Construction of the Emergent Yang-Mills Theory

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Declaration

I declare that this Thesis is my own, unaided work. Chapters 3 and 4 are my original work. They are based on the following papers that I published:

1. S. de Carvalho, R. de Mello Koch and A. L. Mahu, "Anomalous dimensions from boson lattice models," Phys. Rev. D 97, 126004 (2018) [arXiv:1801.02822]

2. S. de Carvalho, R. de Mello Koch and M. Kim, "Central Charges for the Double Coset," [arXiv:2001.10181].

This thesis is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

(Signature of candidate)

14th day of April 2020 in Johannesburg.

Abstract

In this thesis, we focus on the construction of the emergent Yang-Mills theory that is expected to arise at low energy on the world volume of a giant graviton. Our basic approach is to study the operators in $\mathcal{N} = 4$ super Yang-Mills theory dual to excited giant graviton states. The system we work with consists of giant gravitons and open strings connecting between them. They are described in the language of both restricted Schur polynomials and Gauss graph operators, and we study the action of the one loop dilatation operator at large N, including the leading and first subleading terms, in the $SU(3)$ sector. The construction of these operators and computations with them requires sophisticated methods from group representation theory, as well as basic ideas from quantum field theory. Consequently the thesis begins with a careful review of exactly the background that is required. We will consider operators that are a small deformation of a $\frac{1}{2}$-BPS multi-giant graviton state. Our first novel result proves that the subleading matrix elements of the dilatation operator can be interpreted in terms of bosons hopping on a lattice. In this way, we are able to diagonalize the dilatation operator at subleading order. The description and computations are technically challenging, which motivated us to pursue a symmetry based approach to the problem. The second novel result achieved in this thesis, enables us to decompose the state space of excited gaint graviton branes (the Gauss graph operators) into irreducible representations of the $su(2|2)$ global symmetry. As explained by Beisert, this algebra admits central extensions. We argue that in our non-planar setting the global symmetry again is centrally extended, with the charges naturally describing gauge transformations of the emergent gauge theory. Gauge invariance forces these charges to vanish so that we end up with the physically expected result: the full global $su(2|2)$ symmetry is not centrally extended.

Acknowledgements

The biggest acknowledgement and gratitude to be made is to my supervisor, Professor Robert de Mello Koch. He has been a huge support and mentor to me in various ways; from an academic standpoint to a personal standpoint. I highly appreciate the effort, time and patience he has given towards me and this degree. I cannot thank him enough and I will forever be grateful to him. I would like to thank the National Institute for Theoretical Physics (NITheP), Gauteng, South Africa, for funding the first year of my degree of Doctor of Philosophy and to my supervisor funding the remaining two years through his National Research Foundation (NRF) grant. I would like to thank the University of the Witwatersrand for giving me the opportunity to study for the degree. A big thanks to my family: Toy, Grace and Samantha de Carvalho, for their love and support throughout these 3 years - although they did not understand what I was doing in my degree, their unwavering support will always be appreciated. Lastly, I would like to acknowledge and thank my friends: I would like to thank Ashleigh Hutchinson De Almeida, Matthew Unterslak, Kendall Born and Darrian Long for their support academically and personally, especially when my mental health was not at its best, and I would like to thank Nicholas Yannoulis and his wife, Zandalee Yannoulis, for being great friends outside of my academic life and supporting me socially.

Contents

1 Introduction

The proof of the AdS/CFT correspondence remains an open problem in theoretical high energy physics. The AdS/CFT correspondence states that there exists a duality between conformal field theory (CFT) and quantum gravity defined on an asymptotically anti-de Sitter (AdS) spacetime (which is developed in terms of string theory) [1, 2, 3, 4]. The correspondence is also referred to as the gauge/gravity correspondence or the holographic duality. The duality implies a one-to-one correspondence between the states in the Hilbert space of the CFT and the states in the Hilbert space of the quantum gravity. In addition, the state-operator correspondence of the CFT implies that there is a local operator in the CFT for each state in the string theory. Membranes and their excitations, given by open strings, are states that appear in the Hilbert space of quantum gravity described as a string theory. This thesis will study, in particular, the duality between $\mathcal{N} = 4$ SYM theory and type IIB string theory on $AdS_5 \times S^5$. Our goal in this thesis is to shed new light on the AdS/CFT correspondence by studying the CFT realization of excited membrane states. This is a problem of some complexity since even the large N description requires that we sum non-planar contributions. Concretely we aim to construct the emergent Yang-Mills theory that arises at low energy as the brane world volume theory. The states we concentrate on are giant graviton states and we will initially describe these states in terms of restricted Schur polynomials. This provides an orthogonal basis for the state space. We will then develop a description in terms of Gauss graph operators, which have the advantage that these are operators with a good scaling dimension that continue to provide an orthogonal basis. In the case we are studying, the large N and planar limits are not the same and consequently the well-known large N methods are not useful so that new methods need to be utilised. We will make use of group representation theory which will provides the new methods we need. They will allow us to sum the complete set of Feynman diagrams both in the free theory and at one loop.

$\frac{1}{2}$-BPS giant gravitons in $AdS_5 \times S^5$ are $D3$-branes with a spherical worldvolume, embedded into either the S^5 compact space or into AdS_5 ,the latter being known as the dual giant graviton. A D-brane is a non-perturbative object. It has a tension $T \sim \frac{1}{g_s}$, where g_s is the string coupling, which implies that it is very heavy at weak string coupling. This is why it is interesting to study them in CFT. Giant gravitons are stable due to the fact that they couple to the background RR five form flux and carry a non-zero angular momentum, leading to a Lorentz-type force which causes them to puff out and expand. This Lorentz force exactly cancels the force due to their tension which tries to collapse them. When accelerating a giant graviton it grows and the faster you move it, the bigger it gets. There is a limit on how big a giant graviton can get. This limit arises because the giant graviton cannot expand to a size that is bigger than the space (S^5) it lives in. This cut off on the size of the giant translates into a cut off on the angular momentum of the giant. When giant graviton states are excited, they can be described in terms of open strings which end on the brane.

There are a number of questions we might ask. A D-brane is a membrane with the addition of open strings: can we see these elements in the CFT description? We should be able to construct the state space of these excited membranes and do quantum mechanics in the resulting Hilbert space. We might also ask if we can explore the interactions of these open string excitations. It is the lowest modes of these open strings that should give rise to an emergent gauge theory. Understanding the dynamics of this emergent gauge theory seems to be a nice warm up before tackling the problem of understanding how the higher dimensional spacetime and gravitational interactions emerge from the CFT. Answering these questions will lead to important insights into the AdS/CFT correspondence. Further, by studying this problem in the CFT, we gain novel insights into a non-perturbative problem from the perspective of the dual string theory. These are questions, amongst others, that we consider and attempt to answer in this thesis.

When studying the planar limit of the gauge theory we consider a spin chain description. Each field within the operator is identified with a lattice site and the species of field with a spin state. The spin chain was developed to describe CFT operators dual to closed strings, but it is also useful for open strings. Our focus will be on the description of open strings. The spin chain description of the CFT operators is an effective approach to the planar limit of the super Yang-Mills (SYM) theory since it allows the application of powerful integrability techniques. This description maps each CFT operator to a state of a spin chain and it maps the dilatation operator to the Hamiltonian of the spin chain. The dynamics of the spin chain is naturally described in terms of excitations known as magnons. Each magnon will transform as some $SU(2|2) \times SU(2|2)$ representation. Consequently it will be crucial for this work to understand the fundamentals of the $SU(2|2)$ group.

Using the Gauss graph technology we will study the interaction between magnons. We find that the problem can be mapped into bosons hopping on a lattice. The energy of the boson states sets the anomalous dimension of the corresponding CFT operator. This is developed in detail in Section 3.

In the paper [5], Beisert studies irreducible representations of $SU(2|2)$ in which the magnon of the spin chain transform. In Beisert's description, the operator

$$\sum_{n_i=0}^{J} \mathrm{Tr}(Z^{n_1} Y Z^{n_2} X Z^{n_3} Y \cdots) e^{in_1\phi_1 + in_2\phi_2 + in_3\phi_3 + \cdots},$$

where X, Y are distinct impurities, corresponds to a spin chain with impurities that have definite momenta. In the Gauss graph basis description, one of the main questions we want to answer is: how do we understand the irreducible representation in which the magnons transform? One of our goals is to organize the state space of the excited brane system into representations of $su(2|2)$. This is the subject of Section 4.

This thesis is organised as follows: in Section 2 we review the necessary background knowledge that focuses on the large N, non-planar limits of CFT along with methods used for studying matrix models. The majority of the methods and tools are presented in the language of group theory and representation theory of finite groups. The section also covers the action of the dilatation operator in the $SU(3)$ sector, making use of the displaced corners approximation to simplify the result in the large N limit and then finally ends with a description of the Gauss graph basis. There is a pedagogical subsection that demonstrates on how one can compute an exact result in the restricted Schur polynomial basis for a particular selection of Young diagram labels. Section 3 covers the paper [6] which is my original work with de Mello Koch and Mahu. The section covers the diagonalisation of the dilatation operator at one loop including subleading terms. The diagonalisation is achieved by a novel mapping which replaces the problem of diagonalising the dilatation operator with a system of bosons hopping on a lattice. The giant gravitons define the sites of this lattice and the open strings stretching between distinct giant gravitons define the hopping terms of the Hamiltonian. Section 4 covers the paper [7] which is also my original work with de Mello Koch and Kim. The section explains how the excited brane state space is organized into representations of $su(2|2)$. Finally, in Section 5 we review what was achieved in this thesis and draw some conclusions.

2 Background

This chapter reviews the necessary background knowledge required to comprehend and follow the discussions and results presented in this thesis. Topics that are covered are matrix models, ribbon diagrams, factorisation, representation theory, Schur (and restricted Schur) polynomials, the one loop dilatation operator (including an exact one loop dilatation operator computation), the displaced corners approximation and Gauss graphs.

2.1 Matrix Models

The combinatorics of the matrix valued fields of a non-Abelian gauge theory plays an important role in understanding the holography of these theories. Fortunately, these combinatorics can be developed in the setting of matrix models, obtained by reducing non-Abelian gauge theories to zero dimensions. In this subsection we use matrix models to develop the relevant combinatorial results.

Gaussian integrals are good toy models for the path integral. In the same way, matrix models are good models for QCD (gauge theory). When studying a hermitian matrix, i.e. $M = M^\dagger$, $M_{ij} = M^*_{ji}$, where $i, j = 1, \dots, N$, it should be noted that:

- The diagonal elements are in total N real numbers.
- The off-diagonal elements are complex, *but* elements below the diagonal are complex conjugates (c.c.) of elements above the diagonal.
- The number of elements above the diagonal is equal to $(N^2 - N)/2$, so that there are $N^2 - N$ off-diagonal real numbers.

In conclusion, there are N^2 real numbers in total: real diagonal + real off diagonal = $N + N^2 - N = N^2$. The model is defined as follows: any expectation value is given by evaluating the integral

$$\langle \dots \rangle = \int [dM] e^{-\frac{\omega}{2} \operatorname{Tr} M^2} \dots$$

where M_{ii} is real and $M_{ij} = M^r_{ij} + iM^i_{ij}$ if $i > j$ and $M_{ij} = M^*_{ij}$ if $i < j$. Thus the measure for the integral is

$$\int [dM] \dots = \mathcal{N} \prod_{i=1}^{N} \int_{-\infty}^{\infty} dM_{ii} \prod_{k,l=1 \;\, k>l}^{N} \int_{-\infty}^{\infty} dM^r_{kl} \int_{-\infty}^{\infty} dM^i_{kl} \dots$$

Note that this is N^2 real integrals. $\mathcal{N}$ sets the normalisation of the measure. We choose $\mathcal{N}$ so that $\int [dM] e^{-\frac{\omega}{2} \operatorname{Tr} M^2} = \langle 1 \rangle = 1$.

For general N:

$$\begin{aligned}
&\mathcal{N} \int_{-\infty}^{\infty} dM_{11} e^{-\frac{\omega}{2} M_{11}^2} \dots \int_{-\infty}^{\infty} dM_{NN} e^{-\frac{\omega}{2} M_{NN}^2} \int_{-\infty}^{\infty} dM^i_{N1} e^{-\omega M^{i^2}_{N1}} \int_{-\infty}^{\infty} dM^r_{N1} e^{-\omega M^{r^2}_{N1}} \dots \\
&\dots \int_{-\infty}^{\infty} dM^i_{N(N-1)} e^{-\omega M^{i^2}_{N(N-1)}} \int_{-\infty}^{\infty} dM^r_{N(N-1)} e^{-\omega M^{r^2}_{N(N-1)}} = 1 \\
&\Rightarrow \mathcal{N} \left(\sqrt{\frac{2\pi}{\omega}} \right)^N \left(\sqrt{\frac{\pi}{\omega}} \right)^{N^2 - N} = 1 \\
&\Rightarrow \mathcal{N} = \left(\frac{\omega}{2\pi} \right)^{N/2} \left(\frac{\omega}{\pi} \right)^{(N^2-N)/2} = \left(\sqrt{\frac{\omega}{\pi}} \right)^{N^2} \frac{1}{2^{\frac{N}{2}}}.
\end{aligned}$$

We will study correlators of the form

$$\langle M_{ij} M_{kl} \rangle = \int [dM] e^{-\frac{\omega}{2}\operatorname{Tr} M^2} M_{ij} M_{kl}.$$

Towards this end, consider

$$Z[J] = \int [dM] e^{-\frac{\omega}{2}\operatorname{Tr} M^2 + \operatorname{Tr} JM},$$

where $\operatorname{Tr}(JM) = J_{ij} M_{ji}$ (repeated indices are summed). Correlators are given by taking derivatives of $Z[J]$ with respect to the J's. Indeed, noting that

- $\frac{\partial M_{ij}}{\partial M_{kl}} = \delta_{ik}\delta_{jl}$
- $\frac{d}{dM_{ij}} \operatorname{Tr}(JM) = \frac{d}{dM_{ij}} J_{kl} M_{lk} = J_{kl}\delta_{il}\delta_{jk} = J_{ji}$
- $\frac{d}{dJ_{ij}} e^{\operatorname{Tr}(JM)} = e^{\operatorname{Tr}(JM)} \frac{d}{dJ_{ij}} J_{kl} M_{lk} = e^{\operatorname{Tr}(JM)} \delta_{ki}\delta_{lj} M_{lk} = e^{\operatorname{Tr}(JM)} M_{ji}$
- $\frac{d}{dJ_{ij}} \frac{d}{dJ_{kl}} e^{-\frac{\omega}{2}\operatorname{Tr}(M^2) + \operatorname{Tr}(JM)}|_{J=0} = e^{-\frac{\omega}{2}\operatorname{Tr}(M^2)} \frac{d}{dJ_{ij}} \frac{d}{dJ_{kl}} e^{\operatorname{Tr}(JM)}|_{J=0} = e^{-\frac{\omega}{2}\operatorname{Tr}(M^2)} M_{ji} M_{lk}$,

we easily find

$$\langle M_{ij} M_{kl} \rangle = \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} Z[J]|_{J=0}.$$

This can be generalised to

$$\langle M_{ij} M_{kl} \dots M_{rs} \rangle = \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} \dots \frac{d}{dJ_{sr}} Z[J]|_{J=0}.$$

To solve for $Z[J]$, we need to compute a Gaussian integral which amounts to completing the square. Note that $\frac{\omega}{2}\operatorname{Tr}(M^2) - \operatorname{Tr}(JM) = \frac{\omega}{2}\operatorname{Tr}([M - \frac{J}{\omega}]^2) - \frac{1}{2\omega}\operatorname{Tr}(J^2)$. Thus

$$Z[J] = \int [dM] e^{-\frac{\omega}{2}\operatorname{Tr}([M-\frac{J}{\omega}]^2) + \frac{1}{2\omega}\operatorname{Tr}(J^2)}.$$

Changing the integration variables: $M = M' + \frac{J}{\omega} \Rightarrow dM = dM'$ because J is independent of M. Then

$$\begin{aligned} Z[J] &= \int [dM'] e^{-\frac{\omega}{2}\operatorname{Tr}(M'^2) + \frac{1}{2\omega}\operatorname{Tr}(J^2)} \\ &= e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)} \int [dM] e^{-\frac{\omega}{2}\operatorname{Tr}(M^2)} \\ &= e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)}, \end{aligned}$$

since $\int [dM] e^{-\frac{\omega}{2}\operatorname{Tr}(M^2)} = 1$. Now

$$\frac{d}{dJ_{ij}} \operatorname{Tr}(J^2) = \frac{d}{dJ_{ij}} J_{kl} J_{lk} = \delta_{ik}\delta_{lj} J_{lk} + J_{kl}\delta_{il}\delta_{jk} = J_{ji} + J_{ji} = 2J_{ji}.$$

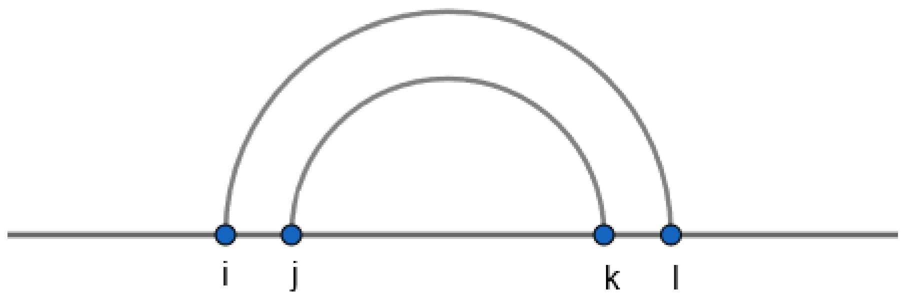
Thus,

$$\begin{aligned}
\langle M_{ij}M_{kl}\rangle &= \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)}|_{J=0} \\
&= \frac{d}{dJ_{ji}}\left[\frac{1}{2\omega}e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)}\frac{d}{dJ_{lk}}\operatorname{Tr}(J^2)\right]_{|J=0} \\
&= \frac{d}{dJ_{ji}}\left[\frac{J_{kl}}{\omega}e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)}\right]_{|J=0} \\
&= \left[\frac{\delta_{jk}\delta_{il}}{\omega}+\frac{J_{kl}}{\omega}\frac{J_{ij}}{\omega}\right]e^{\frac{1}{2\omega}\operatorname{Tr}(J^2)}|_{J=0} \\
&= \frac{1}{\omega}\delta_{jk}\delta_{il}.
\end{aligned}$$

This is the mathematical expression of the propagator in the matrix model.

2.1.1 Ribbon diagrams

The propagator $\langle M_{ij}M_{kl}\rangle = \frac{1}{\omega}\delta_{jk}\delta_{il}$ can be expressed as a diagram:

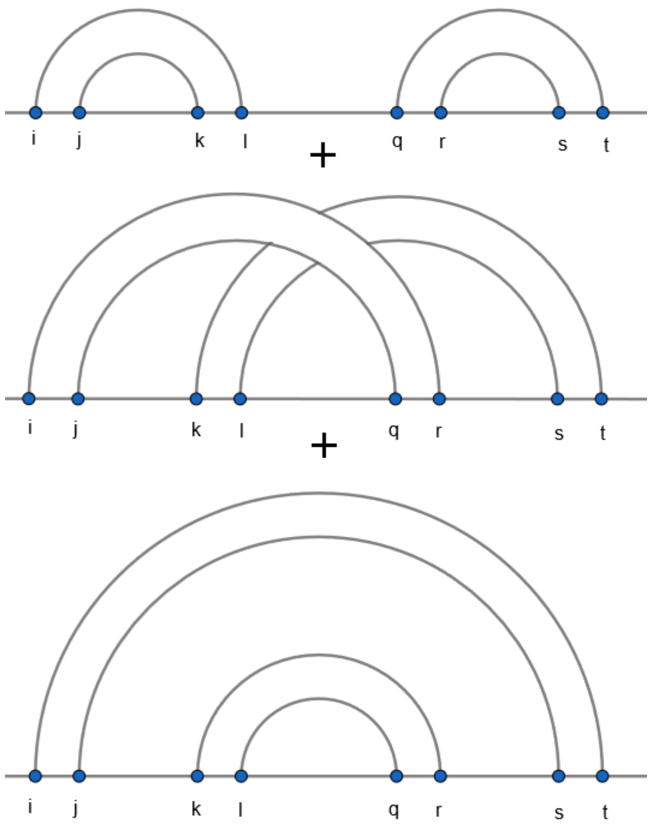


where the indices of each matrix are placed in pairs, in order, on a line. Lines are used to connect the indices of the two matrices in accordance with the indices paired by the Kronecker deltas, i.e. "i" connects to "l" according to the Kronecker delta δ_{il} and "j" connects to "k" according to the Kronecker delta δ_{jk}. Thus an image of a "ribbon" is formed.

The Feynman rules for interpreting ribbon diagrams are as follows:

- Every ribbon comes with a $\frac{1}{\omega}$.
- Every ribbon edge comes with a Kronecker delta.

For example, $\langle M_{ij}M_{kl}M_{qr}M_{st}\rangle$ is represented as the diagrams below:



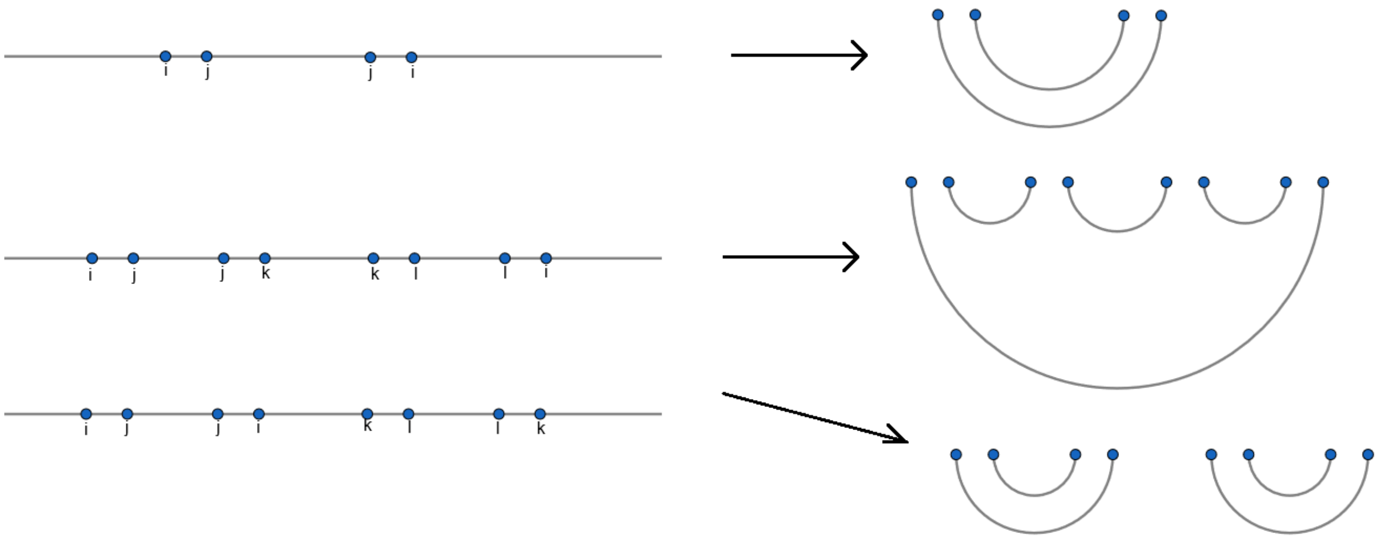
The value of the diagrams (using the Feynman rules) are:

$$\langle M_{ij}M_{kl}M_{qr}M_{st}\rangle = \frac{1}{\omega^2}\delta_{il}\delta_{jk}\delta_{qt}\delta_{rs} + \frac{1}{\omega^2}\delta_{ir}\delta_{jq}\delta_{kt}\delta_{ls} + \frac{1}{\omega^2}\delta_{it}\delta_{js}\delta_{kr}\delta_{lq}.$$

The matrix model is a zero dimensional model. Thus, we can't really sensibly define a local (gauge) symmetry. We simply declare that global $U(N)$ rotations of the hermitian matrix M are a gauge symmetry. In this case, the physical observables are traces of M, since these are the quantities that are gauge invariant.

2.1.2 Improving notation for gauge invariant states

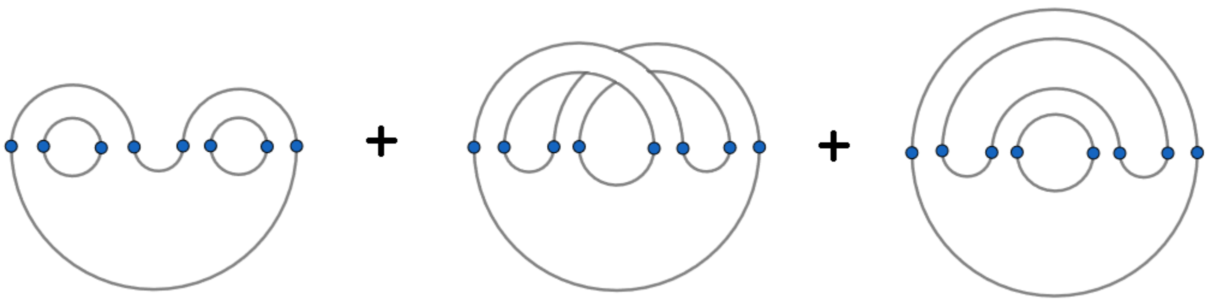
We now simplify our notation. There is no reference to indices in the improved notation. We will connect points that are labelled by the same index since they are summed to give the traces of powers of M_{ij} i.e. there are no free indices. The idea for the diagram corresponding to $\langle M_{ij}M_{ji}\rangle$ is illustrated in the diagram below:



In what follows we will make use of the following correlators

- $\langle \mathrm{Tr}(M^2)\rangle = \frac{1}{\omega}N^2$
- $\langle \mathrm{Tr}(M^4)\rangle = \frac{1}{\omega^2}(2N^3 + N)$
- $\langle \mathrm{Tr}(M^2)\,\mathrm{Tr}(M^2)\rangle = \frac{1}{\omega^2}(N^4 + 2N^2)$
- $\langle \mathrm{Tr}(M^2)\,\mathrm{Tr}(M^4)\rangle = \frac{1}{\omega^3}(2N^5 + 9N^3 + 4N)$
- $\langle \mathrm{Tr}(M^4)\,\mathrm{Tr}(M^4)\rangle = \frac{1}{\omega^4}(4N^6 + 40N^4 + 61N^2)$.

These results were all obtained by drawing ribbon graphs. For example, for $\langle \mathrm{Tr}(M^4)\rangle$ we sum the following three diagrams



Gerard 't Hooft [8] was the first to suggest that it is useful to consider the limit $N \to \infty$. In this limit, the results quoted above can be simplified as follows

- $\langle \mathrm{Tr}(M^2)\rangle = \frac{1}{\omega}N^2$
- $\langle \mathrm{Tr}(M^4)\rangle = \frac{2}{\omega^2}N^3$
- $\langle \mathrm{Tr}(M^2)\,\mathrm{Tr}(M^2)\rangle = \langle \mathrm{Tr}(M^2)\rangle\langle \mathrm{Tr}(M^2)\rangle = \frac{1}{\omega^2}N^4$
- $\langle \mathrm{Tr}(M^2)\,\mathrm{Tr}(M^4)\rangle = \langle \mathrm{Tr}(M^2)\rangle\langle \mathrm{Tr}(M^4)\rangle = \frac{2}{\omega^3}N^5$
- $\langle \mathrm{Tr}(M^4)\,\mathrm{Tr}(M^4)\rangle = \langle \mathrm{Tr}(M^4)\rangle\langle \mathrm{Tr}(M^4)\rangle = \frac{4}{\omega^4}N^6$.

Thus, we learn that

> The expectation value of products is equal to the product of expectation values in the $N \to \infty$ limit.

This is a property of the large N theory and it is called *factorisation*. Although we have only illustrated this in a few examples, the result holds in complete generality.

2.1.3 Factorisation

Our goal now is to understand the physical interpretation and implications of factorisation. Consider a system that can be in a number of different states; these states are labelled with index i. The probability for the system to be in a state i is μ_i. Note that $\sum_i \mu_i = 1,\ \ \mu_i \geq 0\ \forall i$. There is a set of observables O_I and the value of O_I in state i is $O_I(i)$. The expectation value of observables is $\langle O_I \rangle = \sum_i \mu_i O_I(i)$. Factorisation is the statement that

$$\langle O_{I_1} O_{I_2} \dots O_{I_n} \rangle = \langle O_{I_1} \rangle \langle O_{I_2} \rangle \dots \langle O_{I_n} \rangle$$
$$\Rightarrow \sum_i \mu_i O_{I_1}(i) O_{I_2}(i) \dots O_{I_n}(i) = \sum_{i_1} \mu_{i_1} O_{I_1}(i_1) \sum_{i_2} \mu_{i_2} O_{I_2}(i_2) \cdots \sum_{i_n} \mu_{i_n} O_{I_n}(i_n)$$

where we can consider any O's and n can be anything. Interpreted as a system of equations for the probabilities μ_i, the equations are highly over determined. Fortunately there is a solution and it sets $\mu_i = 1$ for $i = i^*$ and $\mu_i = 0$ for $i \neq i^*$. Then both sides of the equation are equal to $O_{I_1}(i^*) O_{I_2}(i^*) \dots O_{I_n}(i^*)$. The system occupies a definite state, so that the sum over states in the path integral reduces to a single term. Only in the classical limit is the system in a definite state, so that this is a classical limit!

This conclusion holds for any theory of matrix fields so that the large N limit of $\mathcal{N} = 4$ SYM theory is equivalent to some classical theory. Maldacena has conjectured that this classical theory is IIB string theory on $AdS_5 \times S^5$, and that $1/N^2$ is equal to $\hbar_{\text{string theory}}$. There is now growing evidence for Maldacena's conjecture.

2.1.4 Interacting theory

The expectation value for the interacting theory is given by

$$\langle \dots \rangle_{\text{int}} = \mathcal{N} \int [dM] e^{-\frac{\omega}{2} \operatorname{Tr}(M^2) - g \operatorname{Tr}(M^4)} \dots$$

The following steps allow us to compute expectation values of traces in the interacting theory:

1. Couple in a source, $e^{\operatorname{Tr}(JM)}$:

   $$Z[J] = \mathcal{N} \int [dM] e^{-\frac{\omega}{2} \operatorname{Tr}(M^2) - g \operatorname{Tr}(M^4) + \operatorname{Tr}(JM)}$$

   with

   $$\langle M_{ji} \rangle = \frac{d}{dJ_{ij}} Z[J]_{|J=0},$$

   or more generally

   $$\langle M_{ij} M_{kl} \dots M_{mn} \rangle = \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} \cdots \frac{d}{dJ_{nm}} Z[J]_{|J=0}.$$

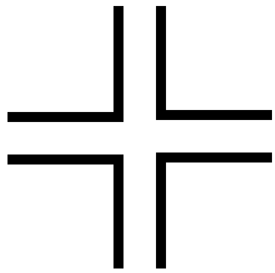
2. Expand the exponential, change M's into $\frac{d}{dJ}$'s and pull them out of the integral:

$$Z[J] = \sum_{q=0}^{\infty} \frac{(-g)^q}{q!} \left(\frac{d}{dJ_{ba}} \frac{d}{dJ_{cb}} \frac{d}{dJ_{dc}} \frac{d}{dJ_{ad}} \right)^q \mathcal{N} \int [dM] e^{-\frac{\omega}{2} \operatorname{Tr}(M^2) + \operatorname{Tr}(JM)}.$$

3. Perform the Gaussian integral by completing the square:

$$Z[J] = \sum_{q=0}^{\infty} \frac{(-g)^q}{q!} \left(\frac{d}{dJ_{ba}} \frac{d}{dJ_{cb}} \frac{d}{dJ_{dc}} \frac{d}{dJ_{ad}} \right)^q e^{\frac{1}{2\omega} \operatorname{Tr}(J^2)}.$$

Since this theory comes with a term that has "$-g \operatorname{Tr}(M^4)$", a new diagram is added with a new rule:



This is a vertex and since M is raised to the power of 4 in the trace, the vertex has 4 legs. Every vertex comes with a "$-g$".

2.1.5 Double scaling limit

Taking the limits $g \to 0$ and $N \to \infty$ such that $\lambda = gN$ is fixed and small [1], we see that

$$\begin{aligned}
\langle \operatorname{Tr}(M^2) \rangle &= (C_0^{(0)} + C_1^{(0)}\lambda + C_2^{(0)}\lambda^2 + \cdots)N^2 + (C_0^{(1)} + C_1^{(1)}\lambda + C_2^{(1)}\lambda^2 + \cdots) + \\
&+ (C_0^{(2)} + C_1^{(2)}\lambda + C_2^{(2)}\lambda^2 + \cdots)\frac{1}{N^2} + \cdots \\
&= N^2 \bigg((C_0^{(0)} + C_1^{(0)}\lambda + C_2^{(0)}\lambda^2 + \cdots) + (C_0^{(1)} + C_1^{(1)}\lambda + C_2^{(1)}\lambda^2 + \cdots)\frac{1}{N^2} + \\
&+ (C_0^{(2)} + C_1^{(2)}\lambda + C_2^{(2)}\lambda^2 + \cdots)\frac{1}{N^4} \bigg) + \cdots .
\end{aligned}$$

Also,

$$\begin{aligned}
\langle \operatorname{Tr}(M^4) \rangle &= (C_0^{(0)} + C_1^{(0)}\lambda + C_2^{(0)}\lambda^2 + \cdots)N^3 + (C_0^{(1)} + C_1^{(1)}\lambda + C_2^{(1)}\lambda^2 + \cdots)N + \\
&+ (C_0^{(2)} + C_1^{(2)}\lambda + C_2^{(2)}\lambda^2 + \cdots)\frac{1}{N} \\
&= N^3 \bigg((C_0^{(0)} + C_1^{(0)}\lambda + C_2^{(0)}\lambda^2 + \cdots) + (C_0^{(1)} + C_1^{(1)}\lambda + C_2^{(1)}\lambda^2 + \cdots)\frac{1}{N^2} + \\
&+ (C_0^{(2)} + C_1^{(2)}\lambda + C_2^{(2)}\lambda^2 + \cdots)\frac{1}{N^4} \bigg)
\end{aligned}$$

This expansion is called the *'t Hooft expansion* [8].

[1] The coupling g is called g_{YM}^2 when you consider Yang-Mills theory.

There are two small numbers, $\frac{1}{N^2}$ (which is $\hbar$ for string theory) and λ (which is $\hbar$ for QFT). λ in this case is for matrix models. Recall that $\hbar$ measures how difficult it is to simultaneously measure x and p. The variable "λ" is related to the string tension in string theory. It is a source of uncertainty and it is a new fundamental constant. These numbers are a hint that you are doing string theory since in string theory we are faced with a new uncertainty principle which is connected with the string's tension. The new uncertainty arises because the string is an extended object and hence is not able to probe position with the same resolution that a point particle can.

Note that factorisation does indeed hold true for the interacting theory, for example:

$$\langle \mathrm{Tr}(M^2)\,\mathrm{Tr}(M^2)\rangle_{\mathrm{int}} = \langle \mathrm{Tr}(M^2)\rangle_{\mathrm{int}}\langle \mathrm{Tr}(M^2)\rangle_{\mathrm{int}}\left(1+O\left(\frac{1}{N^2}\right)\right).$$

As concluded before, from factorisation, a classical limit is achieved in the large N limit. Noting that $\frac{1}{N^2} \to 0$ as $N \to \infty$, we identify $\frac{1}{N^2} \equiv \hbar_{\text{large } N \text{ theory}}$. It will be argued that the large N theory is string theory.

2.1.6 The Genus Expansion

Recall that

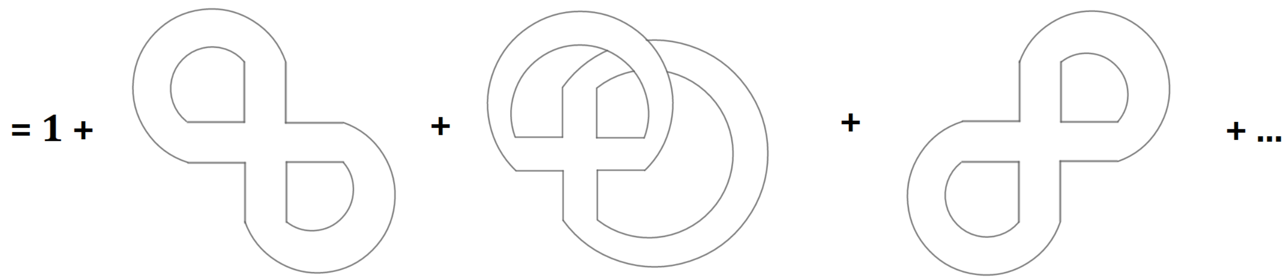
$$\langle \dots \rangle_{\mathrm{int}} = \int [dM] e^{-\frac{\omega}{2}\,\mathrm{Tr}\,(M^2) - g\,\mathrm{Tr}\,(M^4)} \dots .$$

Now, let $M = \sqrt{N} M'$, so that $\frac{\omega}{2}\,\mathrm{Tr}\,(M^2) = \frac{N\omega}{2}\,\mathrm{Tr}\,(M'^2)$ and $g\,\mathrm{Tr}\,(M^4) = gN^2\,\mathrm{Tr}\,(M'^4) = N\lambda\,\mathrm{Tr}\,(M'^4)$. Thus,

$$\begin{aligned} Z[0] = \int [dM] e^{-\frac{\omega}{2}\,\mathrm{Tr}\,(M^2) - g\,\mathrm{Tr}\,(M^4)} &= \int [dM'] e^{-\frac{N\omega}{2}\,\mathrm{Tr}\,(M'^2) - \lambda N\,\mathrm{Tr}\,(M'^4)} \\ &= \int [dM'] e^{-\frac{N\omega}{2}\,\mathrm{Tr}\,(M'^2)} \sum_{n=0}^{\infty} \frac{1}{n!}(-\lambda N\,\mathrm{Tr}\,(M'^4))^n. \end{aligned}$$

Before, we had $\langle M_{ij} M_{kl}\rangle = \frac{\delta_{il}\delta_{jk}}{\omega}$ and for every M vertex we have a "$-g$". Now, we have $\langle M'_{ij} M'_{kl}\rangle = \frac{\delta_{il}\delta_{jk}}{N\omega}$ and for every M' vertex we have a "$-\lambda N$". Consider

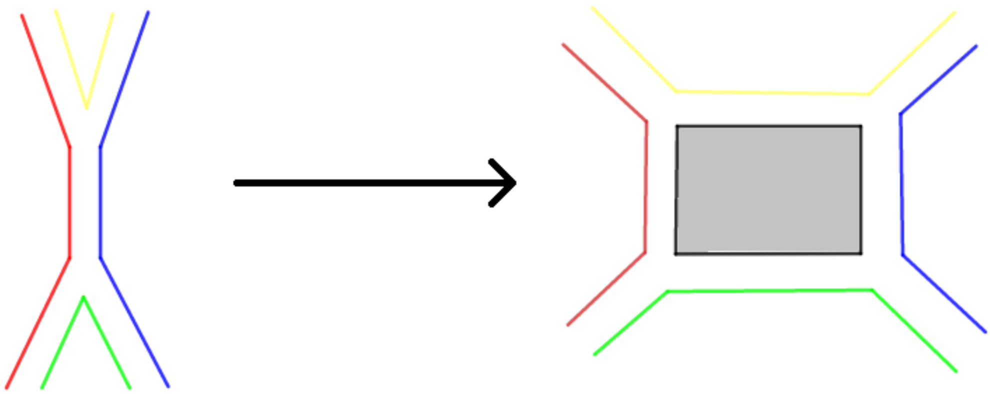
$$Z[0] = \int [dM'] e^{-\frac{N\omega}{2}\,\mathrm{Tr}\,(M'^2)} \left(1 + (-\lambda N)\,\mathrm{Tr}\,(M'^4) + \frac{(-\lambda N)^2}{2}(\mathrm{Tr}\,(M'^4))^2 + \dots\right) =$$



The number of closed loops in the ribbon graph is the number of "faces" in the triangulation. The number of ribbons in the ribbon graph is the number of "edges" in the triangulation that the sheets are joined on. The number of "vertices" in the ribbon graph is the number of vertices in the triangulation. The Feynman diagrams we get from the interacting theory triangulate a surface. Looking at the first diagram, the value of it is $\left(\frac{1}{N\omega}\right)^2(-\lambda N)(N)^3$. Now, the diagram has 2 edges, 1 vertex, and 3 faces, which we observe corresponds to the powers of 2, 1, and 3 that are of the terms $\left(\frac{1}{N\omega}\right)^2$, $(-\lambda N)^1$, and $(N)^3$ respectively. In general, the Feynman diagrams will have the value

$$\left(\frac{1}{N\omega}\right)^E(-\lambda N)^V(N)^F,$$

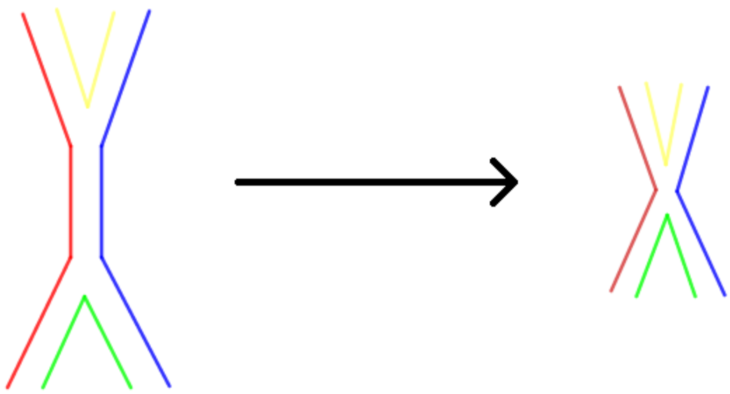
where E, V, and F correspond to the number of edges, vertices, and faces a diagram has respectively. The N dependence of a graph with E edges, F faces, and V vertices is N^{F-E+V}. The number $\chi \equiv F - E + V$ is called the *Euler characteristic* and it is a topological invariant which means that all spaces of the same topology share the same number. For example, modify a triangulation by stretching it such that we create a new face:



After the stretch, we get $F' = F + 1$, $E' = E + 3$, and $V' = V + 2$, and then

$$F' - E' + V' = F + 1 - (E + 3) + V + 2 = F - E + V,$$

which is a concrete demonstration that this quantity is a topological invariant. If we squash the triangulation to get rid of an edge, we find

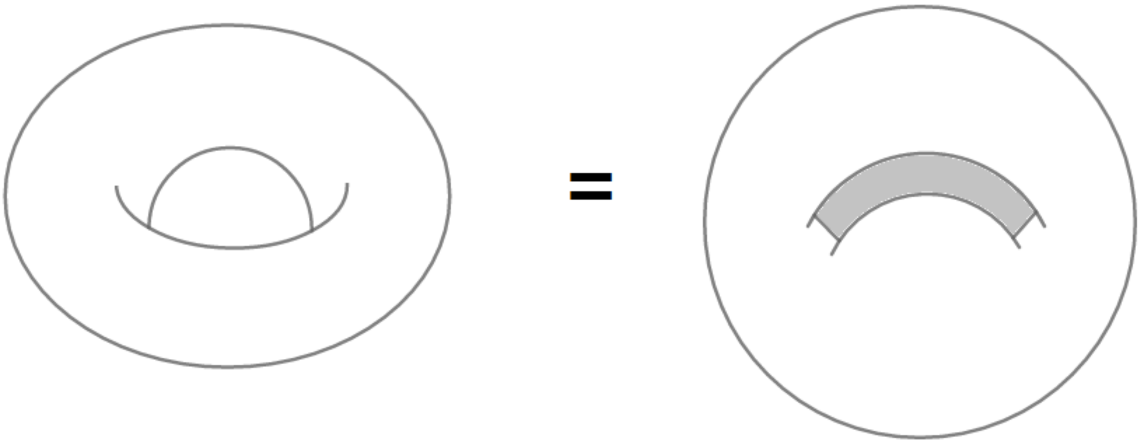


After the squash, we get $F'' = F$, $E'' = E - 1$, and $V'' = V - 1$, and then

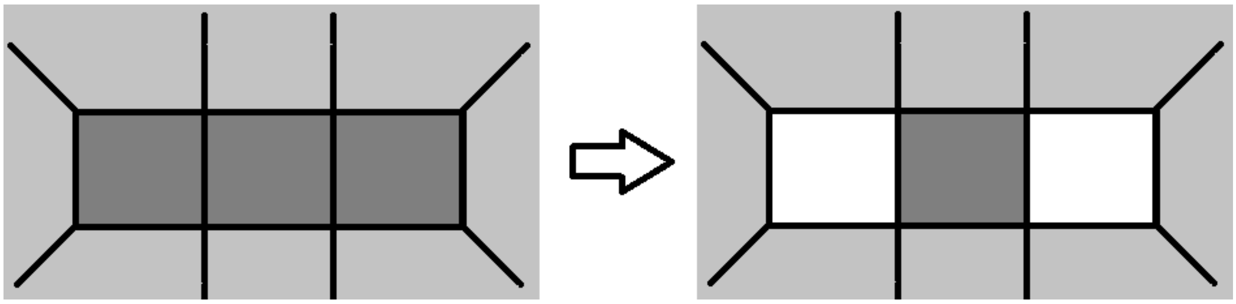
$$F' - E' + V' = F - (E - 1) + (V - 1) = F - E + V,$$

which is also a topological invariant.

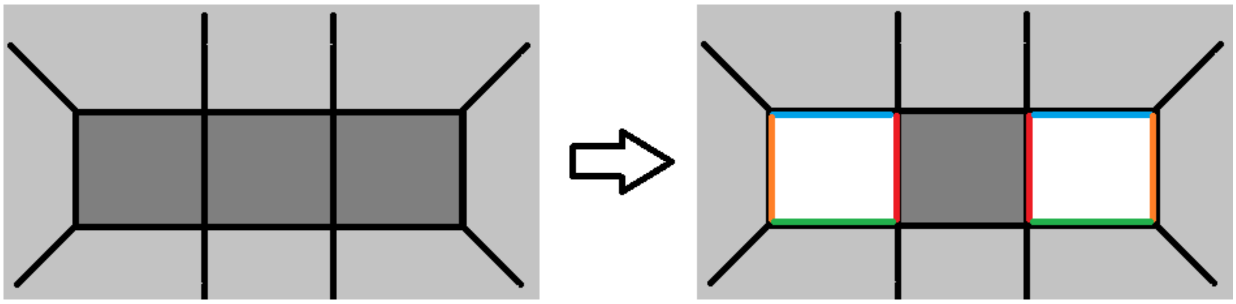
A torus is obtained by modifying a sphere by adding a handle:



For us to add a handle to a sphere, we need to cut 2 holes:



This gives us $F' = F - 2$, $E' = E$, and $V' = V$. Next, add the handle:

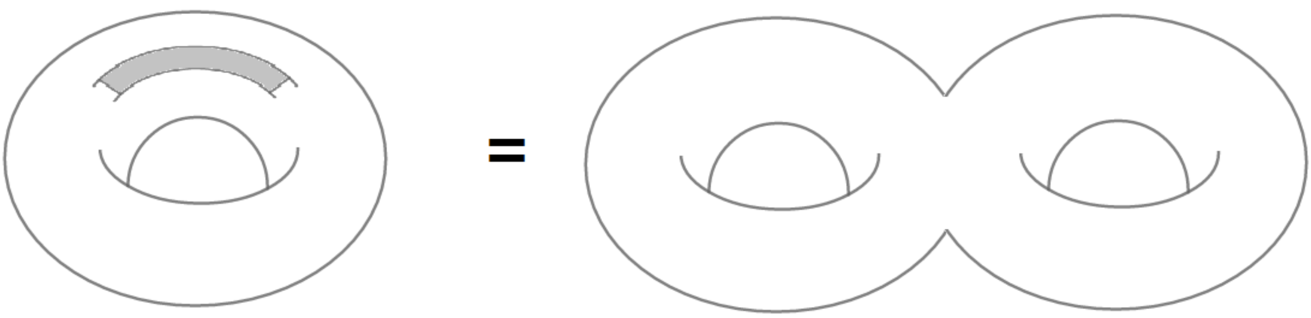


This gives us $F'' = F - 2$, $E'' = E - 4$, and $V'' = V - 4$. Thus,

$$F - E + V \to F'' - E'' + V'' = F - E + V - 2.$$

This tells you that you will always subtract 2 from χ each time you add a handle, just as you do to create a torus from a sphere.

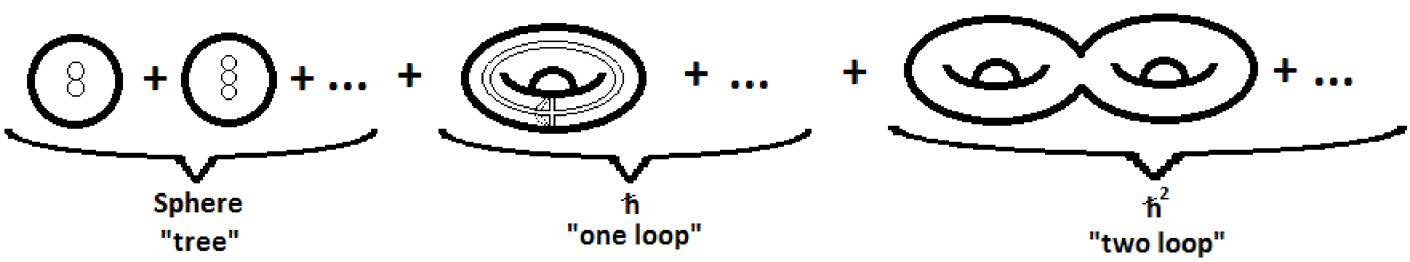
Adding a handle to a torus produces a pretzel:



Note the following:

- $\chi_{\text{sphere}} = F_{\text{sphere}} - E_{\text{sphere}} + V_{\text{sphere}} = 4 - 6 + 4 = 2$
- $\chi_{\text{torus}} = \chi_{\text{sphere}} - 2 = 0$
- $\chi_{\text{pretzel}} = \chi_{\text{torus}} - 2 = -2.$

The value of χ for the surface determines the N dependence of the diagram. Planar diagrams are diagrams you can draw on a plane/sphere; for diagrams that you cannot draw on a plane/sphere we refer to them as non-planar diagrams. Thus, our sum over ribbon graphs becomes a sum over surfaces as illustrated in the figure below.



Our path integral can be reinterpreted as a sum over surfaces. For particles we sum over world lines. For strings we sum over world sheets which is a sum over surfaces. This strongly suggests that the large N limit of a matrix model should be formulated as a string theory.

