma) = \arg\left(\prod_{i=1}^{n} (\xi^{i_m})^q\right).$$

Considering K acting in the representation $\mathrm{ind}_{U(1)}^{SU(2)}(W_p)$, the flux through γ is given by

$$\Phi_p(\gamma) = \arg\left(\prod_{i=1}^{n} (\xi^{i_m})^p\right),$$

which, since $\xi^{2k} = 1$, clearly only depends on the value of $p \mod 2k$ and agrees with $\Phi(\gamma)$ if $q \equiv p \mod 2k$. □

We start by investigating the graphs of Platonic solids with the simplest case, that of the tetrahedron, before moving through the Platonic solids in order of increasing number of vertices.

4.3.1 Tetrahedron

The binary tetrahedral group T^* is listed in Appendix A.1 and has order 24. The stabilizer of a vertex in $T \subset SO(3)$ is a cyclic group of order 3 and this lifts to the binary cyclic group C_3^*, which has order 6 — take for this group the cyclic group H generated by $\frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$.

Lemma 4.3.3 *The space of left cosets of H in T^* is given by*

$$1 := \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & 1-i \\ -1-i & -1+i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}\right\}$$

$$2 := \left\{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1+i & -1-i \\ 1-i & -1-i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & -1-i \\ 1-i & -1+i \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ -1+i & 1+i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix}\right\}$$

$$3 := \left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & -1+i \\ 1+i & -1+i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & 1+i \\ -1+i & -1+i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ -1-i & 1-i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & -1-i \\ 1-i & 1-i \end{pmatrix}\right\}$$

$$4 := \left\{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1+i & 1+i \\ -1+i & -1-i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1+i & 1-i \\ -1-i & -1-i \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & -1-i \\ 1-i & 1+i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & -1+i \\ 1+i & 1+i \end{pmatrix}\right\}$$

Proof. Direct calculation. □

Definition. Define J to be the real Casimir formed by taking the sum of the elements in the conjugacy class of (123) and (132), namely

$$J = \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ -1+i & 1+i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} +$$
$$+\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ -1-i & 1-i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1+i & -1-i \\ 1-i & 1-i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1-i & -1-i \\ 1-i & 1+i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1-i & -1+i \\ 1+i & 1+i \end{pmatrix}$$

and these are labelled as $J_1, \ldots, J_8$ respectively.

Lemma 4.3.4 *The action of the Casimir J on the coset representatives of the tetrahedron may be represented graphically as in Figure 4.3. We see that according to the prescription given in the previous section we generate the tetrahedral graph.*

Proof. The action of each of the J_i's is recorded in Table 4.1. The elements of the table correspond to cosets and elements of each coset, for example, the entry 1,2 refers to the second element of the first coset, as listed in Lemma 4.3.3. □

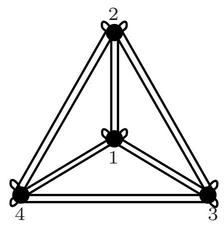

Figure 4.3: The graph generated by the action of J acting on the space of left-cosets that represent the tetrahedron.

J	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
1	1,2	1,6	2,5	2,6	3,5	3,6	4,5	4,6
2	4,2	3,6	3,5	4,3	2,2	1,3	1,2	2,6
3	2,2	4,6	1,2	3,6	4,5	2,3	3,2	1,3
4	3,2	2,6	4,2	1,3	1,2	4,6	2,5	3,3

Table 4.1: Table showing where each summand of the Casimir K for O^* maps the representatives of each left-coset of the octahedron.

The irreducible representations of $\mathbb{Z}_3^* \cong \mathbb{Z}_6 \cong \langle \eta \rangle$ ($\eta = \exp[2\pi i/6]$) are all characters and are indexed by an integer k between 0 and 5 — specifically $\eta \mapsto \eta^k$. The character table of T^* is given in Appendix A.1.

We are now in a position to describe monopoles on the tetrahedral graph: denote by $\eta = \exp[i\pi/3]$. Acting in the representation $\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(k)$, the Casimir J has matrix

$$J^k = \begin{pmatrix} \eta^k + \eta^{5k} & \eta^{4k} + \eta^{5k} & \eta^{4k} + \eta^{5k} & \eta^{4k} + \eta^{5k} \\ \eta^k + \eta^{2k} & \eta^k + \eta^{5k} & \eta^{4k} + \eta^{5k} & \eta^k + \eta^{2k} \\ \eta^k + \eta^{2k} & \eta^k + \eta^{2k} & \eta^{3k} + \eta^{5k} & \eta^{4k} + \eta^{5k} \\ \eta^k + \eta^{2k} & \eta^{4k} + \eta^{5k} & \eta^k + \eta^{2k} & \eta^{3k} + \eta^{5k} \end{pmatrix},$$

To get an adjacency matrix for the tetrahedral graph, first we have to subtract $\eta^k + \eta^{5k}$ from the diagonal. However, it might seem that this is still not the desired object, since each non-zero entry of the matrix is not an element of $U(1)$. However, each entry has the same magnitude, dividing by this magnitude gives a well-defined element of $U(1)$ that turns out to be a twelfth root of unity. Denote the resulting matrix by A_J^k. For example, for $k = 1$, we have that the magnitude of each non-zero entry is $\sqrt{3}$. Dividing by this and setting $\theta = \exp[2\pi i/12]$ gives the matrix

$$A_J^1 = \begin{pmatrix} 0 & \theta^9 & \theta^9 & \theta^9 \\ \theta^3 & 0 & \theta^9 & \theta^3 \\ \theta^3 & \theta^3 & 0 & \theta^9 \\ \theta^3 & \theta^9 & \theta^3 & 0 \end{pmatrix},$$

One computes that the flux through each face, when oriented anti-clockwise, as in Figure 4.3, is given by $\pi/2$ in the representation $\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(1)$. Similarly, one finds that it is given by π for $k = 2, 4$ and $3\pi/2$ for $k = 5$. Computing the Chern number gives 1 for $k = 1$; 2 for $k = 2, 4$; and 3 for $k = 5$. There is an anomaly when $k = 3$, in that the matrix K^3 is identically 0, owing to each entry being a sum of $\theta^3 = -1$ and $\theta^6 = 1$.

Remark. It is worth remarking that by considering the matrix whose entries are obtained by raising the corresponding entry of A_J^1 to the power l, one obtains an adjacency matrix, where the flux through each face is give by $l\pi/2 \mod 2\pi$ for $l = 0, 1, 2, 3, 0, 1$. This thus generates every magnetic charge on the tetrahedral graph.

Having done this, we now move to describe the spectrum of the adjacency matrices corresponding to these monopoles.

Lemma 4.3.5 *The representations of T^* induced from the characters of $\mathbb{Z}_3^*$ decompose into irreducible representations of T^* as*

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{T^*}(0) \cong U \oplus V,$$

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{T^*}(1) \cong S \oplus S',$$

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{T^*}(2) \cong U' \oplus V,$$

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{T^*}(3) \cong S' \oplus S'',$$

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{T^*}(4) \cong U'' \oplus V,$$

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{T^*}(5) \cong S \oplus S''.$$

Proof. This is a direct calculation done by using Frobenius reciprocity in conjunction with the scalar product of characters. □

Theorem 4.3.6 *Denote by A_J^k the matrix obtained by normalising the matrix of K^k, such that every element belongs to $U(1)$ (which is possible for all $k \neq 3$). The Chern numbers of these magnetic fields is given by k. The spectrum of these operators is given in Table 4.2. The corresponding magnetic Laplacian is formed by $\Delta = 3I - \tilde{A_J^k}$.*

Proof. We have that $A_J^0 = \frac{1}{2}J - I$, $A_J^1 = \frac{1}{\sqrt{3}}J^1 - I$, $A_J^2 = J^2 + I$, $A_J^4 = J^4 + I$ and $A_J^5 = \frac{1}{\sqrt{3}}J^5 - I$. Comparing this with the Casimir table for the T^* in Appendix A.1 and the decomposition of the representations $\mathrm{ind}_{\mathbb{Z}_3^*}^{T^*}(k)$ into irreducibles given in Lemma 4.3.5 gives the result. □

Operator	Chern number	Adjacency Spectrum	Laplacian Spectrum
A_J^0	0	$-3, 1^3$	$0, 4^3$
A_J^1	1	$[\sqrt{3}]^2$,$[-\sqrt{3}]^2$	$[3-\sqrt{3}]^2$,$[3+\sqrt{3}]^2$
A_J^2	2	$3, -1^3$	$2^3, 6$
A_J^4	2	$3, -1^3$	$2^3, 6$
A_J^5	3	$[\sqrt{3}]^2$,$[-\sqrt{3}]^2$	$[3-\sqrt{3}]^2$,$[3+\sqrt{3}]^2$

Table 4.2: Chern numbers and spectrum for the operators A_K^k

.

Remark. The spectrum for the operator J^3 acting on $S' \oplus S''$ is -2^4, as can be seen from the Casimir table and Lemma 4.3.13, or by seeing that the matrix J^3 itself is equal to $-2I$. However, one can generate a magnetic field on the graph with a flux through each face of π by taking the matrix A_J^1 and raising each matrix entry to the third power. The spectrum of this matrix is again $-\sqrt{3}^2, \sqrt{3}^2$, as must be the case since its corresponding Chern number is 3 and in view of Lemma 4.2.2.

4.3.2 Octahedron

The binary octahedral group is listed in Appendix A.1 and has order 48. The stabilizer of a vertex for $O \subset SO(3)$ was a cyclic group of order 4 and so lifts to the binary cyclic group C_4^*, which has order 8 — take for this group the cyclic group H generated by $\frac{1}{\sqrt{2}}\begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}$.

