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SPACE-CHARGE IMPEDANCES OF BEAMS WITH NON-UNIFORM TRANSVERSE DISTRIBUTIONS

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Abstract

The longitudinal monopole and transverse dipole space-charge impedances of a round beam with various transverse distributions inside a cylindrical beam pipe are derived analytically. The derivations are then extended to a beam with any distribution in various shapes of the vacuum chamber. The relation with the incoherent self-field space-charge betatron tune shift is mentioned.

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1 INTRODUCTION

For a round beam of radius a with uniform transverse distribution, the longitudinal monopole and transverse dipole space-charge impedances are well-known. They are usually written as

$$\frac{Z_0^{\parallel}}{n} = -j\frac{Z_0}{2\gamma^2\beta}\left(1+2\ln\frac{b}{a}\right) \ , \tag{1.1}$$

and

$$Z_1^{\perp} = -j\frac{Z_0 R}{\gamma^2\beta^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right) \ , \tag{1.2}$$

where γ and β are the nominal relativity factors of the beam particles, R is the mean radius of the accelerator ring, b is the radius of the cylindrical vacuum chamber, and $Z_0 = 376.6\ \Omega$ is the impedance of free-space. Unfortunately, most beams are not uniform in transverse distribution and these formulas therefore required modification. In application, one may continue to use the two above formulas by guessing at an effective equivalent uniform-density radius a. When the cross-section beam is of bi-Gaussian distribution with rms spread σ_r, one may be tempted to substitute $a = \sqrt{6}\sigma_r$, the radius that encloses 95% of the beam. As will be illustrated below, the correct answer should be $a = 1.747\sigma_r$ for the longitudinal impedance and $a = \sqrt{2}\sigma_r$ for the transverse impedance. Thus, this wild guess would have led to an $1+2\ln(b/a)$ which is smaller by 0.676 in absolute value, and an incorrect transverse space-charge impedance which is 3 times too small.

In this note, we list in Sec. 2, the longitudinal monopole and transverse dipole space-charge impedances of beams with some commonly-used transverse distributions, such as the elliptical, parabolic, cosine-square, and bi-Gaussian. In Sec. 3, we first review the derivations of the space-charge impedances for beam with uniform transverse impedance. In Sec. 4, these derivations are extended to beams with non-uniform transverse distributions. The connection with incoherent betatron space-charge tune shift is discussed in Sec. 5. The extension to beams with non-cylindrical distributions inside non-cylindrical beam pipes is discussed in Sec. 6. Conclusions are given in Sec. 7.

2 RESULTS

The longitudinal monopole space-charge impedance in Eq. (1.1) can be rewritten as

$$\frac{Z_0^{\parallel}}{n} = -j\frac{Z_0}{2\gamma^2\beta}\,g_0 \ , \tag{2.1}$$

where g_0 is a geometric factor depending on the geometry of the beam transverse distribution and the geometry of the beam pipe. For round beam of radius a and uniform transverse distribution

$$g_0 = 1 + 2 \ln \frac{b}{a} \ . \tag{2.2}$$

We have computed g_0 and also the transverse space-charge impedances of elliptical distribution, parabolic distribution, cosine-square distribution, and bi-Gaussian distribution. The geometric factor and the transverse impedance can be expressed as Eq. (2.2) and (1.2) with a replaced by the effective uniform-density radii $a_\parallel$ and $a_\perp$. The results are listed in Table I.

Table I: Equivalent uniform-density radii $a_\parallel$ and $a_\perp$ in the space-charge impedances of a beam with linear density λ for various cylindrically symmetric volume particle densities $\rho(r)$. See Sec. 4.1 and 4.2.1 for truncated bi-Gaussian. $\gamma_e = 0.57721$ is Euler's constant.

	Phase space distribution	Longitudinal		Transverse
	$\rho(r)/\lambda$	g_0	$a_\parallel$	$a_\perp$
Uniform	$\dfrac{1}{\pi\hat{r}^2}\,\theta(\hat{r}-r)$	$1+2\ln\dfrac{b}{\hat{r}}$	$\hat{r}$	$\hat{r}$
Elliptical	$\dfrac{3}{2\pi\hat{r}^2}\left(1-\dfrac{r^2}{\hat{r}^2}\right)^{1/2}\theta(\hat{r}-r)$	$\dfrac{8}{3}-2\ln 2+2\ln\dfrac{b}{\hat{r}}$	$0.8692\,\hat{r}$	$\sqrt{\dfrac{2}{3}}\,\hat{r}$
Parabolic	$\dfrac{2}{\pi\hat{r}^2}\left(1-\dfrac{r^2}{\hat{r}^2}\right)\theta(\hat{r}-r)$	$\dfrac{3}{2}+2\ln\dfrac{b}{\hat{r}}$	$0.7788\,\hat{r}$	$\dfrac{\hat{r}}{\sqrt{2}}$
Cosine-square	$\dfrac{2\pi}{(\pi^2-4)\hat{r}^2}\cos^2\dfrac{\pi r}{2\hat{r}}\,\theta(\hat{r}-r)$	$1.9212+2\ln\dfrac{b}{\hat{r}}$	$0.6309\,\hat{r}$	$\dfrac{\sqrt{\pi^2-4}}{\sqrt{2}\pi}\,\hat{r}$
Bi-Gaussian	$\dfrac{1}{2\pi\sigma_r^2}e^{-r^2/(2\sigma_r^2)}$	$\gamma_e+2\ln\dfrac{b}{\sqrt{2}\,\sigma_r}$	$1.747\,\sigma_r$	$\sqrt{2}\,\sigma_r$

3 UNIFORM DISTRIBUTION

Let us first review how the longitudinal and transverse space-charge impedances are derived for a beam with uniform transverse distribution, so that the impedances of other distributions can be derived following the same procedures.

3.1 Longitudinal Monopole Space-charge Impedance

Consider a particle beam with linear density[†] $\lambda(s,t)$ traveling in the positive s-direction with velocity v inside a cylindrical beam pipe of radius b with infinitely-conducting walls. The axis of the beam coincides with the axis of the beam pipe. The beam is assumed to be rigid; therefore $\lambda = \lambda(s-vt)$. We also assume that its radius a does not vary longitudinally. We are interested in the longitudinal electric field E_s seen by the beam particles at the axis of the beam. To compute that, we invoke Faraday's law,

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B} \ , \tag{3.3}$$

or in the integral form,

$$\oint \vec{E}\cdot\overrightarrow{d\ell} = -\frac{\partial}{\partial t}\oint \vec{B}\cdot d\vec{A} \ . \tag{3.4}$$

In above, the closed path of integration of the electric field $\vec{E}$ is along two radii of the beam pipe at s and $s+ds$ together with two length elements at the beam axis and the wall of the beam pipe, as illustrated in Fig. 1. The area of integration of the magnetic flux density $\vec{B}$

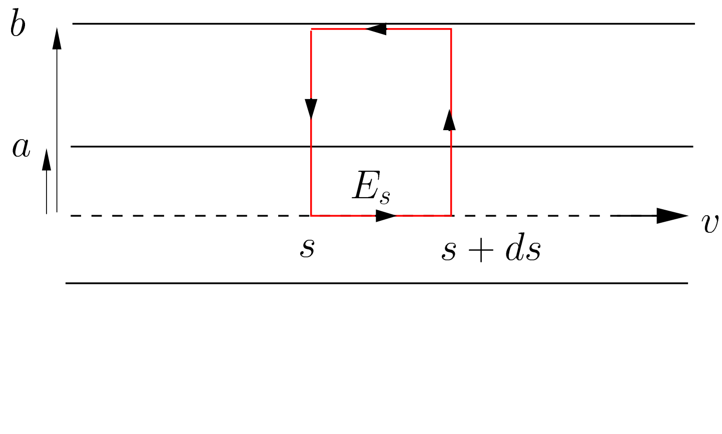

Figure 1: Derivation of the space-charge longitudinal electric field E_s experienced by a beam particle in a beam of radius a in an infinitely conducting beam pipe of radius b.