2.1.7 Complex matrices

Suppose we construct a complex matrix Z using two $N \times N$ Hermitian matrices M_1 and M_2 such that $Z = \frac{1}{\sqrt{2}}(M_1 + iM_2)$ and $Z^\dagger = \frac{1}{\sqrt{2}}(M_1 - iM_2)$. The correlators for this matrix model are defined by

$$\langle \dots \rangle = \int [dZ\ dZ^\dagger] e^{-\omega \operatorname{Tr}(ZZ^\dagger)} \dots = \int [dM_1\ dM_2] e^{-\frac{\omega}{2}\operatorname{Tr}(M_1^2) - \frac{\omega}{2}\operatorname{Tr}(M_2^2)} \dots \ .$$

We can evaluate the following 3 basic correlators:

- $\langle Z_{ij} Z_{kl} \rangle = 0$
- $\langle Z^\dagger_{ij} Z^\dagger_{kl} \rangle = 0$
- $\langle Z_{ij} Z^\dagger_{kl} \rangle = \frac{1}{\omega}\delta_{il}\delta_{jk}.$

Now, computing correlators of traces of Z, where Wick's theorem instructs us to group fields in pairs (indicated using round brackets in the following equation), we see that

$$\begin{aligned}\langle \operatorname{Tr}(Z^2)\operatorname{Tr}(Z^{\dagger^2})\rangle &= \langle (Z_{ij}Z_{ji})(Z^\dagger_{kl}Z^\dagger_{lk})\rangle + \langle (Z_{ij}Z^\dagger_{kl})(Z_{ji}Z^\dagger_{lk})\rangle + \langle (Z_{ji}Z^\dagger_{kl})(Z_{ij}Z^\dagger_{lk})\rangle \\ &= \langle (Z_{ij}Z^\dagger_{kl})(Z_{ji}Z^\dagger_{lk})\rangle + \langle (Z_{ji}Z^\dagger_{kl})(Z_{ij}Z^\dagger_{lk})\rangle,\end{aligned}$$

which produces 2 diagrams. Similarly, $\langle \operatorname{Tr}(Z^3)\operatorname{Tr}(Z^{\dagger^3})\rangle$ will produce 6 graphs. It can be deduced that $\langle \operatorname{Tr}(Z^n)\operatorname{Tr}(Z^{\dagger^n})\rangle$ will produce $n!$ diagrams. The first two correlators are illustrated with diagrams below:

which are equal to $\frac{2N^2}{\omega^2}$ and $\frac{3N^3}{\omega^3} + \frac{3N}{\omega^3}$ respectively. In the large N limit:

$$\begin{aligned}\langle \mathrm{Tr}\,(Z^2)\,\mathrm{Tr}\,(Z^{\dagger^2})\rangle &= \frac{2N^2}{\omega^2}\\ \langle \mathrm{Tr}\,(Z^3)\,\mathrm{Tr}\,(Z^{\dagger^3})\rangle &= \frac{3N^3}{\omega^3}\left(1 + O\left(\frac{1}{N^2}\right)\right).\end{aligned}$$

The general form of the correlator for traces of complex matrices is

$$\langle \mathrm{Tr}\,(Z^J)\,\mathrm{Tr}\,(Z^{\dagger^J})\rangle = \frac{JN^J}{\omega^J}\left(1 + O\left(\frac{1}{N^2}\right)\right).$$

As mentioned before, planar diagrams are diagrams you can draw on a plane/sphere. Only the planar diagrams contribute in the large N limit since $N^{\chi_{\mathrm{sphere}}} = N^2$ dominates in the large N limit. Introducing an operator $\mathcal{O}_J$ such that

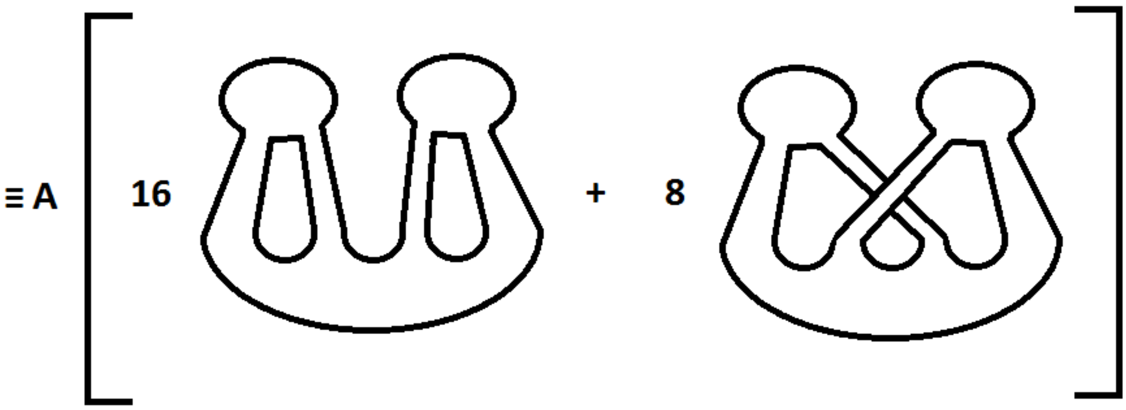
$$\mathcal{O}_J = \frac{\sqrt{\omega^J}\,\mathrm{Tr}\,(Z^J)}{\sqrt{JN^J}},$$

we find the correlator of this operator is

$$\langle \mathcal{O}_J \mathcal{O}_K^\dagger \rangle = \delta_{JK}.$$

We now wish to evaluate a correlator of the form $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_4^\dagger\rangle$:

$$\begin{aligned}\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_4^\dagger\rangle &= \langle \frac{\omega}{\sqrt{2}N}\,\mathrm{Tr}\,(Z^2)\frac{\omega}{\sqrt{2}N}\,\mathrm{Tr}\,(Z^2)\frac{\omega^2}{2N^2}\,\mathrm{Tr}\,(Z^{\dagger^4})\rangle\\ &= \frac{\omega^4}{4N^4}\langle \mathrm{Tr}\,(Z^2)\,\mathrm{Tr}\,(Z^2)\,\mathrm{Tr}\,(Z^{\dagger^4})\rangle\end{aligned}$$



$$
\begin{aligned}
&= A\left[\frac{16N^3}{\omega^3} + \text{non-planar}\right] \\
&= A\left[\frac{2\times 2\times 4N^3}{\omega^3} + \text{non-planar}\right],
\end{aligned}
$$

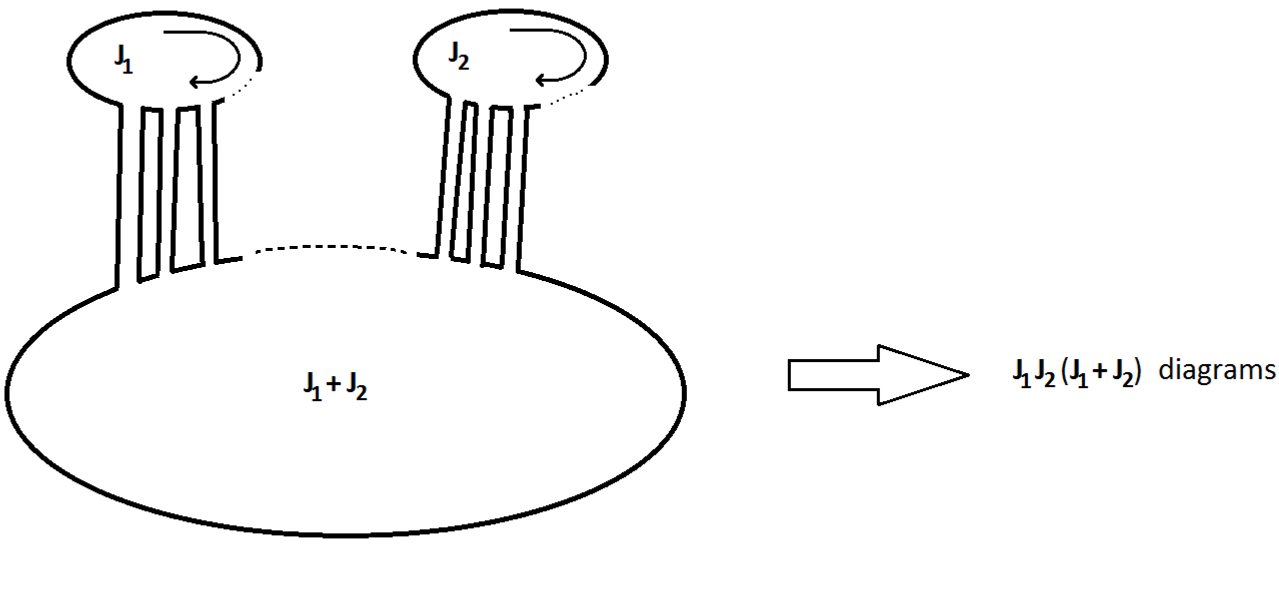
where $A \equiv \frac{\omega^4}{4N^4}$. The general form of the example above is

$$
\langle \mathrm{Tr}\,(Z^{J_1})\,\mathrm{Tr}\,(Z^{J_2})\,\mathrm{Tr}\,(Z^{\dagger^{J_1+J_2}})\rangle = \frac{J_1J_2(J_1+J_2)N^{J_1+J_2-1}}{\omega^{J_1+J_2}}
$$

and

$$
\begin{aligned}
\langle \mathcal{O}_{J_1}\mathcal{O}_{J_2}\mathcal{O}^{\dagger}_{K}\rangle &= \delta_{K,J_1+J_2}\frac{\sqrt{\omega^{J_1+J_2+K}}}{\sqrt{J_1J_2K}\sqrt{N^{J_1+J_2+K}}}\langle \mathrm{Tr}\,(Z^{J_1})\,\mathrm{Tr}\,(Z^{J_2})\,\mathrm{Tr}\,(Z^{\dagger^{K}})\rangle \\
&= \delta_{K,J_1+J_2}\frac{\omega^{J_1+J_2}}{\sqrt{J_1J_2(J_1+J_2)}}\frac{N^{J_1+J_2-1}}{N^{J_1+J_2}}\frac{J_1J_2(J_1+J_2)}{\omega^{J_1+J_2}} \\
&= \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N},
\end{aligned}
$$

where we have used the identity $f(x)\delta_{x,a} = f(a)\delta_{x,a}$. Note that we only sum the planar diagram contribution since it is the only contribution that survives in the large N limit. To illustrate this result, we have two small loops that have J_1 and J_2 legs respectively, and they connect to a bigger loop that has $J_1 + J_2$ legs:



The J_1 and J_2 diagrams can have their legs shifted around the $J_1 + J_2$ diagram producing J_1 and J_2 diagrams respectively. But the $J_1 + J_2$ diagram can also shift its legs resulting in $J_1 + J_2$ diagrams. Therefore, in total, $J_1 J_2 (J_1 + J_2)$ diagrams are produced. This diagram can be analogously thought of as 2 strings with lengths J_1, J_2 joining to produce a third string of length $J_1 + J_2$.

2.1.8 Comparisons

In this section we will compare the results from the matrix model to those of $\mathcal{N} = 4$ SYM theory. To go from the matrix model results to the $\mathcal{N} = 4$ SYM results, we need to append spacetime dependence which is easily determined by dimensional analysis. This section will provide concrete examples which nicely illustrate this.

Matrix Model:

Recall that for the matrix model, the correlator is

$$\langle Z_{ij} Z^\dagger_{kl} \rangle = \frac{\delta_{il}\delta_{jk}}{\omega}$$

and the operator we use is

$$\mathcal{O}_J = \frac{\sqrt{\omega^J}\,\mathrm{Tr}\,(Z^J)}{\sqrt{JN^J}},$$

with the correlators

$$\langle \mathcal{O}_J \mathcal{O}^\dagger_K \rangle = \delta_{JK}$$

and

$$\langle \mathcal{O}_{J_1} \mathcal{O}_{J_2} \mathcal{O}^\dagger_K \rangle = \delta_{K, J_1+J_2} \frac{\sqrt{J_1 J_2 (J_1 + J_2)}}{N}.$$

$\mathcal{N} = 4$ SYM:

In this theory, the action is

$$S = \int d^4x \;\, \mathrm{Tr}\,(\partial_\mu Z \partial^\mu Z^\dagger) + \ldots$$

where $Z = \phi_1 + i\phi_2$, and ϕ_1 and ϕ_2 are hermitian matrices. Thus, the correlator in this theory is

$$\langle Z_{ij}(x_1) Z^\dagger_{kl}(x_2) \rangle = \frac{\delta_{il}\delta_{jk}}{|x_1 - x_2|^2}.$$

The gauge invariant operators for this theory include

$$\mathcal{O}_J(x_1) = \frac{\mathrm{Tr}\,(Z^J(x_1))}{\sqrt{JN^J}},$$

which have the following two point function

$$\langle \mathcal{O}_J(x_1) \mathcal{O}_K(x_2) \rangle = \frac{\delta_{JK}}{|x_1 - x_2|^{2J}}$$

and three point function

$$\langle \mathcal{O}_{J_1}(x_1) \mathcal{O}_{J_2}(x_1) \mathcal{O}^\dagger_K(x_2) \rangle = \delta_{K, J_1+J_2} \frac{\sqrt{J_1 J_2 (J_1 + J_2)}}{N} \frac{1}{|x_1 - x_2|^{2(J_1+J_2)}}.$$

2.1.9 Link between string theory and gauge theory

According to the AdS/CFT correspondence, for IIB strings on $AdS_5 \times S^5$, energy and momentum on S^5 is mapped into the dimension Δ/scaling dimension and R-charge in $\mathcal{N} = 4$ SYM respectively. The scaling dimension of any local operator A is equal to the negative power of the dimension of length at the free field fixed point. To be explicit, $\Delta_A = B \Rightarrow [A] = L^{-B}$.

Here are some more simple examples:

- $[Z] = L^{-1} \Rightarrow \Delta_Z = 1$
- $[Z^2] = L^{-2} \Rightarrow \Delta_{Z^2} = 2$

Thus the scaling dimension for the operator $\mathcal{O}_J$ is

$$\Delta_{\mathcal{O}_J} = J,$$

since

$$[\mathcal{O}_J] = [\mathrm{Tr}\,(Z^J)] = [Z^J] = L^{-J}.$$

The scaling dimension of operator $\mathcal{O}_J$ is in fact the energy of the dual string state as mentioned in the beginning of this subsection 2.1.9, i.e. energy in string theory $\leftrightarrow$ Δ in $\mathcal{N} = 4$ SYM.

A 5-sphere, S^5, defined by $(x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 + (x_5)^2 + (x_6)^2 = R^2$ has an $SO(6)$ symmetry. For the QFT, there are the 6 fields $\phi_1, \ldots, \phi_6$ that also enjoy an $SO(6)$ symmetry which is the R-symmetry. Remember that the angular momentum operator is $L = -i\frac{\partial}{\partial \phi}$ and $e^{-i\theta L} Z = e^{-i\theta \#} Z$, where $Z = \phi_1 + i\phi_2$ and $\#$ is equal to the $SO(6)$ charge that Z carries. $e^{i\theta L} Z$ is nothing but a finite rotation by angle θ, generated by L. This is simple to compute. We rotate Z such that

$$\phi_1 \to \phi_1' = \cos{(\theta)}\phi_1 + \sin{(\theta)}\phi_2$$

and

$$\phi_2 \to \phi_2' = -\sin{(\theta)}\phi_1 + \cos{(\theta)}\phi_2,$$

so that

$$\begin{aligned} \phi_1' + i\phi_2' &= \cos{(\theta)}\phi_1 + \sin{(\theta)}\phi_2 - i\sin{(\theta)}\phi_1 + i\cos{(\theta)}\phi_2 \\ &= e^{-i\theta}\phi_1 + ie^{-i\theta}\phi_2 \\ &= e^{-i\theta}(\phi_1 + i\phi_2) \\ &= e^{-i\theta} Z. \end{aligned}$$

Therefore, we deduce that $\#$ for Z is 1. We have thus deduced that the R-charge of $\mathcal{O}_J$ is

$$R_{\mathcal{O}_J} = -J,$$

which is in fact the momentum of the string as stated before, i.e. momentum in string theory $\leftrightarrow$ R-charge in $\mathcal{N} = 4$ SYM. Now, we can deduce the mass of $\mathcal{O}_J$:

$$\begin{aligned} m^2_{\mathcal{O}_J} &= E^2_{\mathcal{O}_J} - p^2_{\mathcal{O}_J} \\ &= \Delta^2_{\mathcal{O}_J} - R^2_{\mathcal{O}_J} \\ &= J^2 - (-J)^2 \\ &= 0. \end{aligned}$$

Therefore, $\mathcal{O}_J$ corresponds to a massless graviton moving on a 5-sphere.

2.1.10 Fock space in supergravity

Recall that the mode expansion of a field in a Lorentz invariant QFT with relativistic normalisation is given as

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik\cdot x}\alpha(k) + e^{ik\cdot x}\alpha^\dagger(k)),$$

where $\alpha(k) = (2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}a_{\vec{k}}$ and $\alpha^\dagger(k) = (2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}a^\dagger_{\vec{k}}$. For the non-relativistic case, we had $\langle \vec{k}'|\vec{k}\rangle = \delta(\vec{k}-\vec{k}')$, $a^\dagger_{\vec{k}}|0\rangle = |\vec{k}\rangle$, and $a^\dagger_{\vec{k}_1}a^\dagger_{\vec{k}_2}|0\rangle = |\vec{k}_1, \vec{k}_2\rangle$ for continuous momenta, and $\langle n|m\rangle = \delta_{nm}$, $a^\dagger_n|0\rangle = |n\rangle$, and $a^\dagger_n a^\dagger_m|0\rangle = |n, m\rangle$ for the field in a box.

Now, $|J\rangle$ = state of graviton with energy J and momentum $-J$. We also have $\langle J|K\rangle = \delta_{JK}$ and $\langle J_1, J_2|J_1 + J_2\rangle = 0$ (since a one particle state is orthogonal to a two particle state in the free theory). When energies become very large, then there is an interaction between the particle states. This is due to gravity, and the "charge" responsible for generating the gravitational field is the mass. We know how mass and energies are related by Einstein's formula $E = mc^2$. In this case we will see that $\langle J_1, J_2|J_1 + J_2\rangle$ begins to differ from zero. Computing $\langle J_1, J_2|J_1 + J_2\rangle$, we have [9]

$$\langle J_1, J_2|J_1 + J_2\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N},$$

which is obtained by summing only the planar contribution (i.e. this result is not exact). This overlap suggests operators are orthogonal as long as $J \ll N^{\frac{2}{3}}$. Considering the case when $J_1, J_2 = O(1)$, the correlator is

$$\langle J_1, J_2|J_1 + J_2\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N} \to \delta_{K,J_1+J_2}\frac{\#}{N} \to 0, \text{ as } N \to \infty.$$

However, when $J_1, J_2 = O(N)$, then the correlator reads

$$\langle J_1, J_2|J_1 + J_2\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N} \to \delta_{K,J_1+J_2}\frac{\sqrt{N^2}}{N}\left(\#\sqrt{N}\right) = \#\sqrt{N}\delta_{K,J_1+J_2} \neq 0.$$

Thus, the identification of each trace with a particle has broken down. In the matrix model, we have

$$\langle \mathcal{O}_J\mathcal{O}_J^\dagger\rangle = JN^J + J^5N^{J-2} + \dots$$

Note that if $J \sim N^{\frac{1}{2}}$, then the N^{J-2} term is as big as the N^J term. As a result one cannot neglect the non-planar diagrams; we *must* sum everything. This informs us that as we change J, the object we are studying changes. To illustrate what was just stated, the table below summarises the relationship between J (which is a label of QFT) and the object (which lives in string theory):

J	object
$O(1)$	graviton
$O(\sqrt{N})$	string
$O(N)$	giant graviton
$O(N^2)$	new spacetime geometry

To explain this further in detail, the table below explains what each order means in connection to J and the dual string theory object:

N	graviton ($O(1)$)	string ($O(\sqrt{N})$)	giant graviton ($O(N)$)	new $g_{\mu\nu}$ ($O(N^2)$)
10	$\mathrm{Tr}\,(Z^3)$	$\mathrm{Tr}\,(Z^3 Y)$	$\mathrm{Tr}\,(Z^4) + \dots$	$\mathrm{Tr}\,(Z^4) + \dots$
100	$\mathrm{Tr}\,(Z^3)$	$\mathrm{Tr}\,(Z^9 Y)$	$\mathrm{Tr}\,(Z^{40}) + \dots$	$\mathrm{Tr}\,(Z^{400}) + \dots$
1000	$\mathrm{Tr}\,(Z^3)$	$\mathrm{Tr}\,(Z^{27} Y)$	$\mathrm{Tr}\,(Z^{400}) + \dots$	$\mathrm{Tr}\,(Z^{40\ 000}) + \dots$

The fields Z and Y are independent complex fields and are discussed in detail in subsections 2.3 and 2.4 ; we do not need any further details about the fields at this point in this subsection. For a graviton, large N correlators are obtained by summing planar diagrams only. For a string we also sum planar diagrams as long as $\frac{J^2}{N} \ll 1$. For a giant graviton, every diagram contributes; there will be $N!$ diagrams and we take $N \to \infty$. For new spacetime geometry, including black holes we again need to sum all of the diagrams.

2.1.11 Conclusion

When constructing the matrix model, we construct Gaussian integrals that help us to evaluate correlators. This led us to specific properties of correlators which compelled us to think of making use of diagrams when computing correlators; we called these diagrams *ribbon diagrams*. These diagrams came with their own set of Feynman rules and we deduced that traces of powers of our matrices are the physical observables with interesting correlators.

When we take the limit $N \to \infty$, we notice that only the leading term of the correlators survive and that the expectation value of products is equal to the product of expectation values. This result first obtained by [10], is called *factorisation*. We linked this to the conjecture that the large N limit of $\mathcal{N} = 4$ SYM theory, is given by the classical limit of IIB string theory on $AdS_5 \times S^5$, and we argued that $1/N^2$ is equal to $\hbar_{\text{string theory}}$.

We considered the interacting theory. The interaction term "$-g\,\mathrm{Tr}\,(M^4)$" required a new rule: the vertex ribbon diagram that has 4 legs corresponding to M to the power of 4. From there we take the double scaling limit $g \to 0$ and $N \to \infty$ such that $\lambda = gN$ is fixed and small. This double scaling limit is known as the *'t Hooft limit*. The new constant λ along with $1/N^2$ are both related to $\hbar$ (QFT and string theory respectively) and they indicate that studying the large N expansion in the matrix model is equivalent to doing string theory.

We rescaled variables such that ribbon graphs have a N dependence of N^{F-E+V} where $\chi = F - E + V$ is the *Euler characteristic*. It is a topological invariant. We have concluded that the ribbon diagrams of the interacting theory triangulate a surface.

Following that, we investigated complex matrices and the correlators associated with complex matrices. We noted that $\mathrm{Tr}\,(Z^n)\,\mathrm{Tr}\,(Z^{\dagger n})$ will produce $n!$ diagrams, but in the large N limit, we will only get contributions from planar diagrams. Thus $\langle \mathrm{Tr}\,(Z^J)\,\mathrm{Tr}\,(Z^{\dagger J})\rangle = \frac{JN^J}{\omega^J}\left(1 + O\left(\frac{1}{N^2}\right)\right)$. We introduced the operator $\mathcal{O}_J$ which normalises $\mathrm{Tr}\,(Z^J)$ such that $\langle \mathcal{O}_J \mathcal{O}_K\rangle = \delta_{JK}$. Through further investigation, we also found that $\langle \mathcal{O}_{J_1}\mathcal{O}_{J_2}\mathcal{O}_K^\dagger\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N}$, where J_1, J_2, and $J_1 + J_2$ can be thought of as lengths of 3 strings. Comparing this result from the matrix model to $\mathcal{N} = 4$ SYM, we see that in $\mathcal{N} = 4$ SYM theory, $\langle \mathcal{O}_{J_1}(x_1)\mathcal{O}_{J_2}(x_1)\mathcal{O}_K^\dagger(x_2)\rangle = \delta_{K,J_1+J_2}\frac{\sqrt{J_1J_2(J_1+J_2)}}{N}\frac{1}{|x_1-x_2|^{2(J_1+J_2)}}$. It has the additional term of $\frac{1}{|x_1-x_2|^{2(J_1+J_2)}}$ compared to the result of the matrix model correlator. From there, we deduced that the scaling dimension of $\mathcal{O}_J$ is $\Delta_{\mathcal{O}_J} = J$ and the R-charge of $\mathcal{O}_J$ is $R_{\mathcal{O}_J} = J$, which relates to the energy and momentum in the string theory respectively. Thus we determine the mass of $\mathcal{O}_J$ to be $m^2_{\mathcal{O}_J} = E^2_{\mathcal{O}_J} - p^2_{\mathcal{O}_J} = 0$, which shows that $\mathcal{O}_J$ is a massless graviton moving on a 5-sphere.

In the Fock space in supergravity, when the energy is very high, there will be interactions between particle states. The correlator we study becomes $\langle \mathcal{O}_J \mathcal{O}_J^\dagger\rangle = JN^J + J^5N^{J-2} + \dots$, and if $J \sim N^{\frac{1}{2}}$, then the N^{J-2} term is as big as the N^J term, so one cannot neglect the non-planar diagrams. It further shows us that changing J changes the interpretation of the object you study (in string theory). Taken together, this is compelling evidence that matrix models are in fact string theories.

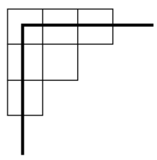
2.2 Young diagrams

For operators of dimension $\Delta = N$ we know that the planar limit is not accurate. The large N limit of these operators is described by an emergent Yang-Mills theory [11]. We would like to study this theory. With the use of group representation theory, we will be able to sum all ribbon graphs and not just planar contributions. Young diagrams play a central role in this discussion.

A Young diagram is an array of n boxes aligned on the left in rows and each consecutive row will have the same or less boxes as the row above it. Young diagrams label all of the possible irreducible inequivalent representations (known as irreps) of the symmetric group S_n. When we list the number of boxes in each row of Young diagram R, it gives a partition of n, so we can write "R is a partition of n", which is denoted as $R \vdash n$. Each representation is a set of matrices acting on a vector space. We can label a basis for this vector space using the Young-Yamanouchi symbols. For us to get a Young-Yamanouchi symbol, we fill in the boxes with the integers $1, 2, 3, \ldots, n$ in the order of which box we could remove such that what is left is still a valid Young diagram. For example, suppose we have the irrep labelled by $\yng(3,1)$ which has the following Young-Yamanouchi symbols:

$$\young(432,1) \quad \young(431,2) \quad \young(421,3) .$$

There is a formula to determine the dimensions of a symmetric group irrep given the Young diagram. This formula includes the notion of a hook length, which is a number that is given to each box in the Young diagram. The hook length of a box b in Young diagram R is the number of boxes that are in the same row to the right of b plus the number of boxes in the same column below b plus one. Inside the column containing box b, draw a line from below the bottom of R to b and then continue that line to the right until you exit R. The hook length is equal to the number of boxes this line (the "hook") passes through. The diagram below illustrates an example of a hook associated to the first box in the first row of the Young diagram $\yng(3,2,1)$ that has a hook length of 5:



As an example, the Young diagram $\yng(3,2,1)$ with the hooks lengths filled in is given as

$$\text{hook lengths} = \young(531,31,1) .$$

Now, the dimension of an irrep labelled by Young diagram R is equal to $n!$ divided by the product of hook lengths, and it is written as

$$d_R = \frac{n!}{\prod_{x \in R} \text{hook}(x)} \equiv \frac{n!}{\text{hooks}_R},$$

where $R \vdash n$. The Young diagram $\yng(3,2,1)$ labels an irrep of S_6 with dimension

$$d_{\yng(3,2,1)} = \frac{6!}{5 \cdot 3 \cdot 1 \cdot 3 \cdot 1 \cdot 1} = 16.$$

Thus, there are 16 valid Young-Yamanouchi symbols that can be drawn. We now introduce the concept of the *content* of a box in a Young diagram. A box x in row i and column j of R has content $j-i$. Here is an example of Young diagram with the content filled in

0	1	2
-1	0	
-2		

A Young-Yamanouchi pattern has, each box in the Young diagram labelled by an integer i that is unique, with $1 \le i \le n$. The content of the box labelled i is c_i. Let $|R_{(k,k+1)}\rangle$ denote the Young-Yamanouchi symbol that is obtained from $|R\rangle$, some Young-Yamanouchi symbol of Young diagram R, by swapping the labels of boxes k and $k+1$. The matrix elements of the adjacent transpositions are now specified by

$$\Gamma_R((k,k+1))|R\rangle = \frac{1}{c_k - c_{k+1}}|R\rangle + \sqrt{1 - \frac{1}{(c_k - c_{k+1})^2}}|R_{(k,k+1)}\rangle.$$

For example

$$\Gamma_{\yng(2,1,1)}((1\ 2))|\young(43,2,1)\rangle = -|\young(43,2,1)\rangle$$

$$\Gamma_{\yng(2,1,1)}((1\ 2))|\young(42,3,1)\rangle = -\frac{1}{3}|\young(42,3,1)\rangle + \frac{\sqrt{8}}{3}|\young(41,3,2)\rangle$$

$$\Gamma_{\yng(2,1,1)}((1\ 2))|\young(41,3,2)\rangle = \frac{1}{3}|\young(41,3,2)\rangle + \frac{\sqrt{8}}{3}|\young(42,3,1)\rangle.$$

By choosing

$$|\young(43,2,1)\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad |\young(42,3,1)\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad |\young(41,3,2)\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

we find the following matrix representation for the permutation (1 2)

$$\Gamma_{\yng(2,1,1)}((1\ 2)) = \begin{bmatrix}-1 & 0 & 0\\ 0 & -\frac{1}{3} & \frac{\sqrt{8}}{3}\\ 0 & \frac{\sqrt{8}}{3} & \frac{1}{3}\end{bmatrix}.$$

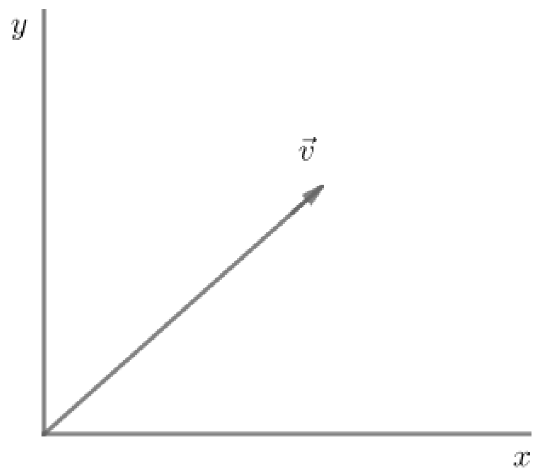
In conclusion, Young diagrams label irreps, they provide us with a formula for the dimensions of irreps, they can be used to label the elements of a basis and they give us the matrices defining the irreps.

2.3 Single Matrix Z

In applying group representation theory to matrix models we make frequent use of projection operators. In this subsection we introduce some of the projection operators that are needed.

2.3.1 Projection operators

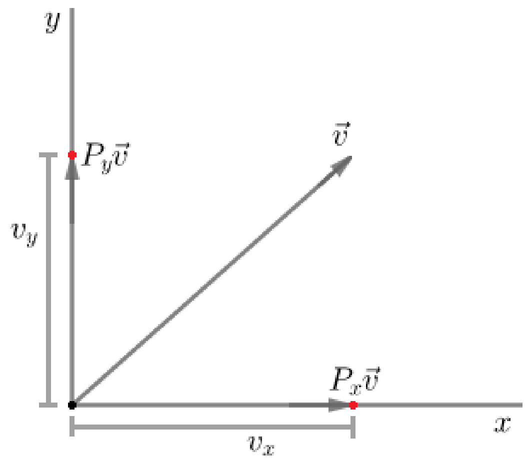
Suppose we have a vector $\vec{v}$ with two components $\vec{v} = (v_x, v_y)$. The diagram below illustrates this arbitrary vector



Introduce the operators $\hat{P}_x$ and $\hat{P}_y$ such that

$$\hat{P}_x \vec{v} = (v_x, 0) \quad , \quad \hat{P}_y \vec{v} = (0, v_y).$$

The diagram below makes this more clear:



$\hat{P}_x$ and $\hat{P}_y$ are projection operators that project an arbitrary vector $\vec{v}$ to its x-component and y-component respectively in the plane. Note that $\hat{P}_x \cdot \hat{P}_x |v\rangle = \hat{P}_x^2 |v\rangle = \hat{P}_x |v\rangle$, which implies

$$\hat{P}_x^2 = \hat{P}_x,$$

and similarly $\hat{P}_y^2 = \hat{P}_y$. Also,

$$\hat{P}_x \cdot \hat{P}_y = \hat{P}_y \cdot \hat{P}_x = 0.$$

This result is simple to interpret: $\hat{P}_x \cdot \hat{P}_y$ says you first project onto the y-axis and then you project onto the x-axis which vanishes. A similar argument is made for $\hat{P}_y \cdot \hat{P}_x$. Finally, note that

$$\hat{P}_x |v\rangle + \hat{P}_y |v\rangle = |v\rangle$$
$$\Rightarrow \hat{P}_x + \hat{P}_y = \mathbb{1}.$$

This is obvious upon noting that $\hat{P}_x\vec{v} = (v_x, 0)$ and $\hat{P}_y\vec{v} = (0, v_y)$. These properties of projection operators, that are obvious in our simple example, hold in general. These projection operators, in general, act in the space $V_N^{\otimes n}$ - we discuss this space in a few pages that are to follow. We can define projection operators that project onto definite irreducible representations of the symmetric group. The projector that projects onto irrep R is

$$(\hat{P}_R)^I_J = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)(\sigma)^I_J,$$

where d_R is the dimension of irrep R, σ is a permutation, S_n is the symmetric group permuting n elements, $\chi_R(\sigma)$ is the character of σ in irrep R, I and J denote $i_1i_2\dots i_n$ and $j_1i_2\dots j_n$ respectively, and $(\sigma)^I_J = \delta^{i_1}_{j_{\sigma(1)}}\dots\delta^{i_n}_{j_{\sigma(n)}}$.

We now consider the cases $n=2$ with $R = \Box\Box$ and $R = \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$:

- $n = 2$ (thus $S_2 = \{\mathbb{1}, (1\ 2)\}$) and $R = \Box\Box$

$$\begin{aligned}
(\hat{P}_{\Box\Box})^{i_1i_2}_{j_1j_2}v^{j_1}w^{j_2} &= \frac{1}{2}\left[\chi_{\Box\Box}(\mathbb{1})(\mathbb{1})^{i_1i_2}_{j_1j_2} + \chi_{\Box\Box}((1\ 2))((1\ 2))^{i_1i_2}_{j_1j_2}\right]v^{j_1}w^{j_2}\\
&= \frac{1}{2}\left[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\right]v^{j_1}w^{j_2}\\
&= \frac{1}{2}\left[v^{i_1}w^{i_2} + w^{i_1}v^{i_2}\right],
\end{aligned}$$

which is the symmetric part of the tensor product of two vectors. Thus $\hat{P}_{\Box\Box}$ projects to the symmetric part of the tensor product of two vectors. Also, note that

$$\begin{aligned}
(\hat{P}_{\Box\Box})^{i_1i_2}_{j_1j_2}(\hat{P}_{\Box\Box})^{j_1j_2}_{k_1k_2} &= \frac{1}{4}\left[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\right]\left[\delta^{j_1}_{k_1}\delta^{j_2}_{k_2} + \delta^{j_1}_{k_2}\delta^{j_2}_{k_1}\right]\\
&= \frac{1}{4}\left[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} + \delta^{i_1}_{k_2}\delta^{i_2}_{k_1} + \delta^{i_1}_{k_2}\delta^{i_2}_{k_1} + \delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\right]\\
&= \frac{1}{2}\left[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} + \delta^{i_1}_{k_2}\delta^{i_2}_{k_1}\right]\\
&= (\hat{P}_{\Box\Box})^{i_1i_2}_{k_1k_2},
\end{aligned}$$

which is the property $\hat{P}^2 = \hat{P}$, with the correct placing of indices.

- $n = 2$ and $R = \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$

$$\begin{aligned}
(\hat{P}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}})^{i_1i_2}_{j_1j_2}v^{j_1}w^{j_2} &= \frac{1}{2}\left[\chi_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}(\mathbb{1})(\mathbb{1})^{i_1i_2}_{j_1j_2} + \chi_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}((1\ 2))((1\ 2))^{i_1i_2}_{j_1j_2}\right]v^{j_1}w^{j_2}\\
&= \frac{1}{2}\left[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\right]v^{j_1}w^{j_2}\\
&= \frac{1}{2}\left[v^{i_1}w^{i_2} - w^{i_1}v^{i_2}\right],
\end{aligned}$$

which is the antisymmetric part of the tensor product of two vectors. Thus $\hat{P}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}$ projects to the antisymmetric part of the tensor product of two vectors. Also, note that

$$
\begin{aligned}
(\hat{P}_{\substack{\square\\\square}})^{i_1 i_2}_{j_1 j_2}(\hat{P}_{\substack{\square\\\square}})^{j_1 j_2}_{k_1 k_2} &= \frac{1}{4}\left[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\right]\left[\delta^{j_1}_{k_1}\delta^{j_2}_{k_2} - \delta^{j_1}_{k_2}\delta^{j_2}_{k_1}\right] \\
&= \frac{1}{4}\left[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} - \delta^{i_1}_{k_2}\delta^{i_2}_{k_1} - \delta^{i_1}_{k_2}\delta^{i_2}_{k_1} + \delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\right] \\
&= \frac{1}{2}\left[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} - \delta^{i_1}_{k_2}\delta^{i_2}_{k_1}\right] \\
&= (\hat{P}_{\substack{\square\\\square}})^{i_1 i_2}_{k_1 k_2},
\end{aligned}
$$

which, once again, is the property $\hat{P}^2 = \hat{P}$.

We also observe that these operators are complete:

$$
(\hat{P}_{\square\square} + \hat{P}_{\substack{\square\\\square}})^I_J = \mathbb{1}^I_J.
$$

Recall that the general form of the projection operator is given by $(\hat{P}_R)^I_J = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)(\sigma)^I_J$. We will compute the product of two projection operators:

$$
\hat{P}_R\hat{P}_S = \frac{d_R}{n!}\frac{d_S}{n!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\chi_R(\sigma_1)\chi_S(\sigma_2)\sigma_1\sigma_2,
$$

where everything on the right hand side of the equation is a number except for $\sigma_1\sigma_2$ which is a matrix, with the property (if we consider the left action):

$$
(\sigma_1)^I_J(\sigma_2)^J_K = (\sigma_2\cdot\sigma_1)^I_K.
$$

We change variables from σ_2 to $\psi = \sigma_2\cdot\sigma_1$. Therefore the product becomes

$$
\hat{P}_R\hat{P}_S = \frac{d_R}{n!}\frac{d_S}{n!}\sum_{\sigma_1\in S_n}\sum_{\psi\in S_n}\chi_R(\sigma_1)\chi_S(\psi\cdot\sigma_1^{-1})\psi.
$$

Multiplying both sides to the right by σ_1^{-1} results in $\psi\cdot\sigma_1^{-1} = \sigma_2\cdot\sigma_1\sigma_1^{-1} = \sigma_2\cdot\mathbb{1} = \sigma_2$. The term $\chi_R(\sigma_1)\chi_S(\psi\cdot\sigma_1^{-1})$ suggests we use the Fundamental Orthogonality relation. Thus, we obtain

$$
\begin{aligned}
\hat{P}_R\hat{P}_S &= \frac{d_R}{n!}\frac{d_S}{n!}\sum_{\psi\in S_n}\left(\sum_{\sigma_1\in S_n}(\Gamma_R(\sigma_1))_{ii}\Gamma_S(\psi)_{jk}\Gamma_S(\sigma_1^{-1})_{kj}\right)\psi \\
&= \delta_{RS}\frac{d_S}{n!}\sum_{\psi\in S_n}\chi_S(\psi)\psi \\
&= \delta_{RS}\hat{P}_S.
\end{aligned}
$$

Therefore, the general result of multiplying two projection operators, of irrep R and S respectively, is given as

$$
\hat{P}_R\hat{P}_S = \delta_{RS}\hat{P}_R,
$$

where $R \vdash n$.

Let V_N denote the N dimensional vector space. This space has a basis given by the N basis vectors

$$|e^1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad |e^2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \ldots \quad |e^N\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

where each vector has N components. This is a complex vector space so that the general vector in this case can be expressed as

$$|v\rangle = \sum_{i=1}^{N} c_i |e^i\rangle$$

with c_i are some complex numbers. The dual vector space is given by row vectors, with basis $\langle e_i|$. To get the dual to $|v\rangle$, we transpose columns into rows and complex conjugate the expansion coefficients

$$\langle v| = \sum_{i=1}^{N} c_i^* \langle e_i|.$$

The inner product is given by

$$\langle v|v\rangle = \sum_{i=1}^{N} |c_i|^2.$$

The group of $N \times N$ unitary matrices, $U(N)$, has the obvious action on this space, acting by matrix multiplication on vectors

$$|v\rangle \to U|v\rangle, \quad \langle v| \to \langle v|U^\dagger.$$

The inner product under the action of $U(N)$

$$\langle v|v\rangle \to \langle v|U^\dagger U|v\rangle = \langle v|v\rangle$$

is invariant. The tensor product of n copies of V_N vector spaces can be written as

$$V_N \otimes V_N \otimes \cdots \otimes V_N \equiv V_N^{\otimes n},$$

where $V_N^{\otimes n}$ is an N^n dimensional space. $V_N^{\otimes n}$ inherits a natural action of the unitary group from the action on V_N:

$$\begin{aligned} &(U^{\otimes n})|v(1)\rangle \otimes |v(2)\rangle \otimes \cdots \otimes |v(n)\rangle = \\ &= (U \otimes U \otimes \cdots \otimes U)|v(1)\rangle \otimes |v(2)\rangle \otimes \cdots \otimes |v(n)\rangle \\ &= U|v(1)\rangle \otimes U|v(2)\rangle \otimes \cdots \otimes U|v(n)\rangle. \end{aligned}$$

U acts in the same way (by matrix multiplication) on every vector in the tensor product. One can view $U^{\otimes n}$ as an N^n dimensional matrix. We now define an action of the symmetric group S_n on $V_N^{\otimes n}$. The symmetric group acts by interchanging the order of the vectors in the tensor product, without shuffling the components of any given vector

$$(\sigma)|v(1)\rangle \otimes |v(2)\rangle \otimes \cdots \otimes |v(n)\rangle = |v(\sigma(1))\rangle \otimes |v(\sigma(2))\rangle \otimes \cdots \otimes |v(\sigma(n))\rangle.$$

An important observation is the fact that the actions of the symmetric group S_n and the unitary group $U(N)$ on $V_N^{\otimes n}$ commute:

$$\begin{aligned} U^{\otimes n}\cdot\sigma\cdot(|v(1)\rangle\otimes|v(2)\rangle\otimes\cdots\otimes|v(n)\rangle) &= U^{\otimes n}\cdot(|v(\sigma(1))\rangle\otimes|v(\sigma(2))\rangle\otimes\cdots\otimes|v(\sigma(n))\rangle)\\ &= U|v(\sigma(1))\rangle\otimes U|v(\sigma(2))\rangle\otimes\cdots\otimes U|v(\sigma(n))\rangle\\ &= \sigma\cdot(U|v(1)\rangle\otimes U|v(2)\rangle\otimes\cdots\otimes U|v(n)\rangle)\\ &= \sigma\cdot U^{\otimes n}\cdot(|v(1)\rangle\otimes|v(2)\rangle\otimes\cdots\otimes|v(n)\rangle). \end{aligned}$$

The fact that there are commuting actions of $U(N)$ and S_n implies deep relations between the representation theory and these two groups. This connection is called Schur-Weyl duality.

The Gelfand-Tsetlin pattern labelling chooses basis states that are simultaneous eigenstates of the Cartan subalgebra of $U(N)$, and further, explicit formulas are known for the matrix elements of the generators of the group with respect to these basis states. The key characteristic of the Gelfand-Tsetlin basis is that states are labelled by the irrep they belong to, for each group in the chain of subgroups

$$U(N)\supset U(N-1)\supset U(N-2)\supset\cdots\supset U(3)\supset U(2)\supset U(1).$$

We will now describe how the Gelfand-Tsetlin patterns are constructed. An inequivalent irreducible representation for $GL(N,\mathbb{C})$ is uniquely given by specifying the sequence of N integers

$$\mathbf{m}=(m_{1N},m_{2N},\ldots,m_{NN}) \tag{1}$$

satisfying $m_{kN}\geq m_{k+1,N}$ for $1\leq k\leq N-1$. Notice that this sequence can be identified with the row lengths of a Young diagram R that has no more than N rows - this is the Young diagram labelling the $GL(N,\mathbb{C})$ irreducible representation.[2] We call the sequence (1) the weight of the irreducible representation. The restriction of this irrep onto the subgroup $GL(N-1,\mathbb{C})$ is reducible. It decomposes into a direct sum of $GL(N-1,\mathbb{C})$ irreps with highest weights

$$\mathbf{m}'=(m_{1,N-1},m_{2,N-1},\ldots,m_{N-1,N-1}),$$

for which the "betweenness" conditions

$$m_{kN}\geq m_{k,N-1}\geq m_{k+1,N}\quad \text{for}\quad 1\leq k\leq N-1$$

hold. This specifies the branching rule for how a $GL(N,\mathbb{C})$ irrep decomposes when we restrict to the $GL(N-1,\mathbb{C})$ subgroup. We can repeat this procedure until we get to $GL(1,\mathbb{C})$ which has one-dimensional carrier spaces. The Gelfand-Tsetlin labelling assembles this sequence of representations of the subgroups into a Gelfand-Tsetlin pattern. There is a unique pattern for each state in the basis of the carrier space of the original $GL(N,\mathbb{C})$ irrep. The pattern can be written as a triangular arrangement of integers, denoted M, with the structure

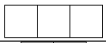
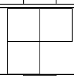
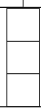
$$M=\begin{bmatrix} m_{1N} & & m_{2N} & \ldots & m_{N-1,N} & & m_{NN}\\ & m_{1,N-1} & & m_{2,N-1} & \ldots & m_{N-1,N-1} & \\ & & \ldots & \ldots & \ldots & & \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{bmatrix}$$

The top row contains the weight that specifies the irrep of the state and the entries of lower rows are subject to the betweenness condition. The lower rows give the sequence of irreps our state

[2] $U(N)$ is a subgroup of $GL(N)$. Each irrep of $GL(N)$ restricts to a unique irrep of $U(N)$ so that the irreps of $GL(N)$ are also labelled by Young diagrams.

belongs to as we pass through successive restrictions from $GL(N,\mathbb{C})$ to $GL(N-1,\mathbb{C})$ to ... to $GL(1,\mathbb{C})$. For each possible pattern we have a distinct vector. The vectors labelled by two distinct patterns are orthogonal and the dimension of the irrep is given by the total number of distinct Gelfand-Tsetlin patterns that can be constructed. As we have already described above, everything we have said about $GL(N,\mathbb{C})$ applies, without any change, to $U(N)$. This orthogonal basis of $U(N)$ is called the Gelfand-Tsetlin basis.

The representations of $U(N)$ are labelled by Young diagrams with no more than N rows. We will denote the dimension of a $U(N)$ irrep by Dim_R. Consider $n = 3$, corresponding to the vector space $V_N^{\otimes 3}$. The table below displays the values of R, d_R, and Dim_R:

R	d_R	Dim_R
	1	$\frac{N(N+1)(N+2)}{6}$
	2	$\frac{N(N+1)(N-1)}{3}$
	1	$\frac{N(N-1)(N-2)}{6}$

Summing up the products of d_R and Dim_R with their respective representations R, we see that

$$\begin{aligned}\sum_R d_R \text{Dim}_R &= \frac{N(N+1)(N+2)}{6} + \frac{2N(N+1)(N-1)}{3} + \frac{N(N-1)(N-2)}{6} \\ &= N^3\left(\frac{1}{6}+\frac{2}{3}+\frac{1}{6}\right) + N^2\left(\frac{3}{6}-\frac{3}{6}\right) + N\left(\frac{2}{6}-\frac{2}{3}+\frac{2}{6}\right) \\ &= N^3,\end{aligned}$$

which is an insightful result since the dimension of $V_N^{\otimes 3}$ is N^3. In fact, $\sum_R d_R \text{Dim}_R$ is equal to the total number of states in the vector space $V_N^{\otimes n}$ in general. This shows each state is labelled by a unique Young-Yamanouchi pattern and a unique Gelfand-Tsetlin pattern.

2.3.2 Equivalence relation and equivalence classes

$\sim$ is an equivalence relation if it obeys the following three conditions:

- $a \sim a$ (reflectivity)
- $a \sim b$ then $b \sim a$ (symmetry)
- $a \sim b$ and $b \sim c$, then it implies that $a \sim c$ (transitivity).

Now, we say g is conjugate to h, written as $g \sim h$ with g,$h \in \mathcal{G}$, if $g = \sigma h \sigma^{-1}$ for some $\sigma \in \mathcal{G}$. We now make and prove the statement that "conjugate to" is an equivalence relation.

Proof:

- Reflexivity is satisfied with the condition $g = \mathbb{1} g \mathbb{1}^{-1}$
- Symmetry is satisfied when $g = \sigma h \sigma^{-1}$ implies $h = \sigma^{-1} g \sigma = \sigma^{-1} g (\sigma^{-1})^{-1}$
- Transitivity is satisfied when we let $g = \sigma_1 h \sigma_1^{-1}$ and $h = \sigma_2 j \sigma_2^{-1}$ which implies $g = \sigma_1(\sigma_2 j \sigma_2^{-1})\sigma_1^{-1} = \sigma_1\sigma_2 j (\sigma_1\sigma_2)^{-1}$ $\square$

This equivalence relation partitions the group into *conjugacy classes* (this is the name of these equivalence classes). Consider the characters of g and h, denoted as $\chi_R(g) = \mathrm{Tr}(\Gamma_R(g))$ and $\chi_R(h) = \mathrm{Tr}(\Gamma_R(h))$ respectively, with $g \sim h$. Then

$$\begin{aligned}\chi_R(h) = \mathrm{Tr}(\Gamma_R(h)) &= \mathrm{Tr}(\Gamma_R(\sigma)\Gamma_R(g)\Gamma_R(\sigma^{-1})) \\ &= \mathrm{Tr}(\Gamma_R(\sigma^{-1})\Gamma_R(\sigma)\Gamma_R(g)) \\ &= \mathrm{Tr}(\Gamma_R(\mathbb{1})\Gamma_R(g)) \\ &= \mathrm{Tr}(\Gamma_R(g)) \\ &= \chi_R(g),\end{aligned}$$

where in the second line the cyclicity of the trace is used, and in the third line the property $\Gamma_R(a) \cdot \Gamma_R(b) = \Gamma_R(a \cdot b)$ for $a,b \in \mathcal{G}$ and the fact that $\sigma^{-1}\sigma = \mathbb{1}$ for $\sigma \in \mathcal{G}$ was used. The result $(\chi_R(g) = \chi_R(h))$ tells us that all elements in a conjugacy class have the same character.

