Newtonian Twistor Theory

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This dissertation is submitted for the degree of Doctor of Philosophy.

May 2017

To my mother and father,

Abstract

In twistor theory the nonlinear graviton construction realises four-dimensional anti-self-dual Ricci-flat manifolds as Kodaira moduli spaces of rational curves in three-dimensional complex manifolds. We establish a Newtonian analogue of this procedure, in which four-dimensional Newton-Cartan manifolds arise as Kodaira moduli spaces of rational curves with normal bundle $\mathcal{O} \oplus \mathcal{O}(2)$ in three-dimensional complex manifolds. The isomorphism class of the normal bundle is unstable with respect to general deformations of the complex structure, exhibiting a jump to the Gibbons-Hawking class of twistor spaces. We show how Newton-Cartan connections can be constructed on the moduli space by means of a splitting procedure augmented by an additional vector bundle on the twistor space which emerges when considering the Newtonian limit of Gibbons-Hawking manifolds. The Newtonian limit is thus established as a jumping phenomenon.

Newtonian twistor theory is extended to dimensions three and five, where novel features emerge. In both cases we are able to construct Kodaira deformations of the flat models whose moduli spaces possess Galilean structures with torsion. In five dimensions we find that the canonical affine connection induced on the moduli space can possess anti-self-dual generalised Coriolis forces.

We give examples of anti-self-dual Ricci-flat manifolds whose twistor spaces contain rational curves whose normal bundles suffer jumps to $\mathcal{O}(2-k) \oplus \mathcal{O}(k)$ for arbitrarily large integers k, and we construct maps which portray these big-jumping twistor spaces as the resolutions of singular twistor spaces in canonical Gibbons-Hawking form. For $k > 3$ the moduli space itself is singular, arising as a variety in an ambient $\mathbb{C}^{k+1}$. We explicitly construct Newtonian twistor spaces suffering similar jumps.

Finally we prove several theorems relating the first-order and higher-order symmetry operators of the Schrödinger equation to tensors on Newton-Cartan backgrounds, defining a Schrödinger-Killing tensor for this purpose. We also explore the role of conformal symmetries in Newtonian twistor theory in three, four, and five dimensions.

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as specified in the text.

James Gundry

19th May 2017

Acknowledgments

None of the work reviewed in this thesis would have occurred without the enjoyable and invigorating oversight of my supervisor Maciej Dunajski. He provided a pleasing and ultimately fruitful initial project and has continued to be an invaluable ally through insightful conversations and guidance in the ways of research. I thank him for the opportunities and support.

In addition I would like to thank Michael Atiyah, Christian Duval, Gary Gibbons, Nick Manton, Lionel Mason, Roger Penrose, David Skinner, George Sparling, and especially Paul Tod for various helpful and stimulating mathematical discussions over the course of the last few years.

Thanks are also due to my parents for their unending encouragement; to my friends in Cambridge, who have brightened the research period with their company; and to my partner for her wonderful and warm-hearted support.

Finally I wish to express my sincere gratitude to STFC and to DAMTP for their tremendously generous research funding.

Contents

Chapter 1

Introduction

Twistor theory takes local solutions of equations in mathematical physics and recasts them as natural global holomorphic data on complex manifolds called *twistor spaces*. This is a reversible procedure, allowing one to take generic holomorphic structures on twistor spaces and construct spacetime models equipped with local geometry automatically obeying familiar integrability conditions. These conditions are often nonlinear partial differential equations and are, on inspection, highly nontrivial to solve; twistor theory thus provides a back-door solution in such situations, and often this solution is general.

One of the field's most important results is the *nonlinear graviton* construction of Penrose [66]. This provides the general solution of the anti-self-dual vacuum Einstein equations by establishing a bijection between four-manifolds equipped with anti-self-dual complexified Riemannian Ricci-flat metrics and cohomology classes of a line bundle over a twistor space. This powerful result can be upgraded to consider Einstein manifolds and a variety of other possibilities [21, 80]. How it works in practice is that the class is used to deform the complex structure of the twistor space, and the spacetime arises as a *Kodaira moduli space* of rational curves embedded in the twistor space [49]. That is to say, we build a spacetime manifold by associating each of its points to a rational curve in the twistor space.

The field began algebraically with Penrose, as a way of unifying the homogeneous and inhomogeneous sectors of the Poincaré group, resulting in a notion of *twistors* as spinors of the conformal group [64]. Soon followed contour integral formulas for linear wave equations [65], the nonlinear graviton construction [66], Ward's solution of the anti-self-dual Yang-Mills equations [78], and the impact of twistor theory on integrability [55, 21]. In the present day twistor methods are finding employment at the forefront of theoretical physics (see, for example, [81, 54, 1, 13]). Faced with such a wide variety of approaches [4], we wish to emphasise at this stage that the aspect of twistor theory with which we are primarily concerned in this thesis is that of constructing Kodaira families of rational curves and their induced local geometry.

It is not only four-dimensional Riemannian geometry which is amenable to this treatment; Hitchin described two additional twistor theories in which one constructs two-dimensional projective surfaces and three-dimensional Einstein-Weyl manifolds as Kodaira moduli spaces of rational curves [42].

The central strand of this thesis establishes several novel twistor theories pertaining to Newton-Cartan geometry in dimensions four, three, and five. Newton-Cartan manifolds are the non-relativistic equivalent of a Riemannian manifold, a setting where gravity is felt via the curvature of a connection [17, 28]. We construct Newton-Cartan manifolds as Kodaira moduli spaces of rational curves in complex twistor spaces in chapter 3, explicitly building the affine connection from twistor data. This *Newtonian twistor theory* contains some surprises, one of which is that the twistorial version of the Newtonian limit is a *jumping* phenomenon in four dimensions. The normal bundles to the family of rational curves suffer a discontinuous change as the speed of light tends to infinity; this makes the four-dimensional theory unstable with respect to Kodaira deformations [22]. Another surprise comes when the theory is extended to three and five dimensions [41]; we find that deformations result not in a jump but in the introduction of torsion.

We study other jumping phenomena in twistor theory and Newtonian twistor theory in chapter 4, where we review the construction of the first examples of twistor spaces containing rational curves suffering arbitrarily large numbers of jumps [23], as well as proving some related theorems regarding the nature of such twistor spaces.

The development of Newtonian twistor theory led the author to prove some theorems on the conformal symmetries of Newton-Cartan geometry [40]. In chapter 5 we establish a correspondence between a non-relativistic analogue of conformal Killing vectors on a curved Newton-Cartan background and the first-order symmetry operators of the Schrödinger equation; we define a Schrödinger analogue of a conformal Killing tensor and identify such tensors as the higher symmetry algebra of the free-particle Schrödinger equation, a non-relativistic lesser cousin of the famous result of Eastwood [31]; and we construct the conformal symmetry algebras induced on Newton-Cartan manifolds by the algebras of global holomorphic vector fields on Newtonian twistor spaces.

Finally in chapter 6 we take the first steps towards building a twistor-theoretic approach to the free-particle Schrödinger equation. We derive contour integral formulas from twistor theory for its solutions in $(3+1)$ and $(2+1)$ dimensions, before proving that all plane-wave states lie within their range.

Conventions

- ◇ We denote the complex projective line $\mathbb{P}^1$ (instead of $\mathbb{CP}^1$); $\mathbb{RP}^1$ appears nowhere in the remainder of this thesis.
- ◇ The complex projective line is a complex manifold coordinatised by two patches U and $\hat{U}$ with respective coordinate functions λ and $\hat{\lambda}$ obeying $\hat{\lambda} = \lambda^{-1}$ on $U \cap \hat{U}$.
- ◇ Standard line bundles on the complex projective line are denoted $\mathcal{O}(n) \to \mathbb{P}^1$ for first Chern class $n \in \mathbb{Z}$ and have transition functions λ^{-n}.
- ◇ The symmetrised tensor product is denoted $\odot$, i.e.

$$A \odot B = \frac{1}{2}\left(A \otimes B + B \otimes A\right).$$

- ◇ The phrases "twistor lines", "projective lines", "rational curves", and (where appropriate) "global sections" are freely interchanged when describing the compact complex submanifolds of twistor spaces constituting Kodaira families.

- ⋄ The abbreviation (A)SD always stands for (anti-)self-dual.

- ⋄ Let M be a $(d+1)$-dimensional moduli space. Lower-case Latin indices $(a, b, c...)$ run from 0 to d; lower-case Latin indices $(i, j, k...)$ run from 1 to d and will usually be spatial in nature.

- ⋄ On a D-dimensional twistor space Z lower-case Greek indices $(\mu, \nu, ...)$ will run from 0 to $D-1$. (The exception is in section 5.2.3, where a second set of spacetime indices is required.)

- ⋄ A vertical slash on the right of a symbol indicates that the quantity represented by that symbol has been restricted to a rational curve in the Kodaira family.

- ⋄ Inhomogeneous twistor variables are preferred and will generally be denoted by upper-case symbols; homogeneous twistor variables will generally be denoted by lower-case symbols. The exceptions are in chapter 4, where ζ denotes an inhomogeneous twistor variable, and in chapter 6, where $\Pi_{A'}$ is homogeneous of weight one.

- ⋄ The description of special kinds of geometry follows the standard syntax so that, for example, a *Newton-Cartan manifold* is a smooth manifold equipped with a *Newton-Cartan structure*.

- ⋄ The word "spacetime" will sometimes be used in place of "moduli space", "Newton-Cartan manifold", or "Riemannian manifold" when we wish to emphasise the role of these spaces in physical models.

- ⋄ Spinor indices are raised and lowered with the standard symplectic forms according to the "down and to the right" convention: $\psi^A = \epsilon^{AB}\psi_B$ and $\psi_A = \psi^B\epsilon_{BA}$.

Chapter 2

Preliminaries

Here we discuss the technical background against which the main body of the thesis is set.

2.1 Kodaira moduli spaces

Twistor theory models spacetimes with local geometry as moduli spaces of rational curves within a complex manifold; the mathematics describing the precise nature of such moduli spaces is due to Kodaira [49, 50, 51]. In particular there is a criterion which must be satisfied for the moduli space to itself be a complex manifold. To frame the theorem we must establish the following definition, following [56].

Let Z and M be complex manifolds. Their product space $Z \times M$ is equipped with the natural projections

$$\Pi_Z \,:\, Z \times M \to Z$$

and

$$\Pi_M \,:\, Z \times M \to M.$$

Definition 2.1.1. *A holomorphic family of compact complex submanifolds of Z with moduli space M is a complex submanifold*

$$F \subset Z \times M$$

such that $\nu := \Pi_M|_F$ is a proper regular map.

By *regular* we mean that the rank of the differential of ν is everywhere equal to the dimension of M.

Writing $\mu := \Pi_Z|_F$ we have a double fibration.

$$\begin{tikzcd} & F \arrow[dl, "\nu"'] \arrow[dr, "\mu"] & \\ M & & Z \end{tikzcd} \tag{2.1.1}$$

In twistor theory we study examples of these families, where Z is the *twistor space*; M is the spacetime; and F is called the *correspondence space.* For $x \in M$ we have an associated submanifold $X_x = \mu \circ \nu^{-1}(x) \subset Z$, which in all cases of interest in this thesis will be $X_x = \mathbb{P}^1$.

Let

$$N_x = \frac{(TZ)\,|_{X_x}}{TX_x}$$

be the normal bundle to a submanifold X_x. There is then a canonical map $\mathcal{K}$ due to Kodaira;

$$\mathcal{K}_x \,:\, T_xM \to \check{H}^0\left(X_x, N_x\right) \ ,$$

and we will explicitly construct this map for the twistor theories under consideration.

Definition 2.1.2. *If $\mathcal{K}_x$ is an isomorphism for all $x \in M$ then the family F is said to be complete.*

There is one more technical point, with which we will not be concerned again in this thesis.

Definition 2.1.3. *A holomorphic family F of compact submanifolds is called maximal if all other holomorphic families of compact submanifolds $F' = Z \times M'$ in Z which have some points $x \in M$ and $x' \in M'$ associated to the same submanifold admit holomorphic maps $j : M' \to M$ in the neighbourhoods of x' such that $\nu'^{-1}(x'') = \nu^{-1}(j(x''))$ for all x'' in the neighbourhood of x'.*

We are now in a position to state the relevant theorem.

Theorem 2.1.4. (Kodaira [49])

Let $X \subset Z$ be a compact complex submanifold with normal bundle $N \to X$. If

$$\check{H}^1(X, N) = 0 \tag{2.1.2}$$

then X is a member of a complete maximal holomorphic family of compact complex submanifolds $\{X_x \mid x \in M\}$ whose moduli space is of dimension $\dim \check{H}^0(X, N)$.

We will say that a compact complex submanifold $X \subset Z$ is *Kodaira* if it satisfies (2.1.2). Note that this theorem implies that M is a complex manifold, as this is contained in the definition of a complete holomorphic family of submanifolds.

Thus one can picture taking a particular Kodaira submanifold $X \subset Z$ and then building out from X a complete holomorphic family labelled by coordinates which come to constitute a chart in M. One reason we study this way of constructing M is that it often comes equipped with canonically-induced geometrical structures such as metrics and/or affine connections which are relevant to physics. Prescribing additional data on Z, such as (unconstrained) vector bundles $E \to Z$ leads to further induced structures on M such as Yang-Mills fields via, for instance, the Ward transform [78].

2.2 The nonlinear graviton construction

Penrose's seminal paper [66] provides the twistor-theoretic context for much of this thesis. In it he described the nonlinear graviton construction, a theorem providing a highly significant family of twistor spaces whose Kodaira moduli spaces of rational curves are relevant to four-dimensional theoretical physics. The class of moduli spaces to be considered were named *gravitons* by Penrose, in the belief that they would come to represent such a particle in the semi-classical description of a future theory of quantum gravity.

2.2.1 Theorems

The moduli spaces are equipped with an anti-self-dual conformal structure, which we'll now define. Let C be the (conformal) Weyl curvature associated to a conformal structure $[g]$ on an n-manifold of Euclidean signature[1], and let

$$\star \,:\, \Lambda^{p}\left(M\right) \rightarrow \Lambda^{n-p}\left(M\right)$$

be the associated Hodge operator on p-forms. For $n = 4$ the Hodge star is an involution on two-forms, and has two eigenspaces with eigenvalues ± 1;

$$\Lambda^{2}\left(M\right) = \Lambda^{+}\left(M\right) \oplus \Lambda^{-}\left(M\right)$$

where the right-hand-side is the direct sum of the self-dual (eigenvalue $+1$) and anti-self-dual (eigenvalue -1) two-forms. The self-duality property extends to C by considering the action of $\star$ on a skew pair of indices. Let ϵ be the volume-form associated to a representative of $[g]$.

Definition 2.2.1. *A conformal structure $[g]$ is said to be anti-self-dual (ASD) iff*

$$C_{abcd} = -\frac{1}{2}\epsilon_{ab}{}^{ef}C_{efcd}.$$

The importance of this definition for twistor theory is exposed by the following theorem.

Theorem 2.2.2. (Penrose [66])

There is a one-to-one correspondence between

- ◇ *three-dimensional complex manifolds Z equipped with a four-parameter family of rational curves X_x with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$*

and

- ◇ *four-dimensional complexified manifolds M equipped with an ASD holomorphic conformal structure $[g]$.*

[1] One could also include neutral signature; see [27] for a discussion of anti-self-dual conformal structures in neutral signature.

This theorem establishes a large class of interesting examples of theorem 2.1.4; note that the isomorphism class $N_x = \mathcal{O}(1) \oplus \mathcal{O}(1)$ of the normal bundles to X_x ensures that $\check{H}^1(X_x, N_x) = 0$. We refer the reader to [66, 21, 55, 46, 80] for various self-contained proofs of this theorem; in this thesis demonstrations of how the theorem works can be found throughout, such is its importance.

Theorem 2.2.2 can then be upgraded by requiring additional structures on the twistor space Z. We'll now detail two essential upgrades.

Theorem 2.2.3. (Penrose [66])

There is a one-to-one correspondence between

- ◇ *three-dimensional complex manifolds Z equipped with*
  - *a fibration $\varrho : Z \to \mathbb{P}^1$,*
  - *a four-parameter family of rational curves X_x with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$,*
  - *and a non-degenerate holomorphic two-form $\{\Sigma\}$ on the fibres of ϱ with values in the pull-back of $\mathcal{O}(2)$ from $\mathbb{P}^1$,*

and

- ◇ *four-dimensional complexified manifolds M equipped with an ASD Ricci-flat metric g.*

The presence of the two-form ensures that the induced ASD conformal structure contains a Ricci-flat representative; moreover it explicitly constructs that representative in the following way. Restricting the two-form to the family $\{X_x\}$ of global sections induces three two-forms $\Sigma^{A'B'}$ on M:

$$\Sigma| = \Sigma^{0'0'}\lambda^2 + 2\Sigma^{0'1'}\lambda + \Sigma^{1'1'}$$

(because Σ is global and takes values in the pull-back of $\mathcal{O}(2)$). Linear combinations of these forms constitute a hyperKähler structure[2] on M, and hence determine a volume-form $\Sigma^{0'1'} \wedge \Sigma^{0'1'}$ on M. This volume-form can then be fixed to be the metric volume-form,

[2]For a definition of a hyperKähler structure see definition 2.5.1 and for a fuller explanation of how $\Sigma^{A'B'}$ determine such a structure see [21].

determining the conformal factor. Ricci-flatness follows from the equivalence of the ASD Ricci-flat condition to the hyperKähler condition [21].

Twistor theory thus provides the general solution of the (complex) ASD Ricci-flat equations, as complex manifolds Z with the required properties as quite straightforward to write down, as we shall see shortly.

The second upgrade allows one to consider real moduli spaces.

Theorem 2.2.4. (Penrose [66, 83, 5])

There is a one-to-one correspondence between

- ◇ *three-dimensional complex manifolds Z equipped with*
  - *a fibration $Z \to \mathbb{P}^1$,*
  - *a four-parameter family of rational curves X_x with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$,*
  - *a non-degenerate holomorphic two-form $\{\Sigma\}$ on the fibres with values in the pull-back of $\mathcal{O}(2)$ from $\mathbb{P}^1$,*
  - *and an involution $\varkappa : Z \to Z$ such that $\varkappa^2 = 1$ on the (projective) twistor space which acts as the antipodal map $\lambda \mapsto -\bar{\lambda}^{-1}$ on the rational curves,*

and

- ◇ *four-dimensional real manifolds M equipped with an ASD Ricci-flat metric g of Euclidean signature.*

The involution allows one to single out a real moduli space M. Rational curves preserved by $\varkappa$ form a (real) four-parameter non-intersecting family: thus Z is fibred by such curves, and we can take the family to be a real slice of the usual Kodaira moduli space [83].

One can upgrade these theorems further to consider, for instance, moduli spaces which are ASD Einstein rather than just Ricci-flat [80], hyper-Hermitian [20], or scalar-flat Kähler [70].

To end this section we will briefly describe how examples of theorem 2.2.3 can be readily constructed by deforming the complex structure of the flat model $Z = \mathcal{O}(1) \oplus \mathcal{O}(1)$ equipped with the natural family of rational curves provided by the global sections of $Z \to \mathbb{P}^1$.

2.2.2 Flat model

Consider $Z = \mathcal{O}(1) \oplus \mathcal{O}(1)$ with patching

$$\hat{\Omega}^A = \lambda^{-1}\Omega^A \qquad \hat{\lambda} = \lambda^{-1} \tag{2.2.1}$$

for $A = 0, 1$ and global sections

$$\Omega^A| = x^{A0'}\lambda + x^{A1'} \tag{2.2.2}$$

where $x^{AA'}$ are coordinates on M parametrising the family of global sections[3]. According to the twistor theory we should label as *null* those vectors tangent to the submanifolds in M (called *alpha-surfaces*) defined by setting Ω^A and λ to be constant in $\Omega^A = \Omega^A|$, i.e. defined to encompass all points x in M whose associated twistor lines X_x contain $\left(\Omega^A, \lambda\right)$. The conformal structure which singles out those same vectors as null is then

$$[g] = dx^{00'}dx^{11'} - dx^{01'}dx^{10'}. \tag{2.2.3}$$

The two-form $\hat{\Sigma} = \lambda^{-2}\Sigma$ is given by $\Sigma = d\Omega^0 \wedge d\Omega^1$, which fixes g to be the representative on the right-hand-side of (2.2.3).

2.2.3 Deformations

Consider instead a complex manifold Z with patching

$$\hat{\Omega}^A = \lambda^{-1}\Omega^A - \epsilon^{AB}\frac{\partial f}{\partial \Omega^B} \qquad \hat{\lambda} = \lambda^{-1} \ ,$$

a Kodaira deformation [50, 51] of (2.2.1), where $f\left(\Omega^A, \lambda\right)$ is a function representing a cohomology class in $\check{H}^1\left(\mathcal{O}(1) \oplus \mathcal{O}(1), \mathcal{O}(2)_{\mathcal{O}(1)\oplus\mathcal{O}(1)}\right)$.

[3]Over $\hat{U}$ the counterparts to (2.2.2) are $\hat{\Omega}^A| = x^{A0'} + x^{A1'}\hat{\lambda}$.

Following [46, 51], $\check{H}^1(X_x, \mathrm{End}(N_x)) = 0$ means that the isomorphism class $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is stable with respect to the deformation, and the four-parameter family of global sections still exists. Thus one can find functions $\Omega^A|(x, \lambda)$ more complicated than (2.2.2) and again single out the vectors tangent to the alpha-surfaces $\Omega^A = \Omega^A|$ in M, the Kodaira moduli space of the sections. The resulting conformal structure will be ASD, and is guaranteed to contain a Ricci-flat metric.

Example: a plane wave

To illustrate how this works in practice we will consider the simple twistor space (due to Sparling [75, 46]) described by the patching

$$\hat{\Omega}^0 = \lambda^{-1}\Omega^0 \qquad \hat{\Omega}^1 = \lambda^{-1}\Omega^1 + \epsilon\lambda^{-2}\left(\Omega^0\right)^3 \tag{2.2.4}$$

and possessing twistor functions

$$\Omega^0| = u\lambda + z \qquad \Omega^1| = x\lambda + y - \epsilon u^3\lambda^2$$

where ϵ is a deformation parameter and $x^a = (u, z, x, y)$ are coordinates on M. Vectors tangent to the alpha surfaces $\Omega^A = \Omega^A|$ lie withing the kernel of a conformal structure admitting the representative

$$g = dxdz - dydu + 3\epsilon u^2 dz^2 \ , \tag{2.2.5}$$

and it is straightforward to check that that this representative is the Ricci-flat one. The metric (2.2.5) is that of a plane wave.

2.3 Induced frames and metrics

Given a Kodaira family of rational curves there are various ways of constructing the induced geometry on M, all relying on the Kodaira isomorphism of definition 2.1.2. In Penrose's original approach one looks at the special surfaces called alpha surfaces induced on M by the pull-back of the twistor coordinates to the correspondence space and then uses the alpha surfaces to single out preferred vectors [66], as described in the previous section.

Alternatively, in the case of the nonlinear graviton, one can directly find the contravariant conformal structure as (the image on M of) the zero section of $\check{H}^0\left(Z, TZ \odot TZ\right)$. A third approach would be to construct an integral formula from twistor cohomology classes and then compute the differential operator(s) on M for which the twistor classes constitute the kernel.

In this section we'll describe another method, in which (families of) frames of one-forms are directly computed on M.

Theorem 2.3.1. [41]

Let M be the moduli space of a complete holomorphic family of rational curves $X_x = \mathbb{P}^1$ in a complex manifold $Z \to \mathbb{P}^1$ with normal bundles $N_x \to \mathbb{P}^1$. M is equipped with a preferred family of one-forms - the frame - defined uniquely up to an invertible element of $\check{H}^0\left(X_x, N_x \otimes N_x^*\right)$ per section X_x.

Proof

Let Z be $(k+1)$-dimensional. The family of one-forms on the moduli space M arises as a section of $N_x \otimes \Lambda^1_x\left(M\right)$ for each $x \in M$. Cover $\mathbb{P}^1$ by two patches U and $\hat{U}$ with coordinates λ and $\hat{\lambda}$ respectively, with holomorphic transition function

$$\hat{\lambda} = \lambda^{-1}$$

on $U \cap \hat{U}$. We can describe $\varrho : Z \to \mathbb{P}^1$ concretely as a complex manifold by exhibiting its patching. If $\hat{w}^\mu\left(w^\nu, \lambda\right)$ is this patching (for $\mu, \nu = 0, 1, ..., k-1$) then we can describe the global section X_x (over U) by the equation $w^\mu = w^\mu|\left(x, \lambda\right)$ for $x \in M$ parametrising the space of sections and where $w^\mu|$ are functions extracted from the patching.

One then finds that

$$d\hat{w}^\mu| = \mathcal{F}^\mu_\nu dw^\nu|$$

where

$$\mathcal{F}^\mu_\nu\left(x, \lambda\right) = \frac{\partial \hat{w}^\mu}{\partial w^\nu}\left(w^\alpha|, \lambda\right)$$

is the patching of the normal bundle N_x. (We recall the use of a vertical slash to denote the restriction to X_x throughout this paper.) By the Birkhoff-Grothendieck theorem [39] we can write

$$\mathcal{F} = \hat{H}\,\mathrm{diag}\left(\lambda^{-n_1}, \lambda^{-n_2}, \dots, \lambda^{-n_k}\right) H^{-1}$$

where H and $\hat{H}$ are holomorphic maps to $\mathrm{GL}\,(k, \mathbb{C})$ from U and $\hat{U}$ respectively, and where the integers $(n_1, n_2, ..., n_k)$ specify the isomorphism class of N_x. We then extract the frame from a section $(H^{-1})^{\mu}_{\nu}\, dw^{\nu}|$ (or equivalently $\left(\hat{H}^{-1}\right)^{\mu}_{\nu} d\hat{w}^{\nu}|$), and the non-uniqueness results from the fact that we can multiply H (and $\hat{H}$) by an invertible global section of $N_x \otimes N_x^*$ (which may vary arbitrarily with $x \in M$). We denote

$$v\,(x) = H^{-1} dw| \in \check{H}^0\left(F|_x, N_x \otimes \Lambda^1_x\,(M)\right)$$

the *frame section.*

The frame section then gives rise to a collection of one-forms $e^{\mu A'B'...C'}$ on M, where the index μ arises from the component-structure of v and $A'B'...C'$ (each running from $0'$ to $1'$) arise from the dependence of v on the base $\mathbb{P}^1$. For example, in the case where

$$N_x = \mathcal{O}(2) \oplus \mathcal{O}(2)$$

we write

$$v^{\mu} = e^{\mu A_1' A_2'} \pi_{A_1'} \pi_{A_2'}\ ,$$

where $[\pi_{A'}]$ are homogeneous coordinates on the base $\mathbb{P}^1$ and where $\mu = 0, 1$. One then extracts the frame $e^{\mu A_1' A_2'} = e_a^{\mu A_1' A_2'}\left(x^b\right) dx^a$. □

In this thesis we will include in v the result of the most general element of $\check{H}^0\,(X_x, N_x \otimes N_x^*)$ per line X_x in the guise of arbitrary functions. (This is analogous to writing a conformal structure $[g]$ as a single metric $g = \alpha g_0$ for some representative $g_0 \in [g]$ and some arbitrary non-vanishing function α.)

In the case of $k = 2$ we often revert to the notation $\mu = A = 0, 1$ and make use of

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.3.1}$$

Definition 2.3.2. [41]. *All tensor fields on M which can be constructed from the frame using only the tensor product and the symplectic forms ϵ_{AB} and $\epsilon_{A'B'}$ are induced on M as the span of v.*

Often, pleasingly, the span of v contains a metric (which may depend on some number of arbitrary functions), and in the nonlinear graviton construction this metric is exactly the conformal structure we could induce in a more traditional way (i.e. via the presence of alpha surfaces in M as is done in [66]).

To see this consider the following example, in which the twistor space is three-dimensional and the normal bundles to the rational curves are $\mathcal{O}\left(2n-1\right) \oplus \mathcal{O}\left(2n-1\right)$ for $n \geq 1$.

Theorem 2.3.3. [41]

Let $Z \to \mathbb{P}^1$ be a complex three-fold containing a rational curve X_0 with normal bundle $N_0 = \mathcal{O}\left(2n-1\right) \oplus \mathcal{O}\left(2n-1\right)$ for some integer $n \geq 1$. Then the Kodaira moduli space of rational curves X_x is a $(4n)$-dimensional complexified conformal manifold.

For $n = 1$ Z is a standard Penrose twistor space described in section 2.2, and in a different context this class of normal bundles is the setting for the heavenly hierarchy described in [24].

Note that theorem 2.3.3 is a different construction of $4n$-dimensional moduli spaces to that in [71, 63], where the authors induce quaternionic structures on Kodaira families of global sections of manifolds with normal bundle $\oplus^{2n}\mathcal{O}\left(1\right)$.

Proof

Since

$$\check{H}^1\left(\mathbb{P}^1, \mathcal{O}\left(2n-1\right) \oplus \mathcal{O}\left(2n-1\right)\right) = 0$$

the rational curve X_0 is a member of a family of dimension

$$\dim \check{H}^0\left(\mathbb{P}^1, \mathcal{O}\left(2n-1\right) \oplus \mathcal{O}\left(2n-1\right)\right) = 4n \ ,$$

and by theorem 2.3.1 we obtain a section of $\Lambda^1_x(M) \otimes N_x$ at each point $x \in M$ which gives rise to a frame $e^{AA'_1...A'_{2n-1}}$ via

$$v^A = e^{AA'_1...A'_{2n-1}} \pi_{A'_1}...\pi_{A'_{2n-1}}$$

and so a metric

$$g = e_{AA'_1...A'_{2n-1}} \otimes e^{AA'_1...A'_{2n-1}}.$$

The non-uniqueness acts acts by multiplication of an invertible global section of $N_x \otimes N_x^*$, which takes

$$\begin{pmatrix} e^{00'...0'}\lambda^{2n-1} + e^{00'...0'1'}\lambda^{2n-2} + ... + e^{01'...1'} \\ e^{10'...0'}\lambda^{2n-1} + e^{10'...0'1'}\lambda^{2n-2} + ... + e^{11'...1'} \end{pmatrix}$$
$$\mapsto \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} e^{00'...0'}\lambda^{2n-1} + e^{00'...0'1'}\lambda^{2n-2} + ... + e^{01'...1'} \\ e^{10'...0'}\lambda^{2n-1} + e^{10'...0'1'}\lambda^{2n-2} + ... + e^{11'...1'} \end{pmatrix}$$

for any four functions $\phi^A_B = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} : \mathbb{C} \to \mathrm{GL}(2, \mathbb{C})$, resulting in

$$e^{AA'_1...A'_{2n-1}} \mapsto \phi^A_B e^{BA'_1...A'_{2n-1}}$$

and so

$$g \mapsto \epsilon_{AD} \phi^D_B \phi^A_C e^B_{A'_1...A'_{2n-1}} \otimes e^{CA'_1...A'_{2n-1}} = (\det \phi)\, g.$$

We thus obtain a conformal transformation. $\square$

Using alpha surfaces to find the induced conformal structure (or otherwise) becomes increasingly complicated in higher dimensions, but the frame method of theorem 2.3.1 does not.

Example: alpha-surfaces for $Z = \mathcal{O}(4)$

Via theorem A.1.1 one can construct some five-dimensional Riemannian manifolds M as the Kodaira families of curves with normal bundle $\mathcal{O}(4)$ in a complex two-fold Z, the flat model being $Z = \mathcal{O}(4)$. In this example we'll consider how the conformal structure on M arises via the presence of alpha surfaces. The construction is considerably more complicated than that of the frame advocated throughout this paper, but its existence is nonetheless reassuring.

The (inhomogeneous) patching for $\mathcal{O}(4)$ is

$$\hat{S} = \lambda^{-4} S$$

and so the global sections are

$$S| = t + 4u\lambda + 6x\lambda^2 + 4v\lambda^3 + w\lambda^4$$

for $x^a = (t, u, x, v, w) \in M$.

The direct calculation of the frame (the case $n = 2$ in theorem A.1.1) is trivial in the flat case and clearly gives us $e^{A'B'C'D'} = \alpha dx^{A'B'C'D'}$ for arbitrary $\alpha : M \to \mathbb{C}^*$, and the span contains the metric

$$g = \alpha^2 dx_{A'B'C'D'} \otimes dx^{A'B'C'D'}.$$

According to the usual way of proceeding we take the null vectors δx^a to be those for which

$$0 = \delta t + 4\delta u\lambda + 6\delta x\lambda^2 + 4\delta v\lambda^3 + \delta w\lambda^4$$

has a unique solution in λ. The classical theory of quartic equations tells us that this happens when δx^a solves simultaneously the three conditions

$$\begin{aligned}\Delta_6 = -\delta t^3\delta w^3 + 12\delta t^2\delta u\delta v\delta w^2 + 27\delta t^2\delta v^4 - 54\delta t^2\delta v^2\delta w\delta x \\ + 18\delta t^2\delta w^2\delta x^2 + 6\delta t\delta u^2\delta v^2\delta w - 54\delta t\delta u^2\delta w^2\delta x - 108\delta t\delta u\delta v^3\delta x \\ + 180\delta t\delta u\delta v\delta w\delta x^2 + 54\delta t\delta v^2\delta x^3 - 81\delta t\delta w\delta x^4 + 27\delta u^4\delta w^2 \\ + 64\delta u^3\delta v^3 - 108\delta u^3\delta v\delta w\delta x - 36\delta u^2\delta v^2\delta x^2 + 54\delta u^2\delta w\delta x^3 = 0\end{aligned}$$

$$\Delta_4 = -\delta t\delta w^3 + 4\delta u\delta v\delta w^2 + 12\delta v^4 - 24\delta v^2\delta w\delta x + 9\delta w^2\delta x^2 = 0$$

$$\Delta_2 = -\delta t\delta w + 4\delta u\delta v - 3\delta x^2 = 0.$$

Claim: The vanishing of Δ_6, Δ_4, and Δ_2 is equivalent to δx^a falling into the union of the kernels of the span of the frame $e^{A'B'C'D'}$.

◇ The latter condition $\Delta_2 = 0$ is exactly what one would expect for the metric, giving us the symmetric two-form g above. (It agrees exactly with the g one would calculate from the direct frame method.) Concretely,

$$[g] = -\delta t \delta w + 4\delta u \delta v - 3\delta x^2.$$

◇ The vanishing of Δ_2 and Δ_6 simultaneously is equivalent to δx^a lying in the kernel of both g and a symmetric three-form $\mathcal{G}_3$ (which is also consistent with the direct frame calculation). In particular

$$\left[\mathcal{G}_3\left(\delta x^a, \delta x^a, \delta x^a\right)\right]^2 \propto \Delta_6 \qquad \text{when} \quad \Delta_2{=}0\,,$$

for

$$\mathcal{G}_3\left(\delta x^a, \delta x^a, \delta x^a\right) = -\delta t \delta v^2 + \delta t \delta w \delta x - \delta u^2 \delta w + 2\delta u \delta v \delta x - \delta x^3.$$

◇ Finally, when $\Delta_2 = \Delta_6 = 0$ the vanishing of Δ_4 is equivalent to having

$$\left[\delta x \delta w - \delta v^2 = 0 \quad \text{and} \quad \delta t \delta x - \delta u^2 = 0\right]$$
$$\text{and either} \quad \delta w \delta t - \delta x^2 = 0 \quad \text{or} \quad \delta u \delta v - \delta x^2 = 0. \tag{2.3.2}$$

These (effectively three) conditions are the requirement that δx^a lie in the kernel of three rank-three symmetric two-forms which we identify as $e^{0'0'}{}_{A'B'} \otimes e^{0'0'A'B'}$, $e^{0'1'}{}_{A'B'} \otimes e^{0'1'A'B'}$, and $e^{1'1'}{}_{A'B'} \otimes e^{1'1'A'B'}$.

◇ The rest of the canonical symmetric two-forms $e^{C'D'}{}_{A'B'} \otimes e^{E'F'A'B'}$ also arise, but as redundant conditions equivalent to (2.3.2). (One could choose to isolate three other conditions, say one rank-three symmetric two-form and two rank-four symmetric two-forms, and fit those to canonical forms instead, but for concreteness we have chosen the three rank-three symmetric two-forms.)

Thus we can obtain the induced metric via either direct calculation or (in a more complicated fashion) by the usual twistor theory arguments. Once one has the frame one has every canonical form discussed above.

2.4 Canonical connections

The construction of affine connections on moduli spaces of complete holomorphic families of submanifolds was considered by Merkulov [56, 57]. This work led to the solution of the holonomy problem [58].

First consider torsion-free affine connections more generally, taking [56] as a reference. Let J_x^k be the ideal of germs of holomorphic functions on M which vanish to order k at $x \in M$. The second-order tangent bundle $T^{[2]}M$ is defined to be the union over all points in M of second-order tangent spaces

$$T_x^{[2]}M = \left({}^{J_x}/_{J_x^3}\right)^* .$$

An element $\left(V^{ab}, V^a\right)$ of $T_x^{[2]}M$ consists of the first two non-vanishing terms of the Taylor expansion of a function vanishing at x; a section of $T^{[2]}M$ gives rise to a second-order linear differential operator

$$V^{[2]} = \left(V^{ab}, V^a\right) \rightsquigarrow V^{ab}\partial_a\partial_b + V^a\partial_a \ ,$$

where for brevity's sake we put $\partial_a = \frac{\partial}{\partial x^a}$. There is a short exact sequence

$$0 \to TM \to T^{[2]}M \to \odot^2 TM \to 0 \tag{2.4.1}$$

with maps

$$\left(V^a\right) \mapsto \left(0, V^a\right) \qquad \text{and} \qquad \left(V^{ab}, V^a\right) \mapsto \left(V^{ab}\right) .$$

A torsion-free affine connection ∇ on TM is then equivalent to a (left) splitting of (2.4.1), i.e. a linear map

$$\gamma \, : \, T^{[2]}M \to TM \tag{2.4.2}$$

acting as

$$\gamma \, : \, \left(V^{ab}, V^a\right) \mapsto \left(V^a + \Gamma^a_{bc}V^{bc}\right)$$

for functions Γ^a_{bc} on M which we identify as Christoffel symbols. We'll now describe (following [56]) two ways of constructing such a map in twistor theory, as well as a generalisation by the present author (following [41]). The original treatment in [56] is considerably more sophisticated than that given here, as Merkulov describes the construction for a general Kodaira moduli space. In the case $Z \to \mathbb{P}^1$ the construction is much simpler.

2.4.1 The Ξ-connection

Cover $\mathbb{P}^1$ with two open sets U and $\hat{U}$ with respective inhomogeneous coordinate functions λ and $\hat{\lambda}$ subject to the transition function $\hat{\lambda} = \lambda^{-1}$ on $U \cap \hat{U}$. Again consider a general fibred twistor space $\varrho : Z \to \mathbb{P}^1$ characterised by the holomorphic patching

$$\hat{w}^\mu = \hat{w}^\mu \left(w^\nu, \lambda\right) \ , \tag{2.4.3}$$

where w^μ are the twistor coordinates on the fibres. Henceforth restrict to cases in which one has

$$\check{H}^1 \left(\mathbb{P}^1, N_x \otimes N_x^*\right) = 0;$$

the reason for this will soon become clear.

A section $V = \left(V^{ab}, V^a\right)$ of $T^{[2]}M$ gives rise to a differential operator $V^{ab}\partial_a\partial_b + V^a\partial_a$ which following [56] we'll now apply to (2.4.3).

$$V^{ab}\partial_a\partial_b\hat{w}^\mu + V^a\partial_a\hat{w}^\mu = V^{ab}\mathcal{F}^\mu_\nu\partial_a\partial_b w^\nu + V^a\mathcal{F}^\mu_\nu\partial_a w^\nu + V^{ab}\mathcal{F}^\mu_{\nu\rho}\partial_a w^\nu\partial_b w^\rho, \tag{2.4.4}$$

where again we have

$$\mathcal{F}^\mu_\nu = \frac{\partial\hat{w}^\mu}{\partial w^\nu}|$$

and

$$\mathcal{F}^\mu_{\nu\rho} = \frac{\partial^\mu\hat{w}}{\partial w^\nu\partial w^\rho}|.$$

(Recall that a vertical slash indicates that one restricts to $w^\nu = w^\nu| \left(x^a, \lambda\right)$.) If we can write (2.4.4) as a global section of N_x then we have (via the Kodaira isomorphism $T_xM = \check{H}^0 \left(\mathbb{P}^1, N_x\right)$) constructed a map γ. The only problematic term is the last one. The Ξ-connection is constructed by splitting

$$\mathcal{F}^\mu_{\nu\rho}\partial_a w^\nu = -\hat{\chi}^\mu_{\alpha\, a}\mathcal{F}^\alpha_\rho + \mathcal{F}^\mu_\alpha\chi^\alpha_{\rho\, a} \tag{2.4.5}$$

for a 0-cochain $\{\chi\}$ of $N_x \otimes N_x^* \otimes T_x^* M$ per point $x \in M$. The left-hand-side of (2.4.5) is always (in the $Z \to \mathbb{P}^1$ case) a 1-cocycle of $N_x \otimes N_x^* \otimes T_x^* M$, and since by assumption $\check{H}^1\left(\mathbb{P}^1, N_x \otimes N_x^*\right) = 0$ the splitting is always possible.

Equation (2.4.4) then becomes

$$V^{ab}\partial_a\partial_b\hat{w}^\mu + V^a\partial_a\hat{w}^\mu + V^{ab}\hat{\chi}^\mu_{\ \alpha\,(a}\partial_{b)}\hat{w}^\alpha = \mathcal{F}^\mu_\nu\left(V^{ab}\partial_a\partial_b w^\nu + V^a\partial_a w^\nu + V^{ab}\chi^\nu_{\ \rho\,(a}\partial_{b)}w^\rho\right)$$

and so we have constructed a global section of N_x per point x. The connection symbols for the map thus constructed can be read off as

$$\partial_a\partial_b w^\nu + \chi^\nu_{\ \rho\,(a}\partial_{b)}w^\rho = \Gamma^c_{\ ab}\partial_c w^\nu.$$

There is, though, a possible source of non-uniqueness. If $\check{H}^0\left(\mathbb{P}^1, N_x \otimes N_x^*\right) \neq 0$ then one is free to add any element of that group to both $\hat{\chi}^\mu_{\alpha\, a}$ and $\chi^\mu_{\alpha\, a}$. Therefore what one obtains is an equivalence class of connections (the Ξ-connection). As with the frame we will choose to describe the equivalence class by a single most-general representative containing arbitrary functions.

Example: $Z = \mathcal{O}(1) \oplus \mathcal{O}(1)$

The flat model in the nonlinear graviton construction was described in section 2.2; inspection of the patching (2.2.1) and twistor lines (2.2.2) reveals

$$\mathcal{F}^A_{BC} = 0 \qquad \mathcal{F}^A_B = \delta^A_B\lambda^{-1}$$

(where we put $\mu = A = 0, 1$ as is conventional for $N_x = \mathcal{O}(1) \oplus \mathcal{O}(1)$).

The Christoffel symbols for the Ξ-connection are then given by

$$\Gamma^{AA'}_{\quad\ CC'DD'} = \chi^A_{D\ CC'}\delta^{A'}_{D'} + \chi^A_{C\ DD'}\delta^{A'}_{C'}\ , \qquad (2.4.6)$$

where χ^A_B are four arbitrary one-forms on M constituting a global section of $N_x \otimes N_x^* \otimes T_x^* M$ per point $x \in M$.

2.4.2 The torsion Ξ-connection

Later in this thesis we will describe the twistor theory of three- and five-dimensional Newton-Cartan manifolds. For certain deformations (see sections 3.2.2 and 3.3.2) of the complex structure the Λ-connections (to be discussed in section 2.4.3) fail to exist and the Ξ-connections cannot be made compatible with the induced frame data. The frame section suggests that the reason for this is that the moduli space's connection possesses torsion. In this section we will generalise the Ξ-connection of [57, 56] to include torsion. This simple generalisation was first made by the author in [41].

Consider a general (possibly torsional) affine connection to be a map

$$\gamma : \Gamma(TM \times TM) \to \Gamma(TM).$$

As in the constructions of the previous two sections, we'll build such a map from the twistor data via the Kodaira isomorphism. Let $V = V^a \partial_a$ and $W = W^a \partial_a$ be vector fields on M. We want to use the complex structure of the twistor space Z to build a torsional connection by directly constructing $\nabla_W V = \gamma(V, W)$.

Apply V to the patching (2.4.3) to obtain

$$V^a \partial_a \hat{w}^\mu = \mathcal{F}^\mu_\nu V^a \partial_a w^\nu \tag{2.4.7}$$

as usual. Then apply W to (2.4.7) to obtain

$$\begin{aligned} &W^b \partial_b V^a \partial_a \hat{w}^\mu + W^b V^a \partial_b \partial_a \hat{w}^\mu \\ &\qquad = W^b \partial_b \mathcal{F}^\mu_\nu V^a \partial_a w^\nu + W^b \mathcal{F}^\mu_\nu \partial_b V^a \partial_a w^\nu + W^b \mathcal{F}^\mu_\nu V^a \partial_b \partial_a w^\nu. \end{aligned}$$

As in the torsion-free case the only obstruction to this line (evaluated at $x \in M$) constituting a global section of N_x is the first term on the right-hand-side, and one must decide what to do with it.

If we can write

$$\partial_b \mathcal{F}^\mu_\nu = -\hat{\rho}^\mu_{\ \alpha b} \mathcal{F}^\alpha_\nu + \mathcal{F}^\mu_\beta \rho^\beta_{\ \nu b} \tag{2.4.8}$$

for some 0-cochain $\{\rho\}$ of $N_x \otimes N_x^* \otimes \Lambda_x^1(M)$ for each $x \in M$ then

$$W^b \partial_b V^a \partial_a \hat{w}^\mu + W^b V^a \partial_b \partial_a \hat{w}^\mu + W^b \hat{\rho}^\mu_{\alpha\, b} V^a \partial_a \hat{w}^\alpha$$
$$= \mathcal{F}^\mu_\beta \left(W^b \rho^\beta_{\nu\, b} V^a \partial_a w^\nu + W^b \partial_b V^a \partial_a w^\beta + W^b V^a \partial_b \partial_a w^\beta \right)$$

constitutes a global section of N_x (for each $x \in M$), and hence a vector field on M via the Kodaira isomorphism. One can then extract the connection symbols from

$$\Gamma^c_{ab} \partial_c w^\mu | = \partial_a \partial_b w^\mu | + \rho^\mu_{\nu\, b} \partial_a w^\nu | \tag{2.4.9}$$

(or its counterpart over $\hat{U}$) just as in the torsion-free case.

