NONCOMMUTATIVE SCALAR FIELD THEORIES, SOLITONS AND SUPERALGEBRA

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2002

I dedicate this work to my family.

ACKNOWLEDGMENTS

When one thinks about his or her success, one would find that an individual's success always relies on the kindness and help from many other people. It is no exception for the PhD degree that I will be receiving. Listed in the following are the persons towards whom I feel deepest appreciation. Certainly they are not the only ones that deserve the credit for my success. With this opportunity I would also like to acknowledge my thanks to all other people who have helped me.

I would like to begin by thanking my parents, Yate Xiong and Qifang Zhou. When I was little and naughty, they always led me through hardship to happiness. When older and ambitious, they were always very supportive of me. They always leave me the best, but never ask me to do anything in return. Every time I succeed, it is always been a great joy to them as well. This time certainly I would like to share the wonderful moment with them again.

A special thank you also goes to my sister Xiaolin Xiong. As an elder sister, she is always ready to look after me. She always walks one step ahead of me, and shares with me her success and failure. I feel very lucky to have such a nice sister. If it was not because of her, I would have experienced much more failure while I was growing up. I am sure she will feel very proud to see her brother to get a PhD one day.

Of course I owe a great debt of gratitude to my advisor, Prof. Pierre Ramond. It has been a great honor and pleasure to have him as my supervisor, and work with him through the PhD. He not only led me through the academic research work, but also from time to time gave me suggestions on mental strength for being a dedicated scientist. Some of his criticisms were even blunt, but I am sure

I will remember all of them because they have contributed a lot to my success in obtaining my PhD and will have a tremendous amount of influence in my future work and life as well.

A lot of thanks go to Prof. Richard Woodard. He is the very first one who introduced me to the field theory, and was always willing to answer my questions in detail. The knowledge I have learned in the field theory class has always been a great support to my PhD research. His principles for physicist and for nerd have always been an encouragement for me to move forwards. I admire his courage and determination in holding his principles, because often I found they are far more easy to understand than to stick to.

Very special thanks go to my very good friend Vivian Guo. She has encouraged me a lot during the hardest time in my PhD work, when I felt exhausted physically and mentally. The time we spent together is always a precious and beautiful memory to me. At this moment of my success I would like to say thanks to her and wish her the best.

Many thanks go to my girlfriend, Qiang Mei. She gave me a lot of encouragements when I met difficulties in the research, and felt that I was not able to finish the research work. It is a great pleasure to talk to her everyday. I am looking forward to continuing to develop our relationship after I finish the PhD.

I would also like to thank Susan Rizzo and Darlene Latimer. Their work has always been, and will always be very helpful to graduate students and faculty in the department. Certainly I appreciate the convenience they bring to me during the long time I have spent here.

Finally a lot of thanks would have to go to the faculty, postdoctorals and fellow students in IFT. They are always very friendly and ready to help and discuss the questions. The research environment here is wonderful. It would be a great memory of mine spending the best time in my life with the people here. Particular thanks go to my other PhD committee members, Dr. Pierre Sikivie, Dr. Charles Thorn, Dr. Zongan Qiu and Dr. Paul Ehrlich. I appreciate their time in guiding my process towards the PhD.

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December 2002

Chair: Pierre Ramond
Major Department: Physics

This dissertation presents perturbative and nonperturbative aspects of noncommutative (NC) field theories, as well as superalgebras in NC field theory and higher dimensional theories. In particular, the perturbative structures of the NC Wess-Zumino model are investigated in detail, as well as the deformed superalgebra relations of the model. NC solitons in scalar field theory are quantized and quantum corrections to the energy are calculated, where UV-IR divergences are found similar to those in the perturbative theory. Kostant equations in higher dimensions are constructed with differential form representations, in which the solutions are also expressed.

CHAPTER 1
INTRODUCTION

This dissertation is based on the following papers on noncommutative(NC) field theory and superalgebra [1–4].

The dissertation research focuses on the quantum behavior of NC field theory, including renormalization of the perturbative and nonperturbative structures in the theory. The chapters are organized as follows.

Following an introduction to NC geometry as certain limit of string theory in the beginning chapter, chapter 2 and 3 cover the work on NC perturbative field theory [1]. In chapter 2, we discuss the perturbative dynamics of NC field theory. Renormalization of the Wess-Zumino model is studied in detail. In chapter 3, a slightly digressed topic, the deformed superPoincaré algebra in NC field theory, is discussed. The representation of the conserved generators is also useful for the soliton theory later.