Consider $R \vdash n$, which means $\hat{P}_R$ acts on $V_N^{\otimes n}$. We wish to evaluate the product of $\hat{P}_R$ and ψ (where $\psi \in S_n$). For the representation of S_n on $V^{\otimes n}$ recall that $(\sigma_1)^I_J(\sigma_2)^J_K = (\sigma_2 \cdot \sigma_1)^I_K$. The multiplication is given as

$$(\hat{P}_R)^I_J(\psi)^J_K = \frac{d_R}{n!}\sum_{\sigma \in S_n} \chi_R(\sigma)(\sigma)^I_J(\psi)^J_K.$$

By changing the summation variable from σ to τ, where

$$\begin{aligned}(\sigma)^I_J(\psi)^J_K &= (\psi)^I_J(\tau)^J_K \\ \Rightarrow (\sigma)^I_L &= (\psi)^I_J(\tau)^J_K(\psi^{-1})^K_L \\ &= (\psi^{-1}\tau\psi)^I_L,\end{aligned}$$

where the right action of the group was used. Then

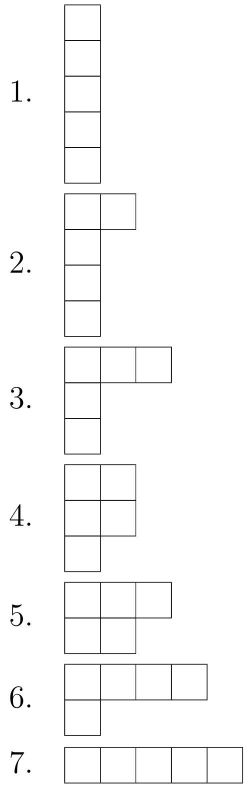
$$\begin{aligned}(\hat{P}_R)^I_J(\psi)^J_K &= \frac{d_R}{n!}\sum_{\tau \in S_n} \chi_R(\psi^{-1}\tau\psi)(\psi)^I_Q(\tau)^Q_L(\psi^{-1})^L_J(\psi)^J_K \\ &= \frac{d_R}{n!}\sum_{\tau \in S_n} \chi_R(\psi^{-1}\tau\psi)(\psi)^I_Q(\tau)^Q_K \\ &= (\psi)^I_Q(\hat{P}_R)^Q_K.\end{aligned}$$

From this result, we conclude that $\hat{P}_R\psi = \psi\hat{P}_R$, so $\hat{P}_R$ and ψ commute with each other, i.e $[\hat{P}_R, \psi] = 0$.

It should be noted that conjugacy classes of the symmetric group correspond to the cycle structure. For example, the S_5 group has the following cycle structures

1. (1)(2)(3)(4)(5)
2. (1 2)(3)(4)(5)
3. (1 2 3)(4)(5)
4. (1 2)(3 4)(5)
5. (1 2 3)(4 5)
6. (1 2 3 4)(5)
7. (1 2 3 4 5)

and they correspond to the following Young diagrams



From this correspondence, we conclude that Young diagrams also label conjugacy classes!

2.3.3 Schur Polynomials

We will now introduce the Schur polynomial [12],[13] which is given as

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)(\sigma)^I_J (Z)^J_I,$$

where Z is a $N \times N$ complex matrix. Consider for example $R = \square\square\square$ and $n = 3$. In this case

$$\begin{aligned}
\chi_{\square\square\square}(Z) &= \frac{1}{3!}(\chi_{\square\square\square}(\mathbb{1}) Z^{i_1}_{i_i} Z^{i_2}_{i_2} Z^{i_3}_{i_3} + \chi_{\square\square\square}((1\ 2)) Z^{i_1}_{i_2} Z^{i_2}_{i_1} Z^{i_3}_{i_3} + \chi_{\square\square\square}((1\ 3)) Z^{i_1}_{i_3} Z^{i_2}_{i_2} Z^{i_3}_{i_1} \\
&\quad + \chi_{\square\square\square}((2\ 3)) Z^{i_1}_{i_1} Z^{i_2}_{i_3} Z^{i_3}_{i_2} + \chi_{\square\square\square}((1\ 2\ 3)) Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} + \chi_{\square\square\square}((1\ 3\ 2)) Z^{i_1}_{i_3} Z^{i_2}_{i_1} Z^{i_3}_{i_2}) \\
&= \frac{1}{6}(Z^{i_1}_{i_i} Z^{i_2}_{i_2} Z^{i_3}_{i_3} + Z^{i_1}_{i_2} Z^{i_2}_{i_1} Z^{i_3}_{i_3} + Z^{i_1}_{i_3} Z^{i_2}_{i_2} Z^{i_3}_{i_1} + Z^{i_1}_{i_1} Z^{i_2}_{i_3} Z^{i_3}_{i_2} + Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} + Z^{i_1}_{i_3} Z^{i_2}_{i_1} Z^{i_3}_{i_2}) \\
&= \frac{1}{6}(\operatorname{Tr}(Z)^3 + 3 \operatorname{Tr}(Z^2) \operatorname{Tr}(Z) + 2 \operatorname{Tr}(Z^3)).
\end{aligned}$$

The relation (for $n = 3$) between the class, the permutation σ and the trace $\operatorname{Tr}(\sigma Z^{\otimes 3})$ is given below

Class	σ	$\operatorname{Tr}(\sigma Z^{\otimes 3})$
1^3	$\mathbb{1}$	$\operatorname{Tr}(Z)^3$
2 1	$(1\ 2), (1\ 3), (2\ 3)$	$\operatorname{Tr}(Z)\operatorname{Tr}(Z^2)$
3	$(1\ 2\ 3), (1\ 3\ 2)$	$\operatorname{Tr}(Z^3)$

We thus conclude from this that conjugacy classes of the symmetric group correspond to trace (physical) operators. Recall that taking a trace produces a gauge invariant operator. We can rewrite the Schur Polynomial as [12],[13]

$$\chi_R(Z) = \frac{1}{d_R}\left[\frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)(\sigma)^I_J(Z)^J_I\right] = \frac{1}{d_R}\,\mathrm{Tr}(\hat{P}_R Z^{\otimes n}).$$

We now wish to evaluate the 2-point function of the Schur polynomial. The correlator is given by [12],[13]

$$\langle\chi_R(Z)\chi_S^\dagger(Z)\rangle = \frac{1}{d_R d_S}(\hat{P}_R)^I_J(\hat{P}_S)^K_L\langle(Z^{\otimes n})^J_I(Z^{\dagger^{\otimes n}})^L_K\rangle,$$

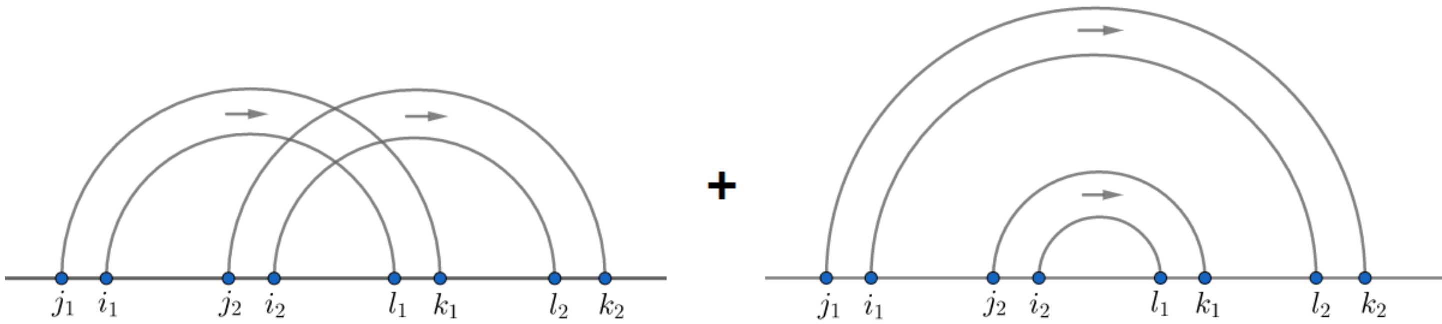
where $(Z^{\otimes n})^I_J = Z^{i_1}_{j_1}\dots Z^{i_n}_{j_n}$ and

$$\langle(Z^{\otimes n})^J_I(Z^{\dagger^{\otimes n}})^L_K\rangle = \sum_{\sigma\in S_n}(\sigma^{-1})^J_K(\sigma)^L_I,$$

and the summation in this case is over all ribbons graphs which is $n!$ diagrams labelled by elements $\sigma\in S_n$. To illustrate this statement, let's take $n=2$ for example. In this case

$$\begin{aligned}\langle Z^{j_1}_{i_1}Z^{j_2}_{i_2}Z^{\dagger l_1}_{k_1}Z^{\dagger l_2}_{k_2}\rangle &= (\mathbb{1})^J_K(\mathbb{1})^L_I + ((1\ 2))^J_K((1\ 2))^L_I\\ &= \delta^{j_1}_{k_1}\delta^{j_2}_{k_2}\delta^{l_1}_{i_1}\delta^{l_2}_{i_2} + \delta^{j_1}_{k_2}\delta^{j_2}_{k_1}\delta^{l_1}_{i_2}\delta^{l_2}_{i_1},\end{aligned}$$

which is indeed the result illustrated in the pairing below to form ribbon diagrams:



Therefore, the 2-point correlator of the Schur polynomials is [12],[13]

$$\begin{aligned}\langle\chi_R(Z)\chi_S^\dagger(Z)\rangle &= \frac{1}{d_R d_S}\sum_{\sigma\in S_n}(\hat{P}_R)^I_J(\hat{P}_S)^K_L(\sigma^{-1})^J_K(\sigma)^L_I\\
&= \frac{1}{d_R d_S}\sum_{\sigma\in S_n}\mathrm{Tr}\left(\hat{P}_R\sigma^{-1}\hat{P}_S\sigma\right)\\
&= \frac{1}{d_R d_S}\sum_{\sigma\in S_n}\mathrm{Tr}\left(\hat{P}_R\hat{P}_S\right),\qquad \because [\hat{P}_S,\sigma]=0\ \&\ \sigma^{-1}\cdot\sigma=\mathbb{1}\\
&= \frac{n!}{d_R d_S}\mathrm{Tr}\left(\hat{P}_R\hat{P}_S\right)\\
&= \frac{n!\delta_{RS}}{d_R d_S}\mathrm{Tr}\left(\hat{P}_R\right),\qquad \because \hat{P}_R\hat{P}_S=\delta_{RS}\hat{P}_R\\
&= \frac{n!\delta_{RS}}{d_R d_S}d_R\mathrm{Dim}_R,\qquad \because \mathrm{Tr}\,(\hat{P}_R)=d_R\mathrm{Dim}_R\\
&= \frac{n!\delta_{RS}}{d_R}\mathrm{Dim}_R\\
&= \delta_{RS}\frac{n!}{\left(\frac{n!}{\mathrm{hooks}_R}\right)}\frac{f_R}{\mathrm{hooks}_R}\\
&= \delta_{RS}f_R,\end{aligned}$$

where $d_R = \frac{n!}{\text{hooks}_R}$ and $\text{Dim}_R = \frac{f_R}{\text{hooks}_R}$, where f_R denotes the product of the weights of boxes in the Young diagram R.

2.4 Two matrices Z and Y

Now, we will consider not just a single matrix Z but two matrices Z and Y, and we use the following results:

$$\langle Z_{ij} Z^\dagger_{kl} \rangle = \delta_{il}\delta_{jk}, \quad \langle Y_{ij} Y^\dagger_{kl} \rangle = \delta_{il}\delta_{jk}, \quad \langle Y_{ij} Z_{kl} \rangle = 0, \quad \langle Y_{ij} Z^\dagger_{kl} \rangle = 0.$$

Extremal correlation functions are correlation functions of operators constructed from a single matrix Z. Extremal correlation functions do not receive quantum corrections. In general, the Z,Y correlators do receive quantum corrections. Suppose we have two Z's and two Y's. The observables we can construct are

$$\text{Tr}(Z^2Y^2),\ \ \text{Tr}(ZYZY),\ \ \text{Tr}(Z^2Y)\,\text{Tr}(Y),\ \ \text{Tr}(Y^2Z)\,\text{Tr}(Z),\ \ \text{Tr}(Z^2)\,\text{Tr}(Y^2),\ \ \text{Tr}(ZY)^2,$$
$$\text{Tr}(Z^2)\,\text{Tr}(Y)^2,\ \ \text{Tr}(Y^2)\,\text{Tr}(Z)^2,\ \ \text{Tr}(ZY)\,\text{Tr}(Z)\,\text{Tr}(Y)\ \ \&\ \ \text{Tr}(Z)^2\,\text{Tr}(Y)^2,$$

whereas if we have four Z's the observables we find are

$$\text{Tr}(Z^4),\ \ \text{Tr}(Z^3)\,\text{Tr}(Z),\ \ \text{Tr}(Z^2)\,\text{Tr}(Z^2),\ \ \text{Tr}(Z^2)\,\text{Tr}(Z)^2\ \ \&\ \ \text{Tr}(Z)^4.$$

Thus, we conclude there will be more observables for two matrices than there would be for a single matrix. This is because Z and Y do not commute, i.e. $ZY \neq YZ$, in general. For operators constructed from nZ's and mY's, we will work in $V_N^{\otimes n+m}$. The tensor product of these matrices is

$$(Z^{\otimes n} Y^{\otimes m})^I_J \equiv Z^{i_1}_{j_1} \dots Z^{i_n}_{j_n} Y^{i_{n+1}}_{j_{n+1}} \dots Y^{i_{n+m}}_{j_{n+m}}.$$

The Bose symmetry associated to these matrices is given by

$$(\sigma^{-1})^I_J (Z^{\otimes n} Y^{\otimes m})^J_K (\sigma)^K_L = (Z^{\otimes n} Y^{\otimes m})^I_L, \quad \sigma \in S_n \times S_m.$$

In the general case, the observables are

$$\begin{aligned}\text{Tr}(\rho Z^{\otimes n} Y^{\otimes m}) &= \text{Tr}(\rho\sigma^{-1} Z^{\otimes n} Y^{\otimes m} \sigma)\ ,\ \rho \in S_{n+m}\\ &= \text{Tr}(\sigma\rho\sigma^{-1} Z^{\otimes n} Y^{\otimes m})\\ &= (\sigma^{-1}\rho\sigma)^I_J (Z^{\otimes n} Y^{\otimes m})^J_I,\end{aligned}$$

which demonstrates that ρ and $\sigma^{-1}\rho\sigma$ correspond to the same physical observable for any $\sigma \in S_n \times S_m$. We say that ρ and τ are *restricted conjugate* if and only if

$$\rho = \sigma^{-1}\tau\sigma$$

for some $\sigma \in S_n \times S_m$. We will now prove the statement that "restricted conjugate" is an equivalence relation:
Proof:

- Reflexivity: $\rho = \mathbb{1}^{-1}\rho\mathbb{1}\ ,\ \mathbb{1} \in S_n \times S_m$.
- Symmetry: $\rho = \sigma^{-1}\tau\sigma \Rightarrow \tau = \sigma\rho\sigma^{-1} = (\sigma^{-1})^{-1}\rho\sigma^{-1}\ ,\ \sigma^{-1} \in S_n \times S_m$.
- Transitivity: $\rho = \sigma_1^{-1}\tau\sigma_1$ and $\tau = \sigma_2^{-1}\beta\sigma_2$, then

$$\rho = \sigma_1^{-1}\tau\sigma_1 = \sigma_1^{-1}\sigma_2^{-1}\beta\sigma_2\sigma_1 = (\sigma_2\sigma_1)^{-1}\beta\sigma_2\sigma_1,$$

  with $\sigma_2 \cdot \sigma_1 \in S_n \times S_m$. □

Therefore restricted conjugate is an equivalence relation and thus we have a notion of *restricted conjugacy classes*. It is a highly non-trivial fact that the number of restricted conjugacy classes is equal to the number of independent gauge invariant operators that can be defined [14].

2.4.1 Intertwining map

An intertwining map is a generalised projection operator in $V^{\otimes n+m}$. It has many of the properties of a projector ($P_A P_B = \delta_{AB} P_{\tilde{A}}$ with A possibly standing for a collection of indices), but does not in general square to itself. They are important in the study of (restricted) Schur polynomials.

Recall for the one matrix observables, we had

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) (\sigma)^I_J (Z)^J_I = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R \operatorname{Tr}(\sigma Z^{\otimes n}) = \operatorname{Tr}(\hat{P}_R Z^{\otimes n}),$$

where $\hat{P}_R = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \sigma$ and σ is defined using the previously introduced action on $V^{\otimes n+m}$. Also recall that for the one matrix problem, the correlators are

$$\langle (Z^{\otimes n})^I_J (Z^{\dagger \otimes n})^K_L \rangle = \sum_{\sigma \in S_n} (\sigma^{-1})^I_L (\sigma)^K_J$$

and

$$\langle \chi_R(Z) \chi^\dagger_S(Z) \rangle = \frac{n!}{d_R d_S} \operatorname{Tr}(\hat{P}_R \hat{P}_S).$$

To obtain this last result, we used the defining properties of the projection operator $\hat{P}_R$. Introducing well chosen projection operators has produced an enormous simplification in the computation of correlation functions.

We now wish to introduce a "projection operator" for the case of two matrices. We have used quotes because we will not introduce a projection operator, but rather we will make use of an intertwining map. We denote this operator as $P_{R,(r,s)\alpha\beta}$ where $R \vdash n+m$ and labels irreps of S_{n+m}. In addition, $r \vdash n$, $s \vdash m$ and (r,s) labels irreps of $S_n \times S_m$. (r,s) is an irrep obtained by restricting irrep R of S_{n+m} to its subgroup $S_n \times S_m$. After restriction, (r,s) may appear more than once. The labels α,β resolve the different copies. Consider $R = \yng(3,2,1)$ for which $\dim_{\yng(3,2,1)} = \frac{6!}{45} = 16$. Now remove m (which we chose to be equal to 3) boxes such that you are still left with a valid Young diagram. If boxes that are removed share sides, we must not disconnect them. Boxes that don't share sides must not be connected. After removing the $m = 3$ boxes, what remains must be a valid Young diagram. There are three possibilities, that we write as ($r \vdash 3$, removed boxes) as follows

$$(\yng(2,1), \square\ \square\ \square), (\yng(1,1,1), \yng(2,1)), (\yng(3), \yng(2,1)).$$

The three disconnected boxes can be arranged into irreps of S_3 as follows

$$\begin{aligned} \square \otimes \square \otimes \square &= \square \otimes (\yng(2) \oplus \yng(1,1)) \\ &= \yng(3) \oplus 2\yng(2,1) \oplus \yng(1,1,1). \end{aligned}$$

Thus, upon restricting to $S_3 \times S_3 \subset S_6$ we find that R returns the following irreps

$$(\yng(2,1), \yng(3)) \oplus (\yng(2,1), \yng(2,1)) \oplus (\yng(2,1), \yng(2,1)) \oplus (\yng(2,1), \yng(1,1,1)) \oplus (\yng(1,1,1), \yng(2,1)) \oplus (\yng(3), \yng(2,1)).$$

We can easily check that the sum of the dimensions of these $S_3 \times S_3$ irreps matches the dimension of our original S_6 irrep as follows

$$\begin{aligned} &\dim_{(\yng(2,1), \yng(3))} + \dim_{(\yng(2,1), \yng(2,1))} + \dim_{(\yng(2,1), \yng(2,1))} + \dim_{(\yng(2,1), \yng(1,1,1))} + \dim_{(\yng(1,1,1), \yng(2,1))} + \dim_{(\yng(3), \yng(2,1))} \\ &= ((2 \times 1) + (2 \times 2) + (2 \times 2) + (2 \times 1)) + (1 \times 2) + (1 \times 2) \\ &= 2 + 4 + 4 + 2 + 2 + 2 \\ &= 16, \end{aligned}$$

Note that there are two copies of $(\yng(2,1), \yng(2,1))$. The Littlewood-Richardson number counts this multiplicity. In our case $g_{\yng(2,1)\,\yng(2,1)\,\yng(3,2,1)} = 2$. The Littlewood-Richardson number g_{rst} is an integer. It counts how many times t appears in the tensor product $r \otimes s$, with r, s and t understood as $U(N)$ irreps. We need to introduce one more concept: $\chi_{R,(r,s)\alpha\beta}(\sigma)$. This is the *restricted character* and it is given by

$$\chi_{R,(r,s)\alpha\beta}(\sigma) = \sum_{i=1}^{d_r d_s} \langle R,(r,s)\alpha; i|\Gamma_R(\sigma)|R,(r,s)\beta; i\rangle,$$

where $\sigma \in S_{n+m}$, i labels states, $R,(r,s)\alpha$ and $R,(r,s)\beta$ denote representations of $S_n \times S_m$. $P_{R,(r,s)\alpha\beta}$ is now given by

$$P_{R,(r,s)\alpha\beta} \equiv \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\sigma,$$

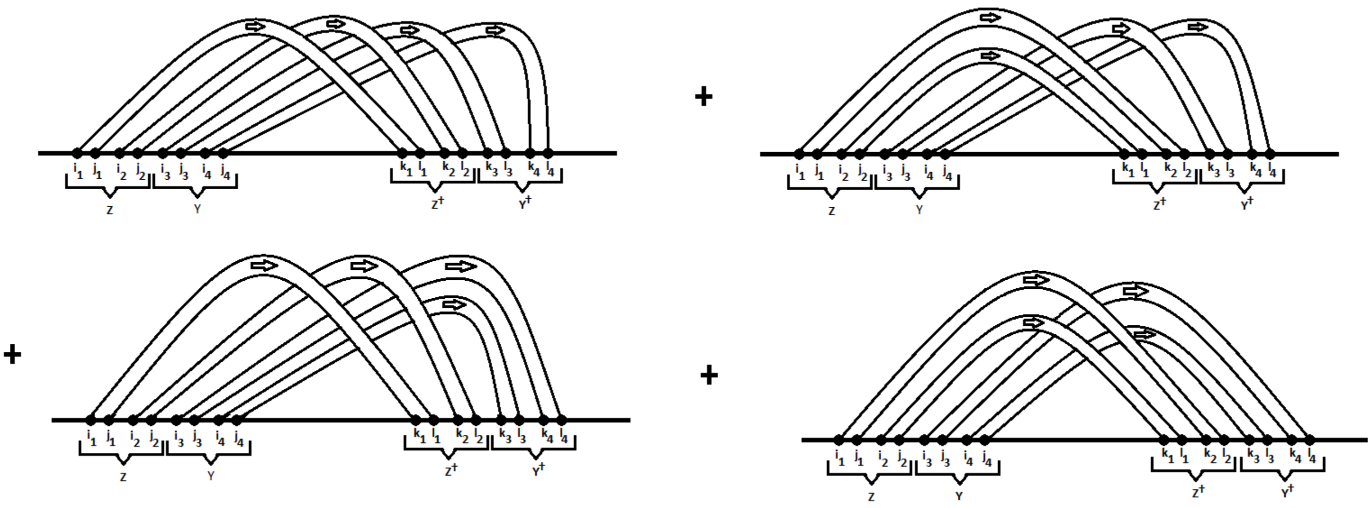
where σ is defined using the familiar action on $V^{\otimes n+m}$. $P_{R,(r,s)\alpha\beta}$ is called an *intertwining map* and it maps you from the α to the β copy of (r,s). For 2 matrices Z and Y, the 2-point function is given as

$$\langle (Z^{\otimes n} Y^{\otimes m})^I_J (Z^{\dagger \otimes n} Y^{\dagger \otimes m})^K_L \rangle = \sum_{\sigma \in S_n \times S_m} (\sigma)^I_L (\sigma^{-1})^K_J.$$

To illustrate this statement, take $n = 2$ and $m = 2$ for example. A simple application of Wick's theorem confirms

$$\begin{aligned}\langle Z^{i_1}_{j_1} Z^{i_2}_{j_2} Y^{i_3}_{j_3} Y^{i_4}_{j_4} Z^{\dagger k_1}_{l_1} Z^{\dagger k_2}_{l_2} Y^{\dagger k_3}_{l_3} Y^{\dagger k_4}_{l_4} \rangle &= (\mathbb{1})^I_L (\mathbb{1})^K_J + ((1\ 2))^I_L ((1\ 2))^K_J \\ &+ ((3\ 4))^I_L ((3\ 4))^K_J + ((1\ 2)(3\ 4))^I_L ((1\ 2)(3\ 4))^K_J \\ &= \delta^{i_1}_{l_1} \delta^{i_2}_{l_2} \delta^{i_3}_{l_3} \delta^{i_4}_{l_4} \delta^{j_1}_{k_1} \delta^{j_2}_{k_2} \delta^{j_3}_{k_3} \delta^{j_4}_{k_4} + \delta^{i_1}_{l_2} \delta^{i_2}_{l_1} \delta^{i_3}_{l_3} \delta^{i_4}_{l_4} \delta^{j_1}_{k_2} \delta^{j_2}_{k_1} \delta^{j_3}_{k_3} \delta^{j_4}_{k_4} \\ &+ \delta^{i_1}_{l_1} \delta^{i_2}_{l_2} \delta^{i_3}_{l_4} \delta^{i_4}_{l_3} \delta^{j_1}_{k_1} \delta^{j_2}_{k_2} \delta^{j_3}_{k_4} \delta^{j_4}_{k_3} + \delta^{i_1}_{l_2} \delta^{i_2}_{l_1} \delta^{i_3}_{l_4} \delta^{i_4}_{l_3} \delta^{j_1}_{k_2} \delta^{j_2}_{k_1} \delta^{j_3}_{k_4} \delta^{j_4}_{k_3},\end{aligned}$$

which is indeed the result obtained using ribbon diagrams:



The restricted Schur polynomial for 2 matrices Z and Y is given as [14],[15]

$$\chi_{R,(r,s)\alpha\beta}(Z,Y) \equiv \mathrm{Tr}(P_{R,(r,s)\alpha\beta} Z^{\otimes n} Y^{\otimes m}),$$

and the 2-point function is given as [14],[15]

$$
\begin{aligned}
\langle \chi_{R,(r,s)\alpha\beta}(Z,Y)\chi_{T,(t,u)\gamma\delta}(Z^\dagger,Y^\dagger)\rangle &= \langle \mathrm{Tr}(P_{R,(r,s)\alpha\beta}Z^{\otimes n}Y^{\otimes m})\,\mathrm{Tr}(P_{T,(t,u)\gamma\delta}Z^{\dagger^{\otimes n}}Y^{\dagger^{\otimes m}})\rangle \\
&= (P_{R,(r,s)\alpha\beta})^I_J(P_{T,(t,u)\gamma\delta})^K_L\langle (Z^{\otimes n}Y^{\otimes m})^J_I(Z^{\dagger^{\otimes n}}Y^{\dagger^{\otimes m}})^L_K\rangle \\
&= \sum_{\sigma\in S_n\times S_m}(P_{R,(r,s)\alpha\beta})^I_J(P_{T,(t,u)\gamma\delta})^K_L(\sigma)^J_K(\sigma^{-1})^L_I \\
&= \sum_{\sigma\in S_n\times S_m}\mathrm{Tr}(P_{R,(r,s)\alpha\beta}\ \sigma P_{T,(t,u)\gamma\delta}\ \sigma^{-1}) \\
&= n!m!\ \mathrm{Tr}(P_{R,(r,s)\alpha\beta}P_{T,(t,u)\gamma\delta}).
\end{aligned}
$$

We should note that the intertwining map $P_{R,(r,s)\alpha\beta}$ has properties that are similar to the properties of a projection operator. These properties are [14],[15]:

- $P_{R,(r,s)\alpha\beta}\cdot P_{T,(t,u)\gamma\delta} = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\beta\gamma}\# P_{R,(r,s)\alpha\delta}$
- $P_{R,(r,s)\alpha\alpha}\cdot P_{R,(r,s)\alpha\alpha} = \# P_{R,(r,s)\alpha\alpha}$
- $[P_{R,(r,s)\alpha\beta},\sigma] = 0, \quad \sigma\in S_n\times S_m$
- $\Gamma_{(r,s)\alpha}(\sigma)P_{R,(r,s)\alpha\beta} = P_{R,(r,s)\alpha\beta}\Gamma_{(r,s)\beta}(\sigma), \quad \sigma\in S_n\times S_m.$

Using these properties together with the value of the trace of $P_{R,(r,s)\alpha\beta}$ we learn that [14],[15]

$$
\langle \chi_{R,(r,s)\alpha\beta}(Z,Y)\chi^\dagger_{T,(t,u)\gamma\delta}(Z,Y)\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma}\frac{f_R\,\mathrm{hooks}_R}{\mathrm{hooks}_r\ \mathrm{hooks}_s}.
$$

2.5 Action of the One Loop Dilatation Operator in the *SU(3)* sector

2.5.1 Restricted Schur Polynomials

The goal of this subsection is to compute the anomalous dimensions for a class of operators in the $SU(3)$ sector in $\mathcal{N}=4$ SYM theory. To achieve this we will compute the matrix elements of the one loop dilatation operator by acting on the restricted Schur polynomial basis in the large N limit.

There are 6 hermitian adjoint scalars ϕ_i that transform as a vector of $SO(6)$. From these fields we introduce the 3 complex Higgs fields

$$
X=\phi_1+i\phi_2 \qquad Y=\phi_3+i\phi_4 \qquad Z=\phi_5+i\phi_6.
$$

The operators we work with are built from p X fields, m Y fields and n Z fields. We are interested in the case p,m,n scale as N in the large N limit, holding $p+m\ll n$ and $\frac{p}{m}\sim 1$. The $\frac{1}{2}$-BPS sector of this theory corresponds to taking $p=m=0$ with $n\neq 0$. We construct the restricted Schur polynomial from these 3 gauge invariant, complex scalars as

$$
\chi_{R,(r,s,t)\vec{\mu}\vec{\nu}} = \frac{1}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}\left(\Gamma^R(\sigma)\right)X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\times
$$
$$
\times Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.
$$

Here (r,s,t) labels an irrep of $S_n\times S_m\times S_p$ and $\vec{\mu}$, $\vec{\nu}$ are again multiplicity labels.

2.5.2 Constructing the Intertwining Map

We can rewrite the restricted trace as follows

$$\mathrm{Tr}_{(t,u,v)\vec{\mu}\vec{\nu}}(\cdots) = \mathrm{Tr}_T(P_{(t,u,v)\vec{\mu}\vec{\nu}}\cdots). \tag{2}$$

The intertwining map $P_{(t,u,v)\vec{\mu}\vec{\nu}}$ has an important role to play in what is to follow and we wish to emphasise this importance by going through the steps on how one constructs it. $T \vdash p+m+n$ labels a Young diagram with $p+m+n$ boxes, $t \vdash n$ labels a Young diagram with n boxes, $u \vdash m$ labels a Young diagram with m boxes, $v \vdash p$ labels a Young diagram with p boxes, $\vec{\mu}$ and $\vec{\nu}$ are multiplicity labels that label the copies of (u, v) obtained upon restricting to $S_n \times S_m \times S_p$.

The intertwining map is a matrix; it has both a row label and a column label. Below are the steps required to build the intertwining map:

1. Start from Young diagram T. Remove p boxes in any desired order such that

   (i) every time a box is removed there is a valid Young diagram left as a result and

   (ii) we remove p_i boxes from row i.

   Collect the p_i into a vector $\vec{p}$. The boxes that are removed are labelled; the first box removed is labelled box 1, the second box removed is labelled box 2, and so on. By performing this procedure in this fashion we end up with a partly labelled Young diagram T.

2. Now remove m boxes in any desired order such that

   (i) every time a box is removed there is a valid Young diagram left as a result and

   (ii) we remove m_i boxes from row i.

   Collect the m_i into a vector $\vec{m}$. Once again, as above, the boxes that are removed are labelled; the first box removed is labelled box $p+1$, the second box removed is labelled box $p+2$, and so on. We end up again with a partly labelled Young diagram T.

3. Collect the vectors with first p boxes into an irrep v of S_p; label the multiplicities of this irrep with ν_1.

4. For each state in a given S_p irrep specified by both v and ν, one has all possible labellings of the next m boxes. Collect these into vectors in an irrep u of S_m; label the multiplicities of this irrep with ν_2.

What we gain as a result of this exercise is a set of vectors labelled with two irreps $v \vdash p$, $u \vdash m$ each with a multiplicity label ν_1 and ν_2, and two state labels, a and b, one for each state $|(v,\nu_1)a;(u,\nu_2)b\rangle$. The unlabelled boxes define vectors that belong to a unique irrep t of S_n. To make this part explicitly clear, we can write our state as $|(v,\nu_1)a;(u,\nu_2)b;(t)c\rangle$. As a consequence of this, the intertwining map can be written as

$$P_{(t,u,v)\vec{\mu}\vec{\nu}} = \sum_{a,b,c} |(v,\mu_1),a;(u,\mu_2),b;(t),c\rangle\langle(v,\nu_1),a;(u,\nu_2),b;(t),c|.$$

Since the S_n, S_m and S_p actions commute, our state can be written as

$$|(v,\nu_1),a;(u,\nu_2),b;(t),c\rangle = |(v,\nu_1),a\rangle \otimes |(u,\nu_2),b\rangle \otimes |(t),c\rangle,$$

where $\otimes$ is the usual tensor product on a vector space. Following from this, the intertwining operators can be written as a tensor product

$$\begin{aligned}P_{(t,u,v)\vec{\mu}\vec{\nu}} &= \sum_a |(v,\mu_1)a\rangle\langle(v,\nu_1)a| \otimes \sum_b |(u,\mu_2)b\rangle\langle(u,\nu_2)b| \otimes \sum_c |(t)c\rangle\langle(t)c| \\ &\equiv p_{v\mu_1\nu_1} \otimes p_{u\mu_2\nu_2} \otimes \mathbb{1}_t,\end{aligned}$$

where $\mathbb{1}_t$ is a genuine projector.

The two point function of the restricted Schur polynomials is [15]

$$\langle \chi_{R,(r,s,t)\vec{\mu}\vec{\nu}} \chi^{\dagger}_{T,(w,x,y)\vec{\beta}\vec{\alpha}} \rangle = \frac{f_R \,\mathrm{hooks}_R}{\mathrm{hooks}_r \,\mathrm{hooks}_s \,\mathrm{hooks}_t} \delta_{RT}\delta_{rw}\delta_{sx}\delta_{ty}\delta_{\vec{\mu}\vec{\beta}}\delta_{\vec{\nu}\vec{\alpha}}. \tag{3}$$

Operators normalised to have a unit two point function $\hat{O}_{R,(r,s,t)\vec{\mu}\vec{\nu}}$ are related to the restricted Schur polynomials $\chi_{R,(r,s,t)\vec{\mu}\vec{\nu}}$ as

$$\hat{O}_{R,(r,s,t)\vec{\mu}\vec{\nu}} = \sqrt{\frac{\mathrm{hooks}_r \,\mathrm{hooks}_s \,\mathrm{hooks}_t}{f_R \,\mathrm{hooks}_R}} \chi_{R,(r,s,t)\vec{\mu}\vec{\nu}}. \tag{4}$$

2.6 Dilatation Operator

The complete one loop dilatation operator has been given in [16]. For our purposes, it is much simpler to start with the one loop dilatation operator in the $SO(6)$ sector which was computed in [17]. Using this starting point, we act on an operator constructed using the three complex matrices X, Y and Z. The result is

$$\begin{aligned}D &= -g_{\mathrm{YM}}^2 \,\mathrm{Tr}\left([Y,Z]\left[\frac{d}{dY},\frac{d}{dZ}\right] + [X,Z]\left[\frac{d}{dX},\frac{d}{dZ}\right] + [Y,X]\left[\frac{d}{dY},\frac{d}{dX}\right]\right) \\ &= -g_{\mathrm{YM}}^2 \,\mathrm{Tr}\left(D^{(1)} + D^{(2)} + D^{(3)}\right),\end{aligned}$$

where

$$\begin{aligned}D^{(1)} &\equiv [Y,Z]\left[\frac{d}{dY},\frac{d}{dZ}\right], \\ D^{(2)} &\equiv [X,Z]\left[\frac{d}{dX},\frac{d}{dZ}\right], \\ D^{(3)} &\equiv [Y,X]\left[\frac{d}{dY},\frac{d}{dX}\right].\end{aligned}$$

The definition for the trace is determined by $U(N)$ invariance. Spelling out the indices we have

$$\mathrm{Tr}\left([Y,X]\left[\frac{d}{dY},\frac{d}{dX}\right]\right) = (Y^i_l X^l_j - X^i_l Y^l_j)\left(\frac{d}{dY^k_j}\frac{d}{dX^i_k} - \frac{d}{dX^k_j}\frac{d}{dY^i_k}\right).$$

2.7 Action of D on Restricted Schur Polynomials

We will first consider the subleading term which mixes the Y and X fields (the action of $D^{(3)}$). This corrects the leading terms (mixing of the X and Z fields and the Y and Z fields). The steps we follow are:

1. Evaluate the derivatives, which generate some Kronecker delta functions.

2. Permute indices to place the delta functions into the first slot.
3. Convert to a sum over $S_{n+m+p-1}$ using the Kronecker deltas. This is done with the help of the reduction rule (derived in [18] by making use of the Jucys-Murphy elements).
4. Express the trace $\mathrm{Tr}(\sigma X^{\otimes p} Y^{\otimes m} Z^{\otimes n})$ in terms of a linear combination of restricted Schur polynomials.
5. Sum over $S_{n+m+p-1}$ using the fundamental orthogonality relation.
6. Consider the large N limit (make use of the displaced corners approximation).
7. Rewrite the result in terms of normalised restricted Schur operators.

These steps have been explicitly listed to guide the reader through the rather technical discussion that follows. It may be useful to refer back to this list as the discussion proceeds.

2.7.1 Derivatives

First, let's consider the explicit action of the derivatives. Only the final results will be given whereas the explicit calculations can be found in Appendix A. The first pair evaluates to

$$\begin{aligned}&\frac{d}{dY^k_j}\frac{d}{dX^i_k}\left(X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right)\\&=\left(\delta^{i_{p+1}}_{i_{\sigma(1)}}\delta^{i_1}_{i}\delta^{j}_{i_{\sigma(p+1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\right.\\&\quad\left.\times\, Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^{j}_{i_{\sigma(p+m)}}\delta^{i_{p+m}}_{i_{\sigma(p)}}\delta^{i_p}_{i}\right)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.\end{aligned}$$

Then

$$\begin{aligned}&[Y,X]^i_j\frac{d}{dY^k_j}\frac{d}{dX^i_k}\left(X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right)\\&=\left([Y,X]^{i_1}_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+[Y,X]^{i_p}_{i_{\sigma(p+m)}}X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\right.\\&\quad\left.\times\, Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^{i_{p+m}}_{i_{\sigma(p)}}\right)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.\end{aligned}$$

The second pair evaluates to

$$\begin{aligned}&-\frac{d}{dX^k_j}\frac{d}{dY^i_k}\left(X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right)\\&=-\left(\delta^{j}_{i_{\sigma(1)}}\delta^{i_1}_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_{i}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\right.\\&\quad\left.\times\, Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^{i_p}_{i_{\sigma(p+m)}}\delta^{i_{p+m}}_{i}\delta^{j}_{i_{\sigma(p)}}\right)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.\end{aligned}$$

Then

$$\begin{aligned}&-[Y,X]^i_j\frac{d}{dX^k_j}\frac{d}{dY^i_k}\left(X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right)\\&=-\left([Y,X]^{i_{p+1}}_{i_{\sigma(1)}}\delta^{i_1}_{i_{\sigma(p+1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+[Y,X]^{i_{p+m}}_{i_{\sigma(p)}}X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\right.\\&\quad\left.\times\, Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^{i_p}_{i_{\sigma(p+m)}}\right)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.\end{aligned}$$

2.7.2 Permutations

One of the terms appearing above is $[Y,X]^{i_1}_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}$. This mixes slots 1 and $p+1$. We'd like to rewrite this so that slots are not being mixed. To do this, make a change of variables $\sigma \to \tau(1,p+1)$, $\tau \in S_{n+m+p}$, where the permutation to the right of τ acts on subscript indices. This change of variables yields

$$[Y,X]^{i_1}_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_{i_{\sigma(1)}} \to [Y,X]^{i_1}_{i_{\tau(1)}}\delta^{i_{p+1}}_{i_{\tau(p+1)}}.$$

Repeating this process for all of the terms appearing we can extract out a Kronecker delta function in the first slot of each term.

To illustrate this, consider the term that has its delta function in the second slot. We make a change of variables such that $\sigma \to (p+2,p+1)\tau(p+2,p+1)$ which results in

$$[Y,X]^{i_{p+2}}_{i_{\sigma(1)}}\delta^{i_1}_{i_{\sigma(p+2)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}} \to [Y,X]^{i_{p+1}}_{i_{\tau(1)}}\delta^{i_1}_{i_{\tau(p+1)}}Y^{i_{p+2}}_{i_{\tau(p+2)}}.$$

We then perform the second change $\tau \to \psi(1,p+1)$, $\psi \in S_{n+m+p}$, to get

$$[Y,X]^{i_{p+1}}_{i_{\tau(1)}}\delta^{i_1}_{i_{\tau(p+1)}}Y^{i_{p+2}}_{i_{\tau(p+2)}} \to [Y,X]^{i_{p+1}}_{i_{\psi(p+1)}}\delta^{i_1}_{i_{\psi(1)}}Y^{i_{p+2}}_{i_{\psi(p+2)}}.$$

Since the first and second terms agree they can be added together. Performing the necessary shifts for all remaining terms and summing, we find an overall contribution of pm due to p X terms and m Y terms. Thus we find

$$\begin{aligned}
&\Bigg[\Bigg([Y,X]^{i_1}_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} + \cdots + [Y,X]^{i_p}_{i_{\sigma(p+m)}}X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\\
&\quad\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^{i_{p+m}}_{i_{\sigma(p)}}\Bigg)\Bigg]Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&= pm[Y,X]^{i_1}_{i_{\sigma(1)}}\delta^{i_{p+1}}_{i_{\sigma(p+1)}}\left[X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right].
\end{aligned}$$

Very similar manipulations give

$$\begin{aligned}
&-\Bigg[\Bigg([Y,X]^{i_{p+1}}_{i_{\sigma(1)}}\delta^{i_1}_{i_{\sigma(p+1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} + \cdots + [Y,X]^{i_{p+m}}_{i_{\sigma(p)}}X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\\
&\quad\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^{i_p}_{i_{\sigma(p+m)}}\Bigg)\Bigg]Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&= -pm[Y,X]^{i_{p+1}}_{i_{\sigma(p+1)}}\delta^{i_1}_{i_{\sigma(1)}}\left[X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right].
\end{aligned}$$

Keep in mind that all changes made to the permutation also change the argument of the restricted character.

Applying these results to evaluate the action of $D^{(3)}$, we obtain

$$D^{(3)}\left[\frac{1}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma))X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right]$$

$$= [Y,X]^i_j\left(\frac{d}{dY^k_j}\frac{d}{dX^i_k}-\frac{d}{dX^k_j}\frac{d}{dY^i_k}\right)\frac{1}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma))X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\times$$
$$\times Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}$$

$$=\frac{mp}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma))\left(\delta^{i_{p+1}}_{i_{\sigma(1)}}[Y,X]^{i_1}_{i_{\sigma(p+1)}}-\delta^{i_1}_{i_{\sigma(p+1)}}[Y,X]^{i_{p+1}}_{i_{\sigma(1)}}\right)\times$$
$$\times X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}$$

$$=\frac{mp}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}\left(\Gamma^R([(1,p+1),\sigma])\right)\delta^{i_1}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}\times$$
$$\times[Y,X]^{i_{p+1}}_{i_{\sigma(p+1)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.$$

2.7.3 Reduction rule

We will now discuss the reduction rule in detail. Introduce the notation $\rho_i = (1,i)\sigma$ and rewrite the above sum as a sum over the subgroup $S_{n+m+p-1}$ and its cosets. The $S_{n+m+p-1}\subset S_{n+m+p}$ subgroup is obtained by retaining the elements that hold 1 fixed, i.e. $\sigma(1) = 1$. The coset decomposition is

$$S_{n+m+p} = \mathbb{1}S_{n+m+p-1}\oplus(1\ 2)S_{n+m+p-1}\oplus(1\ 3)S_{n+m+p-1}\oplus\cdots\oplus(1,p+m+n)S_{n+m+p-1}.$$

To make the discussion clear, consider the example of the symmetric group S_3:

$$S_3 = \{\mathbb{1},(1\ 2),(1\ 3),(1\ 2\ 3),(1\ 3\ 2)\},$$

which can be written as

$$S_3 = \mathbb{1}S_2\oplus(1\ 2)S_2\oplus(1\ 3)S_2,$$

where we have chosen the subgroup

$$S_2 = \{\mathbb{1},(2\ 3)\}.$$

After using the coset decomposition, our result becomes

$$
\begin{aligned}
&\frac{mp}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}\left(\Gamma^R([(1,p+1),\sigma])\right)\delta^{i_1}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}\times\\
&\qquad\times[Y,X]^{i_{p+1}}_{i_{\sigma(p+1)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&=\frac{mp}{p!m!n!}\sum_{\psi\in S_{n+m+p-1}}\Bigg[\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}\left(\Gamma^R([(1,p+1),(1\ 2)\psi])\right)+\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}\left(\Gamma^R([(1,p+1),(1\ 3)\psi])\right)+\cdots+\\
&\quad+\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}\left(\Gamma^R([(1,p+1),(1,p+m+n)\psi])\right)\Bigg]\delta^{i_1}_{i_{\psi(1)}}X^{i_2}_{i_{\psi(2)}}\cdots X^{i_p}_{i_{\psi(p)}}[Y,X]^{i_{p+1}}_{i_{\psi(p+1)}}Y^{i_{p+2}}_{i_{\psi(p+2)}}\cdots Y^{i_{p+m}}_{i_{\psi(p+m)}}\times\\
&\qquad\times Z^{i_{p+m+1}}_{i_{\psi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\psi(p+m+n)}}\\
&=\frac{mp}{n!m!p!}\sum_{\psi\in S_{n+m+p-1}}\sum_{i=1}^{n+m+p}\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}\left(\Gamma^R([(1,p+1),\rho_i])\right)\delta^{i_1}_{i_{\rho_i(1)}}X^{i_2}_{i_{\rho_i(2)}}\cdots X^{i_p}_{i_{\rho_i(p)}}\times\\
&\qquad\times[Y,X]^{i_{p+1}}_{i_{\rho_i(p+1)}}Y^{i_{p+2}}_{i_{\rho_i(p+2)}}\cdots Y^{i_{p+m}}_{i_{\rho_i(p+m)}}Z^{i_{p+m+1}}_{i_{\rho_i(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\rho_i(p+m+n)}}\\
&=\frac{mp}{n!m!p!}\sum_{\psi\in S_{n+m+p-1}}\mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}\left(\Gamma^R\big([(1,p+1),\{N+\sum_{i=2}^{n+m+p}(i,1)\}\psi]\big)\right)X^{i_2}_{i_{\rho_i(2)}}\cdots X^{i_p}_{i_{\rho_i(p)}}\times\\
&\qquad\times[Y,X]^{i_{p+1}}_{i_{\rho_i(p+1)}}Y^{i_{p+2}}_{i_{\rho_i(p+2)}}\cdots Y^{i_{p+m}}_{i_{\rho_i(p+m)}}Z^{i_{p+m+1}}_{i_{\rho_i(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\rho_i(p+m+n)}},
\end{aligned}
$$

where

$$
\sum_{i=1}^{n+m+p}\rho_i=\sum_{i=1}^{n+m+p}(1,i)\psi=\left[(1,1)+\sum_{i=2}^{n+m+p}(1,i)\right]\psi=\left[N+\sum_{i=2}^{n+m+p}(1,i)\right]\psi.
$$

Note that

$$
N+\sum_{i=2}^{p+m+n}(1,i)=N+C_2^{p+m+n}-C_2^{p+m+n-1},
$$

where C_2^{p+m+n} ($C_2^{p+m+n-1}$) is the quadratic Casimir for the symmetric group S_{p+m+n} ($S_{p+m+n-1}$). The difference between the two Casimirs is called the Jucys-Murphy element. Acting with C_2^{p+m+n} on the state $|R,a\rangle$ results in

$$
C_2^{p+m+n}|R,a\rangle=\left(\sum_{i=1}\frac{r_i(r_i-1)}{2}-\sum_{j=1}\frac{c_j(c_j-1)}{2}\right)|R,a\rangle,
$$

where r_i is the length of row i and c_j is the length of column j of Young diagram R. a in $|R,a\rangle$ labels all the different Young-Yamanouchi states you can obtain from R. For example consider the action of the Casimir C_2^{p+m+n} on the Young diagram $\Big|\,\yng(2,1),a\rangle$:

$$
\begin{aligned}
C_2^{p+m+n}\Big|\,\yng(2,1),a\rangle&=\left(\frac{2(2-1)}{2}+\frac{1(1-1)}{2}-\frac{2(2-1)}{2}-\frac{1(1-1)}{2}\right)\Big|\,\yng(2,1),a\rangle\\
&=0\Big|\,\yng(2,1),a\rangle.
\end{aligned}
$$

In what follows, we make use of the branching rule

$$
R=\bigoplus_{R'}R',
$$

where R' is any Young diagram subduced from Young diagram R by removing a single box from R. A simple example of this is provided by the Young diagram :

 $=$  $\oplus$ ,

where the first (second) Young diagram on the right hand side is obtained by removing a single box in the second (first) row of the Young diagram on the left hand side. Using this branching rule we find

$$\begin{aligned}
\Gamma^R\left(N + C_2^{p+m+n} - C_2^{p+m+n-1}\right) &= \left(N + \Gamma^R\left(C_2^{p+m+n}\right)\right) I_{d_R\times d_R} - \Gamma^R\left(C_2^{p+m+n-1}\right) I_{d_{R'}\times d_{R'}} \\
&= \bigoplus_{R'} (N+\lambda_R) I_{d_{R'}\times d_{R'}} - \bigoplus_{R'} \lambda_{R'} I_{d_{R'}\times d_{R'}} \\
&= \bigoplus_{R'} (N+\lambda_R - \lambda_{R'}) I_{d_{R'}\times d_{R'}},
\end{aligned}$$

where $I_{d_R\times d_R}$ ($I_{d_{R'}\times d_{R'}}$) is the identity matrix of irrep R (R') with dimension $d_R\times d_R$ ($d_{R'}\times d_{R'}$), λ_R (λ'_R) is the eigenvalue of C_2^{p+m+n} in irrep R ($C_2^{p+m+n-1}$ in irrep R'), and $\Gamma^R(N\cdot\sigma) = N\cdot\Gamma_R(\sigma)$. It is simple to verify that

$$c_{RR'} \equiv N + \lambda_R - \lambda_{R'} = N + \left[\sum_{i=1} \frac{r_i(r_i-1)}{2} - \sum_{j=1} \frac{c_j(c_j-1)}{2}\right] - \left[\sum_{i'=1} \frac{r_{i'}(r_{i'}-1)}{2} - \sum_{j'=1} \frac{c_{j'}(c_{j'}-1)}{2}\right],$$

where $c_{RR'}$ is the factor of the box removed from R to obtain R' and $\lambda_R - \lambda_{R'}$ is the content of this box. Thus we can write the action of $D^{(3)}$ on the restricted Schur polynomial basis as

$$\begin{aligned}
D^{(3)}\chi_{R,(r,s,t)\vec{\mu}\vec{\nu}} = \frac{mp}{n!m!p!} \sum_{R'} \sum_{\phi\in S_{n+m+p-1}} c_{RR'} \operatorname{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}\left(\Gamma^R([(1,p+1),\phi])\right) X^{i_2}_{i_{\phi(2)}} \cdots X^{i_p}_{i_{\phi(p)}} \times \\
\times [Y,X]^{i_{p+1}}_{i_{\phi(p+1)}} Y^{i_{p+2}}_{i_{\phi(p+2)}} \cdots Y^{i_{p+m}}_{i_{\phi(p+m)}} Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}} \cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}}.
\end{aligned}$$

2.7.4 Expressing $\operatorname{Tr}(\sigma X^{\otimes p} Y^{\otimes m} Z^{\otimes n})$ as a linear combination of restricted Schur polynomials

The product which appears in the expression above can be rewritten as

$$X^{i_2}_{i_{\phi(2)}} \cdots X^{i_p}_{i_{\phi(p)}} (YX)^{i_{p+1}}_{i_{\phi(p+1)}} Y^{i_{p+2}}_{i_{\phi(p+2)}} \cdots Y^{i_{p+m}}_{i_{\phi(p+m)}} Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}} \cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}} = \operatorname{Tr}(\mathbb{1} \otimes X^{p-1} \otimes (YX) \otimes Y^{m-1} \otimes Z^{\otimes n})$$

where the identity sits in the first slot. We can further obtain

$$\begin{aligned}
& X^{i_2}_{i_{\phi(2)}} \cdots X^{i_p}_{i_{\phi(p)}} Y^{i_{p+1}}_{i_{\phi(1)}} X^{i_1}_{i_{\phi(p+1)}} Y^{i_{p+2}}_{i_{\phi(p+2)}} \cdots Y^{i_{p+m}}_{i_{\phi(p+m)}} Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}} \cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}}, \qquad \text{where } \phi(1) = 1 \\
&= (1,p+1) X^{i_1}_{i_{\phi(1)}} \cdots X^{i_p}_{i_{\phi(p)}} Y^{i_{p+1}}_{i_{\phi(p+1)}} \cdots Y^{i_{p+m}}_{i_{\phi(p+m)}} Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}} \cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}} \\
&= \operatorname{Tr}\left(\phi(1,p+1) X^{\otimes p} Y^{\otimes m} Z^{\otimes n}\right),
\end{aligned}$$

where in the first to second line a permutation shift $\phi \to \phi(1,p+1)$ was performed. The trace is over the space $V^{\otimes n+m+p-1}$. The action of $D^{(3)}$ is now

$$D^{(3)}\chi_{R,(r,s,t)\vec{\mu}\vec{\nu}} = \frac{mp}{n!m!p!} \sum_{R'} \sum_{\phi\in S_{n+m+p-1}} c_{RR'} \operatorname{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}}\left(\Gamma^R([(1,p+1),\phi])\right) \operatorname{Tr}\left(\phi(1,p+1) X^{\otimes p} Y^{\otimes m} Z^{\otimes n}\right).$$

By performing the Fourier transform [14]

$$\operatorname{Tr}(\sigma X^{\otimes p} Y^{\otimes m} Z^{\otimes n}) = \sum_{T,(t,u,v)\vec{\mu}\vec{\nu}} \frac{d_T n!m!p!}{d_t d_u d_v (m+n+p)!} \operatorname{Tr}_{(t,u,v)\vec{\mu}\vec{\nu}}\left(\Gamma^T(\sigma^{-1})\right) \chi_{T,(t,u,v)\vec{\nu}\vec{\mu}}(X,Y,Z),$$

we can rewrite the above expression as

$$D^{(3)}\chi_{R,(r,s,t)\vec{\mu}\vec{\nu}} = \sum_{T,(t,u,v)\vec{\mu}\vec{\nu}} \sum_{R'} \sum_{\phi\in S_{n+m+p-1}} \frac{d_T mp\, c_{RR'}}{d_t d_u d_v (m+n+p)!} \mathrm{Tr}_{(r,s,t)\vec{\mu}\vec{\nu}} \left([\Gamma^R(1,p+1), \Gamma^{R'}(\phi)] \right) \times$$

$$\times \mathrm{Tr}_{(t,u,v)\vec{\mu}\vec{\nu}} \left([\Gamma^T(1,p+1), \Gamma^{T'}(\phi^{-1})] \right) \chi_{T,(t,u,v)\vec{\nu}\vec{\mu}}(X,Y,Z). \tag{5}$$

This expresses the action of $D^{(3)}$ on a restricted Schur polynomial as a linear combination of restricted Schur polynomials.