The connection symbols arising from (2.4.9) generically possess torsion, and the torsion-free part of the connection agrees with the torsion-free Ξ-connection exhibited in section 2.4.1. We accordingly call the connection of this section the *torsion Ξ-connection.*

Just like the Ξ-connection the existence is determined by the non-vanishing of

$$\check{H}^1 (X_x, N_x \otimes N_x^*)$$

and the connection is defined up to an element of

$$\check{H}^0 (X_x, N_x \otimes N_x^*) \otimes \Lambda^1_x (M)$$

per $x \in M$, giving us a family of connections induced on M.

2.4.3 The Λ-connection

An alternative twistor construction of a class of maps (2.4.2) gives rise to the so-called Λ-connection, which we now briefly summarise. This class of connections often degenerates into a single affine connection, and in as much as it is necessary and useful to make such a distinction, it is the Λ-connection which should be considered the *physical* connection.

Consider again the patching for a general fibred twistor space $Z \to \mathbb{P}^1$:

$$\hat{w}^\mu = \hat{w}^\mu (w^\nu, \lambda) \ ,$$

where w^μ are the coordinates on the fibres, and again consider the equation (2.4.4) resulting from the action of the section V of $T^{[2]}M$. To construct the Λ-connection we do the splitting differently.

We instead choose to solve

$$\mathcal{F}^\mu_{\alpha\beta} = -\hat{\sigma}^\mu_{\nu\rho}\mathcal{F}^\nu_\alpha\mathcal{F}^\rho_\beta + \mathcal{F}^\mu_\eta\sigma^\eta_{\alpha\beta} \tag{2.4.10}$$

for a 0-cochain $\{\sigma\}$ of $N\otimes(N^*\odot N^*)\to\mathbb{P}^1$, and the Christoffel symbols for the resulting map γ can be read off from

$$\Gamma^a_{bc}\partial_a w^\mu| = \partial_b\partial_c w^\mu| + \sigma^\mu_{\nu\rho}\partial_b w^\nu|\partial_c w^\rho| \tag{2.4.11}$$

(or the equivalent expression over $\hat{U}$). In the $Z\to\mathbb{P}^1$ case the left hand side of (2.4.10) is always a 1-cocycle of $N\otimes(N^*\odot N^*)$.

The difficult part of this construction is the solution of the splitting problem (2.4.10), which in some cases is not possible, and is often not unique.

Uniqueness is determined by whether there are global sections of $N\otimes(N^*\odot N^*)$; if these exist then one is free to add one to $\{\sigma\}$ and so construct a different connection. In Penrose's case we have

$$\check{H}^0\left(\mathbb{P}^1, N\otimes(N^*\odot N^*)\right) = 0$$

and so the connection is always unique. This is unsurprising; we can always call upon the Levi-Civita connection.

There are Kodaira deformations (giving rise to $\mathcal{F}^\mu_{\alpha\beta}$) for which (2.4.10) cannot be solved iff

$$\check{H}^1\left(\mathbb{P}^1, N\otimes(N^*\odot N^*)\right) \neq 0\,,$$

and there are several reasons this may occur. One is that the spacetime suffers a jump in the normal bundle; another is when the torsion-free requirement essential to the construction is broken. In Penrose's case we can calculate that $\check{H}^1\left(\mathbb{P}^1, N\otimes(N^*\odot N^*)\right)$ vanishes, so all Kodaira deformations lead to torsion-free connections, in line with the non-linear graviton construction.

Example: the plane wave revisited

Consider again the plane wave example with patching (2.2.4).

Without reference to the metric (2.2.5), one can construct a unique affine connection on M by calculating

$$\mathcal{F}^{\mu}_{\nu} = \begin{pmatrix} \lambda^{-1} & 0 \\ 3\epsilon\left(u\lambda + z\right)^{2}\lambda^{-2} & \lambda^{-1} \end{pmatrix} \; ; \tag{2.4.12}$$

$$\mathcal{F}^{1}_{00} = 6\epsilon\left(u\lambda + z\right)\lambda^{-2} \; ; \; \text{and all other } \mathcal{F}^{\mu}_{\nu\rho} = 0. \tag{2.4.13}$$

The splitting problem (2.4.10) can be solved uniquely to give

$$\hat{\sigma}^{1}_{00} = -6\epsilon z \; ; \; \sigma^{1}_{00} = 6\epsilon u \; ; \; \text{and all other } \{\sigma^{\mu}_{\nu\rho}\} = 0, \tag{2.4.14}$$

leading to a connection whose only non-vanishing components are

$$\Gamma^{x}_{uz} = \Gamma^{x}_{zu} = \Gamma^{y}_{zz} = 6\epsilon u. \tag{2.4.15}$$

One can check that this agrees with the Levi-Civita connection one would calculate from the Ricci-flat metric (2.2.5).

2.5 Folded hyperKähler geometry

The gravitons constructed by Penrose are ASD and Ricci-flat. In four-dimensional Euclidean signature this means that they are *hyperKähler.*

Definition 2.5.1. *A hyperKähler structure on a smooth Riemannian four-manifold* (M, g) *consists of three complex structures* $J_i = (J_1, J_2, J_3)$ *satisfying*

$$J_1^2 = J_2^2 = J_3^2 = -1$$

and

$$J_i J_j = \epsilon_{ijk} J_k \qquad \textit{for } i \neq j$$

such that g *is hyperHermitian*[4] *with respect to* J_i *and such that the three two-forms* Ω_i *defined by*

$$\Omega_i\left(U, V\right) = g\left(U, J_i V\right) \qquad \left(\forall U, V \in \Gamma\left(TM\right)\right)$$

[4] A Riemannian metric g is Hermitian with respect to a complex structure J if $g(JU, JV) = g(U, V)$ for all vectors U, V, and hyperHermitian with respect to J_i if it is Hermitian with respect to all three.

are symplectic, meaning that they are closed and non-degenerate:

$$d\Omega_i = 0 \qquad \Omega_i \wedge \Omega_i \neq 0 \qquad \textit{(no summation).}$$

For a proof that an ASD and Ricci-flat metric is hyperKähler (and vice-versa) the reader is referred to [21].

In section 4.1 we will be concerned with a generalisation of the above definition, called the *folded* case. Here we allow the Kähler forms to degenerate:

$$\Omega_i \wedge \Omega_i = 0 \qquad \text{(no summation)}$$

on a three-dimensional submanifold called the *fold*, $\mathcal{X}$. On $M\backslash\mathcal{X}$ one obtains hyperKähler metrics either side of the fold, respectively positive-definite and negative-definite. The notion of such folded hyperKähler metrics is due to Hitchin [43]; here we'll follow the definition of Biquard in [12].

Definition 2.5.2. *A folded hyperKähler manifold is a quadruple* $(M, \mathcal{X}, \Omega_i, \iota)$ *consisting of*

- ◇ *a smooth four-dimensional manifold* M*;*
- ◇ *a three-dimensional embedded submanifold* $\mathcal{X}$ *called the fold which divides* M *into two disjoint connected components;*
- ◇ *a triplet of two-forms* Ω_i *which define a hyperKähler structure either side of the fold with metrics* $g_\pm$ *and are such that*

  $$\Omega_1|_{\mathcal{X}} \neq 0 \qquad \Omega_2|_{\mathcal{X}} \neq 0 \qquad \Omega_3|_{\mathcal{X}} = 0$$

  with $\ker \Omega_1|_{\mathcal{X}} \oplus \ker \Omega_2|_{\mathcal{X}}$ *a contact distribution;*
- ◇ *and an involution* $\iota : M \to M$ *which fixes* $\mathcal{X}$*, exchanges the two sides of* $M\backslash\mathcal{X}$ *, and is such that*

  $$\iota g_\pm = -g_\mp \qquad \iota\Omega_3 = -\Omega_3 \qquad \iota\Omega_1 = \Omega_1 \qquad \iota\Omega_2 = \Omega_2.$$

We will call the metrics $g_\pm$ *folded hyperKähler* metrics. The global construction of folded hyperKähler manifolds due to Hitchin [43] suggests that metrics of this type should come in infinite-dimensional families, and Biquard proves in [12] that all infinitesimal deformations of folded hyperKähler metrics can be integrated to give folded hyperKähler metrics, confirming this intuition.

The definition above is that of an *alpha*-folded hyperKähler manifold, where, following Hitchin [43], we must have locally

$$\Omega_3 = dz \wedge \alpha_3 + z\beta_3 \wedge \gamma_3$$

$$\text{and} \quad \Omega_1 = zdz \wedge \alpha_1 + \beta_1 \wedge \gamma_1 \quad \omega_2 = zdz \wedge \alpha_2 + \beta_2 \wedge \gamma_2$$

for a fold $z = 0$ and where $(\alpha_i, \beta_i, \gamma_i)$ possess no dz components.

Alpha-folded hyperKähler manifolds have appeared in several places in theoretical physics: in particular the Gibbons-Hawking [35] metrics

$$g = \frac{1}{V}(d\psi + A) + V\delta_{ij}dx^i dx^j \qquad \text{for} \qquad dV = \star^3 dA \tag{2.5.1}$$

are often examples of folded hyperKähler metrics. These metrics are significant in supergravity [60].

The classic example of a folded hyperKähler metric is a Gibbons-Hawking spacetime with $V = z$,

$$g = \frac{1}{z}(d\psi + \frac{1}{2}xdy - \frac{1}{2}ydx) + z\delta_{ij}dx^i dx^j.$$

Not all metrics (2.5.1) are, strictly speaking, examples of folded hyperKähler metrics. If V has vanishing points then one or more of the Kähler forms may vanish, but key to the notion of a folded metric is that it must be the same Kähler form which vanishes all over the fold. When the form Ω_i which vanishes changes at different points on the fold we have an example of an *ambipolar* metric, a generalisation defined in [60].

Definition 2.5.3. *An ambipolar hyperKähler manifold is a triple $(M, \mathcal{X}, \Omega_i)$ where*

- ◇ *M is a smooth four-dimensional manifold;*
- ◇ *$\mathcal{X}$ is a three-dimensional embedded submanifold called the fold which divides M into two disjoint connected components;*
- ◇ *and Ω_i is a triple of two-forms with the property that span $\{\Omega_i|_{\mathcal{X}}\}$ is two-dimensional, with the corresponding union of kernels being a contact distribution.*

A relevant example of an ambipolar hyperKähler metric which is *not* a folded hyperKähler metric is the Taub-NUT spacetime with negative mass, which will be discussed in section 4.3.

Ambipolar metrics play a role in the *microstate geometries* program [37], where in five-dimensional supergravity they are the four-dimensional base space from which a smooth five-dimensional solution is constructed [60].

The alpha-folds above are to be contrasted, in passing, with a slightly different kind of fold also discussed by Hitchin in [43] called a *beta*-fold. In a beta-fold all three Kähler forms are non-vanishing when restricted to the fold; locally we must have

$$\Omega_i = zdz \wedge \alpha_i + \beta_i \wedge \gamma_i$$

where $z = 0$ is the fold, and where $\beta_i \wedge \gamma_i$ is non-vanishing when restricted to $z = 0$.

2.6 Newton-Cartan geometry

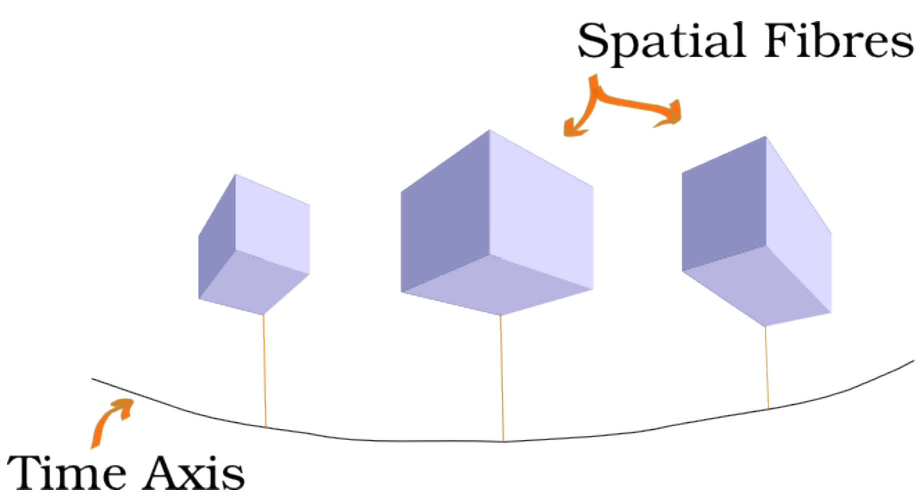
Newton-Cartan spacetimes are the non-relativistic analogues of Lorentzian manifolds in general relativity: they are the geometrical setting for non-relativistic physics [17]. Just like in general relativity we have a four-dimensional manifold playing the role of the spacetime, and particles travel on geodesics of a torsion-free connection. There's a metric too, though unlike in general relativity the connection and the metric are independent quantities. In this section we will describe Newton-Cartan spacetimes in some detail, taking [28] as a reference.

Before introducing the full Newton-Cartan geometry we'll begin with a subordinate definition.

Definition 2.6.1. *A $(d+1)$-dimensional Galilean manifold is a triple $(M,\ h,\ \theta)$ where*

- ◇ *M is a $(d+1)$-dimensional manifold;*
- ◇ *h is a symmetric tensor field of valence $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ with signature $(0++...+)$ (and so has rank d) called the metric;*
- ◇ *and θ is a closed one-form spanning the kernel of h called the clock.*

The pair (h, θ) is called a *Galilean structure*, and the number of spatial dimensions is d. Since θ is closed we can always locally write $\theta = dt$ for some function $t : M \to \mathbb{R}$. This function is then taken as a coordinate on the time axis, a one-dimensional submanifold over which the spacetime M is fibred. We call the fibres *spatial slices* and when restricted to such a slice the metric h is a more familiar signature $(+...++)$ d-metric. (Recall that throughout this thesis the indices a, b, c will run from 0 to d and the spatial indices i, j, k will run from 1 to d.)



Definition 2.6.2. *A $(d+1)$-dimensional Newton-Cartan manifold is a quadruple $(M,\ h,\ \theta,\ \nabla)$ where*

- *M is a $(d+1)$-dimensional manifold;*
- *(h, θ) is a Galilean structure;*
- *and ∇ is a torsion-free connection compatible with the Galilean structure in the sense that $\nabla h = 0$ and $\nabla \theta = 0$.*

We emphasise that ∇ must be specified independently of the metric and clock, and we recall the syntax that a Newton-Cartan manifold $(M,\, h,\, \theta,\, \nabla)$ is a smooth manifold M equipped with a Newton-Cartan structure $(\, h,\, \theta,\, \nabla)$.

Newton-Cartan geometry with some appropriate field equations models Newton-Cartan gravity. The appropriate field equations arise as the Newtonian limit of the Einstein equations [52]. They are

$$R_{ab} = 4\pi G \rho \theta_a \theta_b \tag{2.6.1}$$

where R_{ab} is the Ricci tensor associated to ∇; G is Newton's constant; and $\rho : M \to \mathbb{R}$ is the mass density. Alongside the field equations we have the Trautman condition [28]

$$h^{a[b} R^{c]}_{\ \ (de)a} = 0 \ , \tag{2.6.2}$$

where $R^a_{\ bcd}$ is the Riemann tensor of ∇. This ensures that there always exist potentials for the connection components, which is needed if we are to make contact with Newtonian physics; accordingly connections which satisfy (2.6.2) are referred to as *Newtonian* connections.

The field equations imply that h is flat on spatial slices, and we can always introduce *Galilean* coordinates (t, x^i) such that

$$h = \delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \qquad \text{and} \qquad \theta = dt \tag{2.6.3}$$

for $i = 1, 2, ..., d$. For notational convenience we can then raise and lower purely spatial indices with δ^{ij} and δ_{ij}. We'll refer to (2.6.3) as the *standard* Galilean structure.

Only connections compatible with θ and h are allowed by definition; one can show [19] that the most general such connection has components

$$\Gamma^a_{\ bc} = \frac{1}{2} h^{ad} \left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc} \right) + \partial_{(b} \theta_{c)} U^a + \theta_{(b} F_{c)d} h^{ad} \tag{2.6.4}$$

where

- ◇ U^a is any vector field satisfying $\theta(U) = 1$;
- ◇ F_{ab} is any two-form;
- ◇ and h_{ab} is uniquely determined by $h^{ab}h_{bc} = \delta^a_c - \theta_c U^a$ and $h_{ab}U^b = 0$.

The possible connections are then parametrised by a choice of $(U,\, F)$. The Trautman condition (2.6.2) is equivalent to the statement that F is closed, and hence for a Newtonian connection we can locally write $F = dA$. Thus we could refer to a Newton-Cartan spacetime as a quintuple (M, h, θ, U, A), implicitly considering a Newtonian connection. Clearly there is a gauge symmetry in A, as we can always shift

$$A \longmapsto A + d\chi$$

for any function χ on M.

There is a further redundancy in this description; there exist *Milne boosts* which can be thought of as gauge transformations of $(U,\, F)$ which leave $\Gamma^a{}_{bc}$ unchanged [28, 48]. One can choose to work with a non-trivial vector field U, which is taken up in [73]. Usually we will gauge-fix to $U = \partial_t$, which can be implemented by a Milne boost for any initial choice of (U, F).

With $d = 3$ the most general vacuum Newton-Cartan spacetime satisfying (2.6.1) and (2.6.2) then has

$$\Gamma^i{}_{tt} = \delta^{ij}\partial_j V \qquad \text{and} \qquad \Gamma^i{}_{jt} = \Gamma^i{}_{tj} = \delta_{jl}\epsilon^{ilk}\partial_k\Omega$$

$$\text{where} \qquad \delta^{ij}\partial_i\partial_j V + 2\delta^{ij}\partial_i\Omega\partial_j\Omega = 0 \qquad \text{and} \qquad \delta^{ij}\partial_i\partial_j\Omega = 0, \qquad (2.6.5)$$

with all other connection components vanishing. The corresponding two-form F is given by

$$F = -dV \wedge dt \,+\, \epsilon_{ijk}\delta^{kl}\partial_l\Omega\, dx^i \wedge dx^j.$$

The geodesic equations suggest that we should interpret the function V as the Newtonian (gravitational) potential and the function Ω as a potential for generalised (spatially-varying) Coriolis forces. Note that although the degrees of freedom in a Newton-Cartan connection appear similar to that of an electromagnetic field the equations (2.6.5) governing them are more complicated. This is somewhat analogous to the way in which the Einstein equations in Ashtekar variables reduce to a problem in which Yang-Mills degrees of freedom obey equations more complicated than the Yang-Mills equations [3].

Two harmonic functions are sufficient to solve the vacuum Newton-Cartan fields equations, because (with $\nabla^2 = \delta^{ij}\partial_i\partial_j$) we can rewrite (2.6.5) as

$$\nabla^2\left(V + \Omega^2\right) = 0 \qquad \text{and} \qquad \nabla^2\Omega = 0.$$

Of recent interest has been *torsional* Newton-Cartan geometry [34, 9, 7, 18, 41], in which the connection is allowed to have some torsion. This is manifest in the clock failing to be closed because $d\theta \neq 0$ is incompatible with $\nabla\theta = 0$ for a torsion-free connection. Equation (2.6.4) is modified to include the skew part of $\partial_a\theta_b$ too, giving rise to the torsion. Later in this thesis we will construct Kodaira families with induced torsional Newton-Cartan structures, featuring clocks which aren't closed.

2.7 Non-relativistic symmetries

A menagerie of symmetry algebras relevant in Newton-Cartan geometry is discussed in [28]; we will here provide a brief review of those relevant to this thesis, following that paper. The reason that there are many algebras to describe compared to the relativistic case is that a Newton-Cartan structure (h, θ, ∇) is more intricate than a Riemannian metric g, and so there are a lot of ways of interpreting what the non-relativistic analogue of

$$\mathcal{L}_X g \propto g$$

ought to be.

The first definition is a fairly straightforward one to make; we simply preserve the Galilean structure up to changes of scale.

Definition 2.7.1. *The conformal Galilean algebra* $\mathfrak{cgal}(d)$ *of a* $(d+1)$*-dimensional Galilean structure* (M, h, θ) *is the Lie algebra of vector fields* X *satisfying*

$$\mathcal{L}_X h = fh \tag{2.7.1}$$

and

$$\mathcal{L}_X \theta = g\theta \tag{2.7.2}$$

for some smooth functions f *and* g *on* M*.*

On the standard Galilean structure (2.6.3) the general element $X \in \mathfrak{cgal}(d)$ is

$$X = \alpha(t)\,\partial_t + \left(\omega^i_{\ j}(t)\,x^j + \phi^i(t) + \psi^i(t)\,x_j x^j - 2x^i\psi^j(t)\,x_j + \mu(t)\,x^i\right)\partial_i$$

for arbitrary functions of time $(\omega_{ij}(t) \in \mathfrak{so}(d), \phi^i(t), \psi^i(t), \mu(t), \alpha(t))$. The Lie algebra is thus infinite-dimensional.

Matters become more complicated when we have to decide how X is to preserve ∇, and we'll define several options.

Definition 2.7.2. *The conformal Newton-Cartan algebra* $\mathfrak{cnc}(d)$ *of a Newton-Cartan spacetime* (M, h, θ, ∇) *is the Lie algebra of vector fields* X *which are elements of its conformal Galilean algebra* $\mathfrak{cgal}(d)$ *and permute the null geodesics of* ∇*, satisfying (2.7.1-2.7.2) and*

$$\mathcal{L}_X \Gamma^a_{bc} = -\partial_t f \delta^a_{(b}\theta_{c)} + (\partial_t f + \partial_t g)\, v^a \theta_b \theta_c + (f+g)\, h^{ad}\theta_{(b}F_{c)d}. \tag{2.7.3}$$

The reader is referred to [28] for a motivational derivation of (2.7.3). On (2.6.3) and with $\Gamma^a_{bc} = 0$ a general element of $\mathfrak{cnc}(d)$ is

$$X = \alpha(t)\,\partial_t + \left(\omega^i_{\ j}(t)\,x^j + \phi^i(t) + \mu(t)\,x^i\right)\partial_i$$

for arbitrary functions of time $(\omega_{ij}(t) \in \mathfrak{so}(d), \phi^i(t), \mu(t), \alpha(t))$. We note that

$$\mathfrak{cnc}(d) \subset \mathfrak{cgal}(d),$$

and the algebra is again infinite-dimensional.

Definition 2.7.3. *The expanded Schrödinger algebra* $\widetilde{\mathfrak{sch}}(d)$ *of a Newton-Cartan spacetime* (M, h, θ, ∇) *is the Lie algebra of vector fields* X *which are elements of its conformal Galilean algebra* $\mathfrak{cgal}(d)$ *and effect projective transformations of* ∇*, satisfying (2.7.1-2.7.2) and*

$$\mathcal{L}_X \Gamma^a_{bc} = \delta^a_{(b}\phi_{c)} \tag{2.7.4}$$

with functions (f, g) *and a one-form* ϕ_a *constrained by*

$$\mathcal{L}_X \nabla h = 0 \quad \textit{and} \quad \mathcal{L}_X \nabla \theta = 0. \tag{2.7.5}$$

Condition (2.7.4) ensures that the unparametrised geodesics are unaltered by the transformation.

On (2.6.3) and with $\Gamma^a_{bc} = 0$ we have that $X \in \widetilde{\mathfrak{sch}}(d)$ iff

$$X = \left(\alpha t^2 + \beta t + \gamma\right)\partial_t + \left(\omega^i_{\ j} x^j + \alpha t x^i + \mu x^i + \nu^i t + \rho^i\right)\partial_i \tag{2.7.6}$$

for $(\alpha, \beta, \gamma, \mu, \nu^i, \rho^i) \in \mathbb{R}^{4+2d}$ and $\omega_{kj} \in \mathfrak{so}(d)$. The dimension of $\widetilde{\mathfrak{sch}}(d)$ is therefore finite and given by $\frac{1}{2}\left(d^2 + 3d + 8\right)$.

Definition 2.7.4. *The Schrödinger algebra* $\mathfrak{sch}(d)$ *is the Lie subalgebra of* $\widetilde{\mathfrak{sch}}(d)$ *defined by the additional condition*

$$f + g = 0.$$

This amounts to setting $\beta = 2\mu$ in (2.7.6); in the case (2.6.3) and with $\Gamma^a_{bc} = 0$ we thus have that $X \in \mathfrak{sch}(d)$ iff

$$X = \left(\alpha t^2 + 2\mu t + \gamma\right)\partial_t + \left(\omega^i_{\ j} x^j + \alpha t x^i + \mu x^i + \nu^i t + \rho^i\right)\partial_i. \tag{2.7.7}$$

Physically, this algebra contains translations (γ, ρ^i), spatial rotations $(\omega^i_{\ j})$, boosts (ν^i), a special-conformal transformation (α), and a dilation (μ). The dimension is now $\frac{1}{2}\left(d^2 + 3d + 6\right)$. We also note that

$$\mathfrak{sch}(d) \subset \mathfrak{cnc}(d).$$

This algebra is named *Schrödinger* because of its well-known link (see e.g. [73, 28, 40, 30, 72]) to the free-particle Schrödinger equation: a first-order linear differential operator

$\mathcal{D} = S^a(x)\partial_a + s(x)$ commutes with $\Delta = i\partial_t + \frac{1}{2m}\delta^{ij}\partial_i\partial_j$ in the sense that

$$\Delta\mathcal{D} = \delta\Delta$$

for some linear differential operator δ iff $S^a\partial_a \in \mathfrak{sch}(d)$.

In [33] the authors take a $c^{-1} \to 0$ limit of the conformal algebra, constructing a non-relativistic conformal algebra which they also call the "conformal Galilean algebra" but which is distinct from $\mathfrak{cgal}(d)$. We will refer to their algebra as the CGA and consider it only in $d = 3$.

Definition 2.7.5. *The CGA in* $(3 + 1)$ *dimensions is the fifteen-dimensional Lie algebra obtained by a Wigner contraction of the conformal algebra in four dimensions as the speed of light is taken to infinity.*

The limit is taken by scaling each generator of the conformal algebra by an appropriate factor of c such that the leading order term survives the limit (and is finite); this means that the dimension of the algebra is unchanged by the procedure.

Chapter 3

Newton-Cartan Kodaira families

Penrose's nonlinear graviton construction constructs ASD Ricci-flat manifolds as Kodaira families of rational curves; in this chapter we will describe an analogous construction of Newton-Cartan manifolds, constituting a Newtonian twistor theory.

The strategy is to begin with ordinary four-dimensional twistor theory, take its non-relativistic limit, and explore resulting the complex geometry. Having come to terms with the four-dimensional setting we will then be in a position to extend Newtonian twistor theory to three and five dimensions, finding novel features.

3.1 Newtonian twistor theory in four dimensions

3.1.1 Galilean structures

Recall from definition 2.6.2 that a Newton-Cartan structure consists of a Galilean structure and a connection. It will turn out that the twistor description of the connection is more complicated than that of the Galilean structure, and so this section will be concerned purely with Galilean structures; in section 3.1.2 we will introduce the connection and promote the twistor theory to that of a full Newton-Cartan manifold.

3.1.1.1 The Newtonian limit of twistor space

The following theorem constitutes the construction of a four-complex-dimensional manifold equipped with a Galilean structure arising as a limit of Penrose's twistor theory.

Theorem 3.1.1. [22]

Let $Z_c \to \mathbb{P}^1$ be a one-parameter family of rank-two vector bundles on the Riemann sphere with patching

$$\begin{pmatrix} \hat{T} \\ \hat{Q} \end{pmatrix} = \mathcal{F}_c \begin{pmatrix} T \\ Q \end{pmatrix} \tag{3.1.1}$$

over the intersection, where

$$\mathcal{F}_c = \begin{pmatrix} 1 & -(c\lambda)^{-1} \\ 0 & \lambda^{-2} \end{pmatrix}.$$

The four-complex-dimensional moduli space M of global sections $Z_c \to \mathbb{P}^1$ is always a Kodaira moduli space. The nature of its induced geometry depends on c.

- *When $c > 0$ is finite $Z_c = \mathcal{O}(1) \oplus \mathcal{O}(1)$ and M is equipped is a complexified Riemannian metric g in accordance with theorem* 2.2.3.
- *When c is infinite $Z_\infty = \mathcal{O} \oplus \mathcal{O}(2)$ and M is equipped with a conformal metric $[h]$ and clock θ constituting an equivalence class of Galilean structures.*

We note that the moduli spaces constructed here are complex; real slices will be discussed in section 3.1.1.2. The Galilean structures comprising the equivalence class differ only by a change of conformal factor on the metric h, and the clock is unique (up to usual diffeomorphisms).

Proof

In the case of finite c we can split the patching matrix and write

$$\mathcal{F}_c = \hat{H} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} H^{-1}$$

where

$$H : U \to \mathrm{GL}(2, \mathbb{C}) \qquad \text{and} \qquad \hat{H} : \hat{U} \to \mathrm{GL}(2, \mathbb{C})$$

are holomorphic. Explicitly we have, for instance,

$$H = \begin{pmatrix} 1 & 0 \\ c\lambda & 1 \end{pmatrix} \qquad \text{and} \qquad \hat{H} = \begin{pmatrix} 0 & -c^{-1} \\ c & \lambda^{-1} \end{pmatrix},$$

exhibiting the isomorphism class of Z_c as $\mathcal{O}(1) \oplus \mathcal{O}(1)$. We are thus in the realm of the nonlinear graviton construction [66] and so expect a non-degenerate conformal structure to be induced on the moduli space. To find the four-parameter family of global sections of (3.1.1) we first consider the lower row

$$\hat{Q} = \lambda^{-2} Q \,,$$

which is the patching for $\mathcal{O}(2)$ and thus has a three-parameter family of global sections

$$Q| = \xi\lambda^2 - 2z\lambda - \tilde{\xi} \qquad \hat{Q}| = \xi - 2z\hat{\lambda} - \tilde{\xi}\hat{\lambda}^2 \tag{3.1.2}$$

for $\left(\xi, \tilde{\xi}, z\right) \in \mathbb{C}^3$. Recall that we denote with a vertical slash the restriction of a twistor space quantity to a global section, giving us *twistor functions* on the projective spin bundle $P\mathbb{S}' \to M$. (Recall that the primed spin bundle $P\mathbb{S}'$ is a trivial $\mathbb{P}^1$ bundle on M, to be identified with the correspondence space F in the double fibration picture (2.1.1).) The upper row is then

$$\hat{T} = T - c^{-1}\xi\lambda + 2c^{-1}z + c^{-1}\tilde{\xi}\lambda^{-1}$$

and so we can take the global sections to be

$$T| = t - c^{-1}z + c^{-1}\xi\lambda \qquad \hat{T}| = t + c^{-1}z + c^{-1}\tilde{\xi}\hat{\lambda} \tag{3.1.3}$$

where $t \in \mathbb{C}$ is the fourth parameter enumerating the global sections. We write $x^a = \left(t, \xi, \tilde{\xi}, z\right)$ for the moduli space coordinates. A global section specified by picking x^a is a $\mathbb{P}^1$ submanifold in Z_c, that is to say, a rational curve in Z_c. We refer to the rational curve associated to x^a as X_x in accordance with the machinery introduced in section 2.1. The normal bundles to these rational curves are[1]

$$N_x = {}^{(TZ_c)|_{X_x}}\!/_{TX_x} = \mathcal{O}(1) \oplus \mathcal{O}(1) \ ,$$

which characterises the twistor theory of relativistic spacetimes [66].

We have

$$\check{H}^1 \left(X_x, N_x \right) = 0$$

so the global sections constitute a complete holomorphic family of submanifolds and the moduli space is a Kodaira moduli space, making it a complex manifold.

The twistor correspondence is now that of Penrose.

- ⋄ A point x^a in M corresponds to a rational curve $(T|(\lambda), Q|(\lambda))$ in Z_c and
- ⋄ a point (T, Q, λ) in Z_c corresponds to a two-complex-dimensional plane in M via (3.1.2) and (3.1.3) called an *alpha surface*.

The induced geometry on M arises via the twistor principle: *a vector δx^a is null iff it is tangent to an alpha surface*.

In practice this means that we find the condition(s) on δx^a such that

$$\frac{\partial T|}{\partial x^a}\delta x^a = 0 \qquad \text{and} \qquad \frac{\partial Q|}{\partial x^a}\delta x^a = 0 \tag{3.1.4}$$

have a unique simultaneous solution in λ. (We can equivalently use the sections over the patch $\hat{U}$ instead if we wish.) From (3.1.2) and (3.1.3) we thus calculate

$$\delta t - c^{-1}\delta z + c^{-1}\lambda\delta\xi = 0 \qquad \text{and} \qquad \lambda^2\delta\xi - 2\lambda\delta z - \delta\tilde{\xi} = 0 \ ,$$

which have a unique simultaneous solution iff

$$c^2\delta t^2 - \delta z^2 - \delta\xi\delta\tilde{\xi} = 0.$$

Therefore we conclude that for a vector to be null it must lie in the kernel of a conformal structure

$$[g] = c^2 dt^2 - dz^2 - d\xi d\tilde{\xi}. \tag{3.1.5}$$

[1] One way to see this is to consider the patching of a holomorphic vector field with components only in the Q and T directions; this will again be $\mathcal{F}_c$.

(One could alternatively construct a frame of one-forms via theorem 2.3.1, which reproduces (3.1.5).)

The conformal factor can be fixed using two-form Σ described in theorem 2.2.3:

$$d\hat{T} \wedge d\hat{Q} = \lambda^{-2} dT \wedge dQ \tag{3.1.6}$$

on the fibres of $Z_c \to \mathbb{P}^1$. When restricting $\Sigma := dT \wedge dQ$ to twistor lines one obtains three self-dual two-forms $\Sigma^{A'B'}$ on M arising as

$$\Sigma| = \Sigma^{0'0'}\lambda^2 + 2\Sigma^{0'1'}\lambda + \Sigma^{1'1'}$$

which determine the full conformal factor by ensuring that the metric volume-form is equal to

$$\nu = \Sigma^{0'0'} \wedge \Sigma^{1'1'}.$$

This confirms that one should single out the representative exhibited in (3.1.5) as the Ricci-flat one, though of course this was obvious in the flat case.

When $c = \infty$ we have

$$\mathcal{F}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix},$$

which is simply the patching for $\mathcal{O} \oplus \mathcal{O}(2)$. We henceforth refer to

$$Z_\infty = \mathcal{O} \oplus \mathcal{O}(2)$$

as a *Newtonian twistor space* in the four-dimensional setting. The global sections are

$$T| = t \qquad \hat{T}| = t \tag{3.1.7}$$

and

$$Q| = \xi\lambda^2 - 2z\lambda - \tilde{\xi} \qquad \hat{Q}| = \xi - 2z\hat{\lambda} - \tilde{\xi}\hat{\lambda}^2\,, \tag{3.1.8}$$

which coincide with the $c \to \infty$ limit of the relativistic global sections (3.1.2) and (3.1.3). We continue to refer to the spacetime coordinates as $x^a = \left(t, \xi, \tilde{\xi}, z\right)$. The normal bundle is now

$$N_x = \mathcal{O} \oplus \mathcal{O}(2)$$

for all rational curves, and so we again have

$$\check{H}^1\left(X_x, N_x\right) = 0$$

and thus a Kodaira moduli space. The isomorphism class of the normal bundle can be seen directly from the patching.

The twistor functions (3.1.7) and (3.1.8) again provide a twistor correspondence.

- ◇ A point x^a in M still corresponds to a rational curve $X_x = \mathbb{P}^1$ described by $T|(\lambda)$ and $Q|(\lambda)$, but
- ◇ a point (T, Q, λ) on Z_∞ corresponds to a plane with $t =$ constant, soon to gain an interpretation as a *spatial* plane.

The induced geometry can be constructed in a variety of ways. Using theorem 2.3.1 we can build a frame of one-forms by splitting the patching for the normal bundle in the most general way possible. The splitting problem is

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} H = \hat{H} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} ,$$

which has the general solution

$$H = \begin{pmatrix} m & 0 \\ -a_{\tilde{\xi}} - 2a_z\lambda + a_\xi\lambda^2 & b \end{pmatrix}$$

where $\left(a_{\tilde{\xi}}, a_z, a_\xi, m, b\right)$ are five arbitrary functions on M subject to $m \neq 0$ and $b \neq 0$. The frame section is then

$$v = \frac{1}{mb} \begin{pmatrix} b & 0 \\ a_{\tilde{\xi}} + 2a_z\lambda - a_\xi\lambda^2 & m \end{pmatrix} \begin{pmatrix} dt \\ \lambda^2 d\xi - 2\lambda dz - d\tilde{\xi} \end{pmatrix}$$

$$\Rightarrow \qquad v = \begin{pmatrix} m^{-1}dt \\ b^{-1}\left(\lambda^2\left(d\xi - m^{-1}a_\xi dt\right) - 2\lambda\left(dz - m^{-1}a_z dt\right) - \left(d\tilde{\xi} - m^{-1}a_{\tilde{\xi}}dt\right)\right) \end{pmatrix} ,$$

and the frame one-forms can be read off:

$$\theta = m^{-1}dt \tag{3.1.9}$$

$$e^{00'} = b^{-1}\left(d\xi - m^{-1}a_\xi dt\right) \qquad e^{1'1'} = -b^{-1}\left(d\tilde{\xi} - m^{-1}a_{\tilde{\xi}}dt\right)$$
$$e^{0'1'} = e^{1'0'} = -b^{-1}\left(dz - m^{-1}a_z dt\right).$$

The span of these (see definition 2.3.2) then comprises family of Newton-Cartan clock (3.1.9) parametrised by a choice of non-vanishing function, and a family of degenerate covariant metrics

$$h^{-1} = \frac{1}{2}\epsilon_{A'B'}\epsilon_{C'D'}e^{A'C'} \otimes e^{B'D'}$$
$$\Rightarrow \qquad h^{-1} = e^{0'0'} \odot e^{11'} - e^{0'1'} \odot e^{0'1'}$$
$$\Rightarrow h^{-1} = -b^{-2}\left[\left(d\xi - m^{-1}a_\xi dt\right) \odot \left(d\tilde{\xi} - m^{-1}a_{\tilde{\xi}}dt\right) + \left(dz - m^{-1}a_z dt\right) \odot \left(dz - m^{-1}a_z dt\right)\right] \tag{3.1.10}$$

parametrised by a choice of five functions, two of them non-vanishing. (The significance of $b \neq 0$ and $m \neq 0$ is now revealed by the factors of m^{-1} and b^{-1} in (3.1.9) and(3.1.10).) At this point it's pleasing to reflect that the structures induced by the twistor principle are in agreement with those obtained from theorem 2.3.1: the conditions on δx^a for (3.1.4) to have a unique solution in λ are

$$\delta t = 0 \qquad \text{and} \qquad \delta z^2 + \delta\xi\delta\tilde{\xi} = 0\,.$$

However, the former allows us to modify the latter by adding to $\delta x^i = \left(\delta\xi, \delta z, \delta\tilde{\xi}\right)$ any amounts of δt, giving rise to arbitrary functions a_i. For a vector to be called null it must therefore lie in the kernel of the conformal tensors (3.1.9) and (3.1.10).

One can calculate the *projective inverse* of h^{-1}, despite its degeneracy. This is done by finding the unique vector field U such that

$$\theta\left(U\right) = 1 \qquad \text{and} \qquad h^{-1}\left(\ ,U\right) = 0\,,$$

which here is

$$U = m\frac{\partial}{\partial t} + a_\xi\frac{\partial}{\partial \xi} + a_z\frac{\partial}{\partial z} + a_{\tilde{\xi}}\frac{\partial}{\partial \tilde{\xi}}\ .$$

The projective inverse is then the unique $h \in \Gamma\left(TM \odot TM\right)$ such that

$$h\left(\ ,\theta\right) = 0 \qquad \text{and} \qquad h^{ab}\left(h^{-1}\right)_{bc} = \delta^a_c - U^a\theta_c.$$

We find

$$h = -b^2 \left[\frac{\partial}{\partial z} \odot \frac{\partial}{\partial z} + 4 \frac{\partial}{\partial \xi} \odot \frac{\partial}{\partial \tilde{\xi}} \right] ,$$

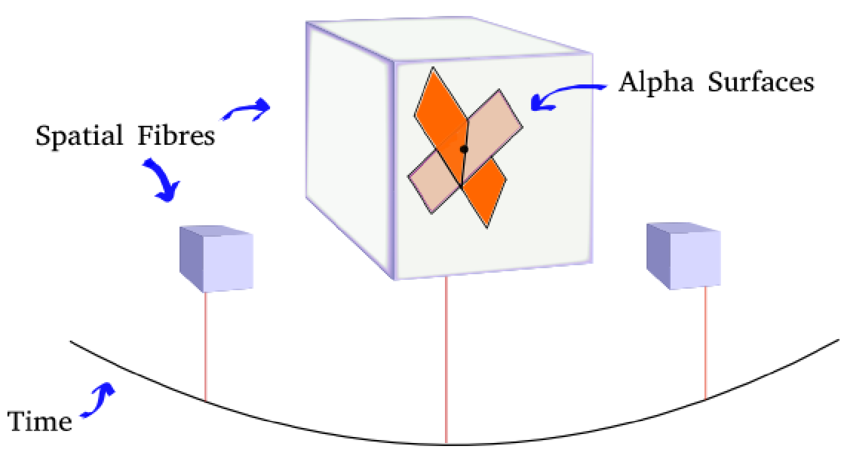
and (h, θ) then constitute a family of Galilean structures on M parametrised by two arbitrary non-vanishing functions m and b. The closure of the clock requires that $m = m(t)$ only.

Note that the significance of m is tied up with diffeomorphisms of the time axis: a diffeomorphic change of time coordinate $t = t(t')$ results in nothing more than a change in m. Thus m isn't a true degree of freedom parametrising the family of Galilean structures; it merely allows for coordinate invariance.

Hence a family of Galilean structures $([h], \theta)$ parametrised by conformal factors $b \neq 0$ is induced on M.

□

In the Newtonian context we see that the concept of a *null* direction has become that of a *spatial* direction. Alpha-surfaces, which in the relativistic case are totally null, are here totally spatial. That is to say, they lie on $t =$ constant fibres of M.



This reinterpretation of nullness is physically intuitive: as $c \to \infty$ the gradients of the light cones decrease until at $c = \infty$ they have entirely flattened out. The speed of propagation of information in non-relativistic physics is infinite; Newtonian interactions are instantaneous.

3.1.1.2 Newtonian spinors, Lax pairs, and real slices

Here we will develop a spinorial version of the twistor theory thus far introduced, finding that the use of a Killing vector naturally allows us to decompose the spinors in a Newtonian way. The Lax pair formulation to be discussed below will give us an alternative way of considering the Newtonian limit which will be of use later in this work.

Spinors

An alternative to the inhomogeneous coordinates $(\lambda, \hat{\lambda})$ on $\mathbb{P}^1$ are the *homogeneous* coordinates $[\pi_{A'}]$ for $A' = 0', 1'$. In this context we have

$$\mathbb{P}^1 = {}^{\left(\mathbb{C}^2 - \{0\}\right)}/_{\Upsilon}$$

for $\pi_{A'} \in \mathbb{C}^2$ and

$$\Upsilon = \pi_{A'} \frac{\partial}{\partial \pi_{A'}}. \tag{3.1.11}$$

The quotient is along the equivalence relation

$$(\pi_{0'}, \pi_{1'}) \sim (\alpha\pi_{0'}, \alpha\pi_{1'})$$

for any non-vanishing $\alpha \in \mathbb{C}$. We can then identify the patches used previously as

$$U = \left\{[\pi_{A'}] \in \mathbb{P}^1 | \pi_{1'} \neq 0\right\} \qquad \text{and} \qquad \hat{U} = \left\{[\pi_{A'}] \in \mathbb{P}^1 | \pi_{0'} \neq 0\right\}$$

and so identify

$$\lambda = {}^{\pi_{0'}}/_{\pi_{1'}} \qquad \text{and} \qquad \hat{\lambda} = {}^{\pi_{1'}}/_{\pi_{0'}}.$$

The fibre coordinate for $\mathcal{O}(n)$ also makes sense in this homogeneous language; sections are represented by functions homogeneous of weight n in $\pi_{A'}$. Taking $\mathcal{O}(2)$ as an example, we write the patching as

$$\hat{q} = q$$

and the global sections as

$$q| = x^{A'B'}\pi_{A'}\pi_{B'} \ ;$$

the inhomogeneous version is then $Q| = {}^{q|}/_{(\pi_{1'})^2}$.

Penrose's twistor theory is more often described using two homogeneous coordinates of weight one called $\omega^A = (\omega^0, \omega^1)$ than the variables (T, Q) introduced in theorem 3.1.1. The latter are more suited to our purposes in this chapter, but we should consider also the former, so we will give a brief outline of this material, following [21]. Let

$$\mathbb{S} \to M \qquad \text{and} \qquad \mathbb{S}' \to M$$

be rank-two symplectic *spin* vector bundles on spacetime, whose sections are two-component spinors $\psi^A(x^a)$ and $\psi^{A'}(x^a)$ for $A = (0,1)$ and $A' = (0', 1')$. The isomorphism

$$TM = \mathbb{S} \otimes \mathbb{S}' \tag{3.1.12}$$

allows us to write vectors on M as $V^{AA'}$ and the metric as

$$g = \epsilon_{AB}\epsilon_{A'B'}e^{AA'} \otimes e^{BB'} \tag{3.1.13}$$

for the symplectic forms ϵ_{AB} and $\epsilon_{A'B'}$ on $\mathbb{S}$ and $\mathbb{S}'$ explicitly given by (2.3.1) and where $e^{AA'}$ is a tetrad of one-forms (such as can be calculated from theorem 2.3.1). The symplectic forms can be used to raise and lower spinor indices according to the conventional rules

$$\psi^A = \epsilon^{AB}\psi_B \qquad \text{and} \qquad \psi_A = \psi^B\epsilon_{BA} \ ,$$

with identical conventions for primed spinors.

A null vector is, as a consequence of (3.1.13), a matrix of determinant zero. Basic linear algebra then implies that null vectors can therefore be written as a pair of spinors:

$$V^{AA'} = \psi^A\pi^{A'}.$$

The properties of the complex conjugation on spinors depends on the signature of g.

