Perturbative decoherence of relativistic entanglement

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Abstract. A consistent theory of quantum entanglement requires that the constituent single-particle states belong to the same Hilbert space, the coherent eigenstates of a complete set of operators in a given representation, defined with respect to a shared continuous parameterization. A covariant theory of relativistic spin and entanglement has been given by Horwitz and Arshansky, in which representations of $O(3,1)$ are induced with respect to an arbitrarily selected timelike unit vector n^μ. In this paper, we construct an induced relativistic representation of spin on an extended phase space $\{(x^\mu, p^\mu), (\zeta^\mu, \pi^\mu)\}$, in which we associate the unit vector n^μ with the momentum π^μ, thus providing a dynamical interpretation for this new quantity. Studying the unitary representations of the Poincaré group on the extended phase space allows us to define basis quantities for quantum states and develop the gauge invariant electromagnetic Hamiltonian in classical and quantum mechanics. We write plane wave solutions for free particles and construct stable singlet states. As in non-relativistic quantum theory the magnetic field couples to the rotation generators of $SU(2) \subset SL(2, C)$, however the electric field couples to the non-compact and anti-Hermitian boost generators. We show that to first order in perturbation theory, a constant external magnetic field will produce a perturbative contribution to the total mass (the eigenvalue of the Lorentz scalar Hamiltonian) or a transition between states in the same representation of $SL(2, C)$, and so will not disrupt a singlet state. However, an electric field normal to π^μ can produce a perturbative transition between states in inequivalent representations of $SL(2, C)$, and therefore potentially cause decoherence of the relativistic entanglement.

1 Introduction

A fully covariant theory of relativistic entanglement and temporal interference for entangled states has been given by Horwitz and Arshansky [1, 2]. Just as a quantum wavefunction impinging on a pair of slits separated in space produces interference fringes, an aperture of ultra-short duration opened and closed at temporal separation was shown by Lindner to produce interference in time [3]. Although the calculation of wavefunction interference is superficially similar for space and time separations, Horwitz has emphasized the requirements imposed on any such treatment by the formal aspects of quantum theory. In particular, entanglement and interference in time must be treated in a fully covariant relativistic framework that places space and time on an equal footing in a consistent manner. Because a state in nonrelativistic quantum theory is given with respect to a Hilbert space defined at a fixed time t, wavefunctions at different times belong to different Hilbert spaces and are not coherent. Moreover, treatment of time interference from an entangled singlet state requires a fully covariant description of relativistic spin. Horwitz has given such a treatment in the framework of Stueckelberg, Horwitz, and Piron (SHP) theory, and extended his analysis of the time interference reported by Lindner [3] to the proposed experiment by Palacios et. al. [4] to observe time interference from an entangled state. In this paper we extend this theoretical framework to explore the possible disruption of the singlet state in perturbation theory.

1.1 The Stueckelberg, Horwitz, Piron (SHP) framework

The Stueckelberg, Horwitz, Piron (SHP) framework is a formulation of special and general relativity in the classical and quantum realms developed to overcome the difficulties associated with the problem of time. This approach is described at length in a series of books [5–8] and only more recent references will be given explicitly here. In describing an antiparticle as a particle moving backward in time, Stueckelberg introduced an external parameter τ, similar to the Newtonian time in nonrelativistic physics, so that the spacetime event $x^\mu(\tau)$ can change direction in coordinate time x^0 while τ advances monotonically. In order to permit the event velocity $\dot{x}^\mu(\tau) = dx^\mu/d\tau$ to evolve continuously through the spacelike region between the forward and reverse lightlike regions, the 8D phase space $(x^\mu, \dot{x}^\mu)$ must be unconstrained, so that the usual constraint $\dot{x}^2 = 1$ is reduced to the status of a conservation law that applies for appropriate interactions. Unlike the convention in SHP theory, we take the flat metric to be $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ for consistency with familiar descriptions of spin. Horwitz and Piron extended the framework into a fully covariant canonical theory with many-body interactions with many applications, including solutions to relativistic bound state and scattering problems under central forces, a microdynamical theory of electrodynamics, and more recently, motion in curved spacetime and evolution of the gravitational field [9–11].

A free classical event in SHP theory is described by the Lagrangian and equivalent Hamiltonian

$$L = \frac{1}{2}M\dot{x}^\mu \dot{x}_\mu \qquad\qquad K = \frac{1}{2M}p^\mu p_\mu \tag{1}$$

where $p_\mu = \partial L/\partial \dot{x}^\mu$, with Euler-Lagrange and Hamilton equations

$$0 = \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} \qquad\qquad \dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial K}{\partial x_\mu} \tag{2}$$

and Stueckelberg-Schrodinger equation

$$i\partial_\tau \Psi(x,\tau) = K\Psi(x,\tau) \tag{3}$$

to which interactions may be added in much the same manner as in nonrelativistic theory. By generalizing the central force problem for particles of spacelike separation $x = (t, \boldsymbol{r})$ as

$$V(r) \longrightarrow V(\rho) \qquad\qquad \rho = \sqrt{\boldsymbol{r}^2 - t^2} \tag{4}$$

Horwitz and Arshansky found relativistic bound state solutions for central forces as solutions to the reduced one-particle Stueckelberg-Schrodinger equation

$$i\partial_\tau \psi = \left[\frac{p^2}{2m} + V(\rho)\right]\psi \tag{5}$$

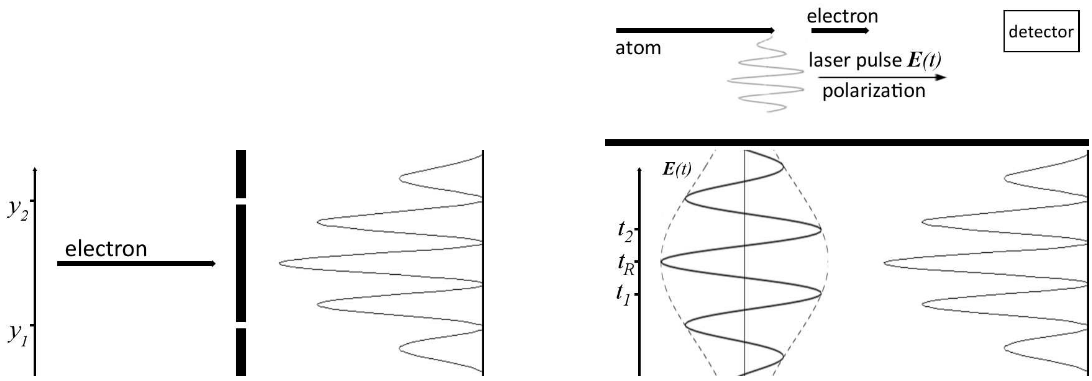
after separation of the free center of mass motion. Solutions with the correct spectrum and multiplicity require that the separation x^μ be restricted to a subspace of the spacelike region, invariant under the $O(2,1) \subset O(3,1)$ subgroup, in which the transverse component of separation $x^\mu_\perp$ with respect to an arbitrary spacelike vector n^μ is always spacelike ($x^2_\perp \leq 0$). As a result, wavefunctions that transform under the full Lorentz group must be constructed as an induced representation with respect to n^μ, leading to generators of $O(3,1)$ in a form that includes n^μ and $\partial/\partial n^\mu$. The Lie algebra for these transformations was found and new wavefunctions were then obtained as eigenstates of the commuting operators formed from these generators, providing an induced representation of $O(3,1)$. Although the value of n^μ was chosen arbitrarily, Land and Horwitz showed that it shifts under radiative transitions. By treating $n^\mu(\tau)$ as a dynamical quantity and extending the classical phase space to include $(n^\mu, \dot{n}^\mu)$, the gauge invariant Hamiltonian acquires an interaction term in which the electromagnetic field is coupled in a natural way to the generators of the induced $O(3,1)$. Thus, the Zeeman and Stark effects were found by Land and Horwitz in manifestly covariant form as a first order perturbation, reproducing the expected lifting of energy degeneracy.

Equations (1) to (3) describe a spinless particle. In standard relativistic quantum theory [12] spin is introduced, following Wigner, by studying the unitary representations of the Poincaré group involving an induced representation of $O(3,1)$ with respect to the particle momentum p^μ. Spin then refers to the eigenvalues of quantum operators derived from the rotation subgroup $SU(2)$ of the induced $SL(2,C)$

covering group for $O(3,1)$. For the SHP theory, Horwitz, Piron, and Reuse modified Wigner's method by specifying a position state and inducing a representation of $SL(2,C)$ with respect to an arbitrarily chosen timelike unit vector designated n^μ. The resulting spin operators act in the spacelike hypersurface normal to n^μ, characterizing the induced $SU(2)$. Because a two-body spin-0 state is an irreducible representation of the direct product of identical spin-1/2 particles in the same representation, the description of a singlet requires that the individual wavefunctions be defined at the same value of τ with the same value of n^μ.

1.2 *Double-slit experiments in space and in time*

The demonstration by Davisson and Germer [13] of the wavelike nature of particles in the double-slit experiment was a foundational contribution to quantum theory. As shown schematically in Figure 1a,



(a) Interference fringes in space **(b)** Interference fringes in time

Figure 1: Interference in space and time

an electron passes through a slit at either $y = y_1$ or $y = y_2$ leading to a spatial superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|y_1\rangle + |y_2\rangle\right) \quad \longrightarrow \quad \langle p|\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{-ipy_1} + e^{-ipy_2}\right) \tag{6}$$

producing interference fringes in space. In the temporal double-slit experiment by Lindner et. al. [3], shown schematically in Figure 1b, an ultra-short laser pulse ionizes atom when $\boldsymbol{E}(t) = \boldsymbol{E}_{max}$ and an electron emitted at either $t = t_1$ or $t = t_2$ is accelerated toward the detector. In analogy to (6) this emission leads to a temporal superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|t_1\rangle + |t_2\rangle\right) \quad \longrightarrow \quad \langle E|\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{-iEt_1} + e^{-iEt_2}\right) \tag{7}$$

and the energy state produces interference fringes in time. A similar temporal double-slit experiment has been proposed by Palacios et. al. [4] involving a sequential double ionization of helium to produce entangled electrons. However, in nonrelativistic quantum theory, the states $|t_1\rangle$ and $|t_2\rangle$ defined in (7) do not belong to the same Hilbert space and do not produce a coherent superposition. A relativistically consistent analysis of these experiments is given in [1].