Lemma 4.3.7 *The set of vertices of the octahedron may be identified with the space of cosets O^*/H.*

$$1 := \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1-i & 0 \\ 0 & -1+i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1+i & 0 \\ 0 & -1-i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}\right\}$$

$$2 := \left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1-i \\ 1-i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1-i \\ -1-i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1+i \\ -1+i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1+i \\ 1+i & 0 \end{pmatrix}\right\}$$

$$3 := \left\{\frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & -1-i \\ 1-i & 1-i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -i \\ -i & i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & 1-i \\ -1-i & -1+i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ -1+i & -1+i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} i & i \\ i & -i \end{pmatrix}\right\}$$

$$4 := \left\{\frac{1}{2}\begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} i & -1 \\ -1 & -i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & -1-i \\ 1-i & 1-i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & 1+i \\ -1+i & -1+i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix}\right\}$$

$$5 := \left\{\frac{1}{2}\begin{pmatrix} -1-i & -1+i \\ 1+i & -1+i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1+i & -1-i \\ 1-i & -1-i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ -i & -i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ -1-i & 1-i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ -1+i & 1+i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ i & i \end{pmatrix}\right\}$$

$$6 := \left\{\frac{1}{2}\begin{pmatrix} -1+i & 1-i \\ -1-i & -1-i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & -1+i \\ 1+i & 1+i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & -1-i \\ 1-i & -1+i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -i \\ -i & -1 \end{pmatrix}\right\}$$

Proof. This is a direct calculation. □

Remark. Using the binary subgroup, instead of the abstract S_4 makes it easy to see the structure of the octahedron. Projecting each matrix to the extended complex plane gives that Cosets $1, 2, 3, 4, 5, 6$ can be identified respectively with the points $\infty, 0, 1, i, -1, -i$, since under the map

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \mapsto \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} \exp[i\theta] & 0 \\ 0 & \exp[-i\theta] \end{pmatrix} = \begin{pmatrix} a\exp[i\theta] & -\overline{b}\exp[-i\theta] \\ b\exp[i\theta] & \overline{a}\exp[-i\theta] \end{pmatrix}$$

the quantity a/b is invariant and gives a well-defined point of $\mathbb{C} \cup \infty$. One may then draw a graph by joining each point to the 'closest' four points.

Definition. Denote by K the Casimir is given by taking the sum of every element in the conjugacy class of (1234) in O^*, namely:

$$K := \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix} +$$
$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and label these elements as $K_1, \ldots K_6$ respectively.

Lemma 4.3.8 *The action of the Casimir K acts on the space $O^*/\mathbb{Z}_4^*$ may be represented graphically as in Figure 4.4. We see that according to the prescription given in the previous section we generate the octahedral graph using the Casimir K.*

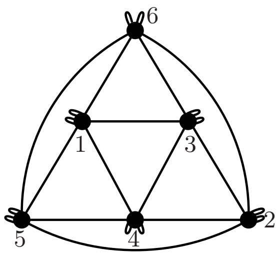

Figure 4.4: The graph generated by the action of K acting on the space of left-cosets that represent the octahedron.

Proof. The action of the elements K_i on the coset representatives, as defined in Lemma 4.3.7, is given in Table 4.3.

The irreducible representations of $\mathbb{Z}_4^* \cong \mathbb{Z}_8 \cong \langle\zeta\rangle$ ($\zeta = \exp[2\pi i/8]$) are all characters and are indexed by an integer k between 0 and 7 — specifically $\zeta \mapsto \zeta^k$. The character table of O^* is given in Appendix A.1.

K	K_1	K_2	K_3	K_4	K_5	K_6
1	1,8	1,2	3,2	4,8	5,6	6,4
2	2,2	2,8	5,2	6,6	3,2	4,2
3	4,2	6,4	2,8	3,2	1,8	3,8
4	5,2	3,8	4,8	2,8	4,2	1,6
5	6,6	4,8	1,4	5,8	2,8	5,2
6	3,6	5,4	6,2	1,6	6,8	2,4

Table 4.3: Table showing where each summand of the Casimir K for O^* maps the representatives of each left-coset of the octahedron.

We can now describe monopoles on the octahedral graph: denote by $\zeta = \exp[i\pi/4]$. Acting in the representation $\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(k)$, the Casimir K has matrix

$$K^k = \begin{pmatrix} \zeta^{3k}+\zeta^{5k} & 0 & \zeta^k & \zeta^{3k} & \zeta^{5k} & \zeta^{3k} \\ 0 & \zeta^{3k}+\zeta^{5k} & \zeta^k & \zeta^k & \zeta^k & \zeta^{5k} \\ \zeta^{7k} & \zeta^{7k} & \zeta^{3k}+\zeta^{5k} & \zeta^k & 0 & \zeta^{3k} \\ \zeta^{5k} & \zeta^{7k} & 0 & \zeta^{3k}+\zeta^{5k} & \zeta^k & 0 \\ \zeta^{3} & \zeta^{7k} & 0 & \zeta^{7k} & \zeta^{3k}+\zeta^{5k} & \zeta^{5k} \\ \zeta^{5k} & \zeta^{3k} & \zeta^{5k} & 0 & \zeta^{3k} & \zeta^{3k}+\zeta^{5k} \end{pmatrix}$$

and so $K^k - (\xi^{3k} + \xi^{5k})I$, gives exactly the adjacency matrix for the octahedron in the presence of a magnetic monopole.

Computing the flux through each face, when oriented anti-clockwise as in Figure 4.4, gives exactly $k\pi/4$ and so the total flux is $8.k\pi/4$ and so the Chern number (as given in equation (4.5)) for the monopole corresponding to $\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(k)$ is exactly k. Thus we have discovered the monopoles with all possible Chern numbers for the octahedral graph. Now look to describe the spectrum of each of these.

Lemma 4.3.9 *The representations of O^* induced from the characters of $\mathbb{Z}_4^*$ decompose into irreducible representations of O^* as*

$$\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(0) \cong U \oplus V' \oplus W,$$

$$\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(1) \cong S \oplus X,$$

$$\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(2) \cong V \oplus V',$$

$$\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(3) \cong X \oplus S',$$

$$\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(4) \cong U' \oplus V \oplus W,$$

$$\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(5) \cong X \oplus S',$$

$$\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(6) \cong V \oplus V',$$

$$\mathrm{ind}_{\mathbb{Z}_4^*}^{O^*}(7) \cong X \oplus S.$$

Operator	Chern number	Adjacency Spectrum	Laplacian Spectrum
A_O^0	0	$4, 0^3, -2^2$	$0, 4^3, 6^2$
A_O^1	1	$[2\sqrt{2}]^2$,$[-\sqrt{2}]^4$	$[4-2\sqrt{2}]^2$,$[4+\sqrt{2}]^4$
A_O^2	2	$2^3, -2^3$	$2^3, 6^3$
A_O^3	3	$-2\sqrt{2}^2, \sqrt{2}^4$	$4-\sqrt{2}^4, 4+2\sqrt{2}^2$
A_O^4	4	2^2,0^3,-4	$2^2, 4^3, 8$
A_O^5	5	$-2\sqrt{2}^2, \sqrt{2}^4$	$4-\sqrt{2}^4, 4+2\sqrt{2}^2$
A_O^6	6	$2^3, -2^3$	$2^3, 6^3$
A_O^7	7	$2\sqrt{2}^2$,$-\sqrt{2}^4$	$[4-2\sqrt{2}]^2$,$[4+\sqrt{2}]^4$

Table 4.4: Chern numbers and spectrum for the operators A_O^k.

Proof. This is a direct calculation done by using the Frobenius Reciprocity Theorem 2.3.2 in conjunction with the scalar product of characters. □

Theorem 4.3.10 *We now have a complete description of monopoles on the octahedral graph. Denote by $A_O^k = K^k - (\xi^{3k} + \xi^{5k})I$ the adjacency matrix for the octahedral graph with Chern number k, as above. The spectrum of these operators is given in Table 4.4. The corresponding magnetic Laplacian is formed by $\Delta = 4I - A_O^k$.*

Proof. This result follows from comparing the Casimir table for O^* described in Appendix A.1 and the decomposition of the representations $\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(k)$ into irreducibles given in Lemma 4.3.9. □

4.3.3 Cube

The binary octahedral group is the binary symmetry group of the cube, which is listed in Appendix A.1 and has order 48. The stabilizer in $O \subset SO(3)$ of a vertex of the cube was a cyclic group of order 3 and so lifts to the binary cyclic group Z_3^*, which has order 6. Take as the identity coset the cyclic subgroup H generated by $\frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$.

Lemma 4.3.11 *The space of left cosets of H in O^* is given by*

$$1 := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & 1-i \\ -1-i & -1+i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} \right\}$$

$$2 := \left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1+i & -1-i \\ 1-i & -1-i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & -1-i \\ 1-i & -1+i \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ -1+i & 1+i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} \right\}$$

$$3 := \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & -1+i \\ 1+i & -1+i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & 1+i \\ -1+i & -1+i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ -1-i & 1-i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & -1-i \\ 1-i & 1-i \end{pmatrix} \right\}$$

$$4 := \left\{ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1+i & 1+i \\ -1+i & -1-i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1+i & 1-i \\ -1-i & -1-i \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & -1-i \\ 1-i & 1+i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-i & -1+i \\ 1+i & 1+i \end{pmatrix} \right\}$$

$$5 := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & 1+i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1-i & 0 \\ 0 & -1+i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}$$

$$6 := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1+i & 0 \\ 0 & -1-i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -i \\ -i & -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -i \\ -i & i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} i & i \\ i & -i \end{pmatrix} \right\}$$

$$7 := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1-i \\ 1-i & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ i & i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1+i \\ -1+i & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ -i & -i \end{pmatrix} \right\}$$

$$8 := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1+i \\ 1+i & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1-i \\ -1-i & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right\}$$

Proof. This is a direct calculation. □

Lemma 4.3.12 *The action of the Casimir K on the space of left cosets may be represented graphically as in Figure 4.5. We see that according to the prescription given in the previous section we generate the graph of the cube using the Casimir K.*

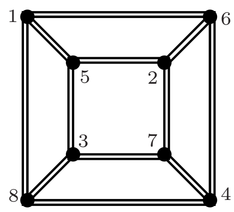

Figure 4.5: The graph generated by the action of K acting on the space of left-cosets that represent the cube.

Proof. The action of the elements of K on the coset representatives of the cube from in Lemma 4.3.11 is given in Table 4.5. □

	K_1	K_2	K_3	K_4	K_5	K_6
1	5,1	5,6	6,4	6,5	8,5	8,6
2	6,1	7,6	5,1	7,5	5,2	6,6
3	7,1	8,6	8,1	5,2	7,2	5,3
4	8,1	6,6	7,4	8,2	6,5	7,3
5	2,1	3,5	1,1	2,6	3,6	1,2
6	1,4	2,2	2,1	4,3	1,3	4,2
7	4,4	4,5	3,1	3,6	2,3	2,2
8	3,1	1,2	4,1	1,3	4,6	3,2

Table 4.5: Table showing where each summand of the Casimir K for O^* maps the representatives of each left-coset of the cube.