- If g is Lorentzian, then the spinor complex conjugation is a map $\mathbb{S} \to \mathbb{S}'$ given by $\psi^A \mapsto \bar{\psi}^{A'} = \begin{pmatrix} \bar{\psi}^0 \\ \bar{\psi}^1 \end{pmatrix}$. Linear $\mathrm{SL}(2,\mathbb{C})$ transformations on $\mathbb{S}$ induce Lorentz transformations on vectors, and

$$\mathrm{SO}(3,1) = {}^{\mathrm{SL}(2,\mathbb{C})}/_{\mathbb{Z}_2}.$$

- ◇ On the other hand if g is Riemannian, then complex conjugation preserves the type of spinors, i.e. it maps sections of $\mathbb{S}$ to sections of $\mathbb{S}$ and sections of $\mathbb{S}'$ to sections of $\mathbb{S}'$. It is given by $\psi^A \mapsto \bar{\psi}^A = \begin{pmatrix} \bar{\psi}^1 \\ -\bar{\psi}^0 \end{pmatrix}$ and $\psi_{A'} \mapsto \bar{\psi}_{A'} = \begin{pmatrix} -\bar{\psi}_{1'} \\ \bar{\psi}_{0'} \end{pmatrix}$. In the Riemannian case the structure group is not simple, and we have

$$\mathrm{SO}\left(4, \mathbb{R}\right) = {}^{\mathrm{SU}(2)\times\mathrm{SU}(2)'}/_{\mathbb{Z}_2} \ ,$$

  where the spin groups $\mathrm{SU}(2)$ and $\mathrm{SU}(2)'$ act linearly on sections of $\mathbb{S}$ and $\mathbb{S}'$ respectively. (See [21, 83] for more details on spinor conjugation in twistor theory.)

One can choose a Killing vector field

$$\mathcal{T} = \mathcal{T}^{AA'} e_{AA'}$$

(where $e_{AA'}$ is the dual tetrad) giving us a preferred map

$$\mathcal{T} \,:\, \mathbb{S} \to \mathbb{S}'$$

which acts as

$$\mathcal{T} \,:\, \omega^A \mapsto \mathcal{T}^{AA'} \epsilon_{AB} \omega^B .$$

When considering the Newtonian limit in theorem 3.1.1 one is forced to make a space-time decomposition and so there is an obvious preferred vector field

$$\mathcal{T} = \frac{\partial}{\partial t}.$$

This allows us to get rid of unprimed spinors all together in the non-relativistic limit, giving us a dual tetrad $e_{A'B'}$ with two primed indices. We thus have

$$TM = \mathbb{S}' \otimes \mathbb{S}' \tag{3.1.14}$$

which decomposes as

$$TM = \mathbb{S}' \odot \mathbb{S}' + \Lambda^2 \left(\mathbb{S}'\right) \,.$$

This decomposition, in terms of the dual tetrad, is a space-time $3+1$ decomposition, with

$$e_{(A'B')} = \frac{\partial}{\partial x^{A'B'}}$$

being the three spatial directions and

$$e_{[A'B']} = c^{-1}\epsilon_{A'B'}\frac{\partial}{\partial t}$$

being the time direction.

Lax pairs

The map $\mu : F \to Z$ featuring in the double-fibration picture (2.1.1) discussed in section 2.1 is the map induced by quotienting the correspondence space F by a distribution of vector fields $\{\mathcal{L}_A\}$ called the *twistor distribution* or, in the context of integrability, the *Lax pair* [21].

In the case of Penrose's nonlinear graviton construction the Lax pair is constructed by first taking the dual tetrad $e_{AA'}$. Via the isomorphism (3.1.12) the Levi-Civita connection induces connections on $\mathbb{S} \to M$ and $\mathbb{S}' \to M$ individually, telling us how to covariantly differentiate spinors. The connection on $\mathbb{S}'$, whose components we write as $\Gamma^{A'}_{B'CC'}$, provides a canonical way of lifting the vectors $e_{AA'}$ to sections of $T\mathbb{S}'$: we lift $e_{AA'}$ to the unique vector $\tilde{e}_{AA'}$ which is *horizontal* with respect to the connection [21]. Concretely we have

$$\tilde{e}_{AA'} = e_{AA'} + \Gamma^{B'}_{C'AA'}\pi^{C'}\frac{\partial}{\partial \pi^{B'}} \tag{3.1.15}$$

and we write

$$\tilde{\mathcal{L}}_A = \pi^{A'}\tilde{e}_{AA'}. \tag{3.1.16}$$

We must also projectivise, which means quotienting $\mathbb{S}' \to M$ by the Euler vector field (3.1.11):

$$\mathbb{S}'/\Upsilon = P\mathbb{S}'$$

and we obtain the projective primed spin bundle, which we take to be the correspondence space $F = P\mathbb{S}'$. The horizontal lifts (3.1.15) descend to $P\mathbb{S}'$to give us the Lax pair

$$\{\mathcal{L}_A\} = \{\tilde{\mathcal{L}}_A\}/\Upsilon. \tag{3.1.17}$$

Penrose considered the Frobenius integrability of this distribution in the nonlinear graviton construction of section 2.2.

Theorem 3.1.2. (Penrose [66])

The distribution (3.1.17) is integrable and hence has a four parameter family of integral surfaces iff $(M, [g])$ is ASD.

Recall the definition 2.2.1 of anti-self-duality. This integrability restriction remains the most significant problem in twistor theory; generically spacetimes are not ASD and extending twistor theory to general spacetimes remains an unsolved problem. (For some recent thoughts on this problem from Penrose see [67].)

For the complexified Minkowski spacetime ($c \neq \infty$) in theorem 3.1.1 we have that

$$\mathcal{L}_0 = 2\frac{\partial}{\partial \xi} - \lambda \left(c^{-1}\frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right)$$

and

$$\mathcal{L}_1 = \left(c^{-1}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) - 2\lambda \frac{\partial}{\partial \tilde{\xi}}$$

for the Lax pair over U. When we take the Newtonian limit we see that the $\frac{\partial}{\partial t}$ parts drop out of the Lax pair entirely, which ties in with the results of theorem 3.1.1: the time coordinate itself becomes a twistor function, so we have

$$T| = t$$

and the remaining spatial derivatives combine to ensure that (3.1.8) is still a twistor function. In terms of the Newtonian spinors we have

$$\tilde{\mathcal{L}}_{A'} = \pi^{B'} e_{A'B'} = \pi^{B'} \left(e_{(A'B')} + c^{-1}\epsilon_{A'B'}\frac{\partial}{\partial t} \right)$$

so the Lax pair post-limit is simply

$$\{\mathcal{L}_{A'}\} = {\left\{\pi^{B'} e_{(A'B')}\right\}}/{\Upsilon}.$$

Real slices

Theorem 3.1.3. [22]

For any $c \neq 0$ (finite or not) there exists an involution $\varkappa : Z_c \to Z_c$ which restricts to an antipodal map on each rational curve. The $\varkappa$-invariant sections form a real four-manifold M_r with real analogues of the induced structures from theorem 3.1.1.

Proof

The Euclidean real slice M_r of the complexified Minkowski space M is characterised by an involution $\varkappa : Z_c \to Z_c$ given by

$$\varkappa : \begin{pmatrix} T \\ Q \\ \lambda \end{pmatrix} \mapsto \begin{pmatrix} -\bar{T} + c^{-1}\bar{\lambda}^{-1}\bar{Q} \\ -\bar{\lambda}^{-2}\bar{Q} \\ -\bar{\lambda}^{-1} \end{pmatrix}$$

so that $\varkappa^2 = 1$ on the (projective) twistor space Z_c. This involution has no fixed points, and so for any $(T, Q, \lambda) \in Z_c$ there is a unique line joining (T, Q, λ) to $\varkappa(T, Q, \lambda)$. These give rise to the real twistor curves, which correspond to points in the Euclidean slice M_r of M. These $\varkappa$-invariant sections are characterised by

$$Q| = \varkappa(Q|) = -\bar{\lambda}^{-2}\bar{Q}$$

$$\Rightarrow \quad \xi\lambda^2 - 2z\lambda - \tilde{\xi} = -\bar{\xi} + 2\bar{z}\bar{\lambda}^{-1} + \bar{\tilde{\xi}}\bar{\lambda}^{-2}$$

which means that z must be real and $\bar{\xi} = \tilde{\xi}$, so we can set

$$\xi = x - iy \qquad \text{and} \qquad \tilde{\xi} = x + iy$$

for $(x, y) \in \mathbb{R}^2$. For T we then have

$$T| = \varkappa(T|) = -\bar{T}| + c^{-1}\bar{\lambda}^{-1}\bar{Q}|$$

which reduces to

$$t = -\bar{t}\ , \tag{3.1.18}$$

so we must set

$$t = i\tau$$

for $\tau \in \mathbb{R}$.

After fixing the conformal factors this yields a real Galilean structure

$$\theta = d\tau \qquad \text{and} \qquad h = \frac{\partial}{\partial x} \odot \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \odot \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \odot \frac{\partial}{\partial z} \tag{3.1.19}$$

in the $c = \infty$ case and a Riemannian metric

$$g = c^2 d\tau^2 + dx^2 + dy^2 + dz^2$$

in the $c \neq \infty$ case, completing the proof.

$\square$

Limit of the null hypersurface

When considering the case of finite c one usually divides Z_c up into two regions separated by a five-real-dimensional hypersurface PN_c called the hypersurface of *null twistors* [46]. This is done by equipping Z_c with an inner product, which in homogeneous coordinates

$$Z_1^\alpha = (T_1, q_1, (\pi_{A'})_1) \qquad Z_2^\alpha = (T_2, q_2, (\pi_{A'})_2)$$

is given by

$$\Sigma_c(Z_1^\alpha, Z_2^\alpha) = \frac{1}{\sqrt{2}}(T_1 + T_2)\left((\pi_{0'})_1 (\pi_{0'})_2 + (\pi_{1'})_1 (\pi_{1'})_2\right) - \frac{\sqrt{2}}{c}\left(q_1 \frac{(\pi_{0'})_2}{(\pi_{1'})_1} + q_2 \frac{(\pi_{0'})_1}{(\pi_{1'})_2}\right).$$

We then have that

$$PN_c := \left\{ Z^\alpha \in Z_c \,|\, \Sigma\left(Z^\alpha, \bar{Z}^\alpha\right) = 0 \right\}.$$

The null twistors are interesting because these are the twistors whose associated alpha surfaces descend to *real* null rays when a real Lorentzian slice is taken.

In the Newtonian limit though, we have

$$\Sigma_\infty(Z^\alpha, \bar{Z}^\alpha) = \frac{1}{\sqrt{2}}\left(T + \bar{T}\right)\left(|\pi_{0'}|^2 + |\pi_{1'}|^2\right) \ ,$$

and so the real twistor curves of theorem 3.1.3 (which satisfy the condition (3.1.18)) happen to lie on the limit of the null hypersurface. The difference between the Riemannian and Lorentzian reality conditions disappears as $c \to \infty$. This makes sense because the difference between the two signatures themselves is destroyed by the limit, with the temporal eigenvalue vanishing in both cases.

3.1.2 Connections and deformations

In the nonlinear graviton construction one moves away from the flat model and introduces curvature by making Kodaira deformations of the complex structure [50, 51, 46]. In this section we will pursue this idea on the Newtonian case, finding (and solving) a serious problem.

3.1.2.1 Kodaira instability and the jump to Gibbons-Hawking

The problem with introducing curvature via Kodaira deformation in the Newtonian setting is that the non-relativistic normal bundle $N_x = \mathcal{O} \oplus \mathcal{O}(2)$ is *unstable* with respect to general deformations and so may experience a discontinuous change in its isomorphism class. That is to say, N_x may experience a *jump*. The relevant fact (see, for example, [46] or [51]) is that the isomorphism class of the normal bundle $N_x \to \mathbb{P}^1$ to a submanifold $X_x \subset Z$ is stable with respect to general Kodaira deformations of X_x iff

$$\check{H}^1\left(\mathbb{P}^1, \mathrm{End}\left(N_x\right)\right) = 0 \ .$$

The sections of End (N_x) can be written as two-by-two matrices of functions whose weights ensure that the matrix is a homogeneous map from N_x to N_x, or more practically we can write

$$\mathrm{End}\left(N_x\right) = N_x \otimes N_x^*$$

where N_x^* is the dual bundle, whose fibres are the dual vector spaces to those of N_x. For $\mathcal{O}(n) \to \mathbb{P}^1$ we have

$$\mathcal{O}(n)^* = \mathcal{O}(-n)$$

as well as

$$\mathcal{O}(n) \otimes \mathcal{O}(m) = \mathcal{O}(n+m).$$

In the relativistic case we have $N_x = \mathcal{O}(1) \oplus \mathcal{O}(1)$, so

$$\check{H}^1\left(\mathbb{P}^1, \left(\mathcal{O}(1) \oplus \mathcal{O}(1)\right) \otimes \left(\mathcal{O}(-1) \oplus \mathcal{O}(-1)\right)\right) = \check{H}^1\left(\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}\right) = 0.$$

Therefore Penrose's normal bundle is stable, and one doesn't have to worry about the isomorphism class jumping.

On the other hand in Newtonian setting we can calculate

$$\check{H}^1\left(\mathbb{P}^1, (\mathcal{O}\oplus\mathcal{O}(2))\otimes(\mathcal{O}\oplus\mathcal{O}(-2))\right) = \check{H}^1\left(\mathbb{P}^1, \mathcal{O}(-2)\oplus\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(2)\right) = \mathbb{C} \ ,$$

so we expect some instability.

Theorem 3.1.4. [22, 23]

Let $Z \to \mathbb{P}^1$ be a Kodaira deformation of the Newtonian twistor space $Z_\infty = \mathcal{O}\oplus\mathcal{O}(2)$ whose patching is

$$\hat{T} = T + f\left(Q, \lambda\right) \tag{3.1.20}$$

$$\hat{Q} = \lambda^{-2} Q \tag{3.1.21}$$

where f represents a cohomology class in $\check{H}^1\left(\mathcal{O}(2), \mathcal{O}_{\mathcal{O}(2)}\right)$. The normal bundle to rational curves is generically $N_x = \mathcal{O}(1)\oplus\mathcal{O}(1)$ and the moduli space of such curves is a complexified Riemannian manifold equipped with a Gibbons-Hawking conformal structure of the form

$$[g] = \frac{1}{V}\left(dt + A\right)^2 + V\left(dx^2 + dy^2 + dz^2\right) \tag{3.1.22}$$

where the Gibbons-Hawking potential is

$$V = \frac{1}{2\pi i}\oint_\Gamma \frac{\partial f}{\partial Q}\left(Q|, \lambda\right) d\lambda. \tag{3.1.23}$$

The normal bundle to twistor lines X_x for $x \in \mathcal{X} = \{x \,|\, V = 0\}$ is

$$N_x = \mathcal{O}\oplus\mathcal{O}(2).$$

The reader is reminded that Gibbons-Hawking metrics [35] are a family of ASD Ricci-flat metrics spanned by a choice of harmonic function V. The one-form A is determined up to gauge equivalence by

$$dV = \star^3\left(dA\right),$$

and we note that the formula (3.1.23) ensures that V is automatically harmonic.

Proof

Consider first the normal bundle, a vector bundle whose patching is

$$\mathcal{F} = \begin{pmatrix} 1 & \frac{\partial f}{\partial Q}| \\ 0 & \lambda^{-2} \end{pmatrix}$$

where as usual the vertical slash denotes the restriction to a twistor line X_x, which for the $\mathcal{O}(2)$ part are again

$$Q| = \xi\lambda^2 - 2z\lambda - \tilde{\xi} \qquad \text{or equivalently} \qquad q| = x^{A'B'}\pi_{A'}\pi_{B'}. \tag{3.1.24}$$

We know from the fact that the undeformed twistor space Z_∞ has rational curves with $N_x = \mathcal{O} \oplus \mathcal{O}(2)$ and the fact that the sum of the degrees $0 + 2 = 2$ is a topological invariant that we must have $N_x = \mathcal{O}(2-k) \oplus \mathcal{O}(k)$ for some integer k [21]. To calculate k we must solve the Riemann-Hilbert splitting problem for $\mathcal{F}$; that is to say we must find holomorphic maps

$$H : U \to \mathrm{GL}(2, \mathbb{C}) \qquad \text{and} \qquad \hat{H} : \hat{U} \to \mathrm{GL}(2, \mathbb{C})$$

such that

$$\mathcal{F} = \hat{H} \begin{pmatrix} \lambda^{k-2} & 0 \\ 0 & \lambda^{-k} \end{pmatrix} H^{-1}$$

for some k (on each rational curve). Write

$$H = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \qquad \text{and} \qquad \hat{H} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_3 & \hat{h}_4 \end{pmatrix}$$

and then consider component-by-component the equation $\mathcal{F}H = \hat{H} \begin{pmatrix} \lambda^{k-2} & 0 \\ 0 & \lambda^{-k} \end{pmatrix}$.

$$\hat{h}_1 = \lambda^{2-k} h_1 + \lambda^{2-k} \frac{\partial f}{\partial Q}| h_3 \tag{3.1.25}$$

$$\hat{h}_2 = \lambda^{k} h_2 + \lambda^{k} \frac{\partial f}{\partial Q}| h_4 \tag{3.1.26}$$

$$\hat{h}_3 = \lambda^{-k} h_3 \tag{3.1.27}$$

$$\hat{h}_4 = \lambda^{k-2} h_4 \tag{3.1.28}$$

We seek global sections of (3.1.25-3.1.28) over each rational curve $X_x = \mathbb{P}^1$. First consider trying to find solutions for $k > 2$. In that case (3.1.28) has no global sections, so $\hat{h}_4 = h_4 = 0$ and so (3.1.26) becomes $\hat{h}_2 = \lambda^k h_2$ which also has no global sections, so $\hat{h}_2 = h_2 = 0$. This means though, that $\det(H) = \det\left(\hat{H}\right) = 0$: there are therefore no invertible solutions for $k > 2$.

This leaves two[2] possibilities; either $k = 1$ or $k = 2$. In the $k = 2$ case (3.1.28) becomes the trivial patching and so we set

$$\hat{h}_4 = h_4 = a_x \in \mathbb{C}$$

for each rational curve. The only potential obstruction to finding global solutions is then in (3.1.26), where we have what looks like the patching for an affine bundle with underlying translation bundle $\mathcal{O}(-2)$. Recall $\check{H}^1\left(\mathbb{P}^1, \mathcal{O}(-2)\right) = \mathbb{C} \neq 0$; the obstruction to finding global sections is to be found in the order-λ term in the expansion of $\lambda^2 \frac{\partial f}{\partial Q}| h_4$. Now note that $\frac{\partial f}{\partial Q}|$ is, for fixed x^a, a function on the annulus $U \cap \hat{U}$: expanding

$$\frac{\partial f}{\partial Q}| = \sum_{i=-\infty}^{\infty} \gamma_i(x^a)\lambda^i$$

we can identify the obstruction as $\gamma_{-1}(x^a)$, which generically does not vanish. If $k \neq 2$ then we must have $k = 1$. Therefore we can conclude the first part of the proof: on generic points where $\gamma_{-1}(x^a) \neq 0$ we have $N_x = \mathcal{O}(1) \oplus \mathcal{O}(1)$ and on special rational curves where $\gamma_{-1}(x^a) = 0$ the isomorphism class jumps to $N_x = \mathcal{O} \oplus \mathcal{O}(2)$.

To calculate the conformal structure we will first calculate the global sections of (3.1.20), restricting to (3.1.24). The quantity $f\left(Q|, \lambda\right)$ is a section of $\mathcal{O} \to \mathbb{P}^1$ and so, because $\check{H}^1\left(\mathbb{P}^1, \mathcal{O}\right) = 0$, we must be able to split $f\left(Q|, \lambda\right)$ into coboundaries:

$$f\left(Q|, \lambda\right) = h\left(x^a, \lambda\right) - \hat{h}(x^a, \lambda).$$

The global sections are then

$$\hat{T}| = t - \hat{h} \qquad \text{and} \qquad T| = t - h.$$

[2] Without loss of generality we can consider $k \geq 1$.

In fact there exist contour-integral formulas for h and $\hat{h}$ (see, for example, [21, 46]), which are most easily manipulated using homogeneous coordinates. We can write

$$h = \frac{1}{2\pi i}\oint_{\Gamma} \frac{(\pi \cdot \alpha)}{(\pi \cdot \rho)\,(\rho \cdot \alpha)} f\left(x^{A'B'}\rho_{A'}\rho_{B'}, \rho_{A'}\right) \rho \cdot d\rho$$

for $[\rho_{A'}] \in \mathbb{P}^1$ and $\Gamma \subset \mathbb{P}^1$ a contour enclosing both $\rho_{0'} = 0$ and $\rho_{A'} = \pi_{A'}$, and for $\alpha_{A'}$ a choice of spinor parametrising the non-uniqueness in the splitting. (This spinor can be absorbed into t but is included here for completeness.)

Rather than directly calculate the conformal structure via the twistor principle we will instead use the Lax pair formulation. We must find vector fields on $P\mathbb{S}'$ whose kernel is spanned by the twistor functions $(T|, q|, \pi_{A'})$. By inspection and direct calculation one can see that

$$L_{A'} = \pi^{B'}\frac{\partial}{\partial x^{A'B'}} - \pi^{B'}\phi_{A'B'}\frac{\partial}{\partial t}$$

where

$$\pi^{B'}\phi_{A'B'} = \pi^{B'}\frac{\partial h}{\partial x^{A'B'}} = \pi^{B'}\frac{1}{2\pi i}\oint_{\Gamma}\frac{\rho_{A'}\alpha_{B'}}{(\rho \cdot \alpha)}\frac{\partial f}{\partial q}\left(x^{A'B'}\rho_{A'}\rho_{B'}, \rho_{A'}\right)\rho \cdot d\rho.$$

By inspection one can then extract the (inverse) conformal structure

$$\left[g^{-1}\right] = \epsilon^{A'B'}\epsilon^{C'D'}\left(\frac{\partial}{\partial x^{A'C'}} - \phi_{A'C'}\frac{\partial}{\partial t}\right) \odot \left(\frac{\partial}{\partial x^{B'D'}} - \phi_{B'D'}\frac{\partial}{\partial t}\right),$$

which is in the Gibbons-Hawking form (3.1.22) with a Gibbons-Hawking potential given by

$$V = \epsilon^{A'B'}\phi_{A'B'} = \frac{1}{2\pi i}\oint_{\Gamma}\frac{\partial f}{\partial q}\left(x^{A'B'}\rho_{A'}\rho_{B'}, \rho_{A'}\right)\rho \cdot d\rho\,, \tag{3.1.29}$$

a naturally harmonic function. Note that the formula (3.1.29) means that $V \propto \gamma_{-1}(x^a)$. Earlier in the proof we noted that γ_{-1} was the obstruction to the normal bundle being isomorphic to $\mathcal{O}\oplus\mathcal{O}(2)$, i.e. to the normal bundle suffering one jump on these lines. Thus the jumping lines are characterised by the vanishing of the Gibbons-Hawking potential. This completes the proof.

□

Other deformations

The above theorem shows why general Kodaira deformations are not a way of introducing curvature in Newtonian twistor theory, but one might ask whether there are other ways of obtaining complex three-folds containing Kodaira families of rational curves with normal bundle bundle $N_x = \mathcal{O} \oplus \mathcal{O}(2)$, and the answer is yes. One such kind involves leaving the $\mathcal{O}$ part unaltered and attaching it to a deformed *minitwistor* space [42, 62]. This will give a curved metric on the spatial fibres of M, which is therefore inappropriate for Newton-Cartan geometry, but may nevertheless constitute an interesting avenue of research. One can also construct jumping Newtonian twistor spaces as Kodaira deformations of $\mathcal{O}(-1) \oplus \mathcal{O}(3)$, as is described in section 4.4.

3.1.2.2 Building connections

We saw in section 2.4 how one can construct canonical connections on Kodaira moduli spaces, and in particular how the so-called Λ-connection is, in the $N_x = \mathcal{O}(1) \oplus \mathcal{O}(1)$ case, the Levi-Civita connection. In this section we will apply this construction to the Newtonian setting. The normal bundle is $N_x = \mathcal{O} \oplus \mathcal{O}(2)$ and so we can calculate that

$$\check{H}^0\left(\mathbb{P}^1, N \otimes (N^* \odot N^*)\right) \neq 0 \tag{3.1.30}$$

and

$$\check{H}^1\left(\mathbb{P}^1, N \otimes (N^* \odot N^*)\right) \neq 0 \ , \tag{3.1.31}$$

so Merkulov's construction doesn't quite work as one might like. There are two failures with which to come to terms.

The non-vanishing (3.1.31) is harmless provided we don't make any Kodaira deformations, so we will restrict attention to $Z_\infty = \mathcal{O} \oplus \mathcal{O}(2)$ only for this section.

The non-vanishing (3.1.30) turns out to be interesting, as it neatly encapsulates the fact that Newton-Cartan connections are not metric and so something must be specified independently: we have a cohomological interpretation of this fact. In the following theorem we find out what can be salvaged from the construction.

Theorem 3.1.5. [22]

Let $Z_\infty = \mathcal{O} \oplus \mathcal{O}(2)$. The moduli space M of its global sections X_x comes equipped with a family of torsion-free affine connections parametrised by five arbitrary functions (ϕ^i, χ, η) on M, whose only non-vanishing Christoffel symbols are

$$\Gamma^i{}_{tt} = \phi^i \qquad \Gamma^i{}_{jt} = \Gamma^i{}_{tj} = \delta^i_j \chi \qquad \Gamma^t{}_{tt} = \eta.$$

Proof

As described in section 2.4.3 we wish to solve (2.4.10) for $\{\sigma\}$. For the undeformed Z_∞ we have

$$\mathcal{F}^\mu_{\alpha\beta} = 0 \qquad \text{and} \qquad \mathcal{F}^\mu_\nu = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix},$$

where $w^\mu = \begin{pmatrix} T \\ Q \end{pmatrix}$, and so (2.4.10) consists of the six equations

$$\begin{aligned} 0 &= -\hat{\sigma}^T_{TT} + \sigma^T_{TT} & 0 &= -\hat{\sigma}^Q_{TT} + \lambda^{-2}\sigma^Q_{TT} \\ 0 &= -\hat{\sigma}^T_{TQ}\lambda^{-2} + \sigma^T_{TQ} & 0 &= -\hat{\sigma}^Q_{TQ}\lambda^{-2} + \lambda^{-2}\sigma^Q_{TQ} \\ 0 &= -\hat{\sigma}^T_{QQ}\lambda^{-4} + \sigma^T_{QQ} & 0 &= -\hat{\sigma}^Q_{QQ}\lambda^{-4} + \lambda^{-2}\sigma^Q_{QQ}. \end{aligned} \tag{3.1.32}$$

These equations are each the patching for $\mathcal{O}(n)$ for some n, and so their general solutions are straightforward to work out. The only non-vanishing parts of $\{\sigma\}$ over U are

$$\sigma^T_{TT} = \eta \qquad \sigma^Q_{TQ} = \chi \qquad \sigma^Q_{TT} = (\phi^x - i\phi^y)\,\lambda^2 - 2\phi^z\lambda - (\phi^x + i\phi^y)$$

for arbitrary functions (ϕ^i, χ, η) on M. (Note that the splitting problem is being solved for *each* twistor line individually, leading to functions on M.) The Christoffel symbols can then be read-off from (2.4.11) to give

$$\Gamma^i{}_{tt} = \phi^i \qquad \Gamma^i{}_{jt} = \Gamma^i{}_{tj} = \delta^i_j \chi \qquad \Gamma^t{}_{tt} = \eta\ ,$$

completing the proof of the theorem.

□

We can give interpretations to each of these five functions.

- ◇ η is a function which must be present in the Newton-Cartan connection to ensure that $\nabla\theta = 0$ for all $\theta \in [\theta]$ before the temporal diffeomorphism factor of the clock has been fixed by global data. Therefore the fixing of $\theta = dt \in [\theta]$ fixes η also.
- ◇ χ plays the same role for $\nabla h = 0$, and so is also determined when the conformal factor on h is fixed.
- ◇ ϕ^i is the Newtonian force of gravity and as discussed in section 2.6 are a part of the connection which is not determined by the Galilean structure. At present then, Z_∞ comes naturally equipped with *all* Newtonian gravitational forces $\boldsymbol{g}(x)$. The next section will be devoted to a natural twistor-theoretic way of fixing ϕ^i.

Coriolis forces

No generalised Coriolis forces have been constructed in theorem 3.1.5; in four dimensions these do not form part of the global $\{\sigma\}$ and so if they are to be constructed at all then they must be either via a special class of Kodaira deformation or via some entirely different method. In section 3.1.3 we will examine a construction which, whilst less satisfying aesthetically, can induce the required data on M to include generalised Coriolis forces.

3.1.2.3 Fixing connections: Gibbons-Hawking revisited

In order to make use of the family of connections naturally induced on Z_∞ we must decide upon a way to fix a preferred $\{\sigma\}$. It turns out that the Newtonian limit of a certain Gibbons-Hawking ansatz provides a natural way to do this.

Theorem 3.1.6. [22]

Let $Z_c \to \mathbb{P}^1$ be the twistor space with patching

$$\hat{T} = T - (c\lambda)^{-1} Q - c^{-3} g\left(Q, \lambda\right)$$

$$\hat{Q} = \lambda^{-2} Q \ ,$$

where g represents a cohomology class in $\check{H}^1\left(\mathcal{O}(2), \mathcal{O}_{\mathcal{O}(2)}\right)$.

1. *The moduli space M is Gibbons-Hawking;*

$$g = \frac{1}{V}\left(cdt + c^{-2}A\right)^2 + V\left(dx^2 + dy^2 + dz^2\right)$$

with potential

$$V = 1 + c^{-2}W$$

where W is the (harmonic) minitwistor transform of g, and where A is determined up to gauge equivalence by $dW = \star^3 dA$.

2. *Taking the limit $c \to \infty$ yields a Newton-Cartan spacetime with the standard Galilean (h, θ) given by (3.1.19) and with a connection ∇ whose only non-vanishing components are*

$$\Gamma^i_{tt} = \frac{1}{2} h^{ij} \partial_j W. \tag{3.1.33}$$

3. *The calculation of $\{\sigma\}_c$ for Z_c yields a 0-cochain which survives the $c \to \infty$ limit and can be used to fix the Newtonian 0-cochain. The resulting $\{\sigma\}_\infty$ is naturally determined entirely by the Ward transform of a line bundle $E \to Z_\infty$.*

Proof

The proof of part one is a straightforward application of theorem 3.1.4. Simply take

$$f = -\left(c\lambda\right)^{-1} Q - c^{-3} g\left(Q, \lambda\right)$$

in (3.1.20) and calculate

$$V = \frac{1}{2\pi i}\oint_\Gamma \left\{-\left(c\lambda\right)^{-1} - c^{-3}\frac{\partial g}{\partial Q}|\right\} d\lambda = -c\left(1 + c^{-2}W\right)$$

$$\Rightarrow \qquad [g] = \frac{1}{1 + c^{-2}W}\left(cdt + c^{-2}A\right)^2 + \left(1 + c^{-2}W\right)\left(dx^2 + dy^2 + dz^2\right). \tag{3.1.34}$$

The global data fixes g to be the representative displayed in (3.1.34).

For the second part with simply calculate the Levi-Civita connection of g and then take the Newtonian limit of the triple $\left(g_{ab}, g^{ab}, \Gamma^a_{\ bc}\right)$. For the sake of brevity we omit the pre-limit Christoffel symbols and proceed to take the limits.

$$\lim_{c\to\infty}\left(c^{-2}g\right) \to dt^2$$

and so we identify $\theta = dt$. Similarly we have

$$\lim_{c\to\infty}\left(g^{-1}\right) \to \frac{\partial}{\partial x}\odot\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\odot\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\odot\frac{\partial}{\partial z}\ ,$$

so we identify the Newton-Cartan metric.

All components of the connection are finite or zero in the Newtonian limit, a fact guaranteed by a theorem of Künzle [52], which requires that the leading order term in g (i.e. the part of order c^2) be the tensor-square of a hypersurface-orthogonal one-form in the limit. The only non-vanishing parts of the connection are the Christoffel symbols (3.1.33).

For the third part we must solve the splitting problem (2.4.10) with

$$\mathcal{F}^T_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & -c^{-3}\frac{\partial^2 g}{\partial Q^2}| \end{pmatrix} \qquad \mathcal{F}^Q_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\mathcal{F}^\mu_{\ \nu} = \begin{pmatrix} 1 & -(c\lambda)^{-1} - c^{-3}\frac{\partial g}{\partial Q}| \\ 0 & \lambda^{-2} \end{pmatrix}.$$

This is possible without the introduction of new ideas, but difficult. The equations to be solved for $\{\sigma\}_c$ are

$$\begin{aligned}\frac{1}{c^3}\frac{\partial^2 g}{\partial Q^2}| = \hat{\sigma}^T_{QQ}\lambda^{-4} - 2\hat{\sigma}^T_{QT}\lambda^{-2}\left(\frac{1}{c\lambda} + \frac{1}{c^3}\frac{\partial g}{\partial Q}|\right) \\ + \hat{\sigma}^T_{TT}\left(\frac{1}{c\lambda} + \frac{1}{c^3}\frac{\partial g}{\partial Q}|\right)^2 - \sigma^T_{QQ} + \left(\frac{1}{c\lambda} + \frac{1}{c^3}\frac{\partial g}{\partial Q}|\right)\sigma^Q_{QQ}\ ;\end{aligned} \tag{3.1.35}$$

$$0 = -\hat{\sigma}^T_{QT}\lambda^{-2} + \hat{\sigma}^T_{TT}\left(\frac{1}{c\lambda} + \frac{1}{c^3}\frac{\partial g}{\partial Q}|\right) + \sigma^T_{QT} - \left(\frac{1}{c\lambda} + \frac{1}{c^3}\frac{\partial g}{\partial Q}|\right)\sigma^Q_{QT}\ ; \tag{3.1.36}$$

$$0 = -\hat{\sigma}^T_{TT} + \sigma^T_{TT} - \left(\frac{1}{c\lambda} + \frac{1}{c^3}\frac{\partial g}{\partial Q}|\right)\sigma^Q_{TT}\ ; \tag{3.1.37}$$

$$0 = -\hat{\sigma}^Q_{TT} + \lambda^{-2}\sigma^Q_{TT}\ ; \tag{3.1.38}$$

$$0 = -\hat{\sigma}^{Q}_{TQ}\lambda^{-2} + \hat{\sigma}^{Q}_{TT}\left(\frac{1}{c\lambda} + \frac{1}{c^3}\frac{\partial g}{\partial Q}|\right) + \lambda^{-2}\sigma^{Q}_{TQ} \ ; \tag{3.1.39}$$

and $$0 = -\hat{\sigma}^{Q}_{QQ}\lambda^{-4} + 2\hat{\sigma}^{Q}_{TQ}\lambda^{-2}\left(\frac{1}{c\lambda} + \frac{1}{c^3}\frac{\partial g}{\partial Q}|\right) - \hat{\sigma}^{Q}_{TT}\left(\frac{1}{c\lambda} + \frac{1}{c^3}\frac{\partial g}{\partial Q}|\right)^2 + \lambda^{-2}\sigma^{Q}_{QQ} \ . \tag{3.1.40}$$

Sparing the reader some arduous steps, the (abridged) solution of (3.1.35-3.1.40) can be written as

$$\sigma^{Q}_{\tau\tau} = \frac{1}{2\pi i}\oint_{\Gamma}(\lambda - \xi)^2\left(\frac{\partial^2 g}{\partial Q^2}|(x^a, \xi)\right)\, d\xi + \mathcal{O}\left(\frac{1}{c}\right) \tag{3.1.41}$$

$$\text{and all other} \quad \sigma^{A}_{BC} = \mathcal{O}\left(\frac{1}{c}\right), \tag{3.1.42}$$

where Γ is a contour enclosing $\xi = 0$ and $\xi = \lambda$. Thus one finds that the only parts of $\{\sigma\}_c$ which don't vanish in the Newtonian limit are $\sigma^{Q}_{\tau\tau}$ and $\hat{\sigma}^{Q}_{\tau\tau}$, constituting a global section of $\mathcal{O}(2)$ and giving rise to a non-zero Γ^{i}_{tt} via the construction above. This provides a way of fixing the Newtonian 0-cochain $\{\sigma\}_\infty$: we simply identify it with the $c \to \infty$ limit of the Gibbons-Hawking 0-cochain.

$$\{\sigma\}_\infty = \lim_{c\to\infty}\left[\{\sigma\}_c\right] \ . \tag{3.1.43}$$

The integral in (3.1.41) has the effect of pulling out the parts of $\frac{\partial^2 g}{\partial Q^2}|$ at orders $\lambda^{-1}, \lambda^{-2}$, and λ^{-3}. These are the parts of $\frac{\partial^2 g}{\partial Q^2}|$ which represent a 1-cocycle of $\mathcal{O}(-4)$; our global section of $\mathcal{O}(2)$ arises from a cohomology class in $\check{H}^1\left(\mathbb{P}^1, \mathcal{O}(-4)\right)$, an instance of Serre duality.

Moreover, the class in $\check{H}^1\left(\mathbb{P}^1, \mathcal{O}(-4)\right)$ is always the the restriction to twistor curves of a second derivative of a function representing a class in $\check{H}^1\left(Z_\infty, \mathcal{O}_{Z_\infty}\right)$. Thus we conclude that the limiting procedure fixes $\{\sigma\}_\infty$ to be determined entirely by a cohomology class $g \in \check{H}^1\left(Z_\infty, \mathcal{O}_{Z_\infty}\right)$, via the Ward transform of its associated line bundle

$$E \to Z_\infty.$$

This fixes the Newtonian potential, via $\{\sigma\}_\infty$, to be given by the minitwistor transform of g, making it naturally harmonic.

□

In light of theorem 3.1.6 it makes sense to state that a Newtonian twistor space is a pair (Z_∞, E) consisting of a complex three-fold Z_∞ containing a four-parameter family of rational curves with normal bundle $\mathcal{O} \oplus \mathcal{O}(2)$ and a line bundle $E \to Z_\infty$ trivial when restricted to the rational curves, since this data is what is obtained from the $c \to \infty$ limit of Gibbons-Hawking twistor spaces. Schematically we can thus describe the Newtonian limit on both sides of the twistor correspondence:

$$\text{Spacetime} \qquad (g) \;\mapsto\; ((h,\theta)\;,\;\nabla)$$

$$\text{Twistor Space} \qquad (N = \mathcal{O}(1) \oplus \mathcal{O}(1)) \;\mapsto\; (N = \mathcal{O} \oplus \mathcal{O}(2)\;,\;E)\,.$$

The twistor data surviving the limit pleasingly mirrors what occurs on the spacetime side of the correspondence.

3.1.2.4 The Ξ-connection for $\mathcal{O} \oplus \mathcal{O}(2)$

It is tangentially interesting to calculate the larger family of torsion-free connections called the Ξ-connection. The construction of section 2.4.1 in this straightforward case amounts to taking a global section

$$\chi^{\mu}_{\;\nu\,a} \in \check{H}^0\left(F|_x, N_x \otimes N_x^* \otimes \Lambda^1_x\,(M)\right)$$

per point $x \in M$ and extracting the connection Γ^a_{bc} from

$$\Gamma^a_{\;bc}\partial_a w^\mu| = \partial_b\partial_c w^\mu| + \chi^{\mu}_{\;\nu\,b}\partial_c w^\nu| + \chi^{\mu}_{\;\nu\,c}\partial_b w^\nu|.$$

For $Z = \mathcal{O} \oplus \mathcal{O}(2)$ we have

$$N_x \otimes N_x^* = \begin{pmatrix} \mathcal{O} & \mathcal{O}(-2) \\ \mathcal{O}(2) & \mathcal{O} \end{pmatrix}$$

so the most general $\chi^{\mu}_{\;\nu\,a}$ is

$$\chi^{T}_{\;T\,a} = A_a \qquad \chi^{T}_{\;Q\,a} = 0 \qquad \chi^{Q}_{\;Q\,a} = E_a$$

$$\chi^{Q}_{\;T\,a} = B_a + \lambda C_a + \lambda^2 D_a$$

for five arbitrary one-forms $(A_a, B_a, C_a, D_a, E_a)$ on M. One can then read off the connection components;

$$\Gamma^t{}_{tt} = 2A_t \qquad \Gamma^t{}_{it} = A_i \qquad \Gamma^t{}_{ij} = 0$$

$$\Gamma^\xi{}_{tt} = 2D_t \qquad \Gamma^z{}_{tt} = -C_t \qquad \Gamma^{\tilde{\xi}}{}_{tt} = -2B_t$$

$$\Gamma^\xi{}_{jt} = D_j + E_t\delta^\xi_j \qquad \Gamma^z{}_{jt} = -\frac{1}{2}C_j + E_t\delta^z_j \qquad \Gamma^{\tilde{\xi}}{}_{jt} = -B_j + E_t\delta^{\tilde{\xi}}_j$$

$$\Gamma^i{}_{jk} = E_j\delta^i_k + E_k\delta^i_j.$$

The connection can therefore be any connection provided that $\Gamma^t{}_{ij} = 0$ and that the spatial sector is that of a flat projective structure in three dimensions. Note that this includes all generalised Coriolis forces.

3.1.3 Vector bundles on twistor space

Via the Penrose-Ward correspondence (see [21]) one can equip twistor spaces with vector bundles which are trivial when restricted to twistor lines and find that the spacetimes are then naturally equipped with ASD gauge fields. In the case of a line bundle this is completely equivalent to a procedure called the *twisted photon*, in which a non-projective twistor space is deformed by a Maxwell field whilst leaving the projective twistor space untouched [80].

3.1.3.1 The twisted corpuscle

We will briefly consider first the Penrose case. The (projective) twistor space $Z = \mathcal{O}(1) \oplus \mathcal{O}(1)$ can be constructed as a quotient of the non-projective twistor space $\mathbb{C}^4 \to \mathbb{C}^2$, a rank-two (trivial) vector bundle on $\mathbb{C}^2$, by the homogeneity operator

$$\Upsilon = \omega^A \frac{\partial}{\partial \omega^A} + \pi_{A'} \frac{\partial}{\partial \pi_{A'}}.$$

Ward showed (see, for example, [80]) how to deform the non-projective twistor space whilst leaving Z untouched; one starts with the homogeneous Lax pair (3.1.16) and adds (a priori) any multiple of $\pi_{A'}\frac{\partial}{\partial \pi_{A'}}$, so that we work, in the flat case, with

$$\tilde{\mathcal{L}}^\Phi_A = \tilde{\mathcal{L}}_A + \Phi_{AB'}\pi^{B'}\pi_{A'}\frac{\partial}{\partial \pi_{A'}}$$

where $\Phi_{AA'}(x^a)$ are four functions on M. The inhomogeneous twistor functions are then the same as in the undeformed case. The Frobenius integrability condition then requires that

$$\partial^{A(A'}\Phi_A^{\ B')} = 0 \tag{3.1.44}$$

(where $\partial_{AA'} = \frac{\partial}{\partial x^{AA'}}$ and indices are raised and lowered with ϵ and ϵ'). Equation (3.1.44) then implies that $\Phi_{AA'}$ is a potential for an ASD Maxwell field

$$\phi_{AB} = \partial_A^{\ A'}\Phi_{BA'}.$$

Thus one can harbour an electromagnetic field by "twisting-up" the non-projective twistor space; this procedure is called the *twisted photon* construction.

In the Newtonian setting a similar procedure makes sense: we add a multiple of $\pi_{A'}\frac{\partial}{\partial\pi_{A'}}$ to the homogeneous Lax pair in the language of the primed Newtonian spinors of section 3.1.1.2.

$$\tilde{\mathcal{L}}_{A'} = \pi^{B'}\frac{\partial}{\partial x^{(A'B')}} + \Phi_{A'B'}\pi^{B'}\pi^{C'}\frac{\partial}{\partial\pi^{C'}}.$$

Integrability now requires that

$$\Phi_{A'B'} = \begin{pmatrix} A_{0'0'} & \frac{1}{2}\left(A_{0'1'} + V\right) \\ \frac{1}{2}\left(A_{0'1'} - V\right) & A_{1'1'} \end{pmatrix} \tag{3.1.45}$$

obey the Abelian monopole equation $dV = \star^3 dA$, where $A_{A'B'} = A_{(A'B')}$, meaning that V must be harmonic. We note that this is the required data for fixing the 0-cochain of theorem 3.1.5, and we call this variant on the twisted photon the *twisted corpuscle*, recalling Newton's name for a particle of light.

The twisted photon (and corpuscle) are entirely equivalent to taking the Ward transform [80, 78] of a line bundle on the (Newtonian) twistor space which is trivial (i.e. has isomorphism class $\mathcal{O} \to \mathbb{P}^1$) when restricted to twistor lines.

More generally Ward considered equipping $Z = \mathcal{O}(1)\oplus\mathcal{O}(1)$ with a rank-k vector bundle $E \to Z$ such that $E|_{X_x} = \mathcal{O} \oplus ... \oplus \mathcal{O}$. The space of global sections of $E|_{X_x}$ defines pointwise a trivial rank-k bundle over M, and the patching gives rise to a way of defining a covariant derivative on that trivial bundle on M. The pleasing result of [78] is that the connection associated to the covariant derivative is automatically an ASD Yang-Mills field, giving the construction enormous application in the field of integrability. (See [55] for further details.)

3.1.3.2 Sparling bundles

Instead of bundles $E \to Z$ which are *trivial* when restricted to (relativistic) twistor lines Sparling considered in [74] vector bundles which have an isomorphism class other than $\mathcal{O} \oplus ... \oplus \mathcal{O}$ when restricted to twistor lines. Here we will discuss the case of a rank-two vector bundle E on the Newtonian twistor space $Z_\infty = \mathcal{O} \oplus \mathcal{O}(2)$ whose restriction to a twistor lines X is

$$E|_{X_x} = \mathcal{O}(n) \oplus \mathcal{O}(m). \tag{3.1.46}$$

A rank-two vector bundle on Z_∞ is characterised by a patching relation

$$\begin{pmatrix} \hat{\psi} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} f_{mm} & f_{mn} \\ f_{nm} & f_{nn} \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

on $U \cap \hat{U}$, where $\left(\hat{\psi}, \hat{\phi}, \psi, \phi\right)$ are homogeneous fibre coordinates and where

$$\mathcal{E} = \begin{pmatrix} f_{mm} & f_{mn} \\ f_{nm} & f_{nn} \end{pmatrix}$$

is a patching matrix of four functions f_{ij} (of weight $i - j$) representing cohomology classes in $\check{H}^1 \left(Z_\infty, \mathcal{O}\left(i-j\right)_{PT_\infty}\right)$. Then the property (3.1.46) is realised if

$$\mathcal{E}|_{X_x} = \hat{H} H^{-1}$$

for holomorphic maps

$$H \,:\, U \to \mathrm{GL}(2, \mathbb{C}) \qquad \text{and} \qquad \hat{H} \,:\, \hat{U} \to \mathrm{GL}(2, \mathbb{C})$$

whose matrix entries have homogeneous weights.