There are comprehensive discussions in literature about NC solitons and their interpretations as D-branes in NC scalar or gauge field theories [5, 6]. However, quantization of NC solitons has only been discussed in limited places [7, 8], where only large NC limit or $1/\theta$ corrections are considered. An important feature of NC field theories, UV/IR mixing, is omitted in the large NC limit. Chapter 4 is dedicated to quantization of NC solitons in the small enough θ limit. In particular quantization of NC Q-ball solitons is discussed in detail. Quantum corrections to the soliton energy are then calculated and UV/IR mixing structures similar to those in perturbation theory are recovered. Energy of NC GMS solitons is found to be UV renormalizable at one-loop with UV/IR terms included in the corrections.

The structure is suggested as the consequence of interactions between D-branes and strings, which could be a future direction to pursue.

Chapter 5 summarizes the work [4] on 11 dimensional superPoincaré algebra, where an alternative representation of the solutions of Kostant equations in coset space $F4/SO(9)$ is given. The final chapter summarizes the results in this dissertation and discusses future research directions.

Recently there has been a revival of interest in the study of noncommutative (NC) geometry, due to the discovery that NC field theory appears to be certain limit of the effective action of the open string modes living on branes [9, 10]. NC geometry has been formulated in strict mathematical fashion [11]. The idea can be captured as follows: On commutative manifold M there exists an algebra $A = C^{\infty}(M)$ of commutative smooth functions, with the product being function multiplication. NC algebra is a deformation of commutative algebra with deformed product, or more specifically, star product for the concerns of this dissertation,

$$f \star g(x) \equiv \exp(\frac{i}{2}\theta^{ij}\frac{\partial}{\partial^i x}\frac{\partial}{\partial^j y})f(x)g(y)\Big|_{x=y} \ , \tag{1.1}$$

where $\theta^{ij} = -\theta^{ji}$ is a non-degenerate constant antisymmetric matrix. NC geometry is then defined in terms of NC generalizations of the algebraic constructs defined in the ordinary commutative geometry.

The motivation for studying NC geometry has been manyfold. The idea can come from the fundamental principle of quantum mechanics, where the phase variables, position and momentum do not commute. One can just conjecture that the positions themselves might not commute, leading to

$$[x^i, x^j] = i\theta^{ij} \ , \tag{1.2}$$

where θ^{ij} is an antisymmetric parameter measuring noncommutativity.

Quantum field theory can be well formulated with NC geometry concept, called NC field theories. One simple reason to investigate NC field theory is that noncommutativity would introduce phase factors that could better regularize ultraviolet (UV) divergence present in ordinary field theories. However, as we will see, often part of UV divergence associated with planar diagrams are still present and other UV divergences associated with nonplanar diagrams become UV/IR divergence, which still needs further interpretation.

Another motivation is from the uncertainty principle in quantum gravity, where position is not expected to be measured accurately at the Planck scale. People also believe quantum gravity should be nonlocal in general. One expects by investigating NC field theory as nonlocal theory that a better understanding of nonlocality can be achieved conceptually and practically.

String theory actually provides stronger motivation for studying NC field theory. Yang-Mills theory is proved to arise in a natural limit in the context of the matrix model of M-theory [9, 12, 13], with the noncommutativity arising from the expectation value of a background field. NC geometry has also been used as a framework for open string field theory [14]. Later Seiberg and Witten studied open strings in the presence of a constant Neveu-Schwarz B field nonzero on the Dp-brane [10]. In the zero slope limit ($\alpha' \to 0$), NC geometry arises as a limit of string theory. The effective action of the open string modes on the brane becomes NC field theory due to the presence of B field. The same paper actually shows the equivalence between NC Yang-Mills theory and ordinary Yang-Mills theory.

NC field theory has also appeared naturally in condensed matter theory. A simple example has been shown [15] in which noncommutativity arises when the theory of electrons moving in a magnetic field is kept in the lowest Landau level in certain limit. The idea is generalized in the theory of the quantum Hall effect [16]. Basically an observable algebra, which is well defined in periodic case, can be generalized to non-periodical background (presence of magnetic field) and actually becomes a NC manifold called NC Brillouin zone.