is the area enclosed by the closed path. Now, the left side of Eq. (3.4) becomes

$$\begin{aligned}\oint \vec{E}\cdot\overrightarrow{d\ell} = E_s ds &+ \frac{e\lambda(s+ds-vt)}{2\pi\epsilon_0}\left[\int_0^a \frac{rdr}{a^2}+\int_a^b\frac{dr}{r}\right] \\ &- \frac{e\lambda(s-vt)}{2\pi\epsilon_0}\left[\int_0^a \frac{rdr}{a^2}+\int_a^b\frac{dr}{r}\right] \ ,\end{aligned} \tag{3.5}$$

[†]Here $\lambda(s,t)$ is the linear particle density. The linear charge density is $e\lambda$ where e is the particle charge.

while the right side

$$-\frac{\partial}{\partial t}\oint \vec{B}\cdot d\vec{A} = -\frac{\partial}{\partial t}\frac{\mu_0 e\lambda(s-vt)v}{2\pi}\left[\int_0^a \frac{rdr}{a^2}+\int_a^b \frac{dr}{r}\right]ds\ , \tag{3.6}$$

where $\epsilon_0 = 1/(Z_0c)$ and $\mu_0 = Z_0/c$ are the electric permittivity and magnetic permeability of free-space with c being the velocity of light. Assumption has been made that the opening angle $1/\gamma$ of the radial electric field is small compared with the distance ℓ over which the linear density changes appreciably, or $b/\gamma \ll \ell$. Here, $\gamma = E_0/(mc^2)$ and m is the rest mass of the beam particle. In terms of the squared-bracketed expressions in Eqs. (3.5) and (3.6), we can define the geometric factor

$$g_0 = 2\left[\int_0^a \frac{rdr}{a^2}+\int_a^b \frac{dr}{r}\right] = 1+2\ln\frac{b}{a}\ . \tag{3.7}$$

Combining the above, we arrive at

$$E_s + \frac{eg_0}{4\pi\epsilon_0}\frac{\partial\lambda}{\partial s} = v^2\frac{e\mu_0 g_0}{4\pi}\frac{\partial\lambda}{\partial s}\ , \tag{3.8}$$

or

$$E_s = -\frac{eg_0}{4\pi\epsilon_0\gamma^2}\frac{\partial\lambda}{\partial s}\ , \tag{3.9}$$

which is the space-charge force experienced by a particle in a beam. In the reduction from Eq. (3.7) to Eq. (3.9), use has been made of the relation $\epsilon_0\mu_0 = c^{-2}$.

Consider a longitudinal harmonic wave

$$\lambda_1(s,t) \propto e^{-j(ns/R-\Omega t)}\ , \tag{3.10}$$

perturbing a coasting beam of uniform linear density λ, where n is a revolution harmonic, R is the radius of the accelerator ring, and Ω is the frequency of the wave. It can be shown that $\Omega \approx n\omega_0 = nv/R$; the difference comes from the perturbation of the coupling impedance. Thus, λ_1 is roughly a function of $s-vt$. Substitution into Eq. (3.9) results in the voltage

$$V = -E_s 2\pi R = -\frac{jneZ_0cg_0}{2\gamma^2}\lambda_1 \tag{3.11}$$

seen by a beam particle per accelerator turn. The perturbing wave constitutes a perturbing current $I_1 = e\lambda_1 v$. The space-charge impedance experienced by a beam particle is just the potential divided by the perturbing current, and we obtain Eq. (2.1). In general, for a beam with linear density λ and cylindrically symmetric distribution density $\rho(r)$, the geometric factor g_0 is therefore given by

$$\frac{g_0}{2} = \int_0^{r_e}\frac{dr}{r}\int_0^r \frac{\rho(r)}{\lambda}2\pi r'dr' + \int_{r_e}^b \frac{dr}{r}\ , \tag{3.12}$$

where r_e is the transverse edge radius of the beam.

3.2 Transverse Dipole Space-Charge Impedance

When the beam is at rest, the particle volume distribution density is

$$\rho(r) = \frac{e\lambda}{\pi a^2}\theta(a-r)\ , \tag{3.13}$$

where λ is the linear particle density. To create a dipole beam, let us displace it by an infinitesimal amount Δ in the vertical direction. The dipole charge density is given by the differential of $\rho(\vec{r})$, or[‡]

$$\Delta\rho(\vec{r}) = -\frac{\partial\rho(r)}{\partial y}\Delta = \frac{e\lambda\Delta\sin\theta}{\pi a^2}\,\delta(a-r)\ , \tag{3.14}$$

which describes a hollow cylinder of charges of radius a with $\sin\theta$ variation, as illustrated in Fig. 2. The electric force acting on a test particle at the center of the *unperturbed beam* in

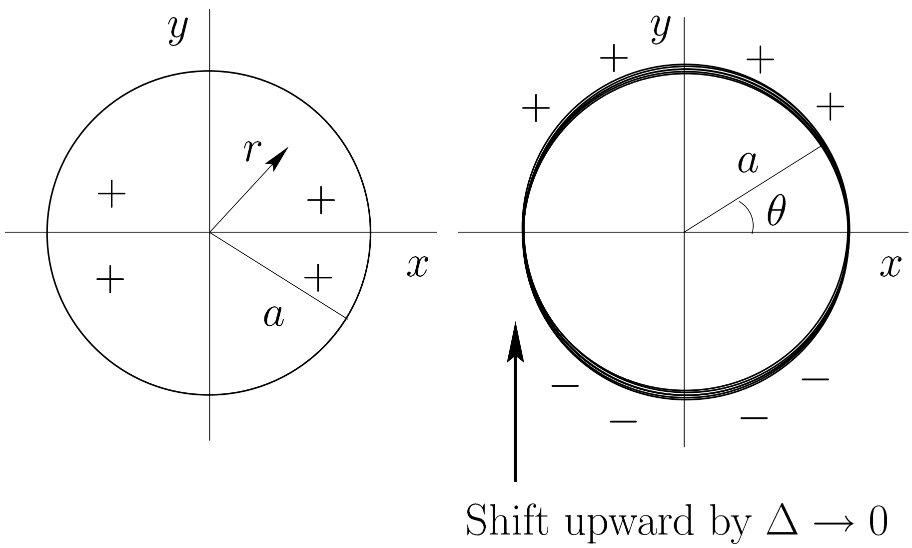

Figure 2: A beam of uniform transverse density and radius a is shifted by an infinitesimal amount Δ vertically to create a dipole beam of hollow cylinder of charges of radius a.

the y-direction is purely dipole in nature and is given by

$$F_{\rm elec} = -\int_0^{2\pi} d\theta \int_0^{\infty} rdr \frac{e^2\lambda\Delta\sin\theta}{\pi a^2}\delta(a-r)\frac{\sin\theta}{2\pi\epsilon_0 r} = -\frac{e^2\lambda\Delta Z_0 c}{2\pi a^2}\ . \tag{3.15}$$

The negative sign in the last expression implies that the net vertical force on a particle at the center of the beam pipe is pointing downward. We now take into account that the beam

[‡]To shift the whole distribution by the amount Δ in the y-direction, the distribution $\rho(x,y)$ becomes $\rho(x,y-\Delta)$, which explains the negative sign after the first equal sign in Eq. (3.13).