The irreducible representations of $\mathbb{Z}_3^* \cong \mathbb{Z}_6 \cong \langle \eta \rangle$ ($\eta = \exp[2\pi i/6]$) are all characters and are indexed by an integer k between 0 and 5 — specifically $\eta \mapsto \eta^k$. The character table of O^* is given in Appendix A.1.

Therefore the matrix of the Casimir K acting in the representation $\mathrm{ind}_{\mathbb{Z}_3}^{O^*}(k)$ is given by K^k, where we denote by $\eta = \exp[2\pi i/6]$

$$K^k = \begin{pmatrix} 0 & 0 & 0 & 0 & 1+\eta^{5k} & \eta^{3k}+\eta^{4k} & 0 & \eta^{4k}+\eta^{5k} \\ 0 & 0 & 0 & 0 & 1+\eta^{k} & 1+\eta^{5k} & \eta^{4k}+\eta^{5k} & 0 \\ 0 & 0 & 0 & 0 & \eta^{k}+\eta^{2k} & 0 & 1+\eta^{k} & 1+\eta^{5k} \\ 0 & 0 & 0 & 0 & 0 & \eta^{4k}+\eta^{5k} & \eta^{2k}+\eta^{3k} & 1+\eta^{k} \\ 1+\eta^{k} & 1+\eta^{5k} & \eta^{4k}+\eta^{5k} & 0 & 0 & 0 & 0 & 0 \\ \eta^{2k}+\eta^{3k} & 1+\eta^{k} & 0 & \eta^{k}+\eta^{2k} & 0 & 0 & 0 & 0 \\ 0 & \eta^{k}+\eta^{2k} & 1+\eta^{5k} & \eta^{3k}+\eta^{4k} & 0 & 0 & 0 & 0 \\ \eta^{k}+\eta^{2k} & 0 & 1+\eta^{k} & 1+\eta^{5k} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It might seem that this is not the desired object, since each non-zero entry of the matrix is not an element of $U(1)$. However, each entry has the same magnitude and dividing by this magnitude gives a well-defined element of $U(1)$ that turns out to be a twelfth root of unity. For example, for $k = 1$, we have that the magnitude of each non-zero entry is $\sqrt{3}$. Dividing by this and setting $\theta = \exp[2\pi i/12]$ gives the matrix

$$A_K^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \theta^{11} & \theta^7 & 0 & \theta^9 \\ 0 & 0 & 0 & 0 & \theta & \theta^{11} & \theta^9 & 0 \\ 0 & 0 & 0 & 0 & \theta^3 & 0 & \theta & \theta^{11} \\ 0 & 0 & 0 & 0 & 0 & \theta^9 & \theta^5 & \theta \\ \theta & \theta^{11} & \theta^9 & 0 & 0 & 0 & 0 & 0 \\ \theta^5 & \theta & 0 & \theta^3 & 0 & 0 & 0 & 0 \\ 0 & \theta^3 & \theta^{11} & \theta^7 & 0 & 0 & 0 & 0 \\ \theta^3 & 0 & \theta & \theta^{11} & 0 & 0 & 0 & 0 \end{pmatrix},$$

One computes that the flux through each face, when oriented anti-clockwise, as in Figure 4.5, is given by $\pi/3$ in the representation $\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(1)$. Similarly, one finds that it is given by $k\pi/3$ for $k = 1, 2, 4, 5, 0$. There is an anomaly when $k = 3$, in that the matrix K^3 is identically 0, owing to each entry being a sum of $\theta^3 = -1$ and $\theta^6 = 1$. Computing the Chern number of each of the monopoles corresponding to the representations $\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(k)$ $(k \neq 3)$ we find that the Chern number is exactly k.

It is worth remarking that by considering the matrix whose entries are obtained by raising the corresponding entry of A_K^1 to the power l, one obtains an adjacency matrix, where the flux through each face is give by $l\pi/3$ for $l = 0, 1, 2, 3, 4, 5$. This thus generates every magnetic charge.

Having done this, we now move to describe the spectrum of the adjacency matrices corresponding to these monopoles.

Operator	Chern number	Adjacency spectrum	Laplacian spectrum
A_K^0	0	$3, 1^3, -1^3, -3^3$	$0, 2^3, 4^3, 6$
A_K^1	1	$[\sqrt{6}]^2$,0^4,$[-\sqrt{6}]^2$	$[3-\sqrt{6}]^2$,3^4,$[3+\sqrt{6}]^2$
A_K^2	2	$2^3, 0^2, -2^3$	$1^3, 3^2, 5^3$
A_K^4	4	$2^3, 0^2, -2^3$	$1^3, 3^2, 5^3$
A_K^5	5	$[\sqrt{6}]^2$,0^4,$[-\sqrt{6}]^2$	$[3-\sqrt{6}]^2$,3^4,$[3+\sqrt{6}]^2$

Table 4.6: Chern numbers and spectrum for the operators A_K^k
.

Lemma 4.3.13 *The representations of O^* induced from the characters of $\mathbb{Z}_3^*$ decompose into irreducible representations of O^* as*

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(0) \cong U \oplus U' \oplus V \oplus V',$$

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(1) \cong S \oplus S' \oplus X,$$

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(2) \cong V \oplus V' \oplus W,$$

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(3) \cong X \oplus X,$$

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(4) \cong V \oplus V' \oplus W,$$

$$\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(5) \cong S \oplus S' \oplus X.$$

Proof. This is a direct calculation done by using Frobenius reciprocity in conjunction with the scalar product of characters. □

Theorem 4.3.14 *Denote by A_K^k the matrix obtained by normalising the matrix of K^k, such that every element belongs to $U(1)$ (which is possible for $k \neq 3$). The Chern numbers of each of these magnetic fields (as defined in equation (4.5)) is given by k. The corresponding magnetic Laplacian is formed by $\Delta = 3I - A_K^k$. The spectrum of these operators is given in Table 4.6.*

Proof. We have that the corresponding adjacency operators are given by $A_K^0 = \frac{1}{2}K^1$, $A_K^1 = \frac{1}{\sqrt{3}}K^1$, $A_K^2 = K^2$, $A_K^4 = K^4$ and $A_K^5 = \frac{1}{\sqrt{3}}K^5$. Comparing this with the Casimir table in Appendix A.1 and the decomposition of the representations $\mathrm{ind}_{\mathbb{Z}_3^*}^{O^*}(k)$ into irreducibles given in Lemma 4.3.13 gives the result. □

Remark. The spectrum for the operator K^3 is identically zero, as can be seen from the Casimir table and Lemma 4.3.13, or by seeing that the matrix K^3 itself is identically zero. However, one can generate a magnetic field on the graph with a flux through each face of π by taking the matrix A_K^1 and raising each matrix entry to the third power.

4.3.4 Icosahedron

The binary icosahedral group is listed in Appendix A.1 and has order 120. The stabilizer of a vertex as a subgroup of I is a cyclic group of order 5 and so this lifts to the binary cyclic group C_5^*, which has order 10.

The stabilizing subgroup is $H \cong \mathbb{Z}_5^* = \mathbb{Z}_{10}$. In the following let $\epsilon = \exp[2\pi i/5]$: moreover, in order not to overwhelm the reader with data only the first half of the elements for cosets 3-12 are displayed. The second half are got by multiplying each element from the first half by $-I$. To keep the elements readable, the following notation is used for the elements in cosets 3–12:

$$\begin{pmatrix} 0-2 & 1-2 \\ 3-4 & 0-3 \end{pmatrix} := \frac{1}{\sqrt{5}} \begin{pmatrix} 1-\epsilon^2 & \epsilon-\epsilon^2 \\ \epsilon^3-\epsilon^4 & 1-\epsilon^3 \end{pmatrix}$$

Lemma 4.3.15 *The space of left cosets of H in I^* is given by*

$$1 := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\epsilon^3 & 0 \\ 0 & -\epsilon^2 \end{pmatrix}, \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^4 \end{pmatrix}, \begin{pmatrix} -\epsilon^4 & 0 \\ 0 & -\epsilon \end{pmatrix}, \begin{pmatrix} \epsilon^2 & 0 \\ 0 & \epsilon^3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^2 \end{pmatrix}, \begin{pmatrix} -\epsilon & 0 \\ 0 & -\epsilon^4 \end{pmatrix}, \begin{pmatrix} \epsilon^4 & 0 \\ 0 & \epsilon \end{pmatrix}, \begin{pmatrix} -\epsilon^2 & 0 \\ 0 & -\epsilon^3 \end{pmatrix} \right\}$$

$$2 := \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\epsilon^2 \\ \epsilon^3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \epsilon^4 \\ -\epsilon & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\epsilon \\ \epsilon^4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \epsilon^3 \\ -\epsilon^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \epsilon^2 \\ -\epsilon^3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\epsilon^4 \\ \epsilon & 0 \end{pmatrix}, \begin{pmatrix} 0 & \epsilon \\ -\epsilon^4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\epsilon^3 \\ \epsilon^2 & 0 \end{pmatrix} \right\}$$

$$3 := \left\{ \begin{pmatrix} 4-1 & 2-3 \\ 2-3 & 1-4 \end{pmatrix}, \begin{pmatrix} 4-2 & 0-4 \\ 1-0 & 1-3 \end{pmatrix}, \begin{pmatrix} 0-2 & 1-2 \\ 3-4 & 0-3 \end{pmatrix}, \begin{pmatrix} 0-3 & 4-3 \\ 2-1 & 0-2 \end{pmatrix}, \begin{pmatrix} 1-3 & 0-1 \\ 4-0 & 4-2 \end{pmatrix}, \dots \right\}$$

$$4 := \left\{ \begin{pmatrix} 2-4 & 0-1 \\ 4-0 & 3-1 \end{pmatrix}, \begin{pmatrix} 2-0 & 3-2 \\ 3-2 & 3-0 \end{pmatrix}, \begin{pmatrix} 3-0 & 4-0 \\ 0-1 & 2-0 \end{pmatrix}, \begin{pmatrix} 3-1 & 2-1 \\ 4-3 & 2-4 \end{pmatrix}, \begin{pmatrix} 4-1 & 3-4 \\ 1-2 & 1-4 \end{pmatrix}, \dots \right\}$$