Despite the reasons above, experimental support for NC field theory as realistic low energy theory is limited, due to nonlocality and violation of Lorentz symmetry introduced by uncertainty relations between the coordinates. An upper bound of NC parameter, $(10Tev)^{-2}$, is obtained in the Lorentz violating extension of standard model [17].

In this dissertation we mainly consider NC field theory from string theory perspective. In the following we briefly illustrate the idea [10, 15] that NC Yang-Mills theory arises from certain limit of open string theory.

The worldsheet action for an open string with nonzero B field on Dp-brane boundary in the Euclidean signature is

$$S = \frac{1}{4\pi\alpha'}\int_{\Sigma} g_{ij}\partial_a x^i \partial^a x^j - \frac{i}{2}\int_{\partial\Sigma} B_{ij} x^i \partial_t x^j \ , \tag{1.3}$$

where ∂_t is a tangential derivative along the worldsheet boundary $\partial\Sigma$. For x^i along the brane, we have the equation of motion at the boundary $\partial\Sigma$,

$$g_{ij}\partial_n x^j + 2\pi i\alpha' B_{ij}\partial_t x^j|_{\partial\Sigma} = 0 \ . \tag{1.4}$$

The above boundary conditions actually interpolate from Neumann boundary conditions ($B = 0$) to Dirichlet boundary conditions ($B \to \infty$ or $g_{ij} \to 0$). With a special boundary condition when the world sheet is a disc conformally mapped to upper half plane, the propagator becomes [18–20]

$$\langle x^i(z)x^j(z)\rangle = -\alpha'[g^{ij}\ln|z-z'| - g^{ij}\ln|z-\bar{z}'| + G^{ij}\ln|z-\bar{z}'|^2 + \frac{1}{2\pi\alpha'}\theta^{ij}\ln\frac{z-\bar{z}'}{\bar{z}-z'} + D^{ij} \ . \tag{1.5}$$

where

$$G^{ij} = \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_S = \left(\frac{1}{g+2\pi\alpha' B} g \frac{1}{g-2\pi\alpha' B}\right)^{ij} , \quad (1.6)$$

$$G_{ij} = g_{ij} - (2\pi\alpha')^2 (Bg^{-1}B)_{ij} , \quad (1.7)$$

$$\theta^{ij} = 2\pi\alpha' \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_A = -(2\pi\alpha')^2 \left(\frac{1}{g+2\pi\alpha' B} B \frac{1}{g-2\pi\alpha' B}\right) . \quad (1.8)$$

The propagator of open string vertex operators inserted on the boundary of Σ is

$$\langle x^i(\tau) x^j(\tau')\rangle = -\alpha' G^{ij} \ln(\tau-\tau')^2 + \frac{i}{2}\theta^{ij}\epsilon(\tau-\tau') , \quad (1.9)$$

where G^{ij} , the coefficient that determines the anomalous dimensions of open string vertex operators, is referred to as open string metric.

To focus on the low energy behavior while decoupling the string behavior, take the zero slope limit ($\alpha' \to 0$) of the open string system,

$$\alpha' \sim \epsilon^{\frac{1}{2}} \to 0 , \qquad g_{ij} \sim \epsilon \to 0 , \quad (1.10)$$

where i, j refer to the directions along the brane. Then G and θ become

$$G^{ij} = \begin{cases} -\frac{1}{(2\pi\alpha')^2}(\frac{1}{B} g \frac{1}{B})^{ij} & \text{for } i,j \text{ along the brane} \\ g^{ij} & \text{otherwise} \end{cases} \quad (1.11)$$

$$G_{ij} = \begin{cases} -(2\pi\alpha')^2 (Bg^{-1}B)_{ij} & \text{for } i,j \text{ along the brane} \\ g_{ij} & \text{otherwise} \end{cases} \quad (1.12)$$

$$\theta^{ij} = \begin{cases} \left(\frac{1}{B}\right)^{ij} & \text{for } i,j \text{ along the brane} \\ 0 & \text{otherwise} \end{cases} \quad (1.13)$$

In this limit, the action has only the topological term,

$$-\frac{i}{2}\int_{\partial\Sigma} B_{ij} x^i \partial_t x^j , \quad (1.14)$$

and the propagator (1.9) becomes

$$\langle x^i(\tau)x^j(\tau')\rangle = \frac{i}{2}\theta^{ij}\epsilon(\tau-\tau') \ . \tag{1.15}$$