Just like with the Ward transform we define a trivial vector bundle $\mathcal{V}$ (of rank $n + m + 2$ for $n, m \geq 0$) on M fibrewise by

$$\mathcal{V}|_x = \check{H}^0 \left(\mathbb{P}^1, E|_{X_x}\right).$$

Taking $\begin{pmatrix} \psi \\ \phi \end{pmatrix} \in \mathbb{C}^2$ at some point x we then have a global section

$$\Psi_x = H^{-1} \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{3.1.47}$$

of $E|_{X_x}$ per $\begin{pmatrix} \psi \\ \phi \end{pmatrix}$. These extend across alpha-surfaces and define a connection on $\mathcal{V}$ by the requirement that the sections (3.1.47) be covariantly constant on alpha surfaces. The connection arising in this way is given by

$$\pi^{B'} A_{A'B'} (x, \pi_{C'}) = H^{-1} \mathcal{L}_{A'} H \ ,$$

which itself possesses weighted matrix entries (which is different to the case of a Ward transform) giving rise to connection fields in each coefficient. (We take $x^{A'B'}$ to be the four coordinates on M using the language of the Newtonian spinor indices from section 3.1.1.2 and $\mathcal{L}_{A'}$ is the weight-one Newtonian Lax pair.)

Example

Now consider taking $n = 0$ and $m = 2$. We then have that H has matrix entries

$$\begin{pmatrix} u_0 & u_{-2} \\ v_0 & v_{-2} \end{pmatrix} \qquad \text{taking values in} \qquad \begin{pmatrix} \mathcal{O} & \mathcal{O}(-2) \\ \mathcal{O} & \mathcal{O}(-2) \end{pmatrix}$$

and $H^{-1}\mathcal{L}_{A'} H$ has entries

$$\frac{1}{u_0 v_{-2} - u_{-2} v_0} \begin{pmatrix} v_{-2}\mathcal{L}_{A'} u_0 - u_{-2}\mathcal{L}_{A'} v_0 & v_{-2}\mathcal{L}_{A'} u_{-2} - u_{-2}\mathcal{L}_{A'} v_{-2} \\ u_0\mathcal{L}_{A'} v_0 - v_0\mathcal{L}_{A'} u_0 & u_0\mathcal{L}_{A'} v_{-2} - v_0\mathcal{L}_{A'} u_{-2} \end{pmatrix} \tag{3.1.48}$$

taking values in

$$\begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(-1) \\ \mathcal{O}(3) & \mathcal{O}(1) \end{pmatrix} .$$

Thus (3.1.48) gives rise to spacetime fields as

$$\begin{pmatrix} \Phi_{A'B'} (x) \, \pi^{B'} & 0 \\ \gamma_{A'B'C'D'} \pi^{B'} \pi^{C'} \pi^{D'} & \Psi_{A'B'} \pi^{B'} \end{pmatrix} .$$

Fields constructed in this way obey zero-rest-mass field equations automatically, and in particular we note that $\Phi_{A'B'}$ and $\Psi_{A'B'}$ constitute a pair of Abelian monopoles (3.1.45). (We can use these, if desired, as an alternative (and somewhat *ad hoc*) way of constructing a connection, since we recall that Newton-Cartan connections depend upon two harmonic functions.)

3.2 Newtonian twistor theory in three dimensions

There has been considerable recent interest in the application of three-dimensional Newton-Cartan geometry to non-relativistic field theory [73, 2, 6, 10]. The twistor theory of complexified three-dimensional manifolds with non-degenerate metrics is called *minitwistor* theory and is well-understood [42]. In this section we will consider the twistor theory of three-dimensional Newton-Cartan manifolds. The relevant twistor spaces are three-dimensional and will be characterised by the normal bundle to twistor lines X_x being

$$N_x = \mathcal{O} \oplus \mathcal{O}(1).$$

Families of rational curves with normal bundles isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$ have been considered in [38], where it is described that the tangent spaces of the three-dimensional moduli space come equipped with preferred one-parameter families of null rays. As we shall see below, this makes the isomorphism class $\mathcal{O} \oplus \mathcal{O}(1)$ well-suited to describing Newton-Cartan structures in three dimensions.

It is straightforward to see that

$$\check{H}^1\left(\mathbb{P}^1, N_x\right) = 0 \qquad \text{and} \qquad \check{H}^0\left(\mathbb{P}^1, N_x\right) = \mathbb{C}^3$$

and so a three-dimensional moduli space is feasible. Additionally

$$\check{H}^1\left(\mathbb{P}^1, N_x \otimes N_x^*\right) = 0 \qquad \text{and} \qquad \check{H}^0\left(\mathbb{P}^1, N_x \otimes N_x^*\right) = \mathbb{C}^4$$

so that the isomorphism class is stable; the Ξ-connection always exists and always depends on four arbitrary one-forms. Finally we see that

$$\check{H}^1\left(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*)\right) = \mathbb{C} \qquad \text{and} \qquad \check{H}^0\left(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*)\right) = \mathbb{C}^4$$

so that the Λ-connection fails to exist for some Kodaira deformations, and when it does exist it is not unique, depending on four arbitrary functions. The case in which the Λ-connection fails to exist is when the deformation introduces torsion.

We'll begin by considering the undeformed case $Z = \mathcal{O} \oplus \mathcal{O}(1)$. After calculating the canonical connections and the Galilean structure we will then proceed to deform Z.

3.2.1 The flat model

In this section we'll discuss the canonical geometry induced on the moduli space of global sections of $Z = \mathcal{O} \oplus \mathcal{O}(1)$. The patching is

$$\hat{T} = T \qquad \hat{\Omega} = \lambda^{-1}\Omega$$

and $\hat{\lambda} = \lambda^{-1}$ as usual for the base $\mathbb{P}^1$. The global sections are

$$w^\mu| = \begin{pmatrix} T| \\ \Omega| \end{pmatrix} = \begin{pmatrix} t \\ y + z\lambda \end{pmatrix}$$

for $x^a = (t, y, z) \in \mathbb{C}^3 = M$. (Recall that a vertical slash indicates the restriction to rational curves.) We'll also wish to use homogeneous coordinates $[\pi_{A'}]$ on the base and ω for the $\mathcal{O}(1)$ fibre, writing

$$\omega| = x^{A'}\pi_{A'}$$

for the global sections.

Theorem 3.2.1. [41]

Let $Z = \mathcal{O} \oplus \mathcal{O}(1)$ with global sections X_x. The moduli space $M \ni x$ of these rational curves is a complex three-dimensional manifold equipped with a family of Newton-Cartan structures parametrised by three arbitrary functions on M and an element of GL $(2, \mathbb{C})$.

Two of these functions determine the gravitational field; the other is a conformal factor for the Galilean metric. The connection is the Λ-connection of section 2.4.3.

Proof

To begin we will construct the frame induced on M using theorem 2.3.1, which is done by splitting the patching $\mathcal{F}$ for the normal bundle; we must solve

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} H = \hat{H} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for

$$H = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \qquad \text{and} \qquad \hat{H} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_3 & \hat{h}_4 \end{pmatrix}$$

as holomorphic maps from U and $\hat{U}$ to $\mathrm{GL}(2, \mathbb{C})$. This amounts to the four individual splitting problems

$$\hat{h}_1 = h_1 \qquad \hat{h}_2 = \lambda h_2$$
$$\hat{h}_3 = \lambda^{-1} h_3 \qquad \hat{h}_4 = h_4$$

whose general solution is

$$H = \begin{pmatrix} m & 0 \\ a_0 + a_1\lambda & k \end{pmatrix}$$

for four arbitrary holomorphic functions (m, k, a_0, a_1) on M constrained only by $m \neq 0$ and $k \neq 0$. The frame section can then be read off from $v = H^{-1}dw|$, giving us a clock

$$\theta = m^{-1}dt \tag{3.2.1}$$

and spatial one-forms

$$e^{0'} = k^{-1}\left(dx^{0'} - m^{-1}a_1 dt\right)$$
$$e^{1'} = k^{-1}\left(dx^{1'} - m^{-1}a_0 dt\right).$$

We thus have arrived at a natural decomposition of the tangent bundle: $TM = \mathbb{C} \oplus \mathbb{S}'$.

To proceed further we must calculate the Λ-connection on M as described in section 2.4.3. The splitting problem to be solved is

$$0 = -\hat{\sigma}^{\mu}_{\ \nu\rho}\mathcal{F}^{\nu}_{\ \alpha}\mathcal{F}^{\rho}_{\ \beta} + \mathcal{F}^{\mu}_{\ \nu}\sigma^{\nu}_{\ \alpha\beta}$$

for a 0-cochain $\{\sigma\}$ valued in $N_x \otimes (N_x^* \odot N_x^*)$ on each twistor line X_x, which amounts to

$$\hat{\sigma}^T_{TT} = \sigma^T_{TT} \qquad \hat{\sigma}^T_{T\Omega} = \lambda\sigma^T_{T\Omega} \qquad \hat{\sigma}^T_{\Omega\Omega} = \lambda^2\sigma^T_{\Omega\Omega}$$

$$\hat{\sigma}^\Omega_{TT} = \lambda^{-1}\sigma^\Omega_{TT} \qquad \hat{\sigma}^\Omega_{T\Omega} = \sigma^\Omega_{T\Omega} \qquad \hat{\sigma}^\Omega_{\Omega\Omega} = \lambda\sigma^\Omega_{\Omega\Omega}$$

and so

$$\sigma^T_{TT} = \Sigma \qquad \sigma^T_{TQ} = 0$$

$$\sigma^T_{\Omega\Omega} = 0 \qquad \sigma^\Omega_{TT} = \phi_0 + \lambda\phi_1$$

$$\sigma^\Omega_{T\Omega} = \chi \qquad \sigma^\Omega_{\Omega\Omega} = 0$$

for any four functions $(\Sigma, \chi, \phi_0, \phi_1)$ on M. The connection symbols are then

$$\Gamma^t{}_{tt} = \Sigma \qquad \Gamma^t{}_{it} = 0 \qquad \Gamma^t{}_{ij} = 0$$

$$\Gamma^y{}_{tt} = \phi_0 \qquad \Gamma^z{}_{tt} = \phi_1$$

$$\Gamma^y{}_{yt} = \chi \qquad \Gamma^y{}_{zt} = 0 \qquad \Gamma^z{}_{yt} = 0 \qquad \Gamma^z{}_{zt} = \chi$$

$$\Gamma^i{}_{jk} = 0.$$

The result is that the moduli space comes equipped with a family of connections containing gravitational forces (described by the functions ϕ_0 and ϕ_1).

The connection allows us to restrict the clock (3.2.1) by imposing $\nabla\theta = 0$, which tells us that

$$\Sigma = -\partial_t \ln m$$

and

$$\partial_{A'} m = 0\ ,$$

so that m is a function of t alone. Given that m is now just a non-vanishing function of t, we see that (3.2.1) is a standard Newton-Cartan clock, where the function m just allows for diffeomorphisms of the time axis, with Σ ensuring that upon such diffeomorphisms the clock remains parallel.

To complete the proof we must construct a family of Newton-Cartan metrics. The data already induced defines a metric as follows, by requiring the usual conditions $h(\theta, \ ) = 0$ and $\nabla h = 0$. These imply that we must have $h^{at} = 0$ and that h^{ij} obeys

$$\partial_t h^{ij} + 2\chi h^{ij} = 0$$

and

$$\partial_k h^{ij} = 0.$$

We deduce that h^{ij} must be any element of $\mathrm{GL}(2, \mathbb{C})$ multiplied by an arbitrary non-vanishing function of t, and we also have that $\chi = \chi(t)$ only. Constant non-degenerate two-by-two metrics are all equal up to (restricted) diffeomorphisms

$$y \mapsto \alpha y + \beta z \qquad z \mapsto \gamma y + \delta z$$

for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$, so we are free to take any such member $\tilde{h}^{ij}$ as our metric, giving us

$$h^{ij} = \kappa(t)\, \tilde{h}^{ij}$$

where

$$\kappa = \exp\left\{-2 \int \chi \, dt\right\}$$

is non-vanishing and determined by the arbitrary function χ.

Thus we have a family of Galilean structures (h, θ) and a family of connections ∇ spanned by a choice of three arbitrary functions, two describing gravitational forces and one (non-vanishing) specifying a conformal factor.

□

To fix a specific gravitational sector for the Newton-Cartan manifold we require, as in [22], some additional data on Z. The following theorem provides one way of specifying this data.

Theorem 3.2.2. [41]

Equip $Z = \mathcal{O} \oplus \mathcal{O}(1)$ with a 1-cocycle taking values in its canonical bundle $K \to Z$. This induces a preferred global section of $\mathcal{O}(1)$ which can be used to fix a complex Newton-Cartan *structure out of the complex Newton-Cartan structure of theorem 3.2.1, where the gravitational sector is locally of the form $\Gamma^i{}_{tt} = g^i$ for $\boldsymbol{g}$ divergence-free and determined uniquely by f.*

Proof

A simple calculation shows that $K = \mathcal{O}(-3)_Z$, and so a 1-cocycle is represented by a function f of weight minus three in homogeneous coordinates, and provides a (Serre-dual) global section of $\mathcal{O}(1)$ by

$$\phi_{A'}(x) = \frac{1}{2\pi i}\oint_\Gamma \pi_{A'} f(T|, \omega|, \pi_{B'})\, \pi \cdot d\pi$$

where $\omega| = x^{A'}\pi_{A'} = y\pi_{1'} + z\pi_{0'}$. We have

$$\frac{\partial}{\partial x^{A'}}\phi^{A'} = 0$$

automatically. Taking this to fix σ^{Ω}_{TT} we have

$$\Gamma^{A'}_{tt}\partial_{A'}\omega| = \phi^{A'}\pi_{A'}$$

and so

$$\Gamma^y_{tt} = \phi^{1'} \qquad \Gamma^z_{tt} = \phi^{0'}.$$

Thus the gravitational sector is fixed to be a unique divergence-free $\boldsymbol{g}$ given by the global $\phi^{A'}$.

□

We note that the divergence-free condition ensures that the Newton-Cartan spacetime is vacuum according to the field equations (2.6.1).

Torsion-free Ξ-connection

Theorem 3.2.1 employed the Λ-connection because it is the most powerful construction available in terms of constraining the moduli space geometry. It is interesting, however, to consider the Ξ-connection also so as to compare the flat model of theorem 3.2.1 with the torsion-inducing deformations of theorem 3.2.3 where the torsion Ξ-connection is the only connection available.

Here we'll calculate the canonical torsion-free Ξ-connection for $Z = \mathcal{O} \oplus \mathcal{O}(1)$. The calculation is straightforward; since $\partial_a \mathcal{F}^\mu_\nu = 0$ one must solve

$$0 = -\hat{\chi}^\mu_{\nu\ a}\mathcal{F}^\nu_\rho + \mathcal{F}^\mu_\nu \chi^\nu_{\rho\ a}$$

for the most general 0-cochain $\{\chi^\mu_{\nu\,a}\}$ of $N_x \otimes N^*_x \otimes \Lambda^1_x(M)$ for each $x \in M$, where

$$\mathcal{F}^\mu_\nu = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

This becomes the four individual splitting problems given by

$$\hat{\chi}^T_{T\ a} = \chi^T_{T\ a} \qquad \hat{\chi}^T_{\Omega\ a} = \lambda\chi^T_{\Omega\ a}$$

$$\hat{\chi}^\Omega_{T\ a} = \lambda^{-1}\chi^\Omega_{T\ a} \qquad \hat{\chi}^\Omega_{\Omega\ a} = \chi^\Omega_{\Omega\ a}\ ,$$

which have the general solution

$$\chi^T_{T\ a} = C_a \qquad \chi^T_{\Omega\ a} = 0$$

$$\chi^\Omega_{T\ a} = A^0_a + \lambda A^1_a \qquad \chi^\Omega_{\Omega\ a} = B_a$$

for arbitrary one-forms (A, B, C, κ) on M. We then read-off the connection from

$$\Gamma^a_{bc}\partial_a w^\mu| = \partial_b\partial_c w^\mu| + \chi^\mu_{\nu\ b}\partial_c w^\nu| + \chi^\mu_{\nu\ c}\partial_b w^\nu|$$

which here leads to

$$\Gamma^t_{tt} = 2C_t \qquad \Gamma^t_{it} = C_i \qquad \Gamma^t_{ij} = 0$$

$$\Gamma^y_{tt} = A^0_t \qquad \Gamma^z_{tt} = A^1_t$$

$$\Gamma^y_{yt} = A^0_y + B_t \qquad \Gamma^y_{zt} = A^0_z \qquad \Gamma^z_{zt} = A^1_z + B_t \qquad \Gamma^z_{yt} = A^1_y$$

$$\Gamma^k_{ij} = 2B_{(i}\delta^k_{j)}.$$

Thus the Ξ-connection comprises all connections which have $\Gamma^t_{ij} = 0$ and have flat projective structures as their spatial sectors. Compatibility with the (closed) clock $\theta = m^{-1}dt$ then imposes

$$\Gamma^t_{ab} = \delta^t_a m\partial_b\left(m^{-1}\right)$$

so (recalling that the connection is torsion-free) one must put $C_i = 0$ and

$$C_t = \frac{1}{2} m \partial_t \left(m^{-1}\right).$$

The remaining freedom in $(\Gamma^c_{ab}, \theta_a)$ is then given by three one-forms and one non-vanishing function on the time axis.

3.2.2 Deformations and torsion

A natural next step is to consider deforming the complex structure of $Z = \mathcal{O} \oplus \mathcal{O}(1)$, and a case of interest is when we write $\mathcal{O} \oplus \mathcal{O}(1)$ as a trivial affine line bundle on $\mathcal{O}(1)$ and then deform the patching for the affine line bundle so that we have

$$\hat{T} = T + f(\Omega, \lambda) \tag{3.2.2}$$

$$\hat{\Omega} = \lambda^{-1} \Omega$$

as the patching for $Z \to \mathcal{O}(1) \to \mathbb{P}^1$. The analogous deformation in four-dimensional Newtonian twistor theory [22] leads to a jump in the isomorphism class of the normal bundle to every[3] twistor line from $\mathcal{O} \oplus \mathcal{O}(2)$ to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. In the three-dimensional case this cannot be what occurs, because the isomorphism class of the normal bundle is stable.

The deformation leading to (3.2.2), when restricted to twistor lines, corresponds exactly to the part of $N_x \otimes (N_x^* \odot N_x^*)$ which causes $\check{H}^1\left(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*)\right)$ to fail to vanish. Thus we expect something to go wrong with the torsion-free affine connection on M; in fact what happens is that the connection fails to be torsion-free.

Theorem 3.2.3. [41]

Let Z be the total space of an affine line bundle on $\mathcal{O}(1)$ with trivial underlying translation bundle whose patching is

$$\hat{T} = T + f$$

[3] *Almost* every twistor line; see section 3.2.3.

where f represents a cohomology class in $\check{H}^1\left(\mathcal{O}(1), \mathcal{O}_{\mathcal{O}(1)}\right)$. The three-parameter family of global sections X_x have normal bundle $X_x = \mathcal{O} \oplus \mathcal{O}(1)$ and the moduli space M of those sections is a complex three-dimensional manifold equipped with a family of torsional Newton-Cartan structures parametrised by two arbitrary one-forms on M, two functions on M, and an element of $GL\left(2, \mathbb{C}\right)$.

Proof

The proof will proceed in stages.

1. Using theorem 2.3.1 we will construct a clock one-form θ on M depending on one non-vanishing function m on M; this clock will not be closed, meaning that it cannot be made compatible with a torsion-free connection.

2. We will then construct the torsion Ξ-connection described in section 2.4.2; its connection symbols will depend on four arbitrary one-forms $\left(A^0, A^1, B, C\right)$ on M.

3. Imposing $\nabla\theta = 0$ for the torsion Ξ-connection is then possible, and results in the fixing of C.

4. The remaining piece of data, the Newton-Cartan metric h, will then be constructed by imposing $h\left(\theta, \ \right) = 0$ and $\nabla h = 0$. This restricts the remaining one-forms by a closure condition and determines the metric up to a choice of constant two-by-two non-degenerate matrix $\tilde{h}$.

To begin we must construct the twistor functions by finding the global sections of $Z \to \mathbb{P}^1$, as these are required for theorem 2.3.1. It'll be useful to expand the representative f when restricted to sections of $\mathcal{O}(1)$. For $\Omega| = y + z\lambda$ (with y and z coordinates on M) we can write

$$f\left(\Omega|, \lambda\right) = \sum_{n=-\infty}^{\infty} \gamma_n \lambda^n$$

where $\gamma_n\left(y, z\right)$ are functions one can extract via integration. (Recall that $f\left(\Omega|, \lambda\right)$ is a function on the annulus $U \cap \hat{U}$.)

We then have

$$\hat{T} = T + \sum_{n=-\infty}^{\infty} \gamma_n \lambda^n$$

and so the twistor functions are

$$T| = t - \sum_{n=1}^{\infty} \gamma_n \lambda^n \qquad \hat{T}| = t + \sum_{n=-\infty}^{0} \gamma_n \lambda^n$$

where we have chosen to put the γ_0 term into $\hat{T}|$ (without loss of generality: we could always effect the diffeomorphism $t \mapsto t - \gamma_0$). Stage one of the proof is then to use these twistor functions to calculate a frame section via theorem 2.3.1. This involves solving the splitting problem

$$\begin{pmatrix} 1 & \frac{\partial f}{\partial \Omega}| \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_3 & \hat{h}_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

where as usual

$$H = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \qquad \text{and} \qquad \hat{H} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_3 & \hat{h}_4 \end{pmatrix}$$

constitute holomorphic maps to $\mathrm{GL}\,(2, \mathbb{C})$ from U and $\hat{U}$ respectively for each global section X_x. Expand the first derivative of f in a similar fashion to above, putting

$$\frac{\partial f}{\partial \Omega}| = \sum_{n=-\infty}^{\infty} \phi_n \lambda^n \tag{3.2.3}$$

for functions $\phi_n(y, z)$. We then have

$$\hat{h}_3 = \lambda^{-1} h_3 \qquad \Rightarrow \qquad h_3 = a + b\lambda$$

and

$$\hat{h}_4 = h_4 \qquad \Rightarrow \qquad h_4 = e$$

for functions (a, b, e) on M. The other two components of the splitting problem are then

$$\hat{h}_2 = \lambda h_2 + \lambda \left(\sum_{n=-\infty}^{\infty} \phi_n \lambda^n \right) e$$

$$\Rightarrow \qquad h_2 = -e \sum_{n=0}^{\infty} \phi_n \lambda^n$$

and

$$\hat{h}_1 = h_1 + \left(\sum_{n=-\infty}^{\infty} \phi_n \lambda^n\right)(a + b\lambda)$$

$$\Rightarrow \qquad h_1 = m - \left(\sum_{n=0}^{\infty} \phi_n \lambda^n\right)(a + b\lambda)$$

where m is a new function on M parametrising the non-uniqueness in the splitting. We have $\det H = em$ so we must impose $e \neq 0$ and $m \neq 0$. The frame section is then

$$v = H^{-1}\begin{pmatrix} dT| \\ d\Omega| \end{pmatrix} = \frac{1}{em}\begin{pmatrix} e & e\sum_{n=0}^{\infty}\phi_n\lambda^n \\ -(a+b\lambda) & m - \left(\sum_{n=0}^{\infty}\phi_n\lambda^n\right)(a+b\lambda) \end{pmatrix}\begin{pmatrix} dt - \sum_{n=1}^{\infty} d\gamma_n \lambda^n \\ dy + \lambda dz \end{pmatrix}$$

$$\Rightarrow \qquad v = \frac{1}{em}\begin{pmatrix} e\,(dt + \phi_0 dy) \\ m\,(dy + \lambda dz) - (a + b\lambda)(dt + \phi_0 dy) \end{pmatrix}$$

and the clock can be read off:

$$\theta = m^{-1}(dt + \phi_0 dy)\,. \tag{3.2.4}$$

Recall that $\phi_0 = \phi_0(y, z)$ and so for any choice of $m \neq 0$ we have that $d\theta \neq 0$ (provided that $d\phi_0 \wedge dy \neq 0$ which is generically true), suggesting that the moduli space possesses Newton-Cartan torsion. The clock (3.2.4) cannot be made compatible with any torsion-free connection as we must have

$$\nabla_b \theta_a = \partial_b \theta_a - \Gamma^c{}_{ab}\theta_c = 0$$

$$\Rightarrow \qquad \Gamma^t{}_{ab} = m\partial_b\theta_a - \Gamma^y{}_{ab}\phi_0. \tag{3.2.5}$$

If $\Gamma^c{}_{ab}$ were torsion-free then skew-symmetrising over ab in (3.2.5) would give $d\theta = 0$.

In stage two of the proof we construct the torsion Ξ-connection of section 2.4.2 with respect to which the clock can be made parallel. We must solve the splitting problem

$$\partial_b \mathcal{F}^\mu_\nu = -\hat{\rho}^\mu{}_{\alpha\, b}\mathcal{F}^\alpha_\nu + \mathcal{F}^\mu_\beta \rho^\beta{}_{\nu\, b}. \tag{3.2.6}$$

in the case

$$\partial_b \mathcal{F}^\mu_\nu = \partial_b \begin{pmatrix} 1 & \frac{\partial f}{\partial \Omega}| \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial^2 f}{\partial \Omega^2}|\left(\delta^y_b + \delta^z_b \lambda\right) \\ 0 & 0 \end{pmatrix}.$$

Equation (3.2.6) constitutes four coupled splitting problems; we must find the global sections of the following four patchings:

$$\hat{\rho}^{T}_{T\,b} = \rho^{T}_{T\,b} + \frac{\partial f}{\partial \Omega}| \rho^{\Omega}_{T\,b} \;\; ; \tag{3.2.7}$$

$$\hat{\rho}^{T}_{\Omega\,b} = \lambda \rho^{T}_{\Omega\,b} + \lambda \frac{\partial f}{\partial \Omega}| \left(\rho^{\Omega}_{\Omega\,b} - \hat{\rho}^{T}_{T\,b} \right) - \lambda \frac{\partial^2 f}{\partial \Omega^2}| \left(\delta^y_b + \delta^z_b \lambda \right) \;\; ; \tag{3.2.8}$$

$$\hat{\rho}^{\Omega}_{T\,b} = \lambda^{-1} \rho^{\Omega}_{T\,b} \;\; ; \tag{3.2.9}$$

$$\hat{\rho}^{\Omega}_{\Omega\,b} = \rho^{\Omega}_{\Omega\,b} - \lambda \hat{\rho}^{\Omega}_{T\,b} \frac{\partial f}{\partial \Omega}| \;\; . \tag{3.2.10}$$

Equations (3.2.7), (3.2.9), and (3.2.10) are immediately tractable if we again make use of the expansion (3.2.3). Their most general global sections are given by

$$\rho^{\Omega}_{T\,b} = A^0_b + A^1_b \lambda \qquad \hat{\rho}^{\Omega}_{T\,b} = \hat{\lambda} A^0_b + A^1_b$$

$$\rho^{T}_{T\,b} = C_b - \sum_{n=0}^{\infty} \phi_n \lambda^n A^0_b - \sum_{n=-1}^{\infty} \phi_n \lambda^{n+1} A^1_b$$

$$\hat{\rho}^{T}_{T\,b} = C_b + \sum_{n=1}^{\infty} \phi_{-n} \hat{\lambda}^n A^0_b + \sum_{n=2}^{\infty} \phi_{-n} \hat{\lambda}^{n-1} A^1_b$$

$$\rho^{\Omega}_{\Omega\,b} = B_b + \sum_{n=1}^{\infty} \phi_n \lambda^n A^0_b + \sum_{n=0}^{\infty} \phi_n \lambda^{n+1} A^1_b$$

$$\hat{\rho}^{\Omega}_{\Omega\,b} = B_b - \sum_{n=0}^{\infty} \phi_{-n} \hat{\lambda}^n A^0_b - \sum_{n=1}^{\infty} \phi_{-n} \hat{\lambda}^{n-1} A^1_b$$

where (A^0, A^1, B, C) are arbitrary one-forms on M carrying the non-uniqueness in the splitting. The remaining equation (3.2.8) is, after a little work, given by

$$\begin{aligned}
\hat{\rho}^{T}_{\Omega\,b} = \lambda \rho^{T}_{\Omega\,b} &+ \sum_{m=-\infty}^{\infty} \phi_m \lambda^{m+1} \left(B_b - C_b \right) \\
&+ \sum_{m=-\infty}^{\infty} \left(\left[\sum_{n=1}^{\infty} - \sum_{n=-\infty}^{-1} \right] \phi_n \phi_m \lambda^{n+m+1} A^0_b + \left[\sum_{n=0}^{\infty} - \sum_{n=-\infty}^{-2} \right] \phi_n \phi_m \lambda^{m+n+2} A^1_b \right) \\
&\qquad - \lambda \frac{\partial^2 f}{\partial \Omega^2}| \left(\delta^y_b + \delta^z_b \lambda \right) .
\end{aligned}$$

This equation is the patching for an affine line bundle on $\mathbb{P}^1$ with underlying translation bundle $\mathcal{O}(-1)$ (for each direction x^b on M) and hence always has a unique solution.

If we expand the second derivative of f such that

$$\frac{\partial^2 f}{\partial \Omega^2}| = \sum_{m=-\infty}^{\infty} \psi_m \lambda^m$$

then (after a calculation) we can write the solution to the splitting problem as

$$\rho^T_{\Omega\, b} = -\sum_{m=0}^{\infty} \phi_m \lambda^m \left(B_b - C_b\right) - \sum_{k=1}^{\infty} \lambda^{k-1} \mathcal{W}_{k\, b} + \sum_{m=0}^{\infty} \left(\delta^y_b \psi_m + \delta^z_b \psi_{m-1}\right) \lambda^m$$

$$\hat{\rho}^T_{\Omega\, b} = \sum_{m=1}^{\infty} \phi_{-m} \hat{\lambda}^{m-1} \left(B_b - C_b\right) + \sum_{k=0}^{\infty} \hat{\lambda}^k \mathcal{W}_{-k\, b} - \sum_{m=1}^{\infty} \left(\delta^y_b \psi_{-m} + \delta^z_b \psi_{-(m+1)}\right) \hat{\lambda}^{m-1}$$

where for convenience we define

$$\mathcal{W}_{k\, b} = \sum_{n=1}^{\infty} \left[A^0_b \left(\phi_n \phi_{k-1-n} - \phi_{-n} \phi_{k-1+n}\right) + A^1_b \left(\phi_n \phi_{k-2-n} - \phi_{-n} \phi_{k-2+n}\right)\right] \\ + A^1_b \left(\phi_0 \phi_{k-2} + \phi_{-1} \phi_{k-1}\right).$$

Having solved the splitting problem it is straightforward to extract $\Gamma^c{}_{ab}$ from (2.4.9); we have

$$\Gamma^c{}_{ab} \left(\delta^t_c - \sum_{n=1}^{\infty} \left(\partial_c \gamma_n\right) \lambda^n\right) = -\sum_{n=1}^{\infty} \left(\partial_a \partial_b \gamma_n\right) \lambda^n + \rho^T_{T\, b} \left(\delta^t_a - \sum_{n=1}^{\infty} \left(\partial_a \gamma_n\right) \lambda^n\right) + \rho^T_{\Omega\, b} \left(\delta^y_a + \delta^z_a \lambda\right)$$

and

$$\Gamma^c{}_{ab} \left(\delta^y_c + \delta^z_c \lambda\right) = \rho^\Omega_{T\, b} \left(\delta^t_a - \sum_{n=1}^{\infty} \left(\partial_a \gamma_n\right) \lambda^n\right) + \rho^\Omega_{\Omega\, b} \left(\delta^y_a + \delta^z_a \lambda\right)$$

from which we can read off

$$\Gamma^y{}_{ab} = \delta^t_a \left[\rho^\Omega_{T\, b}\right]_0 + \delta^y_a \left[\rho^\Omega_{\Omega\, b}\right]_0$$

$$\Gamma^z{}_{ab} = \delta^t_a \left[\rho^\Omega_{T\, b}\right]_1 - \left(\partial_a \gamma_1\right) \left[\rho^\Omega_{T\, b}\right]_0 + \delta^y_a \left[\rho^\Omega_{\Omega\, b}\right]_1 + \delta^z_a \left[\rho^\Omega_{\Omega\, b}\right]_0$$

$$\Gamma^t{}_{ab} = \delta^t_a \left[\rho^T_{T\, b}\right]_0 + \delta^y_a \left[\rho^T_{\Omega\, b}\right]_0$$

where we adopt the notation $[\rho^\mu{}_{\nu\, b}]_n$ for the coefficient of λ^n in $\rho^\mu{}_{\nu\, b}$. The Christoffel symbols hence can be written

$$\Gamma^y{}_{ab} = \delta^t_a A^0_b + \delta^y_a B_b$$

$$\Gamma^z{}_{ab} = \delta^t_a A^1_b + \delta^y_a \phi_0 A^1_b + \delta^z_a \left(B_b - \phi_0 A^0_b\right)$$

$$\Gamma^{t}{}_{ab} = \delta^t_a \left(C_b - \phi_0 A^0_b - \phi_{-1} A^1_b\right) + \delta^y_a \left(-\phi_0 \left(B_b - C_b\right) - A^1_b \phi_0 \phi_{-1} + \delta^y_b \psi_0 + \delta^z_b \psi_{-1}\right).$$

This is the torsion Ξ-connection, a family of connections parametrised by four arbitrary one-forms (A^0, A^1, B, C) on M which generically possess torsion arising from the second derivative of f via the ψ_n terms in $\Gamma^{t}{}_{ab}$. For example,

$$\Gamma^{t}{}_{[yz]} = \frac{1}{2}\left(-\phi_0 \left(B_z - C_z\right) - A^1_z \phi_0 \phi_{-1} + \psi_{-1}\right)$$

cannot be set to zero by a global choice of (A^0, A^1, B, C) provided $\psi_{-1} \neq 0$ and provided ϕ_0 has vanishing points, which is generically the case.

It is precisely the presence of this torsion which allows the above connection to be made compatible with the clock (3.2.4) in stage three of the proof. To carry out this stage we impose $\nabla\theta = 0$ for the torsion Ξ-connection.

$$\nabla\theta = 0 \qquad \Rightarrow \qquad \Gamma^{t}{}_{ab} = m\partial_b\theta_a - \Gamma^{y}{}_{ab}\phi_0$$

which results in the one-form C being fixed to be

$$C_b = m\partial_b \left(m^{-1}\right) + \phi_{-1} A^1_b$$

which simplifies $\Gamma^{t}{}_{ab}$ to

$$\Gamma^{t}{}_{ab} = \delta^t_a \left(m\partial_b \left(m^{-1}\right) - \phi_0 A^0_b\right) + \delta^y_a \left(\phi_0 m \partial_b \left(m^{-1}\right) - \phi_0 B_b + \delta^y_b \psi_0 + \delta^z_b \psi_{-1}\right).$$

Thus the moduli space comes equipped with a family of compatible connections and clocks with torsion parametrised by three arbitrary one-forms (A^0, A^1, B) and one non-vanishing function m.

In stage four the construction is completed with the calculation of a family of Newton-Cartan metrics compatible with the connections and whose kernels are spanned by the clock. The latter condition, $h\left(\theta, \ \right) = 0$, requires

$$h^{at} + h^{ay}\phi_0 = 0 \tag{3.2.11}$$

so it is only necessary to calculate the spatial components h^{ij} of the metric; one can always reconstruct the other components by factors of $-\phi_0$. Moving to the compatibility with the connection,

$$\nabla h = 0 \qquad \Rightarrow \qquad \partial_b h^{ij} + 2h^{ij}\left(B_b - \phi_0 A^0_b\right) = 0. \tag{3.2.12}$$

The Frobenius theorem (see, for example, [21]) tells us that there exists a unique solution $h^{ij}\left(x^b\right)$ for given initial data $h^{ij}\left(x^b_0\right)$ iff the one-form $B - \phi_0 A^0$ is closed. Thus we may henceforth consider A^0 to be free and B to be constrained to be given by

$$B = \phi_0 A^0 + d\mathcal{B} \tag{3.2.13}$$

for an arbitrary function $\mathcal{B}$ on M. (We assume the first De Rham cohomology of the relevant domain in M is trivial; otherwise (3.2.13) is reduced to a local statement.) Equation (3.2.12) then has solutions

$$h^{ij} = \tilde{h}^{ij} \exp\left\{-2\int \left(B_b - \phi_0 A^0_b\right)\, dx^b\right\} = \tilde{h}^{ij}\exp\left\{-2\mathcal{B}\right\}$$

where $\tilde{h}^{ij}$ are constants, and for the purposes of constructing a metric we may choose any $\tilde{h}^{ij} \in \mathrm{GL}\left(2,\mathbb{C}\right)$. Recall that the h^{at} components can then be found from (3.2.11). The final form of the connection is then

$$\Gamma^y{}_{ab} = \delta^t_a A^0_b + \delta^y_a \phi_0 A^0_b + \delta^y_a \partial_b \mathcal{B}$$

$$\Gamma^z{}_{ab} = \delta^t_a A^1_b + \delta^y_a \phi_0 A^1_b + \delta^z_a \partial_b \mathcal{B}$$

$$\Gamma^t{}_{ab} = \left(\delta^t_a + \delta^y_a\phi_0\right)\left(m\partial_b\left(m^{-1}\right) - \phi_0 A^0_b\right) - \delta^y_a\phi_0\partial_b\mathcal{B} + \delta^y_a\delta^y_b\psi_0 + \delta^y_a\delta^z_b\psi_{-1}.$$

This completes the construction of the Newton-Cartan structures; the free data consists of two one-forms (A^0, A^1), two functions $(\mathcal{B}, m)$ (subject to $m \neq 0$), and a choice of $\tilde{h}^{ij} \in \mathrm{GL}\left(2,\mathbb{C}\right)$. $\square$

Note that, compared to the case of theorem 3.2.1, we have a larger family of Newton-Cartan structures (depending on more arbitrary degrees of freedom). This is because we had to use the torsion Ξ-connection rather than the more powerful torsion-free Λ-connection. It would be pleasing to be able to construct an analogue of the Λ-connection which allows torsion; we defer such prospects to future investigations.

3.2.3 On jumping hypersurfaces of Gibbons-Hawking manifolds

We end this section with a tangential result, in which three-dimensional torsional Newton-Cartan structures arise on certain hypersurfaces of Gibbons-Hawking manifolds. Recall that a Penrose twistor space is in the Gibbons-Hawking class if it admits a fibration over $\mathcal{O}(2)$ [35, 77].

Theorem 3.2.4. [41]

Let $Z \to \mathbb{P}^1$ be a twistor space in the Gibbons-Hawking class and let (M, g) be its associated moduli space with

$$g = V^{-1}\left(dt + A\right)^2 + V\left(dz^2 + d\xi d\tilde{\xi}\right)$$

where the Gibbons-Hawking potential V satisfies $dV = \star^3 dA$. On twistor lines satisfying $V = 0$ the normal bundle is $\mathcal{O} \oplus \mathcal{O}(2)$ and the twistor-induced local geometry is that of a (generically-torsional) $(2+1)$-dimensional Newton-Cartan spacetime, provided that the restriction to $V = 0$ of the flat three-metric $dz^2 + d\xi d\tilde{\xi}$ is of rank two.

Proof

Consider first the isomorphism class of the normal bundle to twistor lines. The patching for the normal bundle is

$$\mathcal{F} = \begin{pmatrix} 1 & \frac{\partial f}{\partial Q}| \\ 0 & \lambda^{-2} \end{pmatrix}$$

and we can make an expansion

$$\frac{\partial f}{\partial Q}| = \sum_{n=-\infty}^{\infty} \gamma_n \lambda^n.$$

(The intersection $U \cap \hat{U} \subset \mathbb{P}^1$ is an annulus so this is always possible.) The splitting problem is

$$\begin{pmatrix} 1 & \frac{\partial f}{\partial Q}| \\ 0 & \lambda^{-2} \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_3 & \hat{h}_4 \end{pmatrix} \begin{pmatrix} \lambda^{m-2} & 0 \\ 0 & \lambda^{-m} \end{pmatrix} \tag{3.2.14}$$

and if there exists a holomorphic solution to this for some m on some line X_x then the normal bundle to that line is $\mathcal{O}(2-m) \oplus \mathcal{O}(m)$. We're interested in the exact form of H on the set of twistor lines described by $V = 0$, which is when $\gamma_{-1} = 0$ and where we can solve the splitting problem for $m = 2$. Henceforth assume that $\frac{\partial f}{\partial Q}|$ is restricted to $\gamma_{-1} = 0$.

$$\hat{h}_1 = h_1 + \frac{\partial f}{\partial Q}| h_3$$

$$\hat{h}_2 = \lambda^2 h_2 + \lambda^2 \frac{\partial f}{\partial Q}|h_4$$

$$\hat{h}_3 = \lambda^{-2} h_3$$

$$\hat{h}_4 = h_4.$$

So

$$h_4 = b_0 \qquad h_3 = a_0 + a_1\lambda + a_2\lambda^2$$

for four functions (a_0, a_1, a_2, b_0) on M and

$$h_2 = -b_0 \sum_{n=0}^{\infty} \gamma_n \lambda^n \qquad h_1 = c_0 - a_0 \sum_{n=1}^{\infty} \gamma_n \lambda^n - a_1 \sum_{n=0}^{\infty} \gamma_n \lambda^{n+1} - a_2 \sum_{n=0}^{\infty} \gamma_n \lambda^{n+2}.$$

Then

$$\det H = \det \hat{H} = b_0 c_0.$$

One can now set

$$a_0 = a_1 = a_2 = 0$$

and

$$b_0 = c_0 = 1$$

to get a simple solution to (3.2.14). The twistor functions are

$$Q| = \xi\lambda^2 - 2z\lambda - \tilde{\xi} \qquad \text{for } x^i = \left(\xi, \tilde{\xi}, z\right) \in \mathbb{C}^3$$

and

$$T| = t - h\left(x^i, \lambda\right)$$

where

$$f| = h - \hat{h}$$

for h and $\hat{h}$ holomorphic on U and $\hat{U}$ respectively.

Now let $w = \begin{pmatrix} T \\ Q \end{pmatrix}$ and $\hat{w} = \begin{pmatrix} \hat{T} \\ \hat{Q} \end{pmatrix}$;

$$d\hat{w}| = \mathcal{F} dw|$$

$$\Rightarrow \qquad \hat{H}^{-1}d\hat{w}| = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} H^{-1}dw|$$

is a global section of $N \otimes \Lambda^1(M)$ which determines the frame $\left(\theta, e^{A'B'}\right)$ for the moduli space.

$$H^{-1}dw| = \begin{pmatrix} 1 & \sum_{n=0}^{\infty} \gamma_n \lambda^n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dt - dh \\ d\xi\lambda^2 - 2dz\lambda - d\tilde{\xi} \end{pmatrix}$$

$$\Rightarrow \qquad \begin{pmatrix} \theta \\ e^{0'0'}\lambda^2 - 2e^{0'1'}\lambda - e^{1'1'} \end{pmatrix} = \begin{pmatrix} dt - dh + \left(\sum_{n=0}^{\infty} \gamma_n \lambda^n\right)\left[d\xi\lambda^2 - 2dz\lambda - d\tilde{\xi}\right] \\ d\xi\lambda^2 - 2dz\lambda - d\tilde{\xi} \end{pmatrix}$$

Now consider

$$df| = dh - d\hat{h} = dQ| \sum_{n=-\infty}^{\infty} \gamma_n \lambda^n.$$

$$\Rightarrow \qquad dh = \left[d\xi \sum_{n=0}^{\infty} \gamma_n \lambda^{n+2} - 2dz \sum_{n=0}^{\infty} \gamma_n \lambda^{n+1} - d\tilde{\xi} \sum_{n=1}^{\infty} \gamma_n \lambda^n\right] + \alpha \left[\gamma_{-2} d\xi + \gamma_0 d\tilde{\xi}\right]$$

for any choice of α (parametrising how we choose to share this term between dh and $d\hat{h}$).

The spatial part of the frame defines a conformal structure in the two remaining spatial dimensions provided that the rank of the restriction of $dz^2 + d\xi d\tilde{\xi}$ to $V = 0$ is of rank two, as required in the theorem.

We can now extract the clock:

$$\theta = dt - \alpha\gamma_{-2} d\xi - (1-\alpha)\gamma_0 d\tilde{\xi} \tag{3.2.15}$$

and the triad is the standard flat triad $\left(d\xi, dz, d\hat{\xi}\right)$ restricted to $V = 0$. Choose $\alpha = \frac{1}{2}$ for definiteness. The torsion of the Newton-Cartan connection is determined by

$$d\theta = -\frac{1}{2} d\gamma_{-2} \wedge d\xi - \frac{1}{2} d\gamma_0 \wedge d\hat{\xi}$$

which does not generically vanish. □

Note that the torsion originates from A.

3.3 Newtonian twistor theory in five dimensions

In five dimensions the twistor spaces will be four-dimensional, and the normal bundle to twistor lines will generically be

$$N_x = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1).$$

This is appropriate for a five-dimensional moduli space because

$$\check{H}^0\left(\mathbb{P}^1, N_x\right) = \mathbb{C}^5$$

and in the following section we will show that in the flat model the moduli space comes equipped with a Newton-Cartan structure.

As in three dimensions we have that

$$\check{H}^1\left(\mathbb{P}^1, \text{End}\left(N_x\right)\right) = 0$$

so the isomorphism class is stable with respect to deformations of the complex structure of the twistor space; the consequences of such deformations will be discussed in section 3.3.2.

3.3.1 The flat model

We'll begin by constructing the geometry on the moduli space for the undeformed case $Z = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$, equipped with its canonical Λ-connection.

Theorem 3.3.1. [41]

Let $Z = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$. The induced geometry on the complex five-dimensional moduli space M of global sections of $Z \to \mathbb{P}^1$ is a family of Newton-Cartan structures (h, θ, ∇), where the connection components of ∇ depend upon a choice of one conformal factor and seven arbitrary functions,

- *four of which are the Newtonian gravitational force Γ^i_{tt};*
- *and the remaining three of form an anti-self-dual spatial two-form W_{ij} describing Coriolis forces.*

Proof

The normal bundle to twistor lines is $N_x = \mathcal{O}\oplus\mathcal{O}(1)\oplus\mathcal{O}(1)$, which satisfies $\check{H}^1\left(\mathbb{P}^1, N_x\right) = 0$. By the Kodaira theorem 2.1.4 the moduli space M is a complex manifold, with dimension dim$\check{H}^0\left(\mathbb{P}^1, N_x\right) = 5$.