1.3 n^μ *as a dynamical quantity*

In the SHP theory, a representation of $SL(2,C)$ is induced with respect to an arbitrarily chosen timelike unit vector designated n^μ. In this paper, we treat $n^\mu(\tau)$ as a dynamical quantity, as was done for the induced representation of $O(3,1)$ in the analysis of the Zeeman and Stark effects for relativistic bound states. More specifically, we introduce an auxiliary momentum $\pi^\mu = Mn^\mu$ conjugate to a position ζ^μ and study the classical and quantum dynamics on the extended phase space $\{(x^\mu, p^\nu), (\zeta^\mu, \pi^\nu)\}$. We analyze the classical Lorentz force for a two-body system and identify the conditions under which π_1^μ and π_2^μ may evolve differently in an external electromagnetic field. Applying Wigner's method, we construct a unitary representation of the Poincaré group for quantum states, leading to the representation of spin found by Horwitz et. al. induced on π^μ. We show that for a pair of free particles defined in the same Hilbert space, the values of π_i^μ will generally evolve together, involving no decoherence. Considering a first order perturbation by an external magnetic field in a quantum system, we find transition probabilities for a spin flip affecting each particle in a singlet state, leaving the singlet undisturbed. We show, however, that an external electric field may induce a transition between inequivalent representations of $SL(2,C)$, potentially disrupting the singlet.

In Section 2 we extend the classical phase space to include (ζ^μ, π^μ) and explore the electromagnetic interaction in the extended space. Section 3 reviews the construction of unitary representations of the Poincaré group for quantum states, and presents the induced representation of spin. Quantum mechanics in the extended phase space is then developed in Section 4. In Section 5 we find matrix elements for a constant electromagnetic field as a first order perturbation.

2 Classical extended phase space

We write the classical phase space

$$\left\{ \left(x^\mu, \zeta^\mu \right), \left(\dot{x}^\mu, \dot{\zeta}^\mu \right) \right\} \tag{8}$$

and introduce the fields $A^\mu(x,\zeta)$ and $\chi^\mu(x,\zeta)$ to write the classical Lagrangian

$$L = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{1}{2} M \dot{\zeta}^\mu \dot{\zeta}_\mu + e \dot{x}_\mu A^\mu(x,\zeta) + e \dot{\zeta}_\mu \chi^\mu(x,\zeta) \tag{9}$$

which is invariant under the gauge transformations

$$A^\mu(x,\zeta) \longrightarrow A^\mu(x,\zeta) + \frac{\partial \Lambda}{\partial x^\mu} \qquad \chi^\mu(x,\zeta) \longrightarrow \chi^\mu(x,\zeta) + \frac{\partial \Lambda}{\partial \zeta^\mu} \,. \tag{10}$$

Variation with respect to x^μ and ζ^μ leads to the Lorentz force

$$M\ddot{x}^\mu(\tau) = eF^{\mu\nu}(x,\zeta)\, \dot{x}_\nu(\tau) + eH^{\mu\nu}(x,\zeta)\, \dot{\zeta}_\nu(\tau) \tag{11}$$

$$M\ddot{\zeta}^\mu(\tau) = eG^{\mu\nu}(x,\zeta)\, \dot{\zeta}_\nu(\tau) - eH^{\mu\nu}(x,\zeta)\, \dot{x}_\nu(\tau) \tag{12}$$

where

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \qquad G^{\mu\nu} = \frac{\partial \chi^\nu}{\partial \zeta_\mu} - \frac{\partial \chi^\mu}{\partial \zeta_\nu} \qquad H^{\mu\nu} = \frac{\partial \chi^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial \zeta_\nu} \tag{13}$$

are the field strengths.

We may define the canonical momenta

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = M\dot{x}_\mu + eA_\mu \qquad \longrightarrow \qquad \dot{x}_\mu = \frac{1}{M}\left(p_\mu - eA_\mu\right) \tag{14}$$

$$\pi_\mu = \frac{\partial L}{\partial \dot{\zeta}^\mu} = M\dot{\zeta}_\mu + e\chi^\mu \qquad \longrightarrow \qquad \dot{\zeta}_\mu = \frac{1}{M}\left(\pi_\mu - e\chi_\mu\right) \tag{15}$$

and perform a Legendre transformation on the classical Lagrangian (9) to obtain the classical gauge invariant scalar Hamiltonian

$$K = \frac{1}{2M}\left[\left(p^\mu - eA^\mu\right)\left(p_\mu - eA_\mu\right) + \left(\pi^\mu - e\chi^\mu\right)\left(\pi_\mu - e\chi_\mu\right) \right] \,. \tag{16}$$

In Section 4 we will see how the quantum Hamiltonian acquires a spin interaction term. As discussed in Section 1, Horwitz et. al. found induced a representation of spin on the orbit of an arbitrary timelike unit vector n^μ and argued that a pair of quantum particles must have the same value of n^μ in order to enter a singlet state. In section 3 we will identify this unit vector as the dynamical quantity $n^\mu = \pi^\mu / M$.

To get a sense of how this quantity behaves in an external field, we consider two classical particles with initial conditions

$$\zeta_1^\mu(0) = \zeta_2^\mu(0) \qquad \pi_1^\mu(0) = \pi_2^\mu(0) \;\longrightarrow\; n_1^\mu(0) = n_2^\mu(0) \tag{17}$$

and write the Hamilton equations

$$\dot{p}_\mu = \frac{e}{M}\left[\left(p^\nu - eA^\nu\right) \frac{\partial A_\nu}{\partial x^\mu} + \left(\pi^\nu - e\chi^\nu\right) \frac{\partial \chi_\nu}{\partial x^\mu} \right] \tag{18}$$

$$\dot{\pi}_\mu = \frac{e}{M}\left[\left(p^\nu - eA^\nu\right)\frac{\partial A_\nu}{\partial \zeta^\mu} + \left(\pi^\nu - e\chi^\nu\right)\frac{\partial \chi_\nu}{\partial \zeta^\mu}\right] . \tag{19}$$

In the case that $A^\mu = A^\mu(x)$ and $\chi^\mu = \chi^\mu(\zeta)$ we have

$$\dot{\pi}_1^\mu = \frac{e}{M}\left[\pi_1^\nu - e\chi^\nu(\zeta_1)\right]\frac{\partial \chi_\nu(\zeta_1)}{\partial \zeta_\mu^1} \qquad \dot{\pi}_2^\mu = \frac{e}{M}\left[\pi_2^\nu - e\chi^\nu(\zeta_2)\right]\frac{\partial \chi_\nu(\zeta_2)}{\partial \zeta_\mu^2} \tag{20}$$

and note that by (13), $H^{\mu\nu} = 0$ in this case. Since the forces and initial conditions are the same for each particle, they will evolve without divergence in their values of $\pi^\mu = Mn^\mu$. However, if either potential depends on both x and ζ then we have

$$\begin{aligned}\dot{\pi}_1^\mu &= \frac{e}{M}\left[\left[p_1^\nu - eA^\nu(x_1,\zeta_1)\right]\frac{\partial A_\nu(x_1,\zeta_1)}{\partial \zeta_\mu^1} + \left[\pi_1^\nu - e\chi^\nu(x_1,\zeta_1)\right]\frac{\partial \chi_\nu(x_1,\zeta_1)}{\partial \zeta_\mu^1}\right]\\ \dot{\pi}_2^\mu &= \frac{e}{M}\left[\left[p_2^\nu - eA^\nu(x_2,\zeta_2)\right]\frac{\partial A_\nu(x_2,\zeta_2)}{\partial \zeta_\mu^2} + \left[\pi_2^\nu - e\chi^\nu(x_2,\zeta_2)\right]\frac{\partial \chi_\nu(x_2,\zeta_2)}{\partial \zeta_\mu^2}\right]\end{aligned} \tag{21}$$

so that the forces on π_1 and π_2 may differ at spacetime separated locations x_1 and x_2, leading to a relative change in their values. In the quantum context, such a change may point to a mechanism of decoherence.

3 Unitary representation of the Poincaré group

Basis quantities for quantum states are found as eigenstates of the commuting operators formed from the generators of a unitary representation of the Poincaré group

$$\left|\psi'\right\rangle = U(\Lambda, a)\left|\psi\right\rangle \tag{22}$$

for $\Lambda \in O(3,1)$ and translation a. The spacetime Lorentz transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ and translations $x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu$ have generators

$$L_{\mu\nu} = \left(X_\mu P_\nu - X_\nu P_\mu\right) \qquad P_\mu = i\frac{\partial}{\partial X^\mu} \tag{23}$$

from which one may write unitary transformations

$$U(\Lambda, a) \simeq 1 + ia^\mu P_\mu + i\omega^{\mu\nu} L_{\mu\nu} \tag{24}$$

where P_μ and $L_{\mu\nu}$ satisfy the Lie algebra

$$\left[P_\mu, P_\nu\right] = 0 \qquad \left[L_{\mu\nu}, P_\sigma\right] = i\left(g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu\right) \tag{25}$$

$$\left[L^{\mu\nu}, L^{\rho\sigma}\right] = i\left(g^{\nu\rho}L^{\mu\sigma} + g^{\mu\sigma}L^{\nu\rho} - g^{\mu\rho}L^{\nu\sigma} - g^{\nu\sigma}L^{\mu\rho}\right) \tag{26}$$

leading to spinor representations.

In the extended phase space, Poincaré transformations of ζ^μ and π^μ are generated by

$$N_{\mu\nu} = \left(\zeta_\mu \pi_\nu - \zeta_\nu \pi_\mu\right) \qquad \pi_\mu = i\frac{\partial}{\partial \zeta^\mu} \tag{27}$$

defined in analogy to (23) but independent of X^μ and P_μ. With these, we may consider extended unitary transformations of the quantum states of the type

$$U(\Lambda, a) \simeq 1 + ia^\mu\left(P_\mu + \pi_\mu\right) + i\omega^{\mu\nu}\left(L_{\mu\nu} + N_{\mu\nu}\right) \tag{28}$$

whose generators satisfy the Poincaré algebra. In the SHP framework, the spacetime 4-momentum is unconstrained and $p^\mu p_\mu$ is not necessarily constant (or non-negative definite). But we now stipulate that the momentum π^μ satisfies the mass-shell constraint $\pi^2 = M^2$.