Interpreting τ as time, NC geometry arises by evaluating the commutator,

$$[x^i(\tau)\ ,x^j(\tau)] = T(x^i(\tau)x^j(\tau^-) - x^i(\tau)x^j(\tau^+)) = i\theta^{ij} \ . \tag{1.16}$$

With the above equation, one can then argue that for general operator products $\mathcal{O}(\tau)\mathcal{O}'(\tau')$, the leading terms would be independent of $\tau-\tau'$ for $\tau \to \tau'$, and would have to give star products (1.1) of operators, because of the associativity and translation invariance. Explicitly, normal ordered operators satisfy

$$: e^{ip_ix^i(\tau)} :: e^{iq_ix^i(0)} := e^{-\frac{i}{2}\theta^{ij}p_iq_j\epsilon(\tau)} : e^{ipx(\tau)+iqx(0)} : , \tag{1.17}$$

or more generally,

$$: f(x(\tau)) :: g(x(0)) :=: e^{\frac{i}{2}\epsilon(\tau)\theta^{ij}\frac{\partial}{\partial x^i(\tau)}\frac{\partial}{\partial x^j(0)}} f(x(\tau))g(x(0)) : , \tag{1.18}$$

where the right hand side is exactly the star product (1.1) of the functions on the NC space.

Through the general procedure for reduction of open string field theory with nonzero B field along the brane [10], taking $\alpha' \to 0$ and keeping G and the effective open string coupling G_s fixed, it is suggested that the effective action is gauge invariant NC Yang-Mills theory with field $\hat{F}$,

$$\mathcal{L}(\hat{F}) = \frac{c}{G_s}\sqrt{\det(G + 2\pi\alpha'(\hat{F}+\Phi))} \ . \tag{1.19}$$

where $c = T_p/g_s$ is independent of g_s and T_p is the Dp-brane tension for $B = 0$. In this form θ appears only in the star product affected only by B, and certain degrees of freedom exist in the choice of the parameter Φ, which is given by

$$\frac{1}{G+2\pi\alpha'\Phi} = -\frac{\theta}{2\pi\alpha'} + \frac{1}{g+2\pi\alpha' B} \ . \tag{1.20}$$

To determine the effective open string coupling G_s, take $\hat{F}=0$ to find the constant term,

$$\mathcal{L}(\hat{F}=0) = \frac{c}{G_s}\sqrt{\det(G+2\pi\alpha'\Phi)} \ . \tag{1.21}$$

Also for the Dirac-Born-Infeld Lagrangian (see [21] for a review) for slowing varying fields,

$$\mathcal{L}_{DBI} = \frac{c}{g_s}\sqrt{\det(g+2\pi\alpha'(F+B))} \ , \tag{1.22}$$

take $F=0$,

$$\mathcal{L}(F=0) = \frac{c}{g_s}\sqrt{\det(g+2\pi\alpha' B)} \ . \tag{1.23}$$

The equivalence of the above two terms gives

$$G_s = g_s\left(\frac{\det(G+2\pi\alpha'\Phi)}{\det(g+2\pi\alpha' B)}\right)^{\frac{1}{2}} \ . \tag{1.24}$$

CHAPTER 2
NONCOMMUTATIVE PERTURBATIVE DYNAMICS

NC field theory, after quantization, shows different ultraviolet structures from the ordinary field theory [22]. Basically noncommutativity introduces phase factors in the vertices, which in the loop integration become convergent factors that regularize the UV divergence. However, some UV divergences are still left, and additional UV/IR divergences are introduced. The intriguing UV/IR mixing terms can be reproduced by integrating out some new light degrees of freedom with special propagators and interactions. These new light degrees of freedom can be interpreted as closed string modes with channel duality [23]. Even for field theory concern, renormalization of NC field theory needs to be reexamined because of the change in the divergence structure.

In this chapter following a basic introduction to the NC field theory, perturbation dynamics in NC ϕ^4 theory [22], as well as the implications from string theory, are reviewed. Then we discuss renormalization of supersymmetric NC Wess-Zumino model.

The commutative Wess-Zumino model is the simplest supersymmetrical field theory model in $(3 + 1)$ dimension. It includes a scalar and a fermion field with supersymmetry between them. Because of the supersymmetry, cancelation of the divergence occurs generally. The only mass renormalization is due to wave function renormalization, and the vertex and mass corrections are absent [24].