2.7.5 Fundamental Orthogonality Relation and sum over $S_{n+m+p-1}$

We will use the fundamental orthogonality relation

$$\sum_{\phi\in S_{n+m+p-1}} [\Gamma^{R'}(\phi)]_{ij} [\Gamma^{T'}(\phi^{-1})]_{ab} = \frac{(n+m+p-1)!}{d_{R'}} \delta_{R'T'} [I_{R'T'}]_{ib} [I_{T'R'}]_{aj}, \tag{6}$$

where $I_{R'T'}$ ($I_{T'R'}$) is the intertwiner that maps from the Young diagram R' (T'), subduced from Young diagram R (T), to Young diagram T' (R'), subduced from Young diagram T (R) to perform the sum over ϕ in (5). The intertwiners obey

$$\Gamma^{R'}(\sigma) I_{R'T'} = I_{R'T'} \Gamma^{T'}(\sigma),$$

where $\sigma \in S_{n+m+p-1}$. By making use of (2) we can rewrite our result as

$$D^{(3)}\chi_{R,(r,s,t)\vec{\mu}\vec{\nu}} = \sum_{T,(t,u,v)\vec{\mu}\vec{\nu}} \sum_{R'} \sum_{\phi\in S_{n+m+p-1}} \frac{d_T mp\, c_{RR'}}{d_t d_u d_v (m+n+p)!} \mathrm{Tr} \left(P_{R,(r,s,t)\vec{\mu}\vec{\nu}} [\Gamma^R(1,p+1), \Gamma^{R'}(\phi)] \right) \times$$

$$\times \mathrm{Tr} \left(P_{T,(t,u,v)\vec{\mu}\vec{\nu}} [\Gamma^T(1,p+1), \Gamma^{T'}(\phi^{-1})] \right) \chi_{T,(t,u,v)\vec{\nu}\vec{\mu}}(X,Y,Z),$$

which comprises of 4 terms. Let us for the time being only focus on the first term, i.e.

$$\begin{aligned}
&\sum_{T,(t,u,v)\vec{\mu}\vec{\nu}}\sum_{R'}\sum_{\phi\in S_{n+m+p-1}}\frac{d_T mp\, c_{RR'}}{d_t d_u d_v (m+n+p)!}\,\mathrm{Tr}\left(P_{R,(r,s,t)\vec{\mu}\vec{\nu}}\Gamma^R(1,p+1)\Gamma^{R'}(\phi)\right)\times\\
&\times\mathrm{Tr}\left(P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\Gamma^T(1,p+1)\Gamma^{T'}(\phi^{-1})\right)\chi_{T,(t,u,v)\vec{\nu}\vec{\mu}}(X,Y,Z)\\
&=\sum_{T,(t,u,v)\vec{\mu}\vec{\nu}}\sum_{R'}\sum_{\phi\in S_{n+m+p-1}}\frac{d_T mp\, c_{RR'}}{d_t d_u d_v (m+n+p)!}\left(P_{R,(r,s,t)\vec{\mu}\vec{\nu}}\right)^a_b\left(\Gamma^R(1,p+1)\right)^b_c\left(\Gamma^{R'}(\phi)\right)^c_a\times\\
&\times\left(P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\right)^i_j\left(\Gamma^T(1,p+1)\right)^j_k\left(\Gamma^{T'}(\phi^{-1})\right)^k_i\chi_{T,(t,u,v)\vec{\nu}\vec{\mu}}(X,Y,Z)\\
&=\sum_{T,(t,u,v)\vec{\mu}\vec{\nu}}\sum_{R'}\left[\sum_{\phi\in S_{n+m+p-1}}\left(\Gamma^{R'}(\phi)\right)^c_a\left(\Gamma^{T'}(\phi^{-1})\right)^k_i\right]\frac{d_T mp\, c_{RR'}}{d_t d_u d_v (m+n+p)!}\left(P_{R,(r,s,t)\vec{\mu}\vec{\nu}}\right)^a_b\left(\Gamma^R(1,p+1)\right)^b_c\times\\
&\times\left(P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\right)^i_j\left(\Gamma^T(1,p+1)\right)^j_k\chi_{T,(t,u,v)\vec{\nu}\vec{\mu}}(X,Y,Z)\\
&=\sum_{T,(t,u,v)\vec{\mu}\vec{\nu}}\sum_{R'}\frac{(n+m+p-1)!}{d_{R'}}\delta_{R'T'}\frac{d_T mp\, c_{RR'}}{d_t d_u d_v (m+n+p)!}\left(P_{R,(r,s,t)\vec{\mu}\vec{\nu}}\right)^a_b\left(\Gamma^R(1,p+1)\right)^b_c[I_{R'T'}]^c{}_i[I_{T'R'}]_a{}^k\times\\
&\times\left(P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\right)^i_j\left(\Gamma^T(1,p+1)\right)^j_k\chi_{T,(t,u,v)\vec{\nu}\vec{\mu}}(X,Y,Z)\\
&=\sum_{T,(t,u,v)\vec{\mu}\vec{\nu}}\sum_{R'}\frac{(n+m+p-1)!}{(m+n+p)!}\frac{d_T mp\, c_{RR'}}{d_{R'}d_t d_u d_v}\delta_{R'T'}[I_{T'R'}]_{ka}\left(P_{R,(r,s,t)\vec{\mu}\vec{\nu}}\right)_{ab}\left(\Gamma^R(1,p+1)\right)_{bc}[I_{R'T'}]_{ci}\times\\
&\times\left(P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\right)_{ij}\left(\Gamma^T(1,p+1)\right)_{jk}\chi_{T,(t,u,v)\vec{\nu}\vec{\mu}}(X,Y,Z)\\
&=\sum_{T,(t,u,v)\vec{\mu}\vec{\nu}}\sum_{R'}\frac{(n+m+p-1)!}{(m+n+p)!}\frac{d_T mp\, c_{RR'}}{d_{R'}d_t d_u d_v}\delta_{R'T'}\,\mathrm{Tr}\Big(I_{T'R'}P_{R,(r,s,t)\vec{\mu}\vec{\nu}}\Gamma^R(1,p+1)I_{R'T'}\times\\
&\times P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\Gamma^T(1,p+1)\Big)\chi_{T,(t,u,v)\vec{\nu}\vec{\mu}}(X,Y,Z),
\end{aligned}$$

where in the third to fourth line, the fundamental orthogonality relation (6) was used. Performing a similar procedure for the rest of the terms and summing, our result becomes

$$\begin{aligned}
D^{(3)}\chi_{R,(r,s,t)\vec{\mu}\vec{\nu}}&=\sum_{T,(t,u,v)\vec{\mu}\vec{\nu}}\sum_{R'}\frac{(n+m+p-1)!}{(m+n+p)!}\frac{d_T mp\, c_{RR'}}{d_{R'}d_t d_u d_v}\delta_{R'T'}\times\\
\times\,\mathrm{Tr}\Big(I_{T'R'}[P_{R,(r,s,t)\vec{\mu}\vec{\nu}},\Gamma^R(1,p+1)]&I_{R'T'}[P_{T,(t,u,v)\vec{\mu}\vec{\nu}},\Gamma^T(1,p+1)]\Big)\chi_{T,(t,u,v)\vec{\nu}\vec{\mu}}(X,Y,Z).
\end{aligned}$$

2.8 Exact One-Loop Dilatation Operator

This subsection is a pedagogical section to aid the reader to understand how one can determine the matrix elements discussed throughout this thesis. This subsection discusses the action of the exact one loop dilatation operator in the $SU(2)$ sector, which corresponds to the mixing of Y and Z fields. We illustrate the argument with examples. Our primary goal is to illustrate all the details of the mathematics developed so far in a concrete example. We will return to points 6 and 7 of the list at the start of subsection 2.7 once we have discussed this example.

The action of the one loop dilatation operator on the restricted Schur polynomial is given as

$$D\chi_{R,(r,s)}(Z,Y)=\sum_{T,(t,u)}M_{R,(r,s);T,(t,u)}\chi_{T,(t,u)}(Z,Y), \qquad (7)$$

where

$$M_{R,(r,s);T,(t,u)} = -g_{\rm YM}^2 \sum_{R'} \frac{c_{RR'} d_T(nm)}{d_{R'} d_t d_u (n+m)} \,{\rm Tr}(I_{R'T'}[\Gamma_T((n,n+2)), P_{T,(t,u)}] I_{T'R'}[\Gamma_R((n,n+2)), P_{R,(r,s)}]).$$

We are focusing on Young diagrams that have at most two columns.

There are two main pieces to calculate: the upfront factor to the left of the trace and the trace itself. Calculating these pieces and combining them together determines the matrix element, $M_{R,(r,s);T,(t,u)}$, for a selected $R,(r,s)$ (which we can think of as the row label) and $T,(t,u)$ (which we can think of as the column label).

There are no multiplicity labels in (7) since for Young diagrams with only two columns, each irrep (r,s) arises only once. To obtain the spectrum of anomalous dimensions, we consider the action of the dilatation operator on normalized operators. The two point function for restricted Schur polynomials is

$$\langle \chi_{R,(r,s)}(Z,Y) \chi^\dagger_{T,(t,u)}(Z,Y) \rangle = \delta_{R,(r,s);T,(t,u)} \frac{f_R \, {\rm hooks}_R}{{\rm hooks}_r \, {\rm hooks}_s}.$$

Therefore we can write

$$\chi_{R,(r,s)}(Z,Y) = \sqrt{\frac{f_R \, {\rm hooks}_R}{{\rm hooks}_r \, {\rm hooks}_s}} O_{R,(r,s)}(Z,Y),$$

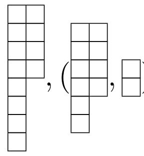
where $O_{R,(r,s)}(Z,Y)$ is a normalised operator. The restricted Schur expression and the two-point function expression before it are written without multiplicity labels. We can now write (7) as

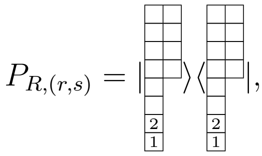
$$\begin{aligned}
D\chi_{R,(r,s)}(Z,Y) &= \sum_{T,(t,u)} M_{R,(r,s);T,(t,u)} \chi_{T,(t,u)}(Z,Y) \\
\Rightarrow D\sqrt{\frac{f_R \, {\rm hooks}_R}{{\rm hooks}_r \, {\rm hooks}_s}} O_{R,(r,s)}(Z,Y) &= \sum_{T,(t,u)} M_{R,(r,s);T,(t,u)} \sqrt{\frac{f_T \, {\rm hooks}_T}{{\rm hooks}_t \, {\rm hooks}_u}} O_{T,(t,u)}(Z,Y) \\
\Rightarrow DO_{R,(r,s)}(Z,Y) &= \sum_{T,(t,u)} M_{R,(r,s);T,(t,u)} \sqrt{\frac{{\rm hooks}_r \, {\rm hooks}_s}{f_R \, {\rm hooks}_R}} \sqrt{\frac{f_T \, {\rm hooks}_T}{{\rm hooks}_t \, {\rm hooks}_u}} O_{T,(t,u)}(Z,Y) \\
&= \sum_{T,(t,u)} M_{R,(r,s);T,(t,u)} \sqrt{\frac{f_T \, {\rm hooks}_T \, {\rm hooks}_r \, {\rm hooks}_s}{f_R \, {\rm hooks}_R \, {\rm hooks}_t \, {\rm hooks}_u}} O_{T,(t,u)}(Z,Y) \\
&\equiv \sum_{T,(t,u)} N_{R,(r,s);T,(t,u)} O_{T,(t,u)}(Z,Y), \qquad (8)
\end{aligned}$$

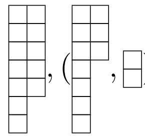
where

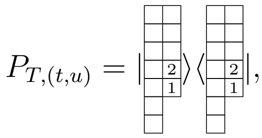
$$\begin{aligned}
N_{R,(r,s);T,(t,u)} &\equiv M_{R,(r,s);T,(t,u)} \sqrt{\frac{f_T \, {\rm hooks}_T \, {\rm hooks}_r \, {\rm hooks}_s}{f_R \, {\rm hooks}_R \, {\rm hooks}_t \, {\rm hooks}_u}} \\
&= -g_{\rm YM}^2 \sum_{R'} \frac{c_{RR'} d_T(nm)}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T \, {\rm hooks}_T \, {\rm hooks}_r \, {\rm hooks}_s}{f_R \, {\rm hooks}_R \, {\rm hooks}_t \, {\rm hooks}_u}} \times \\
&\times {\rm Tr}(I_{R'T'}[\Gamma_T((n,n+2)), P_{T,(t,u)}] I_{T'R'}[\Gamma_R((n,n+2)), P_{R,(r,s)}]).
\end{aligned}$$

Diagonalising the matrix $N_{R,(r,s);T,(t,u)}$ above gives the spectrum of anomalous dimensions.

We will now discuss the projection operators, intertwiners and the partially-labelled Young diagram states used to build them. Suppose we have the example $R,(r,s) =$ : the projection operator corresponding to these labels is



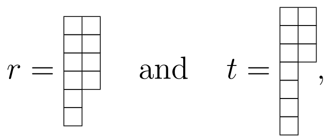
which we denote by $P_a(b_0, b_1)$. We can label the corresponding normalised operator as $O_a(b_0, b_1)$. The variable b_0 is the number of rows in Young diagram r of length 2 and b_1 is the number of rows in Young diagram r of length 1. We can also have, for example, $T,(t,u) =$ : the projection operator corresponding to these labels is



which we denote by $P_b(b_0, b_1)$. We can label the corresponding normalised operator as $O_b(b_0, b_1)$. The computation involves terms of the form

$$DO_a(b_0, b_1) = \dots + \#O_b(b_0 - 1, b_1 + 2) + \dots,$$

since

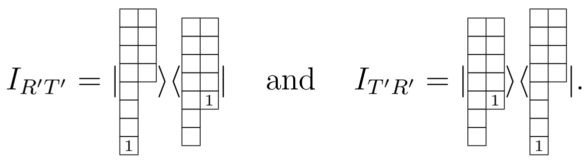


hence there is the following relationship between the labels for r and t

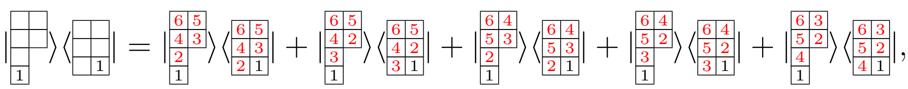
$$(b_0, b_1) = (\tilde{b}_0 + 1, \tilde{b}_1 - 2) \quad \text{or} \quad (\tilde{b}_0, \tilde{b}_1) = (b_0 - 1, b_1 + 2).$$

For the example we are working with $r \to (b_0, b_1) = (4, 2)$ and $t \to (\tilde{b}_0, \tilde{b}_1) = (b_0 - 1, b_1 + 2) = (3, 4)$.

Notice that if $R \neq T$ we must have $R' = T'$, for the intertwiners to be non-vanishing. To obtain R' from R we remove a box from column 1 and to obtain T' from T we remove a box from column 2. The corresponding intertwiners are

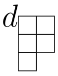


When labelling the remaining boxes one must ensure consistency of the bra and ket labels. In other words, if a ket-state has box 2 labelled in the first column then its partnered bra-state must also have box 2 labelled in the first column. This consistency with the labelling is done for all boxes that are labelled. An illustration of this is



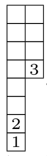
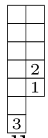
and we can check that the number of terms are correct by computing the dimension of the Young diagram part of the partially-labelled Young diagram that does not have its boxes labelled, i.e.

$$d_{\text{}} = \frac{5!}{1\cdot 3\cdot 1\cdot 4\cdot 2} = \frac{120}{24} = 5.$$



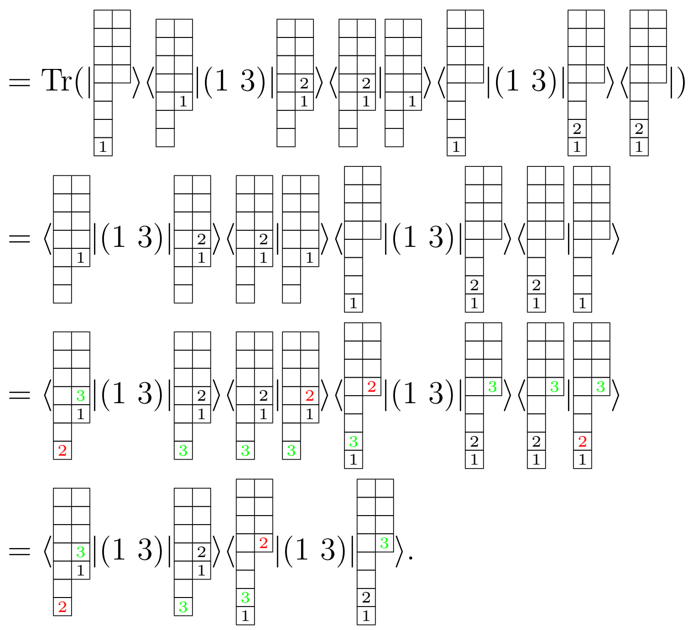
The trace sums over the space defined by Young diagram $R \to (r,s)$. In other words, Young diagrams r and s build Young diagram R; recall $r \vdash n$, $s \vdash 2$ and $R \vdash n+2$. We must consider the space spanned by the labelled boxes, s, and the space spanned by the unlabelled boxes, r. In order to have orthonormal states and thus have inner products of paired states yield a result of 1, we write

$$\begin{aligned}&\mathrm{Tr}_{R\to(r,s)}\left(I_{R'T'}[\Gamma_T((1\ 3)),P_{T,(t,u)}]I_{T'R'}[\Gamma_R((1\ 3)),P_{R,(r,s)}]\right) = \\ &\quad = d_{r'}\,\mathrm{Tr}\left(I_{R'T'}[\Gamma_T((1\ 3)),P_{T,(t,u)}]I_{T'R'}[\Gamma_R((1\ 3)),P_{R,(r,s)}]\right). \qquad (9)\end{aligned}$$

The reason the dimension factor above which is the dimension of Young diagram $r' =$ , is because we will label at most 3 boxes in Young diagram R. This is enough to evaluate the permutation $(1\ 3)$. Two boxes removed from R yields r and one box removed after that yields r'. One could ask why r' takes the form it does and not any other form: the answer is because the inner products that appear in computations arising later in this section force the from for r' given above. There is also the possibility of an inner product of paired states with $t' =$  coming up, but it is easy to see that $d_{r'} = d_{t'}$; we will stick to the label $d_{r'}$ since it is naturally connected to Young diagram R.

Since the trace contains 2 commutators, it has 4 terms in total. We can compute these 4 terms separately and then sum at the end. The **first term** is

$$\mathrm{Tr}\left(I_{R'T'}\Gamma_T((1\ 3))P_{T,(t,u)}I_{T'R'}\Gamma_R((1\ 3))P_{R,(r,s)}\right) =$$



In the second last line, the states have certain boxes colour-labelled to ensure the inner products do not vanish and that the bra-states that act on the ket-states after action by permutation

(1 3) are the correct paired state for the respective ket-state. The action of the representation matrix $\Gamma_R((1\ 3))$ or $\Gamma_T((1\ 3))$ (written in the short form "(1 3)") on the ket-states shown above is

$$
\begin{aligned}
(1\ 3)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&2\\\square&1\\\square&\\3&\end{smallmatrix}\right\rangle &= (1\ 2)(2\ 3)(1\ 2)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&2\\\square&1\\\square&\\3&\end{smallmatrix}\right\rangle \\
&= (1\ 2)(2\ 3)\Big[-\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&2\\\square&1\\\square&\\3&\end{smallmatrix}\right\rangle\Big] \\
&= -(1\ 2)\Big[\frac{1}{b_1+2}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&2\\\square&1\\\square&\\3&\end{smallmatrix}\right\rangle + \frac{\sqrt{(b_1+1)(b_1+3)}}{b_1+2}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&3\\\square&1\\\square&\\2&\end{smallmatrix}\right\rangle\Big] \\
&= -\frac{1}{b_1+2}\Big[(1\ 2)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&2\\\square&1\\\square&\\3&\end{smallmatrix}\right\rangle\Big] - \frac{\sqrt{(b_1+1)(b_1+3)}}{b_1+2}\Big[(1\ 2)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&3\\\square&1\\\square&\\2&\end{smallmatrix}\right\rangle\Big] \\
&= -\frac{1}{b_1+2}\Big[-\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&2\\\square&1\\\square&\\3&\end{smallmatrix}\right\rangle\Big] - \frac{\sqrt{(b_1+1)(b_1+3)}}{b_1+2}\Big[\frac{1}{b_1+1}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&3\\\square&1\\\square&\\2&\end{smallmatrix}\right\rangle + \frac{\sqrt{b_1(b_1+2)}}{b_1+1}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&3\\\square&2\\\square&\\1&\end{smallmatrix}\right\rangle\Big] \\
&= \frac{1}{b_1+2}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&2\\\square&1\\\square&\\3&\end{smallmatrix}\right\rangle - \frac{1}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&3\\\square&1\\\square&\\2&\end{smallmatrix}\right\rangle + \sqrt{\frac{b_1(b_1+3)}{(b_1+1)(b_1+2)}}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&3\\\square&2\\\square&\\1&\end{smallmatrix}\right\rangle,
\end{aligned}
$$

therefore

$$
\left\langle\begin{smallmatrix}\square&\square\\\square&\square\\\square&3\\\square&1\\\square&\\2&\end{smallmatrix}\right|(1\ 3)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&2\\\square&1\\\square&\\3&\end{smallmatrix}\right\rangle = -\frac{1}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}.
$$

We also have

$$
\begin{aligned}
(1\ 3)\left|\begin{smallmatrix} & 3\\ 2 & \\ 1 & \end{smallmatrix}\right\rangle &= (1\ 2)(2\ 3)(1\ 2)\left|\begin{smallmatrix} & 3\\ 2 & \\ 1 & \end{smallmatrix}\right\rangle \\
&= (1\ 2)(2\ 3)\Big[-\left|\begin{smallmatrix} & 3\\ 2 & \\ 1 & \end{smallmatrix}\right\rangle \Big] \\
&= -(1\ 2)\Big[-\frac{1}{b_1+2}\left|\begin{smallmatrix} & 3\\ 2 & \\ 1 & \end{smallmatrix}\right\rangle + \frac{\sqrt{(b_1+1)(b_1+3)}}{b_1+2}\left|\begin{smallmatrix} & 2\\ 3 & \\ 1 & \end{smallmatrix}\right\rangle \Big] \\
&= \frac{1}{b_1+2}\Big[(1\ 2)\left|\begin{smallmatrix} & 3\\ 2 & \\ 1 & \end{smallmatrix}\right\rangle\Big] - \frac{\sqrt{(b_1+1)(b_1+3)}}{b_1+2}\Big[(1\ 2)\left|\begin{smallmatrix} & 2\\ 3 & \\ 1 & \end{smallmatrix}\right\rangle\Big] \\
&= \frac{1}{b_1+2}\Big[-\left|\begin{smallmatrix} & 3\\ 2 & \\ 1 & \end{smallmatrix}\right\rangle\Big] - \frac{\sqrt{(b_1+1)(b_1+3)}}{b_1+2}\Big[-\frac{1}{b_1+3}\left|\begin{smallmatrix} & 2\\ 3 & \\ 1 & \end{smallmatrix}\right\rangle + \frac{\sqrt{(b_1+2)(b_1+4)}}{b_1+3}\left|\begin{smallmatrix} & 1\\ 3 & \\ 2 & \end{smallmatrix}\right\rangle\Big] \\
&= -\frac{1}{b_1+2}\left|\begin{smallmatrix} & 3\\ 2 & \\ 1 & \end{smallmatrix}\right\rangle + \frac{1}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}}\left|\begin{smallmatrix} & 2\\ 3 & \\ 1 & \end{smallmatrix}\right\rangle - \sqrt{\frac{(b_1+1)(b_1+4)}{(b_1+2)(b_1+3)}}\left|\begin{smallmatrix} & 1\\ 3 & \\ 2 & \end{smallmatrix}\right\rangle,
\end{aligned}
$$

so that

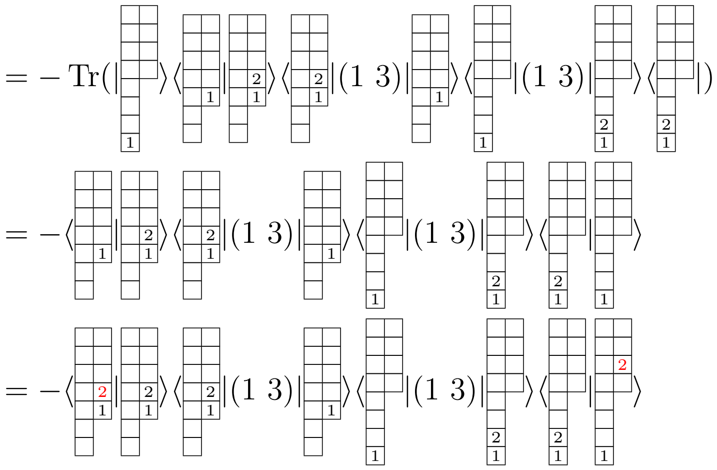
$$
\left\langle\begin{smallmatrix} & 2\\ 3 & \\ 1 & \end{smallmatrix}\right|(1\ 3)\left|\begin{smallmatrix} & 3\\ 2 & \\ 1 & \end{smallmatrix}\right\rangle = \frac{1}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}}.
$$

Finally, we have

$$
\begin{aligned}
\mathrm{Tr}\left(I_{R'T'}\Gamma_T((1\ 3))P_{T,(t,u)}I_{T'R'}\Gamma_R((1\ 3))P_{R,(r,s)}\right) &= \left\langle\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right|(1\ 3)\left|\begin{smallmatrix} & 2\\ & 1\\ 3 & \end{smallmatrix}\right\rangle\left\langle\begin{smallmatrix} & 2\\ 3 & \\ 1 & \end{smallmatrix}\right|(1\ 3)\left|\begin{smallmatrix} & 3\\ 2 & \\ 1 & \end{smallmatrix}\right\rangle \\
&= \left(-\frac{1}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}\right)\left(\frac{1}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}}\right) \\
&= -\frac{1}{(b_1+2)^2}.
\end{aligned}
$$

For the **second term** we have

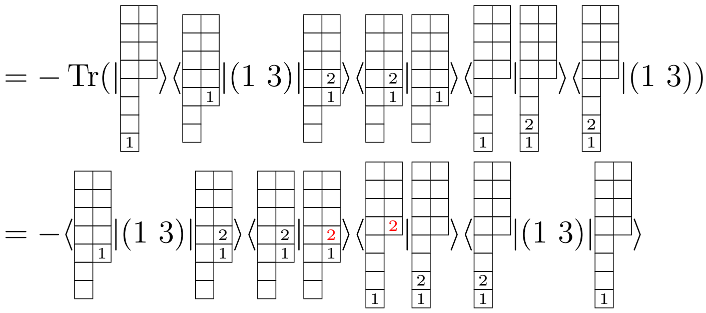
$$-\,\mathrm{Tr}\left(I_{R'T'}P_{T,(t,u)}\Gamma_T((1\ 3))I_{T'R'}\Gamma_R((1\ 3))P_{R,(r,s)}\right) =$$



$$= 0.$$

The second term vanishes because when we chose to ensure one of the inner products does not vanish, the other will vanish as a result; this is forced by consistency of the labelling of boxes. Note that it does not matter which inner product you choose to label initially (to ensure it does not vanish), the other inner product will always vanish. For the **third term** we have

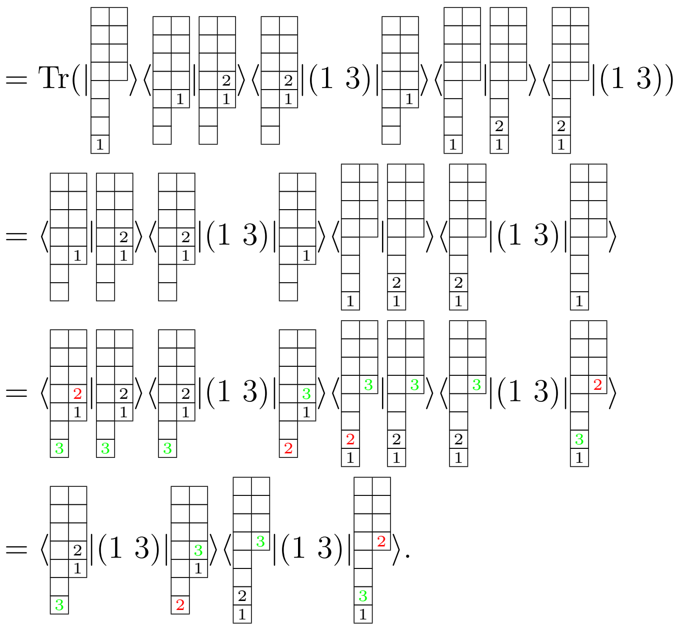
$$-\,\mathrm{Tr}\left(I_{R'T'}\Gamma_T((1\ 3))P_{T,(t,u)}I_{T'R'}P_{R,(r,s)}\Gamma_R((1\ 3))\right) =$$



$$= 0.$$

The third term vanishes by the same reasoning. Lastly, we have the **fourth term** given as

$$\mathrm{Tr}\left(I_{R'T'}P_{T,(t,u)}\Gamma_T((1\ 3))I_{T'R'}P_{R,(r,s)}\Gamma_R((1\ 3))\right) =$$



The action of the representation matrix $\Gamma_R((1\ 3))$ or $\Gamma_T((1\ 3))$ on the ket-states shown above is

$$
\begin{aligned}
(1\ 3)\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle &= (1\ 2)(2\ 3)(1\ 2)\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle \\
&= (1\ 2)(2\ 3)\Big(\frac{1}{b_1+1}\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle + \frac{\sqrt{b_1(b_1+2)}}{b_1+1}\left|\begin{smallmatrix} & 3\\ & 2\\ 1 & \end{smallmatrix}\right\rangle\Big) \\
&= \frac{1}{b_1+1}(1\ 2)\Big((2\ 3)\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle\Big) + \frac{\sqrt{b_1(b_1+2)}}{b_1+1}(1\ 2)\Big((2\ 3)\left|\begin{smallmatrix} & 3\\ & 2\\ 1 & \end{smallmatrix}\right\rangle\Big) \\
&= \frac{1}{b_1+1}(1\ 2)\Big(-\frac{1}{b_1+2}\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle + \frac{\sqrt{(b_1+1)(b_1+3)}}{b_1+2}\left|\begin{smallmatrix} & 2\\ & 1\\ 3 & \end{smallmatrix}\right\rangle\Big) + \frac{\sqrt{b_1(b_1+2)}}{b_1+1}(1\ 2)\Big(-\left|\begin{smallmatrix} & 3\\ & 2\\ 1 & \end{smallmatrix}\right\rangle\Big) \\
&= -\frac{1}{(b_1+1)(b_1+2)}\Big((1\ 2)\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle\Big) + \frac{1}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}\Big((1\ 2)\left|\begin{smallmatrix} & 2\\ & 1\\ 3 & \end{smallmatrix}\right\rangle\Big) - \frac{\sqrt{b_1(b_1+2)}}{b_1+1}\Big((1\ 2)\left|\begin{smallmatrix} & 3\\ & 2\\ 1 & \end{smallmatrix}\right\rangle\Big) \\
&= -\frac{1}{(b_1+1)(b_1+2)}\Big(-\frac{1}{b_1+1}\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle + \frac{\sqrt{b_1(b_1+2)}}{b_1+1}\left|\begin{smallmatrix} & 3\\ & 2\\ 1 & \end{smallmatrix}\right\rangle\Big) + \frac{1}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}\Big(-\left|\begin{smallmatrix} & 2\\ & 1\\ 3 & \end{smallmatrix}\right\rangle\Big) - \\
&\quad - \frac{\sqrt{b_1(b_1+2)}}{b_1+1}\Big(-\frac{1}{b_1+1}\left|\begin{smallmatrix} & 3\\ & 2\\ 1 & \end{smallmatrix}\right\rangle + \frac{\sqrt{b_1(b_1+2)}}{b_1+1}\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle\Big) \\
&= \frac{1}{(b_1+1)^2(b_1+2)}\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle - \frac{1}{(b_1+1)^2}\sqrt{\frac{b_1}{b_1+2}}\left|\begin{smallmatrix} & 3\\ & 2\\ 1 & \end{smallmatrix}\right\rangle - \frac{1}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}\left|\begin{smallmatrix} & 2\\ & 1\\ 3 & \end{smallmatrix}\right\rangle - \\
&\quad + \frac{\sqrt{b_1(b_1+2)}}{(b_1+1)^2}\left|\begin{smallmatrix} & 3\\ & 2\\ 1 & \end{smallmatrix}\right\rangle + \frac{b_1(b_1+2)}{(b_1+1)^2}\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle,
\end{aligned}
$$

so that

$$
\left\langle\begin{smallmatrix} & 2\\ & 1\\ 3 & \end{smallmatrix}\right|(1\ 3)\left|\begin{smallmatrix} & 3\\ & 1\\ 2 & \end{smallmatrix}\right\rangle = -\frac{1}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}.
$$

We also have

$$
\begin{aligned}
(1\ 3)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle &= (1\ 2)(2\ 3)(1\ 2)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle \\
&= (1\ 2)(2\ 3)\Big(-\frac{1}{b_1+3}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle + \frac{\sqrt{(b_1+2)(b_1+4)}}{b_1+3}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&1\\\square&\\\square&\\3&\\2&\end{smallmatrix}\right\rangle\Big) \\
&= -\frac{1}{b_1+3}(1\ 2)\Big((2\ 3)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle\Big) + \frac{\sqrt{(b_1+2)(b_1+4)}}{b_1+3}(1\ 2)\Big((2\ 3)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&1\\\square&\\\square&\\3&\\2&\end{smallmatrix}\right\rangle\Big) \\
&= -\frac{1}{b_1+3}(1\ 2)\Big(\frac{1}{b_1+2}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle + \frac{\sqrt{(b_1+1)(b_1+3)}}{b_1+2}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&3\\\square&\\\square&\\2&\\1&\end{smallmatrix}\right\rangle\Big) + \frac{\sqrt{(b_1+2)(b_1+4)}}{b_1+3}(1\ 2)\Big(-\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&1\\\square&\\\square&\\3&\\2&\end{smallmatrix}\right\rangle\Big) \\
&= -\frac{1}{(b_1+2)(b_1+3)}\Big((1\ 2)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle\Big) - \frac{1}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}}\Big((1\ 2)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&3\\\square&\\\square&\\2&\\1&\end{smallmatrix}\right\rangle\Big) - \frac{\sqrt{(b_1+2)(b_1+4)}}{b_1+3}\Big((1\ 2)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&1\\\square&\\\square&\\3&\\2&\end{smallmatrix}\right\rangle\Big) \\
&= -\frac{1}{(b_1+2)(b_1+3)}\Big(-\frac{1}{b_1+3}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle + \frac{\sqrt{(b_1+2)(b_1+4)}}{b_1+3}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&1\\\square&\\\square&\\3&\\2&\end{smallmatrix}\right\rangle\Big) - \frac{1}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}}\Big(-\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&3\\\square&\\\square&\\2&\\1&\end{smallmatrix}\right\rangle\Big) - \\
&\quad - \frac{\sqrt{(b_1+2)(b_1+4)}}{b_1+3}\Big(\frac{1}{b_1+3}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&1\\\square&\\\square&\\3&\\2&\end{smallmatrix}\right\rangle + \frac{\sqrt{(b_1+2)(b_1+4)}}{b_1+3}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle\Big) \\
&= \frac{1}{(b_1+2)(b_1+3)^2}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle - \frac{1}{(b_1+3)^2}\sqrt{\frac{b_1+4}{b_1+2}}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&1\\\square&\\\square&\\3&\\2&\end{smallmatrix}\right\rangle + \frac{1}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&3\\\square&\\\square&\\2&\\1&\end{smallmatrix}\right\rangle - \\
&\quad - \frac{\sqrt{(b_1+2)(b_1+4)}}{(b_1+3)^2}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&1\\\square&\\\square&\\3&\\2&\end{smallmatrix}\right\rangle - \frac{(b_1+2)(b_1+4)}{(b_1+3)^2}\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle,
\end{aligned}
$$

which implies

$$
\left\langle\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&3\\\square&\\\square&\\2&\\1&\end{smallmatrix}\right|(1\ 3)\left|\begin{smallmatrix}\square&\square\\\square&\square\\\square&\square\\\square&2\\\square&\\\square&\\3&\\1&\end{smallmatrix}\right\rangle = \frac{1}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}}.
$$

Finally, we have

$$\begin{aligned}
\operatorname{Tr}\left(I_{R'T'}P_{T,(t,u)}I_{T'R'}\Gamma_T((1\ 3))P_{R,(r,s)}\Gamma_R((1\ 3))\right) &= \langle \young(\ \ ,\ \ ,\ \ ,\ 2,\ 1,\ ,3) |(1\ 3)| \young(\ \ ,\ \ ,\ \ ,\ 3,\ 1,\ ,2) \rangle \langle \young(\ \ ,\ \ ,\ \ ,\ 3,\ ,\ ,2,1) |(1\ 3)| \young(\ \ ,\ \ ,\ \ ,\ 2,\ ,\ ,3,1) \rangle \\
&= \left(-\frac{1}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}\right)\left(\frac{1}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}}\right) \\
&= -\frac{1}{(b_1+2)^2}.
\end{aligned}$$

Adding these results together, we obtain

$$\begin{aligned}
&\operatorname{Tr}\left(I_{R'T'}[\Gamma_T((1\ 3)), P_{T,(t,u)}]I_{T'R'}[\Gamma_R((1\ 3)), P_{R,(r,s)}]\right) = \\
&= \text{1st term} + \text{2nd term} + \text{3rd term} + \text{4th term} \\
&= -\frac{1}{(b_1+2)^2} - 0 - 0 - \frac{1}{(b_1+2)^2} \\
&= -\frac{2}{(b_1+2)^2}.
\end{aligned}$$

The dimension $d_{r'}$ appearing in (9) is given by

$$d_{r'} = \frac{(n-1)!}{\text{hooks}_{r'}} = \frac{(n-1)!}{\left[\frac{(b_0-1)!((b_0-1)+(b_1+1)+1)!}{(b_1+1)+1}\right]} = \frac{(n-1)!}{\left[\frac{(b_0-1)!(b_0+b_1+1)!}{b_1+2}\right]} = \frac{(n-1)!(b_1+2)}{(b_0-1)!(b_0+b_1+1)!}.$$

Thus we can write

$$d_{r'}\operatorname{Tr}\left(I_{R'T'}[\Gamma_T((1\ 3)), P_{T,(t,u)}]I_{T'R'}[\Gamma_R((1\ 3)), P_{R,(r,s)}]\right) = -\frac{2d_{r'}}{(b_1+2)^2} = -\frac{2}{(b_1+2)^2}\left[\frac{(n-1)!(b_1+2)}{(b_0-1)!(b_0+b_1+1)!}\right]$$

At this point we consider the upfront factor

$$-g_{\text{YM}}^2\sum_{R'}\frac{c_{RR'}d_T(nm)}{d_{R'}d_td_u(n+m)}\sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s}{f_R\,\text{hooks}_R\,\text{hooks}_t\,\text{hooks}_u}} \to -g_{\text{YM}}^2\frac{c_{RR'}d_T(nm)}{d_{R'}d_td_u(n+m)}\sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s}{f_R\,\text{hooks}_R\,\text{hooks}_t\,\text{hooks}_u}},$$

since there is only one way to remove a single box from R to obtain R' in the example we are considering. We first consider the expression $\sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s}{f_R\,\text{hooks}_R\,\text{hooks}_t\,\text{hooks}_u}}$. For the case $R = \yng(2,2,2,2,1,1,1,1)$,

$T = \yng(2,2,2,2,2,1,1)$, $r = \yng(2,2,2,2,1,1)$, $t = \yng(2,2,2,1,1,1,1)$ and $s = u = \yng(1)$, the factors are

$$\begin{aligned}
f_R &= N^2(N+1)(N-1)^2(N-2)^2(N-3)(N-4)(N-5)(N-6)(N-7) \\
f_T &= N^2(N+1)(N-1)^2(N-2)^2(N-3)^2(N-4)(N-5)(N-6) \\
\Rightarrow \frac{f_T}{f_R} &= \frac{N^2(N+1)(N-1)^2(N-2)^2(N-3)^2(N-4)(N-5)(N-6)}{N^2(N+1)(N-1)^2(N-2)^2(N-3)(N-4)(N-5)(N-6)(N-7)} = \frac{N-3}{N-7},
\end{aligned}$$

which can be written in terms of b_0, b_1 as

$$\frac{f_T}{f_R} = \frac{N-b_0+1}{N-b_0-b_1-1}.$$

The hook lengths of the Young diagrams studied are

$$\text{hook lengths of } r = \begin{array}{|c|c|}\hline 7 & 4 \\ \hline 6 & 3 \\ \hline 5 & 2 \\ \hline 4 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},\ \text{hook lengths of } t = \begin{array}{|c|c|}\hline 8 & 3 \\ \hline 7 & 2 \\ \hline 6 & 1 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},\ \text{hook lengths of } R = \begin{array}{|c|c|}\hline 9 & 4 \\ \hline 8 & 3 \\ \hline 7 & 2 \\ \hline 6 & 1 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},$$

$$\text{hook lengths of } T = \begin{array}{|c|c|}\hline 8 & 5 \\ \hline 7 & 4 \\ \hline 6 & 3 \\ \hline 5 & 2 \\ \hline 4 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},\ \text{hook lengths of } s = \text{hook lengths of } u = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$$

and, thus, the hooks (the product of hook lengths) for these Young diagrams are computed, in terms of b_0 and b_1, to be

$$\text{hooks}_r = \frac{b_0!(b_0+b_1+1)!}{b_1+1}, \quad \text{hooks}_t = \frac{(b_0-1)!((b_0-1)+(b_1+2)+1)!}{(b_1+2)+1} = \frac{(b_0-1)!(b_0+b_1+2)!}{b_1+3},$$
$$\text{hooks}_s = \text{hooks}_u = 2 \Rightarrow \frac{\text{hooks}_s}{\text{hooks}_u} = 1, \quad \text{hooks}_R = \frac{b_0!(b_0+(b_1+2)+1)!}{(b_1+2)+1} = \frac{b_0!(b_0+b_1+3)!}{b_1+3},$$
$$\text{hooks}_T = \frac{(b_0+1)!((b_0+1)+b_1+1)!}{b_1+1} = \frac{(b_0+1)!(b_0+b_1+2)!}{b_1+1}.$$

From this we find

$$\begin{aligned}\frac{\text{hooks}_T\,\text{hooks}_r}{\text{hooks}_R\,\text{hooks}_t} &= \frac{\frac{(b_0+1)!(b_0+b_1+2)!}{b_1+1} \times \frac{b_0!(b_0+b_1+1)!}{b_1+1}}{\frac{b_0!(b_0+b_1+3)!}{b_1+3} \times \frac{(b_0-1)!(b_0+b_1+2)!}{b_1+3}} \\ &= \frac{(b_1+3)^2}{(b_1+1)^2}\frac{(b_0+1)!(b_0+b_1+1)!}{(b_0+b_1+3)!(b_0-1)!} \\ &= \frac{(b_1+3)^2}{(b_1+1)^2}\frac{(b_0+1)b_0(b_0-1)!(b_0+b_1+1)!}{(b_0+b_1+3)(b_0+b_1+2)(b_0+b_1+1)!(b_0-1)!} \\ &= \frac{(b_1+3)^2}{(b_1+1)^2}\frac{b_0(b_0+1)}{(b_0+b_1+2)(b_0+b_1+3)},\end{aligned}$$

$$\therefore \sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s}{f_R\,\text{hooks}_R\,\text{hooks}_t\,\text{hooks}_u}} = \frac{b_1+3}{b_1+1}\sqrt{\frac{N-b_0+1}{N-b_0-b_1-1}}\sqrt{\frac{b_0(b_0+1)}{(b_0+b_1+2)(b_0+b_1+3)}}.$$

We will now evaluate $\frac{c_{RR'}}{d_{R'}}$. Recall that for $R' = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ & \ \\ \hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \ & \multicolumn{1}{c}{} \\ \cline{1-1} \ & \multicolumn{1}{c}{} \\ \cline{1-1} * & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$, we have that

$$c_{RR'} = N - b_0 - b_1 - 1$$

and

$$\begin{aligned} d_{R'} &= \frac{(n+m-1)!}{\text{hooks}_{R'}} \\ &= \frac{(n+1)!}{\left(\frac{b_0!(b_0+(b_1+1)+1)!}{(b_1+1)+1}\right)} \\ &= \frac{(n+1)!}{\left(\frac{b_0!(b_0+b_1+2)!}{b_1+2}\right)} \\ &= \frac{(n+1)!(b_1+2)}{b_0!(b_0+b_1+2)!}\end{aligned}$$

Thus,

$$\frac{c_{RR'}}{d_{R'}} = \frac{N-b_0-b_1-1}{\left(\frac{(n+1)!(b_1+2)}{b_0!(b_0+b_1+2)!}\right)} = (N-b_0-b_1-1)\frac{b_0!(b_0+b_1+2)!}{(n+1)!(b_1+2)}.$$

For the remaining dimensions participating in our expression, we have

$$d_T = \frac{(n+m)!}{\text{hooks}_T} = \frac{(n+2)!}{\left(\frac{(b_0+1)!((b_0+1)+b_1+1)!}{b_1+1}\right)} = \frac{(n+2)!}{\left(\frac{(b_0+1)!(b_0+b_1+2)!}{b_1+1}\right)} = \frac{(n+2)!(b_1+1)}{(b_0+1)!(b_0+b_1+2)!},$$

$$d_t = \frac{n!}{\text{hooks}_t} = \frac{n!}{\left(\frac{(b_0-1)!((b_0-1)+(b_1+2)+1)!}{(b_1+2)+1}\right)} = \frac{n!}{\left(\frac{(b_0-1)!(b_0+b_1+2)!}{b_1+3}\right)} = \frac{n!(b_1+3)}{(b_0-1)!(b_0+b_1+2)!},$$

$$d_u = \frac{m!}{\text{hooks}_u} = \frac{2!}{2!} = 1.$$

Thus,

$$\begin{aligned}
\frac{d_T nm}{d_t d_u (n+m)} &= \frac{2n}{n+2}\frac{d_T}{d_t} \\
&= \frac{2n}{n+2}\frac{(n+2)!(b_1+1)}{(b_0+1)!(b_0+b_1+2)!}\frac{(b_0-1)!(b_0+b_1+2)!}{n!(b_1+3)} \\
&= \frac{2n}{n+2}\frac{(n+2)(n+1)n!(b+1)(b_0-1)!(b_0+b_1+2)!}{(b_0+1)b_0(b_0-1)!(b_0+b_1+2)!n!(b_1+3)} \\
&= \frac{2n(n+1)(b_1+1)}{b_0(b_0+1)(b_1+3)}.
\end{aligned}$$

Using these results, the upfront factor is computed to be

$$\begin{aligned}
&-g_{\text{YM}}^2\frac{c_{RR'}d_T(nm)}{d_{R'}d_t d_u(n+m)}\sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s}{f_R\,\text{hooks}_R\,\text{hooks}_t\,\text{hooks}_u}} = \\
&= -g_{\text{YM}}^2(N-b_0-b_1-1)\frac{b_0!(b_0+b_1+2)!}{(n+1)!(b_1+2)}\frac{2n(n+1)(b_1+1)}{b_0(b_0+1)(b_1+3)}\frac{b_1+3}{b_1+1}\sqrt{\frac{N-b_0+1}{N-b_0-b_1-1}}\times \\
&\quad\times\sqrt{\frac{b_0(b_0+1)}{(b_0+b_1+2)(b_0+b_1+3)}} \\
&= -2g_{\text{YM}}^2\sqrt{(N-b_0-b_1-1)(N-b_0+1)}\sqrt{\frac{b_0(b_0+b_1+2)}{(b_0+1)(b_0+b_1+3)}}\frac{(b_0-1)!(b_0+b_1+1)!}{(n-1)!(b_1+2)}.
\end{aligned}$$

Finally, combining the upfront factor and the final trace result, we obtain

$$\begin{aligned}
N_{a;b} &= -g_{\text{YM}}^2\frac{c_{RR'}d_T(nm)}{d_{R'}d_t d_u(n+m)}\sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s}{f_R\,\text{hooks}_R\,\text{hooks}_t\,\text{hooks}_u}}d_{r'}\,\text{Tr}\left(I_{R'T'}[\Gamma_T((1\ 3)),P_{T,(t,u)}]I_{T'R'}[\Gamma_R((1\ 3)),P_{R,(r,\right. \\
&= -2g_{\text{YM}}^2\sqrt{(N-b_0-b_1-1)(N-b_0+1)}\sqrt{\frac{b_0(b_0+b_1+2)}{(b_0+1)(b_0+b_1+3)}}\frac{(b_0-1)!(b_0+b_1+1)!}{(n-1)!(b_1+2)}\times \\
&\quad\times\left(-\frac{2}{(b_1+2)^2}\left[\frac{(n-1)!(b_1+2)}{(b_0-1)!(b_0+b_1+1)!}\right]\right) \\
&= 4g_{\text{YM}}^2\frac{\sqrt{(N-b_0-b_1-1)(N-b_0+1)}}{(b_1+2)^2}\sqrt{\frac{b_0(b_0+b_1+2)}{(b_0+1)(b_0+b_1+3)}}. \qquad (10)
\end{aligned}$$

Since $b_0 = O(N)$ and assuming nothing about b_1, in the large N limit we learn

$$4g_{\rm YM}^2\frac{\sqrt{(N-b_0-b_1-1)(N-b_0+1)}}{(b_1+2)^2}\sqrt{\frac{b_0(b_0+b_1+2)}{(b_0+1)(b_0+b_1+3)}}\to 4g_{\rm YM}^2\frac{\sqrt{(N-b_0-b_1-1)(N-b_0+1)}}{(b_1+2)^2}.$$

It should be clear that (10) is only a single term in the computation of (8) since a single example of the labels $T, (t, u)$ was considered.