3.1 Representations of the Lorentz group

The spinor and vector representations of the Lorentz group are obtained as in standard quantum theory from the combined generators $M_{\mu\nu} = L_{\mu\nu} + N_{\mu\nu}$. The operators are partitioned into boost and rotation generators as

$$K_i = M_{0i} \qquad J_i = \frac{1}{2}\epsilon_{ijk} M_{jk} \qquad i, j, k = 1, 2, 3 \tag{29}$$

with commutation relations

$$[J_i, J_j] = \epsilon_{ijk} J_k \qquad [J_i, K_j] = \epsilon_{ijk} K_k \qquad [K_i, K_j] = -\epsilon_{ijk} J_k \tag{30}$$

decomposing the Lorentz transformation as $\Lambda = \exp\{i(\boldsymbol{\beta} \cdot \boldsymbol{K} + \boldsymbol{\omega} \cdot \boldsymbol{J})\}$. Under space reflection P, $K_i \longrightarrow -K_i$ and $J_i \longrightarrow J_i$ which leads to the left and right handed operators

$$\boldsymbol{N}^L = \frac{1}{2}(\boldsymbol{J} + i\boldsymbol{K}) \qquad \boldsymbol{N}^R = \frac{1}{2}(\boldsymbol{J} - i\boldsymbol{K}) \qquad P\left[\boldsymbol{N}^{L,R}\right] = \boldsymbol{N}^{R,L} \tag{31}$$

with commutation relations

$$\left[N_i^L, N_j^L\right] = \epsilon_{ijk} N_k^L \qquad \left[N_i^R, N_j^R\right] = \epsilon_{ijk} N_k^R \qquad \left[N_i^L, N_j^R\right] = 0\,. \tag{32}$$

The N_i^L and N_i^R generate two inequivalent representations of the $SU(2)$ Lie algebra with Casimir operators

$$\left(\boldsymbol{N}^L\right)^2 = n(n+1) \qquad \left(\boldsymbol{N}^R\right)^2 = m(m+1) \tag{33}$$

so that representations

$$SO(3,1) = SU(2)_L \otimes SU(2)_R \tag{34}$$

are characterized as (n, m). Since $\boldsymbol{J} = \boldsymbol{N}^R + \boldsymbol{N}^L$, the spin of a given representation is given by $n + m$. For the two-component spinor

$$\xi^{L,R} = \left(\xi^{L,R}_{1/2}\,,\ \xi^{L,R}_{-1/2}\right) \tag{35}$$

J_i and K_i are represented by Pauli matrices, and (31) leads to

$$J_3 \xi_\alpha^{L,R} = \frac{1}{2}\sigma_3 \xi_\alpha^{L,R} = \alpha \xi_\alpha^{L,R} \qquad K_3 \xi_\alpha^{L,R} = \mp\frac{i}{2}\sigma_3 \xi_\alpha^{L,R} = \mp i\alpha \xi_\alpha^{L,R}\,. \tag{36}$$

which transform as $\xi_\alpha^{L,R} \longrightarrow A^{L,R}\,\xi_\alpha^{L,R}$ where

$$A^L = \exp\left(\boldsymbol{\beta} \cdot \boldsymbol{\sigma}/2 + i\boldsymbol{\omega} \cdot \boldsymbol{\sigma}/2\right) \qquad A^R = \exp\left(-\boldsymbol{\beta} \cdot \boldsymbol{\sigma}/2 + i\boldsymbol{\omega} \cdot \boldsymbol{\sigma}/2\right) \tag{37}$$

belong to the group $SL(2, C)$, and we have $\left(A^{L,R}\right)^\dagger = \left(A^{L,R}\right)^{-1}$. In particular, pure rotations and boosts are

$$R = \cos\frac{\omega}{2} + i\sin\frac{\omega}{2}\hat{\boldsymbol{\omega}} \cdot \boldsymbol{\sigma} \qquad H = \cosh\frac{\beta}{2} \pm \sinh\frac{\beta}{2}\hat{\boldsymbol{\beta}} \cdot \boldsymbol{\sigma}\,. \tag{38}$$

For $A \in SL(2, C)$ the anti-Hermitian operator $C = i\sigma^2$ acts as

$$C^{-1} A^T C = A^{-1} \tag{39}$$

and taking the complex conjugate of a left handed Lorentz transformation

$$\xi'_L = A_L \xi_L \;\longrightarrow\; \xi'^*_L = A^*_L \xi^*_L = \left(A^\dagger_L\right)^T \xi^*_L = \left(A^{-1}_R\right)^T \xi^*_L = C\Lambda_R C^{-1} \xi^*_L \tag{40}$$

we see that $C^{-1}\xi'^*_L = \Lambda_R\left(C^{-1}\xi^*_L\right)$. Thus, $C^{-1}\xi^*_L$ transforms as ξ_R, implementing the parity transformation $\xi_R = P[\xi_L] = C^{-1}\xi^*_L$. Introducing $\sigma^0 = I = \mathrm{diag}(1, 1)$, the matrices

$$\sigma^\mu = \left\{\sigma^0, \boldsymbol{\sigma}\right\} \qquad \underline{\sigma}^\mu = C\sigma^*_\mu C^\dagger = \left\{\sigma^0, -\boldsymbol{\sigma}\right\} = P[\sigma^\mu] \tag{41}$$

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provide two inequivalent bases for $SL(2,C)$. Deploying the indices of C as

$$C = \left\| C_{\alpha\beta} \right\| \qquad C^{-1} = \left\| C^{-1\ \alpha\beta} \right\| \tag{42}$$

the spinor with upper index is defined as

$$\xi^{\alpha} = C^{-1\ \alpha\beta} \xi_{\beta} \qquad\qquad \xi_{\alpha} = C_{\alpha\beta} \xi^{\beta} \tag{43}$$

with C playing the role of a metric. Under Lorentz transformation

$$\xi'_{\alpha} = A_{\alpha}{}^{\beta} \xi_{\beta} \tag{44}$$

$$\xi'^{\alpha} = \left(A^{-1T}\right)^{\alpha}{}_{\beta} \xi^{\beta} = \left(C^{-1}AC\right)^{\alpha}{}_{\beta} \xi^{\beta} = C^{-1\ \alpha\beta} A_{\beta}{}^{\gamma} C_{\gamma\delta} \xi^{\delta} \tag{45}$$

where here C plays the role of a similarity transformation between the equivalent representations ξ_{β} and ξ^{β}.

The vector representation of $O(3,1)$ by $SL(2,C)$ is found by writing

$$X = x^0\sigma^0 + x^1\sigma^1 + x^2\sigma^2 + x^3\sigma^3 \qquad\qquad X' = AXA^{\dagger} \tag{46}$$

which conserves $\det X' = \det X = x_0^2 - x_1^2 - x_2^2 - x_3^2 = x^{\mu} x_{\mu}$ because $\det A = 1$.

3.2 The little group and Wigner operator

The little group $\mathcal{L}(\pi) \subset O(3,1)$ consists of all Lorentz transformations $\widehat{\Lambda}(\pi)$ for which

$$\pi' = \widehat{\Lambda}(\pi)\,\pi = \pi \tag{47}$$

preserves the momentum π. One may construct the little group for π by choosing a standard vector $\mathring{\pi}$ with known $\mathcal{L}(\mathring{\pi})$ and the Wigner operator $\widehat{\Lambda}(\pi,\mathring{\pi}) \in O(3,1)$ generated by $N_{\mu\nu}$ such that

$$\pi = \widehat{\Lambda}(\pi,\mathring{\pi})\ \mathring{\pi}\ . \tag{48}$$

In the $SL(2,C)$ representation this becomes

$$\Pi = \alpha(\pi)\ \mathring{\Pi}\,\alpha^{\dagger}(\pi) \qquad\qquad \Pi = \sum_{\mu} \pi^{\mu}\sigma^{\mu} \tag{49}$$

where we denote by

$$\alpha(\pi) = \alpha(\pi,\mathring{\pi}) \tag{50}$$

the $SL(2,C)$ matrix associated with the Wigner operator $\widehat{\Lambda}(\pi,\mathring{\pi})$. For any $\widehat{A}_{\pi} \in SL(2,C)$ associated with $\widehat{\Lambda}(\pi) \in \mathcal{L}(\pi)$ we may find the corresponding element of $\mathcal{L}(\mathring{\pi})$

$$\widehat{A}_{\mathring{\pi}} = \alpha^{-1}(\pi)\,\widehat{A}_{\pi}\,\alpha(\pi) \ \longrightarrow\ \widehat{A}_{\mathring{\pi}}\,\mathring{\Pi}\,\widehat{A}^{\dagger}_{\mathring{\pi}} = \mathring{\Pi} \tag{51}$$

and by inversion we may find the little group $\mathcal{L}(\pi)$ from a known little group $\mathcal{L}(\mathring{\pi})$ as

$$\mathcal{L}(\pi) = \alpha(\pi)\ \mathcal{L}(\mathring{\pi})\ \alpha^{-1}(\pi)\ . \tag{52}$$

It follows that every Lorentz transformation $A \in SL(2,C)$ can be expressed as a combination of a little group element $\widehat{A}_{\mathring{\pi}} \in \mathcal{L}(\mathring{\pi})$ and the Wigner operator $\alpha(\pi)$ in the form

$$A = \alpha(\pi')\ \widehat{A}_{\mathring{\pi}}\,\alpha^{\dagger}(\pi) \tag{53}$$

establishing the isomorphism $SL(2,C) \approx \mathcal{L}(\mathring{\pi})$.

For the discussion of spin, it is convenient to choose the standard timelike $\mathring{\pi}$ vector as pure time

$$\mathring{\pi} = M(1,0,0,0) \qquad\qquad \mathring{\Pi} = M\sigma^0 = MI \tag{54}$$

and so we write

$$n^\mu = \frac{1}{M}\pi^\mu \qquad \mathring{n} = (1,0,0,0) \qquad N = \frac{1}{M}\Pi = \sum_\mu n^\mu \sigma^\mu \tag{55}$$

making explicit the connection with the arbitrary timelike unit vector in the work of Horwitz et. al. This leads to the requirement

$$I = \widehat{A}_{\mathring{n}}\; I\; \widehat{A}^\dagger_{\mathring{\pi}} = \widehat{A}_{\mathring{\pi}}\; \widehat{A}^\dagger_{\mathring{\pi}} \;\longrightarrow\; \widehat{A}^\dagger_{\mathring{\pi}} = \widehat{A}^{-1}_{\mathring{\pi}} \tag{56}$$

so that $\mathcal{L}(\mathring{\pi})$ is the group of spatial rotations represented by

$$\mathcal{L}(\mathring{\pi}) = SU(2) \subset SL(2,C) \tag{57}$$

and a general $\mathcal{L}(\pi)$ is the $SU(2)$ rotation group in the spacelike hypersurface normal to π. An explicit form of the standard transformation $\alpha(\pi)$ is the pure boost

$$\alpha(\pi) = \exp\left(\beta\, \hat{\boldsymbol{\beta}} \cdot \boldsymbol{\sigma}/2\right) = \cosh\frac{\beta}{2} + \hat{\boldsymbol{\beta}} \cdot \boldsymbol{\sigma} \sinh\frac{\beta}{2} \tag{58}$$

so that

$$\frac{1}{M}\Pi = \alpha^\dagger(\pi)\, I \alpha(\pi) = [\alpha(\pi)]^2 = \exp\left(\beta\, \hat{\boldsymbol{\beta}} \cdot \boldsymbol{\sigma}\right) = \cosh\beta + \hat{\boldsymbol{\beta}} \cdot \boldsymbol{\sigma} \sinh\beta = \left(\pi^0, \boldsymbol{\pi}\right) \tag{59}$$

where

$$\beta = \tanh^{-1}\frac{|\boldsymbol{\pi}|}{\pi^0} \tag{60}$$

is the velocity parameter.