2.1 Noncommutative Perturbation Theory

As explained in the introduction chapter, a specific NC algebra can be defined by the star product of the commutative functions, Eqn. (1.1). The underlining noncommutative space has phase space quantization structure,

$$[\hat{x}^i \ , \hat{x}^j] = i\theta^{ij} \ , \tag{2.1}$$

where i, j is assumed to label the space dimensions only in this dissertation. Noncommutativity associated with time coordinate brings the problems of causality and unitarity when the theory is canonically quantized [25]. Weyl proposed Eqn. (2.1) as Lie algebra of a group with group elements,

$$U(p) = \exp(ip\hat{x}) \ . \tag{2.2}$$

In the function representation of the NC algebra with Weyl ordering,

$$\hat{f}(\hat{x}) \equiv \int \frac{d^n p}{(2\pi)^n} \int d^n x f(x) \mathrm{e}^{-ipx} U(p) = \int \frac{d^n p}{(2\pi)^n} \tilde{f}(p) U(p) \ , \tag{2.3}$$

where n is the number of the space dimensions, the operator products

$$\begin{aligned} \hat{f}(\hat{x})\hat{g}(\hat{x}) &= \int \frac{d^n p}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \tilde{f}(p)\tilde{g}(q) U(p) U(q) \\ &= \int \frac{d^n p}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \tilde{f}(p-q)\tilde{g}(q) \mathrm{e}^{-\frac{i}{2} p^i q^j \theta^{ij}} U(p) \end{aligned} \tag{2.4}$$

NC star product (1.1) is just the function representation of the above operator products, for

$$f(x) \star g(x) = \int \frac{d^n p}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} e^{-i(p+q)x} \tilde{f}(p) \tilde{g}(q) e^{-\frac{i}{2} p^i q^j \theta^{ij}} \ . \tag{2.5}$$

The Weyl representation is initially used in phase space quantization [26]. NC field theory is first proposed by Filk [27], replacing the ordinary products of the fields by NC star products. The propagator remains the same since

$$\int f(x) \star g(x) = 0 \ . \tag{2.6}$$

The interaction vertices now depend on the external momentum through phase factors, which is induced by the star products. The phase factors, while independent of the overall permutation of the momentum, distinguish the Feynman diagrams to be planar and nonplanar ones. The phase factors for planar diagram depend only on external momenta, and do not affect the UV divergence in the loop integration. But for nonplanar diagram, the dependence of the internal momenta by the phase factor introduces regularization in the momentum integral, and UV divergence is generally converted into UV/IR divergence. Renormalization of the theory needs to be reexamined case by case. It is interesting that renormalization of the NC field theories differs significantly from their commutative analog. For example, NC QED, as a simple extension of NC $U(1)$ YM theory, is renormalizable at one loop due to Slavnov-Taylor identity for $SU(2)$ like symmetry, but the β functions include contributions from electrons in $U(1)$ facet [28]. General consideration of convergence theorem and renormalization in NC field theory has been discussed [29, 30]. The rest of the chapter discusses renormalization of NC Φ^4 theory and Wess-Zumino model in detail.

2.2 Noncommutative ϕ^4 Theory

The NC Φ^4 theory in the four-dimensional space-time, is described by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi \star \partial^\mu\Phi - \frac{1}{2}m^2\Phi \star \Phi - \frac{\lambda}{4!}\Phi \star \Phi \star \Phi \star \Phi. \tag{2.7}$$

It is well known [10, 22, 27] that under the integration the star product of the fields does not affect the quadratic parts of the Lagrangians, whereas it makes the interaction Lagrangian become nonlocal. Hence, the Feynman rules in momentum space of NC field theory are similar to those of commutative theory except that the vertices of the NC theory are modified by a phase factor. For the Lagrangian (2.7), the Feynman rule for the deformed vertex is

$$
\begin{aligned}
&-\frac{i}{3}\lambda(\cos\frac{1}{2}(p_1\times p_2+p_1\times p_3+p_2\times p_3)\\
&\quad+\cos\frac{1}{2}(p_1\times p_2+p_1\times p_3-p_2\times p_3)\\
&\quad+\cos\frac{1}{2}(p_1\times p_2-p_1\times p_3-p_2\times p_3)),
\end{aligned} \tag{2.8}
$$