$$5 := \left\{ \begin{pmatrix} 0-2 & 3-4 \\ 1-2 & 0-3 \end{pmatrix}, \begin{pmatrix} 0-3 & 1-0 \\ 0-4 & 0-2 \end{pmatrix}, \begin{pmatrix} 1-3 & 2-3 \\ 2-3 & 4-2 \end{pmatrix}, \begin{pmatrix} 1-4 & 0-4 \\ 1-0 & 4-1 \end{pmatrix}, \begin{pmatrix} 2-4 & 1-2 \\ 3-4 & 3-1 \end{pmatrix}, \dots \right\}$$

$$6 := \left\{ \begin{pmatrix} 3-0 & 1-2 \\ 3-4 & 2-0 \end{pmatrix}, \begin{pmatrix} 3-1 & 4-3 \\ 2-1 & 2-4 \end{pmatrix}, \begin{pmatrix} 4-1 & 0-1 \\ 4-0 & 1-4 \end{pmatrix}, \begin{pmatrix} 4-2 & 3-2 \\ 3-2 & 1-3 \end{pmatrix}, \begin{pmatrix} 0-2 & 4-0 \\ 0-1 & 0-3 \end{pmatrix}, \dots \right\}$$

$$7 := \left\{ \begin{pmatrix} 1-3 & 4-0 \\ 0-1 & 4-2 \end{pmatrix}, \begin{pmatrix} 1-4 & 2-1 \\ 4-3 & 4-1 \end{pmatrix}, \begin{pmatrix} 2-4 & 3-4 \\ 1-2 & 3-1 \end{pmatrix}, \begin{pmatrix} 2-0 & 1-0 \\ 0-4 & 3-0 \end{pmatrix}, \begin{pmatrix} 3-0 & 2-3 \\ 2-3 & 2-0 \end{pmatrix}, \dots \right\}$$

$$8 := \left\{ \begin{pmatrix} 3-2 & 4-1 \\ 4-1 & 2-3 \end{pmatrix}, \begin{pmatrix} 0-1 & 3-1 \\ 4-2 & 0-4 \end{pmatrix}, \begin{pmatrix} 4-3 & 3-0 \\ 0-2 & 1-2 \end{pmatrix}, \begin{pmatrix} 1-2 & 2-0 \\ 0-3 & 4-3 \end{pmatrix}, \begin{pmatrix} 0-4 & 2-4 \\ 1-3 & 0-1 \end{pmatrix}, \dots \right\}$$

$$9 := \left\{ \begin{pmatrix} 0-4 & 1-3 \\ 2-4 & 0-1 \end{pmatrix}, \begin{pmatrix} 2-3 & 0-3 \\ 2-0 & 3-2 \end{pmatrix}, \begin{pmatrix} 1-0 & 0-2 \\ 3-0 & 4-0 \end{pmatrix}, \begin{pmatrix} 3-4 & 4-2 \\ 3-1 & 2-1 \end{pmatrix}, \begin{pmatrix} 2-1 & 4-1 \\ 4-1 & 3-4 \end{pmatrix}, \dots \right\}$$

$$10 := \left\{ \begin{pmatrix} 2-1 & 3-0 \\ 0-2 & 3-4 \end{pmatrix}, \begin{pmatrix} 4-0 & 2-0 \\ 0-3 & 1-0 \end{pmatrix}, \begin{pmatrix} 3-2 & 2-4 \\ 1-3 & 2-3 \end{pmatrix}, \begin{pmatrix} 0-1 & 1-4 \\ 1-4 & 0-4 \end{pmatrix}, \begin{pmatrix} 4-3 & 1-3 \\ 2-4 & 1-2 \end{pmatrix}, \dots \right\}$$

$$11 := \left\{ \begin{pmatrix} 4-3 & 0-2 \\ 3-0 & 1-2 \end{pmatrix}, \begin{pmatrix} 1-2 & 4-2 \\ 3-1 & 4-3 \end{pmatrix}, \begin{pmatrix} 0-4 & 4-1 \\ 4-1 & 0-1 \end{pmatrix}, \begin{pmatrix} 2-3 & 3-1 \\ 4-2 & 3-2 \end{pmatrix}, \begin{pmatrix} 1-0 & 3-0 \\ 0-2 & 4-0 \end{pmatrix}, \dots \right\}$$

$$12 := \left\{ \begin{pmatrix} 1-0 & 2-4 \\ 1-3 & 4-0 \end{pmatrix}, \begin{pmatrix} 3-4 & 1-4 \\ 1-4 & 2-1 \end{pmatrix}, \begin{pmatrix} 2-1 & 1-3 \\ 4-2 & 3-4 \end{pmatrix}, \begin{pmatrix} 4-0 & 0-3 \\ 2-0 & 1-0 \end{pmatrix}, \begin{pmatrix} 3-2 & 0-2 \\ 3-0 & 2-3 \end{pmatrix}, \dots \right\}$$

Proof. This is a straightforward calculation. □

Remark. Using the same projection as for the octahedron gives that the cosets $1, \dots, 12$ may be identified with the following points of the extended complex plane

1	2	3	4
∞	0	$\frac{1}{2}(-1-\sqrt{5}) \approx -1.6$	$\epsilon^3+\epsilon^4 \approx -0.5-1.5i$

5	6	7	8
$1+\epsilon^4 \approx 1.3-.095i$	$1+\epsilon \approx 1.3+0.95i$	$\epsilon^2+\epsilon^3 \approx -0.5+1.5i$	$\frac{1}{2}(-1+\sqrt{5}) \approx 0.6$

9	10	11	12
$\frac{1}{1+\epsilon+\epsilon^2} \approx 0.2-0.6i$	$-1+\frac{1}{1+\epsilon} \approx -0.5-0.36i$	$\frac{-1}{1+\epsilon} \approx -0.5+0.36i$	$\frac{1}{1+\epsilon^3+\epsilon^4} \approx 0.2+0.6i$

Again, one may then draw a graph between each point and the closest five points in order to obtain the graph of the icosahedron.

Definition. There are four Casimirs made up of elements of order 5 which we label $M, N, -M$ and $-N$. Recalling the notation used in the description of the cosets, the Casimir M is given by the sum of each element in the conjugacy class of (12345), namely:

$$M := \begin{pmatrix} -\epsilon^3 & 0 \\ 0 & -\epsilon^2 \end{pmatrix} + \begin{pmatrix} -\epsilon^2 & 0 \\ 0 & -\epsilon^3 \end{pmatrix} + \begin{pmatrix} 0-2 & 1-2 \\ 3-4 & 0-3 \end{pmatrix} + \begin{pmatrix} 0-3 & 4-3 \\ 2-1 & 0-2 \end{pmatrix} +$$
$$\begin{pmatrix} 0-2 & 2-3 \\ 2-3 & 0-3 \end{pmatrix} + \begin{pmatrix} 0-3 & 0-4 \\ 1-0 & 0-2 \end{pmatrix} \begin{pmatrix} 0-2 & 3-4 \\ 1-2 & 0-3 \end{pmatrix} + \begin{pmatrix} 0-3 & 1-0 \\ 0-4 & 0-2 \end{pmatrix} +$$
$$\begin{pmatrix} 0-2 & 4-0 \\ 0-1 & 0-3 \end{pmatrix} + \begin{pmatrix} 0-3 & 2-1 \\ 4-3 & 0-2 \end{pmatrix} + \begin{pmatrix} 0-2 & 0-1 \\ 4-0 & 0-3 \end{pmatrix} + \begin{pmatrix} 0-3 & 3-2 \\ 3-2 & 0-2 \end{pmatrix}$$

and the Casimir N is given by the sum of each element in the conjugacy class of (12354), namely:

$$N := \begin{pmatrix} -\epsilon^4 & 0 \\ 0 & -\epsilon \end{pmatrix} + \begin{pmatrix} -\epsilon & 0 \\ 0 & -\epsilon^4 \end{pmatrix} + \begin{pmatrix} 1-0 & 1-3 \\ 2-4 & 4-0 \end{pmatrix} + \begin{pmatrix} 4-0 & 4-2 \\ 3-1 & 1-0 \end{pmatrix} +$$

$$\begin{pmatrix} 1-0 & 0-2 \\ 3-0 & 4-0 \end{pmatrix} + \begin{pmatrix} 4-0 & 3-1 \\ 4-2 & 1-0 \end{pmatrix} \begin{pmatrix} 4-0 & 2-0 \\ 0-3 & 1-0 \end{pmatrix} + \begin{pmatrix} 1-0 & 4-1 \\ 4-1 & 4-0 \end{pmatrix} +$$

$$\begin{pmatrix} 1-0 & 3-0 \\ 0-2 & 4-0 \end{pmatrix} + \begin{pmatrix} 4-0 & 1-4 \\ 1-4 & 1-0 \end{pmatrix} + \begin{pmatrix} 1-0 & 2-4 \\ 1-3 & 4-0 \end{pmatrix} + \begin{pmatrix} 4-0 & 0-3 \\ 2-0 & 1-0 \end{pmatrix}.$$

Label the summands as $M_1, \ldots, M_{12}$ and $N_1, \ldots, N_{12}$ respectively. The Casimirs $-M$ and $-N$ are given exactly by $-1 \times M$ and $-1 \times N$.

Now we move to describe the action of these elements on the coset representatives described in Lemma 4.3.15.

Lemma 4.3.16 *The action of the Casimir M on $I^*/\mathbb{Z}_{10}^*$ may be represented graphically as in Figure 4.6. We see that according to the prescription given in the previous section we generate the icosahedral graph using the Casimir M.*

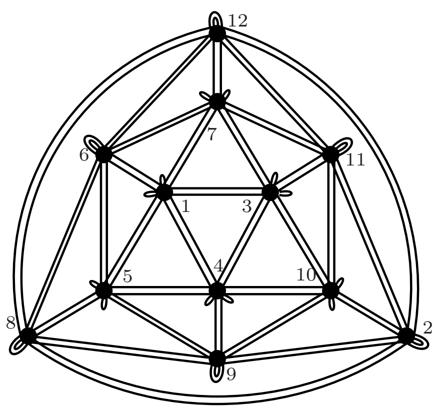

Figure 4.6: The graph generated by the action of M acting on the space of left-cosets that represent the icosahedron.