is moving at a velocity βc forming a current $\lambda\beta c$. A hollow cylinder of current with $\sin\theta$ distribution becomes a dipole current $I_{\rm dipole} = \lambda\Delta\beta c$. The magnetic force on the test particle at the center of the beam is the same as the electric force except that it is multiplied by $-\beta^2$. One of these β comes from the Lorentz force while the other comes from the velocity of the test particle. The total integrated force observed by the test particle in a revolution turn is therefore

$$\int_0^{2\pi R} F_{\rm self} ds = \frac{e^2\lambda\Delta c Z_0 R}{\gamma^2 a^2} \ . \tag{3.16}$$

The dipole current also induces a vertical dipole image current at the wall of the cylindrical beam pipe at radius b. This dipole image current exerts a dipole force at the center of the beam in the y-direction but in the opposite direction of the space-charge self-force. Thus the total force exerted at the beam center in the y-direction is

$$\int_0^{2\pi R} F_{\rm total} ds = -\frac{e^2\lambda\Delta c Z_0 R}{\gamma^2}\left[\frac{1}{a^2} - \frac{1}{b^2}\right] \ . \tag{3.17}$$

The transverse dipole impedance is defined as

$$Z_1^\perp = -\frac{1}{j\beta I_{\rm dipole}} \int_0^{2\pi R} F_{\rm total} ds \tag{3.18}$$

which is just the integrated transverse force normalized to the vertical dipole current. We therefore obtain the total transverse dipole space-charge impedance stated in Eq. (1.2)

4 NON-UNIFORM TRANSVERSE DISTRIBUTIONS

4.1 Transverse Space-Charge Self-Force Impedance

Following the procedure laid out in Sec. 3.2, the transverse space-charge impedance for any cylindrical transverse distribution $\rho(r)$ can be derived. However, a closer examination shows that the derivation can, in fact, be made much simpler. The shifted dipole density is

$$\Delta\rho(\vec{r}) = -\frac{\partial\rho(r)}{\partial y}\,\Delta = -\frac{d\rho(r)}{dr}\,\sin\theta\,\Delta \ . \tag{4.19}$$

When substituted in Eq. (3.15), the dipole electric force in the vertical direction can be written more generally as

$$F_{\rm elec} = -\Delta\int_0^{2\pi} d\theta \int_0^\infty r dr \left[-e^2\frac{d\rho(r)}{dr}\right]\frac{\sin^2\theta}{2\pi\epsilon_0 r} = -\frac{e^2\rho(0)\Delta}{2\epsilon_0} \ , \tag{4.20}$$

where we have used the fact that $\rho(r)$ vanishes when evaluated at the upper limit or the transverse edge of the beam.[§] The space-charge self-force part of the transverse impedance is therefore

$$Z_1^\perp\Big|_{\rm self} = -j\frac{Z_0 R}{\gamma^2\beta^2}\frac{\pi\rho(0)}{\lambda} \ . \tag{4.21}$$

In other words, the self-force part of the transverse impedance of a coasting beam with cylindrical transverse distribution is just proportional to the distribution density at $r = 0$. Or the equivalent uniform distribution radius $a_\perp$ is just

$$a_\perp = \sqrt{\frac{\lambda}{\pi\rho(0)}} \tag{4.22}$$

Thus the last column in Table I can be read out from the particle volume distribution easily. A comment has to be made for the bi-Gaussian distribution if it is truncated at $r = m\sigma_r$, where m need not be an integer, although we must require $m\sigma_r < b$ so that the beam pipe will be large enough to hold the beam. The volume particle distribution density is

$$\rho(r) = \frac{A_N\lambda}{2\pi\sigma_r^2}e^{-r^2/(2\sigma_r^2)}\,\theta(m\sigma_r - r) \ , \tag{4.23}$$

with

$$A_N = \frac{1}{1-e^{-m^2/2}} \ , \tag{4.24}$$

so that it is properly normalized to λ when integrated over the transverse coordinates. Thus the equivalent uniform-density radius is

$$a_\perp = \frac{\sqrt{2}\sigma_r}{\sqrt{1-e^{-m^2/2}}} \ . \tag{4.25}$$

However, the approximation $a_\perp \approx \sqrt{2}\sigma_r$ is usually accurate enough. For example, with a truncation at $3\sigma_r$, the error in the transverse space-charge self-force impedance is only 1%.

4.2 Longitudinal Space-Charge Impedance

The geometric factor g_0 can be derived easily by substituting the particle volume distribution density $\rho(r)$ into Eq. (3.12). Here we will just give some comments for the truncated bi-Gaussian distribution and the cosine-square distribution.

[§]For a distribution density like that uniform distribution, $\rho(r) = \lambda/(\pi a^2)$ for $r < a$, that is not continuous at the edge of the beam, one may be confused whether we should use $\rho(a+) = 0$ or $\rho(a-) = \lambda/(\pi a^2)$ at the beam edge. However, looking into the integration of Eq. (3.15), it is clear that we must use $r = a+$ as the edge, otherwise the contribution of the δ-function will not be included.

4.2.1 Truncated Bi-Gaussian

With the bi-Gaussian truncated at $m\sigma_r$ as given by Eq. (4.23), the geometric factor, according to Eq. (3.12) is given by

$$\frac{g_0}{2} = A_{\scriptscriptstyle N} \left[\int_0^{m\sigma_r} \frac{dr}{r} \left(1 - e^{-r^2/(2\sigma_r^2)} \right) + \int_{m\sigma_r}^{b} \frac{dr}{r} \right] \ , \tag{4.26}$$

where the normalization constant A is given by Eq. (4.24). The result is

$$g_0 = A_{\scriptscriptstyle N} \left[\gamma_e + 2 \ln \frac{b}{\sqrt{2}\sigma_r} + E_1\left(\tfrac{1}{2}m^2\right) \right] \ , \tag{4.27}$$

where $\gamma_e = \lim_{n\to\infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n\right] = 0.57721$ is Euler's constant and the exponential integral is defined as

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n}\, dt \ , \tag{4.28}$$

with the asymptotic expansion

$$E_n(z) = \frac{e^{-z}}{z} \left[1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \cdots \right] \ . \tag{4.29}$$

Thus, the geometric factor can be written as

$$g_0 = \frac{\gamma_e + 2\ln \dfrac{b}{\sqrt{2}\sigma_r} + \dfrac{e^{-m^2/2}}{m^2}\left[1 - \dfrac{2}{m^2} + \mathcal{O}(m^{-4})\right]}{1 - e^{-m^2/2}} \ . \tag{4.30}$$

Notice that with the truncation at 3σ's, terms involving $e^{-m^2/2}$ can be neglected with an error of approximately 1% only. When written in the form of Eq. (2.2), the equivalent uniform-density radius $a_\|$ is given by

$$\ln \frac{a_\|}{\sqrt{2}\sigma_r} = \frac{1-\gamma_e}{2} \ . \tag{4.31}$$

4.2.2 Cosine-Square Distribution

The volume particle distribution density is

$$\rho(r) = A_{\scriptscriptstyle N} \lambda \cos^2 \frac{\pi r}{2\hat{r}}\, \theta(\hat{r} - r) \ , \tag{4.32}$$

where the normalization constant is

$$A_{\scriptscriptstyle N} = \frac{2\pi}{(\pi^2 - 4)\hat{r}^2} \ . \tag{4.33}$$

The integration from 0 to r is

$$\int_0^r \frac{\rho(r')}{\lambda} 2\pi r' dr' = \pi A_{\scriptscriptstyle N} \left[\frac{r}{2} + \frac{r\hat{r}}{\pi} \sin\frac{\pi r}{\hat{r}} + \frac{\hat{r}^2}{\pi^2}\left(\cos\frac{\pi r}{\hat{r}} - 1\right)\right] \ . \tag{4.34}$$

To compute g_0, the line integral inside the beam involves the integral

$$\int_0^{\hat{r}} \frac{dt}{t}(1 - \cos t) = \frac{\pi^2}{2\cdot 2!} - \frac{\pi^4}{4\cdot 4!} + \frac{\pi^6}{6\cdot 6!} - \cdots \ . \tag{4.35}$$

We then obtain

$$g_0 = 2\ln\frac{b}{\hat{r}} + 2\pi A_{\scriptscriptstyle N}\left[\frac{2}{\pi^2} + \frac{\pi^4}{4\cdot 4!} - \frac{\pi^6}{6\cdot 6!} + \frac{\pi^8}{8\cdot 8!} - \cdots\right] = 2\ln\frac{b}{\hat{r}} + 1.921168 \ . \tag{4.36}$$

The equivalent uniform-density radius $a_\parallel = 0.6309\,\hat{r}$ can be derived easily.