where p_i's, $i = 1\ldots 4$, are momenta coming out of the vertex and $p_i \times p_j = p_{i\mu}\Theta^{\mu\nu}p_{j\nu}$. When $\Theta^{\mu\nu} \to 0$, the deformed vertex becomes the non-deformed one. By using the above vertex, one yields a wave function renormalization of the scalar field ϕ at one-loop order that has only one diagram as follows:

$$
\begin{aligned}
&\Gamma^{(\Phi\Phi)}(p^2)\\
=\;& -\frac{\lambda}{6}\int\frac{d^4k}{i(2\pi)^4}\frac{(2+\cos(p\times k))}{(k^2+m^2)}\\
=\;& -\frac{\lambda}{48\pi^2}\int_0^\infty\frac{d\alpha}{\alpha^2}e^{-i\alpha m^2}\left(1+\frac{1}{2}e^{-i\frac{\tilde{p}^2}{4\alpha}}\right)e^{\frac{i}{\Lambda^2\alpha}}\\
=\;& -\frac{\lambda}{48\pi^2}\left(\Lambda^2-m^2ln(\frac{\Lambda^2}{m^2})\right)-\frac{\lambda}{96\pi^2}\left(\Lambda_{eff}^2-m^2ln(\frac{\Lambda_{eff}^2}{m^2})\right)+\cdots .
\end{aligned} \tag{2.9}
$$

The Schwinger parametrization technique to deal with the above integrations can be found in Itzykson and Zuber [31] and Hayakawa [28]. In the second line, the term is proportional to $\exp(-i\tilde{p}^2/4\rho)$, where $\tilde{p} = p_\mu\Theta^{\mu\nu}$, is due to the nonplanar contribution and the $\exp(i/\rho\Lambda^2)$ factor is introduced to regulate the small ρ divergence in the planar contribution. Note that the nonplanar contribution is one-half of the planar one. In the third line, we keep only the divergent terms and the effective cutoff, $\Lambda_{eff}^2 = 1/\left(1/\Lambda^2 + (\tilde{p}^2)/4\right)$, shows the mixing of UV divergence and IR singularity [22]. The above integration can also be done by using dimensional regularization method [32]. In the case that ϕ is a complex scalar field, there are two ways of ordering the fields ϕ and ϕ^* in the quartic interaction $(\phi^*\phi)^2$. So, the most general potential of the NC complex scalar field action is

$$A\phi^* \star \phi \star \phi^* \star \phi + B\phi^* \star \phi^* \star \phi \star \phi.$$

The potential is invariant under global transformation since the star product has nothing to do with the constant phase transformation. It was shown by Aref'eva, Belov and Koshelev [33] that the theory is not generally renormalizable for arbitrary values of A and B and is renormalizable at one-loop level only when $B = 0$ or $A = B$.

The one loop 1PI quadratic effective action is

$$\begin{aligned} S^{(2)}_{1PI} &= \int d^4p \frac{1}{2}\left(p^2 + M^2 + \frac{\lambda}{96\pi^2(\frac{1}{4}\tilde{p}^2 + \frac{1}{\Lambda^2})} \right. \\ &\left. -\frac{\lambda M^2}{96\pi^2}\ln(\frac{1}{M^2(\frac{1}{4}\tilde{p}^2 + \frac{1}{\Lambda^2})}) + \cdots \right)\phi(p)\phi(-p) \ , \end{aligned} \tag{2.10}$$

where

$$M^2 = m^2 + \frac{\lambda\Lambda^2}{48\pi^2} - \frac{\lambda m^2}{48\pi^2}\ln(\frac{\Lambda^2}{m^2}) + \cdots \tag{2.11}$$

is the renormalized mass.

The appearance of UV/IR mixing terms suggests the presence of new degrees of the freedom. Indeed the correct IR singularity in the effective action can be systematically reproduced by the introduction of new light degrees of freedom [22, 23]. A brief review of the idea is as follows.