Proof. The action of the M_i on the coset representatives is given in Table 4.7. □

Lemma 4.3.17 *The action of the Casimir N on $I^*/\mathbb{Z}_{10}^*$ may be represented graphically as in Figure 4.7. We see that according to the prescription given in the previous section we generate the icosahedral graph using the Casimir N.*

Proof. The action of the N_i on the coset representatives is given in Table 4.8. □

M	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}	M_{12}
1	1,10	1,2	3,3	3,4	4,7	4,8	5,1	5,2	6,6	6,5	7,9	7,10
2	2,2	2,10	8,9	8,8	12,5	12,4	11,1	11,10	10,6	10,7	9,3	9,2
3	7,6	4,6	11,8	10,4	3,10	10,3	1,8	4,5	1,9	7,7	11,9	3,2
4	3,6	5,6	10,9	4,2	10,8	9,4	4,10	9,3	5,5	1,4	3,7	1,5
5	4,6	6,6	4,7	1,1	9,9	5,2	9,8	8,4	8,3	5,10	1,10	6,5
6	5,6	7,6	1,6	7,5	5,7	1,7	8,9	6,2	12,4	8,8	6,10	12,3
7	6,6	3,6	7,10	11,3	1,2	3,5	6,7	1,3	7,2	12,9	12,8	11,4
8	9,6	12,6	5,9	6,3	8,2	6,4	2,4	12,7	2,9	9,5	5,8	8,10
9	10,6	8,6	9,2	5,4	2,10	8,7	10,5	2,9	9,10	4,8	4,9	5,3
10	11,6	9,6	2,6	9,7	11,5	2,5	3,8	10,10	9,2	3,9	10,2	4,4
11	12,6	10,1	12,5	2,1	7,8	11,10	7,9	3,3	3,9	11,2	2,2	10,7
12	8,6	11,1	6,8	12,10	6,9	7,3	12,2	7,4	11,7	2,8	8,5	2,7

Table 4.7: Table showing where each summand of the Casimir M for I^* maps the representatives of each left-coset of the icosahedron.

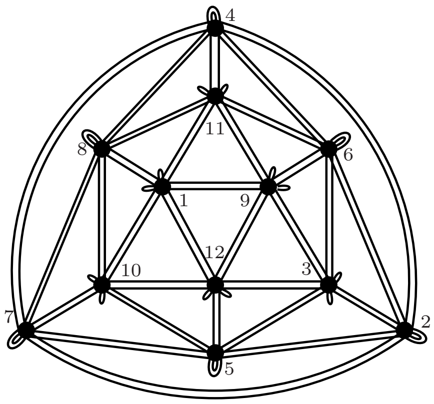

Figure 4.7: The graph generated by the action of N acting on the space of left-cosets that represent the icosahedron.

Remark. Note that the Casimirs $-M$ and $-N$ are given by exactly $-1 \times M$ and $-1 \times N$: therefore their action on the coset representatives is given by adding 5 onto each of the second numbers in each entry of the table and taking the result $\mod 10$. Therefore $-M$ generates the same graph as M and $-N$ as N.

The irreducible representations of $\mathbb{Z}_5^* \cong \mathbb{Z}_{10} \cong \langle \xi \rangle$, where $\xi = \exp[2\pi i/10]$ are all characters and are indexed by an integer k between 0 and 9 — specifically $\xi \mapsto \xi^k$. The character table of I^* is given in Appendix A.1.

In the following, let $\xi = \exp[2\pi i/10]$. The Casimir M represented in the representation $\mathrm{ind}_{\mathbb{Z}_5^*}^{I^*}(k)$ is denoted by M^k and the Casimir N represented in the representation $\mathrm{ind}_{\mathbb{Z}_5^*}^{I^*}(k)$ is denoted by N^k. The following notation is introduced: the matrix entry $ak + bk$ means that that entry is $\xi^{ak} + \xi^{bk}$ — with $0k + ak$ corresponding to $1 + \xi^{ak}$. However, if 0 appears on its own then the corresponding entry is 0 of the matrix.

N	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_10	N_11	N_12
1	1,8	1,4	8,7	8,10	9,6	9,3	10,9	10,2	11,8	11,5	12,1	12,4
2	2,4	2,8	3,10	3,7	7,1	7,4	6,8	6,5	5,9	5,2	4,5	4,3
3	5,6	6,6	12,10	9,2	2,3	5,3	3,4	9,5	3,8	12,7	2,5	6,9
4	6,6	7,6	6,3	2,6	8,5	4,4	11,7	4,8	6,5	2,9	11,10	8,2
5	7,6	3,6	5,4	12,5	5,8	10,7	2,8	3,9	12,2	10,10	7,3	2,10
6	3,6	4,6	9,7	6,8	4,9	2,7	9,10	11,2	2,4	3,3	6,4	11,5
7	4,6	5,6	2,1	5,9	10,2	8,10	4,3	2,8	10,5	7,4	8,7	7,8
8	11,6	10,6	4,7	7,5	1,5	11,9	8,8	7,2	8,4	4,10	1,2	10,3
9	12,6	11,6	1,6	11,3	6,5	3,7	12,9	1,9	6,2	9,8	3,10	9,4
10	8,6	12,6	7,10	10,4	12,3	1,10	7,7	5,5	1,3	8,9	10,8	5,2
11	9,6	8,6	11,8	4,2	11,4	6,10	1,4	8,3	4,5	6,7	9,9	1,7
12	10,6	9,6	10,9	1,1	3,2	12,8	5,10	12,4	9,3	1,8	5,7	3,5

Table 4.8: Table showing where each summand of the Casimir N for I^* maps the representatives of each left-coset of the icosahedron.

This notation is necessary to help make the matrix less cumbersome, by crystallising the important data that it contains. With this in mind, the matrix of the Casimir M acting in the representation $\mathrm{ind}_{\mathbb{Z}_5^*}^{I^*}(k)$ is given by

$$M^k = \begin{pmatrix} k+9k & 0 & 2k+3k & 6k+7k & 4k+5k & 8k+9k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k+9k & 0 & 0 & 0 & 0 & 0 & 7k+8k & k+2k & 5k+6k & 0k+9k & 3k+4k \\ 7k+8k & 0 & k+9k & 4k+5k & 0 & 0 & 5k+6k & 0 & 0 & 2k+3k & 7k+8k & 0 \\ 3k+4k & 0 & 5k+6k & k+9k & 4k+5k & 0 & 0 & 0 & 2k_+3k & 7k+8k & 0 & 0 \\ 0k+9k & 0 & 0 & 5k+6k & k+9k & 4k+5k & 0 & 2k+3k & 7k+8k & 0 & 0 & 0 \\ 5k+6k & 0 & 0 & 0 & 5k+6k & k+9k & 4k+5k & 7k+8k & 0 & 0 & 0 & 2k+3k \\ k+2k & 0 & 4k+5k & 0 & 0 & 5k+6k & k+9k & 0 & 0 & 0 & 2k+3k & 7k+8k \\ 0 & 2k+3k & 0 & 0 & 7k+8k & 2k+3k & 0 & k+9k & 4k+5k & 0 & 0 & 5k+6k \\ 0 & 6k+8k & 0 & 7k+8k & 2k+3k & 0 & 0 & 5k+6k & k+9k & 4k+5k & 0 & 0 \\ 0 & 4k+5k & 7k+8k & 2k+3k & 0 & 0 & 0 & 0 & 5k+6k & k+9k & 4k+5k & 0 \\ 0 & 0k+k & 2k+3k & 0 & 0 & 0 & 7k+8k & 0 & 0 & 5k+6k & k+9k & 4k+5k \\ 0 & 6k+7k & 0 & 0 & 0 & 7k+8k & 2k+3k & 4k+5k & 0 & 0 & 5k+6k & k+9k \end{pmatrix}.$$

and the matrix of the Casimir N acting in the representation $\mathrm{ind}_{\mathbb{Z}_5^*}^{I^*}(k)$ is given by

$$N^k = \begin{pmatrix} 3k+7k & 0 & 0 & 0 & 0 & 0 & 0 & 6k+9k & 2k+5k & k+8k & 4k+7k & 0k+3k \\ 0 & 3k+7k & 6k+9k & 2k+5k & k+8k & 4k+7k & 0k+3k & 0 & 0 & 0 & 0 & 0 \\ 0 & k+4k & 3k+7k & 0 & 2k+5k & 5k+8k & 0 & 0 & k+4k & 0 & 0 & 6k+9k \\ 0 & 5k+8k & 0 & 3k+7k & 0 & 2k+5k & 5k+8k & k+4k & 0 & 0 & 6k+9k & 0 \\ 0 & 2k+9k & 5k+8k & 0 & 3k+7k & 0 & 2k+5k & 0 & 0 & 6k+9k & 0 & k+4k \\ 0 & 3k+6k & 2k+5k & 5k+8k & 0 & 3k+7k & 0 & 0 & 6k+9k & 0 & k+4k & 0 \\ 0 & 0k+7k & 0 & 2k+5k & 5k+8k & 0 & 3k+7k & 6k+9k & 0 & k+4k & 0 & 0 \\ k+4k & 0 & 0 & 6k+9k & 0 & 0 & k+4k & 3k+7k & 0 & 2k+5k & 5k+8k & 0 \\ 5k+8k & 0 & 6k+9k & 0 & 0 & k+4k & 0 & 0 & 3k+7k & 0 & 2k+5k & 5k+8k \\ 2k+9k & 0 & 0 & 0 & k+4k & 0 & 6k+9k & 5k+8k & 0 & 3k+7k & 0 & 2k+5k \\ 3k+6k & 0 & 0 & k+4k & 0 & 6k+9k & 0 & 2k+5k & 5k+8k & 0 & 3k+7k & 0 \\ 0k+7k & 0 & k+4k & 0 & 6k+9k & 0 & 0 & 0 & 2k+5k & 5k+8k & 0 & 3k+7k \end{pmatrix}$$

It might appear as though these matrices are not of the right form, i.e. having entries that do not belong to $U(1)$. Firstly, notice that for each k one may subtract off a constant multiple of the identity matrix to leave a matrix with no non-zero diagonal elements. Having done this, notice that for each fixed k each entry of one of the matrices has the same magnitude. (This can be seen by noticing that each entry is the sum of two tenth roots of unity and, whilst these may change, the distance between them does not, i.e. one may be obtained from the other by multiplying by ξ a fixed number of times.) One can therefore divide by this magnitude and we are left with each non-zero element being of the form $\tau^l = \exp[2l\pi i/20]$, for some $l = 0, 1, \ldots, 19$.