To construct the conformal Galilean structure (h, θ) we first find the twistor lines explicitly. In homogeneous coordinates the patching is

$$\hat{T} = T \qquad \text{(weight zero)}$$

and

$$\hat{\omega}^A = \omega^A \qquad \text{(weight one)},$$

with twistor lines

$$T| = t \qquad \text{and} \qquad \omega^A| = x^{AA'}\pi_{A'}$$

for coordinates $\left(t, x^{AA'}\right)$ on M. To find the frame (via theorem 2.3.1) we need to solve

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 & \hat{h}_3 \\ \hat{h}_4 & \hat{h}_5 & \hat{h}_6 \\ \hat{h}_7 & \hat{h}_8 & \hat{h}_9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

for the most general

$$H : U \to \mathrm{GL}\left(3, \mathbb{C}\right) \quad \text{and} \quad \hat{H} : \hat{U} \to \mathrm{GL}\left(3, \mathbb{C}\right).$$

The solution is

$$H^{-1} = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ \frac{1}{mk}\sum_{p=0}^{1}\left(k_2 b_p - k_4 a_p\right)\lambda^p & \frac{k_4}{k} & -\frac{k_2}{k} \\ \frac{1}{mk}\sum_{p=0}^{1}\left(k_3 a_p - k_1 b_p\right)\lambda^p & -\frac{k_3}{k} & \frac{k_1}{k} \end{pmatrix}.$$

for nine arbitrary functions $(m, a_0, a_1, b_0, b_1, k_1, k_2, k_3, k_4)$ on M which must be chosen such that $m \neq 0$ and $k := (k_1 k_4 - k_2 k_3) \neq 0$ anywhere. These functions will parametrise the family of induced structures on the moduli space. The frame section is therefore given by

$$v = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ \frac{1}{mk}\sum_{p=0}^{1}\left(k_2 b_p - k_4 a_p\right)\lambda^p & \frac{k_4}{k} & -\frac{k_2}{k} \\ \frac{1}{mk}\sum_{p=0}^{1}\left(k_3 a_p - k_1 b_p\right)\lambda^p & -\frac{k_3}{k} & \frac{k_1}{k} \end{pmatrix} \begin{pmatrix} dt \\ dx^{01'} + dx^{00'}\lambda \\ dx^{11'} + dx^{10'}\lambda \end{pmatrix}$$

from which we can read off the frame

$$\begin{aligned}
\theta &= m^{-1}\,dt \\
e^{00'} &= \frac{k_4}{k}dx^{00'} - \frac{k_2}{k}dx^{10'} + \frac{1}{mk}\left(k_2 b_1 - k_4 a_1\right)dt \\
e^{01'} &= \frac{k_4}{k}dx^{01'} - \frac{k_2}{k}dx^{11'} + \frac{1}{mk}\left(k_2 b_0 - k_4 a_0\right)dt \\
e^{10'} &= -\frac{k_3}{k}dx^{00'} + \frac{k_1}{k}dx^{10'} + \frac{1}{mk}\left(k_3 a_1 - k_1 b_1\right)dt \\
e^{11'} &= -\frac{k_3}{k}dx^{01'} + \frac{k_1}{k}dx^{11'} + \frac{1}{mk}\left(k_3 a_0 - k_1 b_0\right)dt\,,
\end{aligned}$$

leading to the natural decomposition of the tangent bundle

$$TM = \mathbb{C} \oplus (\mathbb{S} \otimes \mathbb{S}')\,.$$

The clock is therefore $\theta = m^{-1}\,dt$, and the *covariant* metric is

$$h^{-1} = \epsilon_{AB}\epsilon_{A'B'}e^{AA'} \otimes e^{BB'}$$

$$\begin{aligned}
\Rightarrow \qquad h^{-1} = k^{-1}&\left(dx^{00'}dx^{11'} - dx^{10'}dx^{01'}\right) \\
&+ m^{-1}k^{-1}dt\left(-b_0 dx^{00'} + b_1 dx^{01'} + a_0 dx^{10'} - a_1 dx^{11'}\right) \\
&\qquad\qquad + m^{-2}k^{-1}\left(a_1 b_0 - a_0 b_1\right)dt^2
\end{aligned}$$

The metric is rank-four, as one would expect for a five-dimensional Newton-Cartan space-time. The contravariant metric is obtained as the projective inverse in the following way. First find a vector U such that

$$\theta(U) = 1 \qquad \text{and} \qquad h^{-1}(U,\ ) = 0\ ,$$

which uniquely determines

$$U = m\partial_t + b_0\partial_{11'} + b_1\partial_{10'} + a_0\partial_{01'} + a_1\partial_{00'}.$$

The contravariant metric h is then the unique solution to

$$h^{ab}h_{bc} = \delta^a_c - U^a\theta_c \qquad \text{and} \qquad h(\theta,\ ) = 0\,,$$

which determines

$$h = k^{-1}\epsilon^{AB}\epsilon^{A'B'}\frac{\partial}{\partial x^{AA'}} \otimes \frac{\partial}{\partial x^{BB'}}.$$

Thus we have a Galilean structure (M, h, θ) depending on arbitrary functions. (We could also equivalently have used the more traditional twistor theory method of calculating the null vectors from the twistor functions as was done for the case of four dimensions in [22].)

It only remains to calculate the physical induced connection, i.e. we must construct the Λ-connection. Denote by

$$\hat{w}^\mu = \begin{pmatrix} \hat{T} \\ {}^{\hat{\omega}^A}\!/\!_{\pi_{0'}} \end{pmatrix} \quad \text{and} \quad w^\mu = \begin{pmatrix} T \\ {}^{\omega^A}\!/\!_{\pi_{1'}} \end{pmatrix} \tag{3.3.1}$$

column vectors of inhomogeneous twistor coordinates on the fibres, and for ease of notation set

$$x^{AA'} = x^i = \begin{pmatrix} v & u \\ y & x \end{pmatrix}. \tag{3.3.2}$$

Following the discussion in section 2.4.3 the construction of ∇ occurs in two stages, the first being the solution of the splitting problem

$$\mathcal{F}^\mu_{\nu\rho} = -\hat{\sigma}^\mu_{\ \alpha\beta}\mathcal{F}^\alpha_\nu\mathcal{F}^\beta_\rho + \mathcal{F}^\mu_\gamma\sigma^\gamma_{\ \nu\rho} \tag{3.3.3}$$

for a 0-cochain $\{\sigma\}$ of $N \otimes (N^* \odot N^*) \to \mathbb{P}^1$, where

$$\mathcal{F}^\alpha_\nu = \frac{\partial \hat{w}^\alpha}{\partial w^\nu}| \quad \text{and} \quad \mathcal{F}^\mu_{\nu\rho} = \frac{\partial^2 \hat{w}^\mu}{\partial w^\nu \partial w^\rho}|.$$

The solution of (3.3.3) depends on nine arbitrary functions (on M) because

$$\check{H}^0\left(\mathbb{P}^1, N \otimes (N^* \odot N^*)\right) = \mathbb{C}^9 \ ,$$

and is explicitly given by

$$\hat{\sigma}^T_{\ AB} = \sigma^T_{\ AB} = 0 \qquad \hat{\sigma}^T_{\ AT} = \sigma^T_{\ AT} = 0 \qquad \hat{\sigma}^A_{\ BC} = \sigma^A_{\ BC} = 0$$

$$\hat{\sigma}^T_{\ TT} = \sigma^T_{\ TT} = \Sigma \qquad \hat{\sigma}^A_{\ BT} = \sigma^A_{\ BT} = \chi^A_B$$

$$\hat{\sigma}^A_{TT} = \hat{\lambda}\phi^A + \psi^A \qquad \sigma^A_{TT} = \phi^A + \lambda\psi^A.$$

(The nine functions are Σ, ϕ^A, ψ^A, and χ^A_B.)

The second stage of the construction is the reading-off of Γ^a_{bc} from the map $T^{[2]}M \to TM$ determined by $\{\sigma\}$ via the Kodaira isomorphism $TM = \check{H}^0(\mathbb{P}^1, N)$. Concretely we read off $\Gamma^a_{bc}(x^d)$ from

$$\Gamma^a_{bc}\partial_a w^\mu| = \partial_b\partial_c w^\mu| + \sigma^\mu_{\nu\rho}\partial_b w^\nu|\partial_c w^\rho| \ ,$$

giving us

$$\Gamma^t_{tt} = \Sigma \tag{3.3.4}$$

$$\Gamma^u_{tt} = \phi^0 \quad \Gamma^v_{tt} = \psi^0 \quad \Gamma^x_{tt} = \phi^1 \quad \Gamma^y_{tt} = \psi^1$$

$$\Gamma^u_{ut} = \Gamma^v_{vt} = \chi^0_0 \quad \Gamma^x_{xt} = \Gamma^y_{yt} = \chi^1_1$$

$$\Gamma^u_{xt} = \Gamma^v_{yt} = \chi^0_1 \quad \Gamma^x_{ut} = \Gamma^y_{vt} = \chi^1_0 \tag{3.3.5}$$

with all other components of Γ^a_{bc} vanishing. Two of these functions are related to ones we already have by the compatibility conditions

$$\nabla\theta = 0 \qquad \nabla h = 0 \ ,$$

which give us

$$\Sigma = -\partial_t \ln m \quad \text{and} \quad \text{tr}(\chi) = \chi^0_0 + \chi^1_1 = -\frac{1}{2}\partial_t \ln k \ ,$$

as well as

$$\frac{\partial m}{\partial x^{AA'}} = \frac{\partial k}{\partial x^{AA'}} = 0$$

so these two factors are functions of time only. The function m can be set to one without loss of generality by a diffeomorphism of the time axis.

The four components Γ^i_{tt} are completely arbitrary (given in terms of ϕ^A and ψ^A), whilst the remaining Γ^i_{jt} components depend on three arbitrary functions from the traceless part of χ^A_B. Thus we get only three functions' worth of Γ^i_{jt} instead of the most general case depending on six functions. Given that this is twistor theory it is perhaps no surprise that the three functions form an anti-self-dual two-form on spatial fibres. Concretely we have

$$\Gamma^i_{jt} = \delta^{ik} W_{jk}$$

with

$$W = \left(\chi^0_{\ 0} - \chi^1_{\ 1}\right)[du \wedge dy + dx \wedge dv] + 2\chi^0_{\ 1}[dx \wedge dy] + 2\chi^1_{\ 0}[dv \wedge du]\,. \tag{3.3.6}$$

It is then straightforward to check that (3.3.6) is the most general anti-self-dual two-form on spatial fibres with respect to a volume-form

$$\epsilon_{\text{space}} = dx \wedge dy \wedge dv \wedge du\ ,$$

completing the proof.

□

We thus conclude that Newtonian twistor theory in five dimensions admits generalised Coriolis connection components as arbitrary functions in its Λ-connection, but only *half* of them.

Ξ-connection for $Z = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$

As described in section 2.4.1 the calculation amounts to taking a global section $\chi^{\mu}_{\ \nu\, a}$ of $N_x \otimes N^*_x \otimes \Lambda^1_x(M)$ per point $x \in M$. For $N_x = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ we have

$$N_x \otimes N^*_x = \begin{pmatrix} \mathcal{O} & \mathcal{O}(-1) & \mathcal{O}(-1) \\ \mathcal{O}(1) & \mathcal{O} & \mathcal{O} \\ \mathcal{O}(1) & \mathcal{O} & \mathcal{O} \end{pmatrix}$$

giving us

$$\chi^T_{\ T\,a} = C_a \qquad \chi^T_{\ A\,a} = 0 \qquad \chi^A_{\ T\,a} = A^{AA'}_a \pi_{A'}$$

$$\chi^A_{\ B\,a} = B^A_{\ B\,a}$$

for nine arbitrary one-forms $\left(C, A^{AA'}, B^A_B\right)$ on M constituting forty-five arbitrary functions. We extract the connection symbols by reading off from

$$\Gamma^a_{bc}\partial_a w^\mu| = \partial_b\partial_c w^\mu| + \chi^\mu_{\ \nu\,b}\partial_c w^\nu| + \chi^\mu_{\ \nu\,c}\partial_b w^\nu|\ ,$$

which gives us

$$\Gamma^{t}_{bc} = \delta^{t}_{b} C_c + \delta^{t}_{c} C_b$$

$$\Gamma^{AA'}_{tt} \pi_{A'} = 2 A^{AB'}_{t} \pi_{B'}$$

$$\Gamma^{AA'}_{BB't} \pi_{A'} = A^{AC'}_{BB'} \pi_{C'} + B^{A}_{B\,t} \pi_{B'}$$

$$\Gamma^{AA'}_{BB'CC'} \pi_{A'} = B^{A}_{C\,BB'} \pi_{C'} + B^{A}_{B\,CC'} \pi_{B'}.$$

The Ξ-connection therefore contains every connection which has $\Gamma^{t}_{ij} = 0$ and for which the four-dimensional spatial sector resembles that of the Ξ-connection (2.4.6) for $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

3.3.2 Deformations and torsion

In this section we'll study deformations of the form

$$\hat{T} = T + \epsilon f \left(\Omega^A, \lambda \right) \tag{3.3.7}$$

over the total space of $\mathcal{O}(1) \oplus \mathcal{O}(1)$, where ϵ is a deformation parameter. Of course, from one point of view this is nothing more than a Ward bundle on the twistor space for flat four-dimensional spacetime [80, 78]. The approach adopted in this paper is instead to study the geometry of the full five-dimensional moduli space of global sections.

Given that the normal bundle $N_x = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ is stable with respect to all Kodaira deformations we must therefore possess a five-dimensional Galilean structure, but the connection is more subtle. $\check{H}^1 \left(\mathbb{P}^1, N_x \otimes \left(N_x^* \odot N_x^* \right) \right) \neq 0$, so some deformations will result in a moduli space which does *not* possess a Λ-connection. Like in the three-dimensional case, what is going wrong is that a Λ-connection is, by construction, torsion-free, and deformations of the form (3.3.7) give rise to moduli spaces whose Newton-Cartan structures possess torsion.

Theorem 3.3.2. [41]

Let Z be a complex four-fold fibred over $\mathbb{P}^1$ with patching given by (3.3.7) whose five-parameter family of global sections X_x have normal bundle $X_x = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$. The moduli space M

of those sections is a complex five-dimensional manifold equipped with a Galilean structure with torsion whose clock admits a representative with

$$d\theta = \epsilon \left\{ \frac{1}{2\pi i} \oint \frac{\partial^2 f}{\partial \omega^A \partial \omega^B} | \frac{\pi_{B'}}{\pi_{0'}} \pi \cdot d\pi \right\} dx^{BB'} \wedge dx^{A1'}.$$

(Recall that ω^A are the homogeneous versions of Ω^A.)

Proof

Take the global sections of the base $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to be $x^{AA'}\pi_{A'}$ as in (3.3.2), or in inhomogeneous coordinates

$$\Omega^0| = u + v\lambda \qquad \Omega^1| = x + y\lambda.$$

Now restrict f to these lines and expand it in a Laurent series in λ;

$$f| = \sum_{n=-\infty}^{\infty} \gamma_n \lambda^n \qquad \text{for} \qquad \gamma_n = \frac{1}{2\pi i} \oint f\left(\Omega^A|, \lambda\right) \lambda^{-(1+n)} d\lambda.$$

The global sections of $Z \to \mathbb{P}^1$ are then completed by

$$T| = t - \epsilon \sum_{n=1}^{\infty} \gamma_n \lambda^n.$$

The next task is to calculate the frame section. We must solve

$$\begin{pmatrix} 1 & \epsilon \frac{\partial f}{\partial \Omega^0}| & \epsilon \frac{\partial f}{\partial \Omega^1}| \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 & \hat{h}_3 \\ \hat{h}_4 & \hat{h}_5 & \hat{h}_6 \\ \hat{h}_7 & \hat{h}_8 & \hat{h}_9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

Written out in full we must therefore solve

$$\hat{h}_1 = h_1 + \epsilon \frac{\partial f}{\partial \Omega^0}| h_4 + \epsilon \frac{\partial f}{\partial \Omega^1}| h_7$$

$$\hat{h}_2 = \lambda h_2 + \epsilon\lambda \frac{\partial f}{\partial \Omega^0}| h_5 + \epsilon\lambda \frac{\partial f}{\partial \Omega^1}| h_8$$

$$\hat{h}_3 = \lambda h_3 + \epsilon\lambda \frac{\partial f}{\partial \Omega^0}| h_6 + \epsilon\lambda \frac{\partial f}{\partial \Omega^1}| h_9$$

$$\hat{h}_4 = \lambda^{-1} h_4 \qquad \hat{h}_7 = \lambda^{-1} h_7$$

$$\hat{h}_5 = h_5 \qquad \hat{h}_6 = h_6 \qquad \hat{h}_8 = h_8 \qquad \hat{h}_9 = h_9.$$

Put

$$\begin{pmatrix} h_5 & h_6 \\ h_8 & h_9 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$$

and

$$\begin{pmatrix} h_4 \\ h_7 \end{pmatrix} = \begin{pmatrix} a_0 + a_1\lambda \\ b_0 + b_1\lambda \end{pmatrix}.$$

Now expand the derivatives of f (restricted to twistor lines) in Laurent series;

$$\frac{\partial f}{\partial \Omega^A}| = \sum_{n=-\infty}^{\infty} \phi_{n,A}\lambda^n \qquad \text{for} \qquad \phi_{n,A} = \frac{1}{2\pi i}\oint \frac{\partial f}{\partial \Omega^A}|\lambda^{-(1+n)}d\lambda.$$

We then (uniquely) obtain

$$h_2 = -\epsilon \sum_{n=0}^{\infty} \left(\phi_{n,0}k_1 + \phi_{n,1}k_3\right)\lambda^n$$

$$h_3 = -\epsilon \sum_{n=0}^{\infty} \left(\phi_{n,0}k_2 + \phi_{n,1}k_4\right)\lambda^n$$

and we can solve for the remaining piece h_1 up to the arbitrary function m to obtain

$$h_1 = m - \epsilon \sum_{n=0}^{\infty} \lambda^n \left[\phi_{n,0}\left(a_0 + a_1\lambda\right) + \phi_{n,1}\left(b_0 + b_1\lambda\right)\right].$$

Define $k = k_1k_4 - k_2k_3$. The determinant of H is then given by

$$\det H = mk$$

so we must impose $m \neq 0$ and $k \neq 0$.

We can then calculate

$$\left(H^{-1}\right)^T_{\ T} = m^{-1} \qquad \left(H^{-1}\right)^T_{\ A} = \epsilon m^{-1}\sum_{n=0}^{\infty}\phi_{n,A}\lambda^n$$

$$\left(H^{-1}\right)^A_{\ T} = \begin{pmatrix} \frac{1}{mk}\sum_{p=0}^{1}\left(k_2b_p - k_4a_p\right)\lambda^p \\ \frac{1}{mk}\sum_{p=0}^{1}\left(k_3a_p - k_1b_p\right)\lambda^p \end{pmatrix}$$

$$\left(H^{-1}\right)^0_{\ 0} = \frac{k_4}{k} + \frac{\epsilon}{mk}\sum_{p=0}^{1}\sum_{n=0}^{\infty}\phi_{n,0}\left(k_2b_p - k_4a_p\right)\lambda^{n+p}$$

$$\left(H^{-1}\right)^{0}_{\ 1} = -\frac{k_2}{k} + \frac{\epsilon}{mk}\sum_{p=0}^{1}\sum_{n=0}^{\infty}\phi_{n,1}\left(k_2 b_p - k_4 a_p\right)\lambda^{n+p}$$

$$\left(H^{-1}\right)^{1}_{\ 0} = -\frac{k_3}{k} + \frac{\epsilon}{mk}\sum_{p=0}^{1}\sum_{n=0}^{\infty}\phi_{n,0}\left(k_3 a_p - k_1 b_p\right)\lambda^{n+p}$$

$$\left(H^{-1}\right)^{1}_{\ 1} = \frac{k_1}{k} + \frac{\epsilon}{mk}\sum_{p=0}^{1}\sum_{n=0}^{\infty}\phi_{n,1}\left(k_3 a_p - k_1 b_p\right)\lambda^{n+p}.$$

The clock is therefore given by

$$\theta = m^{-1}dT| + m^{-1}\epsilon\sum_{n=0}^{\infty}\phi_{n,A}\lambda^n d\Omega^A|$$

$$\Rightarrow \qquad \theta = m^{-1}\left(dt - \epsilon\sum_{n=1}^{\infty}d\gamma_n\lambda^n + \epsilon\sum_{n=0}^{\infty}\phi_{n,A}\lambda^n d\Omega^A|\right).$$

Now, we have

$$d\gamma_n = \phi_{n,A}dx^{A1'} + \phi_{n-1,A}dx^{A,0'}$$

so

$$\theta = m^{-1}\left(dt + \epsilon\phi_{0,A}dx^{A1'}\right).$$

As in the five-dimensional case this is not closed for any $m \neq 0$ (and $\epsilon \neq 0$). Now take a representative with $m = 1$; we then have

$$d\theta = \epsilon\partial_{BB'}\phi_{0,A}dx^{BB'} \wedge dx^{A1'}$$

$$\Rightarrow \qquad d\theta = \epsilon\left\{\frac{1}{2\pi i}\oint\frac{\partial^2 f}{\partial\omega^A\partial\omega^B}|\frac{\pi_{B'}}{\pi_{0'}}\pi\cdot d\pi\right\}dx^{BB'} \wedge dx^{A1'}\ .$$

The Newton-Cartan metric arises, as in theorem 3.3.1, from the projective inverse of degenerate covariant metric arising from the frame section, completing the construction of a Galilean structure with torsion.

□

In theorem 3.3.2 we chose to merely construct the torsional Galilean structure, exhibiting the torsion via the non-closure of the clock. We could, however, go further and explicitly construct the torsion Ξ-connection of section 2.4.2 as was done for the three-dimensional case in theorem 3.2.3. In the interests of brevity we omit this cumbersome calculation.

Chapter 4

Jumps, folds, and singularities

It was observed in section 3.1 that the twistor-theoretic Newtonian limit is a jumping phenomenon in which the normal bundles to twistor lines suffer a jump from $\mathcal{O}(1)\oplus\mathcal{O}(1)$ to $\mathcal{O}\oplus\mathcal{O}(2)$ as the speed of light is varied. In this chapter we will study a different kind of jumping phenomenon, in which jumps occur as one moves from twistor line to twistor line.

4.1 Big jumps

Let Z be a three-dimensional complex manifold and $\{X_x\}$ a Kodaira family of rational curves in Z with normal bundles N_x which are generically isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Some lines may suffer single jumps to $N_x = \mathcal{O} \oplus \mathcal{O}(2)$.

Definition 4.1.1. *A rational curve from such a family whose normal bundle is $N_x = \mathcal{O}(2-k)\oplus\mathcal{O}(k)$ for $k > 2$ will be said to have suffered a big jump.*

In section 4.1 we are concerned with the construction of moduli spaces whose twistor spaces possess rational curves suffering big jumps. We already know that a single jump results in a singularity in the induced metric (the Newtonian limit); we will find that for two or more jumps the metric is also singular, and that for three or more jumps the moduli space itself also becomes singular. The first consequences of jumps in twistor theory were discussed in [76].

4.1.1 The normal bundle

Theorem 4.1.2. [23]

Let $k \geq 2$ be an integer and let $Z \xrightarrow{\varrho} \mathbb{P}^1$ be a complex manifold fibred over $\mathcal{O}(k)$ described by two patches $\varrho^{-1}(U)$ and $\varrho^{-1}\left(\hat{U}\right)$ with holomorphic patching

$$\hat{\zeta} = \lambda^{k-2}\zeta + \epsilon\lambda^{-2}S^2 \tag{4.1.1}$$

$$\hat{S} = \lambda^{-k}S \tag{4.1.2}$$

$$\hat{\lambda} = \lambda^{-1}$$

for a deformation parameter $\epsilon \in \mathbb{C}^*$ and for coordinates (ζ, S, λ) and $(\hat{\zeta}, \hat{S}, \hat{\lambda})$.

- *Generic global sections of $Z \xrightarrow{\varrho} \mathbb{P}^1$ have normal bundle $N_x = \mathcal{O}(1) \oplus \mathcal{O}(1)$;*
- *at least one global section has normal bundle $\mathcal{O}(2-k) \oplus \mathcal{O}(k)$.*

For $k \geq 4$ the moduli space M arises as a complex subvariety in the space of global sections of $\mathcal{O}(k)$.

Proof

To find the isomorphism class of the normal bundle to twistor lines one must first calculate the global sections of (4.1.1-4.1.2) which describe those twistor lines. Equation (4.1.2) is nothing more than the patching for $\mathcal{O}(k)$, so we can put

$$S| = x_0 + x_1\lambda + x_2\lambda^2 + ... + x_k\lambda^k$$

for coordinates $(x_0, ..., x_k) \in \mathbb{C}^{k+1}$ which parametrise the space of sections of $\mathcal{O}(k)$. Substituting this into (4.1.1) we have

$$\hat{\zeta} = \lambda^{k-2}\zeta + \epsilon\lambda^{-2}\left(\sum_{i=0}^{k} x_i\lambda^i\right)^2.$$

We need to decompose the squared sum on the left-hand-side into three parts:

$$\lambda^{-2}\left(\sum_{i=0}^{k} x_i\lambda^i\right)^2 = \left(\sum_{i=-2}^{0}\alpha_i\left(x\right)\lambda^i\right) + \left(\sum_{i=1}^{k-3}\alpha_i\left(x\right)\lambda^i\right) + \left(\sum_{i=k-2}^{2k-2}\alpha_i\left(x\right)\lambda^i\right) \tag{4.1.3}$$

where $\alpha_i\left(x\right)$ where are quadratic functions of the coordinates. The terms in the middle sum are those which correspond to elements of $\check{H}^1\left(\mathbb{P}^1, \mathcal{O}(2-k)\right)$, and so must vanish for the existence of global holomorphic sections. This gives $(k-3)$ conditions

$$\alpha_1 = \alpha_2 = ... = \alpha_{k-3} = 0 \tag{4.1.4}$$

on the $(k+1)$ coordinates x_i, which for $k > 3$ defines a four-dimensional subvariety in $\mathbb{C}^{k+1}$ which is the moduli space. The other two parts of the sum in (4.1.3) then fall into ζ and $\hat{\zeta}$, giving us twistor functions

$$\zeta| = -\epsilon\sum_{i=k-2}^{2k-2}\alpha_i\left(x\right)\lambda^{i+2-k} \qquad S| = \sum_{i=0}^{k} x_i\lambda^i$$

over U and

$$\hat{\zeta}| = \epsilon\sum_{i=-2}^{0}\alpha_i\left(x\right)\hat{\lambda}^{-i} \qquad \hat{S}| = \sum_{i=0}^{k} x_i\hat{\lambda}^{k-i}$$

over $\hat{U}$, all subject to (4.1.4).

With these in hand we are in a position to consider the normal bundle to a twistor line $X_x = \mathbb{P}^1$. The patching for the normal bundle is

$$\mathcal{F} = \begin{pmatrix} \frac{\partial\hat{\zeta}}{\partial\zeta}| & \frac{\partial\hat{\zeta}}{\partial S}| \\ \frac{\partial\hat{S}}{\partial\zeta}| & \frac{\partial\hat{S}}{\partial S}| \end{pmatrix} = \begin{pmatrix} \lambda^{k-2} & 2\epsilon\lambda^{-2}S| \\ 0 & \lambda^{-k} \end{pmatrix}.$$

To calculate the isomorphism class on each twistor line one must find holomorphic maps

$$H\,:\,U \to \mathrm{GL}(2,\mathbb{C}) \qquad \text{and} \qquad \hat{H}\,:\,\hat{U} \to \mathrm{GL}(2,\mathbb{C})$$

such that

$$\mathcal{F} = \hat{H}\begin{pmatrix} \lambda^{m-2} & 0 \\ 0 & \lambda^{-m} \end{pmatrix} H^{-1} \tag{4.1.5}$$

for some $m \in \mathbb{Z}$ on each line X_x (which enters the calculation through $S|$). Note that m will generically vary with X_x. Writing

$$H = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \qquad \text{and} \qquad \hat{H} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_3 & \hat{h}_4 \end{pmatrix}$$

as in previous proofs in this thesis we find that (4.1.5) constitutes the four equations

$$\hat{h}_1 = \lambda^{k-m} h_1 + 2\epsilon\lambda^{-m} S| h_3 \tag{4.1.6}$$

$$\hat{h}_2 = \lambda^{m+k-2} h_2 + 2\epsilon\lambda^{m-2} S| h_4 \tag{4.1.7}$$

$$\hat{h}_3 = \lambda^{2-m-k} h_3 \tag{4.1.8}$$

$$\hat{h}_4 = \lambda^{m-k} h_4. \tag{4.1.9}$$

Now we must determine the values of m which permit a solution for a each X_x. First consider $m > k$. In this case we have from (4.1.9) that $\hat{h}_4 = h_4 = 0$ and so (4.1.7) becomes

$$\hat{h}_2 = \lambda^{m+k-2} h_2\,,$$

which, since $m > k \geq 2$, has only the solution $\hat{h}_2 = h_2 = 0$. We then find $\det H = \det \hat{H} = 0$, and so we conclude that there are no jumps to $\mathcal{O}(2-m) \oplus \mathcal{O}(m)$ for $m > k$.

Now fix $1 \leq m \leq k$, so that (4.1.8-4.1.9) have solutions

$$h_3 = a_0(x) + a_1(x)\lambda + ... + a_{m+k-2}(x)\lambda^{m+k-2}$$

$$h_4 = b_0(x) + b_1(x)\lambda + ... + b_{k-m}(x)\lambda^{k-m}$$

for arbitrary functions $a_i(x)$ and $b_i(x)$ on M, and (4.1.6-4.1.7) become

$$\hat{h}_1 = \lambda^{k-m} h_1 + 2\epsilon \sum_{j=0}^{k} \sum_{i=0}^{m+k-2} a_i(x) x_j \lambda^{i+j-m} \tag{4.1.10}$$

and

$$\hat{h}_2 = \lambda^{m+k-2} h_2 + 2\epsilon \sum_{j=0}^{k} \sum_{i=0}^{k-m} b_i(x) x_j \lambda^{i+j+m-2}. \tag{4.1.11}$$

When $x_i = 0\,\forall i$, which we note is a solution of (4.1.4) and so lies within the moduli space, we have $\hat{h}_1 = \lambda^{k-m}h_1$ and $\hat{h}_2 = \lambda^{m+k-2}h_2$. We thus always have $\hat{h}_2 = h_2 = 0$ and then the only way to have $\det H \neq 0$ and $\det \hat{H} \neq 0$ comes from setting $m = k$ so that we have

$$\hat{h}_1 = h_1 = c_0(x)$$

for arbitrary $c_0 : M \to \mathbb{C}$. We can then choose $a_i = 0$ and $h_1 = h_4 = 1$, demonstrating that

$$N_x = \mathcal{O}(2-k) \oplus \mathcal{O}(k)$$

on $x_i = 0$, which is the largest jump possible for a given k.

Consider $m = 1$, for which we have

$$h_3 = \sum_{i=0}^{k-1} a_i \lambda^i \qquad \text{and} \qquad h_4 = \sum_{i=0}^{k-1} b_i \lambda^i \ ,$$

so there are $2k$ free functions. Equations (4.1.10-4.1.11) then look like the patchings for affine line bundles with underlying translation bundle $\mathcal{O}(1-k)$, and so we have to ensure that the parts of the additive deformation corresponding to elements of $\check{H}^1\left(\mathbb{P}^1, \mathcal{O}(1-k)\right)$ vanish. These are the terms proportional to $\lambda, \lambda^2, ..., \lambda^{k-2}$. We can always use up $2k-4$ of the free functions to make these terms vanish. Generically (i.e. assuming the coordinates do not generally vanish) we can put

$$a_2 = -\frac{1}{x_0}\left(x_1 a_1 + x_2 a_0\right)$$

$$a_3 = -\frac{1}{x_0}\left(x_2 a_2 + x_2 a_1 + x_3 a_0\right)$$

$$\vdots$$

$$a_{n>1} = -\frac{1}{x_0}\sum_{l=0}^{n-1} a_l x_{n-l} \tag{4.1.12}$$

and similar solutions for b_i. This then leaves (a_0, a_1, b_0, b_1) free, and we are left with

$$\hat{h}_1 = \hat{\lambda}\left(x_0 a_0\right) + \left(x_1 a_0 + a_1 x_0\right)$$

$$\hat{h}_2 = \hat{\lambda}\left(x_0 b_0\right) + \left(x_1 b_0 + b_1 x_0\right)$$

$$\hat{h}_3 = a_0 \hat{\lambda}^{k-1} + a_1 \hat{\lambda}^{k-2} + ... + a_{k-1}$$

$$\hat{h}_4 = b_0 \hat{\lambda}^{k-1} + b_1 \hat{\lambda}^{k-2} + ... + b_{k-1}.$$

These are global on $\hat{U}$ (and the corresponding expressions for H are global on U). It only remains to check that the determinant is nowhere-vanishing. We calculate

$$\det \hat{H} = \left(b_{k-1}\left(a_0 x_1 + a_1 x_0\right) - a_{k-1}\left(b_0 x_1 + b_1 x_0\right)\right) + \sum_{j=0}^{k-2} \hat{\lambda}^{k-j-1} \mu_j$$

where we define

$$\mu_j = b_{j+1} a_0 x_0 + b_j \left(a_0 x_1 + a_1 x_0\right) - a_{j+1} b_0 x_0 - a_j \left(b_0 x_1 + b_1 x_0\right).$$

For $\det \hat{H}$ to be nowhere-vanishing we require $\mu_j = 0$. This follows from j applications of the relation (4.1.12) to the highest indexed a_i and b_i, leaving only

$$\det \hat{H} = b_{k-1}\left(a_0 x_1 + a_1 x_0\right) - a_{k-1}\left(b_0 x_1 + b_1 x_0\right) \ ,$$

which can be generically made to be non-vanishing by careful choice of (a_0, a_1, b_0, b_1). The equations (4.1.6-4.1.9) imply that

$$\det \hat{H} = \det H \ ,$$

so we also have that H is non-degenerate, exhibiting the normal bundle as $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on generic twistor lines.

$\square$

Note that this theorem merely exhibits that the normal bundle is generically $\mathcal{O}(1) \oplus \mathcal{O}(1)$ but jumps to $\mathcal{O}(2-k) \oplus \mathcal{O}(k)$ at *some* point(s), demonstrating a proof-of-concept for big jumps. In fact the behaviour of the normal bundle is somewhat richer than this. We will see in some examples that the biggest jump occurs at more than just one point, and that we also obtain intermediate jumps to $\mathcal{O}(2-m) \oplus \mathcal{O}(m)$ for $1 < m < k$. Before the examples we must consider another theorem regarding the nature of the moduli spaces arising via the above theorem.

4.1.2 The link to Gibbons-Hawking

It turns out that the moduli spaces arising in theorem 4.1.2 are (almost) in the Gibbons-Hawking class.

Theorem 4.1.3. [23]

The big-jumping twistor spaces with patching (4.1.1-4.1.2) always admit a global $\mathcal{O}(2)$-valued function and their corresponding moduli spaces admit a triholomorphic Killing vector. The twistor spaces admit maps which put their patching in canonical Gibbons-Hawking form (3.1.22), though these are not biholomorphisms and eliminate the sections suffering big jumps. Let the restriction to a twistor line X_x (over U) of the global $\mathcal{O}(2)$-valued function be denoted $Q|(x^i, \lambda)$; the Gibbons-Hawking potential is

$$V = -\frac{1}{2(k-1)!}\frac{\partial^{k-1}}{\partial\lambda^{k-1}}\left[Q|\left(x^i,\lambda\right)^{-\frac{1}{2}}\right]_{\lambda=0}.$$

Recall that a triholomorphic Killing vector on a hyperKähler manifold is a Killing vector which Lie-derives all three Kähler forms.

Proof

Rewriting the patching (4.1.1) as

$$\hat{\zeta} = \lambda^{-2}\left(\lambda^k\zeta + \epsilon S^2\right)$$

reveals the global $\mathcal{O}(2)$-valued function, motivating the change of coordinates

$$\hat{Q}\left(\hat{\zeta},\hat{S},\hat{\lambda}\right) = \hat{\zeta} \qquad Q\left(\zeta,S,\lambda\right) = \lambda^k\zeta + \epsilon S^2. \tag{4.1.13}$$

The $\mathcal{O}(2)$-valued function then serves as a Hamiltonian which gives rise to a Hamiltonian vector field via the canonical symplectic structure on the fibres provided by

$$d\hat{\zeta}\wedge d\hat{S} = \lambda^{-2}d\zeta\wedge dS. \tag{4.1.14}$$

Such a construction makes the vector field on the moduli space naturally triholomorphic with respect to the three two-forms induced by (4.1.14) on M, which are themselves linear combinations of the Kähler forms. (See [21, 23] for more details.)

After using (4.1.13) to remove S we find that $\hat{S} = \lambda^{-k} S$ becomes

$$\hat{S} = \lambda^{-k} \epsilon^{-\frac{1}{2}} \sqrt{Q - \lambda^k \zeta}$$

$$\Rightarrow \qquad \hat{S} = \epsilon^{-\frac{1}{2}} \lambda^{-k} \sqrt{Q} + \epsilon^{-\frac{1}{2}} \sum_{n=1}^{\infty} \beta_n \lambda^{-k} \sqrt{Q} \left(\frac{\lambda^k \zeta}{Q} \right)^n$$

for irrelevant numerical coefficients

$$\beta_n = \frac{(2n)!}{(1-2n)(n!)^2 4^n}$$

coming from the expansion of the square root. The advantage of this expansion is that we can then identify the other change of coordinates

$$\hat{T}\left(\hat{\zeta}, \hat{S}, \hat{\lambda}\right) = \hat{S}$$

$$T\left(\zeta, S, \lambda\right) = \epsilon^{-\frac{1}{2}} \sum_{n=1}^{\infty} \beta_n \lambda^{-k} \sqrt{Q\left(\zeta, S, \lambda\right)} \left(\frac{\lambda^k \zeta}{Q\left(\zeta, S, \lambda\right)} \right)^n$$

so that we have the new patching

$$\hat{T} = T + \epsilon^{-\frac{1}{2}} \lambda^{-k} \sqrt{Q} \qquad \hat{Q} = \lambda^{-2} Q \ ,$$

which is in canonical Gibbons-Hawking form (3.1.20) with

$$f = \epsilon^{-\frac{1}{2}} \lambda^{-k} \sqrt{Q}$$

representing a cohomology class in $\check{H}^1\left(\mathcal{O}(2), \mathcal{O}_{\mathcal{O}(2)}\right)$. The disadvantage of the change of coordinates is that it is not a biholomorphism on each patch: clearly the region $Q = 0$ is singular and the resulting twistor space is undefined there. That the big-jumping twistor lines have been removed from the twistor space by this change of variables now follows from theorem 3.1.4, in which it was shown that a twistor space whose patching is (3.1.20) cannot suffer jumps larger than those to $\mathcal{O} \oplus \mathcal{O}(2)$.

The explicit form of the Gibbons-Hawking potential is now also a simple application of the formula proven in theorem 3.1.4. We calculate

$$V = \frac{1}{2\pi i} \oint_\Gamma \frac{\partial}{\partial Q} \left[\epsilon^{-\frac{1}{2}} \lambda^{-k} \sqrt{Q} \right] |d\lambda.$$

Put $Q| = \lambda^2 (x-y) + 2\lambda z + (x+y)$ for the global sections of the $\mathcal{O}(2)$ part of the patching, so that (x,y,z) are three of the four coordinates on the moduli space. Then we can finally calculate

$$V = -\frac{1}{2(k-1)!}\frac{\partial^{k-1}}{\partial\lambda^{k-1}}\left[\lambda^2 (x-y) + 2\lambda z + (x+y)\right]^{-\frac{1}{2}}_{\lambda=0}\ , \tag{4.1.15}$$

completing the proof.

□

4.1.3 Two illustrative examples

We will now consider two cases of the above theorems in some detail, following the story on the spacetime side of the correspondence rather that the twistor space, explicitly constructing the metric both before and after the change of twistor variables.

Example: $k = 3$

Consider the case $k = 3$ in theorem 4.1.2. We have

$$\hat{\zeta} = \lambda\zeta + \epsilon\lambda^{-2}\left(x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3\right)^2$$

so there is no question of restricting to a subvariety when constructing the moduli space. The twistor functions over $\hat{U}$ are

$$\hat{\zeta}| = \epsilon x_0^2\hat{\lambda}^2 + 2\epsilon x_0 x_1\hat{\lambda} + \epsilon\left(x_1^2 + 2x_0x_2\right)$$

and

$$\hat{S}| = x_0\hat{\lambda}^3 + x_1\hat{\lambda}^2 + x_2\hat{\lambda} + x_3.$$

(Whilst we could equivalently consider the more traditional twistor functions over U the resulting calculations would be more complicated.) The conformal structure arises via the twistor principle that alpha-surfaces should be totally null, which amounts here to taking the resultant of the two polynomials in $\hat{\lambda}$ given by

$$\frac{\partial\hat{\zeta}|}{\partial x^a}\delta x^a \qquad \text{and} \qquad \frac{\partial\hat{S}|}{\partial x^a}\delta x^a\ ,$$

yielding a conformal structure

$$[g] = x_0^3 dx_3^2 + 2x_0^2 x_1 dx_2 dx_3 + x_0\left(x_0x_2 + x_1^2\right) dx_1 dx_3 + x_1\left(3x_0x_2 - x_1^2\right) dx_0 dx_3 + x_0^2 x_1^2 dx_2^2$$
$$+ x_1\left(x_0x_2 + x_1^2\right) dx_1 dx_2 + x_2\left(x_0x_2 + x_1^2\right) dx_0 dx_2 + x_1^2 x_2 dx_1^2 + 2x_1 x_2^2 dx_0 dx_1 + x_2^3 dx_0^2.$$

Multiplying this representative by a conformal factor

$$\frac{2}{x_1^2 - x_0x_2}$$

yields a Ricci-flat metric. The conformal structure is ASD, as of course it must be by the nonlinear graviton theorem [66], and the self-dual parallel two-forms are given by

$$\Sigma^{0'0'} = 2dx_3 \wedge (x_0 dx_2 + x_2 dx_0 + x_1 dx_1)$$

$$\Sigma^{0'1'} = x_0 dx_3 \wedge dx_1 + x_1 dx_3 \wedge dx_0 + x_1 dx_2 \wedge dx_1 + x_2 dx_2 \wedge dx_0$$

$$\Sigma^{1'1'} = 2\left(x_0 dx_3 + x_2 dx_1 + x_1 dx_2\right) \wedge dx_0.$$

These forms are all Lie-derived by the Killing vector

$$\mathcal{K} = \frac{\partial}{\partial x_3} \ ,$$

making the Killing vector triholomorphic and meaning that we can always recast this metric in Gibbons-Hawking form

$$g = \frac{1}{V}\left(dt + A\right)^2 + Vh$$

for some flat three-metric h. (Of course, this is something we already know from the twistor version of the story in theorem 4.1.3.)

One coordinate for the Gibbons-Hawking form will be

$$t = x_3 \ ;$$

to find the other three we have (at least) two options. We could compute a three-dimensional Abelian subalgebra $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$ of the algebra of isometries of the "spatial" part of the Gibbons-Hawking spacetime

$$h = V^{-1}\left(g - V\,\mathcal{K} \otimes \mathcal{K}\right)$$

where

$$V=(g(\mathcal{K},\mathcal{K}))^{-1}=\frac{x_1^2-x_0x_2}{2x_0^3} \tag{4.1.16}$$

is the Gibbons-Hawking potential, and then deduce that these must be translations, giving that $h(\mathcal{X}_i,\ \ )$ must be exact derivatives of the flat coordinates.

Considerably less work, though, is to employ the twistor theory of theorem 4.1.3 and exploit the fact that the "spatial" coordinates there arise via the global weight-two function defined by (4.1.13). For brevity's sake we will henceforth set $\epsilon=1$. The new coordinates can thus be read-off from the components of

$$\hat{Q}|=\hat{\zeta}|=x_0^2\hat{\lambda}^2+2x_0x_1\hat{\lambda}+\left(x_1^2+2x_0x_2\right).$$

We therefore define

$$x+y=x_0^2 \tag{4.1.17}$$

$$z=x_0x_1$$

$$x-y=\left(x_1^2+2x_0x_2\right). \tag{4.1.18}$$

These new coordinates agree with those obtained from the algebra of translations, and the result is a Gibbons-Hawking spacetime in canonical form, with

$$h=dx^2-dy^2-dz^2$$

and

$$V=\frac{x^2-y^2-3z^2}{4(x+y)^{5/2}}.$$

One can easily check that this result is in agreement with the formula (4.1.15). The coordinates used here are naturally adapted to neutral signature, and these manifolds do not admit Euclidean real slices. The extension of this work to Euclidean signature will be considered in section 4.2.

As an aside we note that the metric admits an additional Killing vector

$$\mathcal{K}_2=5x_3\frac{\partial}{\partial x_3}+2x_2\frac{\partial}{\partial x_2}-x_1\frac{\partial}{\partial x_1}-4x_0\frac{\partial}{\partial x_0}$$

which Lie-derives Σ^{01} but merely rotates Σ^{00} and Σ^{11}. The space of orbits of $\mathcal{K}_2$ in M thus admits a Toda Einstein-Weyl structure; the reader is referred to [79] for details.

To complete the discussion of this example we will calculate the precise loci at which different jumps appear, following through the calculation in theorem 4.1.2 in more detail. The patching for the normal bundle is

$$\mathcal{F} = \begin{pmatrix} \lambda & 2\epsilon S| \\ 0 & \lambda^{-3} \end{pmatrix}$$

and the splitting problem to be solved is the four equations (4.1.6-4.1.9) which now read

$$\hat{h}_1 = \lambda^{3-m} h_1 + 2\epsilon\lambda^{-m} \left(x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3\right) h_3 \tag{4.1.19}$$

$$\hat{h}_2 = \lambda^{m+1} h_2 + 2\epsilon\lambda^{m-2} \left(x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3\right) h_4 \tag{4.1.20}$$

$$\hat{h}_3 = \lambda^{-(m+1)} h_3 \tag{4.1.21}$$

$$\hat{h}_4 = \lambda^{m-3} h_4. \tag{4.1.22}$$

We know from the proof of theorem 4.1.2 that the only possible values of m which can give rise to solutions are 1, 2, and 3. For $m = 3$ we have

$$h_3 = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + a_4\lambda^4$$

and

$$h_4 = b_0.$$

The most general solution of (4.1.19) is then

$$\hat{h}_1 = c_0 + 2\epsilon \left[\lambda^{-3}\left(x_0 a_0\right) + \lambda^{-2}\left(x_0 a_1 + x_1 a_0\right) + \lambda^{-1}\left(x_0 a_2 + x_1 a_1 + x_2 a_0\right)\right]$$

for arbitrary c_0. Turning to (4.1.20) we have

$$\hat{h}_2 = \lambda^4 h_2 + 2\epsilon b_0 \left[\lambda x_0 + x_1\lambda^2 + x_2\lambda^3 + x_3\lambda^4\right] \; ,$$

and for this to have a global solution we must impose either $b_0 = 0$ or

$$x_0 = x_1 = x_2 = 0. \tag{4.1.23}$$

In the former case we would then have $h_2 = h_4 = 0$, implying $\det H = 0$, so we must have (4.1.23), leaving

$$h_2 = -2\epsilon b_0 x_3 \qquad \hat{h}_2 = 0$$

and

$$\det \hat{H} = \left(c_0 + 2\epsilon\left[\lambda^{-3}\left(x_0 a_0\right) + \lambda^{-2}\left(x_0 a_1 + x_1 a_0\right) + \lambda^{-1}\left(x_0 a_2 + x_1 a_1 + x_2 a_0\right)\right]\right) b_0 \; .$$

Setting $b_0 = c_0 = 1$ and $a_0 = a_1 = a_2 = a_3 = a_4 = 0$ then yields a valid solution for H and $\hat{H}$. This calculation shows that the minimum requirement necessary for a big jump to $\mathcal{O}(-1) \oplus \mathcal{O}(3)$ is (4.1.23); the big-jumping loci is a line parametrised by x_3. As expected, this is the $Q| = 0$ region from theorem 4.1.3, and corresponds to where the Gibbons-Hawking potential (4.1.16) is undefined.