3.3 Basis quantities for states with spin

Horwitz et. al. defined quantum states in the position representation as eigenstates of X_μ, but here we work in the extended momentum representation and define eigenstates of the operators P_μ and π_μ. With the notation

$$N_\mu = \frac{1}{M}\pi_\mu \tag{61}$$

we introduce the Pauli–Lubanski pseudovector

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} N^{\nu\lambda} N^\sigma \tag{62}$$

where

$$W_\mu N^\mu = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} N^{\nu\lambda} N^\sigma N^\mu = 0 \tag{63}$$

$$\left[W_\mu, N_\rho\right] = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\left[N^{\nu\lambda} N^\sigma, N_\rho\right] = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\, i\left(g^\nu_\rho N^\lambda - g^\lambda_\rho N^\nu\right) N^\sigma = 0 \tag{64}$$

indicate the orthogonality of W_μ to n^μ. The scalar

$$W^\mu W_\mu = \frac{1}{2} N^{\mu\nu} N_{\mu\nu} N^\lambda N_\lambda - N^{\nu\sigma} N_{\mu\sigma} N^\mu N_\nu = \frac{1}{2} N^{\mu\nu} N_{\mu\nu} - N^{\nu\sigma} N_{\mu\sigma} N^\mu N_\nu \tag{65}$$

commutes with the other operators and in the special frame $\mathring{n} = (1,0,0,0)$ takes the form

$$W^\mu W_\mu = \frac{1}{2}\left(N^{0i} N_{0i} + N^{i0} N_{i0} + N^{ij} N_{ij}\right) - N^{0i} N_{0i} = \frac{1}{2} N^{ij} N_{ij} \tag{66}$$

which represents the total spin as the Casimir invariant of the rotation subgroup in the hypersurface normal to $\mathring{\pi} = M\mathring{n}$.

Writing σ for the diagonal spin component, while suppressing the total spin J and any additional eigenvalues associated with internal symmetries, we expect a Lorentz transformation to act on the momentum eigenvalue as

$$p'^{\mu} = \Lambda^{\mu}{}_{\nu} p^{\nu} \qquad \longrightarrow \qquad P^{\mu}\, U(\Lambda)\, |\pi, p, \sigma\rangle = p'^{\mu}\, U(\Lambda)\, |\pi, p, \sigma\rangle \tag{67}$$

where $U(\Lambda)$ is a unitary representation of Λ. This is demonstrated by expressing the transformed operator P' as

$$P'^{\mu} = \Lambda^{\mu}{}_{\nu} P^{\nu} = U^{-1}(\Lambda)\, P^{\mu} U(\Lambda) \tag{68}$$

so that

$$P^{\mu} U(\Lambda)\, |\pi, p, \sigma\rangle = \left[U(\Lambda)\, U^{-1}(\Lambda)\right]\; P^{\mu} U(\Lambda)\, |\pi, p, \sigma\rangle = U(\Lambda)\, P'^{\mu}\, |\pi, p, \sigma\rangle \ . \tag{69}$$

The wavefunction for a spinless momentum state is

$$\psi(\pi, p) = \langle \pi, p | \psi \rangle \tag{70}$$

so that frame covariance requires

$$\psi'(\pi', p') = \psi(\pi, p) \ \longrightarrow \ \psi'(\pi, p) = \psi(\Lambda^{-1} n, \Lambda^{-1} p) \ . \tag{71}$$

The Lorentz transformed momentum state is

$$\psi'(\pi, p) = \langle \pi, p | \psi' \rangle = \langle \pi, p | \left[U(\Lambda) | \psi \rangle \right] = \left[\langle \pi, p | U(\Lambda) \right] | \psi \rangle \tag{72}$$

and so

$$\langle \pi, p | U(\Lambda) = \left[U^{\dagger}(\Lambda) | \pi, p \rangle \right]^{\dagger} = \left[U(\Lambda^{-1}) | \pi, p \rangle \right]^{\dagger} = \left\langle \Lambda^{-1} \pi, \Lambda^{-1} p \right| \tag{73}$$

confirming that

$$\psi'(\pi, p) = \langle \Lambda^{-1} n, \Lambda^{-1} p | \psi \rangle \ . \tag{74}$$

The matrix element of $U(\Lambda)$ for the spinless state is thus

$$\begin{aligned} \left\langle \pi', p' \right| U(\Lambda) | \pi, p \rangle &= \delta^4\left(p - \Lambda^{-1} p'\right) \delta^4\left(\pi - \Lambda^{-1} \pi'\right) &(75)\\ &= \delta^4\left(p' - \Lambda p\right) \delta^4\left(\pi' - \Lambda \pi\right) &(76) \end{aligned}$$

where the delta functions represent the action of $U(\Lambda)$ on the left, taking p' to $\Lambda^{-1} p'$ and π' to $\Lambda^{-1} \pi'$, while the last equality follows from $\det \Lambda = 1$.

For states with spin the identity operator is

$$I = \sum_{\sigma'} \int d\mu\left(p'\right) d\mu\left(\pi'\right) \left|\pi', p', \sigma'\right\rangle \left\langle \pi,' p' \sigma' \right| \tag{77}$$

and so the transformation acts as

$$U(\Lambda)\, |\pi, p, \sigma\rangle = \sum_{\sigma'} \int d\mu\left(p'\right) d\mu\left(\pi'\right) \left|\pi', p', \sigma'\right\rangle \ \left\langle \pi', p' \sigma' \right| U(\Lambda)\, |\pi, p, \sigma\rangle \tag{78}$$

with the matrix element

$$\left\langle \pi', p' \sigma' \right| U(\Lambda)\, |\pi, p, \sigma\rangle = \delta^4\left(p' - \Lambda p\right) \delta^4\left(\pi' - \Lambda \pi\right) \ V_{\sigma' \sigma}(\pi, p, \Lambda) \tag{79}$$

where $V_{\sigma' \sigma}(\pi, p, \Lambda)$ is a discrete matrix representation of the action of the Lorentz transformation on the spin state. Thus

$$U(\Lambda)\, |\pi, p, \sigma\rangle = \sum_{\sigma'} V_{\sigma' \sigma}(\pi, p, \Lambda) \ \left|\Lambda \pi, \Lambda p, \sigma'\right\rangle \tag{80}$$

expresses the action of the unitary Lorentz transformation. Unitarity requires

$$1 = U(\Lambda)\,U(\Lambda)^{\dagger} = \sum_{\sigma}\int d\mu(p)\,d\mu(\pi)\,U(\Lambda)\,|\pi,p,\sigma\rangle\,\langle\pi,p,\sigma|\,U(\Lambda)^{\dagger} \tag{81}$$

and using the matrix element this becomes

$$\sum_{\sigma'} V_{\sigma\sigma'}V^{*}_{\sigma''\sigma'} = \sum_{\sigma'} V_{\sigma\sigma'}\left(V_{\sigma'\sigma''}\right)^{\dagger} = \delta_{\sigma\sigma''} \tag{82}$$

showing that $V_{\sigma\sigma'}(\Lambda)$ is a unitary matrix.

The Wigner operator $\alpha(\pi)$ is a pure boost constructed from the operators N^{0i} and so does not act on p or σ. The unitary representation $U(\alpha(\pi))$ is defined such that

$$U(\alpha(\pi))\,|\mathring{\pi},p,\sigma\rangle = |\pi,p,\sigma\rangle\ . \tag{83}$$

Inserting this definition into (80) and multiplying both sides by $U^{\dagger}(\alpha(\pi'))$ leads to

$$U^{\dagger}\left(\alpha(\pi')\right)\ U(\Lambda)\ U(\alpha(\pi))\ |\mathring{\pi},p,\sigma\rangle = \sum_{\sigma'} V_{\sigma'\sigma}(\pi,p,\Lambda)\,\left|\mathring{\pi},p',\sigma'\right\rangle \tag{84}$$

so that writing $U^{\dagger}(\alpha(\pi')) = U\left(\alpha^{-1}(\pi')\right)$, combining unitary matrices on the LHS, and using (51) this becomes

$$U\left(\widehat{\Lambda}_{\mathring{\pi}}\right)|\mathring{\pi},p,\sigma\rangle = \sum_{\sigma'} V_{\sigma'\sigma}(\pi,p,\Lambda)\,|\mathring{\pi},p,\sigma\rangle\ . \tag{85}$$

The matrix element for $U\left(\tilde{\Lambda}_{\mathring{\pi}}\right)$ is thus

$$\left\langle\pi,p',\sigma'\right|U\left(\tilde{\Lambda}_{\mathring{\pi}}\right)|\mathring{\pi},p,\sigma\rangle\ = V_{\sigma''\sigma}(\pi,p,\Lambda)\,\delta^{4}\left(p-p'\right)\delta^{4}\left(n-\mathring{\pi}\right) \tag{86}$$

showing that $V_{\sigma'\sigma}(\pi,p,\Lambda)$ belongs to the little group $\mathcal{L}(\mathring{\pi})$. This can be written as

$$V_{\sigma'\sigma}(\pi,p,\Lambda) = V_{\sigma'\sigma}\left(\mathring{\pi},\widehat{\Lambda}_{\mathring{\pi}}\right) = V_{\sigma'\sigma}\left(\mathring{\pi},\alpha^{-1}\left(\pi'\right)\ A\,\alpha(\pi)\right) = D^{J}_{\sigma'\sigma}(\Lambda,\pi) \tag{87}$$

where A is the $SL(2,C)$ representation of Λ and $D^{J}_{\sigma'\sigma}(\Lambda,\pi)$ is the Wigner matrix representation of the rotation

$$R = \alpha^{-1}\left(\pi'\right)A\,\alpha(\pi)\ . \tag{88}$$

Since this combination is an element of $SL(2,C)$ and a rotation, it belongs to the subgroup $SU(2)$ and is therefore unitary, and we have simply

$$U\left(\widehat{\Lambda}_{\mathring{\pi}}\right) = U\left(\alpha^{-1}\left(\pi'\right)A\,\alpha(\pi)\right) = \alpha^{-1}\left(\pi'\right)A\,\alpha(\pi)\ . \tag{89}$$

Together, the matrix element $\langle\pi',p'\sigma'|\,U(\Lambda)\,|\pi,p,\sigma\rangle$ in (79) is a combination of a pure boost with a rotation of the spin indices in the hypersurface normal to π^{μ}.

In wavefunction notation with $D^{J}_{\sigma'\sigma}(\Lambda,\pi)$ as defined in (87) the transformation can be written

$$\psi'_{\sigma}(\pi,p) = \sum_{\sigma'}\left[\alpha^{-1}(\pi)\ A\right]_{\sigma'\sigma}\ \left[\alpha(\Lambda^{-1}\pi)\psi\left(\Lambda^{-1}\pi,\Lambda^{-1}p\right)\right]_{\sigma'} \tag{90}$$

and multiplying both sides by $\alpha(\pi)$ this becomes

$$\left[\alpha(\pi)\psi(\pi,p)\right]'_{\sigma} = \sum_{\sigma'} A_{\sigma',\sigma}\left[\alpha(\Lambda^{-1}\pi)\psi\left(\Lambda^{-1}\pi,\Lambda^{-1}p\right)\right]_{\sigma'} \tag{91}$$

showing that $\alpha\psi$ undergoes Lorentz transform as $(\alpha\psi)' = A\,(\alpha\psi)$. As we saw in Section 3.1 there is an inequivalent representation that transforms under the complex conjugate representation as $(\underline{\alpha}\phi)' = \underline{A}\,(\underline{\alpha}\phi)$, so together

$$\alpha\psi \in (1/2,0) \qquad\qquad \underline{\alpha}\phi \in (0,1/2) \tag{92}$$

are the fundamental spinorial bases for the $SL(2,C)$ representation of the Lorentz group.