5 CONNECTION WITH INCOHERENT SPACE-CHARGE TUNE SHIFT

The self-field part of the transverse space-charge impedance resembles the incoherent self-field space-charge betatron tune shift. We are going to show that they are, in fact, proportional to one another. The equation of motion of a beam particle of mass m and angular revolution frequency ω_0 in the y-direction can be written as

$$\frac{d^2y}{dt^2} + \nu_\beta^2\omega_0^2 y = \frac{1}{\gamma m}\Big[F'_{\rm coh}\langle y\rangle + F'_{\rm incoh}(y - \langle y\rangle)\Big] \ , \tag{5.37}$$

where ν_β is the bare betatron tune provided by the external focusing forces like the quadrupoles. The forces arising from the vacuum chamber and other beam particles are on the right and are expanded up to first order in the particle vertical displacement y and first order in the displacement of the beam center $\langle y\rangle$. It is obvious that $F'_{\rm coh}$ is the force gradient responsible for coherent betatron tune shift and $F'_{\rm incoh}$ is the force gradient responsible for incoherent betatron tune shift. Since the wake force $F'_{\rm wake}\langle y\rangle$ on the beam particle is proportional to the displacement of the beam center $\langle y\rangle$, we therefore have

$$F'_{\rm wake} = F'_{\rm coh} - F'_{\rm incoh} \ . \tag{5.38}$$

However, the self-field part of $F'_{\rm coh}$ is exactly zero because the beam center moves with the beam and will not experience any net force from other beam particles inside the beam, which is another statement of Newton's third law of motion. Thus we must have

$$F'_{\rm wake}\Big|_{\rm self} = -F'_{\rm incoh}\Big|_{\rm self} \; . \tag{5.39}$$

For this reason, the self-field part of the transverse space-charge impedance must be proportional to the incoherent self-field betatron tune shift. The equality of the two forces in Eq. (5.39) is not accidental and there is a simple physical reason behind it. A test particle at the center of the beam will not experience any space-charge forces from other beam particles. However, when the beam is displaced vertically upward by an infinitesimal amount Δ, the particle at the unperturbed beam center will be seeing the dipole wake force $F'_{\rm wake}\Delta$. But the test particle will be seeing exactly the same force if the particle itself is shifted downward by Δ instead while the beam is not moved. The latter force is just the defocusing space-charge force experience by a particle while performing betatron oscillation. It is responsible for the incoherent space-charge tune shift and is therefore denoted by $-F'_{\rm incoh}\Delta$. This argument is illustrated in Fig. 3 for clarification.

The equality $F'_{\rm wake}\big|_{\rm self} = -F'_{\rm incoh}\big|_{\rm self}$ provides a direct relation between the self-field part

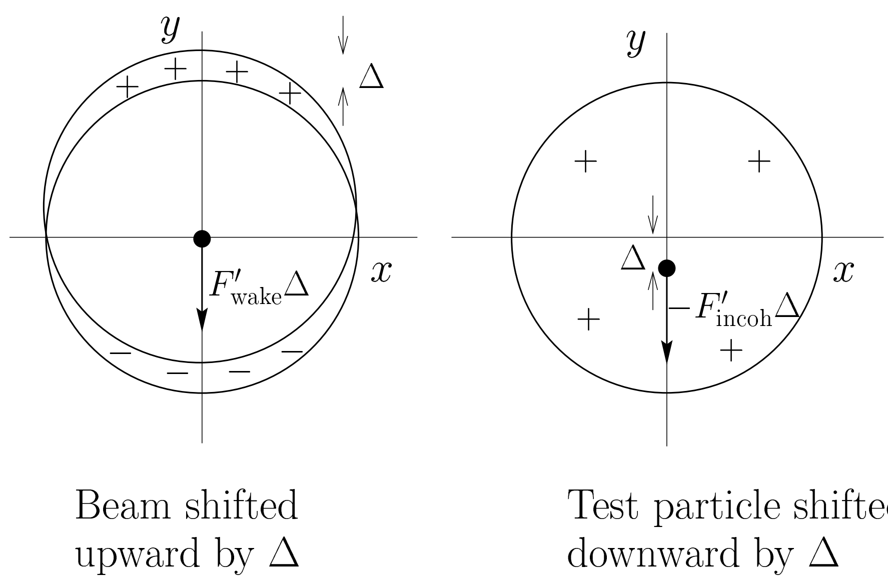

Figure 3: When the beam is shifted upward by Δ, the self-field space-charge force, denoted by $F'_{\rm wake}\Delta$, acting on a test particle at the unperturbed beam center (left) is exactly the same force, denoted by $-F'_{\rm incoh}\Delta$, if the test particle is shifted downward by Δ while the beam is unperturbed (right).

of the transverse space-charge impedance and the incoherent betatron space-charge tune shift. For verification, we are now deriving the relation in another way. Consider a test particle at the beam center making small amplitude betatron oscillation. At a distance r from the center of the beam, the radial repulsive space-charge force is given by

$$F_{\rm incoh} = \frac{e^2}{2\pi\epsilon_0 r}\int_0^r \rho(r)2\pi r'dr' \approx \frac{e^2\rho(0)}{2\epsilon_0}r\ , \tag{5.40}$$

where the small displacement approximation has been made in the last step. The vertical incoherent self-field space-charge tune shift is therefore

$$\Delta\nu_\beta = -\frac{1}{2\nu_\beta\gamma m\omega_0^2}\frac{\partial F_{\rm incoh}}{\partial y} = -\frac{e^2\rho(0)}{4\epsilon_0\nu_\beta\gamma m\omega_0^2}\ . \tag{5.41}$$

Introducing the classical particle radius $r_0 = e^2/(4\pi\epsilon_0 mc^2)$ and the total number of particles in the beam $N = 2\pi R\lambda$, we get

$$\Delta\nu_\beta^{\rm max} = -\frac{r_0 NR}{2\nu_\beta\gamma^3\beta^2}\frac{\rho(0)}{\lambda}\ , \tag{5.42}$$

where an extra γ^2 has been added to the denominator so that both electric and magnetic space-charge forces are included. We have also labeled this tune shift by "max" because the betatron space-charge tune shift of beam particles at the center of the beam is usually at its maximum. Comparing Eq. (5.42) with Eq. (4.21), we obtain the relation between the maximum incoherent betatron space-charge tune shift and the self-field part of the transverse space-charge impedance, which can be written as

$$\Delta\nu_\beta^{\rm max} = -j\frac{eI_0R}{4\pi\nu_\beta\beta E_0}Z_1^\perp\Big|_{\rm self}\ , \tag{5.43}$$

where $I_0 = eN\omega_0/(2\pi)$ is the average beam current.