For example, consider $k = 1$. Each element on the diagonal of matrix N^1 is equal to $\xi^3 + \xi^7$. Subtracting this leaves us with a matrix where each non-zero entry is equal to $\sqrt{2 + 2\cos\left(\frac{3\pi}{5}\right)}$ times a twentieth root of unity. Dividing by this magnitude, gives the matrix

$$A_N^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau^{15} & \tau^7 & \tau^{19} & \tau^{11} & \tau^3 \\ 0 & 0 & \tau^{15} & \tau^7 & \tau^{19} & \tau^{11} & \tau^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau^5 & 0 & 0 & \tau^7 & \tau^{13} & 0 & 0 & \tau^5 & 0 & 0 & \tau^{15} \\ 0 & \tau^{13} & 0 & 0 & 0 & \tau^7 & \tau^{13} & \tau^5 & 0 & 0 & \tau^{15} & 0 \\ 0 & \tau & \tau^{13} & 0 & 0 & 0 & \tau^7 & 0 & 0 & \tau^{15} & 0 & \tau^5 \\ 0 & \tau^9 & \tau^7 & \tau^{13} & 0 & 0 & 0 & 0 & \tau^{15} & 0 & \tau^5 & 0 \\ 0 & \tau^{17} & 0 & \tau^7 & \tau^{13} & 0 & 0 & \tau^{15} & 0 & \tau^5 & 0 & 0 \\ \tau^5 & 0 & 0 & \tau^{15} & 0 & 0 & \tau^5 & 0 & 0 & \tau^7 & \tau^{13} & 0 \\ \tau^{13} & 0 & \tau^{15} & 0 & 0 & \tau^5 & 0 & 0 & 0 & 0 & \tau^7 & \tau^{13} \\ \tau & 0 & 0 & 0 & \tau^5 & 0 & \tau^{15} & \tau^{13} & 0 & 0 & 0 & \tau^7 \\ \tau^9 & 0 & 0 & \tau^5 & 0 & \tau^{15} & 0 & \tau^7 & \tau^{13} & 0 & 0 & 0 \\ \tau^{17} & 0 & \tau^5 & 0 & \tau^{15} & 0 & 0 & 0 & \tau^7 & \tau^{13} & 0 & 0 \end{pmatrix}$$

Computing the flux through each face leads to the answer $2\pi/20$, which gives that the total flux is 2π and so the Chern number (as defined in equation (4.5)) is 1. Similarly, one can compute the flux through each face in the representations $\mathrm{ind}_{\mathbb{Z}_5^*}^{I^*}(l)$, where $2, 3, 4, 6, 7, 8, 9, 0$. This leads to the Chern numbers $2, 3, 4, 16, 17, 18, 19, 0$. If $l = 5$ then the matrix M^5 degnerates to $-2I$ and it is not possible to draw a graph from this matrix.

One could also consider the Casimir $-M$ in each representation, which is obtained by multiplying the matrix M^k by $-I$. This has the effect of multiplying every entry by $\xi^5 = -1$ and so the corresponding τ's are multiplied by τ^{10}. Since each face is a triangle, this has the effect of changing the flux through each face by the addition of π (when considered $\mod 2\pi$). This means that in the representations $\mathrm{ind}_{\mathbb{Z}_5^*}^{I^*}(l)$ for $l = 1, 2, 3, 4, 6, 7, 8, 9, 0$ the corresponding Chern numbers are $11, 12, 13, 14, 6, 7, 8, 9, 0$.

Again there is no way to make sense of $l = 5$.

This method produces natural magnetic potentials on the icosahedral graph for all charges between 0 and 19 except 5 and 15. The spectrum of these graphs may be read off from the decomposition of the induced representations and the Casimir table for I^*.

However, it is worth remarking that one can generate all 20 magnetic charges by raising each element inside the matrix A^1_M to the power l, where $l = 0, 1, \ldots, 19$. Because this is a peculiar operation, it is worth reinforcing that this is emphatically different from considering the matrix M in the representation $\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(l)$. It is also different from considering the matrix M multiplied by itself l times.

Lemma 4.3.18 *The representations of I^* induced from the characters of $\mathbb{Z}^*_5$ decompose into irreducible representations of I^* as*

$$\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(0) \cong U \oplus Y \oplus W \oplus Z,$$

$$\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(1) \cong S \oplus V \oplus X,$$

$$\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(2) \cong Y \oplus W \oplus V',$$

$$\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(3) \cong V \oplus X \oplus S',$$

$$\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(4) \cong W \oplus V' \oplus Z,$$

$$\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(5) \cong X \oplus X,$$

$$\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(6) \cong W \oplus V' \oplus Z,$$

$$\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(7) \cong V \oplus X \oplus S'$$

$$\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(8) \cong Y \oplus W \oplus V',$$

$$\mathrm{ind}^{I^*}_{\mathbb{Z}^*_5}(9) \cong S \oplus V \oplus X.$$

Proof. This is a direct calculation done by using Frobenius reciprocity in conjunction with the scalar product of characters. □

Theorem 4.3.19 *The adjacency operators T that correspond to distinct magnetic fields for the icosahedron are listed in Tables 4.9 and 4.10. The spectrum is obtained by shifting the given spectrum and scaling it by the constants* $\alpha = \sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}$, $\beta = \sqrt{\frac{1}{2}\left(3+\sqrt{5}\right)}$, $\gamma = \sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}$, $\delta = \sqrt{\frac{1}{2}\left(3-\sqrt{5}\right)}$, $\lambda = \frac{1}{2}(1+\sqrt{5})$ *and* $\mu = \frac{1}{2}(-1+\sqrt{5})$.

The corresponding magnetic Laplacian is formed by $\Delta = 5 - T$.

The Chern numbers and spectrum are given in the adjacent columns.

Operator	Chern number	Spectrum
A_M^0	0	$5, \sqrt{5}^3, -\sqrt{5}^3, -1^5$
$\alpha A_M^1 + \lambda$	1	$[2(1+\sqrt{5})]^2$,$[-3]^4$,$[-2]^6$
$\beta A_M^2 + \mu$	2	$[2(1+\sqrt{5})]^3$,$[0]^5$,$[3]^4$
$\gamma A_M^3 - \mu$	3	$[-3]^4$,$[-2]^6$,$[2(1-\sqrt{5}]^2$
$\delta A_M^4 - \lambda$	4	$[0]^5$,$[3]^4$,$[2(1-\sqrt{5})]^3$
$\delta A_M^6 - \lambda$	16	$[0]^5$,$[3]^4$,$[2(1-\sqrt{5})]^3$
$\gamma A_M^7 - \mu$	17	$[-3]^4$,$[-2]^6$,$[2(1-\sqrt{5}]^2$
$\beta A_M^8 + \mu$	18	$[2(1+\sqrt{5})]^3$,$[0]^5$,$[3]^4$
$\alpha A_M^9 + \lambda$	19	$[2(1+\sqrt{5})]^2$,$[-3]^4$,$[-2]^6$
$-A_M^0$	10	$-5, \sqrt{5}^3, -\sqrt{5}^3, 1^5$
$-\alpha A_M^1 - \lambda$	11	$[-2(1+\sqrt{5})]^2$,$[3]^4$,$[2]^6$
$-\beta A_M^2 - \mu$	12	$[-2(1+\sqrt{5})]^3$,$[0]^5$,$[-3]^4$
$-\gamma A_M^3 + \mu$	13	$[3]^4$,$[2]^6$,$[-2(1-\sqrt{5}]^2$
$-\delta A_M^4 + \lambda$	14	$[0]^5$,$[-3]^4$,$[-2(1-\sqrt{5})]^3$
$-\delta A_M^6 + \lambda$	6	$[0]^5$,$[-3]^4$,$[-2(1-\sqrt{5})]^3$
$-\gamma A_M^7 + \mu$	7	$[3]^4$,$[2]^6$,$[-2(1-\sqrt{5}]^2$
$-\beta A_M^8 - \mu$	8	$[-2(1+\sqrt{5})]^3$,$[0]^5$,$[-3]^4$
$-\alpha A_M^9 - \lambda$	9	$[-2(1+\sqrt{5})]^2$,$[3]^4$,$[2]^6$

Table 4.9: Chern numbers and spectrum for the operators M and $-M$

Operator	Chern number	Spectrum
A_N^0	0	$5, \sqrt{5}^3, -\sqrt{5}^3, -1^5$
$\gamma A_N^1 - \mu$	3	$[-3]^4$,$[-2]^6$,$[2(1-\sqrt{5}]^2$
$\delta A_N^2 - \lambda$	16	$[0]^5$,$[3]^4$,$[2(1-\sqrt{5})]^3$
$\alpha A_N^3 + \lambda$	19	$[2(1+\sqrt{5})]^2$,$[-3]^4$,$[-2]^6$
$\beta A_N^4 + \mu$	2	$[2(1+\sqrt{5})]^3$,$[0]^5$,$[3]^4$
$\beta A_N^6 + \mu$	18	$[2(1+\sqrt{5})]^3$,$[0]^5$,$[3]^4$
$\alpha A_N^7 + \lambda$	1	$[2(1+\sqrt{5})]^2$,$[-3]^4$,$[-2]^6$
$\delta A_N^8 - \lambda$	4	$[0]^5$,$[3]^4$,$[2(1-\sqrt{5})]^3$
$\gamma A_N^9 - \mu$	17	$[-3]^4$,$[-2]^6$,$[2(1-\sqrt{5}]^2$
$-A_N^0$	10	$-5, \sqrt{5}^3, -\sqrt{5}^3, 1^5$
$-\gamma A_N^1 + \mu$	13	$[3]^4$,$[2]^6$,$[-2(1-\sqrt{5}]^2$
$-\delta A_N^2 + \lambda$	6	$[0]^5$,$[-3]^4$,$[-2(1-\sqrt{5})]^3$
$-\alpha A_N^3 - \lambda$	9	$[-2(1+\sqrt{5})]^2$,$[3]^4$,$[2]^6$
$-\beta A_N^4 - \mu$	12	$[-2(1+\sqrt{5})]^3$,$[0]^5$,$[-3]^4$
$-\beta A_N^6 - \mu$	8	$[-2(1+\sqrt{5})]^3$,$[0]^5$,$[-3]^4$
$-\alpha A_N^7 - \lambda$	11	$[-2(1+\sqrt{5})]^2$,$[3]^4$,$[2]^6$
$-\delta A_N^8 + \lambda$	14	$[0]^5$,$[-3]^4$,$[-2(1-\sqrt{5})]^3$
$-\gamma A_N^9 + \mu$	7	$[3]^4$,$[2]^6$,$[-2(1-\sqrt{5}]^2$

Table 4.10: Chern numbers and spectrum for the operators N and $-N$

4.3.5 Dodecahedron

It is curious that the construction given in the previous section does not work for the dodecahedral graph, shown in Figure 4.8.