4 Quantum mechanics in the extended phase space

4.1 Bispinors

Horwitz et. al. wrote the Dirac spinor

$$\Psi(n,x) = \frac{1}{\sqrt{2}} \begin{bmatrix} \alpha\psi(n,x) + \underline{\alpha}\phi(n,x) \\ -\alpha\psi(n,x) + \underline{\alpha}\phi(n,x) \end{bmatrix} \tag{93}$$

in which the upper and lower sectors have even and odd parity, respectively. This field must transform as

$$\Psi'(n,x) = S(\Lambda)\Psi(\Lambda^{-1}n, \Lambda^{-1}x) \tag{94}$$

where $S(\Lambda)$ mixes components of the bispinor and acts on the standard gamma matrices as

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = \left(\Lambda^{-1}\right)^\mu{}_\nu \gamma^\nu \tag{95}$$

in analogy to the $SL(2,C)$ transformation. Comparing the forms of the transformation

$$\Lambda^\alpha{}_\beta = \eta^\alpha{}_\beta + \frac{1}{2}\left(M^{\mu\nu}\right)^\alpha{}_\beta \omega_{\mu\nu} \qquad S(\Lambda) = 1 - \frac{i}{2}\Sigma^{\mu\nu}\omega_{\mu\nu} \tag{96}$$

leads to the condition

$$\left[\gamma^\mu, \Sigma^{\alpha\beta}\right] = i\left(\eta^{\mu\alpha}\gamma^\beta - \eta^{\mu\beta}\gamma^\alpha\right) \tag{97}$$

which is satisfied by

$$\Sigma^{\alpha\beta} = \frac{i}{4}\left[\gamma^\alpha, \gamma^\beta\right] . \tag{98}$$

It is convenient here to adopt the Hestenes approach to Clifford algebra [14] and treat γ^μ as the vector basis for 4D spacetime. Thus, we write vectors and bivectors as

$$a = a_\mu \gamma^\mu \qquad a \wedge b = a_\mu b_\nu \, \gamma^\mu \wedge \gamma^\nu \tag{99}$$

with the products

$$ab = a \cdot b + a \wedge b = \frac{1}{2}\{a,b\} + \frac{1}{2}[a,b] \qquad a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b \tag{100}$$

and so

$$\gamma^\mu \cdot \gamma^\nu = \frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} = \eta^{\mu\nu} \qquad \Sigma^{\mu\nu} = \frac{i}{2}\gamma^\mu \wedge \gamma^\nu . \tag{101}$$

The explicit matrix form of $\Sigma^{\mu\nu}$ is

$$\Sigma^{0i} = \frac{i}{4}\left[\gamma^0, \gamma^i\right] = \frac{i}{2}\begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} \qquad \Sigma^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right] = \frac{1}{2}\epsilon^{ij}{}_k \begin{bmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix} \tag{102}$$

so that only Σ^{12} is diagonal. The bivector K^μ is defined as

$$K^\mu = \Sigma^{\mu\nu} n_\nu = \frac{i}{2}\gamma^\mu \wedge \gamma^\nu n_\nu = \frac{i}{2}\gamma^\mu \wedge n \qquad \longrightarrow \qquad K^\mu n_\mu = \frac{i}{2} n \wedge n = 0 \tag{103}$$

and takes the explicit form

$$K^0 = -\frac{i}{2}\begin{bmatrix} 0 & \boldsymbol{n}\cdot\boldsymbol{\sigma} \\ \boldsymbol{n}\cdot\boldsymbol{\sigma} & 0 \end{bmatrix} \qquad K^i = \frac{1}{2}\begin{bmatrix} \epsilon^{ij}{}_k n_j \sigma^k & -in_{(0)}\sigma^i \\ -in_{(0)}\sigma^i & \epsilon^{ij}{}_k n_j \sigma^k \end{bmatrix} . \tag{104}$$

Writing the transverse projection of the basis vectors as $\gamma_\perp^\mu = \gamma^\mu - n(n \cdot \gamma^\mu)$ Horwitz et. al. define

$$\Sigma_\perp^{\mu\nu} = \frac{i}{4}\left[\gamma_\perp^\mu, \gamma_\perp^\nu\right] = \Sigma^{\mu\nu} + K^\nu n^\mu - K^\mu n^\nu \qquad \longrightarrow \qquad \Sigma_\perp^{\mu\nu} n_\nu = 0 . \tag{105}$$

Thus, K^μ and $\Sigma_\perp^{\mu\nu}$ each have 3 independent components, satisfying the $O(3,1)$ Lie algebra and generating boosts and rotations in the spacelike hypersurface transverse to $\pi^\mu = Mn^\mu$.

4.2 Quantum Hamiltonian

Using $n^2 = \pi^2/M^2 = 1$, the projections of spacetime momentum can be expressed as

$$p = n^2 p = n\,(n \cdot p) + n \cdot (n \wedge p) = n\,(n \cdot p) + [p - n\,(n \cdot p)] = p_{\|} + p_{\perp} \tag{106}$$

and the parity transformation is given by

$$npn = n\,(p \cdot n) + n \cdot (p \wedge n) = n\,(p \cdot n) - n \cdot (n \wedge p) = p_{\|} - p_{\perp}\,. \tag{107}$$

Horwitz and Arshansky write the longitudinal and transverse parts of the momentum vector in the form

$$p_{\|} = \frac{1}{2}\,(p + npn) \qquad\qquad p_{\perp} = \frac{1}{2}\,(p - npn) \tag{108}$$

and using

$$\mathring{n} = \mathring{n}_\mu \gamma^\mu = \gamma^0 = \left(\gamma^0\right)^\dagger \tag{109}$$

they obtain

$$p_{\|} = p_0 \gamma^0 = p_{\|}^\dagger \qquad\qquad \gamma^5 p_{\perp} = \gamma^5 \left(p_k \gamma^k\right) = \left(\gamma^5 p_{\perp}\right)^\dagger \tag{110}$$

so that $K_L = p_{\|}$ and $K_T = \gamma^5 p_{\perp}$ may be treated as Hermitian with respect to the standard γ^μ matrices. Using $\{\gamma^5, \gamma^\mu\} = 0$

$$K_L^2 = \left(p_{\|}\right)^2 = (p \cdot n)^2 \qquad\qquad K_T^2 = -\left(\gamma^5\right)^2 (p_{\perp})^2 = -p_{\perp}^2 \tag{111}$$

so that

$$K_L^2 - K_T^2 = p_{\|}^2 + p_{\perp}^2 = p^2 \tag{112}$$

and they take the expression

$$K_0 = \frac{p^2}{2M} = \frac{1}{2M}\left(K_L^2 - K_T^2\right) \tag{113}$$

as the free particle quantum Hamiltonian. Under the minimal gauge substitution $p \longrightarrow p - eA$ the longitudinal component is

$$K_L^2 = \left(p_{\|} - eA_{\|}\right) \cdot \left(p_{\|} - eA_{\|}\right) + \left(p_{\|} - eA_{\|}\right) \wedge \left(p_{\|} - eA_{\|}\right) \tag{114}$$

$$= \left(p_{\|} - eA_{\|}\right)^2 \tag{115}$$

which follows from

$$p_{\|} \wedge A_{\|} = n\,(n \cdot p) \wedge n\,(n \cdot A) = (n \cdot p)\,(n \cdot A)\,(n \wedge n) = 0 \tag{116}$$

and the transverse component

$$K_T^2 = \gamma^5\,(p_{\perp} - eA_{\perp})\,\gamma^5\,(p_{\perp} - eA_{\perp}) \tag{117}$$

$$= -\,(p_{\perp} - eA_{\perp}) \cdot (p_{\perp} - eA_{\perp}) - (p_{\perp} - eA_{\perp}) \wedge (p_{\perp} - eA_{\perp}) \tag{118}$$

$$= -\,(p_{\perp} - eA_{\perp})^2 + e\,(p_{\perp} \wedge A_{\perp}) \tag{119}$$

where

$$p_{\perp} \wedge A_{\perp} = (p_\mu A_\nu)\,\gamma_{\perp}^\mu \wedge \gamma_{\perp}^\nu = -\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\Sigma_{\perp}^{\mu\nu}\,. \tag{120}$$

The Horwitz-Arshansky single particle electromagnetic Hamiltonian is then

$$K = \frac{1}{2M}\left(K_L^2 - K_T^2\right) = \frac{1}{2M}(p - eA)^2 + \frac{e}{2M} F_{\mu\nu} \Sigma_{\perp}^{\mu\nu} \tag{121}$$

and a many-body Hamiltonian may be constructed in a similar usual manner.

In the extended phase space, we treat the generators π_μ and $N_{\mu\nu}$ introduced in (27) as quantum operators, and again separate the longitudinal and transverse components

$$K_L^p = p_\parallel - eA_\parallel \qquad K_T^p = \gamma^5 \left(p_\perp - eA_\perp\right) \tag{122}$$

$$K_L^\pi = \pi_\parallel - e\chi_\parallel \qquad K_T^\pi = \gamma^5 \left(\pi_\perp - e\chi_\perp\right) \tag{123}$$

to write the electromagnetic Hamiltonian as

$$K = \frac{1}{2M}\left[\left(K_L^p\right)^2 - \left(K_T^p\right)^2\right] + \frac{1}{2M}\left[(K_L^\pi)^2 - (K_T^\pi)^2\right] . \tag{124}$$

Following the derivation in equations (114) to (120) we are lead to

$$K = \frac{1}{2M}\left[(p-eA)^2 + (\pi - e\chi)^2\right] + \frac{e}{2M}\left(F_{\mu\nu} + G_{\mu\nu}\right)\Sigma_\perp^{\mu\nu} \tag{125}$$

where we used the commutation relations

$$\left[p_\mu, A_\nu\right] = i\frac{\partial}{\partial x^\mu}A_\nu \qquad \left[\pi_\mu, A_\nu\right] = i\frac{\partial}{\partial n^\mu}\chi_\nu \tag{126}$$

and $F_{\mu\nu}$ and $G_{\mu\nu}$ are as defined in (13). As in Section 2 terms containing $H_{\mu\nu}$ do not directly appear in the Hamiltonian or couple to the spin operator $\Sigma_\perp^{\mu\nu}$.