2.9 Displaced Corners and Gauss Graphs

This subsection continues from the end of subsection 2.7 with focus now on the large N limit, the displaced corners approximation, Gauss operators and Gauss graphs.

2.9.1 Large N limit

Everything up until now has been exact. We now consider the large N limit. Suppose we have a Young diagram R with q rows of $O(1)$ (remember that there are p X fields, m Y fields and n Z fields) and of $O(N)$ columns. The sum of the number of fields $p+m+n$ is $O(N)$ where $m=\eta N$ and $p=\xi N$ with $\eta,\xi\ll 1$. The length of row i is denoted by l_{R_i} and the difference between two distinct row lengths, i.e. $l_{R_i}-l_{R_j}$, for $i\neq j$, is $O(N)$. The action of representation $\Gamma_R(i,i+1)$ on a Young-Yamanouchi state $|YY\rangle$ is given as

$$\Gamma_R(i,i+1)|YY\rangle = \frac{1}{c_i-c_j}|YY\rangle + \sqrt{1-\frac{1}{(c_i-c_j)^2}}|YY_{(i,i+1)}\rangle,$$

where $i\neq j$, c_i is the content of box i and $|YY_{(i,i+1)}\rangle$ is the Young-Yamanouchi state obtained after acting on Young-Yamanouchi state $|YY\rangle$ with permutation $(i,i+1)$. Consider the two cases:

1. For boxes that sit directly next to each other: $c_i - c_j = 1$ always, which implies that the Young-Yamanouchi state $|YY\rangle$ will survive.

2. For boxes that are a row apart: $c_i - c_j = N\to\infty$ in the large N limit, which implies that the Young-Yamanouchi state $|YY_{(i,i+1)}\rangle$ will survive.

This large N limit approximation is known as the *displaced corners approximation.*

If a box in the $i^{\rm th}$ row of Young diagram R is removed to get Young diagram R' and if a box in the $k^{\rm th}$ row of Young diagram T is removed we then get Young diagram T', and $R'=T'$, then our intertwiners are

$$I_{R'T'} = E^{(1)}_{ik} \qquad I_{T'R'} = E^{(1)}_{ki},$$

where the superscript label (1) signifies E acts on a state in the first slot only. $E^{(1)}_{ik}$ is a matrix of zeroes everywhere except for the matrix element in row i and column k which is equal to 1, i.e.

$$\begin{aligned} E^{(1)}_{ik}a^{(1)}_k &= a^{(1)}_i,\\ E^{(1)}_{ik}A^{(1)}_{ka} &= A^{(1)}_{ia},\end{aligned}$$

where $a^{(1)}_k$ is a vector living in slot 1 and $A^{(1)}_{ka}$ is a matrix acting on slot 1. To illustrate the action of E on a vector state explicitly, consider the following example: denote the states

$$|1\rangle\equiv\begin{bmatrix}1\\0\end{bmatrix}\quad,\quad|2\rangle\equiv\begin{bmatrix}0\\1\end{bmatrix}$$

then

$$E^{(1)}_{12}|2\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |1\rangle.$$

Bear in mind that a short hand notation is used above; a more explicit notation is as follows

$$E^{(1)}_{12} \otimes \mathbb{1}^{(2)}(|2\rangle \otimes |1\rangle) = |1\rangle \otimes |1\rangle$$

or

$$E^{(1)}_{12} \otimes \mathbb{1}^{(2)}(|1\rangle \otimes |2\rangle) = 0,$$

since $E^{(1)}_{12}|1\rangle = 0$. From these two examples it is clear that $E_{ij} \equiv |i\rangle\langle j|$. Consider the tensor product of states $|1\rangle \otimes |2\rangle \otimes |3\rangle$ whose positions in the tensor product are permuted by the permutation $(1\ 2)$

$$(1\ 2)|1\rangle \otimes |2\rangle \otimes |3\rangle = |2\rangle \otimes |1\rangle \otimes |3\rangle.$$

We will now argue that

$$\mathrm{Tr}(E^{(1)}_{ab}E^{(2)}_{ba}) = (1\ 2).$$

To check this claim act on a tensor product of states $|1\rangle \otimes |2\rangle \otimes |3\rangle$:

$$\begin{aligned} \mathrm{Tr}(E^{(1)}_{ab}E^{(2)}_{ba})|1\rangle \otimes |2\rangle \otimes |3\rangle &= \sum_a \sum_b \delta_{a2}\delta_{b1}|a\rangle \otimes |b\rangle \otimes |3\rangle \\ &= |2\rangle \otimes |1\rangle \otimes |3\rangle. \end{aligned}$$

The delta functions naturally appear by the usual action of $E^{(i)}_{ab}$

$$E^{(1)}_{ab}|1\rangle = \begin{cases} |a\rangle & \text{if } |1\rangle = |b\rangle \\ 0 & \text{if } |1\rangle \neq |b\rangle \end{cases}$$

and

$$E^{(2)}_{ba}|2\rangle = \begin{cases} |b\rangle & \text{if } |2\rangle = |a\rangle \\ 0 & \text{if } |2\rangle \neq |a\rangle \end{cases}$$

We also have

$$E^{(i)}_{ab}E^{(i)}_{cd} = \delta_{bc}E^{(i)}_{ad},$$

where both E's on the left hand side act on the i^{th} slot. Returning to the trace term, considering the first term again, we see that

$$\begin{aligned} &\mathrm{Tr}\left(I_{T'R'}P_{R,(r,s,t)\vec{\mu}\vec{\nu}}\Gamma^R(1,p+1)I_{R'T'}P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\Gamma^T(1,p+1)\right) \\ &= \mathrm{Tr}\left(\Gamma^T(1,p+1)I_{T'R'}P_{R,(r,s,t)\vec{\mu}\vec{\nu}}\Gamma^R(1,p+1)I_{R'T'}P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\right) \end{aligned}$$

due to the cyclicity of the trace. Consider the trace of the first two objects in the expression above. We find

$$\begin{aligned} \mathrm{Tr}\left(\Gamma^T(1,p+1)I_{T'R'}\right) &= E^{(1)}_{ab}E^{(p+1)}_{ba}E^{(1)}_{ki} \\ &= \delta_{bk}E^{(1)}_{ai}E^{(p+1)}_{ba} \\ &= E^{(1)}_{ai}E^{(p+1)}_{ka} \end{aligned}$$

and similarly

$$\begin{aligned}\operatorname{Tr}\left(\Gamma^R(1,p+1)I_{R'T'}\right) &= E^{(1)}_{cd}E^{(p+1)}_{dc}E^{(1)}_{ik}\\ &= \delta_{di}E^{(1)}_{ck}E^{(p+1)}_{dc}\\ &= E^{(1)}_{ck}E^{(p+1)}_{ic}.\end{aligned}$$

Thus the trace over the first term becomes

$$\begin{aligned}&\operatorname{Tr}\left(\Gamma^T(1,p+1)I_{T'R'}P_{R,(r,s,t)\vec{\mu}\vec{\nu}}\Gamma^R(1,p+1)I_{R'T'}P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\right)\\ &\qquad= \operatorname{Tr}\left(E^{(1)}_{ai}E^{(p+1)}_{ka}P_{R,(r,s,t)\vec{\mu}\vec{\nu}}E^{(1)}_{ck}E^{(p+1)}_{ic}P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\right),\end{aligned}$$

where we note that

$$E^{(1)}_{ai}E^{(p+1)}_{ka} = (E^{(1)}_{ai}\otimes \mathbb{1}^{p-1})\otimes(E^{(p+1)}_{ka}\otimes\mathbb{1}^{m-1})\otimes\mathbb{1}^n$$

and

$$E^{(1)}_{ck}E^{(p+1)}_{ic} = (E^{(1)}_{ck}\otimes \mathbb{1}^{p-1})\otimes(E^{(p+1)}_{ic}\otimes\mathbb{1}^{m-1})\otimes\mathbb{1}^n.$$

We use

$$P_{R,(t,s,r)\vec{\alpha}\vec{\beta}} = P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1:s\alpha_2\beta_2}\otimes\mathbb{1}_r$$

for the form of the projector (intertwining operator). The trace then evaluates to

$$\begin{aligned}&\operatorname{Tr}\left(E^{(1)}_{ai}E^{(p+1)}_{ka}P_{R,(r,s,t)\vec{\mu}\vec{\nu}}E^{(1)}_{ck}E^{(p+1)}_{ic}P_{T,(t,u,v)\vec{\mu}\vec{\nu}}\right)\\ &= \operatorname{Tr}\left(E^{(1)}_{ai}E^{(p+1)}_{ka}\cdot P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1:s\alpha_2\beta_2}\otimes\mathbb{1}_r\cdot E^{(1)}_{ck}E^{(p+1)}_{ic}\cdot P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1:v\mu_2\nu_2}\otimes\mathbb{1}_u\right).\end{aligned}$$

Using the following identity

$$\operatorname{Tr}(A\otimes B\cdot C\otimes D) = \operatorname{Tr}(A\cdot C\otimes B\cdot D) = \operatorname{Tr}(A\cdot C)\operatorname{Tr}(B\cdot D),$$

we can write

$$\begin{aligned}&\operatorname{Tr}\left(E^{(1)}_{ai}E^{(p+1)}_{ka}\cdot P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1:s\alpha_2\beta_2}\otimes\mathbb{1}_r\cdot E^{(1)}_{ck}E^{(p+1)}_{ic}\cdot P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1:v\mu_2\nu_2}\otimes\mathbb{1}_u\right)\\ &\qquad= \operatorname{Tr}\left(E^{(1)}_{ai}E^{(p+1)}_{ka}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1:s\alpha_2\beta_2}E^{(1)}_{ck}E^{(p+1)}_{ic}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1:v\mu_2\nu_2}\right)\operatorname{Tr}(\mathbb{1}_r\mathbb{1}_u),\end{aligned}$$

where $\operatorname{Tr}(\mathbb{1}_r\mathbb{1}_u) = d_r\delta_{ru}$. This result is obtained with the use of the displaced corners approximation.

2.9.2 Rewrite in terms of normalised operators

Recall from subsection 2.5.1 that the two point function of the restricted Schur polynomials is given in (3). To obtain normalised operators rescale the restricted Schur polynomials so that

$$\langle \hat{O}^\dagger_{T,(w,v,u)\vec{\mu}\vec{\nu}}\hat{O}_{R,(t,s,r)\vec{\alpha}\vec{\beta}}\rangle = \delta_{RT}\delta_{ru}\delta_{sv}\delta_{tw}\delta_{\vec{\alpha}\vec{\mu}}\delta_{\vec{\beta}\vec{\nu}}.$$

The factor that multiplies the trace is given by

$$\frac{c_{RR'}d_T pm\delta_{R'T'}\delta_{ru}}{d_{R'}d_u d_v d_w(p+m+n)}\sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s\,\text{hooks}_t}{f_R\,\text{hooks}_R\,\text{hooks}_u\,\text{hooks}_v\,\text{hooks}_w}}d_r$$
$$=\frac{c_{RR'}d_T pm\delta_{R'T'}\delta_{ru}}{d_{R'}d_v d_w(p+m+n)}\sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s\,\text{hooks}_t}{f_R\,\text{hooks}_R\,\text{hooks}_u\,\text{hooks}_v\,\text{hooks}_w}},$$

where

$$\delta_{ru}d_u=\delta_{ru}d_r$$

was used to simplify the result. We make use of the following result

$$c_{RR'}\sqrt{\frac{f_T}{f_R}}=c_{RR'}\sqrt{\frac{c_{TT'}}{c_{RR'}}}=\sqrt{c_{RR'}c_{TT'}}.$$

Another result, that is valid in the large N limit, is

$$\frac{\text{hooks}_R}{\text{hooks}_{R'}}=l_{R_i},$$

where l_{R_i} is the number of boxes in row i of Young diagram R. Then

$$\sqrt{\frac{\text{hooks}_T}{\text{hooks}_R}}=\sqrt{\frac{\text{hooks}_T}{\text{hooks}_{T'}}\frac{\text{hooks}_{T'}}{\text{hooks}_{R'}}\frac{\text{hooks}_{R'}}{\text{hooks}_R}}=\sqrt{l_{T_k}\cdot 1\cdot\frac{1}{l_{R_i}}}=\sqrt{\frac{l_{T_k}}{l_{R_i}}}.$$

Finally, recall that

$$d_T=\frac{(p+m+n)!}{\text{hooks}_T}\quad\text{and}\quad d_{T'}=\frac{(p+m+n-1)!}{\text{hooks}_{T'}}.$$

Plugging these results into the factor we are considering, we obtain

$$\frac{c_{RR'}d_T pm\delta_{R'T'}\delta_{ru}}{d_{R'}d_v d_w(p+m+n)}\sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s\,\text{hooks}_t}{f_R\,\text{hooks}_R\,\text{hooks}_u\,\text{hooks}_v\,\text{hooks}_w}}$$
$$=\sqrt{c_{RR'}c_{TT'}}\frac{d_T pm}{d_{R'}d_v d_w(p+m+n)}\delta_{R'T'}\delta_{ru}\sqrt{\frac{\text{hooks}_s\,\text{hooks}_t}{\text{hooks}_v\,\text{hooks}_w}}\sqrt{\frac{\text{hooks}_T}{\text{hooks}_R}}$$
$$=\sqrt{c_{RR'}c_{TT'}}\sqrt{\frac{l_{T_k}}{l_{R_i}}}\frac{\text{hooks}_{T'}}{\text{hooks}_T}\frac{pm}{d_v d_w}\sqrt{\frac{\text{hooks}_s\,\text{hooks}_t}{\text{hooks}_v\,\text{hooks}_w}}\delta_{R'T'}\delta_{ru}.$$

Lastly, we make the substitutions

$$\text{hooks}_s=\frac{m!}{d_s}\qquad\text{hooks}_t=\frac{p!}{d_t}$$
$$\text{hooks}_v=\frac{p!}{d_v}\qquad\text{hooks}_w=\frac{m!}{d_w}$$

to obtain

$$\sqrt{\frac{\text{hooks}_s\,\text{hooks}_t}{\text{hooks}_v\,\text{hooks}_w}}=\sqrt{\frac{\frac{m!}{d_s}\frac{p!}{d_t}}{\frac{p!}{d_v}\frac{m!}{d_w}}}=\sqrt{\frac{d_v d_w}{d_s d_t}}.$$

Thus, we have

$$\sqrt{c_{RR'}c_{TT'}}\sqrt{\frac{l_{T_k}}{l_{R_i}}\frac{\text{hooks}_{T'}}{\text{hooks}_T}\frac{pm}{d_v d_w}}\sqrt{\frac{\text{hooks}_s\,\text{hooks}_t}{\text{hooks}_v\,\text{hooks}_w}}\delta_{R'T'}\delta_{ru} = \delta_{R'T'}\delta_{ru}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}}\frac{pm}{\sqrt{d_s d_t d_v d_w}}.$$

To complete our result, we write the action of $D^{(3)}$ on the normalised operator as

$$D^{(3)}\hat{O}_{R,(t,s,r)\vec{\alpha}\vec{\beta}}(Z,Y,X) = \sum_{T,(w,v,u)} M^3_{R,(t,s,r)\vec{\alpha}\vec{\beta};T,(w,v,u)\vec{\mu}\vec{\nu}}\hat{O}_{T,(w,v,u)\vec{\mu}\vec{\nu}},$$

where

$$M^3_{R,(t,s,r)\vec{\alpha}\vec{\beta};T,(w,v,u)\vec{\mu}\vec{\nu}} = \sum_{R'}\delta_{R'_iT'_k}\delta_{ru}\frac{pm}{\sqrt{d_s d_t d_v d_w}}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}}\times$$
$$\times\Bigg[\text{Tr}\left(E^{(1)}_{ki}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(p+1)}_{ik}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right) - \text{Tr}\left(E^{(1)}_{ci}E^{(p+1)}_{kc}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(1)}_{ak}E^{(p+1)}_{ia}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right) -$$
$$- \text{Tr}\left(E^{(1)}_{kc}E^{(p+1)}_{ci}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(1)}_{ia}E^{(p+1)}_{ak}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right) + \text{Tr}\left(E^{(p+1)}_{ki}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(1)}_{ik}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right)\Bigg]. \quad (11)$$

The matrix elements connected to the mixing of Y and Z ($M^1_{R,(t,s,r)\vec{\alpha}\vec{\beta};T,(w,v,u)\vec{\mu}\vec{\nu}}$) and X and Z ($M^2_{R,(t,s,r)\vec{\alpha}\vec{\beta};T,(w,v,u)\vec{\mu}\vec{\nu}}$) are evaluated in the same way and the results are

$$M^1_{R,(t,s,r)\vec{\alpha}\vec{\beta};T,(w,v,u)\vec{\mu}\vec{\nu}} = \sum_{R'}\delta_{R'_iT'_k}\delta_{r_iu_k}\frac{m}{\sqrt{d_s d_t d_v d_w}}\sqrt{\frac{l_{r_i}l_{u_k}}{l_{R_i}l_{T_k}}c_{RR'}c_{TT'}}\times$$
$$\times\Bigg[\text{Tr}\left(P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(p+1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right)\delta_{ik} - \text{Tr}\left(E^{(p+1)}_{kk}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(p+1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right) -$$
$$- \text{Tr}\left(E^{(p+1)}_{ii}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(p+1)}_{kk}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right) + \text{Tr}\left(E^{(p+1)}_{ii}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right)\delta_{ik}\Bigg]$$

and

$$M^2_{R,(t,s,r)\vec{\alpha}\vec{\beta};T,(w,v,u)\vec{\mu}\vec{\nu}} = \sum_{R'}\delta_{R'_iT'_k}\delta_{r_iu_k}\frac{p}{\sqrt{d_s d_t d_v d_w}}\sqrt{\frac{l_{r_i}l_{u_k}}{l_{R_i}l_{T_k}}c_{RR'}c_{TT'}}\times$$
$$\times\Bigg[\text{Tr}\left(P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right)\delta_{ik} - \text{Tr}\left(E^{(1)}_{kk}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right) -$$
$$- \text{Tr}\left(E^{(1)}_{ii}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(1)}_{kk}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right) + \text{Tr}\left(E^{(1)}_{kk}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right)\delta_{ik}\Bigg].$$

2.10 Gauss Graph Basis

In this subsection we wish to rewrite the action of the dilatation operator that mixes the X and Y fields in the basis of Gauss operators. The Gauss operators $O_{R,r}(\sigma)$ diagonalise the action of the dilatation operator in the $s \vdash m$, $t \vdash p$ and multiplicity labels. For simplicity we focus on $p = 0$ for now. Consider $\sigma \in H\backslash S_m/H$. σ represents an equivalence class for the equivalence relation

$$\tilde{\sigma} \sim \sigma \quad \text{if} \quad \tilde{\sigma} \equiv \psi\sigma\tau,$$

where $\tau, \psi \in S_{\vec{m}} = H$. Here we have in mind a restricted Schur polynomial $\chi_{R,(r,s)\alpha\beta}(Z,Y)$. As usual we remove boxes from R to obtain r. The vector $\vec{m} = (m_1, m_2, \dots)$ records how many boxes were removed from each row of R to obtain r. The group $S_{\vec{m}}$ is defined to be a product of symmetric groups as follows

$$S_{\vec{m}} = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p}$$

i.e. R has p rows. Consider the vector state

$$|\vec{v}, \vec{m}\rangle = |v_1^{\otimes m_1} \otimes v_2^{\otimes m_2} \otimes \cdots v_p^{\otimes m_p}\rangle.$$

This is a tensor product of m vectors. v_i is zero everywhere except the i^{th} component which is 1, i.e.

$$(v_i)_a = \delta_{ia}, \qquad a = 1, \dots, p.$$

Notice that this non-zero entry matches the row from which the corresponding box was removed. We define an action of S_m on this state as follows

$$\sigma|v, \vec{m}\rangle \equiv |v_\sigma\rangle.$$

The action of σ on the vector space is

$$\sigma|v_{i_1} \otimes \cdots \otimes v_{i_p}\rangle = |v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(p)}}\rangle$$

We want to implement the symmetry under $S_{\vec{m}}$. Towards this end, define

$$|v_\sigma\rangle = \frac{1}{|H|} \sum_{\gamma \in H} |v_{\sigma\gamma}\rangle. \tag{12}$$

This description of $|v_\sigma\rangle$ provides vectors that are not all independent. If we set γ to be an element of the subgroup H, under the action of γ the state is unchanged

$$|v_\sigma\rangle = |v_{\sigma\gamma}\rangle.$$

We will make use of the vector

$$|v_{s,i,j}\rangle = \sum_{\sigma \in S_m} \Gamma_{ij}(\sigma)|v_\sigma\rangle. \tag{13}$$

Substituting (12) into (13) yields

$$\begin{aligned} |v_{s,i,j}\rangle &= \sum_{\sigma \in S_m} \Gamma_{ij}(\sigma)|v_\sigma\rangle \\ &= \sum_{\sigma \in S_m} \Gamma_{ij}(\sigma) \frac{1}{|H|} \sum_{\gamma \in H} |v_{\sigma\gamma}\rangle \\ &= \frac{1}{|H|} \sum_{\sigma \in S_m} \sum_{\gamma \in H} \Gamma_{ij}(\sigma)|v_{\sigma\gamma}\rangle. \end{aligned}$$

Relabelling $\sigma\gamma \to \psi \Rightarrow \sigma \to \psi\gamma^{-1}$ yields

$$\begin{aligned} |v_{s,i,j}\rangle &= \frac{1}{|H|} \sum_{\sigma \in S_m} \sum_{\gamma \in H} \Gamma_{ij}(\sigma)|v_{\sigma\gamma}\rangle \\ &= \frac{1}{|H|} \sum_{\psi \in S_m} \sum_{\gamma \in H} \Gamma_{ij}(\psi\gamma^{-1})|v_\psi\rangle \\ &= \frac{1}{|H|} \sum_{\psi \in S_m} \sum_{\gamma \in H} \Gamma_{ik}(\psi)\Gamma_{kj}(\gamma^{-1})|v_\psi\rangle. \end{aligned}$$

We can write

$$\frac{1}{|H|}\sum_{\gamma\in H}\Gamma^{s}_{kj}(\gamma^{-1}) = \sum_{\mu} B^{s\to 1_H}_{k\mu} B^{s\to 1_H}_{j\mu}, \tag{14}$$

where the left hand side is the projector onto the trivial representation of H, $\gamma = \gamma^{-1}$ when summing over all γ, and $B^{s\to 1_H}_{k\mu}$ and $B^{s\to 1_H}_{j\mu}$ are branching coefficients that take the irrep s to 1_H, the trivial of subgroup H. Plugging this into our previous result gives

$$\begin{aligned}
|v_{s,i,j}\rangle &= \frac{1}{|H|}\sum_{\psi\in S_m}\sum_{\gamma\in H}\Gamma_{ik}(\psi)\Gamma_{kj}(\gamma^{-1})|v_\psi\rangle \\
&= \sum_{\psi\in S_m}\sum_{\mu}\Gamma_{ik}(\psi)B^{s\to 1_H}_{k\mu}B^{s\to 1_H}_{j\mu}|v_\psi\rangle \\
&= \sum_{\mu} B^{s\to 1_H}_{k\mu}\left[\sum_{\psi\in S_m}\Gamma^{(s)}_{ik}(\psi)|v_\psi\rangle\right]B^{s\to 1_H}_{j\mu}.
\end{aligned}$$

We introduce the Fourier transformation

$$|\vec{m}, s, \mu; i\rangle \equiv \sum_{k} B^{s\to 1_H}_{k\mu}\sum_{\psi\in S_m}\Gamma^{(s)}_{ik}(\psi)|v_\psi\rangle$$

whose inverse is

$$|v_{s,i,j}\rangle = \sum_{\mu} B^{s\to 1_H}_{j\mu}|\vec{m}, s, \mu; i\rangle.$$

Since we wish to work with normalised operators we will need to make use of the following equations

$$\sum_{s\vdash m}\frac{d_s}{m!}\Gamma^{(s)}_{ij}(\sigma)\Gamma^{(s)}_{ij}(\tau) = \delta(\sigma\tau^{-1}), \tag{15}$$

$$O_\tau = \frac{1}{|H|^2}\sum_{\gamma_1,\gamma_2\in H} O_{\gamma_1\tau\gamma_2}, \tag{16}$$

$$O^{s}_{ij} = \sum_{\sigma\in S_m}\Gamma^{(s)}_{ij}(\sigma)O_\sigma. \tag{17}$$

With the use of (15) and (17) we see that

$$\begin{aligned}
\sum_{s\vdash m}\frac{d_s}{m!}\Gamma^{(s)}_{ij}(\tau)O^{s}_{ij} &= \sum_{\sigma\in S_m}\sum_{s\vdash m}\frac{d_s}{m!}\Gamma^{(s)}_{ij}(\tau)\Gamma^{(s)}_{ij}(\sigma)O_\sigma \\
&= \sum_{\sigma\in S_m}\delta(\sigma\tau^{-1})O_\sigma \\
&= O_\tau.
\end{aligned} \tag{18}$$

Substituting (18) into the right hand side of (16), and then using (14) yields

$$\begin{aligned}
O_\tau &= \frac{1}{|H|^2}\sum_{s\vdash m}\sum_{\gamma_1,\gamma_2\in H}\frac{d_s}{m!}\Gamma^{(s)}_{ij}(\gamma_1\tau\gamma_2)O^s_{ij} \\
&= \frac{1}{|H|^2}\sum_{s\vdash m}\sum_{\gamma_1,\gamma_2\in H}\frac{d_s}{m!}\Gamma^{(s)}_{ik}(\gamma_1)\Gamma^{(s)}_{kl}(\tau)\Gamma^{(s)}_{lj}(\gamma_2)O^s_{ij} \\
&= \sum_{\mu_1,\mu_2}\sum_{s\vdash m}\frac{d_s}{m!}\Gamma^{(s)}_{kl}(\tau)B^{s\to 1_H}_{i\mu_1}B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2}B^{s\to 1_H}_{j\mu_2}O^s_{ij} \\
&= \sum_{\mu_1,\mu_2}\sum_{s\vdash m}\sqrt{\frac{d_s}{m!}}\Gamma^{(s)}_{kl}(\tau)B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2}\left(\sqrt{\frac{d_s}{m!}}B^{s\to 1_H}_{i\mu_1}B^{s\to 1_H}_{j\mu_2}O^s_{ij}\right) \\
&= \sum_{\mu_1,\mu_2}\sum_{s\vdash m}\sqrt{\frac{d_s}{m!}}\Gamma^{(s)}_{kl}(\tau)B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2}O^s_{\mu_1\mu_2},
\end{aligned}$$

where $\sum_{\mu_1,\mu_2}\sum_{s\vdash m}\sqrt{\frac{d_s}{m!}}\Gamma^{(s)}_{kl}(\tau)B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2}$ is connected to the double coset. We now define

$$C^s_{\mu_1\mu_2}(\tau) \equiv |H|\sum_{s\vdash p}\sqrt{\frac{d_s}{m!}}\Gamma^{(s)}_{kl}(\tau)B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2}$$

such that

$$\begin{aligned}
C^s_{\mu_1\mu_2}(\tau)C^s_{\mu_1\mu_2}(\sigma) &= |H|^2\sum_{s\vdash p}\frac{d_s}{m!}\Gamma^{(s)}_{kl}(\tau)\Gamma^{(s)}_{ab}(\sigma)B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2}B^{s\to 1_H}_{a\mu_1}B^{s\to 1_H}_{b\mu_2} \\
&= \sum_{s\vdash p}\sum_{\gamma_1,\gamma_2\in H}\frac{d_s}{m!}\Gamma^{(s)}_{kl}(\tau)\Gamma^{(s)}_{ab}(\sigma)\Gamma^{(s)}_{ka}(\gamma_1)\Gamma^{(s)}_{lb}(\gamma_2) \\
&= \sum_{s\vdash p}\sum_{\gamma_1,\gamma_2\in H}\frac{d_s}{m!}\Gamma^{(s)}_{kb}(\tau\gamma_2)\Gamma^{(s)}_{kb}(\gamma_1\sigma) \\
&= \sum_{\gamma_1,\gamma_2\in H}\delta(\gamma_1\sigma\gamma_2^{-1}\tau^{-1}),
\end{aligned}$$

where $\gamma_1\sigma\gamma_2^{-1}\in H\backslash S_m/H$. Thus

$$O_{R,r}(\sigma) = \frac{|H|}{\sqrt{m!}}\sum_{j,k}\sum_{s\vdash m}\sum_{\mu_1,\mu_2}\sqrt{d_s}\Gamma^{(s)}_{jk}(\sigma)B^{s\to 1_H}_{j\mu_1}B^{s\to 1_H}_{k\mu_2}O_{R,(s,r)\mu_1\mu_2},$$

where $O_{R,r}(\sigma)$ is the new basis, $O_{R,(s,r)\mu_1\mu_2}$ is the old basis and $\frac{|H|}{\sqrt{m!}}\sum_{j,k}\sum_{s\vdash m}\sum_{\mu_1,\mu_2}\sqrt{d_s}\Gamma^{(s)}_{jk}(\sigma)B^{s\to 1_H}_{j\mu_1}B^{s\to 1_H}_{k\mu_2}$ is the map from the old basis to the new basis. This description is for the $SU(2)$ sector, i.e. the sector where two fields mix. When we extend to the $SU(3)$ sector (mixing of X, Y and Z) the paper [19] is useful. Consider the tensor product

$$|t,\alpha_1,x;s,\alpha_2,y;r,z\rangle = |t,\alpha_1,x;s,\alpha_2,y\rangle\otimes|r,z\rangle,$$

where $t\vdash p$ ($s\vdash m$ and $r\vdash n$), α_1 (α_2) is the multiplicity of the the irreps of t (s), and x (y and z) labels the state of the respective vector. We are only interested in the vector $|t,\alpha_1,x;s,\alpha_2,y\rangle$ relevant for mixing the X and Y fields - it is the subleading piece. This leads us to introduce another intertwining map, expressed as

$$\begin{aligned}
P_{T,(t,u,v)\vec{\mu}\vec{\nu}} &= \sum_a|(v,\mu_1),a\rangle\langle(v,\nu_1),a|\otimes\sum_b|(u,\mu_2),b\rangle\langle(u,\nu_2),b|\otimes\sum_c|(t),c\rangle\langle(t),c| \\
&\equiv P^{(\vec{p})}_{v\mu_1\nu_1}\otimes P^{(\vec{m})}_{u\mu_2\nu_2}\otimes\mathbb{1}_t
\end{aligned}$$

and

$$P^{(\vec{p})}_{v\mu_1\nu_1} = |\vec{p}, v, \mu_1; a\rangle\langle\vec{p}, v, \nu_1; a|, \qquad P^{(\vec{m})}_{u\mu_2\nu_2} = |\vec{m}, u, \mu_2; b\rangle\langle\vec{m}, u, \nu_2; b|.$$

Note the definition of the vectors

$$\begin{aligned} |\vec{p}, t, \alpha_1; a\rangle &= \sum_{\psi_x \in S_p} \Gamma^{(t)}_{ax_1}(\psi_x) B^{t\to 1_{H_p}}_{x_1\alpha_1} \psi_x |\vec{v}, \vec{p}\rangle, \\ |\vec{m}, s, \alpha_2; b\rangle &= \sum_{\psi_y \in S_m} \Gamma^{(s)}_{bx_2}(\psi_y) B^{s\to 1_{H_m}}_{x_2\alpha_2} \psi_y |\vec{v}, \vec{m}\rangle. \end{aligned}$$

The tensor product between these two vectors can be written as

$$|\vec{p}, t, \alpha_1; a\rangle \otimes |\vec{m}, s, \alpha_2; b\rangle = \sum_{\psi_x \in S_p} \sum_{\psi_y \in S_m} \left(\Gamma^{(t)}_{ax_1}(\psi_x) \cdot B^{t\to 1_{H_p}}_{x_1\alpha_1} \cdot \psi_x |\vec{v}, \vec{p}\rangle \otimes \Gamma^{(s)}_{bx_2}(\psi_y) \cdot B^{s\to 1_{H_m}}_{x_2\alpha_2} \cdot \psi_y |\vec{v}, \vec{m}\rangle \right).$$

Using the identity

$$A \cdot B \otimes C \cdot D = A \otimes C \cdot B \otimes D$$

the tensor product can be written as

$$|\vec{p}, t, \alpha_1; a\rangle \otimes |\vec{m}, s, \alpha_2; b\rangle = \sum_{\psi_x \in S_p} \sum_{\psi_y \in S_m} \left(\Gamma^{(t)}_{ax_1}(\psi_x) \otimes \Gamma^{(s)}_{bx_2}(\psi_y) \cdot B^{t\to 1_{H_p}}_{x_1\alpha_1} \otimes B^{s\to 1_{H_m}}_{x_2\alpha_2} \cdot \psi_x \otimes \psi_y |\vec{v}, \vec{p}\rangle \otimes |\vec{v}, \vec{m}\rangle \right).$$

By making the following definitions

$$\begin{aligned} \psi &\equiv \psi_x \otimes \psi_y, \\ \Gamma^{(t,s)}_{ab;x_1x_2}(\psi) &\equiv \Gamma^{(t)}_{ax_1}(\psi_x) \otimes \Gamma^{(s)}_{bx_2}(\psi_y), \\ B^{(t,s)\to 1_{H_p \times H_m}}_{x_1x_2;\alpha_1\alpha_2} &\equiv B^{t\to 1_{H_p}}_{x_1\alpha_1} \otimes B^{s\to 1_{H_m}}_{x_2\alpha_2}, \\ \psi|\vec{v}, \vec{p}, \vec{m}\rangle &\equiv \psi_x \otimes \psi_y |\vec{v}, \vec{p}\rangle \otimes |\vec{v}, \vec{m}\rangle. \end{aligned}$$

we can simplify the tensor product to

$$|\vec{p}, t, \alpha_1; a\rangle \otimes |\vec{m}, s, \alpha_2; b\rangle = \sum_{\psi \in S_p \times S_m} \left(\Gamma^{(t,s)}_{ab;x_1x_2}(\psi) \cdot B^{(t,s)\to 1_{H_p \times H_m}}_{x_1x_2;\alpha_1\alpha_2} \cdot \psi|\vec{v}, \vec{p}, \vec{m}\rangle \right). \tag{19}$$

Similarly, the bra vector tensor product is given as

$$\langle\vec{p}, t, \beta_1; a| \otimes \langle\vec{m}, s, \beta_2; b| = \sum_{\phi \in S_p \times S_m} \frac{d_t d_s}{p!m!|H_p \times H_m|} \left(\langle\vec{v}, \vec{p}, \vec{m}|\phi^{-1} \Gamma^{(t,s)}_{ab;y_1y_2}(\phi) \cdot B^{(t,s)\to 1_{H_p \times H_m}}_{y_1y_2;\beta_1\beta_2} \right). \tag{20}$$

Note that (19) and (20) are Fourier transform pairs with the corresponding delta function of the form

$$\sum_{t\vdash p, s\vdash m} \frac{d_t d_s}{p!m!} \Gamma^{(t,s)}_{ab;x_1x_2}(\sigma) \Gamma^{(t,s)}_{ab;x_1x_2}(\tau) = \delta(\sigma\tau^{-1}). \tag{21}$$

To make the connection to the Fourier transform more explicit, we now make a small digression to develop a pedagogical example. Consider a simpler example of a Fourier transform pair and the corresponding delta function to illustrate the statement made above. Consider the functions

$$\tilde{f}(k) = \int dx\ e^{ikx} f(x) \quad \text{and} \quad f(x) = \mathcal{N} \int dk\ e^{-ikx} \tilde{f}(k),$$

where $\tilde{f}(k)$ is the function in momentum space, $f(x)$ is the function in position space and $\mathcal{N}$ is a normalisation to be determined. Plugging in the form of $f(x)$ into the form of $\tilde{f}(k)$, one can see that

$$\begin{aligned}\tilde{f}(k) = \int dx\ e^{ikx} f(x) &= \mathcal{N}\int dp \left(\int dx\ e^{i(k-p)x}\right)\tilde{f}(p)\\ &= \mathcal{N}\int dp\ 2\pi\delta(k-p)\tilde{f}(p)\\ &= 2\pi\mathcal{N}\tilde{f}(k),\end{aligned}$$

which is only true if $\mathcal{N} = \frac{1}{2\pi}$. From the third to fourth line, the integral representation of the delta function

$$\int dx\ e^{i(k-p)x} = 2\pi\delta(k-p)$$

was used. In our case, the normalisation in (20) was determined with the use of (19) and (21). We can now express our intertwining map as

$$\begin{aligned}P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2} &= |\vec{p},t,\alpha_1;a\rangle\otimes|\vec{m},s,\alpha_2;b\rangle\langle\vec{p},t,\beta_1;a|\otimes\langle\vec{m},s,\beta_2;b|\\ &= \sum_{\psi,\phi\in S_p\times S_m}\frac{d_t d_s}{p!m!|H_p\times H_m|}\left(\psi|\vec{v},\vec{p},\vec{m}\rangle\langle\vec{v},\vec{p},\vec{m}|\phi^{-1}\right)\Gamma^{(t,s)}_{ab;x_1x_2}(\psi)\Gamma^{(t,s)}_{ab;y_1y_2}(\phi)\times\\ &\times B^{(t,s)\to 1_{H_p\times H_m}}_{x_1x_2;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_p\times H_m}}_{y_1y_2;\beta_1\beta_2},\end{aligned}$$

where $P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}$ and $\psi|\vec{v},\vec{p},\vec{m}\rangle\langle\vec{v},\vec{p},\vec{m}|\phi^{-1}$ are the Fourier transform pair and

$$\Gamma^{(t,s)}_{ab;x_1x_2}(\psi)\Gamma^{(t,s)}_{ab;y_1y_2}(\phi)B^{(t,s)\to 1_{H_p\times H_m}}_{x_1x_2;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_p\times H_m}}_{y_1y_2;\beta_1\beta_2}$$

acts as "e^{ikx}". This operator has appeared in the action of the dilatation operator on the restricted Schur, i.e.

$$\mathrm{Tr}(A\cdot P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}\cdot B\cdot P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}),$$

where $A,B\sim E^{(1)},E^{(2)}$ respectively. We can now define the normalised states

$$\begin{aligned}|O_{R,r}(\sigma_x)\rangle &= \frac{|H_p|}{\sqrt{p!}}\sum_{i,j}\sum_{t\vdash p}\sum_{\alpha_1,\beta_1}\sqrt{d_t}\Gamma^{(t)}_{ij}(\sigma_x)B^{t\to 1_{H_p}}_{i\alpha_1}B^{t\to 1_{H_p}}_{j\beta_1}|O_{R,(t,s,r)\alpha_1\beta_1}\rangle,\\ |O_{R,r}(\sigma_y)\rangle &= \frac{|H_m|}{\sqrt{m!}}\sum_{k,l}\sum_{s\vdash m}\sum_{\alpha_2,\beta_2}\sqrt{d_s}\Gamma^{(s)}_{kl}(\sigma_y)B^{s\to 1_{H_m}}_{k\alpha_2}B^{s\to 1_{H_m}}_{l\beta_2}|O_{R,(t,s,r)\alpha_2\beta_2}\rangle,\end{aligned}$$

and define the tensor product of them as

$$\begin{aligned}|O_{R,r}(\sigma)\rangle &\equiv |O_{R,r}(\sigma_x)\rangle\otimes|O_{R,r}(\sigma_y)\rangle\\ &= \frac{|H_p\times H_m|}{\sqrt{p!m!}}\sum_{i,j,k,l}\sum_{t\vdash p,s\vdash m}\sum_{\alpha_1,\alpha_2,\beta_1,\beta_2}\sqrt{d_td_s}\Gamma^{(t,s)}_{ij;kl}(\sigma)B^{(t,s)\to 1_{H_p\times H_m}}_{ik;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_p\times H_m}}_{jl;\beta_1\beta_2}|O_{R,((t,s),r)\vec{\alpha}\vec{\beta}}\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle O^\dagger_{T,u}(\tau^{-1})| &= \frac{|H_{p'}\times H_{m'}|}{\sqrt{p'!m'!}}\sum_{a,b,c,d}\sum_{w\vdash p',v\vdash m'}\sum_{\mu_1,\mu_2,\nu_1,\nu_2}\sqrt{d_wd_v}\Gamma^{(w,v)}_{ac;bd}(\tau^{-1})B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2}B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2}\times\\ &\times\langle O^\dagger_{T,((w,v),u)\vec{\mu}\vec{\nu}}|.\end{aligned}$$

Finally, we state the matrix element of the dilatation operator acting on the normalised state to be

$$\begin{aligned}M^{(3)\ (\sigma_1;\sigma_2)}_{T,u;R,r} &\equiv \langle O^\dagger_{T,u}(\sigma_2^{-1})|D^{(3)}|O_{R,r}(\sigma_1)\rangle \\ &= \frac{|H_p\times H_m||H_{p'}\times H_{m'}|}{\sqrt{p!m!p'!m'!}} \sum_{i,j,k,l,a,b,c,d}\sum_{t\vdash p,s\vdash m}\sum_{w\vdash p',v\vdash m'}\sum_{\vec\alpha,\vec\beta,\vec\mu,\vec\nu}\sqrt{d_td_sd_wd_v}\times \\ &\times \Gamma^{(t,s)}_{ij;kl}(\sigma_1)B^{(t,s)\to 1_{H_p\times H_m}}_{ik;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_p\times H_m}}_{jl;\beta_1\beta_2}\Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2}B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2}\times \\ &\times \langle O^\dagger_{T,((w,v),u)\vec\mu\vec\nu}|D^{(3)}|O_{R,((t,s),r)\vec\alpha\vec\beta}\rangle, \end{aligned} \tag{22}$$

where

$$\langle O^\dagger_{T,((w,v),u)\vec\mu\vec\nu}|D^{(3)}|O_{R,((t,s),r)\vec\alpha\vec\beta}\rangle = \sum_{T,((w,v),u)} M^{(3)}_{R((t,s),r)\vec\alpha\vec\beta:T((w,v),u)\vec\mu\vec\nu}\delta_{RT}\delta_{(t,s)}\delta_{(w,v)}\delta_{ru}\delta_{\vec\alpha\vec\mu}\delta_{\vec\beta\vec\nu},$$

with $M^{(3)}_{R((t,s),r)\vec\alpha\vec\beta:T((w,v),u)\vec\mu\vec\nu}$ described in (11). We wish to explicitly compute the first term in (22). This term is given as

$$\begin{aligned}M^{(3,1^{\rm st})\ (\sigma_1;\sigma_2)}_{T,u;R,r} &= \frac{|H_p\times H_m||H_{p'}\times H_{m'}|}{(p-1)!(m-1)!}\sum_{R'}\delta_{R'_iT'_k}\delta_{ru}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}}\sum_{i,j,k,l,a,b,c,d}\sum_{t\vdash p,s\vdash m}\sum_{w\vdash p',v\vdash m'}\sum_{\vec\alpha,\vec\beta,\vec\mu,\vec\nu}\times \\ &\times \Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2}B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2}\Gamma^{(t,s)}_{ik;jl}(\sigma_1)B^{(t,s)\to 1_{H_p\times H_m}}_{ik;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_p\times H_m}}_{jl;\beta_1\beta_2}\times \\ &\times \mathrm{Tr}\left(E^{(1)}_{ki}P^{(\vec p,\vec m)}_{t\alpha_1\beta_1;s\alpha_2\beta_2}E^{(p+1)}_{ik}P^{(\vec p',\vec m')}_{w\mu_1\nu_1;v\mu_2\nu_2}\right) \\ &= \sum_{R'}\frac{\delta_{R'_iT'_k}\delta_{ru}}{(p-1)!(m-1)!}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}}\sum_{i,j,k,l,a,b,c,d}\sum_{t\vdash p,s\vdash m}\sum_{w\vdash p',v\vdash m'}\sum_{\vec\alpha,\vec\beta,\vec\mu,\vec\nu}\frac{d_td_sd_wd_v}{p!m!p'!m'!}\times \\ &\times \Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2}B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2}\Gamma^{(t,s)}_{ik;jl}(\sigma_1)B^{(t,s)\to 1_{H_p\times H_m}}_{ik;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_p\times H_m}}_{jl;\beta_1\beta_2}\times \\ &\times \mathrm{Tr}\Bigg(E^{(1)}_{ki}\sum_{\phi_1,\psi_1\in S_p\times S_m}\psi_1|v,\vec p,\vec m\rangle\langle v,\vec p,\vec m|\phi_1^{-1}\Gamma^{(t,s)}_{rm;x_1x_2}(\psi_1)\Gamma^{(t,s)}_{rm;y_1y_2}(\phi_1)\times \\ &\times B^{(t,s)\to 1_{H_p\times H_m}}_{x_1x_2;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_p\times H_m}}_{y_1y_2;\beta_1\beta_2}E^{(p+1)}_{ik}\sum_{\phi_2,\psi_2\in S_p\times S_m}\psi_2|v,\vec p',\vec m'\rangle\langle v,\vec p',\vec m'|\phi_2^{-1}\times \\ &\times \Gamma^{(w,v)}_{np;x_3x_4}(\psi_2)\Gamma^{(w,v)}_{np;y_3y_4}(\phi_2)B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{x_3x_4;\mu_1\mu_2}B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{y_3y_4;\nu_1\nu_2}\Bigg).\end{aligned}$$

We will use the identity

$$\frac{1}{|H_p\times H_m|}\sum_{\gamma_1\in H_p\times H_m}\Gamma^{(t,s)}_{ai;bj}(\gamma_1) = \sum_{\mu,\nu}B^{(t,s)\to 1_{H_p\times H_m}}_{ai;\mu\nu}B^{(t,s)\to 1_{H_p\times H_m}}_{bj;\mu\nu}.$$