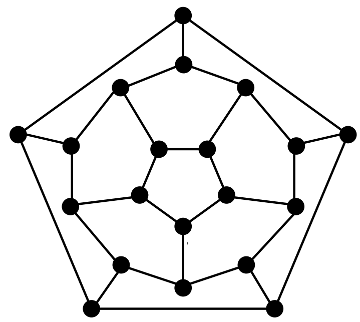

Figure 4.8: The dodecahedral graph.

Theorem 4.3.20 *The dodecahedral graph cannot be constructed using the method described in the previous section.*

Proof. A priori, we know that the spectrum of adjacency matrix for the dodecahedral graph is given by $-\sqrt{5}^3, -2^4, 0^4, 1^5, \sqrt{5}^3, 3^1$, see e.g. [50]. The stabilizer of a vertex of the dodechadron in $SO(3)$ is a cyclic group of order 3, therefore in $SU(2)$ it is a cyclic group H of order 6. Computing the decomposition of $\mathrm{ind}_H^{I^*}(\mathbf{1})$ into irreducible representations of I^* we find that

$$\mathrm{ind}_H^{I^*}(\mathbf{1}) \cong \mathcal{F}(D) \cong U \oplus V \oplus V \oplus W \oplus Y \oplus Z \tag{4.20}$$

as can be easily checked using the Frobenius Reciprocity Theorem 2.3.2.

The crucial point being that the representation V (which has dimension 4) occurs twice and so if there were an element in the centre of the group algebra of A_5 that generated the dodecahedral graph then it would act on V as multiplication by a constant and so one of the eigenvalues of the dodecahedral graph would have multiplicity 8. However, the eigenvalues -2 and 0 each have multiplicity 4, whilst they must both come from a copy of V — as can be seen by relating the multiplicities of each eigenvalue to each representation occuring in the decomposition (4.20).

This observation points to the fact that there is no element in $Z(\mathbb{C}[A_5])$ that generates the dodecahedral graph. The fact that there was a corresponding element for each of the other Platonic solids can perhaps be viewed as a coincidence relating to the lack of complexity of the solid and its symmetry group. Moreover, if there is an element that generates the dodecahedral graph it will not be invariant under the action of A_5. □

Chapter 5

Conclusion

Let us summarise here the main results of the thesis.

In Chapter 2 we considered the geometric quantization of the classical phase space of an electron orbiting a Dirac magnetic monopole at fixed distance. This phase space was identified as a coadjoint orbit of the Euclidean group $E(3) \cong SO(3) \ltimes \mathbb{R}^3$ in [43] and we show that some of the considerations that they make are natural on the quantum level. The main insight is that the space that geometric quantization identifies as the 'Quantum Hilbert Space' may be considered as an induced representation of $SO(3)$, or its universal cover $SU(2)$. The solution of the corresponding spectral problem is essentially a corollary of this fact.

In Chapter Three we looked at a broad class of examples that are generalizations of the case considered by Dirac, namely the classical phase space is given by a 'Magnetic Cotangent Bundle' $(T^*M, dp \wedge dq + \pi^*(\omega))$, where the 'magnetic term' $\pi^*(\omega)$ is the pullback of a closed 2-form on M to T^*M. Examples with a particularly rich geometry were considered in [4], [5] and [14] with M being a coadjoint orbit of a compact Lie group G equipped with the normal metric. We study the geometric quantization of the magnetic geodesic flow on such spaces: the 'quantum Hilbert space' may be identified with an induced representation of G and the quantum Hamiltonian turns out to be the Bochner Laplacian. However, a direct description of the spectrum of the corresponding Schrödinger equation is only possible in general in terms of the Kostant Branching Formula, which is highly non-trivial to compute.

Lastly, in Chapter 4 we consider a discretization of Chapters 2 and 3, namely we address the question of whether if the group G is allowed to be finite, representations of G that are induced from a character of a subgroup H can still be interpreted in terms of magnetic monopoles. In trying to make sense of this question we give a construction of magnetic monopoles on regular graphs, which shares many of the characteristics of monopoles considered in the continuous case. The spectral problem is essentially trivial to solve and this is demonstrated specifically for graphs of the Platonic solids, where in most cases the construction works effectively.

There are several further directions in which it would be interesting to travel.

As is well-known, the story of electromagnetism may be phrased geometrically in terms of connections on $U(1)$-bundles. Physicists are also interested in gauge groups other than $U(1)$ and it would be desirable to try and describe analogues of monopoles for these groups for both coadjoint orbits and graphs.

On the level of graphs there are lots of questions that need addressing. Initially, it would be nice to have interesting examples of Dirac magnetic monopoles on graphs that are not just approximations of S^2, as the Platonic polyhedra are. It would also be satisfying to 'complete the set' and be able to see magnetic monopoles on the dodecahedral graph; however, if possible, this is likely to require a more sophisticated analysis. On a more general level, it is clear that in the construction given we can consider G and H to be discrete groups and not just finite. If H is a cofinite subgroup of G then the construction carries over without issue, but if H does not have finite index in G then there are an infinite number of vertices on the corresponding graph: we would have to use techniques of analysis to make sensible statements. First examples of such graphs that it would be interesting to investigate is for the square and hexagonal lattices on $\mathbb{R}^2$: then the constant flux through each face would make this problem a discrete analogue of the Landau problem on the plane.

In Chapter Four we gave a construction of a graph Γ together with a magnetic field from the data of a group G, a subgroup H and a good Casimir element K. It is intriguing to consider this as a discrete analogue of a coadjoint orbit with a magnetic form and the normal metric, with the orbit being given by the space $X = G/H$, the magnetic field being given by a character of H and the normal metric being given by K. One could consider the general inverse problem, given a graph Γ and a magnetic field together with a transitive action of a group G on the vertices and edges of Γ, that preserves the flux through each cycle, can this be realised by our construction? It is tempting to think of this in terms of a discrete analogue of the theorem that realises homogeneous symplectic G-manifolds as coadjoint orbits of G.

Appendix A

Appendix

A.1 Binary symmetry groups

The purpose of this section is to compute the Character Tables and Casimir Tables of the binary subgroups of $SU(2,\mathbb{C})$. These are the preimages of the finite subgroups of $SO(3,\mathbb{R})$ under the well-known double covering.

The orientation-preserving symmetry group T of the tetrahedron is given abstractly as the alternating group A_4, this can be seen by noting that any of the four vertices of the tetrahedron may be rotated into any other. The character table for A_4 is given in Table A.1 and is taken from [17].

12	1	4	4	3
$T \cong A_4$	1	(123)	(132)	(12)(34)
U	1	1	1	1
U'	1	ω	ω^2	1
U''	1	ω^2	ω	1
V	3	0	0	-1

Table A.1: The character table of A_4, where $\omega = \exp[i\frac{2\pi}{3}]$.

The orientation preserving symmetry group O of the octahedron, or equivalently the cube, can be abstractly identified with the symmetric group S_4. This can be seen by noting that there are four long diagonals on the cube and these may be permuted freely amongst themselves. The character table for S_4 is given in Table A.2.

The orientation-preserving symmetry group I of the icosahedron, or equivalently the dodecahedron, can be abstractly identified with the alternating group A_5. This can be seen by noting that there are five inscribed tetrahedra that can be switched freely by any even permutation. The character table of A_5 is given in Table A.3.

The double covering is most easily explained as follows. The Lie group $SU(2)$

24	1	6	8	6	3
$O \cong S_4$	1	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
W	2	0	-1	0	2

Table A.2: The character table of S_4.

60	1	20	15	12	12
$I \cong A_5$	1	(123)	(12)(34)	(12345)	(21345)
U	1	1	1	1	1
V	4	1	0	-1	-1
W	5	-1	1	0	0
Y	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
Z	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

Table A.3: The character table of A_5.

is given by $\{A \in GL(2,\mathbb{C}) \mid \overline{A^t}A = I, \ \det(A) = 1\}$ and its Lie algebra is given by $\mathfrak{su}_2 = \{X \in \mathfrak{gl}(2,\mathbb{C}) \mid \overline{X^t} + X = 0, \ \operatorname{tr}(X) = 1\}$.

The adjoint action lets $SU(2)$ act on $\mathfrak{su}_2$ by conjugation. The Lie algebra is a three-dimensional real vector space and any element is of the form

$$X = \begin{pmatrix} x & y+iz \\ -y+iz & -x \end{pmatrix}.$$

One can define a Euclidean inner-product by $|X|^2 = -\det(X) = x^2+y^2+z^2$. Therefore, $\mathfrak{su}_2$ is also acted on by $SO(3)$ and so there is a surjective group homomorphism $\pi : SU(2) \to SO(3)$ with kernel $\{\pm I\}$, which is exactly the centre of $SU(2)$.

The binary subgroups G^* of $SU(2)$ are those finite subgroups of $SU(2)$ that are the preimage of a finite subgroup G of $SO(3)$ under π. This might sound like something of a triviality, but in fact the binary symmetry groups can be viewed as more fundamental than the ordinary symmetry groups. This is down to the famous McKay correspondence [40], in which the binary symmetry groups can be used to 'derive' the simple Lie algebras — this has led to deep work in algebraic geometry, see [47] for a survey. (Note that the only finite subgroups of $SU(2)$ that are not the preimage of a $G \subset SO(3)$ are the cyclic subgroups of odd order.)

Before deducing the character tables of the binary groups some remarks about the conjugacy classes of the binary groups are needed. If $C \subset G^*$ is a conjugacy class, then it is clear that $\pi(C) \subset G$ is also a conjugacy class. Moreover, the preimage of a conjugacy class $C \subset G$ is a union of conjugacy classes in G^*: specifically, if C is made up of elements of order 2 then $\pi^{-1}(C)$ is a single conjugacy class and if C is made up of elements of order greater than 2 then $\pi^{-1}(C)$ is the union of two conjugacy classes.