4.3 *Plane wave solutions*

Taking $A = \chi = 0$ in the Hamiltonian (125), the free particle Stueckelberg-Schrodinger equation

$$i\partial_\tau \Psi\left(\zeta, x, \tau\right) = \left(\frac{p^2}{2M} + \frac{\pi^2}{2M}\right)\Psi\left(\zeta, x, \tau\right) \tag{127}$$

admits the plane wave solution

$$\Psi\left(\zeta, x, \tau\right) = \begin{bmatrix} \chi^{(1)}(\pi) \\ \chi^{(2)}(\pi) \\ \chi^{(3)}(\pi) \\ \chi^{(4)}(\pi) \end{bmatrix} \exp\left[i\left(p\cdot x + \pi\cdot\zeta - \frac{p^2+\pi^2}{2M}\tau\right)\right] \tag{128}$$

where $\chi^{(\sigma)}(\pi)$ is a constant amplitude. In the special frame $\pi = \mathring{\pi} = M\left(1,0,0,0\right)$ we can write the four independent amplitudes as

$$\chi^{(\sigma)}(\pi) = \mathcal{N}\,\psi^{(\sigma)} \tag{129}$$

where $\mathcal{N}$ is some normalization and

$$\psi^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \psi^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \psi^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \psi^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} . \tag{130}$$

The general plane wave solution (128) is found by boosting $\mathring{\pi}^\mu$ to

$$\pi = \Lambda\mathring{\pi} = \exp\left(i\beta^k M_{0k}\right)\mathring{\pi} = M\left(\cosh\beta, \sinh\beta\hat{\boldsymbol{\beta}}\right) \tag{131}$$

and transforming the state as

$$\Psi^{(\sigma)}(\zeta, x, \tau) = S(\Lambda)\Psi^{(\sigma)}(\Lambda^{-1}\mathring{\zeta}, \Lambda^{-1}x) = S(\Lambda)\Psi^{(\sigma)}(\mathring{\zeta}, x, \tau) \tag{132}$$

where the phase of the plane wave is a Lorentz invariant and

$$S\left(\Lambda\right) = \exp\left(-i\Sigma^{0k}\beta_k\right) = \begin{bmatrix} \cosh\frac{\beta}{2}\,\sigma^0 & \sinh\frac{\beta}{2}\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} \\ \sinh\frac{\beta}{2}\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} & \cosh\frac{\beta}{2}\,\sigma^0 \end{bmatrix} . \tag{133}$$

We can now write the four independent solutions as

$$\Psi^{(\sigma)}(\zeta,x,\tau)=\mathcal{N}\,u^{(\sigma)}\exp\left[i\left(p\cdot x+\pi\cdot\zeta-\frac{p^2+\pi^2}{2M}\tau\right)\right] \tag{134}$$

where

$$u^{(\sigma)}=S\left(\Lambda\right)\psi^{(\sigma)} \tag{135}$$

so that using

$$\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma}=\begin{pmatrix}\hat{\beta}^3 & \hat{\beta}^1-i\hat{\beta}^2\\ \hat{\beta}^1+i\hat{\beta}^2 & -\hat{\beta}^3\end{pmatrix} \tag{136}$$

we find the transformed amplitudes as

$$u^{(1)}=\begin{bmatrix}\cosh\frac{\beta}{2}\\ 0\\ \sinh\frac{\beta}{2}\hat{\beta}^3\\ \sinh\frac{\beta}{2}\left(\hat{\beta}^1+i\hat{\beta}^2\right)\end{bmatrix}\qquad u^{(2)}=\begin{bmatrix}0\\ \cosh\frac{\beta}{2}\\ \sinh\frac{\beta}{2}\left(\hat{\beta}^1-i\hat{\beta}^2\right)\\ -\sinh\frac{\beta}{2}\hat{\beta}^3\end{bmatrix} \tag{137}$$

$$u^{(3)}=\begin{bmatrix}\sinh\frac{\beta}{2}\hat{\beta}^3\\ \sinh\frac{\beta}{2}\left(\hat{\beta}^1+i\hat{\beta}^2\right)\\ \cosh\frac{\beta}{2}\\ 0\end{bmatrix}\qquad u^{(4)}=\begin{bmatrix}\sinh\frac{\beta}{2}\left(\hat{\beta}^1-i\hat{\beta}^2\right)\\ -\sinh\frac{\beta}{2}\hat{\beta}^3\\ 0\\ \cosh\frac{\beta}{2}\end{bmatrix}. \tag{138}$$

Using (45) the conjugate bispinor is

$$\overline{\Psi}^{(\sigma)}(\zeta,x,\tau)=\mathcal{N}\,\overline{u}^{(\sigma)}\;\exp\left[-i\left(p\cdot x+\pi\cdot\zeta-\frac{p^2+\pi^2}{2M}\tau\right)\right] \tag{139}$$

where

$$\overline{u}^{(\sigma)}=\overline{\left[S\left(\Lambda\right)\psi^{(\sigma)}\right]}=\left[\left(C^{-1}S\left(\Lambda\right)C\right)\psi^{(\sigma)}\right]^{\dagger}=\left[S^{-1}\left(\Lambda\right)\psi^{(\sigma)}\right]^{\dagger} \tag{140}$$

so that

$$\overline{u}^{(1)}=\left[\cosh\frac{\beta}{2}\qquad 0\qquad -\sinh\frac{\beta}{2}\hat{\beta}^3\qquad -\sinh\frac{\beta}{2}\left(\hat{\beta}^1-i\hat{\beta}^2\right)\right] \tag{141}$$

$$\overline{u}^{(2)}=\left[0\qquad \cosh\frac{\beta}{2}\qquad -\sinh\frac{\beta}{2}\left(\hat{\beta}^1+i\hat{\beta}^2\right)\qquad \sinh\frac{\beta}{2}\hat{\beta}^3\right] \tag{142}$$

$$\overline{u}^{(3)}=\left[-\sinh\frac{\beta}{2}\hat{\beta}^3\qquad -\sinh\frac{\beta}{2}\left(\hat{\beta}^1-i\hat{\beta}^2\right)\qquad \cosh\frac{\beta}{2}\qquad 0\right] \tag{143}$$

$$\overline{u}^{(4)}=\left[-\sinh\frac{\beta}{2}\left(\hat{\beta}^1+i\hat{\beta}^2\right)\qquad \sinh\frac{\beta}{2}\hat{\beta}^3\qquad 0\qquad \cosh\frac{\beta}{2}\right] \tag{144}$$

leading to

$$\overline{\Psi}^{(\sigma)}(\zeta,x,\tau)\Psi^{(\sigma)}(\zeta,x,\tau)=\mathcal{N}^2\,\overline{u}^{(\sigma)}u^{(\sigma)}=\mathcal{N}^2\,. \tag{145}$$

4.4 The spin operator

For the bispinor representation, the Pauli–Lubanski pseudovector (62) takes the form of the four matrices

$$W_\mu=-\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\Sigma_\perp^{\nu\lambda}N^\sigma=-\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\Sigma^{\nu\lambda}N^\sigma \tag{146}$$

where the second equality follows from (105). The scalar product

$$W^\mu W_\mu = \frac{1}{2}\Sigma^{\nu\lambda}\Sigma_{\nu\lambda}n^\sigma N_\sigma - \Sigma_{\mu\sigma}\Sigma^{\nu\sigma}N^\mu N_\nu = \frac{1}{2}\Sigma^{\nu\lambda}\Sigma_{\nu\lambda} \tag{147}$$

is independent of N^μ and commutes with all the other generators. By direct calculation we find the explicit forms

$$W_0 = -\frac{1}{2}\begin{bmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{n} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{n} \end{bmatrix} \qquad W_i = \frac{1}{2}\begin{bmatrix} n^{(0)}\delta_{ik}\sigma^k & i\,(\boldsymbol{\sigma}\times\boldsymbol{n})_i \\ i\,(\boldsymbol{\sigma}\times\boldsymbol{n})_i & n^{(0)}\delta_{ik}\sigma^k \end{bmatrix} \tag{148}$$

with the 3D inner product $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \delta_{ij}n^i\sigma^j$ and cross product $(\boldsymbol{\sigma}\times\boldsymbol{n})_i = \epsilon_{ijk}\sigma^j n^k$, from which we obtain

$$-W^\mu W_\mu = -W_0^2 - \eta^{ii'}W_iW_{i'} = \frac{3}{4}\begin{bmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{bmatrix} = \frac{1}{2}\left(\frac{1}{2}+1\right)I \tag{149}$$

describing the bispinor as a spin-1/2 state. In the special frame $n = \mathring{n} = (1,0,0,0)$ we have

$$W_0 = 0 \qquad W_i = \frac{1}{2}\delta_{ik}\begin{bmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix} \tag{150}$$

so that acting on the plane wave solutions, W^3 has eigenvalue $+1$ on $\Psi^{(1)}$ and $\Psi^{(3)}$, and eigenvalue -1 on $\Psi^{(2)}$ and $\Psi^{(4)}$.

Since W_μ is orthogonal to n^μ it has three independent components in the spacelike hypersurface, for which we may find an orthonormal basis $\boldsymbol{e}^i$ for $i = 1,2,3$. Expanding

$$W = W_\mu\gamma^\mu = J_k\,\boldsymbol{e}^k \quad \longrightarrow \quad W^\mu = W\cdot\gamma^\mu = J_k\,\left(\boldsymbol{e}^k\right)^\mu \tag{151}$$

where $\left(\boldsymbol{e}^k\right)^\mu$ is the μ component of $\boldsymbol{e}^k$ in the γ^μ basis. From (149) we see that the components J_k satisfy

$$-\eta_{\mu\nu}W^\mu W^\nu = -\eta^{kk'}J_kJ_{k'} = \boldsymbol{J}^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)I \tag{152}$$

as expected for the 3-vector spin operator $\boldsymbol{J}$.

For simplicity we may consider a boost along one of the three space axes a so that

$$n = n_{(0)}\gamma^0 + n_{(a)}\gamma^a = n^{(0)}\gamma^0 - n^{(a)}\gamma^a = \sqrt{1+n_{(a)}^2}\gamma^0 - n^{(a)}\gamma^a \tag{153}$$

leading to

$$W_0 = -\frac{1}{2}\begin{bmatrix} n^{(a)}\sigma^{(a)} & 0 \\ 0 & n^{(a)}\sigma^{(a)} \end{bmatrix} \qquad W_j = \frac{1}{2}\begin{bmatrix} n^{(0)}\delta_{jk}\sigma^k & i\epsilon_{jik}\sigma^i n^{(a)}\delta^{ak} \\ i\epsilon_{jik}\sigma^i n^{(a)}\delta^{ak} & n^{(0)}\delta_{jk}\sigma^k \end{bmatrix}. \tag{154}$$

Taking the boost along the 3-axis this becomes

$$W_0 = -\frac{1}{2}\begin{bmatrix} n^{(3)}\sigma^{(3)} & 0 \\ 0 & n^{(3)}\sigma^{(3)} \end{bmatrix} \qquad W_j = \frac{1}{2}\begin{bmatrix} n^{(0)}\delta_{jk}\sigma^k & in^{(3)}\epsilon_{ji3}\sigma^i \\ in^{(3)}\epsilon_{ji3}\sigma^i & n^{(0)}\delta_{jk}\sigma^k \end{bmatrix} \tag{155}$$

so that W_0 and W_3 are diagonal while W_1 and W_2 are off-diagonal.