6 EXTENSION TO NON-CYLINDRICAL TRANSVERSE DISTRIBUTIONS

Although the transverse cross-section of a particle beam may not have cylindrical symmetry, however, most of the time it does have elliptical symmetry. In other words, the volume particle distribution density ρ is a function of u where

$$u^2 = \frac{x^2}{a_x^2} + \frac{y^2}{a_y^2}\ , \tag{6.44}$$

where a_x and a_y are measures of the horizontal and vertical beam sizes. Since ρ is now a function of one variable only, many of the above procedures can be repeated by making the substitution

$$\begin{cases} x = a_x u \cos\theta \\ y = a_y u \sin\theta \ . \end{cases} \tag{6.45}$$

6.1 Transverse Space-Charge Impedance

The dipole particle density of Eq. (4.19) resulting from shifting the beam vertically by the amount Δ now becomes

$$\Delta\rho(\vec{r}) = -\frac{\partial\rho(r)}{\partial y}\Delta = -\frac{d\rho(u)}{du}\frac{y}{b^2 u} = -\frac{d\rho(u)}{du}\frac{\Delta}{b}\sin\theta \ . \tag{6.46}$$

The electric force in Eq. (4.20) in the vertical direction becomes

$$F^V_{\rm elec} = -\Delta\int dx\,dy\left[-e^2\frac{d\rho(u)}{du}\right]\sin\theta\frac{y}{2\pi\epsilon_0 r^2} \ . \tag{6.47}$$

Changing the variables from (x, y) to (u, θ), this electric force takes the form

$$F^V_{\rm elec} = \frac{e^2\Delta a_x a_y}{2\pi\epsilon_0}\int_0^{2\pi}d\theta\int_0^\infty udu\frac{d\rho(u)}{du}\,\frac{\sin^2\theta}{a_x^2\cos^2\theta + a_y^2\sin^2\theta} = \frac{e^2}{\epsilon_0}\frac{a_x\Delta}{a_x + a_y}\rho(0) \ , \tag{6.48}$$

where the formula

$$\int_0^{\pi/2}d\theta\,\frac{\sin^2\theta}{a_x^2\cos^2\theta + a_y^2\sin^2\theta} = \frac{\pi}{2a_y(a_x + a_y)} \tag{6.49}$$

has been used. Plugging into the definition of Eq. (3.18), we arrive at the self-field part of the vertical and horizontal dipole space-charge impedances

$$Z_1^V\Big|_{\rm self} = -j\frac{Z_0 R}{\gamma^2\beta^2}\,\frac{2\pi\rho(0)/\lambda}{1 + a_y/a_x} \quad \text{and} \quad Z_1^H\Big|_{\rm self} = -j\frac{Z_0 R}{\gamma^2\beta^2}\,\frac{2\pi\rho(0)/\lambda}{1 + a_x/a_y} \ , \tag{6.50}$$

where a_y/a_x represents the aspect ratio of the cross-section of the beam.

The equality of the self-field part of transverse wake force and the linear defocusing force in Eq. (5.39) does not depend on the cross-section of the beam and will therefore hold when the cross-section of the beam is elliptical. Thus, the relation between incoherent space-charge tune shift and self-field part of the transverse space-charge impedance of Eq. (5.43) will hold as well. From this, we obtain

$$\Delta\nu_\beta^{V\,\max} = -\frac{r_0 NR}{\nu_\beta^V\gamma^3\beta^2}\,\frac{\rho(0)/\lambda}{1 + a_y/a_x} \quad \text{and} \quad \Delta\nu_\beta^{H\,\max} = -\frac{r_0 NR}{\nu_\beta^H\gamma^3\beta^2}\,\frac{\rho(0)/\lambda}{1 + a_x/a_y} \tag{6.51}$$

For uniform distribution

$$\rho(u) = \frac{2\lambda}{\pi a_x a_y}\theta(1-u) \quad \text{with } u = \sqrt{\frac{x^2}{a_x^2} + \frac{y^2}{a_y^2}}\ , \tag{6.52}$$

we have

$$Z_1^{V,H}\Big|_{\text{self}} = -j\frac{Z_0 R}{\gamma^2\beta^2}\,\frac{2}{a_{y,x}(a_x+a_y)} \quad \text{and} \quad \Delta\nu_\beta^{V,H\,\text{max}} = -\frac{r_0 N R}{\pi\nu_\beta^{V,H}\gamma^3\beta^2 a_{y,x}(a_x+a_y)}\ . \tag{6.53}$$

For all other distributions, the results can be expressed in exactly the same forms as Eq. (6.53) by defining the equivalent uniform-density radii a_x and a_y. These are listed in Table II for the elliptical, parabolic, cosine-square, and bi-Gaussian distributions.

For the bi-Gaussian distribution truncated at $m\sigma_x$ and $m\sigma_y$,

$$\rho(u) = \frac{A_N\lambda}{2\pi\sigma_x\sigma_y}e^{-u^2/2}\,\theta(m-u) \qquad \text{with } \ u = \sqrt{\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}}\ , \tag{6.54}$$

Table II: Equivalent uniform-density radii a_x and a_y in the transverse space-charge impedance and incoherent space-charge tune shift of a beam with linear density λ and elliptical cross-section for various particle volume densities $\rho(u)$ with $u = \sqrt{(x/\hat{x})^2 + (y/\hat{y})^2}$ except for the bi-Gaussian distribution where $u = \sqrt{(x/\sigma_x)^2 + (y/\sigma_y)^2}$.

	Phase space distribution $\rho(r)/\lambda$	Equivalent radii a_x	Equivalent radii a_y
Uniform	$\frac{1}{\pi\hat{x}\hat{y}}\,\theta(1-u)$	$\hat{x}$	$\hat{x}$
Elliptical	$\frac{3}{2\pi\hat{x}\hat{y}}\left(1-u^2\right)^{1/2}\theta(1-u)$	$\sqrt{\frac{2}{3}}\,\hat{x}$	$\sqrt{\frac{2}{3}}\,\hat{y}$
Parabolic	$\frac{2}{\pi\hat{x}\hat{y}}\left(1-u^2\right)\theta(1-u)$	$\frac{\hat{x}}{\sqrt{2}}$	$\frac{\hat{y}}{\sqrt{2}}$
Cosine-square	$\frac{2\pi}{(\pi^2-4)\hat{x}\hat{y}}\ \cos^2\frac{\pi u}{2}\,\theta(1-u)$	$\sqrt{\frac{\pi^2-4}{2\pi^2}}\,\hat{x}$	$\sqrt{\frac{\pi^2-4}{2\pi^2}}\,\hat{y}$
Bi-Gaussian	$\frac{1}{2\pi\sigma_x\sigma_y}\,e^{-u^2/2}$	$\sqrt{2}\,\sigma_x$	$\sqrt{2}\,\sigma_y$

where $A_N^{-1} = 1 - e^{-m^2/2}$, the equivalent uniform-density radii are

$$a_{x,y} = \frac{\sqrt{2}\sigma_{x,y}}{\sqrt{1 - e^{-m^2/2}}} \ . \tag{6.55}$$

The denominator can be neglected with less than 1% error in the impedance or incoherent space-charge tune shift if the bi-Gaussian is truncated at $m \gtrsim 3$.