Consider the modified effective action

$$\begin{aligned} S'_{eff}(\Lambda) &= S_{eff} + \int d^4p \left[\frac{1}{2}\left(\frac{4}{\tilde{p}^2} - \frac{1}{\frac{1}{4}\tilde{p}^2 + \frac{1}{\Lambda^2}}\right)^{-1}\chi_1(p)\chi_1(-p) \right. \\ &+ \frac{1}{2}\left(\ln(\frac{1}{4}\tilde{p}^2 + \frac{1}{\Lambda^2}) - \ln(\frac{1}{4}\tilde{p}^2)\right)^{-1}\chi_2(p)\chi_2(-p) \\ &\left. + \frac{1}{\sqrt{96\pi^2}}\sqrt{\lambda}(i\chi_1 + M^2\chi_2)\phi\right] \ . \end{aligned} \tag{2.12}$$

Upon integrating out χ in the above action, the correct quadratic and logarithmic divergences can be reproduced, and the UV/IR mixing terms are cancelled by the χ exchange diagrams.

χ's have special propagators,

$$\langle\chi_1(p)\chi_1(-p)\rangle \sim \frac{4}{\tilde{p}^2} - \frac{1}{\frac{1}{4}\tilde{p}^2 + \frac{1}{\Lambda^2}} \tag{2.13}$$

$$\langle\chi_2(p)\chi_2(-p)\rangle \sim \ln(\frac{1}{4}\tilde{p}^2 + \frac{1}{\Lambda^2}) - \ln(\frac{1}{4}\tilde{p}^2) \tag{2.14}$$

A possible interpretation to the presence of those new degrees of freedom χ is that they are actually transverse modes of particles ψ's which propagate freely in more dimensions. For example, a particle ψ propagates in two extra dimensions and couples linearly to ϕ on the brane will produce the logarithmic propagator. Define $\chi(x) = \psi(x, x_\perp = 0)$ and write the action of ψ with a Lagrange multiplier $\lambda(x)$,

$$\begin{aligned}
&\exp\left(-\int d^4x\phi(x)\psi(x,x_\perp=0) - \int d^4x d^2x_\perp \frac{1}{2}(\partial\psi)^2\right) \\
= &\int [d\lambda][d\chi]\exp\left(-\int d^4x\{\phi(x)\chi(x) + i\lambda(x)[\chi(x) - \psi(x,x_\perp=0)]\}\right. \\
&\left. - \int d^4x d^2x_\perp \frac{1}{2}(\partial\psi)^2\right) \\
= &\int [d\lambda][d\chi]\exp\left(-\int d^4p[\phi(p)\chi(-p) + i\lambda(-p)\chi(p)]\right. \\
&\left. - \int d^4p d^2q[\frac{1}{2}\psi(-p,-q)(\frac{1}{4}\tilde{p}^2 + q^2)\psi(p,q) - i\lambda(-p)\psi(p,q)]\right)
\end{aligned} \tag{2.15}$$

Integrate out ψ first, leaving

$$\exp\left(-\int d^4p\left[\phi(p)\chi(-p) + i\lambda(p)\chi(-p) + \frac{1}{2}\lambda(p)\lambda(-p)\int\frac{d^q}{1/4\tilde{p}^2 + q^2}\right]\right) . \tag{2.16}$$

Then integrate out the Lagrange multiplier λ, giving the desired action,

$$\exp\left(-\int d^4p\left[\phi(p)\chi(-p) + \frac{1}{2}\chi(-p)\chi(p)\ln^{-1}\left(\frac{1/4\tilde{p}^2 + \frac{1}{\Lambda^2}}{1/4\tilde{p}^2}\right)\right]\right) . \tag{2.17}$$

The duality between the high momentum degrees of freedom in ϕ and propagation of ψ in the extra dimensions suggests that ϕ's are associated with open strings modes and χ's or ψ's are associated with closed string modes, since in the string theory the low energy closed string modes are related to the high energy modes of the open strings by channel duality. A nonplanar loop diagram is topologically equivalent to a string diagram in which a number of open strings becomes a closed string that freely propagates in the bulk and turns back into open strings. Connections between NC field theories and string theories suggested by those analogies have not been clearly understood yet.