Using the Gram-Schmidt method we can define the orthonormal basis $\boldsymbol{e}^\mu$ starting with $\boldsymbol{e}^0 = n$ and leading to

$$\begin{aligned} \boldsymbol{e}^0 &= n = n^{(0)}\gamma^0 - n^{(3)}\gamma^3 = \sqrt{1+n_{(3)}^2}\gamma^0 - n^{(3)}\gamma^3 \\ \boldsymbol{e}^{(1)} &= \gamma^1 \\ \boldsymbol{e}^{(2)} &= \gamma^2 \\ \boldsymbol{e}^{(3)} &= n^{(0)}\gamma^3 - n^{(3)}\gamma^0 = -n^{(3)}\gamma^0 + \sqrt{1+n_{(3)}^2}\gamma^3 \end{aligned} \tag{156}$$

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so that although $\boldsymbol{e}^{(3)}$ has a 0-component, it is spacelike in the 4D spacetime and normal to n. Using

$$J^k = W \cdot \boldsymbol{e}^k = \left(W_0\gamma^0 + W_i\gamma^i\right) \cdot \boldsymbol{e}^k \tag{157}$$

we find

$$J^1 = \left(W_0\gamma^0 + W_i\gamma^i\right) \cdot \gamma^1 = W_i\eta^{i1} \tag{158}$$

$$J^2 = \left(W_0\gamma^0 + W_i\gamma^i\right) \cdot \gamma^2 = W_i\eta^{i2} \tag{159}$$

$$J^3 = \left(W_0\gamma^0 + W_i\gamma^i\right) \cdot \left(\gamma^3 n^{(0)} - n^{(3)}\gamma^0\right) = n^{(0)}W_i\eta^{i3} - n^{(3)}W_0 \tag{160}$$

so that

$$J_1 = \frac{1}{2}\begin{bmatrix} n^{(0)}\sigma^1 & in^3\sigma^2 \\ in^3\sigma^2 & n^{(0)}\sigma^1 \end{bmatrix} \qquad J_2 = \frac{1}{2}\begin{bmatrix} n^{(0)}\sigma^2 & -in^3\sigma^1 \\ -in^3\sigma^1 & n^{(0)}\sigma^2 \end{bmatrix} \tag{161}$$

and

$$J_3 = \frac{1}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix} \tag{162}$$

which is diagonal and independent of n. Using the n-dependent bispinors (137) and (138) we see

$$J_3\Psi^{(1)} = \frac{1}{2}\Psi^{(1)} \qquad J_3\Psi^{(3)} = \frac{1}{2}\Psi^{(3)} \tag{163}$$

$$J_3\Psi^{(2)} = -\frac{1}{2}\Psi^{(2)} \qquad J_3\Psi^{(4)} = -\frac{1}{2}\Psi^{(4)} \tag{164}$$

as expected.

4.5 A singlet state

We write the plane wave state (134) as

$$\Psi^{(\sigma)}(\zeta, x, \tau) = \varphi(\zeta, x, \tau)\, u^{(\sigma)}(\pi) \tag{165}$$

where

$$\varphi(\zeta, x, \tau) = \mathcal{N} \exp\left[i\left(p \cdot x + \pi \cdot \zeta - \frac{p^2 + \pi^2}{2M}\tau\right)\right] \tag{166}$$

describes a spinless plane wave on the extended spacetime. A two-body plane wave state is the direct product

$$\Psi^{(\sigma_1,\sigma_2)}(\zeta_1, \zeta_2, x_1, x_2, \tau) = \varphi_1(\zeta_1, x_1, \tau)\,\varphi_2(\zeta_2, x_2, \tau)\, u_1^{(\sigma_1)}(\pi)\, u_2^{(\sigma_2)}(\pi) \tag{167}$$

and so the singlet state is

$$\Psi^{(0)}(\zeta_1, \zeta_2, x_1, x_2, \tau) = \varphi(\zeta_1, \zeta_2, x_1, x_2, \tau)\, u^{(0)}(\pi) \tag{168}$$

where the spacetime part

$$\varphi(\zeta_1, \zeta_2, x_1, x_2, \tau) = \frac{1}{\sqrt{2}}\left[\varphi_1(\zeta_1, x_1, \tau)\,\varphi_2(\zeta_2, x_2, \tau) + \varphi_2(\zeta_1, x_1, \tau)\,\varphi_1(\zeta_2, x_2, \tau)\right] \tag{169}$$

is symmetric under $(\zeta_1, x_1) \leftrightarrow (\zeta_2, x_2)$ while the spin part

$$u^{(0)}(\pi) = \frac{1}{\sqrt{2}}\left[u_1^{(\sigma_1)}(\pi)\, u_2^{(\sigma_2)}(\pi) - u_1^{(\sigma_2)}(\pi)\, u_2^{(\sigma_1)}(\pi)\right] \tag{170}$$

requires

$$J_3\, u^{(\sigma_1)} + J_3\, u^{(\sigma_2)} = 0 \tag{171}$$

and is antisymmetric under $\sigma_1 \leftrightarrow \sigma_2$. Since the one-particle states must transform under the same representation of $SL(2, C)$, we may form singlets from the pairs $\{u^1, u^2\}$ and $\{u^3, u^4\}$.

5 First order perturbation in a constant field

To first order in e the Hamiltonian (125) is

$$K = \frac{1}{2M}\left(p^2 + \pi^2\right) - \frac{e}{M}\left(A \cdot p + \chi \cdot \pi\right) + \frac{e}{2M}\left(F_{\mu\nu} + G_{\mu\nu}\right)\Sigma_\perp^{\mu\nu} \tag{172}$$

where we imposed the Lorenz condition

$$\frac{\partial}{\partial x^\mu}A^\mu = 0 \qquad \frac{\partial}{\partial \zeta^\mu}\chi^\mu = 0 . \tag{173}$$

Taking the field strengths $F_{\mu\nu} = G_{\mu\nu}$ to be constant, the potentials can be written

$$A^\mu = -\frac{1}{2}F^{\mu\nu}x_\nu \qquad \chi^\mu = -\frac{1}{2}F^{\mu\nu}\zeta_\nu \tag{174}$$

so that

$$-\frac{e}{M}\left(A \cdot p + \chi \cdot \pi\right) = \frac{e}{2M}F^{\mu\nu}\left(x_\mu p_\nu + \zeta_\mu \pi_\nu\right) . \tag{175}$$

Using the antisymmetry of the field strengths we have

$$K = K_0 - \frac{e}{4M}F_{\mu\nu}\left(L^{\mu\nu} + N^{\mu\nu}\right) + \frac{e}{M}F_{\mu\nu}\Sigma_\perp^{\mu\nu} \tag{176}$$

where the generators $L^{\mu\nu}$ and $N^{\mu\nu}$ of Lorentz transformation on the extended spacetime (x^μ, ζ^μ) are defined in (23) and (27).

In a frame for which the field is purely magnetic so that

$$F_{0\nu} = 0 \qquad F_{ij} = \varepsilon_{ijk}B^k \tag{177}$$

we can write

$$\frac{1}{2}F_{\mu\nu}\left(L^{\mu\nu} + N^{\mu\nu}\right) = B \cdot J \qquad F_{\mu\nu}\Sigma_\perp^{\mu\nu} = B \cdot \hat{\Sigma}_\perp \tag{178}$$

where

$$J_k = \frac{1}{2}\epsilon_{ijk}M^{ij} = (\boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{\zeta} \times \boldsymbol{\pi})_k \qquad \hat{\Sigma}_\perp^i = \epsilon^i{}_{jk}\Sigma_\perp^{jk} \tag{179}$$

are the orbital angular momentum in the extended spacetime (x, ζ) and the spin. Since J_k vanishes for the plane wave solution (134), the perturbation is just the spin interaction term

$$K_{spin} = \frac{e}{M}F_{\mu\nu}\Sigma_\perp^{\mu\nu} = \frac{e}{M}B_k\hat{\Sigma}_\perp^k \tag{180}$$

which couples the magnetic field to the spin generator in the spacelike hypersurface orthogonal to π^μ. In the frame defined in (156) by a pure boost of $\mathring{\pi}$ along the 3-axis, we obtain

$$K_{spin} = \frac{e}{2M}\begin{bmatrix} B^3\sigma^3 + n_{(0)}^2\left(B^1\sigma^1 + B^2\sigma^2\right) & in_{(0)}n^{(3)}\left(B^1\sigma^2 - B^2\sigma^1\right) \\ in_{(0)}n^{(3)}\left(B^1\sigma^2 - B^2\sigma^1\right) & B^3\sigma^3 + n_{(0)}^2\left(B^1\sigma^1 + B^2\sigma^2\right) \end{bmatrix} \tag{181}$$

using (102) and (104) for $\Sigma^{\mu\nu}$ and K^μ.

Thus, also taking the magnetic field along the 3-axis as

$$\boldsymbol{B} = \left(0, 0, B^3\right) \tag{182}$$

$$K_{spin} = \frac{e}{2M}B^3\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix} \tag{183}$$

which is diagonal, produces

$$K_{spin}\,\Psi^{(\sigma)}(\zeta, x, \tau) = \pm\frac{eB^3}{2M}\Psi^{(\sigma)}(\zeta, x, \tau) \tag{184}$$

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where the upper sign holds for $\sigma = 1,3$ and the lower sign for $\sigma = 2,4$. This term then describes a perturbative contribution to the total mass whose matrix element is

$$\langle p,\pi,\sigma' | \, K \, | p,\pi,\sigma \rangle = \mathcal{N}^2 \left(\frac{p^2+\pi^2}{2M} \pm \frac{eB^3}{2M} \right) \delta^{\sigma\sigma'} \; . \tag{185}$$

However, if we take the magnetic field along the 1-axis, orthogonal to the 3-axis,

$$\boldsymbol{B} = \left(B^1, 0, 0 \right) \tag{186}$$

we find

$$K_{spin} = \frac{eB^1}{2M} \begin{bmatrix} n^2_{(0)}\sigma^1 & n_{(0)}n^{(3)}i\sigma^2 \\ n_{(0)}n^{(3)}i\sigma^2 & n^2_{(0)}\sigma^1 \end{bmatrix} \tag{187}$$

which is real but entirely off-diagonal, producing a transition in spin states, but not a shift in π^μ that could disrupt the singlet. Acting on the bispinor amplitudes (137) and (138) with $\hat{\beta}^3 = 1$ and $\hat{\beta}^1 = \hat{\beta}^2 = 0$ and using (131) to write $n^{(0)} = \cosh\beta$ and $n^{(3)} = \sinh\beta$, we find the non-zero transition amplitudes

$$\begin{aligned} \langle p,\pi,2 | \, K_{spin} \, | p,\pi,1 \rangle = \langle p,\pi,1 | \, K_{spin} \, | p,\pi,2 \rangle = \frac{eB^1}{2M} \cosh\beta \\ \langle p,\pi,4 | \, K_{spin} \, | p,\pi,3 \rangle = \langle p,\pi,3 | \, K_{spin} \, | p,\pi,4 \rangle = \frac{eB^1}{2M} \cosh\beta \end{aligned} \tag{188}$$

with no transitions between the parity symmetric states for $\sigma = 1,2$ and the parity antisymmetric states for $\sigma = 3,4$. For a singlet state with spin part

$$u^{(0)}(\pi) = \frac{1}{\sqrt{2}} \left[u_1^{(1)}(\pi) \, u_2^{(2)}(\pi) - u_1^{(2)}(\pi) \, u_2^{(1)}(\pi) \right] \tag{189}$$

the perturbation thus induces, with probability $\left(eB^1/2M\right)^2 \cosh^2\beta$, the transition

$$u^{(0)}(\pi) \;\longrightarrow\; \frac{1}{\sqrt{2}} \left[u_1^{(2)}(\pi) \, u_2^{(1)}(\pi) - u_1^{(1)}(\pi) \, u_2^{(2)}(\pi) \right] = -u^{(0)}(\pi) \tag{190}$$

equivalent to an exchange of particles.