6.1.1 Effect of Beam Pipe

In studying the self-field portion of the transverse space-charge impedance, nothing has been said about the beam pipe. This is because this self-field portion of the impedance represents the dipole space-charge force acting on a test particle while the whole beam is making dipole oscillation of infinitesimal amplitude, and it is not affected by the existence of the beam pipe. Although the beam forms images in the walls of the beam pipe, the image force acting on the test particle does not depend on the transverse distribution of the beam. As a result, the contribution of the images to the transverse impedance will not affect the contribution of the self-field part. From Eq. (5.38), the contribution of the images to the transverse impedance is proportional to the difference of image coherent betatron tune shift and the image incoherent betatron tune shift. From the definition of transverse dipole impedance in Eq. (3.18), we obtain

$$Z_1^V\Big|_{\rm image} = -\frac{2\pi R}{j\beta e I_0 \Delta}\, F'_{\rm wake}\Big|_{\rm image}\Delta = \frac{4\pi R^2}{e^2 N\beta^2 c}\Big[F'_{\rm coh} - F'_{\rm incoh}\Big]_{\rm image} \ , \tag{6.56}$$

where Eq. (5.38) has been used. Now Eq. (5.37) gives the relation between tune shift and force gradient:

$$-2\omega_0^2\nu_\beta\Delta\nu_\beta = \frac{F'}{\gamma m} \ . \tag{6.57}$$

Eliminating the force gradient, we obtain the relation between the wall-image portion of the transverse impedance and the corresponding tune shifts, or

$$Z_1^{V,H}\Big|_{\rm image} = -j\frac{2\pi\nu_\beta\gamma Z_0}{r_0 N}\Big[\Delta\nu_{\rm coh}^{V,H} - \Delta\nu_{\rm incoh}^{V,H}\Big]_{\rm image} \ . \tag{6.58}$$

The wall-image effects are best represented by Laslett electric coherent image coefficient $\xi_1^{V,H}$ and electric incoherent coefficient $\epsilon_1^{V,H}$, which are defined as [1, 2, 3, 4]

$$\Delta\nu_{\rm coh}^{V,H} = -\frac{r_0 N R}{\pi\nu_\beta^{V,H}\gamma\beta^2}\frac{\xi_1^{V,H}}{h^2} \quad \text{and} \quad \Delta\nu_{\rm incoh}^{V,H} = -\frac{r_0 N R}{\pi\nu_\beta^{V,H}\gamma\beta^2}\frac{\epsilon_1^{V,H}}{h^2} \ , \tag{6.59}$$

where h is the half height of the vacuum chamber. We finally obtain

$$Z_1^V\Big|_{\rm image} = j\frac{2Z_0R}{\gamma^2\beta^2}\,\frac{\xi_1^{V,H}-\epsilon_1^{V,H}}{h^2}\ , \tag{6.60}$$

where an extra γ^2 has been introduced in the denominator to include the effect of the magnetic force. The image coefficients for a centered beam in cylindrical, elliptical, rectangular, and parallel-plate beam pipes are listed in Table III.

Table III: Image coefficients for centered beam in a cylindrical, elliptical, rectangular, and parallel-plate beam pipes. $K(k)$ is the complete elliptic integral of the first kind. k is determined from $(w-h)/(w+h) = \exp(-\pi K'/K)$ for the elliptical cross section but $w/h = K'/(2K)$ for the rectangular cross section, where w and h are the half width and half height, with $\varepsilon = \sqrt{w^2-h^2}$, and $K' = K(k')$ with $k' = \sqrt{1-k^2}$.

Coeff.	Cylindrical	Elliptical	Rectangular	Parallel Plates
ϵ_1^V	0	$\dfrac{h^2}{12\varepsilon^2}\left[(1+k'^2)\left(\dfrac{2K}{\pi}\right)^2-2\right]$	$\dfrac{K^2(k)}{12}\big(1-6k+k^2\big)$	$\dfrac{\pi^2}{48}$
ϵ_1^H	0	$\dfrac{-h^2}{12\varepsilon^2}\left[(1+k'^2)\left(\dfrac{2K}{\pi}\right)^2-2\right]$	$\dfrac{-K^2(k)}{12}\big(1-6k+k^2\big)$	$-\dfrac{\pi^2}{48}$
ξ_1^V	$\dfrac{1}{2}$	$\dfrac{h^2}{4\varepsilon^2}\left[\left(\dfrac{2K}{\pi}\right)^2-1\right]$	$\dfrac{K^2(k)}{4}\,(1-k)^2$	$\dfrac{\pi^2}{16}$
ξ_1^H	$\dfrac{1}{2}$	$\dfrac{h^2}{4\varepsilon^2}\left[1-\left(\dfrac{2Kk'}{\pi}\right)^2\right]$	$K^2(k)k$	0

6.2 Longitudinal Space-Charge Impedance

In the derivation of the longitudinal space-charge impedance, there is the integration of the electric field $\vec{E}$ from the beam center to the wall of the vacuum chamber in Eq. (3.7), which is just the potential difference $\phi_{\rm sp}$ between the beam center and the wall of the vacuum chamber, and we refer it as the space-charge potential. In fact, we can write

$$\phi_{\rm sp} = \frac{e\lambda}{4\pi\epsilon_0}\,g_0\ . \tag{6.61}$$

Thus the derivation of the longitudinal space-charge impedance reduces to the computation of the space-charge potential between the beam center and the walls of the vacuum chamber.

Such a computation involves the solution of the Poisson equation with the beam particle distributed inside the vacuum chamber. It is well-known that this problem is nontrivial because it depends not only on the distribution of the particles in the beam but also the cross-section of the vacuum chamber.¶ The situation of a cylindrical beam at the center of a cylindrical beam pipe has been simple because the equipotential surfaces of a cylindrical beam are cylindrical and the beam pipe just coincides with one of them. Besides this special situation, we are not aware of any that can provide a simple analytic formula.‖

Although most of the time numerical solution is required to evaluate the space-charge potential, however, there exist some semi-analytic approximations. One of them is for a rectangular beam inside a rectangular beam pipe derived by Gröbner and Hübner.[5] Here, the particle distribution is uniform in the y-direction between $\pm a_y$, but is not restricted in the x-direction. The coordinates of the beam pipe and the beam are shown in Fig. 4, with

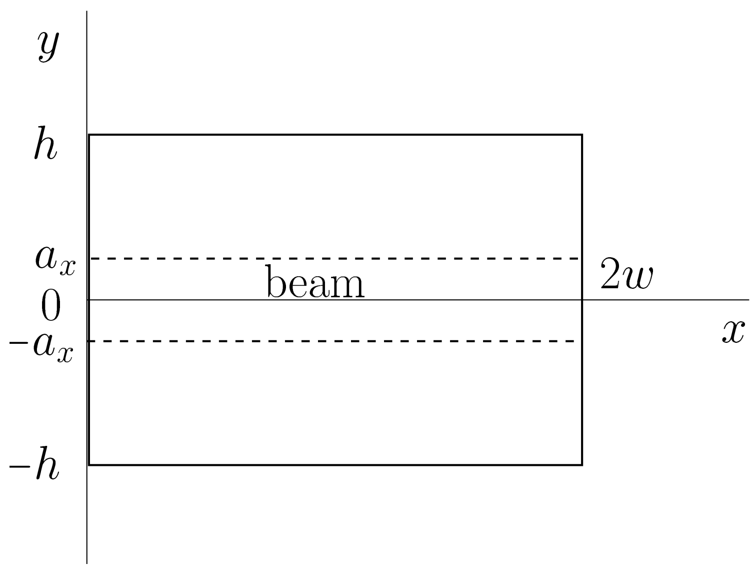

Figure 4: Illustration showing the rectangular beam (enclosed by dashes) inside the rectangular beam pipe.

w and h denoting the half width and half height of the rectangular beam pipe. The particle

¶Even in the cylindrically symmetric case, g_0 depends on the beam size $a_\parallel$ and the radius of the beam pipe b through $\ln(b/a)$, unlike the transverse dipole space-charge impedance where the parts involving a and b are separated.