For $m = 2$ we instead have

$$\hat{h}_3 = a_0\lambda^{-3} + a_1\lambda^{-2} + a_2\lambda^{-1} + a_3 \qquad ; \qquad \hat{h}_4 = b_0\lambda^{-1} + b_1 \qquad ;$$

$$\text{and} \qquad \hat{h}_1 = 2\epsilon\left[\lambda^{-2}\left(x_0 a_0\right) + \lambda^{-1}\left(x_0 a_1 + x_1 a_0\right)\right] .$$

This leaves

$$\hat{h}_2 = \lambda^3 h_2 + 2\epsilon\left[\left(x_0 b_0\right) + \lambda\left(x_0 b_1 + x_1 b_0\right) + \lambda^2\left(x_1 b_1 + x_2 b_0\right) + \lambda^3\left(x_2 b_1 + x_3 b_0\right) + \lambda^4\left(x_3 b_1\right)\right] \tag{4.1.24}$$

and so for a global solution we must ensure that the additive terms of order λ and λ^2 vanish. This can always be done by choosing b_0 and b_1 appropriately. Assuming such choices have been made the determinant is then

$$\det \hat{H} = -2\epsilon x_0 b_0 a_3 + 2\epsilon\lambda^{-1}\left[b_1\left(x_0 a_1 + x_1 a_0\right) - x_0 b_0 a_2\right] .$$

At this juncture we can see that for $N_x = \mathcal{O} \oplus \mathcal{O}(2)$ we require $x_0 \neq 0$ and $b_0 \neq 0$, meaning that we can put $b_1 = -x_0^{-1} x_1 b_0$. The λ^{-1} term in $\det \hat{H}$ can then be set to zero using a_2 and the vanishing of the λ^2 term in (4.1.24) then requires

$$x_2 x_0 - x_1^2 = 0 \; ,$$

which (along with $x_0 \neq 0$) is the minimum requirement for a single jump to $N_x = \mathcal{O} \oplus \mathcal{O}(2)$. All other points must therefore, by a process of elimination, have $N_x = \mathcal{O}(1) \oplus \mathcal{O}(1)$. We note that such regions correspond to $V = 0$ in (4.1.16), in line with the proof of theorem 3.1.4.

Example: $k = 4$

Here the situation is complicated by the fact that the twistor space is an affine bundle on $\mathcal{O}(4)$ whose underlying translation group $\mathcal{O}(-2)$ has non-vanishing

$$\check{H}^1\left(\mathbb{P}^1, \mathcal{O}(-2)\right) = \mathbb{C}.$$

This means that the moduli space must arise as a four-dimensional subvariety in the five-dimensional space of global sections of $\mathcal{O}(4)$. Locally, the different ways of restricting to the subvariety provide a natural construction of coordinate patches on M.

First write the global sections of $\mathcal{O}(4)$ as

$$S| = x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + x_4\lambda^4 \tag{4.1.25}$$

so that (4.1.1) becomes

$$\hat{\zeta} = \lambda^2\zeta + 2\epsilon\lambda^{-2}\left(x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + x_4\lambda^4\right)^2.$$

For this to have a global solution we must set the coefficient of λ on the right-hand-side to zero, which requires

$$x_0x_3 + x_1x_2 = 0. \tag{4.1.26}$$

This is the equation defining the moduli space M as a subvariety in $(x_0, x_1, x_2, x_3, x_4) \in \mathbb{C}^5$. We see that M is a cone. With (4.1.26) satisfied a global solution is possible, giving us

$$\zeta| = -2\epsilon\left[\left(x_2^2 + 2x_1x_3 + 2x_0x_4\right) + \lambda\left(2x_1x_4 + 2x_2x_3\right) + \lambda^2\left(x_3^2 + 2x_2x_4\right) + \lambda^3\left(2x_3x_4\right) + \lambda^4\left(x_4^2\right)\right] \tag{4.1.27}$$

and $$\hat{\zeta}| = 2\epsilon\left[\lambda^{-2}\left(x_0^2\right) + \lambda^{-1}\left(2x_0x_1\right) + \left(x_1^2 + 2x_0x_2\right)\right]$$

over U and $\hat{U}$ respectively. The twistor functions (4.1.25) and (4.1.27) should be understood as being subject to (4.1.26). In solving (4.1.26) we have to assume that one of the coordinates $(x_0, ..., x_3)$ is non-zero, and each one gives rise to a different patch on M.

To obtain the moduli space metric in a patch we can explicitly eliminate one coordinate, say x_3, from the twistor functions using (4.1.26) and then apply the twistor principle, finding the conformal structure for which the alpha surfaces are totally null. If we do that with

$$x_3 = -\frac{x_1 x_2}{x_0}$$

then we obtain a conformal structure containing a Ricci-flat metric

$$\begin{aligned} g = \frac{2}{x_0^2 x_1 \left(3x_0x_2 - x_1^2\right)} &(x_0^6 dx_4^2 - x_4^3 x_1 \left(x_0x_2 + x_1^2\right) dx_4 dx_1 + 2x_0^4 \left(x_0x_2 - x_1^2\right) dx_2 dx_4 \\ &+ x_0^2 \left(2x_0^2x_2^2 - 5x_0x_2x_1^2 + x_1^4\right) dx_4 dx_0 - 2x_0x_2x_1^2 \left(x_0x_2 - x_1^2\right) dx_1^2 \\ &- x_0x_1 \left(x_0^2x_2^2 - x_1^4\right) dx_1 dx_2 - x_2x_1 \left(x_0^2x_2^2 - 6x_0x_2x_1^2 + x_1^4\right) dx_1 dx_0 + x_0^2 \left(x_0x_2 - x_1^2\right) dx_2^2 \\ &+ 2x_0^2x_2^2 \left(x_0x_2 + x_1^2\right) dx_2 dx_0 + x_2^2 \left(x_0^2x_2^2 + 4x_0x_2x_1^2 - x_1^4\right) dx_0^2) , \end{aligned}$$

valid for $x_0 \neq 0$.

(As an alternative calculational trick we could could construct the conformal structure *before* eliminating a coordinate, in which case we obtain a cubic condition via the twistor principle and thus a symmetric three-form on the five-dimensional space. On (any choice of) restriction to M the symmetric three-form then tensor-factorises into the conformal structure on M and a one-form.)

The self-dual two-forms are

$$\Sigma^{0'0'} = 2dx_4 \wedge (x_0 dx_2 + x_2 dx_0 + x_1 dx_1)$$

$$\begin{aligned} \Sigma^{0'1'} = x_0 dx_4 \wedge dx_1 + x_2 dx_4 \wedge dx_0 - x_0^{-1} \left(x_2x_0 - x_1^2\right) dx_1 \wedge dx_2 \\ - x_0^{-2} x_2 \left(x_2x_0 + x_1^2\right) dx_1 \wedge dx_0 - 2x_0^{-1} x_2 x_1 dx_2 \wedge dx_0 \end{aligned}$$

$$\Sigma^{1'1'} = dx_0 \wedge \left(4x_0^{-1} x_2 x_1 dx_1 - 2x_0 dx_4 - 2x_0^{-1} \left(x_2x_0 - x_1^2\right) dx_2\right)$$

and the triholomorphic Killing vector is

$$\mathcal{K} = \frac{\partial}{\partial x_4}.$$

The Gibbons-Hawking potential before the change of coordinates is

$$V = \frac{x_1\left(3x_0x_2 - x_1^2\right)}{2x_0^4}.$$

One can now proceed as with the $k = 3$ example above, finding the transformation to Gibbons-Hawking form. The change of coordinates can again be read off from the global weight-two twistor function

$$\hat{Q} = \hat{\zeta} = 2\lambda^{-2}\left(x_0^2\right) + 2\lambda^{-1}\left(2x_0x_1\right) + 2\left(x_1^2 + 2x_0x_2\right)$$

where we have again set $\epsilon = 1$. The change of coordinates is therefore the same (4.1.17-4.1.18) as it was for $k = 3$, yielding

$$V = \frac{z\left(3x^2 - 3y^2 - 5z^2\right)}{4\left(x+y\right)^{7/2}} \tag{4.1.28}$$

and

$$h = dx^2 - dy^2 - dz^2$$

(with fourth coordinate $t = x_4$ as before). As expected (4.1.28) agrees with the general result of theorem 4.1.3.

To conclude this example we will again discuss the precise jumping loci, beginning with the jump to $\mathcal{O}(-2) \oplus \mathcal{O}(4)$. In this case the splitting problem is the four equations

$$\hat{h}_1 = h_1 + 2\epsilon\lambda^{-4}\left(x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + x_4\lambda^4\right)h_3 \tag{4.1.29}$$

$$\hat{h}_2 = \lambda^6 h_2 + 2\epsilon\lambda^2\left(x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + x_4\lambda^4\right)h_4 \tag{4.1.30}$$

$$\hat{h}_3 = \lambda^{-6}h_3 \tag{4.1.31}$$

$$\hat{h}_4 = h_4. \tag{4.1.32}$$

and we can solve (4.1.31-4.1.32) to obtain

$$h_3 = \sum_{n=0}^{6} a_n\lambda^n \qquad \text{and} \qquad h_4 = b_0.$$

For a global solution of (4.1.30) we then require

$$x_0 = x_1 = x_2 = x_3 = 0 \tag{4.1.33}$$

(since $b_0 = 0$ would lead to $\det H = 0$), and we note that (4.1.33) lies within the moduli space defined by (4.1.26). The solution is then

$$\hat{h}_2 = 0 \qquad \text{and} \qquad h_2 = -2\epsilon b_0 x_4$$

On (4.1.33) the splitting of (4.1.29) becomes straightforward:

$$\hat{h}_1 = h_1 + 2\epsilon x_4 \sum_{n=0}^{6} a_n \lambda^n$$

$$\Rightarrow \qquad \hat{h}_1 = c_0 \qquad \text{and} \qquad h_1 = c_0 - 2\epsilon x_4 \sum_{n=0}^{6} a_n \lambda^n.$$

The determinant is

$$\det \hat{H} = c_0 b_0 \ ,$$

which we can thus always set to be a constant. The minimum condition for the normal bundle to jump to $\mathcal{O}(-2) \oplus \mathcal{O}(4)$ is therefore (4.1.33); the biggest-jumping region is a line parametrised by x_4, intersecting the vertex of the cone (4.1.26).

We can also consider intermediate jumps. Jumps to $\mathcal{O}(-1) \oplus \mathcal{O}(3)$ occur when we can solve the splitting problem

$$\hat{h}_1 = \lambda h_1 + 2\epsilon \lambda^{-3} \left(x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + x_4\lambda^4 \right) h_3 \tag{4.1.34}$$

$$\hat{h}_2 = \lambda^5 h_2 + 2\epsilon \lambda \left(x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + x_4\lambda^4 \right) h_4 \tag{4.1.35}$$

$$\hat{h}_3 = \lambda^{-5} h_3 \tag{4.1.36}$$

$$\hat{h}_4 = \lambda^{-1} h_4. \tag{4.1.37}$$

The procedure should now be clear: (4.1.36-4.1.37) have general solutions given by

$$h_3 = \sum_{n=0}^{5} a_n \lambda^n \qquad \text{and} \qquad h_4 = b_0 + b_1 \lambda,$$

which then allow us to split (4.1.35) as

$$\hat{h}_2 = 0 \qquad \text{and} \qquad h_2 = -2\epsilon\left[(x_3b_1 + x_4b_0) + \lambda\left(x_4b_1\right)\right]$$

provided we ensure

$$x_0b_0 = x_0b_1 + x_1b_0 = x_1b_1 + x_2b_0 = x_2b_1 + x_3b_0 = 0. \tag{4.1.38}$$

Equation (4.1.34) can then be solved (with no further requirements) to give

$$\hat{h}_1 = 2\epsilon\lambda^{-3}\left(x_0a_0\right) + 2\epsilon\lambda^{-2}\left(x_0a_1 + x_1a_0\right) + 2\epsilon\lambda^{-1}\left(x_0a_2 + x_1a_1 + x_2a_0\right) \\ + 2\epsilon\left(x_0a_3 + x_1a_2 + x_2a_1 + x_3a_0\right)$$

and

$$\det \hat{H} = 2\epsilon\left[b_1\left(x_0a_3 + x_1a_2 + x_2a_1 + x_3a_0\right)\right].$$

Thus to find the jumping loci we need to solve (4.1.38) as well as $\det \hat{H} \neq 0$. Some inspection reveals that resulting region is

$$x_0 = x_1 = x_2 = 0 \qquad ; \qquad x_3 \neq 0 \qquad ; \qquad \text{and} \qquad x_4 = \text{anything}$$

(which satisfies the moduli space's equation (4.1.26) as a subvariety in $\mathbb{C}^5$).

Finally there are single jumps to $\mathcal{O} \oplus \mathcal{O}(2)$ to find, for which the splitting problem is

$$\hat{h}_1 = \lambda^2 h_1 + 2\epsilon\lambda^{-2}\left(x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + x_4\lambda^4\right)h_3 \tag{4.1.39}$$

$$\hat{h}_2 = \lambda^4 h_2 + 2\epsilon\left(x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + x_4\lambda^4\right)h_4 \tag{4.1.40}$$

with

$$h_3 = \sum_{n=0}^{4} a_n\lambda^n \qquad \text{and} \qquad h_4 = \sum_{n=0}^{2} b_n\lambda^n.$$

For global solutions of (4.1.39) we need

$$x_3a_0 + x_2a_1 + x_1a_2 + x_0a_3 = 0$$

and for global solutions of (4.1.40) we need

$$x_0b_1 + x_1b_0 = x_0b_2 + x_1b_1 + x_2b_0 = x_1b_2 + x_2b_1 + x_3b_0 = 0.$$

The determinant is

$$\det \hat{H} = 2\epsilon \left(b_2 x_0 a_2 + b_2 x_1 a_1 + b_2 x_2 a_0 - x_0 b_0 a_4\right),$$

which we require to be a non-zero constant. We seek solutions to these conditions which intersect the moduli space hypersurface (4.1.26).

Some thought reveals that the resulting set of points where the normal bundle jumps to $\mathcal{O} \oplus \mathcal{O}(2)$ are

$$\left\{x_0 \neq 0 \qquad ; \qquad x_1\left(x_1^2 - 3x_0 x_2\right) = 0\right\}$$

and

$$\{x_2 \neq 0 \qquad ; \qquad x_0 = x_1 = 0\}.$$

At all points other than those mentioned we have $N_x = \mathcal{O}(1) \oplus \mathcal{O}(1)$.

4.1.4 Euclidean signature and reality conditions

In all of the big-jumping examples so far considered the obvious real slices have been of neutral signature rather than the perhaps more desirable Euclidean signature. In this section this will be rectified.

The Euclidean reality condition for a global section $S|(\lambda)$ of $\mathcal{O}(k)$ requires k to be even; the real structure is induced by the involution

$$\overline{S|\left(\lambda\right)} = (-1)^{k/2}\,\overline{\lambda}^k S|\left(-\overline{\lambda}^{-1}\right).$$

For $k = 4$ with

$$S| = x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + x_4\lambda^4 \tag{4.1.41}$$

we therefore must take x_2 to be real and $(x_0, x_1, x_3, x_4) \in \mathbb{C}^4$ with

$$x_4 = \overline{x_0}$$

and

$$x_3 = -\overline{x_1}.$$

Unfortunately this means that the equation (4.1.26) defining the moduli space as a hypersurface becomes the real-codimension-two condition

$$x_0\overline{x_1} + x_1 x_2 = 0$$

rather than a real-codimension-one condition. This means that the $k = 4$ metric above cannot admit a Euclidean real slice.

To get around this issue we must engineer a deformation of $\mathcal{O}(-2) \oplus \mathcal{O}(4)$ for which the condition defining the moduli space hypersurface happens to be a real-codimension-one condition, which is accomplished in the following example [23].

Example

Let $Z \to \mathbb{P}^1$ be a twistor space fibred over $\mathcal{O}(4)$ with patching

$$\hat{\zeta} = \lambda^2 \zeta + S^2 \left(1 - \lambda^{-6}\right) \tag{4.1.42}$$

$$\hat{S} = \lambda^{-4} S \ . \tag{4.1.43}$$

As above the moduli space arises as the subvariety in the space of global sections of $\mathcal{O}(4)$ described by the vanishing of the term of order λ in (4.1.42). On a real slice this subvariety is

$$x_0 x_1 + \overline{x_0 x_1} = 2\Re\left(x_0 x_1\right) = 0\,,$$

which is a real-codimension-one condition.

One could proceed just as before and construct the metric via the principle that alpha surfaces are totally null; the resulting metric is, however, quite complicated and the calculation is not illuminating. Instead we will in the next section discuss a powerful general framework which will be useful for discussing further the above example.

4.2 Self-dual two forms and the Legendre transformation

For a twistor space constructed as the total space of an affine bundle on $\mathcal{O}(k)$ there is an alternative way of constructing the metric on the moduli space of twistor lines [24]. This is done by directly constructing the self-dual two-forms $\Sigma^{A'B'}$. In what follows we will make use of the homogeneous coordinates described in section 3.1.1.2. The patching is

$$\hat{\nu} = \nu + f\left(s, \pi_{A'}\right) \qquad \hat{s} = s$$

where $(s, \hat{s})$ are of weight k; $(\nu, \hat{\nu})$ are of weight $2-k$; and f is a function of weight $2-k$ representing a cohomology class in $\check{H}^1\left(\mathcal{O}(k), \mathcal{O}(2-k)_{\mathcal{O}(k)}\right)$.

Everything characterising the spacetime must now be obtainable from f, and there are several ways of extracting functions on the space of global sections of $\mathcal{O}(k)$ from f. The first is to restrict s to

$$s| = x^{A'_1 \dots A'_k}\pi_{A'_1} \dots \pi_{A'_k}$$

and then multiply $f|$ by $k-4$ spinors $\pi_{A'}$. One can then contour-integrate out the remaining dependence on $\pi_{A'}$. We therefore define

$$\phi_{A'_1 \dots A'_{k-4}}\left(x^{B'_1 \dots B'_k}\right) = \frac{1}{2\pi i}\oint_\Gamma \pi_{A'_1} \dots \pi_{A'_{k-4}} f\left(s|, \pi_{A'}\right)\pi \cdot d\pi\ ,$$

and note that the conditions

$$\phi_{A'_1 \dots A'_{k-4}} = 0$$

define a four-dimensional subvariety in the $(k+1)$-dimensional space of global sections of $\mathcal{O}(k)$ which we can identify as the moduli space M.

Now define an extra set of quantities from f:

$$\psi_{A'_1 \dots A'_{2k-4}}\left(x^{B'_1 \dots B'_k}\right) = \frac{1}{2\pi i}\oint_\Gamma \pi_{A'_1} \dots \pi_{A'_{2k-4}} \frac{\partial f}{\partial s}\left(s|, \pi_{A'}\right)\pi \cdot d\pi. \tag{4.2.1}$$

A calculation shows that the self-dual two-forms $\Sigma^{A'B'}$ arising from the global $\mathcal{O}(2)$-valued form

$$d\hat{\nu} \wedge d\hat{s} = d\nu \wedge ds$$

by the restriction to twistor lines such that

$$d\nu| \wedge ds| = \Sigma^{A'B'}\pi_{A'}\pi_{B'}$$

are given in terms of (4.2.1) by

$$\Sigma^{A'B'} = \frac{1}{8}\psi^{A'B'}{}_{B'_1\ldots B'_{k-3}C'_1\ldots C'_{k-3}} dx_{P'Q'R'}{}^{B'_1\ldots B'_{k-3}} \wedge dx^{P'Q'R'C'_1\ldots C'_{k-3}} + \frac{3}{2}\psi_{B'_1\ldots B'_{k-2}C'_1\ldots C'_{k-2}} dx_{P'}{}^{B'_1\ldots B'_{k-2}(A'} \wedge dx^{C'_1)C'_2\ldots C'_{k-2}B'P'}. \quad (4.2.2)$$

(Recall that spinor indices can be raised and lowered using $\epsilon_{A'B'}$ and $\epsilon^{A'B'}$.) The hyper-Kähler metric can then be reconstructed from the self-dual two-forms by extracting from them the Kähler potential and writing the metric locally in terms of that potential. See, for instance, [21] for details. Note that in the case $k = 3$ this formalism is simplified by the fact that there are no constraints (ϕ does not exist) and $\psi^{A'B'}$ is a self-dual Maxwell field.

Example

Consider now the Euclidean signature $k = 4$ example above, which has

$$f(s, \pi_{A'}) = s^2 \left(\frac{1}{\pi_{0'}^2 \pi_{1'}^8} - \frac{1}{\pi_{0'}^8 \pi_{1'}^2} \right).$$

The global sections of $\mathcal{O}(4)$ are, on a real slice, given by

$$s| = x^{0'0'0'0'}\pi_{0'}^4 + 4x^{0'0'0'1'}\pi_{0'}^3\pi_{1'} + 6x^{0'0'1'1'}\pi_{0'}^2\pi_{1'}^2 + 4x^{0'1'1'1'}\pi_{0'}\pi_{1'}^3 + x^{1'1'1'1'}\pi_{1'}^4$$

with

$$x^{0'0'0'0'} = \overline{x_0} \qquad 4x^{0'0'0'1'} = -\overline{x_1} \qquad 6x^{0'0'1'1'} = x_2 \qquad 4x^{0'1'1'1'} = x_1 \qquad x^{1'1'1'1'} = x_0.$$

We then have

$$\phi = \frac{1}{2\pi i}\oint_\Gamma \left(x^{A'_1\ldots A'_4}\pi_{A'_1}\ldots\pi_{A'_4} \right)^2 \left(\frac{1}{\pi_{0'}^2\pi_{1'}^8} - \frac{1}{\pi_{0'}^8\pi_{1'}^2} \right) \pi\cdot d\pi$$

$$\Rightarrow \quad \phi = \frac{1}{2\pi i}\oint_\Gamma \left(\overline{x_0}\pi_{0'}^4 - \overline{x_1}\pi_{0'}^3\pi_{1'} + x_2\pi_{0'}^2\pi_{1'}^2 + x_1\pi_{0'}\pi_{1'}^3 + x_0\pi_{1'}^4 \right)^2 \left(\frac{1}{\pi_{0'}^2\pi_{1'}^8} - \frac{1}{\pi_{0'}^8\pi_{1'}^2} \right) \pi\cdot d\pi$$

$$\Rightarrow \quad \phi = 2x_1x_0 + 2\overline{x_0x_1} = 4\Re(x_0x_1).$$

So the moduli space is, as was discussed earlier, given by the real-codimension-one condition $\phi = 0$.

Now calculate $\psi_{A'_1 \dots A'_4}$.

$$\psi_{A'B'C'D'} = \frac{1}{2\pi i} \oint_\Gamma \pi_{A'}\pi_{B'}\pi_{C'}\pi_{D'} 2\left(\overline{x_0}\pi_{0'}^4 - \overline{x_1}\pi_{0'}^3\pi_{1'} + x_2\pi_{0'}^2\pi_{1'}^2 + x_1\pi_{0'}\pi_{1'}^3 + x_0\pi_{1'}^4\right) \times \left(\frac{1}{\pi_{0'}^2\pi_{1'}^8} - \frac{1}{\pi_{0'}^8\pi_{1'}^2}\right)\pi \cdot d\pi$$

$$\Rightarrow \quad \psi_{0'0'0'0'} = 2\overline{x_1} \qquad \psi_{0'0'0'1'} = -2\overline{x_0}$$

$$\psi_{0'0'1'1'} = 0$$

$$\psi_{0'1'1'1'} = 2x_0 \qquad \psi_{1'1'1'1'} = 2x_1.$$

The self-dual two-forms are then given via (4.2.2) by

$$\Sigma^{0'0'} = 2x_0 d\overline{x_0} \wedge dx_0 + \frac{1}{2}x_0 dx_1 \wedge d\overline{x_1} - \frac{1}{2}x_1 d\overline{x_1} \wedge dx_0 - \frac{3}{2}\overline{x_1} d\overline{x_0} \wedge d\overline{x_1} - 2\overline{x_0} d\overline{x_0} \wedge dx_2$$

$$\Sigma^{0'1'} = \frac{1}{2}\overline{x_0}\left(d\overline{x_0} \wedge dx_0 - \frac{1}{8}dx_1 \wedge d\overline{x_1}\right) - \left(\frac{1}{8}x_1 + \frac{9}{8}x_0\right) d\overline{x_1} \wedge dx_0 + \frac{1}{2}\left(x_1 - \overline{x_1}\right) dx_2 \wedge d\overline{x_0} - \frac{7}{16}\overline{x_0} dx_2 \wedge d\overline{x_1} - \left(\frac{7}{16}x_0 - \frac{1}{16}x_1\right) dx_1 \wedge dx_2$$

$$\Sigma^{1'1'} = \frac{1}{2}\left(\overline{x_1} - 4\overline{x_0}\right) d\overline{x_0} \wedge dx_0 - \frac{1}{16}\left(\overline{x_1} + 8\overline{x_0}\right) dx_1 \wedge d\overline{x_1} - \frac{3}{2}x_1 dx_0 \wedge dx_1 + \frac{1}{16}\overline{x_1} dx_2 \wedge d\overline{x_1} - \frac{3}{8}\overline{x_1} dx_1 \wedge d\overline{x_0} - \frac{9}{8}x_0 dx_0 \wedge dx_2\,,$$

restricted to the moduli space $\phi = 0$. From these we could reconstruct the metric, but this is not illuminating and so we will not reproduce it here.

This setup allows us to make contact with [59], in which a generalised Legendre transformation is used to generate hyperKähler manifolds. Using the Kodaira isomorphism we can identify the gradient $d\phi$ with a binary quartic:

$$d\phi \mapsto \mathcal{Q} = \mu_4\lambda^4 + 4\mu_3\lambda^3 + 6\mu_2\lambda^2 + 4\mu_1\lambda + \mu_0$$

where

$$\mu_a = \frac{\partial \phi}{\partial x_a}$$

and x_a are as in (4.1.41). The two classical invariants associated to the binary quartic are

$$\mathcal{I} = \mu_4\mu_0 - 4\mu_3\mu_1 + 3\mu_2^2$$

and

$$\mathcal{J} = \det \begin{pmatrix} \mu_4 & \mu_3 & \mu_2 \\ \mu_3 & \mu_2 & \mu_1 \\ \mu_2 & \mu_1 & \mu_0 \end{pmatrix} .$$

The jumping loci can then be identified by the vanishing or otherwise of $\mathcal{I}$, $\mathcal{J}$, and $d\phi$.

- ⋄ Iff $d\phi = 0$ we have $N_x = \mathcal{O}(-2) \oplus \mathcal{O}(4)$. For the Euclidean example we find that this occurs when $x_0 = x_1 = 0$, and so is a line parametrised by $x_2 \in \mathbb{R}$.

Now consider $d\phi \neq 0$. For the Euclidean example we have

$$\mathcal{I} = 4x_0\overline{x_0} + x_1\overline{x_1}$$

and

$$\mathcal{J} = -2\Re\left(x_0^2\overline{x_1}\right) .$$

- ⋄ Iff $\mathcal{I} = \mathcal{J} = 0$ then we have $N_x = \mathcal{O}(-1) \oplus \mathcal{O}(3)$. This occurs nowhere on the real slice, a consequence of the fact that for odd k we can never have a Euclidean real slice.

- ⋄ Iff $\mathcal{J}$ =0 and $\mathcal{I} \neq 0$ then we have $N_x = \mathcal{O} \oplus \mathcal{O}(2)$, which occurs on the intersection of

$$\Re\left(x_0^2\overline{x_1}\right) = 0 \quad \text{and} \quad \Re\left(x_0x_1\right) = 0.$$

- ⋄ Finally at generic points with $\mathcal{I} \neq 0$ and $\mathcal{J} \neq 0$ we have the usual normal bundle $N_x = \mathcal{O}(1) \oplus \mathcal{O}(1)$.

The generalised Legendre transform of [59] can be used in this context to construct the Kähler potential for the moduli space metric. Following that paper we define G by

$$\frac{\partial G\left(S, \lambda\right)}{\partial S} = \pi_{0'}^{k-2} f \lambda^{-2} \ ,$$

where $\pi_{0'}^{k-2} f$ is the inhomogeneous version of f. For the Euclidean example we have

$$G = \frac{1}{3} S^3 \left(\lambda^{-2} - \lambda^{-8}\right) .$$

The next step is to define

$$F = \frac{1}{2\pi i} \oint_\Gamma \lambda^{-2} G\left(S|, \lambda\right) d\lambda \ ,$$

which in the example is

$$F = -x_0^2 \overline{x_1} + 2x_0 x_1 x_2 - \overline{x_0}^2 x_1 + 2\overline{x_0 x_1} x_2 + \frac{1}{3} x_1^3 + \frac{1}{3} \overline{x_1}^3 .$$

In this context the real moduli space M arises as the subvariety defined by

$$\phi = \frac{\partial F}{\partial x_2} = 4\Re\left(x_0 x_1\right) \ ,$$

in agreement with the earlier treatment. We then perform the generalised Legendre transform of $F\left(x_0, x_1, \overline{x_0}, \overline{x_1}, x_2\right)$, defining

$$u = \frac{\partial F}{\partial x_1} = 2x_0 x_2 - \overline{x_0}^2 + x_1^2 \tag{4.2.3}$$

as the new coordinate and

$$\Omega\left(x_0, \overline{x_0}, u, \overline{u}\right) = F - x_1 u - \overline{x_1 u} \ ,$$

eliminating $\left(x_1, \overline{x_1}, x_2\right)$ in favour of $\left(u, \overline{u}\right)$ using (4.2.3) and $\phi = 0$. The surprising result of [59] is that Ω is the Kähler potential for a hyperKähler metric g on M. In the example we have been following we find

$$\Omega = -2i\left(x_0^3 - \overline{x_0}^3\right) \left(-1 - \frac{\overline{u} x_0 - u \overline{x_0}}{3\left(x_0^3 - \overline{x_0}^3\right)}\right)^{3/2} ,$$

which satisfies the first heavenly equation [69]

$$\frac{\partial^2 \Omega}{\partial x_0 \partial \overline{x_0}} \frac{\partial^2 \Omega}{\partial u \partial \overline{u}} - \frac{\partial^2 \Omega}{\partial x_0 \partial \overline{u}} \frac{\partial^2 \Omega}{\partial u \partial \overline{x_0}} = 1.$$

The metric, if desired, can be explicitly computed from

$$g = \frac{\partial^2 \Omega}{\partial u \partial \overline{u}} du d\overline{u} + \frac{\partial^2 \Omega}{\partial u \partial \overline{x_0}} du d\overline{x_0} + \frac{\partial^2 \Omega}{\partial x_0 \partial \overline{u}} dx_0 d\overline{u} + \frac{\partial^2 \Omega}{\partial x_0 \partial \overline{x_0}} dx_0 d\overline{x_0}.$$

We note that the big-jumping line parametrised by x_2 has been blown-down to a point $u = x_0 = 0$ in the Legendre transformation.

4.3 Physics on folds

The jumping moduli spaces discussed in this section are generically singular at the jumping points. This makes it somewhat remarkable that in [36] Gibbons and Manton, whilst pursuing an unrelated strand of research on the motion of monopoles, make use of smooth normalisable solutions to the time-independent Schrödinger equation on a background whose metric is that of Taub-NUT with negative mass. This background is an example of a jumping spacetime, being in the Gibbons-Hawking class (3.1.22) with

$$V = 1 - \frac{1}{(x^2+y^2+z^2)^{1/2}}.$$

By the proof of theorem 3.1.4 we know that this metric will suffer a jump to $\mathcal{O} \oplus \mathcal{O}(2)$ when $V=0$, i.e. on the submanifold defined by

$$x^2+y^2+z^2 = 1.$$

Taub-NUT with negative mass is an example of an ambipolar metric [60] (see definition 2.5.3) rather than a full folded hyperKähler manifold [43, 12].

The existence of normalisable solutions to the time-independent Schrödinger equation in [36] motivates the question of whether jumping spacetimes generically admit normalisable solutions, potentially making them more relevant to theoretical physics.

Gibbons-Hawking spacetimes with linear potentials are the standard example of a folded hyperKähler manifold, so we here address the question for these spacetimes. Without loss of generality we can consider $V = z$,

$$g = \frac{1}{z}\left(d\psi + \frac{1}{2}xdy - \frac{1}{2}ydx\right)^2 + z\left(dx^2+dy^2+dz^2\right).$$

The determinant and inverse are respectively given by

$$|g| = z^2 \quad \text{and} \quad g^{\mu\nu} = \begin{pmatrix} \left(\frac{(x^2+y^2)}{4z}+z\right) & \frac{y}{2z} & -\frac{x}{2z} & 0 \\ \frac{y}{2z} & \frac{1}{z} & 0 & 0 \\ -\frac{x}{2z} & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & \frac{1}{z} \end{pmatrix}.$$

We aim to investigate how solutions of the time-independent Schrödinger equation

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right) = E\phi$$

behave with respect to the fold and determine whether there are any smooth L^2 solutions.

Written out in full, the Schrödinger equation is

$$\frac{1}{z}(\frac{1}{4}(x^2+y^2)+z^2)\partial_\psi\partial_\psi\phi - \frac{x}{z}\partial_y\partial_\psi\phi + \frac{y}{z}\partial_x\partial_\psi\phi + \frac{1}{z}\delta^{ij}\partial_i\partial_j\phi = E\phi. \tag{4.3.1}$$

To begin tackling this problem we will consider first the simple ψ-independent ansatz

$$\phi(\psi,x,y,z) = \varphi(x,y,z)$$

giving us

$$\frac{1}{z}\delta^{ij}\partial_i\partial_j\varphi = E\varphi. \tag{4.3.2}$$

We can legitimately consider ψ-independent solutions if we take ψ to be an angular coordinate, making normalisability still possible. The equation (4.3.2) separates if we write $\varphi(x,y,z) = f(z)B(x,y)$ to give

$$f_{zz} - Ezf = \lambda f \tag{4.3.3}$$

$$\text{and } \; B_{xx} + B_{yy} = -\lambda B \; , \tag{4.3.4}$$

where λ is the separation constant, and could be complex. The equations (4.3.3) and (4.3.4) are respectively the Airy equations and the two-dimensional Klein-Gordon equation up to coordinate shifts. Consider the latter first.

Equation (4.3.4) is nothing more that a two-dimensional time-independent free-particle Schrödinger equation. This has no smooth normalisable solutions; the stationary states are "scattering" states representing free particles, expressed as plane waves.

Now consider the Airy equation (4.3.3). There are two possibilities: either $E = 0$ or $E \neq 0$. If $E = 0$ then we have

$$f_{zz} - \lambda f = 0 \;\; \Rightarrow \;\; f = a_1 e^{\sqrt{\lambda}z} + a_2 e^{-\sqrt{\lambda}z}.$$

For normalisability we then require λ to have non-vanishing real part, allowing us the continuous solution $f \sim e^{-\sqrt{\lambda}|z|}$. This, however, is not smooth at the fold, and so we can reject it. There are then no smooth normalisable solutions with $E = 0$ which are independent of ψ.

If $E \neq 0$ then we can make the change of variable $\xi = \frac{1}{E}z + \frac{\lambda}{E^2}$ and $f(z) = h(\xi)$ to arrive at the canonical manifestation of the Airy equation,

$$\frac{d^2h}{d\xi^2} - \xi h = 0.$$

The two linearly-independent solutions to this are known as the Airy functions $\mathrm{Ai}(\xi)$ and $\mathrm{Bi}(\xi)$. For negative ξ they both oscillate (out of phase with each other) and decay as ξ becomes more negative, but for positive ξ they're quite different: $\mathrm{Ai}(\xi)$ decays exponentially and $\mathrm{Bi}(\xi)$ diverges exponentially. Thus to have a chance of a normalisable solution we must consider $\mathrm{Ai}(\xi)$.

Unfortunately, for large negative ξ we have

$$\mathrm{Ai}(\xi) \sim |\xi|^{-\frac{1}{4}} \sin\left(\frac{2}{3}|\xi|^{\frac{3}{2}} + \frac{\pi}{4}\right).$$

On a flat back-ground this $|\xi|^{-\frac{1}{4}}$ fall-off is sufficient to ensure that $\mathrm{Ai}(\xi)$ is normalisable, but in our case we have the "extra" factor of $\sqrt{|g|} = |z|$ to consider in our integrals. The relevant part of the integral is

$$\int_{-\infty}^{+\infty} dz\,|z|\,|\mathrm{Ai}\left(\frac{1}{E}z + \frac{\lambda}{E^2}\right)|^2 .$$

At large negative z the integrand goes as $|z|^{\frac{1}{2}}$, and so on the folded background the Airy functions can not give us a normalisable solution, unless one is prepared to suffer a discontinuity in the first derivative, just as in the $E = 0$ case. Thus there are no separable smooth ψ-independent L^2 solutions.

Consider now solutions of the form

$$\phi(\psi, x, y, z) = e^{is\psi}\varphi(x, y, z)$$

for s a non-zero integer. This is a sensible ansatz because ψ is an angular coordinate on the folded background. The Schrödinger equation (4.3.1) becomes

$$-\frac{s^2}{z}(\frac{1}{4}(x^2 + y^2) + z^2)\varphi - \frac{isx}{z}\partial_y\varphi + \frac{isy}{z}\partial_x\varphi + \frac{1}{z}\delta^{ij}\partial_i\partial_j\varphi = E\varphi\ ,$$

which separates as $\varphi = G(x,y)F(z)$ into

$$\frac{d^2F}{dz^2} - (s^2z^2 + Ez + \lambda)F = 0$$

and

$$-\frac{1}{4}s^2(x^2+y^2) - \frac{isx}{G}\partial_y G + \frac{isy}{G}\partial_x G + \frac{1}{G}(\partial_x^2 + \partial_y^2)G + \lambda = 0. \qquad (4.3.5)$$

We'll focus first on the $F(z)$ equation, which has the form of the Schrödinger equation describing a displaced harmonic oscillator. This is readily solved to give

$$F(z) = H_\gamma\left(\sqrt{s}\left(z + \frac{E}{2s^2}\right)\right)\exp\left\{-\frac{1}{2}s\left(z+\frac{E}{2s^2}\right)^2\right\},$$

where $H_\gamma(\xi)$ solves the Hermite equation

$$\frac{d^2H}{d\xi^2} - 2\xi\frac{dH}{d\xi} + 2\gamma H = 0$$

with

$$\gamma = \frac{1}{2s}\left(\frac{E^2}{4s^2} - (\lambda + s)\right).$$

If γ is a non-negative integer then H_γ is a Hermite polynomial and thus $F(z)$ is clearly normalisable for $s > 0$ (even with the folded background's factor of $\sqrt{|g|} = z$) due to the exponential fall-off at large z.

Let us now proceed to consider the $G(x,y)$ equation (4.3.5). This has the form of the Schrödinger equation describing motion in a constant magnetic field (see for example [11]). In the usual manner let us then define the canonical (Hermitian) momenta

$$\Pi_x = -i\partial_x + \frac{1}{2}sy \qquad \Pi_y = -i\partial_y - \frac{1}{2}sx$$

and ladder operators

$$a = \Pi_x + i\Pi_y \qquad a^\dagger = \Pi_x - i\Pi_y.$$

The $G(x,y)$ equation is then

$$(a^\dagger a + s - \lambda)G = 0$$

and we can construct some solutions (choosing $\lambda = s$) by solving $aG_0(x,y) = 0$, and then applying copies of $a^\dagger$ to G_0.

For example, one solution is

$$G(x,y) \propto \exp\left\{-\frac{1}{4}s(x^2+y^2)\right\},$$

and thus we conclude that there do exist normalisable solutions. One class of normalisable solutions is

$$\phi = H_\gamma\left(\sqrt{s}\left(z+\frac{E}{2s^2}\right)\right)\exp\left\{-\frac{1}{2}s\left(z+\frac{E}{2s^2}\right)^2\right\}\exp\left\{-\frac{1}{4}s(x^2+y^2)\right\}\exp\{is\psi\}$$

with s a positive non-zero integer and E chosen such that

$$\gamma = \frac{E^2}{8s^3} - 1$$

is a positive integer, which is always realisable.

4.4 Jumps in Newtonian twistor theory

Jumping phenomena are not constrained to the twistor theory of (complexified) Riemannian spacetimes. In section 4.4 we will examine a novel construction of jumping Newton-Cartan spacetimes.

Theorem 4.4.1. [41]

Let $Z \to \mathbb{P}^1$ be a three-dimensional complex manifold fibred over $\mathcal{O}(3)$ with coordinates (ζ, S, λ) and $(\hat{\zeta}, \hat{S}, \hat{\lambda})$ such that

$$\hat{\zeta} = \lambda\zeta + f(S)$$

$$\hat{S} = \lambda^{-3}S$$

$$\hat{\lambda} = \lambda^{-1}$$

on the intersection of patches, where $f(S)$ is a polynomial of at least quadratic order. Z is a Newtonian twistor space: the Kodaira moduli space M of global sections is locally Newton-Cartan.

- ◇ *The normal bundle is generically $N_x = \mathcal{O} \oplus \mathcal{O}(2)$.*

- *At special points the normal bundle jumps to $\mathcal{O}(-1)\oplus\mathcal{O}(3)$; at these jumping points the Newton-Cartan clock vanishes.*

The special points are characterised by the vanishing of the three two-forms induced on M by the global two form $d\hat{\zeta}\wedge d\hat{S}$ on the fibres of $Z\to\mathbb{P}^1$, or equivalently by the vanishing of the constant term γ_0 in the Laurent expansion of $\frac{\partial f}{\partial S}|$.

Proof

Write the global sections of $\mathcal{O}(3)$ as

$$S| = x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3.$$

First we will consider the normal bundle; we will show that generically the isomorphism class of N_x is $\mathcal{O}\oplus\mathcal{O}(2)$. The patching for N_x is

$$\mathcal{F} = \begin{pmatrix} \lambda & \frac{\partial f}{\partial S}| \\ 0 & \lambda^{-3} \end{pmatrix}$$

and the splitting problem to be solved is

$$\hat{h}_1 = \lambda h_1 + \frac{\partial f}{\partial S}|h_3 \tag{4.4.1}$$

$$\hat{h}_2 = \lambda^3 h_2 + \lambda^2\frac{\partial f}{\partial S}|h_4 \tag{4.4.2}$$

$$\hat{h}_3 = \lambda^{-3}h_3$$

$$\hat{h}_4 = \lambda^{-1}h_4.$$

As usual we put

$$h_3 = \sum_{n=0}^{3} a_n\lambda^n \qquad \text{and} \qquad h_4 = b_0 + b_1\lambda.$$

To proceed further we make an expansion:

$$\frac{\partial f}{\partial S}| = \gamma_0 + \sum_{n=1}^{\infty}\gamma_n\lambda^n$$

where γ_0 is a function of x_0 only. For a global solution of (4.4.2) we then require

$$b_0\gamma_0 = 0 \ , \tag{4.4.3}$$

leaving

$$\hat{h}_2 = 0.$$

We also have

$$\hat{h}_1 = \gamma_0 a_0.$$

The determinant is then

$$\det \hat{H} = \gamma_0 a_0 b_1 \ ,$$

and we conclude that $N_x = \mathcal{O} \oplus \mathcal{O}(2)$ for $\gamma_0 \neq 0$. A straightforward calculation then shows that $N_x = \mathcal{O}(-1) \oplus \mathcal{O}(3)$ for any twistor line X_x with $\gamma_0 = 0$.

Since we generically have $N_x = \mathcal{O} \oplus \mathcal{O}(2)$ we anticipate that the moduli space will be a Newton-Cartan spacetime; to see this in detail we must construct the twistor functions. In this case it is very straightforward: over $\hat{U}$ we have

$$\hat{\zeta} = f\left(S| \left(\lambda = 0\right)\right) := \mathcal{T}(x_0)$$

and

$$\hat{S}| = x_0\hat{\lambda}^3 + x_1\hat{\lambda}^2 + x_2\hat{\lambda} + x_3.$$

We note that $\gamma_0 = \frac{d\mathcal{T}}{dx_0}$. The geometry induced on the moduli space is, as usual, found by identifying null vectors as those tangent to alpha surfaces. The conformal clock is therefore

$$[\theta] = \alpha\left(x_0\right)\gamma_0\left(x_0\right)dx_0$$

for any non-vanishing α and the conformal covariant Galilean metric is

$$\left[h^{-1}\right] = \beta\left(dx_2^2 - 4dx_1dx_3\right)$$

for any non-vanishing β. The geometry is therefore Galilean at generic points (and indeed Newton-Cartan if one chooses to include the canonical Λ-connections discussed in section 2.4.3). At the jumping points with $\gamma_0 = 0$, though, the clock vanishes.

One can construct a map

$$x_0 \mapsto t(x_0) = \mathcal{T}(x_0) \tag{4.4.4}$$

taking M to a more usual non-jumping Newton-Cartan spacetime but the map is not a diffeomorphism; it eliminates the jumping points. This situation is entirely analogous to the map taking the jumping spacetimes of section 4.1 to Gibbons-Hawking form.

The three two-forms arising from the restriction to twistor lines of

$$d\hat{\zeta} \wedge d\hat{S} = \lambda^{-2} d\zeta \wedge dS$$

clearly vanish on and only on the jumping points. □

Example

The simplest example of a jumping Newtonian twistor space is when $f(S) = \frac{1}{2}S^2$. The conformal clock admits a representative

$$\theta = x_0 dx_0 \ ,$$

vanishing at one point $x_0 = 0$. The map (4.4.4) is

$$t = \frac{1}{2}x_0^2 \ ,$$

and so the Newton-Cartan spacetime is thus a 2-fold cover of the standard Newton-Cartan spacetime (with time coordinate t), branched over the spatial fibre $t = 0$.

Chapter 5

Non-relativistic conformal symmetries

Twistor theory and conformal symmetries are closely linked [64, 68]. It is therefore natural when constructing a twistor theory of Newton-Cartan manifolds to consider some related questions to do with the analogues of conformal series in non-relativistic geometry. The reader is reminded that, following [28], some details regarding such symmetries can be found in section 2.7.