In a frame for which the field is purely electric

$$F_{0i} = E_i \qquad\qquad F_{ij} = 0 \tag{191}$$

the perturbative terms are

$$\frac{e}{4M} F_{\mu\nu} M^{\nu\mu} = -\frac{e}{2M} E_i \left(L^{0i} + N^{0i} \right) \qquad\qquad \frac{e}{M} F_{\mu\nu} \Sigma_\perp^{\mu\nu} = \frac{2e}{M} E_i \Sigma_\perp^{0i} \tag{192}$$

where the expectation values of the boost operators in spacetime

$$L^{0i} = x^0 p^i - x^i p^0 \qquad\qquad L_n^{0i} = \zeta^0 \pi^i - \zeta^i \pi^0 \tag{193}$$

vanish. The boost operator spin space is

$$\Sigma_\perp^{0i} = \Sigma^{0i} - \Sigma^{0i} n_{(0)} n^{(0)} - \Sigma^{0j} n_j n^i + \Sigma^{ij} n_j n^{(0)} \tag{194}$$

and using the explicit matrix forms (102) in the frame defined in (156) by a pure boost of $\mathring{\pi}$ along the 3-axis, we find

$$\frac{2e}{M} E_i \, \Sigma_\perp^{0i} = \frac{e}{M} \begin{bmatrix} 0 & T(\boldsymbol{\sigma}) \\ T(\boldsymbol{\sigma}) & 0 \end{bmatrix} \tag{195}$$

where

$$T(\boldsymbol{\sigma}) = -in^2_{(3)}\left(E_1\sigma^1 + E_2\sigma^2\right) - n^{(0)}n^{(3)}\left(E^1\sigma^2 - E^2\sigma^1\right) . \tag{196}$$

Since the boost generators are anti-Hermitian and non-compact, we expect the perturbations to produce complex matrix elements. In the case that the electric field is $E = \left(0, 0, E^3\right)$, parallel to the boost of $\mathring{\pi}$, we find $T(\boldsymbol{\sigma}) = 0$ and so the perturbation vanishes. But in the case that the electric field is $E = \left(E^1, 0, 0\right)$ along the 1-axis, we have

$$K_{spin} = i\frac{eE^1}{M}n^{(3)}\begin{bmatrix} 0 & n^{(3)}\sigma^1 + n^{(0)}i\sigma^2 \\ n^{(3)}\sigma^1 + n^{(0)}i\sigma^2 & 0 \end{bmatrix} \tag{197}$$

which is again entirely off-diagonal. Acting on the bispinor amplitudes (137) and (138) as for the magnetic field, with $\hat{\beta}^3 = 1$ and $\hat{\beta}^1 = \hat{\beta}^2 = 0$ and again using (131) to write $n^{(0)} = \cosh\beta$ and $n^{(3)} = \sinh\beta$, we find the non-zero transition amplitudes (suppressing p and π)

$$\begin{aligned}
\langle 2|\, K_{spin}\, |1\rangle &= i\frac{eE^3}{M}e^{-\beta}\sinh\beta & \langle 1|\, K_{spin}\, |2\rangle &= i\frac{eE^3}{M}e^{\beta}\sinh\beta = \langle 2|\, K_{spin}\, |1\rangle^*_{\beta\to-\beta} \\
\langle 4|\, K_{spin}\, |1\rangle &= i\frac{eE^3}{M}e^{-\beta}\cosh\beta & \langle 1|\, K_{spin}\, |4\rangle &= -i\frac{eE^3}{M}e^{\beta}\cosh\beta = \langle 4|\, K_{spin}\, |1\rangle^*_{\beta\to-\beta} \\
\langle 2|\, K_{spin}\, |3\rangle &= i\frac{eE^3}{M}e^{-\beta}\cosh\beta & \langle 3|\, K_{spin}\, |2\rangle &= -i\frac{eE^3}{M}e^{\beta}\cosh\beta = \langle 2|\, K_{spin}\, |3\rangle^*_{\beta\to-\beta} \\
\langle 4|\, K_{spin}\, |3\rangle &= i\frac{eE^3}{M}e^{-\beta}\sinh\beta & \langle 3|\, K_{spin}\, |4\rangle &= i\frac{eE^3}{M}e^{\beta}\sinh\beta = \langle 4|\, K_{spin}\, |3\rangle^*_{\beta\to-\beta}
\end{aligned} \tag{198}$$

which are pure imaginary, suggesting a decaying transition. We generally have

$$\langle\sigma|\, K_{spin}\, |\sigma'\rangle = \langle\sigma'|\, K_{spin}\, |\sigma\rangle^*_{\beta\to-\beta} \tag{199}$$

for the transitions $1 \leftrightarrow 2, 4$ and $3 \leftrightarrow 2, 4$. We notice that the state transitions $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ preserve parity (+1 for the former and −1 for the latter), while the transitions $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ reverse parity.

For a singlet state with spin part

$$u^{(0)}(\pi) = \frac{1}{\sqrt{2}}\left[u_1^{(1)}(\pi)\, u_2^{(2)}(\pi) - u_1^{(2)}(\pi)\, u_2^{(1)}(\pi)\right] \tag{200}$$

the possible transitions induced by this perturbation are

$$\begin{aligned}
u^{(0)}(\pi) &\longrightarrow \frac{1}{\sqrt{2}}\left[u_1^{(2)}(\pi)\, u_2^{(1)}(\pi) - u_1^{(1)}(\pi)\, u_2^{(2)}(\pi)\right] \\
u^{(0)}(\pi) &\longrightarrow \frac{1}{\sqrt{2}}\left[u_1^{(4)}(\pi)\, u_2^{(3)}(\pi) - u_1^{(3)}(\pi)\, u_2^{(4)}(\pi)\right]
\end{aligned} \tag{201}$$

each of which forms a possible spin-0 singlet state, and the pairs

$$\begin{aligned}
u^{(0)}(\pi) &\longrightarrow \frac{1}{\sqrt{2}}\left[u_1^{(2)}(\pi)\, u_2^{(3)}(\pi) - u_1^{(3)}(\pi)\, u_2^{(2)}(\pi)\right] \\
u^{(0)}(\pi) &\longrightarrow \frac{1}{\sqrt{2}}\left[u_1^{(4)}(\pi)\, u_2^{(1)}(\pi) - u_1^{(1)}(\pi)\, u_2^{(4)}(\pi)\right] .
\end{aligned} \tag{202}$$

Although the latter pairs combine a spin up state and a spin down state, these states have opposite parity, transform under inequivalent representations of $SL(2, C)$, and do not belong to the same Hilbert space of coherent states. We conclude that these pairs cannot form singlet states, and thus the perturbation may disrupt an existing singlet.

6 Summary

The phenomenon of quantum entanglement is an increasingly important topic in contemporary physics, from the continuing debate of philosophical questions deriving from the papers of Einstein, Podolsky, and Rosen [15] to new technological applications in communications and radar [16, 17]. As emphasized by Horwitz and Arshansky, a proper treatment of this subject requires a fully covariant relativistic framework that places space and time on an equal footing in a consistent manner. This is particularly true regarding the representation of spin, which in relativistic quantum theory appears through the $SU(2)$ subgroup of the $SL(2, C)$ covering group for $O(3, 1)$. Horwitz and Arshansky have developed a such a representation, induced on the orbit of an arbitrary timelike unit vector n^μ, thus circumventing issues arising in the Wigner representation induced on the spacetime momentum p^μ. They showed that the description of a singlet state must be constructed from spin-1/2 states in the same Hilbert space, characterized in part by the $SU(2)$ representation induced on a single value of the unit vector n^μ.

In this paper, we constructed an induced relativistic representation of spin on an extended phase space $(x^\mu, p^\mu), (\zeta^\mu, \pi^\mu)$, in which we associate the unit vector n^μ with the momentum π^μ, thus providing an dynamical interpretation for this new quantity. We constructed a classical Lagrangian and Hamiltonian gauge theory on the extended phase space and considered studied the classical equations of motion in the extended electromagnetic field. We showed that when the extended gauge potentials A_μ and χ_μ depend on both x^μ and ζ^μ, then π_1^μ and π_2^μ for a two body system may diverge, indicating a possible mode of disruption of a quantum singlet.

Defining the generators of Poincaré transformation on the extended phase space, we developed the spinor representations of the Lorentz group from the left and right handed representations. A Lorentz transformation can be written as a combination of a pure boost and a rotation, leading us to define the Wigner operator for the boost and the little group of rotations that leaves π^μ invariant. From these generators we constructed basis quantities for representations of spin states, and obtained the matrix elements for a Lorentz transformation of these states, which include a rotation of the spin indices in the spacelike hypersurface orthogonal to π^μ.

Following Horwitz and Arshansky, we defined a Dirac bispinor with two even parity components and two odd parity components, associated with the inequivalent representations of $SL(2, C)$. Projecting onto the longitudinal and transverse components of the momentum $\{p^\mu, \pi^\mu\}$ we found the electromagnetic quantum Hamiltonian, and wrote plane wave solutions on the extended phase space. In the absence of an external field, these free particle states evolve with constant $\{p^\mu, \pi^\mu\}$ and permitting the construction of singlet states defined in the same representation of $SU(2)$. Such states contain a spacetime part symmetric under exchange of particles, and an antisymmetric spin part.

Turning to the electromagnetic Hamiltonian, we studied the behavior of the plane wave solutions in a constant external field, to first order in perturbation theory. As in nonrelativistic quantum theory, the magnetic field couples to the angular momentum generators. But the electric field couples to the boost generators, which are anti-Hermitian and non-compact. In the case of a magnetic field oriented parallel to π^μ we found a perturbative contribution to the particle mass (the eigenvalue of the $O(3, 1)$ scalar Hamiltonian), while the perturbation from an electric field parallel to π^μ vanishes identically. For a field oriented normal to π^μ, the magnetic field produces a perturbative transition between the spin states $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. Because these transitions preserve the parity, and thus the $SL(2, C)$ representation of the states, a singlet state will be unaffected by this perturbation. However, an electric field normal to π^μ produces a purely imaginary transition element, suggesting a decay. Moreover, in addition to to transitions between spin states $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, this perturbation leads to transitions $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$. Since these latter transitions may produce a pair of particles in different representations of $SL(2, C)$, they may disrupt an existing singlet state.

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