Pairing multiple branching coefficients and summing over their multiplicities, we can write our result (with the use of the formula above) as

$$M^{(3,1^{\rm st})\ (\sigma_1;\sigma_2)}_{T,u;R,r} = \frac{1}{|H_p\times H_m|^2|H_{p'}\times H_{m'}|^2}\sum_{R'}\frac{\delta_{R'_iT'_k}\delta_{ru}}{(p-1)!(m-1)!}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}}\sum_{i,j,k,l,a,b,c,d}\sum_{t\vdash p,s\vdash m}\sum_{w\vdash p',v\vdash m'}\times$$
$$\times\frac{d_td_sd_wd_v}{p!m!p'!m'!}\Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})\Gamma^{(t,s)}_{ik;jl}(\sigma_1)\sum_{\phi_1,\psi_1\in S_p\times S_m}\sum_{\phi_2,\psi_2\in S_p\times S_m}\sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\times$$
$$\times\,{\rm Tr}\Bigg(E^{(1)}_{ki}\psi_1|v,\vec p,\vec m\rangle\langle v,\vec p,\vec m|\phi_1^{-1}\Gamma^{(t,s)}_{rm;x_1x_2}(\psi_1)\Gamma^{(t,s)}_{rm;y_1y_2}(\phi_1)\Gamma^{(t,s)}_{ik;x_1x_2}(\gamma_1)\Gamma^{(t,s)}_{jl;y_1y_2}(\gamma_2)\times$$
$$\times E^{(p+1)}_{ik}\psi_2|v,\vec p',\vec m'\rangle\langle v,\vec p',\vec m'|\phi_2^{-1}\Gamma^{(w,v)}_{np;x_3x_4}(\psi_2)\Gamma^{(w,v)}_{np;y_3y_4}(\phi_2)\Gamma^{(w,v)}_{ac;x_3x_4}(\gamma_3)\Gamma^{(w,v)}_{bd;y_3y_4}(\gamma_4)\Bigg),$$

where $\gamma_1,\gamma_2\in H_p\times H_m$ and $\gamma_3,\gamma_4\in H_{p'}\times H_{m'}$. Note the identity

$$\Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})\Gamma^{(w,v)}_{np;x_3x_4}(\psi_2)\Gamma^{(w,v)}_{ac;x_3x_4}(\gamma_3)=\Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})\Gamma^{(w,v)}_{np;ac}(\psi_2\gamma_3^{-1})=\Gamma^{(w,v)}_{np;bd}(\psi_2\gamma_3^{-1}\sigma_2^{-1})=\Gamma^{(w,v)}_{np;bd}(\psi_2\gamma_3\sigma_2^{-1}),$$

where $\sum_{\gamma_3}f(\gamma_3)=\sum_{\gamma_3}f(\gamma_3^{-1})$. Using the above manipulation our result becomes

$$M^{(3,1^{\rm st})\ (\sigma_1;\sigma_2)}_{T,u;R,r} = \frac{1}{|H_p\times H_m|^2|H_{p'}\times H_{m'}|^2}\sum_{R'}\frac{\delta_{R'_iT'_k}\delta_{ru}}{(p-1)!(m-1)!}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}}\sum_{i,j,k,l,a,b,c,d}\sum_{t\vdash p,s\vdash m}\sum_{w\vdash p',v\vdash m'}\times$$
$$\times\frac{d_td_sd_wd_v}{p!m!p'!m'!}\sum_{\phi_1,\psi_1\in S_p\times S_m}\sum_{\phi_2,\psi_2\in S_p\times S_m}\sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4}{\rm Tr}\Bigg(E^{(1)}_{ki}\psi_1|v,\vec p,\vec m\rangle\langle v,\vec p,\vec m|\phi_1^{-1}\times$$
$$\times\Gamma^{(t,s)}_{rm;jl}(\psi_1\gamma_2\sigma_1)\Gamma^{(t,s)}_{rm;jl}(\phi_1\gamma_1)E^{(p+1)}_{ik}\psi_2|v,\vec p',\vec m'\rangle\langle v,\vec p',\vec m'|\phi_2^{-1}\Gamma^{(w,v)}_{np;bd}(\psi_2\gamma_3\sigma_2^{-1})\Gamma^{(w,v)}_{np;bd}(\phi_2\gamma_4)\Bigg).$$

Using (21) we can write

$$M^{(3,1^{\rm st})\ (\sigma_1;\sigma_2)}_{T,u;R,r} = \frac{1}{|H_p\times H_m|^2|H_{p'}\times H_{m'}|^2}\sum_{R'}\frac{\delta_{R'_iT'_k}\delta_{ru}}{(p-1)!(m-1)!}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}}\times$$
$$\times\sum_{\phi_1,\psi_1\in S_p\times S_m}\sum_{\phi_2,\psi_2\in S_p\times S_m}\sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4}{\rm Tr}\Bigg(E^{(1)}_{ki}\psi_1|v,\vec p,\vec m\rangle\langle v,\vec p,\vec m|\phi_1^{-1}\times$$
$$\times\,\delta(\psi_1\gamma_2\sigma_1\gamma_1^{-1}\phi_1^{-1})E^{(p+1)}_{ik}\psi_2|v,\vec p',\vec m'\rangle\langle v,\vec p',\vec m'|\phi_2^{-1}\delta(\psi_2\gamma_3\sigma_2^{-1}\gamma_4^{-1}\phi_2^{-1})\Bigg).$$

Since

$$\delta(\sigma_1\sigma_2^{-1})\neq 0\ \ \text{for}\ \ \sigma_1\sigma_2^{-1}=\mathbb{1}\Rightarrow\sigma_1=\sigma_2,$$

then

$$\delta(\psi_1\gamma_2\sigma_1\gamma_1^{-1}\phi_1^{-1})\neq 0\ \ \text{for}\ \ \psi_1\gamma_2\sigma_1\gamma_1^{-1}\phi_1^{-1}=\mathbb{1}\Rightarrow\phi_1^{-1}=\gamma_1\sigma_1^{-1}\gamma_2^{-1}\psi_1^{-1},$$
$$\delta(\psi_2\gamma_3\sigma_2^{-1}\gamma_4^{-1}\phi_2^{-1})\neq 0\ \ \text{for}\ \ \psi_2\gamma_3\sigma_2^{-1}\gamma_4^{-1}\phi_2^{-1}=\mathbb{1}\Rightarrow\phi_2^{-1}=\gamma_4\sigma_2\gamma_3^{-1}\psi_2^{-1}.$$

Thus, our result becomes

$$M^{(3,1^{\text{st}})\ (\sigma_1;\sigma_2)}_{T,u;R,r} = \frac{1}{|H_p \times H_m|^2 |H_{p'} \times H_{m'}|^2} \sum_{R'} \frac{\delta_{R'_i T'_k} \delta_{ru}}{(p-1)!(m-1)!} \sqrt{\frac{c_{RR'} c_{TT'}}{l_{R_i} l_{T_k}}} \times$$
$$\times \sum_{\psi_1 \in S_p \times S_m} \sum_{\psi_2 \in S_p \times S_m} \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} \text{Tr} \Bigg(E^{(1)}_{ki} \psi_1 |v, \vec{p}, \vec{m}\rangle \langle v, \vec{p}, \vec{m}| \times$$
$$\times \gamma_1 \sigma_1^{-1} \gamma_2^{-1} \psi_1^{-1} E^{(p+1)}_{ik} \psi_2 |v, \vec{p}', \vec{m}'\rangle \langle v, \vec{p}', \vec{m}'| \gamma_4 \sigma_2 \gamma_3^{-1} \psi_2^{-1} \Bigg).$$

Our result can be rewritten as

$$M^{(3,1^{\text{st}})\ (\sigma_1;\sigma_2)}_{T,u;R,r} = \frac{1}{|H_p \times H_m||H_{p'} \times H_{m'}|} \sum_{R'} \frac{\delta_{R'_i T'_k} \delta_{ru}}{(p-1)!(m-1)!} \sqrt{\frac{c_{RR'} c_{TT'}}{l_{R_i} l_{T_k}}} \times$$
$$\times \sum_{\psi_1 \in S_p \times S_m} \sum_{\psi_2 \in S_p \times S_m} \sum_{\gamma_2,\gamma_3} \text{Tr} \Bigg(\langle v, \vec{p}', \vec{m}'| \sigma_2 \gamma_3^{-1} \psi_2^{-1} E^{(1)}_{ki} \psi_1 |v, \vec{p}, \vec{m}\rangle \times$$
$$\times \langle v, \vec{p}, \vec{m}| \sigma_1^{-1} \gamma_2^{-1} \psi_1^{-1} E^{(p+1)}_{ik} \psi_2 |v, \vec{p}', \vec{m}'\rangle \Bigg)$$
$$= \frac{1}{|H_p \times H_m||H_{p'} \times H_{m'}|} \sum_{R'} \frac{\delta_{R'_i T'_k} \delta_{ru}}{(p-1)!(m-1)!} \sqrt{\frac{c_{RR'} c_{TT'}}{l_{R_i} l_{T_k}}} \times$$
$$\times \sum_{\psi_1 \in S_p \times S_m} \sum_{\psi_2 \in S_p \times S_m} \sum_{\gamma_2,\gamma_3} \text{Tr} \Bigg(\langle v, \vec{p}', \vec{m}'| \gamma_3^{-1} \gamma_3 \sigma_2 \gamma_3^{-1} \psi_2^{-1} E^{(1)}_{ki} \psi_1 |v, \vec{p}, \vec{m}\rangle \times$$
$$\times \langle v, \vec{p}, \vec{m}| \gamma_2^{-1} \gamma_2 \sigma_1^{-1} \gamma_2^{-1} \psi_1^{-1} E^{(p+1)}_{ik} \psi_2 |v, \vec{p}', \vec{m}'\rangle \Bigg),$$

where $\mathbb{1} = \gamma_3^{-1}\gamma_3$ was inserted to the left of σ_2 and $\mathbb{1} = \gamma_2^{-1}\gamma_2$ was inserted to the left of σ_1^{-1}. This is done so that we have the conjugacy

$$\gamma_3 \sigma_2 \gamma_3^{-1} \to \tilde{\sigma}_2 \in H \backslash S_m / H,$$
$$\gamma_2 \sigma_1 \gamma_2^{-1} \to \tilde{\sigma}_1 \in H \backslash S_m / H.$$

This helps to simplify our result to obtain

$$M^{(3,1^{\text{st}})\ (\sigma_1;\sigma_2)}_{T,u;R,r} = \frac{1}{|H_p \times H_m||H_{p'} \times H_{m'}|} \sum_{R'} \frac{\delta_{R'_i T'_k} \delta_{ru}}{(p-1)!(m-1)!} \sqrt{\frac{c_{RR'} c_{TT'}}{l_{R_i} l_{T_k}}} \times$$
$$\times \sum_{\psi_1 \in S_p \times S_m} \sum_{\psi_2 \in S_p \times S_m} \sum_{\gamma_2,\gamma_3} \text{Tr} \Bigg(\langle v, \vec{p}', \vec{m}'| \gamma_3^{-1} \tilde{\sigma}_2 \psi_2^{-1} E^{(1)}_{ki} \psi_1 |v, \vec{p}, \vec{m}\rangle \times$$
$$\times \langle v, \vec{p}, \vec{m}| \gamma_2^{-1} \tilde{\sigma}_1^{-1} \psi_1^{-1} E^{(p+1)}_{ik} \psi_2 |v, \vec{p}', \vec{m}'\rangle \Bigg).$$

To complete the simplification of our result we use the fact that $\sum_{\gamma_3} f(\gamma_3^{-1}) = \sum_{\gamma_3} f(\gamma_3)$ (and a similar formula for γ_2^{-1}) and the fact that we can relabel $\tilde{\sigma}_1 \leftrightarrow \sigma_1$ (and $\tilde{\sigma}_2 \leftrightarrow \sigma_2$) to obtain

$$M^{(3,1^{\rm st})\ (\sigma_1;\sigma_2)}_{T,u;R,r} = \sum_{R'} \frac{\delta_{R'_i T'_k}\delta_{ru}}{(p-1)!(m-1)!}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}} \sum_{\psi_1\in S_p\times S_m}\sum_{\psi_2\in S_p\times S_m} \times$$
$$\times\, \mathrm{Tr}\left(\langle v,\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{ki}\psi_1|v,\vec{p},\vec{m}\rangle\langle v,\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ik}\psi_2|v,\vec{p}',\vec{m}'\rangle\right).$$

Performing similar steps for the remaining terms and summing, we can write the explicit form for the matrix element of the dilatation operator acting on the normalised state as

$$M^{(3)\ (\sigma_1;\sigma_2)}_{T,u;R,r} = \sum_{R'} \frac{\delta_{R'_i T'_k}\delta_{ru}}{(p-1)!(m-1)!}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}} \sum_{\psi_1\in S_p\times S_m}\sum_{\psi_2\in S_p\times S_m} \times$$
$$\times\, \mathrm{Tr}\Big(\langle v,\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{ki}\psi_1|v,\vec{p},\vec{m}\rangle\langle v,\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ik}\psi_2|v,\vec{p}',\vec{m}'\rangle -$$
$$- \langle v,\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{kc}E^{(p+1)}_{ci}\psi_1|v,\vec{p},\vec{m}\rangle\langle v,\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ia}E^{(p+1)}_{ak}\psi_2|v,\vec{p}',\vec{m}'\rangle -$$
$$- \langle v,\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{ci}E^{(p+1)}_{kc}\psi_1|v,\vec{p},\vec{m}\rangle\langle v,\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ak}E^{(p+1)}_{ia}\psi_2|v,\vec{p}',\vec{m}'\rangle +$$
$$+ \langle v,\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(p+1)}_{ki}\psi_1|v,\vec{p},\vec{m}\rangle\langle v,\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ik}\psi_2|v,\vec{p}',\vec{m}'\rangle \Big).$$

One can compute the matrix element above explicitly for given examples.

3 Anomalous dimensions from boson lattice models

This chapter is based on [6] and it is my original work with de Mello Koch and Mahu.

3.1 Introduction

Motivated by the AdS/CFT correspondence[2, 20, 21], there has been dramatic progress in computing the planar spectrum of anomalous dimensions in $\mathcal{N} = 4$ super Yang-Mills theory. The planar spectrum is now known, in principle, to all orders in the 't Hooft coupling [22]. This has been possible thanks to the discovery of integrability[17, 23] in the planar limit of the theory. This spectrum of anomalous dimensions reproduces classical string energies on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime, in the dual string theory[24].

As mentioned in subsection 2.1, much less is known about $\mathcal{N} = 4$ super Yang-Mills theory outside the planar limit. There are many distinct large N but non-planar limits of the theory that could be considered and these correspond to a variety of fascinating physical problems. For example, the problem of considering new spacetime geometries (including black hole solutions) corresponds to considering operators with a bare dimension of order N^2[25], while giant graviton branes[26, 27, 28] are dual to operators with a bare dimension of order N. The planar limit does not correctly capture the dynamics of these operators[29, 30].

Although much less is known about these large N but non-planar limits, some progress has been made. Approaches based on group representation theory provide a powerful tool, essentially because they allow us to map the problem of the dynamics of the non-planar limit - summing the ribbon graphs contributing to correlation functions - into a purely algebraic problem in group theory. Typically, it can be phrased as the construction of a collection of projection operators and their properties. Once the algebraic problem is properly formulated, systematic approaches to it can be developed. As an example of this approach, bases of local gauge invariant operators have been given[12, 15, 31, 32, 33, 34, 35, 36]. These bases provide a good starting point from which the anomalous dimensions can be studied. This is basically because they diagonalize the free field two point function and, at weak coupling, operator mixing is highly constrained[37, 38, 39, 40, 41]. The resulting operators have a complicated multi-trace structure, quite different to the single trace structure relevant for the planar limit and its mapping to an integrable spin chain. The spectrum of anomalous dimensions has been computed for operators that are small deformations of $\frac{1}{2}$-BPS operators. Problems with two distinct characters have been solved: It is possible to simply treat all fields in the operator on the same footing, construct the basis and then diagonalize [42, 43, 44, 45] or alternatively, one can build operators that realize a spacetime geometry or a giant graviton brane and use words constructed from the fields of the CFT to describe string excitations[37, 46, 47]. In the approach that treats all fields on the same footing, one simply defines the operators of the basis and considers the diagonalization of the dilatation operator with no physical input from the dual gravity description. When considering states dual to systems of giant gravitons, the Gauss Law of the dual giant world volume gauge theory emerges, so that in this approach we see open string and membranes are present in the CFT Hilbert space. When using words to describe string excitations, computations in the CFT reproduce the classical values of energies computed in string theory[46, 47], the worldsheet S-matrix[5] and has lead to the discovery of integrable subsectors for string excitations of certain LLM backgrounds[47]. Clearly, this is a rich problem with hidden simplicity, so that further study of these limits are bound to be fruitful. The existence of this hidden simplicity is not unexpected: conventional lore of the large N limit identifies $1/N$ as the gravitational interaction, so that the $N \to \infty$ limit, in which this interaction is turned off, should be a simple limit.

One next step that can be contemplated, is to go beyond small perturbations of the $\frac{1}{2}$-BPS sector. This problem is our main motivation in this study, and we will take a small step in this direction. We will study operators constructed from three complex adjoint scalars X, Y, Z of $\mathcal{N}=4$ super Yang-Mills theory, reviewed above.

Operators that are a small perturbation of a $\frac{1}{2}$-BPS operator are constructed using mainly Z fields. For these operators, interactions between the X, Y fields are subdominant to interactions between X, Z and between Y, Z fields and can hence be neglected. As we move further from the original $\frac{1}{2}$-BPS operator, more and more X, Y fields are added. At some point the interactions between the X, Y fields can no longer be neglected. Dealing with these interactions is the focus of our study. We will argue that this is a well defined problem, that can be solved, often explicitly. This is accomplished by phrasing the new X, Y interactions as a lattice model, for essentially free bosons. Thus, we finally land up with a simple problem that is familiar and can be solved. This is the basic achievement of [6].

Our results show a fascinating structure that deserves to be discussed. The mapping to the lattice model associates a harmonic oscillator to both the X field and to the Y field. Earlier results [44] treating the leading term, performed the diagonalization by associating a harmonic oscillator to the Z field, so that in the end we seem to be seeing an equality in the description of the three scalar fields. An even-handed treatment of all three fields is a big step towards being able to treat operators constructed with equal numbers of X, Y and Z fields. This would most certainly go beyond the $\frac{1}{2}$-BPS sector, the main motivation for our study.

In the next subsection we review the action of the one loop dilatation operator D_2. The action of D_2 in the $SU(3)$ sector, in the restricted Schur polynomial basis, has been evaluated previously[19] and we simply quote and use the result. We then move to the Gauss graph basis of [45], in which the terms in D_2 arising from Z, Y or Z, X interactions are diagonal. Again, this is a known result and we simply use it. The Gauss graph basis has a natural interpretation in terms of giant graviton branes and their open string excitations. We will often use this language of branes and strings. We then come to the central term of interest: the term in D_2 arising from X, Y interactions. Denote this term by D_2^{XY}. We will carefully evaluate this term, arriving at a rather simple formula, which is the starting point for subsection 3.3. The explicit expression for D_2^{XY} can easily be identified with a lattice model for a collection of bosons. The giant gravitons define the sites of this lattice, and the open string excitations determine the lattice Hamiltonian. Subsection 3.4 diagonalizes the dilatation operator for a number of giants plus open string configurations, arriving at detailed and explicit expressions both for the anomalous dimensions and for the operators of a definite scaling dimension. Our conclusions and some discussion are given in subsection 3.5.

3.2 Action of the One Loop Dilatation Operator

We combine the 6 hermitian adjoint scalars of $\mathcal{N}=4$ super Yang-Mills theory into three complex combinations, denoted X, Y, Z. The operators we consider are constructed using n Zs, m Ys and p Xs. Operators that are dual to giant graviton branes are constructed using $n+m+p \sim N$ fields. We will focus on operators that are small deformations of $\frac{1}{2}$-BPS operators, achieved by choosing $n \gg m+p$. We will fix $\frac{m}{p} \sim 1$ as $N \to \infty$ and treat $\frac{m}{n}$ as a small parameter. The collection of operators constructed using X, Y, Z fields are often referred to as the $SU(3)$ sector. This is not strictly speaking correct since these operators do mix with operators containing fermions. At one loop however, this is a closed sector.

Our starting point is the action of the one loop dilatation operator of the $SU(3)$ sector

$$D_2 = D_2^{YZ} + D_2^{XZ} + D_2^{XY}, \tag{23}$$

where

$$D_2^{AB} \equiv g_{YM}^2 \operatorname{Tr}\left([A, B][\partial_A, \partial_B]\right) \tag{24}$$

on the restricted Schur polynomial basis. This has been evaluated in [19]. Further, the terms D_2^{YZ} and D_2^{XZ} have been diagonalized. The operators of a definite scaling dimension $O_{R,r}(\sigma)$, called Gauss graph operators[43, 45], are labelled by a pair of Young diagrams $R \vdash n+m+p$ and $r \vdash n$ as well as a permutation $\sigma \in S_m \times S_p$. Although these labels arise when diagonalizing D_2^{YZ} and D_2^{XZ} in the CFT, they have a natural interpretation in the dual gravitational description in terms of giant graviton branes plus open string excitations. A Young diagram R that has q rows corresponds to a system of q dual giant gravitons or giant gravitons in the AdS_5 space.. The Y and X fields describe the open string excitations of these giants, so that there are $m+p$ open strings in total. We can describe the state of the system using a graph, with nodes of the graph representing the branes (and hence rows of R) and directed edges of the graph describing the open string excitations (represented by X and Y fields in the CFT). Each directed edge ends on any two (not necessarily distinct) of the q branes. The only configurations that appear when D_2^{YZ} and D_2^{XZ} are diagonalized have the same number of strings starting or terminating on any given giant, for the X and Y strings separately[19, 45]. Thus the Gauss Law of the brane world volume theory implied by the fact that the giant graviton has a compact world volume[11] emerges rather naturally in the CFT description. Since every terminating edge endpoint can be associated to a unique emanating endpoint, we can give a nice description of how the open strings are connected to the giants by specifying how the terminating and emanating endpoints are associated. The permutation $\sigma \in S_m \times S_p$ describes how the m Y's and the p X's are draped between the q giant gravitons by describing this association[19, 45]. The explicit form of the Gauss graph operators is[19, 45]

$$O_{R,r}^{\vec{m},\vec{p}}(\sigma) = \frac{|H_X \times H_Y|}{\sqrt{p!m!}} \sum_{j,k} \sum_{s\vdash m} \sum_{t\vdash p} \sum_{\vec{\mu}_1,\vec{\mu}_2} \sqrt{d_s d_t}\,\Gamma_{jk}^{(s,t)}(\sigma) \times B_{j\vec{\mu}_1}^{(s,t)\to 1_{H_X\times H_Y}} B_{k\vec{\mu}_2}^{(s,t)\to 1_{H_X\times H_Y}} O_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}. \quad (25)$$

Each box in R is associated with one of the complex fields. r is a label for the Z fields. The graph σ encodes important information. The number of Y (or X) strings terminating on the ith node which equals the number of Y (or X) strings emanating from the ith node is denoted by m_i (or p_i). m_i (or p_i) also counts the number of boxes in the ith row of R that correspond to Y (or X) fields. We will often assemble m_i and p_i into the vectors $\vec{m}$ and $\vec{p}$. The number of Y (or X) strings stretching between nodes i and k is denoted m_{ik} (or p_{ik}), while the number of strings stretching from node i to node k is denoted $m_{i\to k}$ (or $p_{i\to k}$). A Young diagram with k boxes $a \vdash k$ labels an irreducible representation of S_k with dimension d_a. The branching coefficients $B_{j\vec{\mu}_1}^{(s,t)\to 1_{H_X\times H_Y}}$ resolve the operator that projects from (s,t), with $s \vdash m$, $t \vdash p$, an irreducible representation of $S_m \times S_p$, to the trivial (identity) representation of the product group $H_Y \times H_X$ with $H_Y = S_{m_1} \times S_{m_2} \times \cdots S_{m_q}$ and $H_X = S_{p_1} \times S_{p_2} \times \cdots S_{p_q}$, i.e.

$$\frac{1}{|H_X \times H_Y|} \sum_{\gamma \in H_X \times H_Y} \Gamma_{ik}^{(s,t)}(\gamma) = \sum_{\vec{\mu}} B_{i\vec{\mu}}^{(s,t)\to 1_{H_X\times H_Y}} B_{k\vec{\mu}}^{(s,t)\to 1_{H_X\times H_Y}}. \quad (26)$$

$\Gamma_{jk}^{(s,t)}(\sigma)$ is a matrix (with row and column indices jk) representing $\sigma \in S_m \times S_p$ in irreducible representation (s,t) - these matrices are reviewed in subsection 2.9. The operators $O_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}$ are normalized versions of the restricted Schur polynomials [15]

$$\chi_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}(Z,Y,X) = \frac{1}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \chi_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}(\sigma)\,\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m} X^{\otimes p}), \quad (27)$$

which themselves provide a basis for the gauge invariant operators of the theory. The restricted characters $\chi_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}(\sigma)$ are defined by tracing the matrix representing group element σ in representation R over the subspace giving an irreducible representation (t,s,r) of the $S_n\times S_m\times S_p$ subgroup. There is more than one choice for this subspace and the multiplicity labels $\vec{\mu}_1\vec{\mu}_2$ resolve this ambiguity, for the row and column index of the trace. The operators $O_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}$ given by

$$O_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}=\sqrt{\frac{\text{hooks}_r\text{hooks}_s\text{hooks}_t}{\text{hooks}_R f_R}}\chi_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2} \tag{28}$$

have unit two point function. hooks_r stands for the product of hook lengths of Young diagram r and f_R stands for the product of the factors of Young diagram R - as defined in subsection 2.2. The action of the dilatation operator on the Gauss graph operators is [19, 43, 45]

$$\begin{aligned} D_2^{YZ}O_{R,r}^{\vec{m},\vec{p}}(\sigma) &= -g_{YM}^2\sum_{i<j}m_{ij}(\sigma)\Delta_{ij}O_{R,r}^{\vec{m},\vec{p}}(\sigma),\\ D_2^{XZ}O_{R,r}^{\vec{m},\vec{p}}(\sigma) &= -g_{YM}^2\sum_{i<j}p_{ij}(\sigma)\Delta_{ij}O_{R,r}^{\vec{m},\vec{p}}(\sigma), \end{aligned} \tag{29}$$

where $\Delta_{ij}=\Delta_{ij}^-+\Delta_{ij}^0+\Delta_{ij}^+$[44]. We will now spell out the action of the operators Δ_{ij}^+, Δ_{ij}^0 and Δ_{ij}^-. Denote the row lengths of r by l_{r_i}. The Young diagram r_{ij}^+ is obtained by deleting a box from row j and adding it to row i. The Young diagram r_{ij}^- is obtained by deleting a box from row i and adding it to row j. In terms of these Young diagrams we have

$$\Delta_{ij}^0 O_{R,r}^{\vec{m},\vec{p}}(\sigma)=-(2N+l_{r_i}+l_{r_j})O_{R,r}^{\vec{m},\vec{p}}(\sigma), \tag{30}$$

$$\Delta_{ij}^+ O_{R,r}^{\vec{m},\vec{p}}(\sigma)=\sqrt{(N+l_{r_i})(N+l_{r_j})}O_{R_{ij}^+,r_{ij}^+}^{\vec{m},\vec{p}}(\sigma), \tag{31}$$

$$\Delta_{ij}^- O_{R,r}^{\vec{m},\vec{p}}(\sigma)=\sqrt{(N+l_{r_i})(N+l_{r_j})}O_{R_{ij}^-,r_{ij}^-}^{\vec{m},\vec{p}}(\sigma). \tag{32}$$

Notice that D_2^{YZ} and D_2^{XZ} in (29) are not yet diagonal: they still mix operators with different R,r labels. This last diagonalization however, is rather simple: it maps into diagonalizing a collection of decoupled oscillators as demonstrated in [44]. We will call these Z oscillators, since they are associated to the r label which organizes the Z fields. It is clear that D_2^{XY} does not act on the r label so that in the end, the contribution from D_2^{XY} simply shifts the ground state eigenvalue of the Z oscillators.

We will now focus on the term D_2^{XY}. Recall that our operators are built with many more Z fields, than X or Y fields ($n\gg p+m$). Since this term contains no derivatives with respect to Z it is subleading (of order $\frac{m}{n}$) when compared to D_2^{YZ} and D_2^{XZ}. Diagonalizing this operator is the main goal of this chapter, so it is useful to sketch the derivation of the matrix elements of D_2^{XY} in the Gauss graph basis. We will simply quote existing results that we need, giving complete details only for the final stages of the evaluation, which are novel. The reader will find useful background material in [19]. The action of this term on the restricted Schur polynomial basis was computed in [19]. The result is

$$D_2^{XY}O_{R,(t,s,r)\vec{\mu}\vec{\nu}}=\sum_{R'}\sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}}\mathcal{C}\,\text{Tr}_{R\oplus T}\left(\left[P_1,\Gamma^R(1,p+1)\right]I_{R'T'}\left[P_2,\Gamma^T(1,p+1)\right]I_{T',R'}\right)O_{T,(y,x,w)\vec{\beta}\vec{\alpha}},$$

where

$$\mathcal{C} \;=\; -g_{YM}^2c_{RR'}\frac{d_Tmp}{d_xd_yd_w(n+m+p)d_{R'}}\sqrt{\frac{f_T\text{hooks}_T\text{hooks}_r\text{hooks}_s\text{hooks}_t}{f_R\text{hooks}_R\text{hooks}_w\text{hooks}_x\text{hooks}_y}},$$

$$P_1 = P_{R,(t,s,r)\vec{\mu}\vec{\nu}}, \qquad P_2 = P_{T,(y,x,w)\vec{\alpha}\vec{\beta}}. \tag{33}$$

$\Gamma^S(\sigma)$ is the matrix representing $\sigma \in S_{n+m+p}$ in irreducible representation $S \vdash n+m+p$. Young diagram R' is obtained from Young diagram R by dropping a single box, with $c_{RR'}$ denoting the factor of this box. $I_{T'R'}$, $I_{R'T'}$, P_1 and P_2 are intertwining maps. $I_{T'R'}$ maps from the carrier space of R' to the carrier space of T'. It is only non-vanishing if T' and R' are equal as Young diagrams implying that operators labelled by R and T can only mix if they differ by the placement of a single box. The operators P_1 and P_2 are the intertwining maps used in the construction of the restricted Schur polynomials. These expressions and definitions are reviewed earlier in this thesis - we quote them to remind the reader of their meaning and significance. It is challenging to evaluate the above expression explicitly, basically because it is difficult to construct P_1 and P_2. However, the above expression has not yet employed the simplifications of large N. To do this, following [43] we will use the displaced corners approximation. This approximation assumes that the difference of the number of boxes in any two rows of R is of order N, as described in subsection 2.9. In this situation the action of the $S_m \times S_p$ subgroup simplifies so much that the relevant restricted characters can be computed and a complete explicit characterization of the multiplicity labels on the restricted Schur polynomials is possible. The corrections to the displaced corners approximation are suppressed by the inverse of the difference in length of rows of R. After applying the approximation we obtain[19]

$$D_2^{XY} O_{R,(t,s,r)\vec{\mu}\vec{\nu}} = \sum_{T,(w,v,u)\vec{\alpha}\vec{\beta}} \tilde{M}_{R,(t,s,r)\vec{\mu}\vec{\nu}\ T,(w,v,u)\vec{\alpha}\vec{\beta}} O_{T,(w,v,u)\vec{\alpha}\vec{\beta}}, \tag{34}$$

where

$$\begin{aligned} \tilde{M}_{R,(t,s,r)\vec{\mu}\vec{\nu}\ T,(w,v,u)\vec{\alpha}\vec{\beta}} = -g_{YM}^2 \sum_{R'} \delta_{R'_i T'_k} \delta_{ru} \frac{pm}{\sqrt{d_s d_t d_w d_v}} \sqrt{\frac{c_{RR'} c_{TT'}}{l_{R_i} l_{T_k}}} \times \\ \mathrm{Tr}\Big[E^{(1)}_{ki} P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2} E^{(p+1)}_{ik} P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2} - E^{(1)}_{ci} E^{(p+1)}_{kc} P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2} E^{(1)}_{ak} E^{(p+1)}_{ia} P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2} \\ - E^{(1)}_{kc} E^{(p+1)}_{ci} P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2} E^{(1)}_{ia} E^{(p+1)}_{ak} P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2} + E^{(p+1)}_{ki} P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2} E^{(1)}_{ik} P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2} \Big]. \end{aligned} \tag{35}$$

The trace in this expression is over the tensor product $V_q^{\otimes n+m}$ where V_q is the fundamental representation of $U(q)$. The intertwining maps used to define the restricted Schur polynomials (P_1 and P_2 above) factor into an action on the boxes associated to the Z fields, an action on the boxes associated to the Y fields and an action on the boxes associated to the X fields. The intertwining maps[3] $P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}$ and $P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}$ are the actions of the intertwining maps on the X and Y fields only. This happens because the trace over the Z field indices, which is simple as the dilatation operator D_2^{XY} does not act on the Z fields, has been performed. Young diagram R'_i is obtained from R by dropping a single box from row i and T'_k from T by dropping a single box from row k.

The result (35) gives the D_2^{XY} term in the dilatation operator, as a matrix that must be diagonalized. As we will see, all three terms in D_2 are simultaneously diagonalizable at large N so that it is convenient to employ the Gauss graph basis which already diagonalizes both D_2^{ZY} and D_2^{ZX}. The problem of diagonalizing D_2^{XY} then amounts to a diagonalization on degenerate subspaces of D_2^{ZY} and D_2^{ZX}. Thus, the original diagonalization of an enormous matrix is replaced by diagonalizing a number of smaller matrices - a significant simplification. Applying the results of [19], we find that, after the change in basis

$$D_2^{XY} \hat{O}^{\vec{m},\vec{p}}_{R,r}(\sigma_1) = M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2} \hat{O}^{\vec{m},\vec{p}}_{T,t}(\sigma_2), \tag{36}$$

[3] A very explicit algorithm for the construction of these maps has been given in [43].

where

$$\begin{aligned}
M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2} &= -g_{YM}^2 \frac{1}{\sqrt{|O^{\vec{m},\vec{p}}_{R,r}(\sigma_1)|^2 |O^{\vec{m},\vec{p}}_{T,t}(\sigma_2)|^2}} \times \\
&\sum_{R'} \frac{\delta_{R'_i T'_k}\delta_{ru}}{(p-1)!(m-1)!}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}} \sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}}\ \sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}'}} \\
&\Big[\langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{ki}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ik}\psi_2|\vec{p}',\vec{m}'\rangle \\
&-\langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{ci}E^{(p+1)}_{kc}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ak}E^{(p+1)}_{ia}\psi_2|\vec{p}',\vec{m}'\rangle \\
&-\langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{kc}E^{(p+1)}_{ci}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ia}E^{(p+1)}_{ak}\psi_2|\vec{p}',\vec{m}'\rangle \\
&+\langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(p+1)}_{ki}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ik}\psi_2|\vec{p}',\vec{m}'\rangle\Big]. \qquad (37)
\end{aligned}$$

Here the Gauss graph operators $\hat{O}^{\vec{m},\vec{p}}_{R,r}(\sigma_1)$ are normalized to have a unit two point function. They are related to the operators introduced in (25) as follows

$$O^{\vec{m},\vec{p}}_{R,r}(\sigma) = \sqrt{\prod_{i=1}^{q} m_{ii}(\sigma)!p_{ii}(\sigma)! \prod_{k,l,k\neq l} m_{k\to l}(\sigma)!p_{k\to l}(\sigma)!}\ \hat{O}^{\vec{m},\vec{p}}_{R,r}(\sigma). \qquad (38)$$

Introduce the vectors $(v^{(i)})_a = \delta_{ia}$ which form a basis for V_q. The vector $|\vec{p},\vec{m}\rangle$ is defined as follows

$$|\vec{p},\vec{m}\rangle = |\vec{p}\rangle \otimes |\vec{m}\rangle, \qquad (39)$$

where

$$\begin{aligned}
|\vec{p}\rangle &= (v^{(1)})^{\otimes p_1}\otimes\cdots\otimes(v^{(q)})^{\otimes p_q}, \\
|\vec{m}\rangle &= (v^{(1)})^{\otimes m_1}\otimes\cdots\otimes(v^{(q)})^{\otimes m_q}. \qquad (40)
\end{aligned}$$

We will now explain how the sums over ψ_1 and ψ_2 in (37) can be evaluated. This discussion is novel and is one of the new contributions in [6]. Consider the term

$$T_1 = \sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}}\ \sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}'}} \langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{ki}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ik}\psi_2|\vec{p}',\vec{m}'\rangle.$$

The dependence on the permutations σ_1,σ_2 can be simplified with the following change of variables: replace ψ_2 with $\tilde{\psi}_2$ where

$$\tilde{\psi}_2 = \psi_2\sigma_2^{-1} \qquad \Rightarrow \qquad \tilde{\psi}_2^{-1} = \sigma_2\psi_2^{-1}. \qquad (41)$$

After relabelling $\tilde{\psi}_2 \to \psi_2$ and taking the transpose of the first factor which is a real number, we find

$$T_1 = \sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}}\ \sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}'}} \langle\vec{p},\vec{m}|\psi_1^{-1}E^{(1)}_{ik}\psi_2|\vec{p}',\vec{m}'\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ik}\psi_2\sigma_2|\vec{p}',\vec{m}'\rangle.$$

If $i\neq k$, the matrix element $\langle\vec{p},\vec{m}|\psi_1^{-1}E^{(1)}_{ik}\psi_2|\vec{p}',\vec{m}'\rangle$ is only non-vanishing if $\vec{p}\neq\vec{p}'$ and $\vec{m}=\vec{m}'$, while the matrix element $\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ik}\psi_2\sigma_2|\vec{p}',\vec{m}'\rangle$ is only non-vanishing if $\vec{p}=\vec{p}'$ and $\vec{m}\neq\vec{m}'$. Thus, T_1 vanishes for $i\neq k$. Indicate this explicitly as follows

$$T_1 = \delta_{ik}\sum_{\psi_1,\psi_2\in S_{\vec{p}}\times S_{\vec{m}}} \langle\vec{p},\vec{m}|\psi_1^{-1}E^{(1)}_{ii}\psi_2|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ii}\psi_2\sigma_2|\vec{p},\vec{m}\rangle.$$

To simplify this expression further, note that $E^{(1)}_{ii}|\vec{p},\vec{m}\rangle$ is only non-zero if vector $v^{(i)}$ occupies slot one in the vector $|\vec{p}\rangle$. In this case $E^{(1)}_{ii}|\vec{p},\vec{m}\rangle = |\vec{p},\vec{m}\rangle$. Since ψ_1 and ψ_2 shuffle the vectors in $|\vec{p},\vec{m}\rangle$ into all possible locations, $E^{(1)}_{ii}$ will in the end count how many times the vector $v^{(i)}$ appears in $|\vec{p},\vec{m}\rangle$. This is given by p_i introduced above. A similar argument applies to $E^{(p+1)}|\vec{p},\vec{m}\rangle$. Thus, we obtain

$$\begin{aligned}
T_1 &= \delta_{ik}\frac{p_i}{p}\frac{m_i}{m}\sum_{\psi_1,\psi_2\in S_{\vec{p}}\times S_{\vec{m}}}\langle\vec{p},\vec{m}|\psi_1^{-1}\psi_2|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}\psi_2\sigma_2|\vec{p},\vec{m}\rangle \\
&= \delta_{ik}\frac{p_i}{p}\frac{m_i}{m}\sum_{\psi_1,\psi_2\in S_{\vec{p}}\times S_{\vec{m}}}\sum_{h_1,h_2\in H_X\times H_Y}\delta(\psi_1^{-1}\psi_2h_1)\delta(\sigma_1^{-1}\psi_1^{-1}\psi_2\sigma_2h_2).
\end{aligned}$$

Now, perform the following change of summation variables $\psi_1\to\tilde{\psi}_1$ with

$$\psi_1 = \psi_2\tilde{\psi}_1. \tag{42}$$

The summand is now independent of ψ_2 so that after summing over ψ_2 and relabelling $\tilde{\psi}_1\to\psi_1$ we find

$$T_1 = \delta_{ik}(p-1)!(m-1)!p_im_i\sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}}\sum_{h_1,h_2\in H_X\times H_Y}\delta(\psi_1h_1)\delta(\sigma_1^{-1}\psi_1\sigma_2h_2).$$

Summing over ψ_1 now gives

$$T_1 = \delta_{ik}(p-1)!(m-1)!p_im_i\sum_{h_1,h_2\in H_X\times H_Y}\delta(\sigma_1^{-1}h_1^{-1}\sigma_2h_2). \tag{43}$$

We also need to consider the term

$$\begin{aligned}
T_4 &= \sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}}\sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}'}}\langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(p+1)}_{ki}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ik}\psi_2|\vec{p}',\vec{m}'\rangle \\
&= \sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}}\sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}'}}\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ik}\psi_2|\vec{p}',\vec{m}'\rangle\langle\vec{p},\vec{m}|\psi_1^{-1}E^{(p+1)}_{ik}\psi_2\sigma_2^{-1}|\vec{p}',\vec{m}'\rangle.
\end{aligned}$$

Changing variables $\psi_1^{-1}\to\sigma_1^{-1}\psi_1^{-1}$ shows that $T_4 = T_1$ and hence

$$T_1 + T_4 = 2\delta_{ik}(p-1)!(m-1)!p_im_i\sum_{h_1,h_2\in H_X\times H_Y}\delta(\sigma_1^{-1}h_1^{-1}\sigma_2h_2). \tag{44}$$

The next sum we consider is

$$T_2 = \sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}}\sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}'}}\langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{ci}E^{(p+1)}_{kc}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ak}E^{(p+1)}_{ia}\psi_2|\vec{p}',\vec{m}'\rangle.$$

Changing variables $\psi_2^{-1}\to\tilde{\psi}_2^{-1}$ with

$$\tilde{\psi}_2^{-1} = \sigma_2\psi_2^{-1} \qquad \Rightarrow \qquad \tilde{\psi}_2 = \psi_2\sigma_2^{-1}, \tag{45}$$

the sum becomes

$$\begin{aligned}
T_2 = &\sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}}\sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}'}}\langle\vec{p}',\vec{m}'|\psi_2^{-1}E^{(1)}_{ci}E^{(p+1)}_{kc}\psi_1|\vec{p},\vec{m}\rangle \\
&\times\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ak}E^{(p+1)}_{ia}\psi_2\sigma_2|\vec{p}',\vec{m}'\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\psi_1\in S_{\vec p}\times S_{\vec m}} \sum_{\psi_2\in S_{\vec p'}\times S_{\vec m'}} \langle \vec p', \vec m'|\psi_2^{-1}\psi_1 E_{ci}^{\psi_1^{-1}(1)} E_{kc}^{\psi_1^{-1}(p+1)}|\vec p,\vec m\rangle \\
&\quad \times\langle \vec p,\vec m|\sigma_1^{-1}E_{ak}^{\psi_1^{-1}(1)}E_{ia}^{\psi_1^{-1}(p+1)}\psi_1^{-1}\psi_2\sigma_2|\vec p',\vec m'\rangle.
\end{aligned}
$$

Change variables $\psi_2\to\rho$ with $\rho=\psi_1^{-1}\psi_2$ and relabel $\rho\to\psi_2$ to find

$$
\begin{aligned}
T_2 &= \sum_{\psi_1\in S_{\vec p}\times S_{\vec m}} \sum_{\psi_2\in S_{\vec p'}\times S_{\vec m'}} \langle \vec p', \vec m'|\psi_2^{-1} E_{ci}^{\psi_1^{-1}(1)} E_{kc}^{\psi_1^{-1}(p+1)}|\vec p,\vec m\rangle \\
&\quad \times\langle \vec p,\vec m|\sigma_1^{-1}E_{ak}^{\psi_1^{-1}(1)}E_{ia}^{\psi_1^{-1}(p+1)}\psi_2\sigma_2|\vec p',\vec m'\rangle.
\end{aligned}
$$

We will use $\hat b$ to denote the q dimensional vector that has all entries zero except the bth entry which is 1. For a non-zero contribution the first factor requires

$$
\begin{aligned}
\vec p-\hat i+\hat c &= \vec p', \\
\vec m-\vec c+\vec k &= \vec m'
\end{aligned}
\tag{46}
$$

and the second factor requires

$$
\begin{aligned}
\vec m-\hat i+\hat a &= \vec m', \\
\vec p-\vec a+\vec k &= \vec p'.
\end{aligned}
\tag{47}
$$

There are two solutions:

Case 1: $\hat c=\hat i$ and $\hat a=\hat k$. In this case $\vec p=\vec p'$ and $\vec m-\hat i+\hat k=\vec m'$.
Case 2: $\hat c=\hat k$ and $\hat a=\hat i$. In this case $\vec m=\vec m'$ and $\vec p-\hat i+\hat k=\vec p'$.

For case 1

$$
\begin{aligned}
T_2 &= \sum_{\psi_1\in S_{\vec p}\times S_{\vec m}} \sum_{\psi_2\in S_{\vec p'}\times S_{\vec m'}} \langle \vec p', \vec m'|\psi_2^{-1} E_{ii}^{\psi_1^{-1}(1)} E_{ki}^{\psi_1^{-1}(p+1)}|\vec p,\vec m\rangle \\
&\quad \times\langle \vec p,\vec m|\sigma_1^{-1}E_{kk}^{\psi_1^{-1}(1)}E_{ik}^{\psi_1^{-1}(p+1)}\psi_2\sigma_2|\vec p',\vec m'\rangle.
\end{aligned}
$$

Consider the sum over ψ_1. Due to the factor $E_{ki}^{\psi_1^{-1}(p+1)}$ we get a non-zero contribution from the slots $p+1,p+2,\cdots,p+m$ (a Y string) if a string starts from node k and ends at node i. Thus, the sum over ψ_1 gives

$$
\begin{aligned}
T_2 &= (p-1)!(m-1)!p_{i\to k}m_{ii}\sum_{\psi_2\in S_{\vec p}\times S_{\vec m'}}\langle \vec p,\vec m'|\psi_2^{-1}|\vec p,\vec m'\rangle\langle\vec p,\vec m'|\sigma_1^{-1}\psi_2\sigma_2|\vec p,\vec m'\rangle \\
&= (p-1)!(m-1)!p_{i\to k}m_{ii}\sum_{\psi_2\in S_{\vec p}\times S_{\vec m'}}\sum_{h_1,h_2\in H_X\times H_Y}\delta(\psi_2^{-1}h_1)\delta(\sigma_1^{-1}\psi_2\sigma_2h_2) \\
&= (p-1)!(m-1)!p_{i\to k}m_{ii}\sum_{h_1,h_2\in H_X\times H_Y}\delta(\sigma_1^{-1}h_1\sigma_2h_2).
\end{aligned}
\tag{48}
$$

For case 2

$$
\begin{aligned}
T_2 &= \sum_{\psi_1\in S_{\vec p}\times S_{\vec m}} \sum_{\psi_2\in S_{\vec p'}\times S_{\vec m}} \langle \vec p', \vec m|\psi_2^{-1} E_{ki}^{\psi_1^{-1}(1)} E_{kk}^{\psi_1^{-1}(p+1)}|\vec p,\vec m\rangle \\
&\quad \times\langle \vec p,\vec m|\sigma_1^{-1}E_{ik}^{\psi_1^{-1}(1)}E_{ii}^{\psi_1^{-1}(p+1)}\psi_2\sigma_2|\vec p',\vec m\rangle.
\end{aligned}
$$

Consider the sum over ψ_1. We get a non-zero contribution for each Y string starting from node k which ends at node i. After summing over ψ_1 we have

$$
T_2 = (p-1)!(m-1)!p_{ii}m_{k\to i}\sum_{\psi_2\in S_{\vec p'}\times S_{\vec m}}\langle \vec p',\vec m|\psi_2^{-1}|\vec p,\vec m'\rangle\langle\vec p',\vec m|\sigma_1^{-1}\psi_2\sigma_2|\vec p',\vec m\rangle
$$

$$
\begin{aligned}
&= (p-1)!(m-1)!p_{ii}m_{k\to i}\sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}}}\ \sum_{h_1,h_2\in H_X\times H_Y}\delta(\psi_2^{-1}h_1)\delta(\sigma_1^{-1}\psi_2\sigma_2h_2)\\
&= (p-1)!(m-1)!p_{ii}m_{k\to i}\sum_{h_1,h_2\in H_X\times H_Y}\delta(\sigma_1^{-1}h_1\sigma_2h_2). \qquad (49)
\end{aligned}
$$

Armed with these sums, we now obtain a rather explicit expression for the matrix elements of D_2^{XY} in the Gauss graph basis

$$
\begin{aligned}
M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2} &= -g_{YM}^2\frac{\delta_{ru}}{\sqrt{|O^{\vec{m},\vec{p}}_{R,r}(\sigma_1)|^2|O^{\vec{m},\vec{p}}_{T,t}(\sigma_2)|^2}}\sum_{R'}\delta_{R_i'T_k'}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}}\\
&\times\ [2\delta_{ik}p_im_i-p_{ki}m_{ii}-p_{ii}m_{ik}]\sum_{h_1,h_2\in H_X\times H_Y}\delta(\sigma_1^{-1}h_1\sigma_2h_2). \qquad (50)
\end{aligned}
$$

This is the key result of this subsection and one of the key results of [6]. We will now describe how the above matrix can be diagonalized.

3.3 Boson Lattice

Our goal in this subsection is to diagonalize (50). This is achieved by interpreting (50) as the matrix elements of a Hamiltonian for bosons on a lattice. Towards this end, first note that the matrix elements $M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2}$ are only non-zero if we can choose coset representatives such that σ_1 and σ_2 describe the same element of $S_m\times S_p$. This implies that the brane-string systems described by σ_1 and σ_2 differ only in the number of strings with both ends attached to the same brane, but not in the number of string stretching between distinct branes. This already implies that the contribution D_2^{XY} only mixes eigenstates of D_2^{XZ} and D_2^{YZ} that are degenerate and hence that all three are simultaneously diagonalizable. In this case the matrix element in (50) simplifies to

$$
\begin{aligned}
M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2} &= -g_{YM}^2\sqrt{\frac{|O^{\vec{m},\vec{p}}_{R,r}(\sigma_1)|^2}{|O^{\vec{m},\vec{p}}_{T,t}(\sigma_2)|^2}}\delta_{ru}\delta_{R_i'T_k'}\sqrt{\frac{(N+l_{R_i})(N+l_{T_k})}{l_{R_i}l_{T_k}}}\times\\
&\times\ \Big[2\delta_{ik}p_i(\sigma_2)m_i(\sigma_2)-p_{ki}m_{ii}(\sigma_2)-p_{ii}(\sigma_2)m_{ik}\Big]. \qquad (51)
\end{aligned}
$$

The number of strings stretching between the branes m_{ik} (for Y strings) and p_{ki} (for X strings) are the same for both systems so that

$$
m_{ik}(\sigma_1)=m_{ik}(\sigma_2)\equiv m_{ik},\qquad p_{ik}(\sigma_1)=p_{ik}(\sigma_2)\equiv p_{ik}. \qquad (52)
$$

It is the number of closed loops (m_{ii} for Y loops and p_{ii} for X loops) that can differ between the operators that mix. Finally, we have introduced the notation

$$
p_i(\sigma)=\sum_{k\neq i}p_{ik}+p_{ii}(\sigma),\qquad m_i(\sigma)=\sum_{k\neq i}m_{ik}+m_{ii}(\sigma). \qquad (53)
$$

From the structure of the operator mixing problem, we would expect that $M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2} = M^{\vec{m},\vec{p}}_{T,t,\sigma_2\ R,r,\sigma_1}$. This is indeed the case, as a consequence of the easily checked identity

$$
\sqrt{\frac{|O^{\vec{m},\vec{p}}_{R,r}(\sigma_1)|^2}{|O^{\vec{m},\vec{p}}_{T,t}(\sigma_2)|^2}}\Big[2\delta_{ik}p_i(\sigma_2)m_i(\sigma_2)-p_{ki}m_{ii}(\sigma_2)-p_{ii}(\sigma_2)m_{ik}\Big]
$$

$$= \sqrt{\frac{|O^{\vec{m},\vec{p}}_{T,t}(\sigma_2)|^2}{|O^{\vec{m},\vec{p}}_{R,r}(\sigma_1)|^2}}\Big[2\delta_{ik}p_i(\sigma_1)m_i(\sigma_1) - p_{ki}m_{ii}(\sigma_1) - p_{ii}(\sigma_1)m_{ik}\Big], \tag{54}$$

which holds for any i, k.