5.1 First-order symmetries and Killing vectors

In this section we will construct a duality between the conformal Schrödinger symmetries of a curved Newton-Cartan spacetime and the first-order symmetries of a Schrödinger equation coupled to a potential and magnetic field.

5.1.1 Schrödinger-Killing vectors on curved spacetimes

The equations defining the expanded Schrödinger algebra $\widetilde{\mathfrak{sch}}(d)$ and Schrödinger algebra $\mathfrak{sch}(d)$ in section 2.7 make sense for a curved Newton-Cartan spacetime as well as a flat one: we simply use

$$h = \delta^{ij}\partial_i\partial_j \qquad \theta = dt \qquad \Gamma^i_{tt} = \delta^{ij}\partial_j V \qquad \Gamma^i{}_{jt} = \Gamma^i{}_{tj} = \delta_{jl}\epsilon^{ilk}\partial_k\Omega \tag{5.1.1}$$

with all other connection components vanishing instead of (2.6.3) and $\Gamma^a_{bc} = 0$.

In order to prove theorem 5.1.1 it will be useful to write out in more detail the equations defining the Schrödinger algebra on (5.1.1). Thus we collect for reference

$$\partial_i X^t = 0 \tag{5.1.2}$$

$$\partial^i X^j + \partial^j X^i = \partial_t X^t \delta^{ij} \tag{5.1.3}$$

$$\partial_t \partial_t X^i + X^j \partial_j \partial^i V + 2\partial^i V \partial_t X^t + 2\epsilon^i{}_{jk} \partial^k \Omega \partial_t X^j - \partial^j V \partial_j X^i = 0 \tag{5.1.4}$$

$$\epsilon_{ijk} \partial^j \partial_t X^i + 2X^j \partial_j \partial_k \Omega + 2\partial_k \Omega \partial_t X^t + \partial_k X^j \partial_j \Omega - \partial^j X_k \partial_j \Omega = 0. \tag{5.1.5}$$

(Recall that spatial indices are raised and lowered throughout with Kronecker deltas because (h, θ) is the standard Galilean structure.)

Example

Take the $(3+1)$-dimensional Newton-Cartan spacetime with the linear Newtonian potential $V = z$, adopting $x^i = (x, y, z)$. The Riemann tensor vanishes, so we expect the symmetry group to be of maximal dimension. Solving (5.1.2-5.1.5) yields

$$X = \left(\alpha t^2 + 2\mu t + \gamma\right) \partial_t + \left(\omega^i{}_j x^j + \alpha t x^i + \mu x^i + \nu^i t + \rho^i\right) \partial_i \\ + \frac{1}{2}\omega^{xz} t^2 \partial_x + \frac{1}{2}\omega^{yz} t^2 \partial_y - \left(\frac{2}{3}\alpha t^3 + 2\mu t^2\right) \partial_z.$$

We thus indeed find a twelve-dimensional algebra, though the vectors come with some additional terms which result from the strange choice of coordinates.

Example

The Schrödinger-Killing vectors of the $(3+1)$-dimensional Newton-Cartan spacetime with $V = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ and $\Omega = 0$ are

$$X = \gamma \partial_t + \omega^i{}_j x^j \partial_i$$

for γ a constant and $\omega_{jk} \in \mathfrak{so}(3)$. The presence of a point mass at the origin has reduced the symmetry algebra to just time translations and spatial rotations.

5.1.2 Symmetries of the covariant Schrödinger operator

In this section we will consider the first-order symmetries of the operator

$$\hat{\Delta} = i\partial_t - \frac{1}{2m}\delta^{jk}\left(-i\partial_j + mA_j\right)\left(-i\partial_k + mA_k\right) - mV,$$

where V and A_i depend on space only. That is to say, we will seek first-order linear differential operators

$$\mathcal{D} = S^a(x^b)\partial_a + s(x^b)$$

which obey

$$\hat{\Delta}\mathcal{D} = \delta\hat{\Delta} \tag{5.1.6}$$

for some (otherwise irrelevant) linear differential operator δ.

We will now consider the well-known Schrödinger algebra spanned by vectors which are the non-relativistic analogue of the conformal Killing vectors of flat spacetime and take the definition of these *Schrödinger-Killing* vectors in a general Newton-Cartan spacetime, proceeding to prove the following theorem.

Theorem 5.1.1. [40]

The first-order symmetries of the Schrödinger equation

$$\hat{\Delta}\psi := i\partial_t\psi - \frac{1}{2m}\delta^{jk}\left(-i\partial_j + mA_j\right)\left(-i\partial_k + mA_k\right)\psi - mV\psi = 0 \tag{5.1.7}$$

have the Schrödinger-Killing vectors of the Newton-Cartan spacetime with Galilean coordinates (t, x^i) *and non-vanishing connection components*

$$\Gamma^i{}_{tt} = \delta^{ij}\partial_j V \qquad \textit{and} \qquad \Gamma^i{}_{jt} = \Gamma^i{}_{tj} = \delta_{jl}\epsilon^{ilk}\partial_k\Omega$$

as their principal symbols, where $\Omega(x^j)$ *is a function satisfying* $d\Omega = \star^3 dA$*.*

Equation (5.1.7) is the covariant Schrödinger equation exhibited in [29].

Proof

If we calculate the left-hand-side of (5.1.6) then we get $\mathcal{D}\hat{\Delta}$, which is already in the right form, and some additional operator terms. These additional terms arrange themselves into $\hat{\Delta}$ iff

$$\partial_i S^t \tag{5.1.8}$$

$$\partial^i S^j + \partial^j S^i = \delta^{ij}\partial_t S^t \tag{5.1.9}$$

$$-im\partial_t S^i - imA^j\partial_j S^i + imS^j\partial_j A^i + imA^i\partial_t S^t = \partial^i s \tag{5.1.10}$$

$$\frac{i}{2m}\partial_i\partial^i s - A^i\partial_i s - \left(iS^j\partial_j + i\partial_t S^t\right)\left(\frac{i}{2}\partial_i A^i - \frac{m}{2}A^i A_i - mV\right) = \partial_t s. \tag{5.1.11}$$

In order to prove theorem 5.1.1 we must find the conditions on S^a such that one can always find s solving these equations. To that end we use (5.1.10) to rewrite $\frac{i}{2m}\partial_i\partial^i s - A^i\partial_i s$ in (5.1.11) in terms of S^a only. Then (5.1.10-5.1.11) have the form

$$\Sigma = ds$$

for $\Sigma_a = \Sigma_a(S^b, A^i, V)$. By the Poincaré lemma the conditions we are looking for are

$$d\Sigma = 0. \tag{5.1.12}$$

Explicit calculation reveals that (5.1.8,5.1.9,5.1.12) are then exactly the equations (5.1.2-5.1.5) defining Schrödinger-Killing vectors with $d\Omega = \star^3 dA$, completing the proof of theorem 5.1.1.

$\square$

Note that the gauge symmetry

$$A_i \longmapsto A_i + \partial_i\chi$$

has not here been fixed. The Schrödinger-Killing vectors of the curved Newton-Cartan spacetime are the symmetries of the whole gauge equivalence class of operators $\hat{\Delta}$.

5.2 Higher symmetries and Killing tensors

In this section we will define Newton-Cartan analogues of Killing tensors and conformal Killing tensors and prove that the latter, *Schrödinger-Killing* tensors, are the symbols of the higher symmetry operators of the free-particle Schrödinger operator.

5.2.1 Non-relativistic Killing tensors and conserved quantities

In this section we will define the non-relativistic analogues of Killing tensors by exhibiting Newton-Cartan geodesics as the projection of the integral curves of a Hamiltonian vector field on the cotangent bundle. This Hamiltonian formalism is an intrinsic Newton-Cartan analogue of the Eisenhart-Duval lift discussed in, say, [16].

Lemma 5.2.1. [40]

Geodesics of the Newton-Cartan spacetime (M, h, θ, A, U) with connection components

$$\Gamma^a_{bc} = \frac{1}{2}h^{ad}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bd}\right) + \partial_{(b}\theta_{c)}v^a + \theta_{(b}F_{c)d}h^{ad} \tag{5.2.1}$$

and with $F = dA$ are the projection from T^*M to M of the integral curves of the geodesic spray

$$\mathcal{G} = \left(\frac{1}{2}\partial_a h^{cd}\Pi_c\Pi_d + h^{cd}\Pi_c\partial_a A_d - \partial_a U^b\Pi_b - U^b\partial_a A_b\right)\frac{\partial}{\partial p_a} + \left(U^a - h^{ab}\Pi_b\right)\frac{\partial}{\partial x^a}$$

(where $\Pi_a := p_a + A_a$ and $(x^a, p_b) \in T^*M$), which is the Hamiltonian vector field associated to

$$\mathcal{H} = \frac{1}{2}h^{ab}\Pi_a\Pi_b - U^a\Pi_a.$$

The proof of this lemma is straightforward (but tedious); we omit it for brevity.

This Hamiltonian (and therefore also the following definitions) are Milne-boost invariant.

Definition 5.2.2. [40]. *A rank-n Killing tensor of a Newton-Cartan spacetime (M, h, θ, ∇) is a symmetric contravariant tensor field $X^{a_1 \ldots a_n}$ such that functions $\chi_m^{a_1 \ldots a_m}$ on M can be found obeying*

$$\left\{X^{a_1 \ldots a_n}p_{a_1}\ldots p_{a_n} + \sum_{m=0}^{n-1}\chi_m^{a_1 \ldots a_m}p_{a_1}\ldots p_{a_m}\ ,\ \mathcal{H}\right\} = 0\ , \tag{5.2.2}$$

where $\{\ ,\ \}$ is the canonical Poisson structure on T^*M.

The quantity

$$X^{a_1\ldots a_n}p_{a_1}\ldots p_{a_n} + \sum_{m=0}^{n-1} \chi_m^{a_1\ldots a_m} p_{a_1}\ldots p_{a_m}$$

is a constant of motion.

Here we have provided an intrinsic Newton-Cartan definition of the usual concept of a hidden symmetry, entirely in line with the familiar concept from classical dynamics [21].

Taking $n = 1$ in (5.2.2) we arrive at the conditions

$$\mathcal{L}_X h = 0 \tag{5.2.3}$$

$$\mathcal{L}_X U - h\left(\mathcal{L}_X A\,,\ \right) = -h\left(d\chi_0,\ \right) \tag{5.2.4}$$

$$\left(\mathcal{L}_X A\right)(U) = d\chi_0(U). \tag{5.2.5}$$

Solving (5.2.3-5.2.5) on a given Newton-Cartan spacetime will give us the Killing vectors of that spacetime.

Example

$X = X^a\partial_a$ solves (5.2.3-5.2.5) with

$$h = \delta^{ij}\partial_i\partial_j \quad \theta = dt \quad U = \partial_t \quad A = 0 \tag{5.2.6}$$

and is thus a non-relativistic Killing vector of the flat Newton-Cartan spacetime iff

$$X = \gamma\partial_t + \left(\omega^i_{\ j}x^j + \nu^i t + \rho^i\right)\partial_i$$

for any ten constants $(\gamma, \nu^i, \rho^i,\, \omega_{ij} \in \mathfrak{so}(3))$.

For the boosts (ν^i) we have $\chi_0 \neq 0$ and thus boosts generate gauge transformations of the flat Newton-Cartan spacetime.

Example

The $(3+1)$-dimensional Newton-Cartan spacetime

$$h = \delta^{ij}\partial_i\partial_j \quad \theta = dt \quad U = \partial_t \quad A = -(\delta_{lk}x^l x^k)^{-\frac{1}{2}}dt$$

modelling the Kepler problem admits the following three rank-two non-relativistic Killing tensors

$$X^{ij} = \lambda^l x^k \delta_{lk} \delta^{ij} - \lambda^{(i} x^{j)} \qquad X^{it} = X^{tt} = 0$$

(for $\lambda^i \in \mathbb{R}^3$). The lower order terms are

$$\chi_1^a = 0 \qquad \text{and} \qquad \chi_0 = \frac{\lambda^i \delta_{ij} x^j}{(\delta_{lk} x^l x^k)^{\frac{1}{2}}} \,,$$

and the three associated conserved quantities together form the famous Laplace–Runge–Lenz vector (see e.g. [15]).

5.2.2 Schrödinger-Killing tensors

In generalising the Schrödinger algebra $\mathfrak{sch}(d)$ to the case of Schrödinger-Killing tensors we will again make use of the Hamiltonian formalism introduced above. The following definition is, in the Hamiltonian formalism, a natural way to define a notion of a conformal Killing tensor.

Definition 5.2.3. [40]. *A Schrödinger-Killing tensor of a Newton-Cartan spacetime (M, h, θ, ∇) is a symmetric contravariant tensor field $X^{a_1 \ldots a_n}$ for which functions $\chi_m^{a_1 \ldots a_m}$ on M can be found obeying*

$$\left\{ X^{a_1 \ldots a_n} p_{a_1} \ldots p_{a_n} + \sum_{m=0}^{n-1} \chi_m^{a_1 \ldots a_m} p_{a_1} \ldots p_{a_m} \,,\, \mathcal{H} \right\} = \sum_{m=0}^{n-1} \left(f_m^{a_1 \ldots a_m} p_{a_1 \ldots} p_{a_m} \right) \mathcal{H} \,, \tag{5.2.7}$$

for some symmetric tensor fields $f_m^{a_1 \ldots a_m}$.

A Killing tensor as defined above is a special case of a Schrödinger-Killing tensor.
If $n = 1$ we have

$$\mathcal{L}_X h = f_0 h \tag{5.2.8}$$

$$\mathcal{L}_X U - h\left(\mathcal{L}_X A \,,\ \right) = f_0 U - h\left(d\chi_0 ,\ \right) \tag{5.2.9}$$

$$\left(\mathcal{L}_X A\right)(U) = d\chi_0(U). \tag{5.2.10}$$

Using the flat Galilean Newton-Cartan spacetime (5.2.6) reduces this definition to that of $\mathfrak{sch}(d)$ above.

In order to prove theorem 5.2.4 we will display in more detail the conditions describing the Schrödinger-Killing tensors of the flat Newton-Cartan spacetime. The defining condition (5.2.7) becomes the coupled family of equations

$$-\partial^i X^{a_1\ldots a_n} p_i p_{a_1}\ldots p_{a_n} = \frac{1}{2}\delta^{ij} f_{n-1}^{a_1\ldots a_{n-1}} p_i p_j p_{a_1}\ldots p_{a_{n-2}} \tag{5.2.11}$$

$$\begin{aligned}
&\partial_t X^{a_1\ldots a_n} p_{a_1}\ldots p_{a_n} - \partial^i \chi_{n-1}^{a_1\ldots a_{n-1}} p_i p_{a_1}\ldots p_{a_{n-1}} \\
&\qquad = -f_{n-1}^{a_1\ldots a_{n-1}} p_t p_{a_1}\ldots p_{a_{n-1}} + \frac{1}{2}\delta^{ij} f_{n-2}^{a_1\ldots a_{n-2}} p_i p_j p_{a_1}\ldots p_{a_{n-2}}
\end{aligned}$$

$$\begin{aligned}
&\partial_t \chi_{n-1}^{a_1\ldots a_{n-1}} p_{a_1}\ldots p_{a_{n-1}} - \partial^i \chi_{n-2}^{a_1\ldots a_{n-2}} p_i p_{a_1}\ldots p_{a_{n-2}} \\
&\qquad = -f_{n-2}^{a_1\ldots a_{n-2}} p_t p_{a_1}\ldots p_{a_{n-2}} + \frac{1}{2}\delta^{ij} f_{n-3}^{a_1\ldots a_{n-3}} p_i p_j p_{a_1}\ldots p_{a_{n-3}}
\end{aligned}$$

$$\vdots$$

$$\partial_t \chi_2^{a_1 a_2} p_{a_1} p_{a_2} - \partial^i \chi_1^{a_1} p_i p_{a_1} = -f_1^{a_1} p_t p_{a_1} + \frac{1}{2}\delta^{ij} f_0 p_i p_j$$

$$\partial_t \chi_1^{a_1} p_{a_1} - \partial^i \chi_0 p_i = -f_0 p_t$$

$$\partial_t \chi_0 = 0\ . \tag{5.2.12}$$

We can rewrite these concisely using the Schouten brackets of X with h and U, denoted $\mathcal{L}_X h$ and $\mathcal{L}_X U$. They become

$$\mathcal{L}_X h = f_{n-1} h \tag{5.2.13}$$

$$\mathcal{L}_{\chi_{n-1}} h - 2\mathcal{L}_X U = f_{n-2} h - 2 f_{n-1} U \tag{5.2.14}$$

$$\mathcal{L}_{\chi_{n-2}} h - 2\mathcal{L}_{\chi_{n-1}} U = f_{n-3} h - 2 f_{n-2} U \tag{5.2.15}$$

$\vdots$ etc.

with this pattern continuing on the understanding that for negative m we have $f_m = \chi_m = 0$, and where all indices on the right-hand-side products are symmetrised.

5.2.3 Higher symmetry operators

The higher symmetries of the Laplacian and of various Schrödinger operators have been calculated and are to be found in the literature [8, 31, 61]. In this section we will define such symmetries, following those papers, and then proceed to prove theorem 5.2.4, identifying the higher symmetries of the free Schrödinger operator with the Schrödinger-Killing tensors of the flat Newton-Cartan spacetime.

The Laplacian

In [31] Eastwood finds the *higher symmetries* of the Laplacian. These are linear differential operators

$$\mathcal{D} = V_n^{\mu_1 \dots \mu_n} \frac{\partial^n}{\partial x^{\mu_1} \partial x^{\mu_2} \dots \partial x^{\mu_n}} + V_{n-1}^{\mu_1 \dots \mu_{n-1}} \frac{\partial^{n-1}}{\partial x^{\mu_1} \partial x^{\mu_2} \dots \partial x^{\mu_{n-1}}} + \dots + V_1^{\mu_1} \frac{\partial}{\partial x^{\mu_1}} + V_0$$

which commute with the Laplacian Δ_L in the sense that

$$\Delta_L \mathcal{D} = \delta \Delta_L$$

for some linear differential operator δ (determined by $\mathcal{D}$). The functions $V_p^{\mu_1 \dots \mu_p}$ (for $0 \leq p \leq n$) are the components of totally symmetric rank-p tensor fields on flat space, and the tensor of highest rank is called the symbol of the symmetry operator. Eastwood finds that if $\mathcal{D}$ is a symmetry of the Laplacian then its symbol is a conformal Killing tensor on flat spacetime, i.e.

$$\partial^{(\mu_0} V_n^{\mu_1 \dots \mu_n)} = g^{(\mu_0 \mu_1} k^{\mu_2 \dots \mu_n)} \tag{5.2.16}$$

for some rank-$(n-1)$ tensor field k (which itself is determined from (5.2.16)) and inverse (flat) metric $g^{\mu\nu}$. Furthermore, when (5.2.16) is satisfied for some symbol $V_n^{\mu_1 \dots \mu_n}$ one can uniquely solve for lower order operators $(V_{n-1}^{\mu_1 \dots \mu_{n-1}}, V_{n-2}^{\mu_1 \dots \mu_{n-2}}, \dots, V_0)$ determined in terms of the symbol such that $\mathcal{D}$ is a symmetry of the Laplacian [31].

The free Schrödinger operator

The analogous higher symmetries of the free-particle Schrödinger operator

$$\Delta = i\partial_t + \frac{1}{2m} \delta^{ij} \partial_i \partial_j$$

can be found in the literature [61]. Here we will summarise and make use of the approach of [8], where the symmetries of Δ in $d+1$ dimensions arise as the light-cone reduction of conformal Killing tensors in $d+2$ dimensions.

Consider the wave equation in $d{+}2$ dimensions, written in light-cone coordinates (x^i, x^+, x^-):

$$\Delta_L \phi = \left(\delta^{ij}\partial_i\partial_j - 2\partial_+\partial_-\right)\phi = 0.$$

Restricting to fields of the form

$$\phi(x^i, x^+, x^-) = \psi(x^+, x^i)\exp\left\{-imx^-\right\} \tag{5.2.17}$$

reduces the wave equation to

$$\left(i\partial_+ + \frac{1}{2m}\delta^{ij}\partial_i\partial_j\right)\psi(x^+, x^i) = 0\ ,$$

which is just $\Delta\psi = 0$ if we identify x^+ with time.

Let $\mathcal{D}$ be a symmetry of the Laplacian, allowing us to write

$$\Delta_L \mathcal{D}\phi = \delta\Delta_L\phi.$$

Restricting to the ansatz (5.2.17) reduces this to

$$\Delta_L \mathcal{D}\left(e^{-imx^-}\psi\right) = \delta e^{-imx^-}\Delta\phi. \tag{5.2.18}$$

Applying $\mathcal{D}$ to $e^{-imx^-}\psi$ results in a new symmetry operator $\tilde{D}$:

$$\mathcal{D}\left(e^{-imx^-}\psi\right) = e^{-imx^-}\tilde{\mathcal{D}}\psi.$$

The left-hand-side of (5.2.18) rearranges into Δ iff $\partial_-\tilde{D} = 0$, giving us

$$\Delta\tilde{\mathcal{D}}\psi = \tilde{\delta}\Delta\psi \qquad \text{for } \ \partial_-\tilde{D} = 0.$$

We can reverse these steps, giving us the statement that the higher symmetries of Δ are the operators $\tilde{\mathcal{D}}$, arising from conformal Killing tensors.

Theorem 5.2.4. [40]

The higher symmetries of the free Schrödinger equation

$$i\partial_t\psi = -\frac{1}{2m}\delta^{ij}\partial_i\partial_j\psi$$

are linear differential operators which have the Schrödinger-Killing tensors of the flat Galilean Newton-Cartan spacetime

$$h = \delta^{ij}\partial_i\partial_j \qquad \theta = dt \qquad \Gamma^a_{bc} = 0$$

as their principal symbols.

Proof

To prove theorem 5.2.4 we will consider the conformal Killing equation in $d+2$ dimensions with coordinates $x^\mu = (x^i, x^+, x^-)$. We will calculate the resulting conditions on $\mathcal{D}$ and compare them to the equations (5.2.11-5.2.12) characterising Schrödinger-Killing tensors.

For a tensor of rank n the conformal Killing equation is

$$\partial^{(\mu_0}S^{\mu_1\ldots\mu_n)} = g^{(\mu_0\mu_1}k^{\mu_2\ldots\mu_n)}$$

where the only non-vanishing components of the metric are

$$g_{ij} = \delta_{ij} \qquad g_{+-} = g_{-+} = -1\,.$$

Recall that we are only interested in solutions which do not depend on x^-.

Consider first the case $(\mu_0 \ldots \mu_n) = (a_0 \ldots a_n)$, i.e. no $(-)$ indices are included. We can then identify

$$g^{ab} = h^{ab} \qquad \partial^a = h^{ab}\partial_b\ ,$$

giving us

$$h^{b(a_0}\partial_b S^{a_1\ldots a_n)} = h^{(a_0a_1}k^{a_2\ldots a_n)}.$$

Writing $X^{a_1\ldots a_n} = S^{a_1\ldots a_n}$ and $k^{a_1\ldots a_{n-1}} = -\frac{1}{2}f_{n-1}^{a_1\ldots a_{n-1}}$, we then have

$$\mathcal{L}_X h = f_{n-1} h\ ,$$

the first of the equations (5.2.11-5.2.12) characterising the Schrödinger-Killing tensor $X^{a_1...a_n}$. Similarly we can start to set one index to $(-)$ and the rest to $(a_1...a_n)$, giving

$$\partial^- S^{a_1...a_n} + n\partial^{(a_1} S^{a_2...a_n)-} = 2g^{-(a_1} k^{a_2...a_n)} + (n-1)g^{(a_1 a_2} k^{a_3...a_n)-}$$

$$\Rightarrow \ \partial_+ S^{a_1...a_n} - nh^{b(a_1}\partial_b S^{a_2...a_n)-} = 2\delta_+^{(a_1} k^{a_2...a_n)} - (n-1)h^{(a_1 a_2} k^{a_3...a_n)-}.$$

Again, this equation is the same as (5.2.14) with the identifications

$$\chi_{n-1}^{a_1...a_{n-1}} = nS^{-a_1...a_{n-1}} \qquad f_{n-2}^{a_1...a_{n-2}} = -2(n-1)k^{-a_1...a_{n-2}}. \tag{5.2.19}$$

In fact, all of the equations (5.2.11-5.2.12) can be reproduced in this manner from the conformal Killing equation, one for each number of indices set to $(-)$, and with similar identifications to (5.2.19). The equation with q indices set to $(-)$ and the rest set to $(a_1...a_{n+1-q})$ is

$$\begin{aligned} &q\partial^- S^{-...-a_1...a_{n+1-q}} + (n+1-q)\partial^{(a_1} S^{a_2...a_{n+1-q})-...-} \\ &\quad = 2\frac{q}{n}(n+1-q)g^{-(a_1} k^{a_2...a_{n+1-q})-...-} + \frac{1}{n}(n+1-q)(n-q)g^{(a_1 a_2} k^{a_3...a_{n+1-q})-...-} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad &q\partial_+ S^{-...-a_1...a_{n+1-q}} - (n+1-q)h^{b(a_1}\partial_b S^{a_2...a_{n+1-q})-...-} \\ &\quad = 2\frac{q}{n}(n+1-q)\delta_+^{(a_1} k^{a_2...a_{n+1-q})-...-} - \frac{1}{n}(n+1-q)(n-q)h^{(a_1 a_2} k^{a_3...a_{n+1-q})-...-}. \end{aligned}$$

We can then identify

$$\chi_{n-q}^{a_1...a_{n-q}} = \binom{n}{q} S^{a_1...a_{n-q}-...-} \qquad \text{for } q \geq 1$$

$$\text{and} \quad f_{n-q}^{a_1...a_{n-q}} = -2\binom{n-1}{q-1} k^{a_1...a_{n-q}-...-} \qquad \text{for } q \geq 1.$$

These equations are now exactly those above characterising Schrödinger-Killing tensors, completing the proof of theorem 5.2.4.

□

With this achieved, a natural question to ask is whether this result extends to curved Newton-Cartan spacetimes and the covariant Schrödinger equation. The situation here remains unclear, just as it does in the case of the curved Riemannian manifold and the Laplacian, and we defer this question to future investigations.

5.3 Symmetries and twistor spaces

In the nonlinear graviton construction the conformal symmetries of the spacetime are in one-to-one correspondence with holomorphic vector fields on twistor space[1], that is to say the conformal symmetries on M arise as global sections of $TZ \to Z$.

It is natural then to ask whether anything similar can be said in Newtonian twistor theory, and if so, to ask which of the many non-relativistic conformal symmetry algebras are singled out.

5.3.1 Four dimensions

We'll focus mainly on the case of Newtonian twistor theory in four dimensions, where we'll first identify the algebra of global vector fields and then characterise two interesting subalgebras.

5.3.1.1 Global holomorphic vector fields

Let $Z = \mathcal{O} \oplus \mathcal{O}(2)$ in accordance with the twistor theory of section 3.1.

Theorem 5.3.1. [40]

The global sections of $TZ \to Z$ are in one-to-one correspondence with the vectors of the conformal Newton-Cartan algebra $\mathfrak{cnc}(3)$ associated to the Newton-Cartan spacetime M with standard Galilean structure (2.6.3) and $\Gamma^a_{bc} = 0$.

The algebra $\mathfrak{cnc}(3)$ is defined in section 2.7. Both $\mathfrak{cnc}(3)$ and $\check{H}^0(Z, TZ)$ are infinite-dimensional Lie algebras.

[1]This fact is "well-known", and is proven carefully in [53].

Proof

We will prove theorem 5.3.1 by directly calculating the holomorphic vector fields $\beta \in \check{H}^0(Z, TZ)$ on. Write the inhomogeneous twistor coordinates as column vectors

$$Z^\rho = \begin{pmatrix} T \\ Q \\ \lambda \end{pmatrix} \qquad \text{and} \qquad \hat{Z}^\rho = \begin{pmatrix} \hat{T} \\ \hat{Q} \\ \hat{\lambda} \end{pmatrix}.$$

The patching

$$\hat{\beta}^\rho = \frac{\partial \hat{Z}^\rho}{\partial Z^\sigma} \beta^\sigma$$

can be expanded to give

$$\hat{\beta}^T = \beta^T$$
$$\hat{\beta}^Q = \lambda^{-2}\beta^Q - 2\lambda^{-3} Q \beta^\lambda$$
$$\hat{\beta}^\lambda = -\lambda^{-2}\beta^\lambda.$$

By considering an ansatz in which β^α are arbitrary polynomials in (Q, λ) whose coefficients are arbitrary holomorphic functions of the trivial coordinate T we find that $\beta \in \check{H}^0\left(Z, TZ\right)$ iff

$$\beta = h(T)\frac{\partial}{\partial T} + \left(a(T) + b(T)Q + c(T)\lambda + d(T)\lambda Q + e(T)\lambda^2\right)\frac{\partial}{\partial Q} + \left(f(T) + g(T)\lambda + \frac{1}{2}\lambda^2 d(T)\right)\frac{\partial}{\partial \lambda} \qquad (5.3.1)$$

for $(a, b, c, ..., h)$ any eight holomorphic functions of T. These sections form an infinite-dimensional Lie algebra (under the usual commutator).

Pushing this algebra to M is a two-stage procedure. First we consider an arbitrary vector $\Lambda \in T(P\mathbb{S}')$ and it's push-forward to Z:

$$(\mu_\star \Lambda)^\alpha = \frac{\partial \left(Z^\alpha|\right)}{\partial x^\Sigma}\Lambda^\Sigma,$$

where $x^\Sigma = (x^a, \lambda)$ are coordinates on $\mu^{-1}(U) \subset P\mathbb{S}'$. Thus setting $\beta^\alpha = (\mu_\star\Lambda)^\alpha$ we have

$$\beta^T = \Lambda^t \qquad \beta^Q - \frac{\partial (Q|)}{\partial \lambda}\beta^\lambda = \Lambda^i \frac{\partial (Q|)}{\partial x^i} \qquad \beta^\lambda = \Lambda^\lambda,$$

and we can uniquely determine a vector Λ such that Λ^a does not depend on λ (necessary for the next step). The second half of the procedure is to simply push-down Λ to $Y = \nu_\star \Lambda$ on M, giving

$$Y = \Lambda^a(x^i, t)\frac{\partial}{\partial x^a}.$$

Doing this for the general global vector (5.3.1) yields

$$Y = h(t)\frac{\partial}{\partial t} + \left(\omega^i_{\ j}(t)x^j + \chi(t)x^i + \eta^i(t)\right)\frac{\partial}{\partial x^i}$$

where

$$\chi(t) = b(t) - g(t)$$
$$\omega^x_{\ y}(t) = ig(t)$$
$$\omega^z_{\ x}(t) = f(t) + \frac{1}{2}d(t)$$
$$\omega^z_{\ y}(t) = i\left(\frac{1}{2}d(t) - f(t)\right)$$
$$\eta^i(t)\frac{\partial(Q|)}{\partial x^i} = a(t) + c(t)\lambda + e(t)\lambda^2,$$

revealing Y to be an arbitrary element of $\mathfrak{cnc}(3)$. The procedure is reversible: the ∂_λ component of the lift of Y to $P\mathbb{S}'$ is calculated by requiring that the resulting vector field should descend to a vector field on Z. This completes the proof of theorem 5.3.1.

□

Note that the factors of i above do not prevent Y from being real; it is possible to choose the real and imaginary parts of $(a, b, c, ..., h)$ such that Z is any element of the real $\mathfrak{cnc}(3)$.

5.3.1.2 The expanded Schrödinger algebra

The expanded Schrödinger algebra $\widetilde{\mathfrak{sch}}(3)$ in a finite-dimensional subalgebra of $\mathfrak{cnc}(3)$, and so it is to natural to ask what characterises the corresponding holomorphic vector fields on Z. On M we pick out the subalgebra (as described in section 2.7) by a requirement that the vector must generate projective transformations of a connection. In this section we will describe an analogous procedure which takes place in twistor space.

In order to have a notion of a projective holomorphic vector field on Z we must first establish some kind of affine connection on the twistor space. Considering *local* geometry on twistor spaces is an eccentric maneuver; usually one wishes to emphasise that *global* data on Z reduces to interesting local information on M.

Using the standard law for transforming connection components we can write down the patching for a new bundle $\mathcal{G} \to Z$, whose sections are (the components of) torsion-free affine connections on twistor space. The patching is

$$\hat{\Gamma}^{\alpha}_{\beta\gamma} = \frac{\partial \hat{Z}^{\alpha}}{\partial Z^{\mu}} \frac{\partial Z^{\nu}}{\partial \hat{Z}^{\beta}} \frac{\partial Z^{\rho}}{\partial \hat{Z}^{\gamma}} \Gamma^{\mu}_{\nu\rho} - \frac{\partial Z^{\nu}}{\partial \hat{Z}^{\beta}} \frac{\partial Z^{\rho}}{\partial \hat{Z}^{\gamma}} \frac{\partial^2 \hat{Z}^{\alpha}}{\partial Z^{\nu} \partial Z^{\rho}}. \tag{5.3.2}$$

One might expect that we could now identify $\check{H}^0(Z, \mathcal{G})$ and then proceed to consider the projective vectors of global connections. Unfortunately there is an issue: the bundle $\mathcal{G}$ admits no global sections, as can be seen from, for example, the ${}^{\alpha}_{\beta\gamma} = {}^{\lambda}_{\lambda\lambda}$ component of the patching,

$$\hat{\Gamma}^{\lambda}_{\lambda\lambda} = -\lambda^2 \Gamma^{\lambda}_{\lambda\lambda} - 4\lambda Q \Gamma^{\lambda}_{Q\lambda} - 4Q^2 \Gamma^{\lambda}_{QQ} - 2\lambda. \tag{5.3.3}$$

The final term here cannot be (holomorphically) included in any of the other terms, so there can be no global solutions to (5.3.3); $\check{H}^0(Z, \mathcal{G}) = 0$.

Inspection of (5.3.2) reveals that if one considers only vectors in the $Z^A = (T, Q)$ directions then the patching for the relevant connection components *does* admit global sections. Thus it is sensible to decompose the tangent bundle as

$$T(PT_\infty) = \mathfrak{h} \oplus \mathfrak{v}$$

with respect to the fibration $Z \to \mathbb{P}^1$, i.e. such that

$$\beta \in \mathfrak{v} \text{ iff } d\lambda(\beta) = 0 \ .$$

The general global section of the reduced bundle $\mathcal{G}_{\mathfrak{v}}$ is then

$$\Gamma^{T}_{TT} = \Sigma(T) \qquad \Gamma^{Q}_{TQ} = \Gamma^{Q}_{QT} = \Xi(T)$$

$$\Gamma^{Q}_{TT} = \Phi_0(T) + \Phi_1(T)\lambda + \Phi_2(T)\lambda^2 + \Psi(T)Q \,, \tag{5.3.4}$$

with all other components Γ^A_{BC} set to zero, and where $(\Sigma, \Xi, \Phi_0, \Phi_1, \Phi_2, \Psi)$ are six arbitrary holomorphic functions of T.

In the spirit of the spacetime characterisation of the expanded Schrödinger algebra we will consider, out of all the possibilities in (5.3.4), the case in which *all* $\Gamma^A_{BC} = 0$. We then define the subalgebra $S_{\mathfrak{v}} \subset \mathfrak{v}$ to be the algebra of vertical holomorphic vector fields β obeying

$$\mathcal{L}_\beta \Gamma^A_{BC} = \delta^A_{(B}\kappa_{C)}\,, \tag{5.3.5}$$

for κ a one-form on Z to be determined in solving (5.3.5). Not unexpectedly, κ is always an element of $\check{H}^0(Z, T^*Z)$, which is populated only by one-forms $k(T)dT$ for holomorphic k corresponding to the conformal clock on M.

To get the twistorial analogue $\tilde{S}$ of the expanded Schrödinger algebra we must then re-include the ∂_λ parts of the vectors. This is done by taking the closure under Lie bracket of

$$S_{\mathfrak{v}} \oplus \left\{ \left(f(T) + g(T)\lambda + \frac{1}{2}\lambda^2 d(T) \right) \frac{\partial}{\partial \lambda} \right\}.$$

The thirteen-dimensional Lie algebras $\tilde{S}$ on Z and $\widetilde{\mathfrak{sch}}(3)$ on M are then in one-to-one correspondence, a subcorrespondence of that in theorem 5.3.1.

5.3.1.3 The CGA

In [33] the authors discuss a particular non-relativistic limit of the conformal algebra, in which one sends $c \to \infty$ but scales each generator by an appropriate factor of c such that the leading term survives. The number of generators is therefore unchanged, and the resulting fifteen-dimensional algebra is known as the CGA (conformal Galilean algebra).

We can realise Z as the $c \to \infty$ limit of the twistor space Z_c associated to Minkowski space as in theorem 3.1.1, and so we can take a limit of the (fifteen) holomorphic vector fields on Z_c in the CGA style to give a representation of the CGA on Z. The conformal Killing vectors on Minkowski space M_c and their resulting limits on Z are shown in the following table.

	Vector on M_c	Limit on PT_∞
Translations	$\frac{\partial}{\partial t}$	$\frac{\partial}{\partial T}$
	$\frac{\partial}{\partial x}$	$(\lambda^2 - 1)\frac{\partial}{\partial Q}$
	$\frac{\partial}{\partial y}$	$-i(\lambda^2 + 1)\frac{\partial}{\partial Q}$
	$\frac{\partial}{\partial z}$	$-2\lambda\frac{\partial}{\partial Q}$
Dilation	$t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$	$T\frac{\partial}{\partial T} + Q\frac{\partial}{\partial Q}$
Rotations	$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$	$iQ\frac{\partial}{\partial Q} + i\lambda\frac{\partial}{\partial \lambda}$
	$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$	$-i\lambda Q\frac{\partial}{\partial Q} - \frac{i}{2}(\lambda^2 - 1)\frac{\partial}{\partial \lambda}$
	$z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$	$-\lambda Q\frac{\partial}{\partial Q} - \frac{1}{2}(1 + \lambda^2)\frac{\partial}{\partial \lambda}$
Boosts	$t\frac{\partial}{\partial x} + \frac{x}{c^2}\frac{\partial}{\partial t}$	$(\lambda^2 - 1)T\frac{\partial}{\partial Q}$
	$t\frac{\partial}{\partial y} + \frac{y}{c^2}\frac{\partial}{\partial t}$	$-i(1 + \lambda^2)T\frac{\partial}{\partial Q}$
	$t\frac{\partial}{\partial z} + \frac{z}{c^2}\frac{\partial}{\partial t}$	$-2\lambda T\frac{\partial}{\partial Q}$
Special	$-2t\,(x \cdot \partial) - (x \cdot x)\,\frac{1}{c^2}\frac{\partial}{\partial t}$	$-T^2\frac{\partial}{\partial T} - 2TQ\frac{\partial}{\partial Q}$
	$\frac{2}{c^2}x\,(x \cdot \partial) - \frac{1}{c^2}\,(x \cdot x)\,\frac{\partial}{\partial x}$	$(\lambda^2 - 1)T^2\frac{\partial}{\partial Q}$
	$\frac{2}{c^2}y\,(x \cdot \partial) - \frac{1}{c^2}\,(x \cdot x)\,\frac{\partial}{\partial y}$	$-i(\lambda^2 + 1)T^2\frac{\partial}{\partial Q}$
	$\frac{2}{c^2}z\,(x \cdot \partial) - \frac{1}{c^2}\,(x \cdot x)\,\frac{\partial}{\partial z}$	$-2\lambda T^2\frac{\partial}{\partial Q}$

The CGA on Z is a finite-dimensional subalgebra of $\check{H}^0(Z, TZ)$, giving us another sub-correspondence of that in theorem 5.3.1.

5.3.2 Three and five dimensions

In this section we'll identify the algebra of global vector fields for the flat models of three- and five-dimensional Newtonian twistor theory.

Three dimensions: $Z = \mathcal{O} \oplus \mathcal{O}(1)$

We now compute $\check{H}^0\,(Z, TZ)$ for the flat model $Z = \mathcal{O} \oplus \mathcal{O}(1)$ and pull the elements back to sections of $TM \to M$. The patching for $TZ \to Z$ is

$$\hat{\beta}^{\tilde{\alpha}} = \frac{\partial \hat{Z}^{\tilde{\alpha}}}{\partial Z^{\tilde{\beta}}}\beta^{\tilde{\beta}} \tag{5.3.6}$$

where

$$Z^{\tilde{\alpha}} = (T, \Omega, \lambda)^T \qquad \hat{Z}^{\tilde{\alpha}} = \left(\hat{T}, \hat{\Omega}, \hat{\lambda}\right)^T$$

and where we write β for a section of TZ. The components of (5.3.6) read

$$\hat{\beta}^T = \beta^T \qquad \hat{\beta}^\Omega = \lambda^{-1}\beta^\Omega - \lambda^{-2}\Omega\beta^\lambda \qquad \hat{\beta}^\lambda = -\lambda^{-2}\beta^\lambda.$$

The most general global section is then given by

$$\beta^T = h_0$$

$$\beta^\Omega = d_0 + d_1\lambda + e_0\Omega + a_2\Omega\lambda + b_1\Omega^2$$

$$\beta^\lambda = a_0 + a_1\lambda + a_2\lambda^2 + b_0\Omega + b_1\Omega\lambda$$

for nine arbitrary holomorphic functions $(a_0, a_1, a_2, b_0, b_1, d_0, d_1, e_0, h_0)$ of T.

Now let Λ be a vector field on $P\mathbb{S}' \to M$, so we have

$$\Lambda = \Lambda^\Lambda \frac{\partial}{\partial x^\Lambda} = \Lambda^\lambda \frac{\partial}{\partial \lambda} + \Lambda^a \frac{\partial}{\partial x^a}$$

over $U \subset \mathbb{P}^1$. Then $\beta = \mu_* \Lambda$ is a vector field on Z with components

$$\beta^\mu = \frac{\partial Z^\mu|}{\partial x^\Lambda}\Lambda^\Lambda$$

so we have

$$\Lambda^t = h_0$$

$$\Lambda^\lambda = a_0 + a_1\lambda + a_2\lambda^2 + b_0\left(y + z\lambda\right) + b_1\left(y + z\lambda\right)\lambda$$

$$\Rightarrow \qquad \Lambda^y = d_0 + e_0 y + b_1 y^2 - z a_0 - b_0 y z$$

$$\text{and} \qquad \Lambda^z = d_1 + \left(e_0 - a_1\right) z + a_2 y + b_1 y z - b_0 z^2.$$

We can then push Λ down to M to obtain

$$Y := \nu_* \Lambda = h_0 \partial_t + \left(A^i + B^i_j x^j + x^i C_j x^j\right)\partial_i$$

where

$$A^y = d_0 \qquad A^z = d_1$$

$$B^i_{\ j} = \begin{pmatrix} e_0 & -a_0 \\ a_2 & e_0 - a_1 \end{pmatrix}$$
$$C_y = b_1 \qquad C_z = -b_0.$$

This is a nine-dimensional Lie algebra under the usual bracket; it does not fit automatically into any of the algebras discussed in section 2.7.

We can interpret this algebra heuristically as follows. Take the eight-dimensional algebra of projective vector fields on the two-dimensional spatial fibres, add time translations, and then promote the nine-dimensional algebra to an infinite-dimensional one by allowing the nine components to carry arbitrary holomorphic functions on the time axis. We can write

$$\check{H}^0\left(Z, TZ\right) = \left\{ \underset{\text{eight}}{\mathfrak{p}\left(2, \mathbb{C}\right)} \oplus \underset{\text{one}}{\left\{ \frac{\partial}{\partial T} \right\}} \right\} \otimes H\left(\mathcal{O}_{\mathcal{T}}\right)$$

where $\mathfrak{p}\left(2, \mathbb{C}\right)$ is the algebra of projective vector fields on the (flat) two-dimensional spatial slices, and where $H\left(\mathcal{O}_{\mathcal{T}}\right)$ are the holomorphic functions on the time axis.

Five dimensions: $Z = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$

Now we will study the image on M of $\check{H}^0\left(Z, TZ\right)$ for $Z = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$.

As usual let Z^α run over w^μ and λ, with analogous definitions for $\hat{U}$. The patching for $TZ \to Z$ is

$$\hat{V}^\alpha = \frac{\partial \hat{Z}^\alpha}{\partial Z^\beta} V^\beta \quad \Rightarrow \quad \begin{aligned} \hat{\beta}^T &= \beta^T \\ \hat{\beta}^A &= \lambda^{-1}\beta^A - \lambda^{-2} w^A \beta^\lambda \\ \hat{\beta}^\lambda &= -\lambda^{-2}\beta^\lambda \end{aligned} \quad , \tag{5.3.7}$$

and there is one global function of weight zero to consider, $\hat{T} = T$. The general global section of (5.3.7) is

$$\begin{aligned} \beta = a(T)\frac{\partial}{\partial T} &+ \left(h^A(T) + g^A(T)\lambda + j^A_B(T) w^B + d(T)\lambda w^A + f_B(T) w^B w^A\right) \frac{\partial}{\partial w^A} \\ &+ \left(b(T) + c(T)\lambda + d(T)\lambda^2 + e_A(T) w^A + f_A(T) w^A \lambda\right) \frac{\partial}{\partial \lambda} \ , \end{aligned} \tag{5.3.8}$$

depending on sixteen (holomorphic) functions of T. Whilst we won't explicitly include the calculation of the image on M, it is clear that the global vector algebra (5.3.8) has the decomposition

$$\check{H}^0\left(Z, TZ\right) = \left\{ \underset{\text{fifteen}}{\mathfrak{sl}(4,\mathbb{C})} \oplus \underset{\text{one}}{\left\{ \frac{\partial}{\partial T} \right\}} \right\} \otimes H\left(\mathcal{O}_{\mathcal{T}}\right)$$

into conformal symmetries of the flat degenerate metric and time translations, where $H\left(\mathcal{O}_{\mathcal{T}}\right)$ are the holomorphic functions on the time axis.

Chapter 6

Twistor theory and the Schrödinger equation

One of the early successes of twistor theory was Penrose's solution of the zero-rest-mass equations by contour integrals on twistor lines [65]. Later formulas could handle massive states by means of more complicated integrands and domains [44, 45, 46, 47].

In this chapter we will begin to develop analogous contour integral formulas in the Newtonian setting. For four dimensions we'll coordinatise $Z = \mathcal{O} \oplus \mathcal{O}(2)$ homogeneously over U by $(T, q, \pi_{A'})$ as in section 3.1.1.2. The obvious starting point is to integrate a function f (homogeneous of weight -2) representing a cohomology class in $\check{H}^1\left(Z, \mathcal{O}(-2)_Z\right)$ around a contour γ homologous to the equator in $\mathbb{P}^1$. We then find, however, that

$$\frac{1}{2\pi i} \oint_\gamma f\left(T|, q|, \pi_{A'}\right) \pi \cdot d\pi$$

merely solves the spatial Laplace equation as in minitwistor theory [21], with the extra twistor coordinate T playing no role.