In Chapter 1 of his masterpiece [28] Klein explicitly explains how a rotation $X \in SO(3)$ lifts to two elements of $SU(2)$. Specifically, if X is a rotation by an angle θ about a vector (x, y, z) then

$$\pi^{-1}(X) = \begin{pmatrix} t + iu & v + iw \\ -v + iw & t - iu \end{pmatrix},$$

where $t = \cos(\theta/2)$ or $\cos(\theta/2 + \pi)$ and $(u, v, w) = \sin(\theta/2)(-z, -y, -x)$ or $(u, v, w) = \sin(\theta/2 + \pi)(-z, -y, -x)$

The binary cyclic group C_n^* is the cyclic group of order $2n$ given by

$$C_n^* = \left\{ \begin{pmatrix} \exp[ik\pi/n] & 0 \\ 0 & \exp[-ik\pi/n] \end{pmatrix}, \quad k = 0, 1, \ldots, 2n-1 \right\}$$

The binary dihedral group D_n^* is the group with $4n$ elements given by

$$D_n^* = \left\{ \begin{pmatrix} \exp[ik\pi/n] & 0 \\ 0 & \exp[-ik\pi/n] \end{pmatrix}, \begin{pmatrix} 0 & i\exp[-ik\pi/n] \\ i\exp[ik\pi/n] & 0 \end{pmatrix} \right\},$$

where again, $k = 0, 1, \ldots, 2n-1$.

The binary tetrahedral group T^* is the group of 24 elements given by

$$T^* = \left\{ \begin{pmatrix} i^k & 0 \\ 0 & -i^k \end{pmatrix}, \begin{pmatrix} 0 & -(-i)^k \\ i^k & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} i^k(\pm 1 + i) & -i^k(1 - i) \\ (-i)^k(1 + i) & (-i)^k(\pm 1 - i) \end{pmatrix}, \right.$$
$$\left. \frac{1}{2} \begin{pmatrix} -(-i)^k(1 + i) & -(-i)^k(\pm 1 - i) \\ i^k(\pm 1 + i) & i^k(-1 + i) \end{pmatrix}, \text{ with } k = 0, 1, 2, 3 \right\}$$

The binary octahedral group O^* is the group of 48 elements containing T^* as a subgroup and obtained by extending T^* by multiplying each element of T^* by $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i & 0 \\ 0 & 1 - i \end{pmatrix}$.

The binary icosahedral group is the group of 120 elements of the following form,

$$I^* = \left\{ \begin{pmatrix} \pm\epsilon^{3\mu} & 0 \\ 0 & \pm\epsilon^{2\mu} \end{pmatrix}, \begin{pmatrix} 0 & \mp\epsilon^{2\mu} \\ \pm\epsilon^{3\mu} & 0 \end{pmatrix}, \frac{\pm 1}{\sqrt{5}} \begin{pmatrix} (\epsilon^4 - \epsilon)\epsilon^{3\mu+3\nu} & (\epsilon^2 - \epsilon^3)\epsilon^{2\mu+3\nu} \\ (\epsilon^2 - \epsilon^3)\epsilon^{3\mu+2\nu} & (\epsilon - \epsilon^4)\epsilon^{2\mu+2\nu} \end{pmatrix}, \right.$$
$$\left. \frac{\pm 1}{\sqrt{5}} \begin{pmatrix} (\epsilon^3 - \epsilon^2)\epsilon^{3\mu+2\nu} & (\epsilon^4 - \epsilon)\epsilon^{2\mu+2\nu} \\ (\epsilon^2 - \epsilon^3)\epsilon^{3\mu+3\nu} & (\epsilon - \epsilon^4)\epsilon^{2\mu+3\nu} \end{pmatrix} \quad \mu, \nu = 0, 1, 2, 3, 4 \right\}$$

As an example of how to compute the character tables, the character table of T^* will be derived here. Firstly, the character table of $T \cong S^4$ is extended to include the extra conjugacy classes of the binary group, leading to Table A.4. If a conjugacy class in T lifts to two separate classes in T^*, which are demarcated by a $-$ sign for the one that only exists in T^*. Obviously, acting on representations of T, both conjugacy classes have the same character. This leads to Table A.4.

	1	1	12	8	8	6	6
	I	$-I$	(123)	$-$(123)	(132)	$-$(132)	(12)(34)
U	1	1	1	1	1	1	1
U'	1	1	ω	ω	ω^2	ω^2	1
U''	1	1	ω^2	ω^2	ω	ω	1
V	3	3	0	0	0	0	-1

Table A.4: Character table of $S^4 \cong T$, extended to include the binary classes.

However, the lifting to $SU(2)$ has furnished the group with a new representation — namely the restriction S of the standard representation of $SU(2)$ on $\mathbb{C}^2$ to the binary group. One easily computes the characters of the conjugacy classes on S to be as in Table A.5.

	1	1	12	8	8	6	6
	I	$-I$	(123)	$-$(123)	(132)	$-$(132)	(12)(34)
S	2	-2	-1	1	-1	1	0

Table A.5: Characters of T^* acting on S.

The remaining irreducible representations are given by $S' = S \otimes U'$ and $S'' = S \otimes U''$. This gives the full character table of T^* to be as in Table A.6.

	1	1	4	4	4	4	6
T^*	I	$-I$	(123)	$-$(123)	(132)	$-$(132)	(12)(34)
U	1	1	1	1	1	1	1
U'	1	1	ω	ω	ω^2	ω^2	1
U''	1	1	ω^2	ω^2	ω	ω	1
S	2	-2	-1	1	-1	1	0
S'	2	-2	$-\omega$	ω	$-\omega^2$	ω^2	0
S''	2	-2	$-\omega^2$	ω^2	$-\omega$	ω	0
V	3	3	0	0	0	0	-1

Table A.6: Character table of the binary tetrahedral group.

The character tables of the other binary groups may be constructed similarly, by considering the tensor product of S with existing irreducible representations and decomposing this product into irreducibles.

Using these character tables, we can now compute the Casimir tables for these groups. The Casimir table for T^* is given in Table A.9; for O^* in Table A.10; and for I^* in Table A.11.

O^*	1 I	1 $-I$	12 (12)	8 (123)	8 $-$(123)	6 (1234)	6 $-$(1234)	6 (12)(34)
U	1	1	1	1	1	1	1	1
U'	1	1	-1	1	1	-1	-1	1
V	3	3	1	0	0	-1	-1	-1
V'	3	3	-1	0	0	1	1	-1
W	2	2	0	-1	-1	0	0	2
S	2	-2	0	-1	1	$\sqrt{2}$	$-\sqrt{2}$	0
S'	2	-2	0	-1	1	$-\sqrt{2}$	$\sqrt{2}$	0
X	4	-4	0	1	-1	0	0	0

Table A.7: Character table of the binary octahedral group.

I^*	1 I	1 $-I$	20 (123)	20 -(123)	30 (12)(34)	12 (12345)	12 $-$(12345)	12 (12354)	12 $-$(12354)
U	1	1	1	1	1	1	1	1	1
V	4	4	1	1	0	-1	-1	-1	-1
W	5	5	-1	-1	1	0	0	0	0
Y	3	3	0	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
Z	3	3	0	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
S	2	-2	-1	1	0	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
S'	2	-2	-1	1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$
V'	4	-4	1	-1	0	1	-1	1	-1
X	6	-6	0	0	0	-1	1	-1	1

Table A.8: Character table of the binary icosahedral group.

	T^*	1 I	1 $-I$	4 (123)	4 $-$(123)	4 (132)	4 $-$(132)	6 (12)(34)
1	U	1	1	1	1	1	1	1
1	U'	1	1	4ω	4ω	$4\omega^2$	$4\omega^2$	6
1	U''	1	1	$4\omega^2$	$4\omega^2$	4ω	4ω	6
2	S	1	-1	-2	2	-2	2	0
2	S'	1	-1	-2ω	2ω	$-2\omega^2$	$2\omega^2$	0
2	S''	1	-1	$-2\omega^2$	$2\omega^2$	-2ω	2ω	0
3	V	1	1	0	0	0	0	-2

Table A.9: Casimir table of the binary tetrahedral group.

	O^*	1 I	1 $-I$	12 (12)	8 (123)	8 $-(123)$	6 (1234)	6 $-(1234)$	6 (12)(34)
1	U	1	1	12	8	8	6	6	6
1	U'	1	1	-12	8	8	-6	-6	6
3	V	1	1	4	0	0	-2	-2	-2
3	V'	1	1	-4	0	0	2	2	-2
2	W	1	1	0	-4	-4	0	0	6
2	S	1	-1	0	-4	4	$3\sqrt{2}$	$-3\sqrt{2}$	0
2	S'	1	-1	0	-4	4	$-3\sqrt{2}$	$3\sqrt{2}$	0
4	X	1	-1	0	2	-2	0	0	0

Table A.10: Casimir table of the binary octahedral group.

	I^*	1 I	1 $-I$	20 (123)	20 -(123)	30 (12)(34)	12 (12345)	12 $-(12345)$	12 (12354)	12 $-(12354)$
1	U	1	1	20	20	30	12	12	12	12
4	V	1	1	5	5	0	-3	-3	-3	-3
5	W	1	1	-4	-4	6	0	0	0	0
3	Y	1	1	0	0	-10	$2(1+\sqrt{5})$	$2(1+\sqrt{5})$	$2(1-\sqrt{5})$	$2(1-\sqrt{5})$
3	Z	1	1	0	0	-10	$2(1-\sqrt{5})$	$2(1-\sqrt{5})$	$2(1+\sqrt{5})$	$2(1+\sqrt{5})$
2	S	1	-1	-10	10	0	$2(1+\sqrt{5})$	$-2(1+\sqrt{5})$	$2(1-\sqrt{5})$	$2(-1+\sqrt{5})$
2	S'	1	-1	-10	10	0	$2(1-\sqrt{5})$	$2(-1+\sqrt{5})$	$2(1+\sqrt{5})$	$-2(1+\sqrt{5})$
4	V'	1	-1	5	-5	0	3	-3	3	-3
6	X	1	-1	0	0	0	-2	2	-2	2

Table A.11: Casimir table of the binary icosahedral group.

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