The lattice model consists of two distinct species of bosons, one for X and one for Y, hopping on a lattice, with a site for every brane, or equivalently, a site for every row in the Young diagram R labelling the Gauss graph operator $\hat{O}^{\vec{m},\vec{p}}_{R,r}(\sigma)$. The bosons are described by the following commuting sets of operators

$$\begin{aligned} \big[a_i, a_j^\dagger\big] &= \delta_{ij}, \qquad \big[a_i^\dagger, a_j^\dagger\big] = 0 = \big[a_i, a_j\big], \\ \big[b_i, b_j^\dagger\big] &= \delta_{ij}, \qquad \big[b_i^\dagger, b_j^\dagger\big] = 0 = \big[b_i, b_j\big]. \end{aligned} \tag{55}$$

Using these boson oscillators, we have

$$m_{ii} = a_i^\dagger a_i, \qquad p_{ii} = b_i^\dagger b_i, \tag{56}$$

$$m_i = \sum_k m_{ik} + a_i^\dagger a_i, \qquad p_i = \sum_k p_{ik} + b_i^\dagger b_i. \tag{57}$$

The vacuum of the Fock space $|0\rangle$ obeys

$$a_i|0\rangle = 0 = b_i|0\rangle, \qquad i = 1, 2, \cdots, q. \tag{58}$$

The Hamiltonian of the lattice model is given by

$$\begin{aligned} H \;=\; \sum_{i,j=1}^{q} \sqrt{\frac{(N + l_{R_i})(N + l_{R_j})}{l_{R_i} l_{R_j}}} &\Bigg(2\delta_{ij}\Big(\sum_{l\neq i} p_{il} + b_i^\dagger b_i\Big)\Big(\sum_{l\neq i} m_{il} + a_i^\dagger a_i\Big) \\ &-p_{ji} a_j^\dagger a_i - m_{ji} b_j^\dagger b_i\Bigg). \end{aligned} \tag{59}$$

Notice that this Hamiltonian is quadratic in each type of oscillator. It has a nontrivial repulsive interaction given by the $\sum_i a_i^\dagger a_i b_i^\dagger b_i$ term, which makes it energetically unfavorable for a and b type particles to sit on the same site. Also, the full Fock space is a tensor product between the Fock space for the a oscillator and the Fock space for the b oscillator. We will use the occupation number representation to describe the boson states. To complete the mapping to the lattice model, we need to explain the correspondence between Gauss graph operators and states of the boson lattice. This map is given by reading the boson occupation numbers for each site from the number of open strings with coincident end points with both ends attached to the node corresponding to that site. In the next mini-subsection we consider an example which nicely illustrates this map.

Finally, lets make an important observation regarding (59). Although the eigenvalues of this Hamiltonian are subleading contributions to the anomalous dimension, there is an important situation in which this correction is highly significant: for BPS states the leading contribution to the anomalous dimension vanishes and this subleading correction is important. The BPS operators are labelled by Gauss graphs that have $p_{ik} = m_{ik} = 0$ whenever $i \neq k$, i.e. there are no strings stretching between branes. In this case, it is clear that (59) vanishes so that the BPS operators remain BPS when the subleading interactions are included.

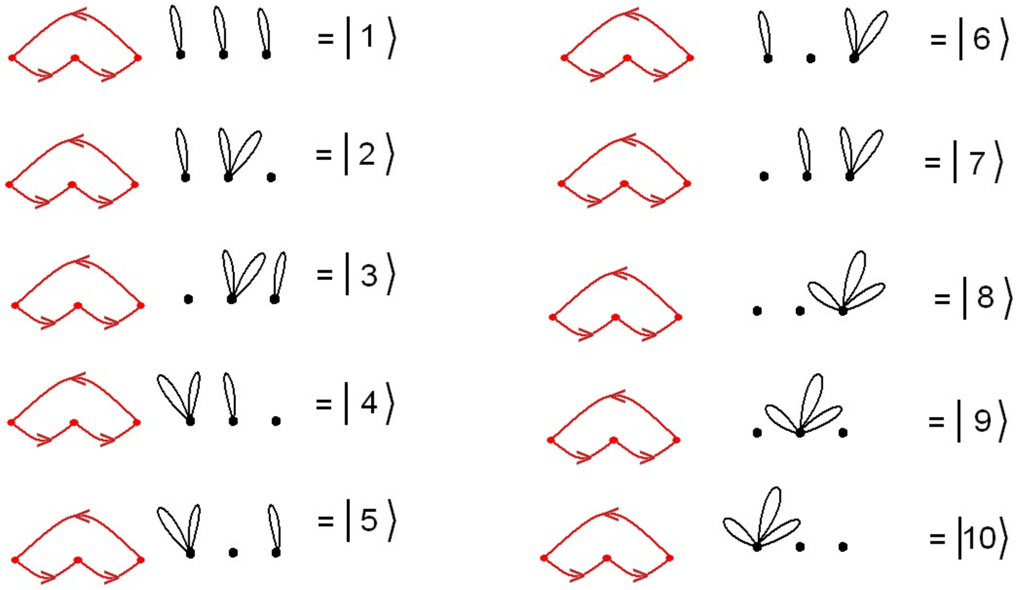


Figure 1: Each Gauss graph label is composed of two graphs, the first for the X strings and the second for the Y strings. Each graph has 3 nodes (because $q = 3$). There are no b type particles because there are no closed X strings. There are 3 a type particles because there are three closed Y strings. All operators share the same r label.

3.3.1 Example

In this mini-subsection we will consider an example for which R has $q = 3$ rows and $p = m = 3$. In this problem, 10 operators mix. The Gauss graph labels for the operators that mix are displayed in Figure 1.

For the Gauss graph operators shown, we have the following correspondence with boson lattice states

$$\begin{aligned}
|1\rangle &= a_1^\dagger a_2^\dagger a_3^\dagger |0\rangle, \qquad |2\rangle = a_1^\dagger \frac{(a_2^\dagger)^2}{\sqrt{2!}}|0\rangle, \\
|3\rangle &= a_3^\dagger \frac{(a_2^\dagger)^2}{\sqrt{2!}}|0\rangle, \qquad |4\rangle = a_2^\dagger \frac{(a_1^\dagger)^2}{\sqrt{2!}}|0\rangle, \\
|5\rangle &= a_3^\dagger \frac{(a_1^\dagger)^2}{\sqrt{2!}}|0\rangle, \qquad |6\rangle = a_1^\dagger \frac{(a_3^\dagger)^2}{\sqrt{2!}}|0\rangle, \\
|7\rangle &= a_2^\dagger \frac{(a_3^\dagger)^2}{\sqrt{2!}}|0\rangle, \qquad |8\rangle = \frac{(a_3^\dagger)^3}{\sqrt{3!}}|0\rangle, \\
|9\rangle &= \frac{(a_2^\dagger)^3}{\sqrt{3!}}|0\rangle, \qquad |10\rangle = \frac{(a_1^\dagger)^3}{\sqrt{3!}}|0\rangle.
\end{aligned} \tag{60}$$

It is now rather straight forwards to compute matrix elements of the lattice Hamiltonian. For example

$$\langle 1|H|2\rangle = -\sqrt{\frac{(N + l_{R_3})(N + l_{R_2})}{l_{R_2} l_{R_3}}}\sqrt{2}. \tag{61}$$

It is instructive to compare this to the answer coming from (50). To move from state 2 to state 1, a string must detach from node 2 and reattach to node 3. Thus, we should plug $i = 2$ and $k = 1$ into (50). The Gauss graph σ_1 corresponds to $|1\rangle$ while σ_2 corresponds to $|2\rangle$. In addition

$R'_2 = T'_1$ and from the Gauss graphs we read off $p_{32} = 1$ and $m_{22}(\sigma_1) = 1$. It is now simple to see that (50) is in complete agreement with the above matrix element.

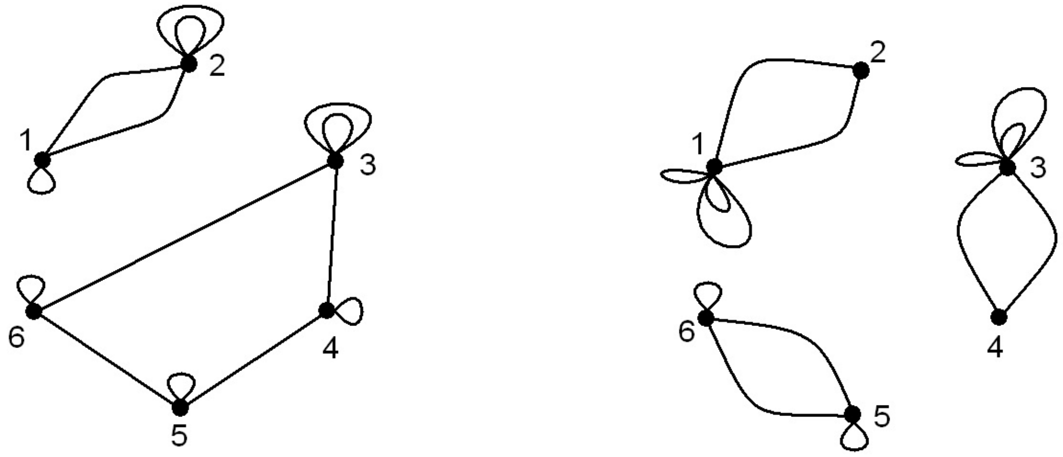


Figure 2: An example of a Gauss graph with non-zero a and b occupation numbers.

Finally, the state corresponding to the Gauss graph in Figure 2 is

$$\frac{(a_1^\dagger)^3}{\sqrt{3!}}\frac{(a_3^\dagger)^3}{\sqrt{3!}}a_5^\dagger a_6^\dagger b_1^\dagger \frac{(b_2^\dagger)^2}{\sqrt{2!}}\frac{(b_3^\dagger)^2}{\sqrt{2!}} b_4^\dagger b_5^\dagger b_6^\dagger |0\rangle. \tag{62}$$

3.4 Diagonalization

In this subsection we will consider a class of examples that can be diagonalized explicitly. Our main motivation is to show that working with the lattice is simple, so the mapping we have found is useful.

3.4.1 Exact Eigenstates

For these examples take

$$p_{ki} = p_{ik} = \delta_{k,i+1}B, \qquad m_{ki} = m_{ik} = \delta_{k,i+1}A \tag{63}$$

with A and B two positive integers. For examples of Gauss graphs that obey this condition, see Figure 3. There are two cases we will consider: we will fix the number of a particles to zero and leave the number of b particles arbitrary, or, fix the number of b particles to zero and leave the number of a particles arbitrary. We will also specialize to labels R that have the difference between any two row lengths $l_{R_i} - l_{R_j} \sim N$, but $\frac{l_{R_i} - l_{R_j}}{l_{R_i}} \approx 0$. In this case our lattice Hamiltonian simplifies to

$$H = \frac{(N + l_{R_1})}{l_{R_1}} \sum_{i=1}^{q} \Big(2\left(B + b_i^\dagger b_i\right)\left(A + a_i^\dagger a_i\right) \\ -B(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) - A(b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i)\Big). \tag{64}$$

This Hamiltonian is easily diagonalized by going to Fourier space. Indeed, in terms of the new oscillators

$$\tilde{a}_n = \frac{1}{\sqrt{q}}\sum_{k=1}^{q} e^{i\frac{2\pi kn}{q}} a_k, \qquad \tilde{b}_n = \frac{1}{\sqrt{q}}\sum_{k=1}^{q} e^{i\frac{2\pi kn}{q}} b_k, \qquad n = 0, 1, \cdots, q-1, \tag{65}$$

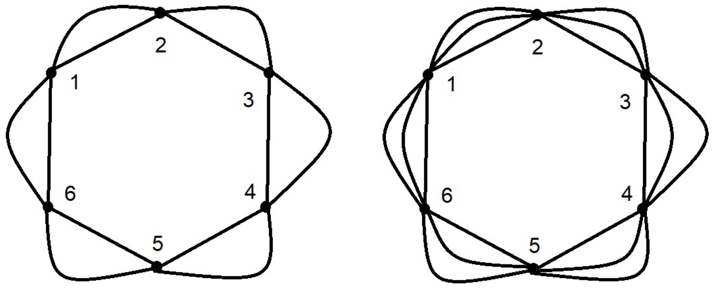


Figure 3: An example of a Gauss graph that is easily solvable. The example shown has $A = 2$ and $B = 3$.

the Hamiltonian becomes (we have set the number of a particles to zero)

$$H = A\frac{(N + l_{R_1})}{l_{R_1}} \sum_{n=0}^{q-1} \left(2 - 2\cos\left(\frac{2\pi n}{q}\right)\right) \tilde{b}_n^\dagger \tilde{b}_n + 2ABq\frac{(N + l_{R_1})}{l_{R_1}}. \tag{66}$$

Eigenstates of the lattice Hamiltonian are given by arbitrary momentum space excitations

$$\prod_{n=0}^{q-1} \frac{(\tilde{a}_n^\dagger)^{\alpha_n}}{\sqrt{\alpha_n!}}|0\rangle \qquad \text{or} \qquad \prod_{n=0}^{q-1} \frac{(\tilde{b}_n^\dagger)^{\beta_n}}{\sqrt{\beta_n!}}|0\rangle, \tag{67}$$

where the occupation numbers α_n, β_n are arbitrary. This state can be translated back into the Gauss graph language to give operators of a definite scaling dimension.

3.4.2 General Properties of Low Energy Eigenstates

In this subsection we will sketch the features of generic low energy states of the lattice Hamiltonian. We begin by relaxing the constraint that only one species is hopping. In the end we will also make comments valid for the general Gauss graph configuration. The Hamiltonian becomes

$$H = H_a + H_b + H_{ab} + E_0, \tag{68}$$

$$H_a = \frac{(N + l_{R_1})}{l_{R_1}} B \sum_{i=1}^{q} \left(2a_i^\dagger a_i - a_i^\dagger a_{i+1} - a_{i+1}^\dagger a_i\right), \tag{69}$$

$$H_b = \frac{(N + l_{R_1})}{l_{R_1}} A \sum_{i=1}^{q} \left(2b_i^\dagger b_i - b_i^\dagger b_{i+1} - b_{i+1}^\dagger b_i\right), \tag{70}$$

$$H_{ab} = \frac{(N + l_{R_1})}{l_{R_1}} \sum_{i=1}^{q} 2b_i^\dagger b_i a_i^\dagger a_i. \tag{71}$$

The constant $E_0 = 2ABq\frac{(N+l_{R_1})}{l_{R_1}}$ is not important for the dynamics but must be included to obtain the correct anomalous dimensions. To start, consider H_a which is a kinetic term for the a particles. The first term in the Hamiltonian implies that it costs energy to have an a particle occupying a site, while the second and third terms tell us this energy can be lowered by hopping between sites i and $i+1$. Consequently, to minimize H_a, the a particles will spread out as much as is possible. This is in perfect accord with the results of the last subsection. The lowest energy single particle state is the zero momentum state, which occupies each site with the same probability: the particle spreads out as much as is possible. Very similar reasoning for H_b implies that the b particles will also spread out as much as is possible. Finally, the term H_{ab} is a repulsive interaction, telling us that it costs energy to have as and bs occupying the same site. So there is a competition going on: The terms H_a and H_b want to spread the as and bs uniformly on the lattice which would certainly distribute as and bs to the same site. The term H_{ab} wants to ensure that any particular site will have only as or bs but not both. Who wins?

Consider a thermodynamic like limit where we consider a very large number of both species of particles, n_a and n_b. In the end, the low energy state will be a "demixed" state with no sites holding both as and bs. To see this, note that H_a grows like n_a and H_b like n_b. This is much smaller than the growth of the term H_{ab} which grows like $n_a n_b$, so the repulsive interaction wins. This conclusion is nicely borne out by numerical results for the two component Bose-Hubbard model[48, 49]. The ground state phase diagram of the Hamiltonian of [48], shows four distinct phases: double super fluid phase, supercounterflow phase, demixed Mott insulator phase and a demixed superfluid phase. Comparing our Hamiltonian to that of [48], we are always in the demixed superfluid phase: the a and b particles do not mix, but are free to move in their respective domains. Thus we have a collection of two species of particles that demix, but are free on their respective domains. It is in this sense that we have an essentially free system.

For the generic Gauss graph, with any choices for the values of m_{ik} and p_{ik}, it is clear that H_a and H_b will still cause the a and b particles to spread out as much as possible. The term H_{ab} will again dominate when we have large numbers of as and bs so we again expect a demixed gas. We can translate this structure of the generic state back into the language of the giant graviton description. Up to now we have considered dual giant gravitons which correspond to operators labelled by Young diagrams with long rows. Recall that dual giant gravitons wrap an $\mathrm{S}^3 \subset \mathrm{AdS}_5$. In this context, l_{R_1} is the momentum of each giant and $N + l_{R_1}$ is the radius on the LLM plane at which the giant orbits. The Hamiltonian for giant gravitons, which wrap $\mathrm{S}^3 \subset \mathrm{S}^5$ is given by

$$H = \frac{(N - l_{R_1})}{l_{R_1}} \sum_{i=1}^{q} \Big(2 \left(B + b_i^\dagger b_i \right) \left(A + a_i^\dagger a_i \right) \\ -B(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) - A(b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) \Big). \tag{72}$$

This is a novel result in [6]. These operators are labelled by Young diagrams with long columns. The giants orbit on the LLM plane with a radius of $N - l_{R_1}$. The X and Y fields are each charged under different $U(1)$s of the $\mathcal{R}$-symmetry group. The $\mathcal{R}$-symmetry of the CFT translates into angular momentum of the dual string theory, so that attaching the particles to a given giant corresponds to giving the giant angular momentum. The lowest energy giant graviton states are obtained by distributing the momenta carried by the X and Y fields evenly between the giants with the condition that any particular giant carries only X or Y momenta, but not both. These conclusions hold for the generic state where there are enough p_{ik} and m_{ik} non-zero, allowing the Xs and Ys to hop between any two giants, possibly by a complicated path. Thus in the end we see that the mapping to the boson lattice model has allowed a rather detailed understanding of the operator mixing problem.

3.5 Conclusions

In this chapter we have studied the action of the one loop dilatation operator D_2 on Gauss graph operators $O_{R,r}^{\vec{m},\vec{p}}(\sigma)$ which belong to the $SU(3)$ sector. The term we have studied, D_2^{XY}, is diagonal in the r label, mixing operators labelled by distinct graphs. It makes a subleading contribution as compared to D_2^{XZ} and D_2^{YZ} when $n \gg m + p$. The two leading terms mix operators labelled by distinct rs. Diagonalizing the action of D_2^{XZ} and D_2^{YZ} on r leads to a collection of decoupled harmonic oscillators, which we refer to as the Z oscillators, since the r label is associated with Z. The spectrum of the Z oscillators gives the leading contribution to the anomalous dimensions. The new contribution that we have studied here can also be mapped to a collection of oscillators, describing a lattice boson model. This is done by introducing two sets of oscillators, the X and Y oscillators associated to the X and Y fields. Diagonalizing the X and Y oscillators breaks degeneracies among different copies of Z oscillators and leads to a constant addition to their ground state energy. This is then a constant shift of the anomalous dimension. Although this shift is subleading (it is of order $\frac{m}{n}$), it could potentially show that certain states are not in fact BPS. This was investigated in detail and it turns out that states that are BPS (their leading order anomalous dimension vanishes) at leading order, remain BPS when the subleading correction is computed (it too vanishes).

The mapping that we have found to a lattice boson model has achieved an enormous simplification of the operator mixing problem and we have managed to understand it in some detail. Indeed, using the lattice boson model, we have argued that the lowest energy giant graviton states are obtained by distributing the momenta carried by the X and Y fields evenly between the giants with the condition that any particular giant carries only X or Y momenta, but not both. Since states with two charges are typically $\frac{1}{4}$-BPS while states with 3 charges are typically $\frac{1}{8}$-BPS, it may be that the solution is locally trying to maximize SUSY. It would be interesting to arrive at the same picture, employing the dual string theory description.

Perhaps the most interesting consequence of our results is that they suggest ways in which one can go beyond the $\frac{1}{2}$-BPS sector. Indeed, all three types of fields considered have been mapped to oscillators, so perhaps there is a more general description of this sector that treats all three types of oscillators on the same footing. This would relax the constraint $n \gg p+m$ which allows for operators that are far from the $\frac{1}{2}$-BPS limit. Deriving this picture is a fascinating open problem, since it will require that we go beyond the displaced corners approximation, or alternatively, that we generalize it.

As a final comment, recall that Mikhailov [50] has constructed an infinite family of $\frac{1}{8}$-BPS giant graviton branes in $\mathrm{AdS}_5\times\mathrm{S}^5$. Quantizing the space of Mikhailov's solutions leads to N non-interacting bosons in a harmonic oscillator[51, 52, 53]. It is tempting to speculate that it is precisely these oscillators that we are uncovering in our study; for evidence in harmony with this suggestion see [54]. It would be interesting to make this speculation precise.

4 Central Charges for the Double Coset

This chapter is based on [7] and it is my original work with de Mello Koch and Kim.

4.1 Introduction

There is by now exquisite evidence of the AdS/CFT correspondence [2, 20, 21]. Many of the precision tests carried out are possible because summing the planar diagrams leads to an integrable model for anomalous dimensions of single trace operators which are dual to closed string states [17, 23]. The integrable model describes defects (magnons) which are excitations of an infinitely long "ferromagnetic ground state". The ground state preserves half the supersymmetries. There are finite size corrections when the chain is finite in length.

The magnon excitations scatter with each other. A significant insight is that the S-matrix of these magnon excitations is completely determined, by symmetry, up to an overall phase [5, 55]. To simplify the description, consider an infinite spin chain which allows us to study excitations individually. The full $PSU(2,2|4)$ symmetry is broken to $SU(2|2) \times SU(2|2) \ltimes \mathbb{R}$. Excitations carry the quantum numbers of a central extension of this subalgebra with the central charge measuring the quasi-momentum of the excitation[5, 55]. The original $PSU(2,2|4)$ does not admit a central extension and for a closed string the net central charge vanishes by level matching constraints.

There are many states in the string theory Hilbert space that are not closed strings. The theory has D-brane excitations which support open strings. These D-branes are dual to CFT operators that have a bare dimension of order N, so that their large N dynamics is not captured by summing planar diagrams [11, 12, 29, 36, 37, 38, 56, 57]. In this setting powerful methods based on group representation theory are effective tools with which to attack the large N limit [15, 31, 32, 33, 58]. A relevant result for us is the diagonalization of the one loop dilatation operator, using a double coset ansatz [43, 44, 45]. This model describes excitations of background branes, with the background branes described using a Young diagram with long[4] rows (for dual giant gravitons) or columns (for giant gravitons). The interactions of these excitations have not been explored in much detail yet [6, 19, 59]. The calculations that are required are technical and quickly become unmanageable. Given the remarkable success in the planar limit, of a symmetry based approach, it is natural to develop a symmetry analysis applicable in this setting[5]. The main goal of this section is to study the $su(2|3)$ sector of the complete theory and show how the global $su(2|2)$ symmetry is realized in the resulting Hilbert space of giant graviton branes and their open string excitations. This result is important since experience from the planar limit suggests that constraints from the global symmetry provide powerful insights with which to study excitations of the background branes. Further, the details are rather intricate so that in the end we arrive at a non-trivial extension of the discussion of [5, 55].

The fact that we are considering open strings has some interesting implications, already explored in [61]. Since this discussion is highly relevant for what follows, we will review the key ideas. To start, consider open superstrings in flat Minkowski spacetime. The lowest lying string modes of a string stretching between two D-branes (flat, parallel and separated) fill out a massive short representation of the unbroken supersymmetry of the D-brane system. The existence of these representations requires a central charge extension of the unbroken supersymmetry algebra. The central extension is needed to get a short multiplet. This additional central charge is an electric charge carried by the string end-points. Closed string states are not charged so that the central charge is only physical in the open string sector or when we compactify the closed string theory on a circle. It is measurable in the field theory limit when we spontaneously

[4]Here long means there are order N boxes in the row/column.

[5]For an early attempt, using a small fraction of the possible symmetries, see [60] .

break the non-abelian gauge symmetry on the stack of branes, corresponding to the Coulomb branch of the Yang-Mills theory living on the world volume of the D-branes.

The key conclusion of [61] is that the central charge of the Coulomb branch is a limit of the central charge extension of [5, 55]. Our analysis supports this conclusion. Note that [61] is not using the language of the double coset ansatz, but instead employs a collective coordinate approach [62, 63, 64] which is well suited to the semi-classical limit. Although the collective coordinate and double coset ansatz are rather different descriptions, their conclusions are in good agreement [62].

We begin in subsection 4.2 with a review of the background needed to understand the double coset ansatz. Our goal is to provide a gentle introduction with enough details to develop the Hilbert space of states of the excited giant graviton brane system. Concretely we explain the change of basis from restricted Schur polynomials to Gauss graph operators which are the eigenoperators of dilatations. Gauss graph operators are labelled with a graph that has a vertex for each brane in the giant graviton brane system. These vertices are decorated with directed edges that describe open string excitations. Our description of the complete state space is novel and in particular we develop the structure of the fermionic states which has not previously been carried out. We then turn to a discussion of the asymptotic symmetries in subsection 4.3. By asymptotic we mean the situation in which impurities are well separated and hence are not interacting. The discussion is necessarily more complicated than the discussion in [5, 55] because we have a far bigger space of possible impurities. The action of the generators of the global symmetry algebra is rather complicated in the restricted Schur polynomial basis. Reorganizing the basis into irreducible representations of the global symmetry appears to be a formidable problem. Remarkably, we find that the basis provided by Gauss graph operators achieves this reorganization! Further we find that excitations again carry a charge under the central extension, echoing what happens in the planar limit. In the (planar) closed string case the central extension measures the quasi-momentum of the excitations and due to cyclicity of the trace (which corresponds to level matching in the string description) the total central extension vanishes. This vanishing of the central extension is necessary, since the algebra on physical states is not centrally extended. We find an equally compelling description in our non-planar setting. Giant graviton branes have a compact world volume, so that the Gauss Law constraint of the brane world volume gauge theory forces the total charge on the world volume to vanish. This is manifested in the fact that there must be the same number of directed edges leaving each node as there are edges terminating on each node. This condition - which is the requirement that the physical state is gauge invariant - ensures that the total central extension vanishes. Further the action of the central charges on the Gauss graph operators has a natural interpretation as a gauge transformation. We end with some conclusions and discussion in subsection 4.4 including speculations on how the global symmetry might be used to study interactions between excitations. The Appendices collect technical details that are used to develop the arguments of the section.

4.2 State space

The operators we consider are built from three complex bosonic matrices X, Y, Z and two complex fermionic matrices ψ_1, ψ_2. These fields all transform in the adjoint of the $U(N)$ gauge group. This sector of the theory is a closed subsector and it enjoys an $su(2|3)$ supergroup global symmetry. We will construct the branes in our giant graviton brane system using only the Z field. The brane system without excitations is a $\frac{1}{2}$-BPS operator. A linear basis for the brane system without excitations is provided by the Schur polynomials, which are labelled by a single Young diagram. Each giant graviton brane corresponds to a long column and each dual giant graviton to a long row, of the Young diagram label. Excitations are described using X, Y and

ψ_1, ψ_2. Generic excited brane states do not preserve any supersymmetry. A linear basis for the excited brane system is provided by the restricted Schur polynomials, which have a number of Young diagram labels, as well as multiplicity labels. The global $su(2|3)$ symmetry of this subsector is not very useful as it relates operators with different numbers of excitations. For this reason, following [65], we will restrict our attention to the $su(2|2)$ subgroup which does preserve the number of excitations. In this subsection we will give a complete description of the excited giant graviton brane state space that will be organized, in the next subsection, by the global $su(2|2)$ symmetry.

4.2.1 Restricted Schur Polynomials

The restricted Schur polynomials provide a linear basis[6] for the gauge invariant operators of a generic multi-matrix model. They correctly account for all constraints following from cyclicity or finite N (trace) relations.

In what follows, we use $b^{(0)}$ to denote the number of Z fields. Consequently, $b^{(0)} = O(N)$. We also use $b^{(1)}, b^{(2)}, f^{(1)}$ and $f^{(2)}$ to denote the number of Y, X, ψ_1 and ψ_2 fields respectively. The integers $b^{(1)}, b^{(2)}, f^{(1)}, f^{(2)}$ are at most $O(\sqrt{N})$. The total number of fields is denoted $n_T = b^{(1)} + b^{(2)} + f^{(1)} + f^{(2)}$.

A restricted Schur polynomial is constructed by tracing a projection operator with the multi-linear operator constructed from a tensor product of matrices. The projection operator projects both the collection of row indices and the collection of column indices, onto a definite representation of $U(N)$, and therefore, by Schur-Weyl duality, onto a definite representation of the permutation group which permutes indices of different fields. The projector first places the complete set of n_T indices into a definite representation, labelled by Young diagram R with n_T boxes. It then places each of the $b^{(i)}$ indices, for each species of bosonic field, into a definite representation labelled by a Young diagram b_i, which has $b^{(i)}$ boxes. Finally, it places the $f^{(i)}$ row indices of each fermion species into the representation f_i and the column indices into the representation f_i^T, each of which have $f^{(i)}$ boxes. s^T is obtained from s by flipping the Young diagram so that rows and columns are exchanged. The reason why bosonic row and column indices are placed into the same representation, is so that the trivial representation of the symmetric group (labelled by a Young diagram with a single row) appears in the tensor product of row and column indices. The trace projects to this trivial representation which is necessary since it follows from bosonic statistics. Further, the reason why fermionic row and column indices are projected as they are, is so that the antisymmetric representation of the symmetric group (labelled by a Young diagram with a single column) appears in the tensor product of row and column indices. The trace projects to this antisymmetric representation which is necessary since it follows from fermionic statistics. For a technical derivation of these facts see [66, 67]. Thus, R is an irreducible representation of S_{n_T}, while the collection of five Young diagram $(\{b_i\}, \{f_i\})$ label an irreducible representation of the subgroup $S_{b^{(0)}} \times S_{b^{(1)}} \times S_{b^{(2)}} \times S_{f^{(1)}} \times S_{f^{(2)}} \subset S_{n_T}$. The representation $(\{b_i\}, \{f_i\})$ of the subgroup may appear more than once upon restricting the representation R of the group. For that reason we need multiplicity labels. Following the construction presented in [66], we need a label for each of the four Young diagrams b_1, b_2, f_1, f_2. The Young diagram b_0 appears without multiplicity. We write these multiplicity labels as a vector $\vec{\mu}$. To get a non-zero trace, the Young diagram labels for the row and column labels must match as explained above. Multiplicity labels can differ. Consequently we can write the restricted Schur polynomials as $\chi_{R,(\{b_i\},\{f_i\})\vec{\mu}_r\vec{\mu}_c}$. Rescaling to produce an operator with unit two point function we obtain $O_{R,(\{b_i\},\{f_i\})\vec{\mu}_r\vec{\mu}_c}$. In what follows, any operator denoted with a capital letter O has been rescaled so that it has a unit two point function.

[6]Here by linear basis we simply mean that any local gauge invariant operator can be expressed as a sum of restricted Schur polynomials. There is never a need, for example, to square a restricted Schur polynomial.

A useful approach towards the construction of the restricted Schur polynomial entails starting with R and then peeling off $f^{(i)}$ boxes, which are then reassembled to produce f_i with multiplicity labels, and then peeling $b^{(i)}$ boxes, which are then reassembled to produce b_i. After peeling off $f^{(1)} + f^{(2)} + b^{(1)} + b^{(2)}$ boxes from R we are left with b_0. This makes it clear that b_0 appears without multiplicity and that the excitations live at the right most corners of R, something we will need below. Further, it is clear that every box in the Young diagram R is associated with a definite species of field.

Any multitrace operator can be written as a linear combination of restricted Schur polynomials. In the free field theory limit, the two point function boils down to computing the trace of a product of two projection operators. This can be done exactly and one finds that the restricted Schur polynomials diagonalize the free field two point function. Finally, the finite N (trace) relations are simply recovered as the statement that the restricted Schur polynomial vanishes whenever any of the Young diagrams labelling the polynomial has more than N rows.

A key fact that we will need below to understand the state space of the excited brane system, concerns the number of values a pair of multiplicity labels $\vec{\mu}_r, \vec{\mu}_c$ can take. This is expressed in terms of the Littlewood-Richardson number $g(r_1, \cdots, r_k; R)$ which is a non-negative integer counting how many times $U(N)$ representation R appears in the tensor product $r_1 \otimes \cdots \otimes r_k$ of $U(N)$ representations. For the restricted Schur polynomial $\chi_{R,(\{b_i\},\{f_i\})\vec{\mu}_r\vec{\mu}_c}$ we find that $\vec{\mu}_r, \vec{\mu}_c$ takes

$$g(b_0, b_1, b_2, f_1, f_2; R)g(b_0, b_1, b_2, f_1^T, f_2^T; R) \tag{73}$$

values [66]. Since the Littlewood-Richardson number also counts the multiplicity of representations of the symmetric group after restriction [68], this formula is not too surprising.

Our discussion in the subsection above aims to give the reader an understanding of the labels of the restricted Schur polynomials. This is essentially all we use below. For a detailed technical derivation of the results reviewed the reader should consult [15, 58, 66].

4.2.2 Double Coset Ansatz

The restricted Schur polynomials do not have a definite scaling dimension. However, they only mix weakly under the action of the dilatation operator: at order g_{YM}^{2L} it is possible for two operators to mix if and only if they differ at most by moving L boxes in any of their Young diagram labels [43, 41]. We want to solve the mixing problem which amounts to finding linear combinations of restricted Schur polynomials that are eigenoperators of the dilatation operator, and finding their eigenvalues. There is a limit in which the mixing problem simplifies dramatically. Recall from the previous subsection that excitations are located at the right hand corners of the Young diagram R. We expect that the excitations are essentially free if they are well separated, which leads to the displaced corners approximation [43, 42]. The displaced corners approximation holds for a specific shape of the Young diagram R. Imagine that R has order 1 long rows. Starting from the right most box in any row of R and moving to the right most box in any other row, along the shortest path in the Young diagram R, if we always need to move through $O(N)$ boxes, then the displaced corners approximation can be used. In the displaced corners approximation there is major simplification in the action of the symmetric group: permutations acting on the impurities simply swap the boxes associated to the excitation. Without the displaced corners approximation, the result of a permutation is a linear combination of the original state and the state with the impurities swapped [43, 42]. This simplified action has two important consequences:

1. There is a new symmetry: restricted Schur polynomials are invariant (up to a sign - for fermions) under swapping impurities that belong to a given row. There is an independent symmetry for the row and column indices.

2. This symmetry results in a new "conservation law": restricted Schur polynomials can only mix if they have the same number and type of excitations in each row. Consequently the number of each species of excitation in each row is conserved[43].

This conservation law holds only at the leading order at large N. There is a compelling physical interpretation of the new conservation law: each row in R is identified with a giant graviton brane. Identifying the excitations as open strings we have recovered the statement that Chan-Paton factors are conserved at zero string coupling.

The mixing problem can be solved by making maximal use of the extra symmetry present in the displaced corners approximation. Let H denote the permutation group that swaps indices of excitations belonging to the same row. Another copy of the same group will swap indices of excitations belonging to the same column. H is a product of symmetric groups, one for each excitation species and for each row (or column) of R. The group or permutations acting on the impurities is given by $S_{\rm exc} = S_{b^{(1)}} \times S_{b^{(2)}} \times S_{f^{(1)}} \times S_{f^{(2)}}$. The extra symmetry implies that we have an operator for each element in the double coset

$$H \setminus S_{\rm exc}/H. \tag{74}$$

The elements of this double coset correspond to graphs, with vertices representing branes (one for each row of R) and directed edges representing oriented strings (one for each excitation field). We will sometimes draw one graph for each species of excitation to unclutter the description. The graphs can be described using some numbers. Focus on a single species of excitation and imagine there are a total of m excitations of this species and that R has p rows. Each excitation corresponds to an edge. Divide each edge into two halves and label each half. Use the orientation of the edges to distinguish out going and in going ends and label the out going ends with numbers $\{1, \cdots, m\}$ and the in going ends with the same numbers. It is natural to specify how the halves are joined by a permutation $\sigma \in S_m$. Let $(m_1, m_2, \cdots, m_p)$ record the number of excitations in each row of R so that $m_1 + m_2 + \cdots m_p = m$. By the Gauss law, the numbers of edges leaving or ending at each vertex are given by the same ordered sequence of integers $(m_1, m_2, \cdots, m_p)$. Choose the labels of the half-edges such that the ones emanating from the first vertex are labelled $\{1, 2, \cdots, m_1\}$, those emanating from second vertex are labelled $\{m_1 + 1, \cdots m_1 + m_2\}$ and so on. Likewise the half-edges incident on the first vertex are labelled $\{1, 2, \cdots, m_1\}$, those incident on the second vertex are labelled $\{m_1 + 1, \cdots m_1 + m_2\}$ etc. The structure of the graph is specified by the permutation $\sigma \in S_m$ which describes how the m out going half-edges are joined with the m in going half-edges. A single graph corresponds to many possible permutations because the m_i strings emanating from the i'th vertex are indistinguishable, as are the m_i strings terminating on the i'th vertex. Thus permutations which differ only by swapping end points that connect to the same vertex do not describe distinct configurations. This symmetry group is nothing but the group H introduced above which makes it clear why the double coset (74) describes the space of restricted Schur polynomials in the displaced corners limit.

The most direct and natural use of the double coset which appears above, is through a Fourier transform. Remarkably, it turns out that the Fourier transform of the restricted Schur polynomial defines an eigenoperator of the dilatation operator [45]. The transformation from the restricted Schur polynomials to the Gauss graph operators replaces the Young diagram and multiplicity labels for each species of excitation with a permutation σ. Consequently, since the transformation works separately for each species, we can simplify the discussion and focus on a single species at a time. The transformation for bosonic excitations was worked out in [45] and is as follows

$$O_{R,r}(\sigma) = \sum_{s\vdash m} \sum_{\mu_1,\mu_2} C^{(s)}_{\mu_1\mu_2}(\sigma) O_{R,(r,s)\mu_1\mu_2}. \tag{75}$$

Here our bosonic excitation is organized by Young diagram s with multiplicity labels μ_1, μ_2 in the restricted Schur basis. After transformation, the state of the excitations is described by permutation σ. Denote the matrix representing $\tau \in S_m$, in the irreducible representation labelled by Young diagram s, by $\Gamma^s(\tau)$. The transformation coefficient is given by

$$C^{(s)}_{\mu_1\mu_2}(\tau) = |H|\sqrt{\frac{d_s}{m!}}\,(\Gamma^s(\tau))_{km}\, B^{s\to 1_H}_{k\mu_1} B^{s\to 1_H}_{m\mu_2}, \tag{76}$$

where we have made use of the branching coefficient defined by

$$\sum_{\mu} B^{s\to 1_H}_{k\mu} B^{s\to 1_H}_{l\mu} = \frac{1}{|H|}\sum_{\gamma\in H}\Gamma^s(\gamma)_{kl}. \tag{77}$$

The branching coefficients $B^{s\to 1_H}_{l\mu}$ resolve the multiplicities that arise when we restrict irrep s of S_m to the identity representation 1_H of H for which $\Gamma^{1_H}(\gamma) = 1\ \forall\gamma$. The transformation for fermionic excitations was worked out in [66] and is as follows

$$O_{R,r}(\sigma) = \sum_{s\vdash m}\sum_{\mu_1,\mu_2} \tilde{C}^{(s)}_{\mu_1\mu_2}(\sigma) O_{R,(r,s)\mu_1\mu_2}, \tag{78}$$

where the transformation coefficient is given by

$$\tilde{C}^{(s)}_{\mu_1\mu_2}(\tau) = |H|\sqrt{\frac{d_s}{m!}}\left(\Gamma^s(\tau)\hat{O}\right)_{km} B^{s\to 1_H}_{k\mu_1} B^{s^T\to 1^m}_{m\mu_2}, \tag{79}$$

where we have made use of the branching coefficient defined by

$$\sum_{\mu} B^{s^T\to 1^m}_{k\mu} B^{s^T\to 1^m}_{l\mu} = \frac{1}{|H|}\sum_{\gamma\in H}\mathrm{sgn}(\gamma)\Gamma^{s^T}(\gamma)_{kl}. \tag{80}$$

The branching coefficients $B^{s^T\to 1^m}_{l\mu}$ resolve the multiplicities that arise when we restrict irrep s^T of S_m to the representation 1^m of H for which $\Gamma^{1^m}(\gamma) = \mathrm{sgn}(\gamma)\ \forall\gamma$. Here $\mathrm{sgn}(\gamma)$ is the sign of the permutation σ. The operator $\hat{O}$ appearing in (79) is defined by

$$\hat{O}_{jl} = S^{[1^n]\, s\, s^T}{}_{j\, l}, \tag{81}$$

where $S^{[1^n]\, s\, s^T}{}_{j\, l}$ is the Clebsch-Gordon coefficient, moving between states in the tensor product $s\times s^T$ and the state spanning 1^m. To get some feeling for $\hat{O}$ note that it satisfies

$$\Gamma^s_{ij}(\sigma)\hat{O}_{jp} = \mathrm{sgn}(\sigma)\hat{O}_{ik}\Gamma^{s^T}_{kp}(\sigma) \tag{82}$$

and hence $\hat{O}_{jl}$ is a map from s^T to s. $\hat{O}^T\hat{O}$ maps from s^T to s^T and it commutes with all elements of the group. Thus, by Schur's Lemma, it is proportional to the identity. $\hat{O}\hat{O}^T$ maps from s to s and it commutes with all elements of the group. Thus it is also proportional to the identity. By normalizing correctly we can choose

$$\hat{O}^T\hat{O} = \mathbf{1}_{s^T} \qquad \hat{O}\hat{O}^T = \mathbf{1}_s. \tag{83}$$

We will use the transformation formulas (75) and (78) extensively in what follows. See Appendix E for technical details of how to applying these transformations.

The Gauss graph operators we consider will, in general, have all four species of excitations participating. The operator is written as $O^{b^{(1)},b^{(2)},f^{(1)},f^{(2)}}_{R,b_0}(\sigma)$. If it is clear from context, we will suppress the $b^{(1)}, b^{(2)}, f^{(1)}, f^{(2)}$ superscript. The permutation $\sigma\in S_{\mathrm{exc}}$ describes how the half edges for all excitations are joined. As mentioned above, these operators have a good scaling dimension. From the formula (2.1) of [69], or the H_2 piece of Table 1 of [65], we have the one loop dilatation operator

$$\begin{aligned} D = & - g_{YM}^2 \left(\sum_{i>j=1}^{3} \mathrm{Tr}\left([\phi_i, \phi_j] \left[\partial_{\phi_i}, \partial_{\phi_j} \right] \right) + \sum_{i=1}^{3} \sum_{a=1}^{2} \mathrm{Tr}\left([\phi_i, \psi_a] \left[\partial_{\phi_i}, \partial_{\psi_a} \right] \right) \right. \\ & + \left. \mathrm{Tr}\left(\{\psi_1, \psi_2\} \{\partial_{\psi_1}, \partial_{\psi_2}\} \right) \right), \end{aligned} \tag{84}$$

where ϕ_i has $i = 1, 2, 3$ and stands for Z, Y, X. Since the number of excitations is much smaller than the number of Z fields, interactions between excitations is subleading and we can work with the simplified expression

$$\begin{aligned} D \;=\; & -g_{YM}^2 \left(\mathrm{Tr}\left([Z, Y] \left[\partial_Z, \partial_Y \right] \right) + \mathrm{Tr}\left([Z, X] \left[\partial_Z, \partial_X \right] \right) \right. \\ & \left. + \sum_{a=1}^{2} \mathrm{Tr}\left([Z, \psi_a] \left[\partial_Z, \partial_{\psi_a} \right] \right) \right) . \end{aligned} \tag{85}$$

The action of the dilatation operator on this Gauss graph operator is given by

$$DO_{R,r}(\sigma_1) = -g_{YM}^2 \sum_{i<j} n_{ij}(\sigma_1) \Delta_{ij} O_{R,r}(\sigma_1), \tag{86}$$

where Δ_{ij} acts only on the Young diagrams R, r. The integer n_{ij} counts the total number of directed edges (both directions counted) stretched between nodes i and j. The operator Δ_{ij} splits into three terms

$$\Delta_{ij} = \Delta_{ij}^+ + \Delta_{ij}^0 + \Delta_{ij}^-. \tag{87}$$

To describe the action of these three pieces, we will need a little more notation. Denote the row lengths of r by r_i. The Young diagram r_{ij}^+ is obtained by removing a box from row j and adding it to row i and r_{ij}^- is obtained by removing a box from row i and adding it to row j. See Appendix B for examples of this notation. We now have

$$\Delta_{ij}^0 O_{R,(r,s)\mu_1\mu_2} = -(2N + r_i + r_j) O_{R,(r,s)\mu_1\mu_2}, \tag{88}$$

$$\Delta_{ij}^+ O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + r_i)(N + r_j)} O_{R_{ij}^+,(r_{ij}^+,s)\mu_1\mu_2}, \tag{89}$$

$$\Delta_{ij}^- O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + r_i)(N + r_j)} O_{R_{ij}^-,(r_{ij}^-,s)\mu_1\mu_2}. \tag{90}$$

Note that R and r change in exactly the same way so that the number of excitations in each row is preserved by the dilatation operator. The operators of definite scaling dimension now follow by diagonalizing the action of Δ_{ij}. This problem was studied in detail in [44, 70], where in a suitable scaling limit, the problem was reduced to the diagonalization of decoupled oscillators.

4.2.3 Bosonic State Space

To specify the states of the Y and X excitations, we must specify the permutation that joins the half edges of these excitations, or equivalently, we need to give the graph that the permutation describes. For the sake of clarity, we will draw the X and Y edges as separate graphs. The reader should bear in mind that corresponding nodes are to be identified, since they correspond to the same row in R. The X and Y impurities populate neighboring boxes in R. There is a distinct (orthogonal) state for each choice of the pair of Young diagrams R and r and the X and Y graphs. The rules for drawing a valid graph for a given excitation species are

1. There is a graph for each type of excitation. The nodes in the graph correspond to the rows in R. Each excitation field appearing in the operator corresponds to a directed edge in the graph. There is no upper limit on the number of edges.

2. The number of edges emanating from a given node is equal to the number of edges terminating on the node which is also equal to the number of excitation boxes (of the given species) in the corresponding row of R.

4.2.4 Fermionic State Space

There is one additional rule that must be applied when drawing the graphs for fermionic excitations. To motivate the rule, lets consider the simplest case in which we have ψ_1 excitations, but no X, Y or ψ_2 excitations. We can simplify the general counting formula appearing in (73) to

$$n_{\text{graphs}} = g(b_0, f_1; R) g(b_0, f_1^T; R). \tag{91}$$

If we have a single excitation $f_1 = f_1^T = \Box$. In this case, $n_{\text{graphs}} = 1$ and we simply have a closed loop on the node corresponding to the row from which a box is removed from R to produce b_0. Now, imagine removing two impurities from a single row. In this case we have $f_1 = \Box\Box$ and $f_1^T = \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ and we find

$$g(b_0, f_1; R) = 1 \qquad\qquad g(b_0, f_1^T; R) = 0 \tag{92}$$

so that $n_{\text{graphs}} = 0$ - there is no restricted Schur polynomial! If we have two fermionic excitations, they can't be removed from the same row. Removing the two excitations from two distinct rows and again taking $f_1 = \Box\Box$ and $f_1^T = \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ we find

$$g(b_0, f_1; R) = 1 \qquad\qquad g(b_0, f_1^T; R) = 1. \tag{93}$$

We could also have taken $f_1 = \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ and $f_1^T = \Box\Box$, so that there are two Gauss graph operators that can be defined. A little work (see Appendix E for useful details) shows that the resulting graphs have two edges, with opposite orientation either (i) stretched between the two nodes or (ii) forming closed loops on each node. If we remove three excitations, two from a single row and then the third from a distinct row, we find that there are three possibilities. First, $s = \Box\Box\Box$ and $s^T = \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}$, or second $s = \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}$ and $s^T = \Box\Box\Box$, or third $s = \begin{smallmatrix}\Box&\Box\\\Box&\end{smallmatrix} = s^T$. It is simple to demonstrate that

$$g(b_0, \Box\Box\Box; R) = 1 \qquad\qquad g(b_0, \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}; R) = 0 \tag{94}$$

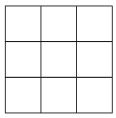
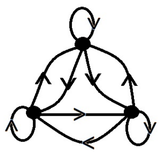
so that the first and second possibilities do not lead to any restricted Schur polynomials and hence no Gauss graph operators. For the third possibility we have

$$g(b_0, \begin{smallmatrix}\Box&\Box\\\Box&\end{smallmatrix}; R) = 1 \tag{95}$$

so that we can define a singe Gauss graph operator. In Appendix E we show that the resulting graph has three edges. There is a closed loop attached to the node corresponding to the row with two impurities removed, as well as two edges with opposite orientation, stretched between the two nodes. There is a simple rule that explains which fermion graphs are possible:

There is at most a single oriented edge with given end points and orientation. Thus, we can't "put two edges into the same state" as a consequence of Fermi statistics.

If R has p rows its easy to check that the largest Young diagram that contributes is a block with p columns and p rows. This corresponds to the Gauss graph with every possible fermion line present. For example, for $p = 3$ we have

$s =$  $\longleftrightarrow$ $\sigma =$  (96)

Notice that there is often a unique Gauss graph σ for each fermionic restricted Schur polynomial, that is the restricted Schur polynomial and the Gauss graph bases often coincide. This is in complete harmony with the results given in [67], which demonstrate that in the context of a single fermionic matrix, the Schur polynomial basis and the trace basis coincide.

4.3 Asymptotic Symmetries

In this subsection we will work out the action of the generators of the $su(2|2)$ global symmetry. We work in the displaced corners approximation so that impurities located at distinct corners are well separated and consequently, at large N, they are not interacting. This is the sense in which we mean "asymptotic" symmetries. A nice conclusion of this analysis is that the Gauss graph operators very naturally fall into representations of $su(2|2)$. Further, we will demonstrate that excitations again carry charges under a central extension of the algebra, generalizing what is known about the planar limit.