‖The potential of a one-dimension beam (planar beam with only variation vertically) between two horizontal parallel plates can be solved exactly. However, this problem may not be of interest in practice, because the horizontal width of a beam is usually not very much larger than the separation between the plates making the one-dimension approximation invalid.

density of the beam is expanded as a sine series:

$$\rho(x,y)=\frac{\lambda}{4a_yw}\sum_{n=1}^{\infty}g_n\sin\eta_nx\,\theta(a_y^2-y^2)\ , \tag{6.62}$$

with

$$\eta_n=\frac{n\pi}{2w}\ . \tag{6.63}$$

The electrostatic potential $U(x,y)$ satisfies the Poisson equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)U(x,y)=\begin{cases}-\dfrac{e\lambda}{4\epsilon_0a_yw}\displaystyle\sum_{n=1}^{\infty}g_n\sin\eta_nx & |y|<a_y\\ \\ 0 & a_y<|y|<h\ ,\end{cases} \tag{6.64}$$

The solution is a linear combination of hyperbolic sine and cosine. After satisfying the boundary condition at the walls of the beam pipe and after matching U and $\partial U/\partial y$ at the edges of the beam, it is not difficult to arrive at the final solution

$$U(x,y)=\begin{cases}\dfrac{e\lambda}{4\epsilon_0a_yw}\displaystyle\sum_{n=1}^{\infty}\left[1-\frac{\cosh\eta_n(h-a_y)\cosh\eta_ny}{\cosh\eta_nh}\right]\frac{g_n\sin\eta_nx}{\eta_n^2} & |y|<a_y\\ \dfrac{e\lambda}{4\epsilon_0a_yw}\displaystyle\sum_{n=1}^{\infty}\left[\frac{\sinh\eta_na_y\sinh\eta_n(h-|y|)}{\cosh\eta_nh}\right]\frac{g_n\sin\eta_nx}{\eta_n^2} & a_y<|y|<h\ ,\end{cases} \tag{6.65}$$

We see that U vanishes at the walls of the beam pipe. Thus the space-charge potential is the negative of the potential at the center of the beam $(w,0)$, or

$$\phi_{\rm sp}=\frac{e\lambda}{4\epsilon_0a_yw}\sum_{n=1}^{\infty}\left[1-\frac{\cosh\eta_n(h-a_y)}{\cosh\eta_nh}\right]\frac{g_n\sin\eta_nw}{\eta_n^2}\ . \tag{6.66}$$

For a uniformly distributed beam within $w-a_x<x<w+a_x$ and $|y|<a_y$, the spectrum g_n is given by

$$g_n=\frac{1}{\eta_na_x}\Big[\cos\eta_n(w-a)-\cos\eta_n(w+a)\Big]=\frac{2}{a_x\eta_n}\sin\eta_nw\sin\eta_na_x\ . \tag{6.67}$$

Thus the geometric factor g_0 can be expressed as

$$g_0=\frac{4\pi}{a_xa_yw}\sum_{n=1,3,\cdots}\frac{\sin\eta_na_x\sinh\frac{1}{2}\eta_na_y\ \sinh\eta_n(h-\frac{1}{2}a_y)}{\eta_n^3\cosh\eta_nh}\ . \tag{6.68}$$

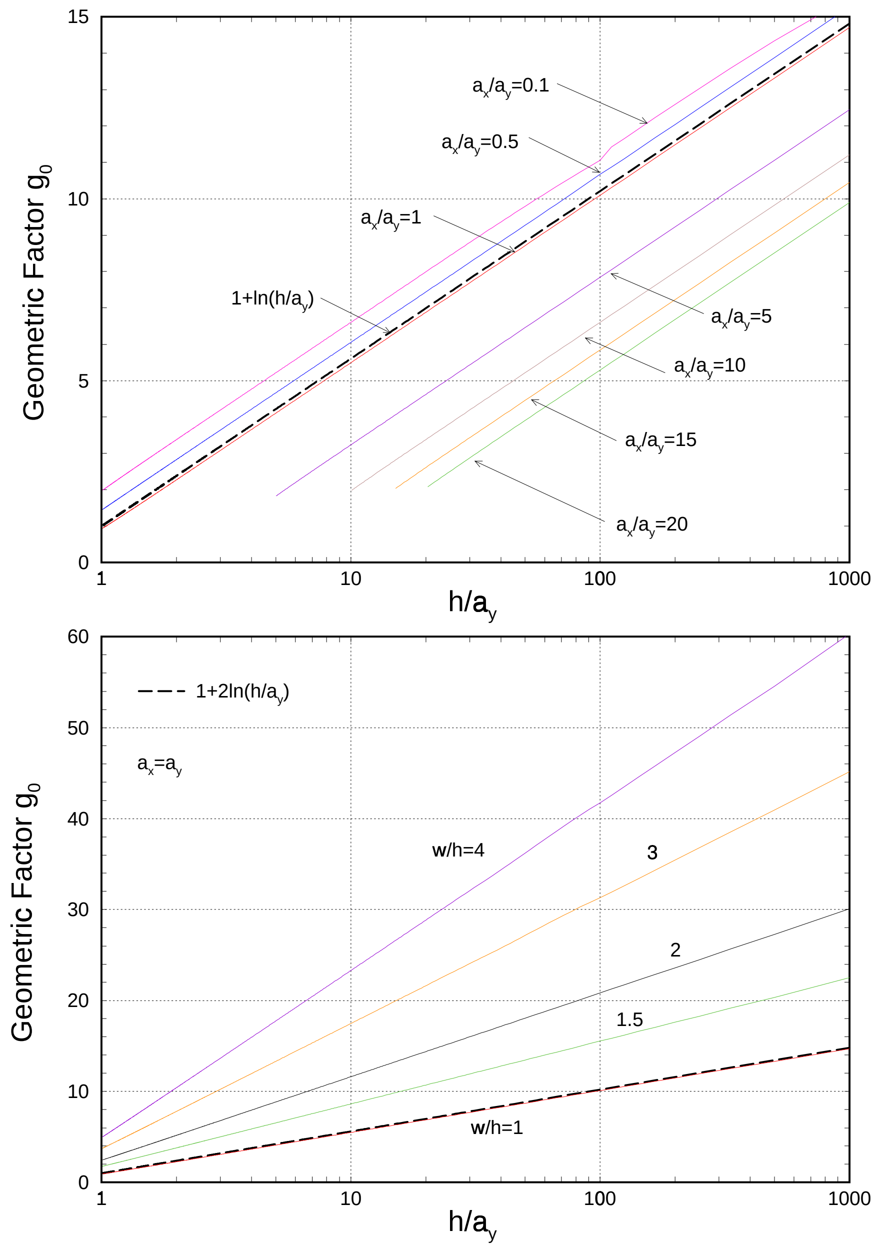

Figure 5: (color) Top: A rectangular beam at the center of a squared beam pipe. We see that g_0 is almost indistinguishable from $1 + 2\ln(h/a_y)$ (plotted in dashes) when the beam is square, but decreases as the beam spreads out horizontally, because the charges spread out and the vertical electric field becomes less intense. However, the $2\ln(h/a_y)$ behavior is preserved, where h and a_y are half heights of beam pipe and beam. Bottom: A squared beam at the center of a rectangular beam pipe. As ratio of width to height w/h of the beam pipe increases, g_0 increases and deviates from the $2\ln(h/a_y)$ behavior, because the electric field from the beam is more concentrated.