The situation becomes more interesting when one considers the free-particle Schrödinger equation.

6.1 Integral formulas and derivations

Here we will derive contour integral formulas which provide solutions to the free-particle Schrödinger equations in (2+1) and (3+1) dimensions.

6.1.1 (2+1) dimensions

Let $(\omega^A, \Pi_{A'})$ be homogeneous coordinates[1] on $Z = \mathcal{O}(1) \oplus \mathcal{O}(1)$ which restrict to

$$\omega^0| = w\Pi_{0'} + u\Pi_{1'} \qquad \text{and} \qquad \omega^1| = v\Pi_{0'} + \tau\Pi_{1'}$$

on global sections, and let $G \in H^1\left(Z, \mathcal{O}(-2)_Z\right)$ be represented by a function homogeneous of weight -2.

Penrose's seminal paper [65] unveiled the integral formula

$$\phi(w,\tau,u,v) = \frac{1}{2\pi i}\oint_\gamma G(\omega^0|, \omega^1|, \Pi_{A'})\,\Pi\cdot d\Pi \tag{6.1.1}$$

giving rise to solutions to the wave equation

$$\left(\partial_w\partial_\tau - \partial_u\partial_v\right)\phi = 0\ , \tag{6.1.2}$$

where γ is a choice of contour homologous to the equator in $\mathbb{P}^1$.

Under the ansatz

$$\phi(w,\tau,u,v) = \psi(\tau,u,v)\exp\{iw\} \tag{6.1.3}$$

equation (6.1.2) reduces to the two-dimensional free-particle Schrödinger equation

$$\left(i\partial_\tau - \partial_u\partial_v\right)\psi = 0. \tag{6.1.4}$$

Now note that

$$\phi = \psi e^{iw} \quad \Leftrightarrow \quad \left(\partial_w - i\right)\phi = 0$$

and so apply $(\partial_w - i)$ to (6.1.1):

$$\left(\partial_w - i\right)\phi = \frac{1}{2\pi i}\oint_\gamma \left(\Pi_{0'}\left(\partial_{\omega^0}G\right)| - iG|\right)\,\Pi\cdot d\Pi.$$

[1] Since the formula for $(2+1)$ dimensions will play a role in the derivation of a formula for $(3+1)$ dimensions we choose to reserve the usual notation $\pi_{A'}$ for the latter case and instead switch to $\Pi_{A'}$ as homogeneous coordinates on $\mathbb{P}^1$ in this section.

Thus $(\partial_w - i)\,\phi = 0$ if and only if

$$\Pi_{0'}\partial_{\omega^0} G - iG = h(\omega^A, \Pi_{A'}) \tag{6.1.5}$$

where h is any function (also of weight minus two) satisfying

$$\oint_\gamma h|\,\Pi \cdot d\Pi = 0. \tag{6.1.6}$$

There are lots of non-vanishing functions h with this property; for instance we can take $h = {}^{\Pi_{0'}}/_{\Pi_{1'}^3}$. Of course, one such function is $h = 0$. We choose to proceed by taking $h = 0$ and the justification for this will be that the resulting integral formula has a significant range, to be shown in section 6.2. The solution to (6.1.6) is then

$$G(\omega^A, \Pi_{A'}) = F(\omega^1, \Pi_{A'}) \exp\left\{\frac{i\omega^0}{\Pi_{0'}}\right\}$$

yielding the integral formula

$$\psi(\tau, u, v) = \frac{1}{2\pi i}\oint_\gamma F(\omega^1|, \Pi_{A'}) \exp\left\{\frac{iu\Pi_{1'}}{\Pi_{0'}}\right\} \Pi \cdot d\Pi \tag{6.1.7}$$

previously presented in an appendix of [22]. By construction $\psi(\tau, u, v)$ solves the $(2+1)$-dimensional Schrödinger equation (6.1.4), but if desired it is straightforward to verify this is the case from (6.1.7).

Example: an elementary state

Let $F = \frac{1}{\omega^1 \Pi_{1'}}$. We then have

$$\psi = \frac{1}{2\pi i}\oint_\gamma \frac{e^{\frac{iu\Pi_{1'}}{\Pi_{0'}}}}{(v\Pi_{0'} + \tau\Pi_{1'})\,\Pi_{1'}} \Pi \cdot d\Pi$$

and putting $\eta = \frac{\Pi_{0'}}{\Pi_{1'}}$ in the patch U we then have

$$\psi = \frac{1}{2\pi i}\oint_\gamma \frac{e^{\frac{iu}{\eta}}}{(v\eta + \tau)}\, d\eta.$$

Now let γ be any contour centred on the origin in U and excluding $\eta = -\tau v^{-1}$; using the residue theorem we then have

$$\psi = v^{-1}\left(e^{-\frac{iuv}{\tau}} - 1\right) \tag{6.1.8}$$

which does indeed solve (6.1.4). The region $\tau = 0$ is explicitly excluded by our choice of contour, so the singularity there is of no concern.

Putting in $u = \frac{1}{2}(x+iy)$ and $v = \frac{1}{2}(x-iy)$ and taking (x,y,τ) to be real spacetime coordinates then gives

$$\psi = \frac{2}{x-iy}\left(\exp\left\{-\frac{i}{4\tau}\left(x^2+y^2\right)\right\}-1\right)$$

$$\Rightarrow \qquad \psi^*\psi \propto \frac{1}{x^2+y^2}\sin^2\left(-\frac{(x^2+y^2)}{8\tau}\right).$$

Thus this *elementary state*[2] arising has the quantum mechanical interpretation of a state localised around the origin at early times whose probability density diffuses radially in the plane.

6.1.2 (3+1) dimensions

To solve the Schrödinger equation in (3+1) dimensions we employ a similar strategy, the difference being that we start with a two-twistor formalism in the style of [44, 47]. Henceforth in this chapter the material presented is original to this thesis.

Consider a function

$$g : Z_\infty \times Z_\infty \to \mathbb{C}$$

where $Z_\infty = \mathcal{O} \oplus \mathcal{O}(2)$ is Newtonian twistor space, the total space of $\mathcal{O} \oplus \mathcal{O}(2)$. We will label the two copies of the twistor space by $a = 0, 1$ and use homogeneous coordinates

$$(T_a,\, q_a,\, (\pi_a)_{A'}) \in Z_\infty \times Z_\infty$$

where T_a and q_a are respectively of weight zero and two, pulling-back to

$$T_a| = it \qquad q_a| = (x-iy)\,(\pi_a)_{0'}^2 - 2z\,(\pi_a)_{0'}\,(\pi_a)_{1'} - (x+iy)\,(\pi_a)_{1'}^2$$

(for the same $(t, x^i) \in M$.) If we take any such g of weight[3] $(-2,-2)$ then we can construct a function

$$\psi(t,x^i) = \frac{1}{(2\pi i)^2}\oint_{\Gamma_0}\pi_0\cdot d\pi_0\oint_{\Gamma_1}\pi_1\cdot d\pi_1 g\left(T_0|, T_1|, q_0|, q_1|, (\pi_0)_{A'}, (\pi_1)_{A'}\right) \tag{6.1.9}$$

[2]Penrose referred to the solutions of the zero-rest-mass equations arising from the simplest twistor functions (like that of this example) as elementary states [65].

[3]We'll refer to the weights m_0 with respect to $a = 0$ and m_1 with respect to $a = 1$ as (m_0, m_1).

on M by restricting both twistor spaces to the twistor lines corresponding to the same point in M. We take the contours Γ_a to be homologous to the equators in the two $\mathbb{P}^1$ bases.

Equation (6.1.9) describes a completely arbitrary function ψ on M. One can convince oneself of this by considering the temporal and spatial dependence separately. The time coordinate is obviously arbitrary: we have two coordinates which both pull back to $T| = it$ and nothing else.

The spatial dependence is a little more subtle. A function on Z_∞ (when pulled-back) assigns a complex number to every real straight line (null ray) on the spatial fibres of M. Integrating out the π-dependence sums up the values assigned to each of the straight lines passing through the point at which the function is being evaluated. This is clearly insufficient to define an arbitrary function; the value at one point is to some extent determined by values at other points. Of course we know that this is not so: a function defined this way is automatically harmonic.

Now consider pulling-back a function on $Z_\infty \times Z_\infty$. What this does (for spatial dependence) is to assign a complex number to every *pair* of real straight lines on the spatial fibres of M. If we restrict to the twistor lines corresponding to the same point $(t, x^i) \in M$ and integrate out the (two sets of) π-dependence then we are effectively summing up the values assigned to each pair of lines both passing through (t, x^i). However in this case the value $\psi(t, x^i)$ is completely unrelated to the value at any other point because there is always a *unique* straight line joining two points in Euclidean $\mathbb{R}^3$; none of the pairs of lines passing through one point can possibly both pass through any other point.

Now apply the Schrödinger operator

$$\Delta_S = i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

to (6.1.9): we find that

$$\Delta_S \psi = 0 \quad \Rightarrow \quad \oint_{\Gamma_0} \pi_0 \cdot d\pi_0 \oint_{\Gamma_1} \pi_1 \cdot d\pi_1 \left(-2\frac{\partial}{\partial \mathcal{T}} - 4(\pi_0 \cdot \pi_1)^2 \frac{\partial^2}{\partial q_0 \partial q_1} \right) g| = 0 \qquad (6.1.10)$$

where $\mathcal{T} := T_0 + T_1$ pulls back to $\mathcal{T}| = 2it$. The situation is now similar to that in (6.1.5); we require the integrand in (6.1.10) to lie in the kernel of $\oint_{\Gamma_0} \pi_0 \cdot d\pi_0 \oint_{\Gamma_1} \pi_1 \cdot d\pi_1$. As in the case of $(2+1)$ dimensions we'll choose to proceed by taking the zero of that kernel.

Thus (after multiplying by i for convenience) we must solve

$$\left(i\frac{\partial}{\partial \mathcal{T}} + 2i(\pi_0 \cdot \pi_1)^2 \frac{\partial^2}{\partial q_0 \partial q_1} \right) g = 0 \ .$$

Fortunately this is an equation we learned how to solve twistorially in the previous section: it a free-particle Schrödinger equation in $(2+1)$ dimensions. By careful identification of the spacetime coordinates (τ, u, v) in the previous section with the twistor variables of this section we will be able to use the contour integral (6.1.7). We must choose (τ, u, v) such that they are weight zero functions of the homogeneous twistor variables, which involves factorising

$$(\pi_0 \cdot \pi_1)^2 = p_0(\pi_0, \pi_1) p_1(\pi_0, \pi_1)$$

where p_0 and p_1 are respectively sections[4] of $\mathcal{O}(2,0)$ and $\mathcal{O}(0,2)$. This factorisation is highly non-unique. One choice is to take

$$p_0 = (\pi_0)_{0'}^2 \quad \text{and} \quad p_1 = \left((\pi_1)_{1'} - (\pi_1)_{0'} \frac{(\pi_0)_{1'}}{(\pi_0)_{0'}} \right)^2 . \tag{6.1.11}$$

With a factorisation chosen it is straightforward to express the "old" twistor functions from the $(2+1)$ setup in terms of the "new" $(3+1)$ twistor variables; we write

$$\omega^0| = w\Pi_{0'} - i\frac{q_0}{2p_0}\Pi_{1'}$$
$$\omega^1| = \frac{q_1}{p_1}\Pi_{0'} - \mathcal{T}\Pi_{1'} \ ,$$

and the resulting formula for solutions of the $(3+1)$ Schrödinger equation (using (6.1.7)) is

$$\psi(t, x^i) = \frac{1}{(2\pi i)^3} \oint_{\Gamma_1} \oint_{\Gamma_0} \oint_{\gamma} f\left(\frac{q_1|}{p_1}\Pi_{0'} - 2it\Pi_{1'},\ (\pi_0)_{A'}\,,\ (\pi_1)_{A'}\,,\ \Pi_{A'} \right) \times \exp\left\{ \frac{\Pi_{1'} q_0|}{2\Pi_{0'} p_0} \right\} d^3\mathcal{P}\,, \tag{6.1.12}$$

[4]We denote by $\mathcal{O}\,(m,n)$ the line bundle whose sections are represented by functions of weight m in $(\pi_0)_{A'}$ and weight n in $(\pi_1)_{A'}$.

where we have defined the weight $(2, 2, 2)$ measure

$$d^3\mathcal{P} = \pi_0 \cdot d\pi_0 \, \pi_1 \cdot d\pi_1 \, \Pi \cdot d\Pi \, .$$

One can then verify directly that (6.1.12) is in the kernel of Δ_S.

We note that a fuller cohomological understanding of the integral formulas in this chapter remains an avenue for future research.

6.2 Plane waves

The integral formulas of the previous section give solutions to the right equations, but it is not yet clear how many such solutions are in their range. In this section we will prove that any plane wave solution lies within this range, meaning that at least all Fourier analysable solutions are available.

Twistor functions giving rise to plane waves will turn out to be unwieldy, containing divergent series. They do, however, make sense under the integral and it is reassuring to see in appendix B that similarly difficult twistor functions arise when considering plane waves in minitwistor theory.

6.2.1 (2+1) dimensions

Theorem 6.2.1.

All plane wave solutions

$$\psi = \exp\{iE\tau + ik_x x + ik_y y\}$$

to the $(2+1)$-dimensional free-particle Schrödinger equation are in the range of the integral formula (6.1.7), where k_x and k_y are real and $E = k_x^2 + k_y^2$.

Proof

Consider the two-parameter family of twistor functions

$$F(\omega^1, \Pi_{A'}) = \frac{e^{\frac{a\omega^1}{\Pi_{0'}}}}{\Pi_{0'}\Pi_{1'}} \sum_{n=0}^{\infty} \left(b \frac{\Pi_{0'}}{\Pi_{1'}} \right)^n \quad \text{for} \quad (a, b) \in \mathbb{C}^2 \tag{6.2.1}$$

and a contour γ enclosing $\Pi_{0'} = 0$. In the $(2+1)$ integral formula (6.1.7) these twistor functions yield

$$\psi(\tau, u, v) = \frac{1}{2\pi i}\oint_\gamma \frac{\exp\left\{av + (a\tau + iu)\frac{\Pi_{1'}}{\Pi_{0'}}\right\}}{\Pi_{0'}\Pi_{1'}} \sum_{n=0}^{\infty}\left(b\frac{\Pi_{0'}}{\Pi_{1'}}\right)^n \Pi \cdot d\Pi \ .$$

This integral is most easily computed in the patch U, where $\eta = \frac{\Pi_{0'}}{\Pi_{1'}}$ is a good coordinate. We then have

$$\psi = \frac{1}{2\pi i}e^{av}\oint_\gamma d\eta \frac{\exp\left\{(a\tau + iu)\frac{1}{\eta}\right\}}{\eta}\sum_{n=0}^{\infty}(b\eta)^n$$

$$\Rightarrow \quad \psi = \frac{1}{2\pi i}e^{av}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!}\oint_\gamma \frac{d\eta}{\eta}(a\tau + iu)^m (b)^n \eta^{n-m}$$

$$\Rightarrow \quad \psi = \exp\left\{av + ibu + ab\tau\right\} \ ;$$

the family of twistor functions (6.2.1) leads to a generic plane wave solution of the $(2+1)$ Schrödinger equation with complex momenta. Restricting to the case of real momenta is achieved by writing

$$u = \frac{1}{2}(x + iy) \qquad v = \frac{1}{2}(x - iy)$$

for real x and y, as well as

$$a = i(k_x + ik_y) \qquad b = (k_x - ik_y)$$

for real k_x and k_y. Therefore we conclude, by admitting the possibility of integrating (6.2.1) over k_x and k_y with some momentum density, that the range of the twistor integral formula (6.1.7) includes all Fourier-analysable states.

□

6.2.2 (3+1) dimensions

Theorem 6.2.2.

All plane wave solutions

$$\psi = \exp\left\{-iEt + ik_x x + ik_y y + ik_z z\right\}$$

to the $(3+1)$-dimensional free-particle Schrödinger equation are in the range of the integral formula (6.1.12), where $(k_x, k_y, k_z) \in \mathbb{R}^3$ and $E = k_x^2 + k_y^2 + k_z^2$.

Proof

Here the situation is more complicated: use the factorisation (6.1.11) and use inhomogeneous coordinates

$$\eta = \frac{\Pi_{0'}}{\Pi_{1'}} \qquad \lambda_0 = \frac{(\pi_0)_{0'}}{(\pi_0)_{1'}} \qquad \lambda_1 = \frac{(\pi_1)_{0'}}{(\pi_1)_{1'}} \ .$$

The integral formula (6.1.12) then becomes

$$\psi = \frac{1}{(2\pi i)^3} \oint_{\Gamma_1} \oint_{\Gamma_0} \oint_{\gamma} f\left(\frac{Q_1|\eta\lambda_0^2}{(\lambda_0 - \lambda_1)^2} - 2it,\ \lambda_0,\ \lambda_1,\ \eta \right) \exp\left\{ \frac{Q_0|}{2\eta\lambda_0^2} \right\} d\eta d\lambda_0 d\lambda_1 \qquad (6.2.2)$$

where

$$Q_a| = \xi\lambda_a^2 - 2z\lambda_a - \tilde{\xi}\ ;$$

γ encloses $\eta = 0$; and Γ_a encloses $\lambda_a = 0$.

Take the ansatz

$$f = \frac{\tilde{f}(\lambda_0, \lambda_1)}{\eta} \exp\left\{ \frac{\mathcal{E}}{2\eta} \left(\frac{Q_1|\eta\lambda_0^2}{(\lambda_0 - \lambda_1)^2} - 2it \right) (\lambda_0 - \lambda_1)^2 \right\} \sum_{n=0}^{\infty} (\alpha\eta)^n \qquad (6.2.3)$$

for complex parameters $(\alpha, \mathcal{E})$ and a function $\tilde{f}$ left arbitrary at present. On computing the η integral in (6.2.2) we find

$$\psi = \frac{1}{(2\pi i)^2} \oint_{\Gamma_1} \oint_{\Gamma_0} \tilde{f}(\lambda_0, \lambda_1) \exp\left\{ \frac{\alpha Q_0|}{2\lambda_0^2} + \frac{\mathcal{E}Q_1|\lambda_0^2}{2} - i\alpha\mathcal{E}t(\lambda_0 - \lambda_1)^2 \right\} d\lambda_0 d\lambda_1$$

$$\Rightarrow \psi = \frac{1}{(2\pi i)^2} \oint_{\Gamma_1} \oint_{\Gamma_0} d\lambda_0 d\lambda_1\, \tilde{f}(\lambda_0, \lambda_1)$$
$$\times\ \exp\left\{ \lambda_1^2 \left(\frac{\mathcal{E}\xi\lambda_0^2}{2} - i\alpha\mathcal{E}t \right) + \lambda_1 \left(-\mathcal{E}z\lambda_0^2 + 2i\alpha\mathcal{E}t\lambda_0 \right) + \left(\frac{\alpha Q_0|}{2\lambda_0^2} - \frac{\mathcal{E}\tilde{\xi}\lambda_0^2}{2} - i\alpha\mathcal{E}t\lambda_0^2 \right) \right\} .$$

Any singularities in λ_1 must thus come from $\tilde{f}$. Now choose $\tilde{f}$ to be

$$\tilde{f} = \frac{1}{\lambda_0\lambda_1} \left[\sum_{n=-\infty}^{\infty} (\mu\lambda_1)^n \right] \left[\sum_{m=-\infty}^{\infty} (\beta\lambda_0)^n \right]$$

for two non-vanishing complex parameters (μ, β). The remaining two integrals are readily computed using the lemma

$$\sum_{n=-\infty}^{\infty} \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} \exp\left\{ \sum_{i=1}^{N} a_i\lambda^{b_i} \right\} (\mu\lambda)^n = \exp\left\{ \sum_{i=1}^{N} \frac{a_i}{\mu^{b_i}} \right\}$$

for any contour enclosing $\lambda = 0$ and any $2N$ complex parameters (a_i, b_i), which follows from the residue theorem. Thus we compute first the λ_1 integral, finding

$$\psi = \frac{1}{2\pi i} \oint_{\Gamma_0} e^{\left\{ \frac{\mathcal{E}\xi\lambda_0^2}{2\mu^2} - \frac{i\alpha\mathcal{E}t}{\mu^2} - \frac{\mathcal{E}z\lambda_0^2}{\mu} + \frac{2i\alpha\mathcal{E}t\lambda_0}{\mu} + \frac{\alpha Q_0|}{2\lambda_0^2} - \frac{\mathcal{E}\tilde{\xi}\lambda_0^2}{2} - i\alpha\mathcal{E}t\lambda_0^2 \right\}} \left[\sum_{m=-\infty}^{\infty} (\beta\lambda_0)^n \right] \frac{d\lambda_0}{\lambda_0},$$

and then the λ_0 integral, finding

$$\psi = \exp\left\{ -it\alpha\mathcal{E}\left(\frac{1}{\beta} - \frac{1}{\mu}\right)^2 + \xi\left(\frac{\mathcal{E}}{2\mu^2\beta^2} + \frac{\alpha}{2}\right) + z\left(-\frac{\mathcal{E}}{\mu\beta^2} - \alpha\beta\right) + \tilde{\xi}\left(-\frac{\mathcal{E}}{2\beta^2} - \frac{\beta^2\alpha}{2}\right) \right\}$$

which for

$$\xi = x - iy \qquad \tilde{\xi} = x + iy$$

can be written as

$$\psi = \exp\left\{-iEt + ik_x x + ik_y y + ik_z z\right\} \tag{6.2.4}$$

where

$$k_x = -\frac{i\mathcal{E}}{2\beta^2}\left(\mu^{-2} - 1\right) - \frac{i\alpha}{2}\left(1 - \beta^2\right) \ ; \tag{6.2.5}$$

$$k_y = -\frac{\mathcal{E}}{2\beta^2}\left(\mu^{-2} + 1\right) - \frac{\alpha}{2}\left(1 + \beta^2\right) \ ;$$

$$k_z = \frac{i\mathcal{E}}{\mu\beta^2} + i\alpha\beta \ ; \tag{6.2.6}$$

$$\text{and} \qquad E = k_x^2 + k_y^2 + k_z^2 = \alpha\mathcal{E}\left(\frac{1}{\beta} - \frac{1}{\mu}\right)^2.$$

We have arrived at a four-parameter family of plane wave solution of the free-particle Schrödinger equation, where the parameters are $(\alpha, \beta, \mu, \mathcal{E})$. In order for this family to range over all plane waves we must ensure that we can find values of the parameters for any $\boldsymbol{k} = (k_x, k_y, k_z) \in \mathbb{R}^3$.

If we invert (6.2.5-6.2.6) for $(\alpha, \mu, \mathcal{E})$ then we find

$$\mu = \frac{(k_x - \beta k_z) - ik_y}{(\beta k_x + k_z) + i\beta k_y} \tag{6.2.7}$$

$$\mathcal{E} = \frac{i\beta^2\left(\beta k_z - k_x + ik_y\right)^2}{\left(\beta^2 - 1\right)k_x + 2\beta k_z + i\left(1 + \beta^2\right)k_y}$$

$$\alpha = -\frac{i\left(k_x^2+k_y^2+k_z^2\right)}{\left(\beta^2-1\right)k_x+2\beta k_z+i\left(1+\beta^2\right)k_y} \tag{6.2.8}$$

where $\beta \neq 0$ has been left free. Generically the ansatz thus constitutes a one-parameter family of twistor functions giving rise to a given plane wave. The only possible cases for concern are then $\alpha^{-1} = \mathcal{E}^{-1} = 0$, $\mu = 0$, and $\mu^{-1} = 0$. (Recall that $\mu \neq 0$ was required when μ was introduced.) However inspection of (6.2.7-6.2.8) reveals that if a given $\boldsymbol{k}$ were to fall onto one of these special cases then β can be adjusted so as to remove that $\boldsymbol{k}$ from the special case. (The only remaining case is the trivial $\boldsymbol{k} = \mathbf{0}$ for which we can take $\alpha = \mathcal{E} = 0$ with $\beta \neq 0$ and $\mu \neq 0$.)

We therefore conclude that any plane wave solution (6.2.4) of the free-particle Schrödinger equation lies within the range of the integral formula (6.1.12), and hence so does any Fourier-analysable solution.

□

The twistor functions (6.2.1) and (6.2.3) giving rise to plane waves are unwieldy sums which are generically divergent. They nevertheless lead to finite solutions and make sense under an integral. One might hope that this is revealing how in twistor theory plane waves are a poor basis choice, and that elementary states such as (6.1.8) can replace them.

Appendix A

Line bundles on $\mathbb{P}^1$

Twistor spaces with families of submanifolds X_x having $N_x = \mathcal{O}(k)$ for some $k \geq 1$ are about as simple as it gets; they are, though, sufficiently sophisticated as to require some careful treatment of their canonical connections, particularly when k is odd.

Applied in these cases theorem 2.3.1 amounts to the construction of a *paraconformal* structure on M, i.e. a bundle isomorphism

$$TM = \odot^k \mathbb{S}'$$

as is studied in [26, 32, 14], concretely given by the frame $e_a^{A'_1 \ldots A'_k}$.

A.1 Odd dimensions

When k is even the treatment is relatively straightforward: the span of the frame will give us a conformal structure and the Λ-connection can pick out a preferred representative.

Theorem A.1.1. *Let $Z \to \mathbb{P}^1$ be a complex two-fold containing a rational curve X_0 with normal bundle $N_0 = \mathcal{O}(2n)$ for some $n \geq 1$. The Kodaira moduli space of rational curves X_x is a $(2n+1)$-dimensional complex conformal manifold.*

For $n = 1$ Z is a minitwistor space [42].

Proof

Since $\check{H}^1\left(\mathbb{P}^1, \mathcal{O}\left(2n\right)\right) = 0$ the rational curve X_0 is a member of a $\dim \check{H}^0\left(\mathbb{P}^1, \mathcal{O}\left(2n\right)\right) = (2n+1)$-dimensional family of rational curves, and by theorem 2.3.1 we obtain a section of $\Lambda^1_x\left(M\right) \otimes N_x$ at each point $x \in M$ which gives rise to a frame via

$$v = e^{A'_1 \dots A'_{2n}} \pi_{A'_1} \dots \pi_{A'_{2n}}$$

and so a metric

$$g = e_{A'_1 \dots A'_{2n}} \otimes e^{A'_1 \dots A'_{2n}} \tag{A.1.1}$$

of maximal rank in the span of v. The redundancy acts as

$$e^{A'_1 \dots A'_{2n}} \mapsto \alpha e^{A'_1 \dots A'_{2n}}$$

for any $\alpha : M \to \mathbb{C}^*$, resulting in conformal transformations $g \mapsto \alpha^2 g$. □

In the case for which the patching for Z is that of $\mathcal{O}(2n)$ (even if the sections are deformed) one may fix a particular metric from the conformal class by constructing the Λ-connection, which is in this case unique and exists for the $\mathcal{O}(2n)$ patching.

We can equip Z with an involution which singles out Euclidean signature metrics. (See, for example, [23, 21].) The metric (A.1.1) is the same as that arising from the classical invariant theory described in [25].

A.2 Even dimensions

When k is odd the situation is more complicated because the span contains no (non-degenerate) metric. In the following theorem we consider one option of what one *can* do with the frame, though this is by no means the only geometry induced on M.

Theorem A.2.1. [41]

Let $Z \to \mathbb{P}^1$ be a complex two-fold containing a rational curve X_0 with normal bundle $N_0 = \mathcal{O}\left(2n-1\right)$ for some $n \geq 1$. Then the Kodaira moduli space M of rational curves X_x is a $(2n)$-dimensional complex torsional projective manifold.

Restricting to torsion-free connections only, for $n = 1$ this is the standard twistor theory of projective surfaces due to Hitchin [42] and for $n = 2$ the normal bundle $\mathcal{O}(3)$ is that associated to exotic holonomies in the work of Bryant [14], whose twistor theory is described in terms of solutions spaces of ODEs in [26].

Proof

Since $\check{H}^1\left(\mathbb{P}^1, \mathcal{O}\left(2n-1\right)\right) = 0$ (for $n \geq 1$) the rational curve X_0 is a member of a

$$\dim \check{H}^0\left(\mathbb{P}^1, \mathcal{O}\left(2n-1\right)\right) = (2n)$$

-dimensional family of rational curves, and by theorem 2.3.1 we obtain a section of $\Lambda^1_x\left(M\right) \otimes N_x$ at each point $x \in M$ which gives rise to a frame via

$$v = e^{A'_1 \ldots A'_{2n-1}} \pi_{A'_1} \ldots \pi_{A'_{2n-1}}.$$

Unlike the case of odd dimensions the span does not contain a metric of maximal rank. We can, though, construct a family of connections out of the frame. A change of global section of $N_x \otimes N^*_x$ acts as

$$v \mapsto \alpha v \tag{A.2.1}$$

for $\alpha : M \to \mathbb{C}^*$, so write the frame as

$$e^{A'_1 \ldots A'_{2n-1}} = \alpha\left(x\right) \varsigma_a^{A'_1 \ldots A'_{2n-1}}\left(x\right) dx^a.$$

We can construct a canonical family of affine connections on M by requiring $\nabla e^{A'_1 \ldots A'_{2n-1}} = 0$. Concretely, this gives us

$$\Gamma^c_{ab} = \varsigma^c_{A'_1 \ldots A'_{2n-1}} \partial_a \varsigma_b^{A'_1 \ldots A'_{2n-1}} + \delta^c_a \partial_b \ln \alpha \tag{A.2.2}$$

where $\varsigma^a_{A'_1 \ldots A'_{2n-1}}$ is the inverse of $\varsigma_a^{A'_1 \ldots A'_{2n-1}}$. The connections described in (A.2.2) possess torsion whenever $de^{A'_1 \ldots A'_{2n-1}} \neq 0$; their torsion-free parts (and hence their geodesics) constitute a projective structure, in that a change of α leaves the unparametrised geodesics unaltered.

□

Now consider the canonical connections induced on M without reference to the frame section. We have $N_x \otimes N_x^* = \mathcal{O}$, so the torsion Ξ-connection always exists and depends on a single one-form on M. In the case $Z = \mathcal{O}(1)$ the torsion-free Ξ-connection is a standard flat projective structure.

On the other hand we have

$$N_x \otimes (N_x^* \odot N_x^*) = \mathcal{O}(1 - 2n)$$

so

$$\check{H}^0\left(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*)\right) = 0$$

and

$$\check{H}^1\left(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*)\right) = \mathbb{C}^{2n-2}.$$

Thus the Λ-connection, when it exists, is unique.

For $Z = \mathcal{O}(1)$ we find that $\Gamma^a_{bc} = 0$, so the moduli space comes equipped with a preferred representative of the projective structure, and moreover one which is metrisable. There is thus in this case an important corollary: M is equipped with a flat metric h_{ab}. We simply impose $\nabla h = 0$ and the torsion-free condition (by analogy with the existence of the Levi-Civita connection), giving us a metric with constant coefficients (which is unique up to diffeomorphisms in two dimensions). This is be important for theorem 3.2.1. (Note that this does not imply that all such Λ-connections give rise to metrics: the connection is not guaranteed to be metrisable.)

In theorem A.2.1 we chose to make the whole frame parallel, but we had other options. Another would be to construct a family of connections by declaring the form $e_{A'_1 \cdots A'_{2n-1}} \otimes e^{A'_1 \cdots A'_{2n-1}}$ to be parallel. In two dimensions this form is complex symplectic, and the connection is known as a symplectic connection.

A.3 Limits in $4n$ dimensions

Let $D = 4n$ where $n > 0$ is an integer.

Theorem A.3.1. [41]

Let $Z_c \to \mathbb{P}^1$ be a one-parameter family of vector bundles with patching

$$\hat{T} = T + \frac{S}{c\lambda^{2n-1}}$$

$$\hat{S} = \lambda^{2-D} S.$$

For $c \neq \infty$ the normal bundle to all rational curves X_x is $N_x = \mathcal{O}(2n-1) \oplus \mathcal{O}(2n-1)$ and the homogeneous frame section is

$$v = \begin{pmatrix} v^0 \\ v^1 \end{pmatrix} \qquad \textit{where} \qquad v^A = e^{AA'_1 \ldots A'_{2n-1}} \pi_{A'_1 \ldots A'_{2n-1}}$$

giving rise to a non-degenerate metric

$$g = e_{AA'_1 \ldots A'_{2n-1}} \otimes e^{AA'_1 \ldots A'_{2n-1}}$$

on M. For $c = \infty$ the normal bundle to all rational curves X_x is $N_x = \mathcal{O} \oplus \mathcal{O}(4n-2)$ and the homogeneous frame section is

$$v = \begin{pmatrix} \theta \\ e^{A'_1 \ldots A'_{4n-2}} \pi_{A'_1} \ldots \pi_{A'_{4n-2}} \end{pmatrix}$$

giving rise to a Galilean structure with clock θ and

$$h^{-1} = e_{A'_1 \ldots A'_{4n-2}} \otimes e^{A'_1 \ldots A'_{4n-2}}.$$

The induced geometry is subject to a redundancy, which in the $c \neq \infty$ case amounts to a conformal ambiguity and in the $c = \infty$ constitutes the non-metric nature of the connection's gravitational sector. For $n = 1$ this is the standard Newtonian limit of twistor theory presented in [22], and for $c = \infty$ the manifold is a Newton-Cartan manifold with arbitrary gravitational sector.

Proof

We need to begin by identifying the isomorphism class of N_x, which will be the same for all $x \in M$ because Z_c is the total space of a vector bundle. For $c = \infty$ the patching for N_x

is

$$\mathcal{F} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{2-D} \end{pmatrix}$$

and so the isomorphism class is obvious: $N_x = \mathcal{O} \oplus \mathcal{O}(D-2)$. We thus obtain a frame section

$$v = H^{-1} \begin{pmatrix} dT| \\ dS| \end{pmatrix}$$

where H^{-1} is a general global section of $N_x \otimes N_x^*$. The rational curves are given by

$$T| = t \qquad S| = x_0 + x_1\lambda + ... + x_{D-2}\lambda^{D-2}$$

and span of v is the Galilean structure which was advertised.

For $c \neq \infty$ the patching of N_x splits as

$$\mathcal{F} = \hat{H} \begin{pmatrix} \lambda^{1-2n} & 0 \\ 0 & \lambda^{1-2n} \end{pmatrix} H^{-1}$$

where (for instance)

$$H = \begin{pmatrix} 1 & 0 \\ -c\lambda^{2n-1} & 1 \end{pmatrix} \qquad \text{and} \qquad \hat{H} = \begin{pmatrix} 0 & c^{-1} \\ -c & \hat{\lambda}^{2n-1} \end{pmatrix}.$$

This exhibits the normal bundle's isomorphism class as $\mathcal{O}(2n-1) \oplus \mathcal{O}(2n-1)$, and the frame section is

$$v = H_0 \begin{pmatrix} 1 & 0 \\ c\lambda^{2n-1} & 1 \end{pmatrix} \begin{pmatrix} dT| \\ dS| \end{pmatrix} = H_0 \begin{pmatrix} dT| \\ dS| + c\lambda^{2n-1} dT| \end{pmatrix}$$

for an arbitrary non-degenerate matrix of functions H_0. The rational curves are given by

$$T| = t - \frac{1}{c} \sum_{m=0}^{2-2n} x_{m+2n} \lambda^{1+m}$$

$$S| = x_0 + x_1\lambda + ... + x_{D-2}\lambda^{D-2},$$

which results in a frame section of the advertised form. □

Appendix B

Plane waves and minitwistors

When constructing the plane wave solutions of the free-particle Schrödinger equation in section 6.2 a useful preliminary exercise is to construct the analogous solutions to the three-dimensional Laplace equation from minitwistor theory.

Let $f \in \check{H}^1\left(\mathcal{O}(2), \mathcal{O}(-2)_{\mathcal{O}(2)}\right)$ be a function of weight -2 modulo coboundaries on the minitwistor space $\mathcal{O}(2) \to \mathbb{P}^1$ equipped with homogeneous coordinates q (of weight two) and $\pi_{A'}$ (of weight one), where q pulls back to $\mathbb{S}' \to \mathbb{R}^3$ as

$$q| = \xi \pi_{0'}^2 - 2z\pi_{0'}\pi_{1'} - \tilde{\xi}\pi_{1'}^2 \tag{B.0.1}$$

for coordinates $x^i = (\xi, \tilde{\xi}, z) \in \mathbb{R}^3$. The minitwistor distribution $\mathcal{L}_A$ is

$$\mathcal{L}_0 = 2\pi_{1'}\frac{\partial}{\partial \xi} + \pi_{0'}\frac{\partial}{\partial z} \qquad \mathcal{L}_1 = \pi_{1'}\frac{\partial}{\partial z} - 2\pi_{0'}\frac{\partial}{\partial \tilde{\xi}}\ . \tag{B.0.2}$$

It is then well-known (see, for example, [21]) that

$$\phi(\xi, \tilde{\xi}, z) = \frac{1}{2\pi i}\oint_{\Gamma} f(q|, \pi_{A'})\pi \cdot d\pi \tag{B.0.3}$$

solves the Laplace equation

$$\left(4\frac{\partial^2}{\partial\xi\partial\tilde{\xi}} + \frac{\partial^2}{\partial z^2}\right)\phi = 0\ .$$

The contour Γ is homologous to the equator in $\mathbb{P}^1$.

What are the twistor functions $f(q, \pi_{A'})$ which gives rise to plane wave solutions of the Laplace equation via (B.0.3)?

A plane wave solution to the Laplace equation is of the form

$$\phi = \exp\left\{k_\xi \xi + k_{\tilde{\xi}} \tilde{\xi} + k_z z\right\} \tag{B.0.4}$$

where

$$4k_\xi k_{\tilde{\xi}} + k_z^2 = 0$$

constrains the three complex parameters (momenta) $k_i = (k_\xi, k_{\tilde{\xi}}, k_z)$. Thus we expect a two-parameter family of twistor functions spanning the possible plane waves.

Explicitly constructing the field ϕ when given a twistor function f is straightforward: we plug f into (B.0.3) and integrate. The inverse process is, however, more complicated[1]. We will here only need to adapt a concrete construction [82] due to Woodhouse.

A function $g(x^i, \pi_{A'})$ of weight -2 on $\mathbb{S}'$ which gives rise to a particular $\phi(x^i)$ is straightforward to find: we can always write

$$g(x^i, \pi_{A'}) = \frac{\phi(x^i)}{\pi_{0'}\pi_{1'}} \quad \Rightarrow \quad \frac{1}{2\pi i}\oint_\Gamma g\, \pi \cdot d\pi = \phi\ . \tag{B.0.5}$$

This useful function g does not descend to a minitwistor function (on $\mathcal{O}(2)$) though, because $\mathcal{L}_A g \neq 0$. The strategy is to add to g a coboundary $h + \hat{h}$ such that

$$f := g + h + \hat{h} \tag{B.0.6}$$

satisfies $\mathcal{L}_A f = 0$ and hence descends to a minitwistor function. Since adding a coboundary can't alter the resulting contour integral we will then have that f is the twistor function corresponding to ϕ. Thus we need to solve

$$\mathcal{L}_A h + \mathcal{L}_A \hat{h} = -\mathcal{L}_A g \tag{B.0.7}$$

for coboundaries h and $\hat{h}$, and these equations are integrable iff ϕ is harmonic.

The calculation of f for fields of the form (B.0.4) goes as follows. General coboundaries h and $\hat{h}$ on $\mathbb{S}'$ are

$$h = \sum_{n=0}^{\infty} \frac{a_n(x^i)}{\pi_{1'}^2}\left(\frac{\pi_{0'}}{\pi_{1'}}\right)^n \quad \text{and} \quad \hat{h} = \sum_{n=0}^{\infty} \frac{b_n(x^i)}{\pi_{0'}^2}\left(\frac{\pi_{1'}}{\pi_{0'}}\right)^n$$

[1] See [83] for a fuller treatment of the inverse problem.

for arbitrary functions $a_n(x^i)$ and $b_n(x^i)$, and so (B.0.7) becomes

$$\sum_{n=0}^{\infty} \frac{\mathcal{L}_A a_n(x^i)}{\pi_{1'}^2} \left(\frac{\pi_{0'}}{\pi_{1'}}\right)^n + \sum_{n=0}^{\infty} \frac{\mathcal{L}_A b_n(x^i)}{\pi_{0'}^2} \left(\frac{\pi_{1'}}{\pi_{0'}}\right)^n = -\frac{\mathcal{L}_A \phi}{\pi_{0'}\pi_{1'}} .$$

Consider the $A = 0$ equation first.

$$\sum_{n=0}^{\infty} \left(\frac{2}{\pi_{1'}}\frac{\partial a_n}{\partial \xi} + \frac{\pi_{0'}}{\pi_{1'}^2}\frac{\partial a_n}{\partial z}\right) \left(\frac{\pi_{0'}}{\pi_{1'}}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2\pi_{1'}}{\pi_{0'}^2}\frac{\partial b_n}{\partial \xi} + \frac{1}{\pi_{0'}}\frac{\partial b_n}{\partial z}\right) \left(\frac{\pi_{1'}}{\pi_{0'}}\right)^n = -\left(\frac{2k_\xi}{\pi_{0'}} + \frac{k_z}{\pi_{1'}}\right) e^{k_i x^i} .$$

Assuming $k_i \neq 0$ the particular (as opposed to the complementary) solution to this equation is

$$h = e^{k_i x^i} \sum_{n=0}^{\infty} \left(-\frac{k_z}{2k_\xi}\right)^{n+1} \frac{1}{\pi_{1'}^2} \left(\frac{\pi_{0'}}{\pi_{1'}}\right)^n \qquad \hat{h} = e^{k_i x^i} \sum_{n=0}^{\infty} \left(-\frac{2k_\xi}{k_z}\right)^{n+1} \frac{1}{\pi_{0'}^2} \left(\frac{\pi_{1'}}{\pi_{0'}}\right)^n ,$$

for any k_i. This then solves the $A = 1$ equation iff $4k_\xi k_{\tilde{\xi}} + k_z^2 = 0$, giving us

$$f = \frac{e^{k_i x^i}}{\pi_{0'}\pi_{1'}} \sum_{n=-\infty}^{\infty} \left(-\frac{k_z \pi_{0'}}{2k_\xi \pi_{1'}}\right)^n . \tag{B.0.8}$$

Since $\mathcal{L}_A f = 0$ we must be able to write $f = f(q|, \pi_{A'})$, and a short calculation shows that (B.0.8) is the same as

$$f = \frac{1}{\pi_{0'}\pi_{1'}} \exp\left\{\frac{k_\xi q|}{\pi_{0'}^2}\right\} \sum_{n=-\infty}^{\infty} \left(-\frac{k_z \pi_{0'}}{2k_\xi \pi_{1'}}\right)^n , \tag{B.0.9}$$

which is the answer to our preliminary question. Note that the process of writing $f = f(q|, \pi_{A'})$ given $f(x^i, \pi_{A'})$ satisfying $\mathcal{L}_A f = 0$ does not *appear* to be unique. For example, we could instead of (B.0.9) take

$$f = \frac{1}{\pi_{0'}\pi_{1'}} \exp\left\{-\frac{k_z q|}{2\pi_{0'}\pi_{1'}}\right\} \sum_{n=-\infty}^{\infty} \left(-\frac{k_z \pi_{0'}}{2k_\xi \pi_{1'}}\right)^n . \tag{B.0.10}$$

Whilst these two ways of writing f look different they must (by the one-to-one nature of the Penrose transform) represent the same cohomology class: their equality is simply being obscured by the infinite series.

To finish this section we will perform the contour integral to check that (B.0.10) does indeed lead to a plane wave solution to the Laplace equation. The integral is most easily done in the patch U where $\lambda = \frac{\pi_{0'}}{\pi_{1'}}$ is a good coordinate.

$$\phi = \frac{1}{2\pi i} \oint_\Gamma \frac{1}{\pi_{0'}\pi_{1'}} \exp\left\{-\frac{k_z q|}{2\pi_{0'}\pi_{1'}}\right\} \sum_{n=-\infty}^{\infty} \left(-\frac{k_z \pi_{0'}}{2k_\xi \pi_{1'}}\right)^n \pi \cdot d\pi \tag{B.0.11}$$

$$\Rightarrow \quad \phi = \frac{1}{2\pi i}\oint_{\Gamma}\frac{d\lambda}{\lambda}\exp\left\{-\frac{k_z}{2\lambda}\left(\lambda^2\xi - 2\lambda z - \tilde{\xi}\right)\right\}\sum_{n=-\infty}^{\infty}\left(-\frac{k_z\lambda}{2k_\xi}\right)^n \tag{B.0.12}$$

$$\Rightarrow \quad \phi = \frac{1}{2\pi i}e^{k_z z}\sum_{n=-\infty}^{\infty}\left(-\frac{k_z}{2k_\xi}\right)^n\oint_{\Gamma}\frac{d\lambda}{\lambda}e^{-\frac{k_z\xi\lambda}{2}}e^{\frac{k_z\tilde{\xi}}{2\lambda}}\lambda^n \tag{B.0.13}$$

$$\Rightarrow \quad \phi = \frac{1}{2\pi i}e^{k_z z}\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{m!l!}\left(-\frac{k_z}{2k_\xi}\right)^n\left(\frac{k_z\tilde{\xi}}{2}\right)^m\left(-\frac{k_z\xi}{2}\right)^l\oint_{\Gamma}\frac{d\lambda}{\lambda}\lambda^{n+l-m}\,. \tag{B.0.14}$$

Since the contour is homologous to the equator we always enclose $\lambda = 0$ and so the integral imposes $n = m - l$, removing the sum over n,

$$\Rightarrow \quad \phi = e^{k_z z}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{m!l!}\left(-\frac{k_z}{2k_\xi}\right)^{m-l}\left(\frac{k_z\tilde{\xi}}{2}\right)^m\left(-\frac{k_z\xi}{2}\right)^l \tag{B.0.15}$$

$$\Rightarrow \quad \phi = e^{k_z z}\sum_{m=0}^{\infty}\frac{1}{m!}\left(-\frac{k_z^2\tilde{\xi}}{4k_\xi}\right)^m\sum_{l=0}^{\infty}\frac{1}{l!}\left(k_\xi\xi\right)^l \tag{B.0.16}$$

$$\Rightarrow \quad \phi = e^{k_z z}e^{-\frac{k_z^2\tilde{\xi}}{4k_\xi}}e^{k_\xi\xi} = \exp\left\{k_\xi\xi + k_{\tilde{\xi}}\tilde{\xi} + k_z z\right\} \tag{B.0.17}$$

for $k_{\tilde{\xi}} = -\frac{k_z^2}{4k_\xi}$. So the twistor function (B.0.10) does indeed yield a plane wave solution (with $k_i \neq 0$).

Notice that the twistor function (B.0.10) is expressed as a series which is in general divergent and is only defined under the contour integral.

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