4.3.1 Algebra

The bosonic $su(2) \times su(2)$ subalgebra is generated by $R^a{}_b$ and $L^\alpha{}_\beta$. The $R^a{}_b$ rotate the bosonic fields Y, X (which are in the $(\mathbf{2}, \mathbf{0})$ of the subalgebra) while $L^\alpha{}_\beta$ rotate the fermionic fields ψ_1, ψ_2 (which are in the $(\mathbf{0}, \mathbf{2})$). We will refer to these two $su(2)$s as $su(2)_R$ and $su(2)_L$. In terms of raising and lowering operators

$$R^1{}_2 = R_+ \qquad R^2{}_1 = R_- \qquad 2R^1{}_1 = -2R^2{}_2 = R_3 \tag{97}$$

$$L^1{}_2 = L_+ \qquad L^2{}_1 = L_- \qquad 2L^1{}_1 = -2L^2{}_2 = L_3 \tag{98}$$

we have

$$[R_3, R_-] = -2R_- \qquad [R_3, R_+] = 2R_+ \qquad [R_+, R_-] = R_3 \tag{99}$$

and

$$[L_3, L_-] = -2L_- \qquad [L_3, L_+] = 2L_+ \qquad [L_+, L_-] = L_3. \tag{100}$$

The algebra also has supersymmetry generators $Q^\alpha{}_a$ and $S^a{}_\alpha$. These generators obey

$$[R^a{}_b, Q^\gamma{}_c] = -\delta^a_c Q^\gamma{}_b + \frac{1}{2}\delta^a_b Q^\gamma{}_c \qquad [R^a{}_b, S^c{}_\gamma] = \delta^c_b S^a{}_\gamma - \frac{1}{2}\delta^a_b S^c{}_\gamma \tag{101}$$

$$[L^\alpha{}_\beta, Q^\gamma{}_c] = \delta^\gamma_\beta Q^\alpha{}_c - \frac{1}{2}\delta^\alpha_\beta Q^\gamma{}_c \qquad [L^\alpha{}_\beta, S^c{}_\gamma] = -\delta^\alpha_\gamma S^c{}_\beta + \frac{1}{2}\delta^\alpha_\beta S^c{}_\gamma \tag{102}$$

as well as

$$\{Q^\alpha{}_a, S^b{}_\beta\} = \delta^\alpha_\beta R^b{}_a + \delta^b_a L^\alpha{}_\beta + \delta^b_a \delta^\alpha_\beta C, \tag{103}$$

$$\{Q^\alpha{}_a, Q^\beta{}_b\} = \epsilon^{\alpha\beta}\epsilon_{ab} P, \qquad \{S^a{}_\alpha, S^b{}_\beta\} = \epsilon_{\alpha\beta}\epsilon^{ab} K. \tag{104}$$

Our goal in the subsections that follow is to argue that the state space of the Gauss graph operators are organized into representations of this algebra, to determine the values of the central charges P, K and C and finally, to demonstrate that when acting on physical states, the central charges P and K vanish.

4.3.2 $SU(2)_R$

The general state in an $su(2)$ representation can be labelled with a pair of quantum numbers, j_R, m_R. The action of the lowering operator is

$$R_-|j_R\, m_R\rangle = \sqrt{j_R(j_R+1) - m_R(m_R-1)}|j_R\, m_R - 1\rangle. \tag{105}$$

To determine the representation that a given Gauss graph corresponds to, we identify

$$R_- = \mathrm{Tr}\left(X\frac{d}{dY}\right). \tag{106}$$

We then act with R_- on a given Gauss graph operator and compare to (105). This analysis is presented in detail in Appendix F. Our conclusion is the following

1. Each node of the Gauss graph belongs to a definite $SU(2)_R$ representation. If the number of closed Y loops attached to node k is $b_k^{(1)}$ and the number of closed X loops is $b_k^{(2)}$, then node k is in the spin $j_R = b_k^{(1)} + b_k^{(2)}$ representation.

2. The specific state in the representation that node k occupies is determined by $m_R = b_k^{(1)} - b_k^{(2)}$. Note that we are using conventions in which m_R jumps in units of two, and j_R and m_R are always integer.

3. The action of R_- on the kth node replaces a single directed Y edge with a single directed X edge, with an overall coefficient given by (105).

4. The generators $R^a{}_b$ do not act on edges that travel between nodes.

From the action defined for R_- above we can work out the action of R_+ (by hermittian conjugation) and the action of R_3 (by using the $su(2)_R$ algebra).

The complete action of the $su(2)_R$ generators follows by summing the result of acting on each node in the graph. This corresponds to the usual co-product action.

Notice that in moving to the Gauss graph basis, we have in fact organized the state space into $su(2)_R$ multiplets!

4.3.3 $SU(2)_L$

In this case we identify

$$L_- = \mathrm{Tr}\left(\psi_2 \frac{d}{d\psi_1}\right). \tag{107}$$

We again find that the $su(2)$ generators (now the $L^\alpha{}_\beta$) do not act on edges that travel between nodes. Each node is again in a definite state. We find four possibilities

1. A node that has no closed ψ_1 loops and no closed ψ_2 loops is in the one dimensional representation with $j_L = 0$.
2. A node with a single closed ψ_1 loop is in the representation $j_L = 1$, and in state $m_L = 1$. L_- acting on this node replaces the ψ_1 loop with a ψ_2 loop and L_+ annihilates the node.
3. A node with a single closed ψ_2 loop is in the representation $j_L = 1$, and in state $m_L = -1$. L_- annihilates the node while L_+ acting on this node replaces the ψ_2 loop with a ψ_1 loop.
4. A node that has both a closed ψ_1 loop and a closed ψ_2 loops is in the one dimensional representation with $j_L = 0$.

As in the previous subsection, the complete action of the $su(2)_L$ generators follows by summing the result of acting on each node in the graph. Further, in moving to the Gauss graph basis, we have in fact organized the state space into $su(2)_L$ multiplets!

4.3.4 Supercharges

When the supercharges act we will again assume that there is an action on each node of the graph and that the total action is the sum of actions on each node. In what follows it is more convenient to specify the Gauss graph by stating how many closed loops of each species there are at each node and how many edges (with orientation) there are stretching between nodes. The numbers $b_k^{(a)}$ count the number of closed bosonic edges at node k, while $f_k^{(\alpha)}$ count the number of closed fermionic edges at node k. The numbers $b_{ij}^{(a)}$ count the number of bosonic edges moving from node i to node j, while $f_{ij}^{(\alpha)}$ count the number of fermionic edges moving from node i to node j. We will assume the following action for the supercharges, acting on node i

$$\begin{aligned}(Q^\alpha{}_a)_i O_{R,r}(\{\cdots, b_i^{(c)}, f_i^{(\gamma)}, \cdots\}) &= c_a(1 - f_i^{(\alpha)})\sqrt{b_i^{(a)}}\, O_{R,r}(\{\cdots, b_i^{(c)} - \delta^c_a, f_i^{(\gamma)} + \delta^\gamma_\alpha, \cdots\}) \\ &+ c_b \sum_{b=1}^{2}\sum_{\beta=1}^{2} f_i^{(\beta)} \epsilon^{\alpha\beta}\epsilon_{ab}\sqrt{b_i^{(b)} + 1}\, O_{R_i^+, r_i^+}(\{\cdots, b_i^{(c)} + \delta^c_b, f_i^{(\gamma)} - \delta^\gamma_\beta, \cdots\}),\end{aligned} \tag{108}$$

$$\begin{aligned}(S^a{}_\alpha)_i O_{R,r}(\{\cdots, b_i^{(c)}, f_i^{(\gamma)}, \cdots\}) &= c_d\, f_i^{(\alpha)}\sqrt{b_i^{(a)} + 1}\, O_{R,r}(\{\cdots, b_i^{(c)} + \delta^c_a, f_i^{(\gamma)} - \delta^\gamma_\alpha, \cdots\}) \\ &+ c_c \sum_{b=1}^{2}\sum_{\beta=1}^{2} (1 - f_i^{(\beta)}) \epsilon_{\alpha\beta}\epsilon^{ab}\sqrt{b_i^{(b)}}\, O_{R_i^-, r_i^-}(\{\cdots, b_i^{(c)} - \delta^c_b, f_i^{(\gamma)} + \delta^\gamma_\beta, \cdots\}).\end{aligned} \tag{109}$$

In the argument of $O_{R,r}$ we have only explicitly specified quantum numbers of the state that change under the action of the supercharge. Notice that both supercharges change the shape of the Young diagram labels R and r; see Appendix B for an explanation of this notation. The two labels R and r change in precisely the same way. The coefficients c_a, c_b, c_c and c_d are constants that will be determined by requiring that $Q^\alpha{}_a$ and $S^a{}_\alpha$ close the correct algebra. The factor of $f_i^{(\alpha)}$ and $(1 - f_i^{(\alpha)})$ are there to ensure that we don't put two fermions into one state or remove a fermion from a state that doesn't contain any. The factors of $\sqrt{b_i^{(a)}}$ and $\sqrt{b_i^{(a)} + 1}$ are there for convenience. With these factors, the coefficients c_a, c_b, c_c and c_d are independent of $b_i^{(a)}$. The factors of ϵ^{ab} and $\epsilon^{\alpha\beta}$ are determined by $su(2)_R \times su(2)_L$ covariance.

The above ansatz is strongly motivated by the action of the supercharges worked out in [5]. The key differences are

1. The excitations of [5] are either a single Y or a single X field. Here we can have an arbitrary number of both. The only effect is that we now need to include the $\sqrt{b_i^{(a)}}$ and $\sqrt{b_i^{(a)} + 1}$ factors.
2. The fermionic states can have any occupancy. This is why we need the $f_i^{(\alpha)}$ and $(1 - f_i^{(\alpha)})$ factors.
3. The action of [5] was written down using markers $\mathcal{Z}^{\pm}$, which insert or remove Zs from the single trace operator, leading to a dynamic lattice with a time dependent number of sites. Here we have a truly non-planar generalization of this action: a box is added or deleted to the Young diagram labels. It appears to be highly non-trivial to describe this operation in terms of traces.

Our next task is to show that these supercharges close the correct algebra and, in the process, determine the coefficients c_a, c_b, c_c and c_d, as well as the values of the central extensions.

4.3.5 Representation

To begin we require that

$$\{(Q^{\alpha}{}_a)_i, (S^b{}_\beta)_i\} = \delta^\alpha_\beta (R^b{}_a)_i + \delta^b_a (L^\alpha{}_\beta)_i + \delta^b_a \delta^\alpha_\beta C_i. \tag{110}$$

This forces

$$c_a c_d - c_b c_c = 1 \tag{111}$$

and the central charge is

$$C_i = \frac{1}{2}(b^{(1)} + b^{(2)} + f^{(1)} + f^{(2)}). \tag{112}$$

The central extension vanishes

$$\{(Q^{\alpha}{}_a)_i, (Q^{\beta}{}_b)_i\} = 0, \tag{113}$$

$$\{(S^{a}{}_\alpha)_i, (S^{b}{}_\beta)_i\} = 0. \tag{114}$$

This is the correct description of the free theory. In particular, we find that there are no anomalous dimensions. This is not correct: the Gauss graph operators are not in general BPS and they will develop non-zero anomalous dimensions. Indeed, looking at the one loop result (86) it is clear that this is the case. Studying (86) leads to a second puzzle: at least at one loop, the anomalous dimension depends only on $n_{ij} = b_{ij}^{(1)} + b_{ij}^{(2)} + f_{ij}^{(1)} + f_{ij}^{(2)}$. These are quantum numbers associated to edges that stretch between different nodes. This dependence appears puzzling because our analysis thus far has demonstrated that the global symmetry generators leave these edges inert!

It is not hard to appreciate why the global symmetry generators do not act on these edges. An edge forming a closed loop at a node is automatically gauge invariant. In contrast to this, edges going between nodes are constrained by the requirement of gauge invariance to form closed paths that respect the orientation of each edge. Replacing one edge with another edge of a different species spoils the Gauss law constraint so that we land up with a state that is not gauge invariant. Thus, the edges that straddle nodes are not transformed by the global symmetry generators because there is no gauge invariant state that they could be transformed into. If, however, we act with a pair of supercharges (for example) we can change the species of an edge with the first action and restore it with the second. Consequently, the edges straddling nodes can give rise to the central extensions introduced below

$$\{(Q^\alpha{}_a)_i,(Q^\beta{}_b)_j\}=\epsilon^{\alpha\beta}\epsilon_{ab}P_{ij},\qquad \{(S^a{}_\alpha)_i,(S^b{}_\beta)_j\}=\epsilon_{\alpha\beta}\epsilon^{ab}K_{ij}. \tag{115}$$

The action of the central extensions on the Gauss graph operators takes the following form

$$P_{ij}O_{R,r}(\sigma)=\alpha\sqrt{N+r_i}n^+_{ij}O_{R^+_i,r^+_i}(\sigma)-\alpha\sqrt{N+r_j}n^+_{ij}O_{R^+_j,r^+_j}(\sigma), \tag{116}$$

$$K_{ij}O_{R,r}(\sigma)=\beta\sqrt{N+r_i}n^+_{ij}O_{R^-_i,r^-_i}(\sigma)-\beta\sqrt{N+r_j}n^+_{ij}O_{R^-_j,r^-_j}(\sigma). \tag{117}$$

These formulas are the natural generalization of the action of the central extension obtained in [5]. Indeed, the markers $\mathcal{Z}^\pm$ are again replaced by an action that adds or removes a box from the Young diagram. Further, these actions again reveal the nature of the central extension as a gauge transformation, exactly as was observed in the planar limit. An important consistency condition is that these central extensions must vanish when acting on physical states. In the planar limit this follows from cyclicity of the trace. In the non-planar problem we study here we find that

$$\sum_{i,j}P_{ij}=0=\sum_{i,j}K_{ij} \tag{118}$$

holds as a consequence of the Gauss Law constraint. The fact that the Gauss graph operators are gauge invariant physical states implies that they are annihilated by the total central extension.

Using the above central extension we obtain the following formula for the anomalous dimension γ

$$\gamma=\frac{1}{2}\sqrt{1+\sum_{ij}P_{ij}K_{ij}}. \tag{119}$$

This correctly reproduces the one loop anomalous dimension. Indeed, evaluating

$$\begin{aligned}P_{ij}K_{ij}O_{R,r}(\sigma) \;&=\; \alpha\beta n^+_{ij}\Big[(N+r_i)O_{R,r}(\sigma)+(N+r_j)O_{R,r}(\sigma)\\ &\quad-\sqrt{(N+r_i)(N+r_j)}\,\Big(O_{R^+_{ij},r^+_{ij}}(\sigma)+O_{R^-_{ij},r^-_{ij}}(\sigma)\Big)\Big],\end{aligned} \tag{120}$$

which, after summing over i and j, is nothing but (86).

4.4 Discussion

Our main result is the decomposition of the state space of CFT operators dual to excited giant graviton branes into irreducible representations of the $su(2|2)\ltimes\mathbb{R}$ global symmetry. There are a number of positive features of our results which support their validity:

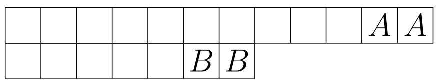
1. Our analysis shows that the state space of restricted Schur polynomials is not organized into irreducible representations of the $su(2|2) \ltimes \mathbb{R}$ global symmetry. However, after transforming to the Gauss graph operator basis, we do indeed have a transparent $su(2|2) \ltimes \mathbb{R}$ structure. Indeed, it is a simple matter to read off the $su(2)_R \times su(2)_L$ quantum numbers from the graph.

2. We have managed to reproduce the one loop anomalous dimension of the Gauss graph operator from the $su(2|2) \ltimes \mathbb{R}$ central charge. This central charge makes a prediction for the higher loop anomalous dimensions. It would be interesting to check these predictions.

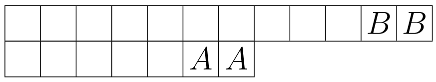
3. Further, excitations are again charged under a central extension of global symmetry. Since the original global symmetry is not centrally extended, the action of the central extension must vanish on physical states. In the planar limit the central extension generates gauge transformations and hence the central extension vanishes when acting on physical states which are gauge invariant. In our case the central charge is again set to zero by gauge invariance: the constraint enforced by the Gauss Law ensures that the central extension vanishes. Further, the central extension again generates gauge transformations.

This is compelling evidence in support of our results.

There are a number of directions in which our study can be extended. One could for example try to formulate a more complete description of excited gaint graviton states, by relaxing the restriction to the $su(2|3)$ sector. In this case the global symmetry algebra is $su(2|2) \times su(2|2) \ltimes \mathbb{R}$. This has proved to be a very fruitful direction in the planar limit of the theory. Another fascinating direction would be to use the global symmetry to study interactions of the excitations. Following [5], a productive way forwards may be to introduce an S-matrix and to use the global symmetry to constrain its form. The Gauss graph operators are natural asymptotic states that might be used to define an S-matrix. For example, consider the following (schematic) state

$|\text{in}\rangle =$  (121)

which we will treat as an "in state". Under time evolution by the dilatation operator, the lengths of the rows can change. When the row lengths are comparable the two impurities can interact, and possibly even swap the row they belong to or rearrange in even more complicated ways. The rows lengths will then continue to evolve until the impurities are again well separated, defining an "out state" of the schematic form

$|\text{out}\rangle =$  (122)

The map from the in state to the out state

$$|\text{out}\rangle = S|\text{in}\rangle \tag{123}$$

defines an S-matrix as usual. In the planar case there is a lot one can do with the S-matrix. The powerful methods of integrability can be applied thanks to the fact that the S-matrix satisfies a Yang-Baxter equation, which expresses the equality of two particle scattering between three particles, with the two particle scattering taking place in different orders. Here there is a natural analogue of this setup: consider a Young diagram R with three rows, and a Gauss graph operator that has excitations on each row. One can ask if there is equality between the different orders in which the excitations on the different rows can scatter. Do we obtain something like the Yang-Baxter equation? Is it possible to generalize something of the powerful integrability machinery? This is the subject of work in progress.

5 Conclusion

In this thesis the goal we set out to achieve was the construction of an emergent Yang-Mills theory, arising from the low energy description of branes and open strings. We have worked directly in the CFT with operators dual to excited giant graviton branes. This is a highly nontrivial goal because this new Yang-Mills theory has no connection to the original gauge symmetry of the CFT. We have answered a number of questions brought forth in this thesis. In this section we will draw together conclusions that have already be made in the thesis, in order to develop a final concluding discussion. We will then end with an appraisal of our success and by outlining directions for further study.

Our first novel result is a mapping from magnon excitations of a giant graviton to a lattice boson model; "magnons" are excitations in an $SU(2|2)$ representation - this is the algebraic definition that encompasses both spin chain magnons and our magnons. This mapping achieves an enormous simplification of the operator mixing problem and we have managed to understand it in some detail. Indeed, using the lattice boson model, we have argued that the lowest energy giant graviton states are obtained by distributing the momenta carried by the X and Y fields evenly between the giants with the condition that any particular giant carries only X or Y momenta, but not both. Since states with two charges are typically $\frac{1}{4}$-BPS while states with 3 charges are typically $\frac{1}{8}$-BPS, it may be that the solution is locally trying to maximize SUSY. It would be interesting to arrive at the same picture, employing the dual string theory description. A physical description along these lines would certainly help to clarify the physical meaning of our result.

Our second novel result is the decomposition of the state space of CFT operators dual to excited giant graviton branes into irreducible representations of the $su(2|2)\ltimes\mathbb{R}$ global symmetry. There are a number of positive features of our results which support their validity:

1. Our analysis shows that the state space of restricted Schur polynomials is not organized into irreducible representations of the $su(2|2)\ltimes\mathbb{R}$ global symmetry. However, after transforming to the Gauss graph operator basis, we do indeed have a transparent $su(2|2)\ltimes\mathbb{R}$ structure. Indeed, it is a simple matter to read off the $su(2)_R\times su(2)_L$ quantum numbers from the graph.

2. We have managed to reproduce the one loop anomalous dimension of the Gauss graph operator from the $su(2|2)\ltimes\mathbb{R}$ central charge. This central charge makes a prediction for the higher loop anomalous dimensions. It would be interesting to check these predictions.

3. Further, excitations are again charged under a central extension of global symmetry. Since the original global symmetry is not centrally extended, the action of the central extension must vanish on physical states. In the planar limit the central extension generates gauge transformations and hence the central extension vanishes when acting on physical states with are gauge invariant. In our case the central charge is again set to zero by gauge invariance: the constraint enforced by the Gauss Law ensures that the central extension vanishes. Further, the central extension again generates gauge transformations.

Given these results it is interesting to ask how much of the dynamics of interactiong magnons we can understand. Indeed, by using a symmetry based approach Beisert was able to determine the two magnon S-matrix up to an overall phase. Is the $su(2|2)$ symmetry analysis carried out in this thesis the first step towards understanding magnon excitations in large N but non-planar limits of the theory? This is a fascinating avenue for further study.

A Action of derivatives

This Appendix evaluates the explicit action of the derivatives considered in the subsection 2.7.1. The first pair evaluates to

$$
\begin{aligned}
&\frac{d}{dY^k_j}\frac{d}{dX^i_k}\left(X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right)\\
&=\left[\frac{d}{dX^i_k}\left(X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}\right)\frac{d}{dY^k_j}\left(Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\right)\right]Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&=\left(\delta^k_{i_{\sigma(1)}}\delta^{i_1}_i X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}+X^{i_1}_{i_{\sigma(1)}}\delta^k_{i_{\sigma(2)}}\delta^{i_2}_i X^{i_3}_{i_{\sigma(3)}}\cdots X^{i_p}_{i_{\sigma(p)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\delta^k_{i_{\sigma(p)}}\delta^{i_p}_i\right)\times\\
&\quad\times\left(\delta^j_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_k Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+Y^{i_{p+1}}_{i_{\sigma(p+1)}}\delta^j_{i_{\sigma(p+2)}}\delta^{i_{p+2}}_k Y^{i_{p+3}}_{i_{\sigma(p+3)}}\delta^j_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+\right.\\
&\quad\left.+Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^j_{i_{\sigma(p+m)}}\delta^{i_{p+m}}_k\right)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&=\left(\delta^k_{i_{\sigma(1)}}\delta^{i_1}_i\delta^j_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_k X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\right.\\
&\quad\left.\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^j_{i_{\sigma(p+m)}}\delta^{i_{p+m}}_k\delta^k_{i_{\sigma(p)}}\delta^{i_p}_i\right)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&=\left(\delta^{i_{p+1}}_{i_{\sigma(1)}}\delta^{i_1}_i\delta^j_{i_{\sigma(p+1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\right.\\
&\quad\left.\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^j_{i_{\sigma(p+m)}}\delta^{i_{p+m}}_{i_{\sigma(p)}}\delta^{i_p}_i\right)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&[Y,X]^i_j\frac{d}{dY^k_j}\frac{d}{dX^i_k}\left(X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right)\\
&=[Y,X]^i_j\left(\delta^{i_{p+1}}_{i_{\sigma(1)}}\delta^{i_1}_i\delta^j_{i_{\sigma(p+1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\right.\\
&\quad\left.\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^j_{i_{\sigma(p+m)}}\delta^{i_{p+m}}_{i_{\sigma(p)}}\delta^{i_p}_i\right)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&=\left([Y,X]^{i_1}_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+[Y,X]^{i_p}_{i_{\sigma(p+m)}}X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\right.\\
&\quad\left.\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^{i_{p+m}}_{i_{\sigma(p)}}\right)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.
\end{aligned}
$$

The second pair evaluates to

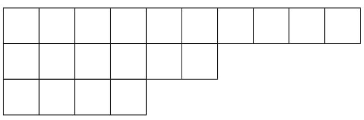
$$
\begin{aligned}
&-\frac{d}{dX^k_j}\frac{d}{dY^i_k}\left(X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right)\\
&=-\left[\frac{d}{dX^k_j}\left(X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}\right)\frac{d}{dY^i_k}\left(Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\right)\right]Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&=-\left(\delta^j_{i_{\sigma(1)}}\delta^{i_1}_k X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}+X^{i_1}_{i_{\sigma(1)}}\delta^j_{i_{\sigma(2)}}\delta^{i_2}_k X^{i_3}_{i_{\sigma(3)}}\cdots X^{i_p}_{i_{\sigma(p)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\delta^j_{i_{\sigma(p)}}\delta^{i_p}_k\right)\times\\
&\quad\times\Big(\delta^k_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_i Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+Y^{i_{p+1}}_{i_{\sigma(p+1)}}\delta^k_{i_{\sigma(p+2)}}\delta^{i_{p+2}}_i Y^{i_{p+3}}_{i_{\sigma(p+3)}}\delta^j_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+\\
&\quad+Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^k_{i_{\sigma(p+m)}}\delta^{i_{p+m}}_i\Big)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&=-\Big(\delta^j_{i_{\sigma(1)}}\delta^{i_1}_k\delta^k_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_i X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\\
&\quad\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^k_{i_{\sigma(p+m)}}\delta^{i_{p+m}}_i\delta^j_{i_{\sigma(p)}}\delta^{i_p}_k\Big)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&=-\Big(\delta^j_{i_{\sigma(1)}}\delta^{i_1}_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_i X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\\
&\quad\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^{i_p}_{i_{\sigma(p+m)}}\delta^{i_{p+m}}_i\delta^j_{i_{\sigma(p)}}\Big)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&-[Y,X]^i_j\frac{d}{dX^k_j}\frac{d}{dY^i_k}\left(X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\right)\\
&=-[Y,X]^i_j\Big(\delta^j_{i_{\sigma(1)}}\delta^{i_1}_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_i X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\\
&\quad\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^{i_p}_{i_{\sigma(p+m)}}\delta^{i_{p+m}}_i\delta^j_{i_{\sigma(p)}}\Big)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&=-\Big([Y,X]^{i_{p+1}}_{i_{\sigma(1)}}\delta^{i_1}_{i_{\sigma(p+1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+\cdots+[Y,X]^{i_{p+m}}_{i_{\sigma(p)}}X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\times\\
&\quad\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^{i_p}_{i_{\sigma(p+m)}}\Big)Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.
\end{aligned}
$$

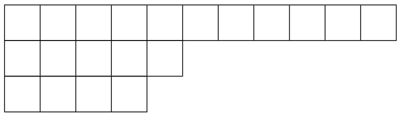
B Young Diagram Notations

The dilatation operator D, central charges C, P_{ij} and K_{ij} as well as the supercharges $Q^\alpha{}_a$ and $S^a{}_\alpha$, when acting on the Gauss graph operator $O_{R,r}(\sigma)$, have a non-trivial action on the Young diagram labels R and r. In this Appendix we will briefly spell out the notation we use, with a few examples to illustrate the ideas. Consider the Young diagram r given by

$r =$  (124)

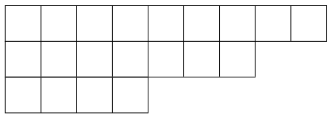
The dilatation operator can transport a box from row i to row j. We use the notation r^+_{ij} to describe the Young diagram obtained from r by deleting a box from row j and adding a box to row i. As an example, we give

$$r^+_{12} = \quad (125)$$



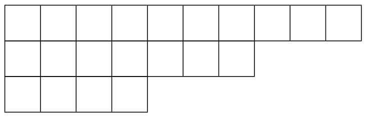
We will also find it convenient to use the notation r^-_{ij} to describe the Young diagram obtained from r by deleting a box from row i and adding a box to row j. As an example of this notation, consider

$$r^-_{12} = \quad (126)$$



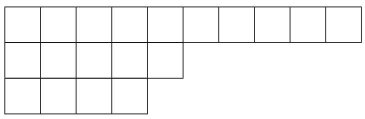
Notice that r_{ij}, r^+_{ij} and r^-_{ij} all have the same number of boxes. The supercharges change the number of boxes in the Young diagram. For example, $Q^\alpha{}_a$ can add a box to a given row. We use r^+_i to denote the Young diagram obtained from r by adding a single box to row i. For example

$$r^+_2 = \quad (127)$$



Notice the the number of boxes is not preserved: r^+_2 has one more box that r. The supercharge $S^a{}_\alpha$ can remove a box from a given row. We use r^-_i to denote the Young diagram obtained from r by deleting a single box from row i. As an example of this notation, we quote

$$r^-_2 = \quad (128)$$



Finally, although we have illustrated the notation using Young diagram r, the discussion also holds for R.

C Restricted Schur Polynomials with 2 rows

A simple setting in which to test the formulas and ideas developed in this study, is to consider Young diagrams R that have two rows. The problem with two rows (or columns) is particularly simple because upon restricting an irreducible representation of S_n to any subgroup $S_k \times S_{n-k}$, irreducible representations of the subgroup appear without multiplicity. In Appendix D we evaluate the action of $su(2)$ rotations on restricted Schur polynomials with bosonic excitations only. Since there are no mulitplicities, the relevant restricted Schur polynomials are $\chi_{R,(b_0,b_1,b_2)}(Z,Y,X)$. There is a $S_{b^{(1)}} \times S_{b^{(2)}}$ symmetry that is Schur Weyl dual to $U(2)$. Consequently, the projection operators needed to construct the restricted Schur polynomials are easily determined in terms of well known $SU(2)$ Clebsch-Gordan coefficients[42]. We use the quantum numbers j, j_3 for the $SU(2)$ used to organize the Y fields and k, k_3 for the $SU(2)$ used to organize the X fields.

Let $(b_i)_k$ denote the number of boxes in row k of Young diagram b_i. The translation of the restricted Schur polynomial $\chi_{R,(b_0,b_1,b_2)}(Z,Y,X)$ to $SU(2)$ state labels is as follows

$$\begin{aligned}
(b_2)_1 &= \tfrac{p}{2} + k & \qquad (b_2)_2 &= \tfrac{p}{2} - k \\
(b_1)_1 &= \tfrac{m}{2} + j & \qquad (b_1)_2 &= \tfrac{m}{2} - j \\
R_1 &= (b_0)_1 + \tfrac{m+p}{2} + j_3 + k_3 & \qquad R_2 &= (b_0)_2 + \tfrac{m+p}{2} - j_3 - k_3
\end{aligned} \qquad (129)$$

j_3 is equal to the number of Y boxes in the first row of R minus the number of Y boxes in the second. k_3 is defined in the same way, but for the X boxes. The above labels may appear to be over complete: given $b^{(0)}, b^{(1)}, b^{(2)}$ as well as b_0, k, j, k_3+j_3 we can reconstruct the Young diagram labels R, b_0, b_1 and b_2. It seems that we need only the sum k_3+j_3 and not the individual values j_3, k_3. The point is that, even when R has two rows, when we restrict S_{a+b+c} to $S_a \times S_b \times S_c$ we do need a multiplicity label. Specifying k_3 and j_3 independently resolves the multiplicity - its tells us which boxes in R are Y boxes and which are X boxes. The simplest way to see this is to note that we can first restrict $S_{b^{(0)}+b^{(1)}+b^{(2)}}$ to $S_{b^{(2)}} \times S_{b^{(0)}+b^{(1)}}$ without multiplicity, and then restrict $S_{b^{(0)}+b^{(1)}}$ to $S_{b^{(1)}} \times S_{b^{(0)}}$, again without multiplicity. The first restriction introduces (k, k_3) and the second (j, j_3).

D Rotating Restricted Schur Polynomials

In this Appendix we review results that were obtained in [71]. We would like to obtain the action of the following $su(2)_R$ generators

$$
\begin{aligned}
R_- &= \mathrm{Tr}\left(X\frac{d}{dY}\right), \qquad R_+ = \mathrm{Tr}\left(Y\frac{d}{dX}\right) \\
R_3 &= [R_+, R_-] = \mathrm{Tr}\left(Y\frac{d}{dY} - X\frac{d}{dX}\right).
\end{aligned} \tag{130}
$$

Once we have evaluated the action of R_+, the action of R_- follows by hermittian conjugation, and the action of R_3 then follows by using the $su(2)$ algebra. Consequently, we only need the action of $R_- = \mathrm{Tr}\left(X\frac{d}{dY}\right)$. The computation is carried out by allowing R_- to act on the restricted Schur polynomial. The result can then be expressed as a linear combination of restricted Schur polynomials, since the restricted Schur operators provide a basis. The coefficients of this linear expansion are given by the trace of a product of projection operators. In the distant corners approximation, the computation of the traces that need to be computed is reduced to the evaluation of $su(2)$ Clebsch-Gordan coefficients. The result is [71]

$$
\begin{aligned}
&\mathrm{Tr}\left(X\frac{d}{dY}\right) O^{(n,m,p)}_{R,r,j,j_3,k,k_3} \\
&= \frac{j+j_3}{2j}\frac{k+k_3+1}{2k+1}\sqrt{\left(\frac{m}{2}+j+1\right)\frac{2j}{2j+1}}\sqrt{\left(\frac{p}{2}+k+2\right)\frac{2k+1}{2k+2}}O^{(n,m-1,p+1)}_{R,r,j-\frac{1}{2},j_3-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2}} \\
&\quad + \frac{j+j_3}{2j}\frac{k-k_3}{2k+1}\sqrt{\left(\frac{m}{2}+j+1\right)\frac{2j}{2j+1}}\sqrt{\left(\frac{p}{2}-k+1\right)\frac{2k+1}{2k}}O^{(n,m-1,p+1)}_{R,r,j-\frac{1}{2},j_3-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2}} \\
&+ \frac{j-j_3+1}{2j+2}\frac{k+k_3+1}{2k+1}\sqrt{\left(\frac{m}{2}-j\right)\frac{2j+2}{2j+1}}\sqrt{\left(\frac{p}{2}+k+2\right)\frac{2k+1}{2k+2}}O^{(n,m-1,p+1)}_{R,r,j+\frac{1}{2},j_3-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2}} \\
&\quad + \frac{j-j_3+1}{2j+2}\frac{k-k_3}{2k+1}\sqrt{\left(\frac{m}{2}-j\right)\frac{2j+2}{2j+1}}\sqrt{\left(\frac{p}{2}-k+1\right)\frac{2k+1}{2k}}O^{(n,m-1,p+1)}_{R,r,j+\frac{1}{2},j_3-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2}} \\
&+ \frac{j-j_3}{2j}\frac{k-k_3+1}{2k+1}\sqrt{\left(\frac{m}{2}+j+1\right)\frac{2j}{2j+1}}\sqrt{\left(\frac{p}{2}+k+2\right)\frac{2k+1}{2k+2}}O^{(n,m-1,p+1)}_{R,r,j-\frac{1}{2},j_3+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2}} \\
&\quad + \frac{j-j_3}{2j}\frac{k+k_3}{2k+1}\sqrt{\left(\frac{m}{2}+j+1\right)\frac{2j}{2j+1}}\sqrt{\left(\frac{p}{2}-k+1\right)\frac{2k+1}{2k}}O^{(n,m-1,p+1)}_{R,r,j-\frac{1}{2},j_3+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2}} \\
&+ \frac{j+j_3+1}{2j+2}\frac{k-k_3+1}{2k+1}\sqrt{\left(\frac{m}{2}-j\right)\frac{2j+2}{2j+1}}\sqrt{\left(\frac{p}{2}+k+2\right)\frac{2k+1}{2k+2}}O^{(n,m-1,p+1)}_{R,r,j+\frac{1}{2},j_3+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2}}
\end{aligned}
$$

$$+\frac{j+j_3+1}{2j+2}\frac{k+k_3}{2k+1}\sqrt{\left(\frac{m}{2}-j\right)\frac{2j+2}{2j+1}}\sqrt{\left(\frac{p}{2}-k+1\right)\frac{2k+1}{2k}}O^{(n,m-1,p+1)}_{R,r,j+\frac{1}{2},j_3+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2}}\,. \tag{131}$$

These are not exact expressions - there are corrections of order $\frac{b^{(1)}}{b^{(0)}}$ and $\frac{b^{(2)}}{b^{(0)}}$, which are subleading at large N. Notice that there is a complicated mixing of the restricted Schur polynomials under $su(2)_R$. The restricted Schur polynomials are not organized into multiplets of $su(2)_R$.

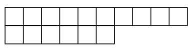
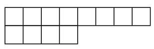
E Gauss Graph Transformations

In this Appendix we will derive explicit formulas for the transformation from the restricted Schur polynomial basis to the Gauss graph basis. These transformation formulas are needed to

1. Construct the Hilbert space of the excited giant graviton brane system.
2. Translate the action of $su(2)$ generators from the restricted Schur basis to the Gauss graph basis.

E.1 Bosonic Operators

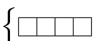
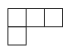
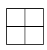
As a non-trivial example of how we move from the restricted Schur to the Gauss graph basis, consider an excitation constructed using 4 bosonic Y fields. Assume that we study a 2 brane system so that both R and r have two rows. We remove two excitations from each row so that

$R =$  $\qquad r =$  (132)

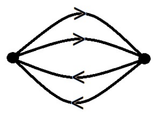
Denoting the excitations removed from row 1 by $1,2$ and the excitations removed from row 2 by $3,4$ we have

$$H=\{1,(12),(34),(12)(34)\}. \tag{133}$$

In the restricted Schur basis, the possible representation that the excitations can be arranged into are

$s \in \{$  ,  ,  $\}$. (134)

We choose our permutation so that we are describing a pair of strings stretched between nodes 1 and 2

$\sigma = (13)(24) =$  (135)

We would like to compute the transformation coefficients, given by

$$C^{(s)}\left((13)(24)\right)=\frac{|H|}{\sqrt{b^{(1)}!}}\sqrt{d_s}\Gamma^{(s)}\left((13)(24)\right)_{km}B^{s\to 1_H}_k B^{s\to 1_H}_m . \tag{136}$$

There are no multiplicity labels on the branching coefficient because R has 2 rows. The branching coefficient is determined by

$$\frac{1}{|H|}\sum_{\gamma\in H}\Gamma^{(s)}(\gamma)_{km}=B_kB_m. \tag{137}$$

For $s = \yng(4)$ the representation is one dimensional, $\Gamma^{(\yng(4))}(\sigma) = 1$ for any σ and the branching coefficient $B = 1$. Consequently

$$C^{(\yng(4))}((13)(24)) = \frac{4}{\sqrt{24}} \cdot \sqrt{1} \cdot 1 = \sqrt{\frac{2}{3}}. \tag{138}$$

For $s = \yng(3,1)$ the representation $\Gamma^{(\yng(3,1))}(\sigma)$ is three dimensional. The branching coefficient is determined to be

$$B = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \end{bmatrix} \tag{139}$$

and consequently

$$C^{(\yng(3,1))}((13)(24)) = \frac{4}{\sqrt{24}} \cdot \sqrt{3} \cdot \Gamma^{(\yng(3,1))}_{km} B_k B_m = -\sqrt{2}. \tag{140}$$

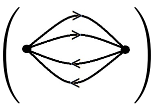
Finally for $s = \yng(2,2)$ the representation $\Gamma^{(\yng(2,2))}(\sigma)$ is two dimensional. The branching coefficient is determined to be

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{141}$$

and consequently

$$C^{(\yng(3,1))}((13)(24)) = \frac{4}{\sqrt{24}} \cdot \sqrt{2} \cdot \Gamma^{(\yng(2,2))}_{km} B_k B_m = \frac{2}{\sqrt{3}}. \tag{142}$$

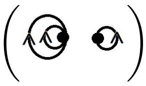
Thus, we find that



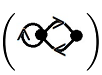
$$O_{R,r}\left(\ \right) = \sqrt{\frac{2}{3}} O_{R,(r,\yng(4))} - \sqrt{2} O_{R,(r,\yng(3,1))} + \frac{2}{\sqrt{3}} O_{R,(r,\yng(2,2))}. \tag{143}$$

We did not explicitly specify that we remove two impurities from the first row and two from the second row on the right hand side of this equation, but it can be read off of the graph appearing on the left hand side.

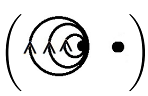
Here are a few more examples of transformations between the restricted Schur and Gauss graph bases



$$O_{R,r}\left(\ \right) = \sqrt{\frac{2}{3}} O_{R,(r,\yng(3))} + \frac{2}{\sqrt{3}} O_{R,(r,\yng(2,1))}, \tag{144}$$



$$O_{R,r}\left(\ \right) = \sqrt{\frac{2}{3}} O_{R,(r,\yng(3))} - \frac{1}{\sqrt{3}} O_{R,(r,\yng(2,1))}, \tag{145}$$



$$O_{R,r}\left(\ \right) = \sqrt{6} O_{R,(r,\yng(3))}. \tag{146}$$

The last example above generalizes very nicely: for m loops attached to the first node, we replace s by a Young diagram that is a single row with m boxes. These expression will be very useful in Appendix F when we study the action of rotations on Gauss graph operators, using the known action of rotations on restricted Schur polynomials.

E.2 Fermionic Operators

The structure of the state space of the fermionic Gauss graphs depends crucially on the properties of the transformation from restricted Schur polynomials to Gauss graph operators. For that reason we will work out a few carefully chosen examples in this Appendix.

Imagine that we have an excitation constructed from the ψ_1 field. The formula for the transformation coefficients from the representation s and multiplicity labels μ_1, μ_2 that organize the fermionic excitations is given by

$$\tilde{C}^{(s)}_{\mu_1\mu_2}(\tau) = |H|\sqrt{\frac{d_s}{f^{(1)}!}}\left(\Gamma^{(s)}(\tau)\hat{O}\right)_{km} B^{s\to 1_H}_{k\mu_1} B^{s^T\to 1^{f^{(1)}}}_{m\mu_2}. \tag{147}$$

Notice that two distinct branching coefficients appear. Before evaluating any examples of the coefficients $\tilde{C}^{(s)}_{\mu_1\mu_2}(\tau)$ we will relate the two branching coefficients that appear. Starting from the definition of the branching coefficient $B^{s^T\to 1^{f^{(1)}}}_{m\mu}$ we easily find

$$\begin{aligned}
\sum_\mu B^{s^T\to 1^{f^{(1)}}}_{k\mu} B^{s^T\to 1^{f^{(1)}}}_{m\mu} &= \frac{1}{|H|}\sum_{\gamma\in H}\mathrm{sgn}(\gamma)\Gamma^{(s^T)}(\gamma)_{km} \\
&= \frac{1}{|H|}\sum_{\gamma\in H}\mathrm{sgn}(\gamma)(\hat{O}\Gamma^{(s)}(\gamma)\hat{O})_{km} \\
&= \frac{1}{|H|}\sum_{\gamma\in H}\Gamma^{(s)}(\gamma)_{km} \\
&= \sum_\mu B^{s\to 1_H}_{k\mu} B^{s\to 1_H}_{m\mu},
\end{aligned} \tag{148}$$

which proves that the two branching coefficients are in fact equal! Consequently the formula for the transformation coefficients can be simplified to

$$\tilde{C}^{(s)}_{\mu_1\mu_2}(\tau) = |H|\sqrt{\frac{d_s}{f^{(1)}!}}\left(\Gamma^{(s)}(\tau)\hat{O}\right)_{km} B^{s\to 1_H}_{k\mu_1} B^{s\to 1_H}_{m\mu_2}. \tag{149}$$

In what follows we again restrict our attention to examples for which R has two rows. Thus, we can again drop the multiplicity labels.

To begin, consider an excitation constructed using three ψ_1s. For simplicity again consider a Young diagram R with two rows. Two of the ψ_1 impurities live in the first row of R and one in the second row. The only possible representation that leads to a non-zero restricted Schur polynomial is $s = \yng(2,1)$ as has already been explained in subsection 4.2.4. A straight forward computation shows that

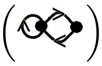
$$\hat{O} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{150}$$

The group $H = \{1, (12)\}$ and the branching coefficient is easily determined to be

$$B = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}. \tag{151}$$

It is now straight forward to verify that

$$\tilde{C}^{(\yng(2,1))}\left(\text{[graph]} \right) = \tilde{C}^{(\yng(2,1))}(1) = \tilde{C}^{(\yng(2,1))}(\,(12)\,) = 0, \tag{152}$$

$$\tilde{C}^{(\yng(2,1))}\left(\right.$$  $$\left.\right) = \tilde{C}^{(\yng(2,1))}(\,(13)\,) = -\tilde{C}^{(\yng(2,1))}(\,(23)\,)$$
$$= \tilde{C}^{(\yng(2,1))}(\,(132)\,) = -\tilde{C}^{(\yng(2,1))}(\,(123)\,) = 1. \quad (153)$$

The negative signs which appear above are exactly what we expect. They reflect an odd number of swaps of fermion fields.

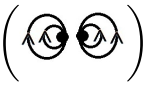
For the second example, consider an excitation constructed using four ψ_1s and again consider a Young diagram R with two rows. Two of the ψ_1 impurities live in the first row of R and two in the second row. The only possible representation that leads to a non-zero restricted Schur polynomial is $s = \yng(2,2)$, which was also explained in subsection 4.2.4. A straight forward computation shows that we again have

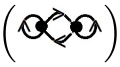
$$\hat{O} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (154)$$

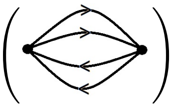
The group $H = \{1, (12), (34), (12)(34)\}$ and the branching coefficient is easily determined to be

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (155)$$

It is now straight forward to verify that

$$\tilde{C}^{(\yng(2,2))}\left(\right.$$  $$\left.\right) = 0, \quad (156)$$

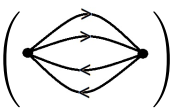
$$\tilde{C}^{(\yng(2,2))}\left(\right.$$  $$\left.\right) = 0, \quad (157)$$

$$\tilde{C}^{(\yng(2,2))}\left(\right.$$  $$\left.\right) = 1. \quad (158)$$

F Rotating Gauss graph operators

In this section we will use the action of the $su(2)_R$ generators on restricted Schur polynomials given in Appendix D, and the translation between restricted Schur polynomials and Gauss graphs worked out in Appendix E, to determine the action of the $su(2)_R$ generators on the Gauss graph operators.

To begin we will work out an example which demonstrates that the $su(2)_R$ generators leave the edges in a Gauss graph, that stretch between distinct nodes, inert. The computation is most easily phrased using the notation introduced in Appendix C. Consider a two giant system constructed using $b^{(0)}$ Z fields, 4 Y fields and no X, ψ_1 or ψ_2 fields. Two Y fields belong to the first row of R and two to the second row. Our starting point is the formula

$$O_{R,r}\left(\right.$$  $$\left.\right) = \sqrt{\frac{2}{3}}O_{R,r,2,0,0,0} - \sqrt{2}O_{R,r,1,0,0,0} + \frac{2}{\sqrt{3}}O_{R,r,0,0,0,0}. \quad (159)$$

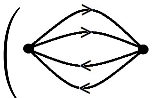
A simple application of the formula in Appendix D leads to

$$\mathrm{Tr}\left(X\frac{d}{dY}\right)O_{R,r,2,0,0,0} = O_{R,r,\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}} + O_{R,r,\frac{3}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}}, \quad (160)$$

$$\mathrm{Tr}\left(X\frac{d}{dY}\right)O_{R,r,1,0,0,0} = \sqrt{\frac{2}{3}}O_{R,r,\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}} + \frac{1}{\sqrt{3}}O_{R,r,\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}} + \sqrt{\frac{2}{3}}O_{R,r,\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}} + \frac{1}{\sqrt{3}}O_{R,r,\frac{3}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}}, \tag{161}$$

$$\mathrm{Tr}\left(X\frac{d}{dY}\right)O_{R,r,0,0,0,0} = O_{R,r,\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}} + O_{R,r,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}}. \tag{162}$$

It is now trivial to verify that

$\mathrm{Tr}\left(X\frac{d}{dY}\right)O_{R,r}\Big($  $\Big) = 0.$ (163)

The second example we consider illustrates the usual co-product action of the $su(2)_R$ generators. We will use black edges to denote Y excitations and grey edges to denote X excitations. Starting from

$O_{R,r}\Big($  $\Big) = \frac{1}{\sqrt{2}}O_{R,r,1,0,0,0} + \frac{1}{\sqrt{2}}O_{R,r,0,0,0,0}$ (164)

and using

$$\mathrm{Tr}\left(X\frac{d}{dY}\right)O_{R,r,1,0,0,0} = \frac{1}{\sqrt{2}}O_{R,r,\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}} + \frac{1}{\sqrt{2}}O_{R,r,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}}, \tag{165}$$

$$\mathrm{Tr}\left(X\frac{d}{dY}\right)O_{R,r,0,0,0,0} = \frac{1}{\sqrt{2}}O_{R,r,\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}} + \frac{1}{\sqrt{2}}O_{R,r,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}}, \tag{166}$$

as well as

$O_{R,r}\Big($  $\Big) = O_{R,r,\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}},$ (167)

$O_{R,r}\Big($  $\Big) = O_{R,r,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}},$ (168)

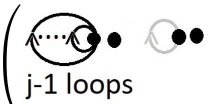
we find

$\mathrm{Tr}\left(X\frac{d}{dY}\right)O_{R,r}\Big($  $\Big) = O_{R,r}\Big($  $\Big)$

$+O_{R,r}\Big($  $\Big).$ (169)

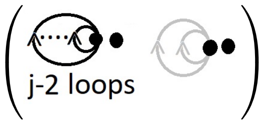
This clearly illustrates that the generator acts on each node individually, turning a black (Y) edge into a grey (X) edge when it acts.

In our final example, we would like to test that the coefficient in (105) comes out correctly. Assume that the excitation is built from $j-1$ Y fields and one X field, which all come from the first row of R. In this case we have

$O_{R,r}\Big($ 

$\Big) = O_{R,r,\frac{j-1}{2},\frac{j-1}{2},\frac{1}{2},\frac{1}{2}}$ (170)

and

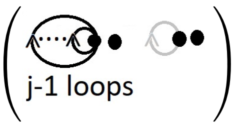
$O_{R,r}$ 

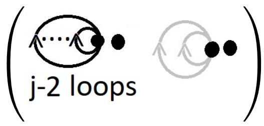
$= O_{R,r,\frac{j-2}{2},\frac{j-2}{2},1,1}.$ (171)

The equation

$$\mathrm{Tr}\left(X\frac{d}{dY}\right)O_{R,r,\frac{j-1}{2},\frac{j-1}{2},\frac{1}{2},\frac{1}{2}} = \sqrt{2(j-1)}O_{R,r,\frac{j-2}{2},\frac{j-2}{2},1,1} \tag{172}$$

implies

$\mathrm{Tr}\left(X\frac{d}{dY}\right)O_{R,r}$ 

$= \sqrt{2(j-1)}O_{R,r}$ 

, (173)

which beautifully matches the expected result of the action of the lowering operator on state $|j,m\rangle$

$$R_{-}|j,j-1\rangle = \sqrt{2(j-1)}|j,j-2\rangle. \tag{174}$$

A node with n_Y closed Y loops and n_X closed X loops is in the representation $j = n_Y + n_X$ and has $m = n_Y - n_X$.

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