In particular, we have computed g_0 for a rectangular beam inside a squared beam pipe with results shown in the top plot of Fig. 5 for various aspect ratios of the beam as function of h/a_y. We see that when the beam is square, g_0 is almost indistinguishable from that of a circular beam inside a circular beam pipe, which is also plotted in dashes for comparison. As the beam spreads out horizontally, g_0 decreases. However, its $2\ln(h/a_y)$ behavior is unchanged. The reduction in g_0 is due to the fact that the charges spread out horizontally so that the electric field becomes relatively smaller.

We have also computed the situation of a square beam at the center of a rectangular beam pipe. We see that as the beam pipe is elongated horizontally, g_0 increases because the electric field coming from the beam becomes more concentrated vertically. We also note that g_0 no longer follows the $2\ln(h/a_y)$ behavior. For example, $g_0 \approx 2.5 + 4\ln(h/a_y)$ when $w/h = 2$, and $g_0 \approx 5 + 7.8\ln(h/a_y)$ when $w/h = 4$.

6.2.1 Bi-Gaussian Distribution

With a beam of non-uniform transverse distribution, it is difficult to solve the Poisson equation in each part of the space inside the beam pipe and obtain the final solution through boundary matching. Here, let us start out with the Bassetti-Erskine formula for the potential of a bi-Gaussian distribution [6]

$$\phi(x,y) = \frac{e\lambda}{4\pi\epsilon_0}\int_0^\infty dt\,\frac{1-\exp\left[-\frac{x^2}{2\sigma_x^2+t}-\frac{y^2}{2\sigma_y^2+t}\right]}{(2\sigma_x^2+t)^{1/2}(2\sigma_x^2+t)^{1/2}}\ , \tag{6.69}$$

where we have set the potential to zero at the center of the beam. A set of equipotential surfaces is described by

$$\phi(x,y) = \text{constant}\ , \tag{6.70}$$

and we assume that the wall of the beam pipe just coincides with one of these surfaces that passes through the point $x = 0$, $y = h$. Then the geometric factor is just given by Eq. (6.61), or

$$g_0 = \int_0^\infty dt\,\frac{1-\exp\left[-\frac{h^2}{2\sigma_y^2+t}\right]}{(2\sigma_x^2+t)^{1/2}(2\sigma_x^2+t)^{1/2}}\ , \tag{6.71}$$

Let us make a transformation

$$R^2 = \frac{2\sigma_y^2+t}{2\sigma_y^2+t}\ , \tag{6.72}$$

to obtain

$$g_0 = \int_{\sigma_y^2/\sigma_x^2}^{1} \frac{2dR}{1-R^2}\left[1-\exp\left(-\frac{(1-R^2)h^2}{2R^2\Delta^2}\right)\right] , \tag{6.73}$$

where $\Delta^2 = \sigma_x^2 - \sigma_y^2$. The next transformation is

$$u = \frac{(1-R^2)h^2}{2R^2\Delta^2} , \tag{6.74}$$

from which we obtain

$$g_0 = \int_0^{h^2/(2\sigma_y^2)} \frac{du}{u\sqrt{1+\frac{2\Delta^2}{h^2}u}}\left[1-e^{-u}\right] . \tag{6.75}$$

We have been successful in limiting the horizontal-vertical asymmetry of the beam cross section to appear only under the square-root in the denominator. Also there is no need to know whether $\sigma_x > \sigma_y$ or $\sigma_x < \sigma_y$. In the situation of a cylindrical symmetric beam, we have $\Delta = 0$ and we recover the integral in Eq. (4.26). When $\Delta \neq 0$, the integral cannot be performed in the closed form. If the asymmetry in the beam cross-section is small, or more precisely $\Delta^2/\sigma_y^2 \ll 1$, the square-root in the denominator can be expanded and then g_0 can be obtained as a series in Δ^2. Unfortunately, such an expansion is usually not possible because most of the time $\sigma_x \gg \sigma_y$ leading to $\Delta^2/\sigma_y^2 \gg 1$. Thus, numerical evaluation is necessary. We have computed g_0 as function of $h/(\sqrt{2}\sigma_y)$ for various aspect ratios of the beam and the results are plotted in Fig. 6. As expected, the curve for the special case of $\sigma_x = \sigma_y$, a cylindrical beam inside a cylindrical beam pipe, coincides exactly with what we derived before in Eq. (4.27), i.e.,

$$g_0 = \gamma_e + 2\ln\frac{h}{\sqrt{2}\sigma_y} + E_1\left(\frac{h^2}{2\sigma_y^2}\right) , \tag{6.76}$$

where $\gamma_e = 0.57721$ is Euler's constant, E_1 is the exponential function. Here, no truncation of the distribution has been considered. The function E_1 compensates for the non-uniform distribution of the beam and is significant when $h \approx 2\sigma_y$ as expected. Except for this behavior, the curves for g_0 do not deviate much from what we have computed for a rectangular beam inside a square beam pipe, and the $2\ln(h/\sqrt{2}\sigma_y)$ behavior is preserved.

7 CONCLUSION

We prove explicitly that the self-field part of the transverse dipole space-charge impedance is proportional to the incoherent betatron space-charge tune shift of a particle at the center of

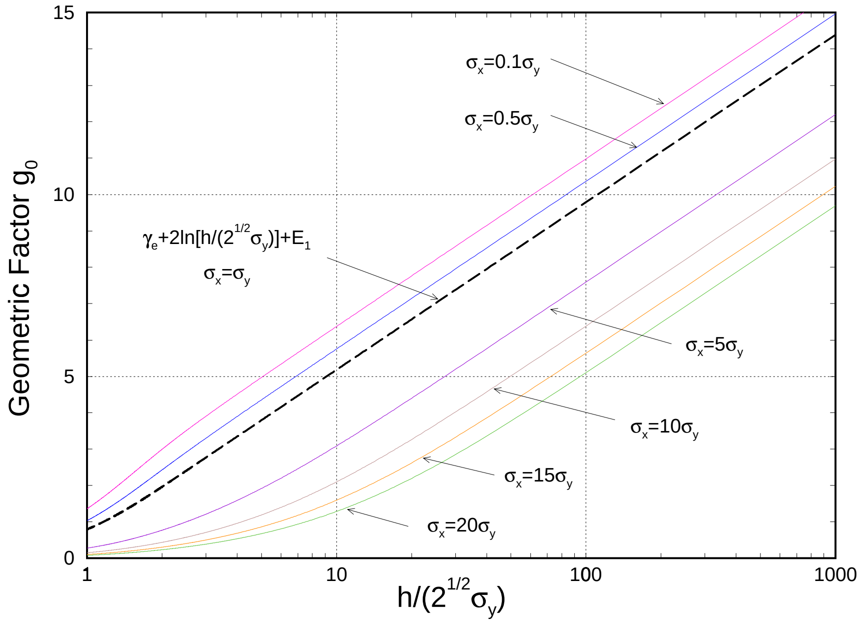

Figure 6: (color) Plot of g_0 as functions of $h/(\sqrt{2}\sigma_y)$ for various aspect ratios of a bi-Gaussian beam using the Bassetti-Erskine formula. The one for a cylindrical beam in a cylindrical beam pipe is shown in dashes.

the beam. A simple formula is derived to compute this part of the transverse dipole space-charge impedance seen by a beam with any transverse distribution. The part coming from the image effects in the walls of the beam pipe has been related to the difference between the coherent and incoherent electric image coefficients defined by Laslett.

As for the longitudinal space-charge impedance, the geometric coefficient g_0 is essentially the potential difference between the walls of the beam pipe and the center of the beam. This is the solution of the Poisson equation with the beam. As a result, the effects of the beam and and beam pipe are not separable. This leads to the impossibility of deriving simple analytic formulas for the geometric factor except in the simple situation of a cylindrical beam inside a cylindrical beam pipe.

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