2.3 Renormalization in Wess-Zumino Model

In this section, we investigate renormalization at one loop in the NCWZ theory. NCWZ Lagrangian is given by introducing star products in the interaction terms and permutating those terms to preserve supersymmetry transformations. Here we follow the conventions by Sohnius [34]. The NCWZ model is described by the sum of the free off-shell Lagrangian and of the two invariants,

$$\mathcal{L}_{tot} = \mathcal{L}_0 + \mathcal{L}_m + \mathcal{L}_g, \tag{2.18}$$

where

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu A \partial^\mu A + \partial_\mu B \partial^\mu B + i\bar{\Psi}\ \partial\!\!\!/ \Psi + F^2 + G^2\right), \tag{2.19}$$

$$\mathcal{L}_m = -m(FA + GB + \frac{1}{2}\bar{\Psi}\Psi), \tag{2.20}$$

$$\mathcal{L}_g = -\frac{g}{3}\left[A \star A \star F - B \star B \star F + A \star B \star G + \bar{\Psi} \star (A - \gamma_5 B) \star \Psi \right.$$
$$\left. + \text{permutation terms}\right]. \tag{2.21}$$

The off-shell Lagrangians $\mathcal{L}_0$, $\mathcal{L}_m$ and $\mathcal{L}_g$ are separately invariant under the supersymmetry transformations:

$$\delta A = \bar{\alpha}\Psi, \quad \delta B = \bar{\alpha}\gamma_5\Psi, \quad \delta F = i\bar{\alpha} \not{\partial}\Psi, \quad \delta G = i\bar{\alpha}\gamma_5 \not{\partial}\Psi,$$
$$\delta\psi = -(F + \gamma_5 G)\alpha - i \not{\partial}(A + \gamma_5 B)\alpha, \tag{2.22}$$

where α and $\bar{\alpha}$ are the global infinitesimal Majorana spinor parameters.

The Feynman rules in momentum space can be extracted out directly from the Lagrangians (2.18). One gets as follows:

1. Propagators

   The propagators of the fields and the mixed fields on the NC space are the same as those on the commutative one.

2. Deformed vertices

   - $-\frac{g}{3}(A \star A \star F +$ permutation terms)

     $$-2ig\cos(\frac{1}{2}p_{A_i} \times p_{A_f}).$$

   - $\frac{g}{3}(B \star B \star F +$ permutation terms)

     $$ig\cos(\frac{1}{2}p_{B_i} \times p_{B_f}).$$

   - $-\frac{g}{3}(A \star B \star G +$ permutation terms)

     $$-2ig\cos(\frac{1}{2}p_A \times p_B).$$

   - $-\frac{g}{3}(\bar{\Psi} \star A \star \Psi +$ permutation terms)

     $$-igI\cos(\frac{1}{2}p_i \times p_f).$$

   - $\frac{g}{3}(\bar{\Psi} \star \gamma_5 B \star \Psi +$ permutation terms)

     $$2ig\gamma_5\cos(\frac{1}{2}p_i \times p_f).$$

The deformed vertices we obtain differ from the non-deformed ones by a factor $\cos(\frac{1}{2}p_i \times p_f)$. By using the above Feynman rules, one can study the renormalization of the NC Wess-Zumino model. The results are summarized as follows:

1. Wave function renormalization

   - Majorana field Ψ

     For Majorana field, at one loop there are two diagrams. The sum of them gives a contribution

$$\begin{aligned} \Gamma^{(\bar{\Psi}\Psi)}(\not{p}) &= 8g^2 \int \frac{d^4k}{i(2\pi)^4} \cos^2(\frac{1}{2}p \times k) \frac{\not{k}}{(k^2-m^2)((k+p)^2-m^2)} \\ &= -\not{p}\frac{g^2}{4\pi^2} \int_0^1 d\alpha(1-\alpha) \int_0^\infty \frac{d\rho}{\rho} e^{-\rho i\left(m^2-\alpha(1-\alpha)p^2\right)} \left(1+e^{-\frac{1}{\rho}\frac{i\tilde{p}^2}{4}}\right) e^{\frac{1}{\rho}\frac{i}{\Lambda^2}} \\ &= -\not{p}\frac{g^2}{8\pi^2}\left(ln(\frac{\Lambda^2}{m^2}) + ln(\frac{\Lambda^2_{eff}}{m^2})\right) + \ldots . \end{aligned} \quad (2.23)$$

   - Scalar fields $A,\ B$

     For each scalar field, at one loop there are five diagrams. The sum of them gives a contribution

$$\begin{aligned} & \Gamma^{(AA)}(p^2) = \Gamma^{(BB)}(p^2) \\ &= 8g^2 \int \frac{d^4k}{i(2\pi)^4} \cos^2(\frac{1}{2}p \times k) \frac{k \cdot p}{(k^2-m^2)((k+p)^2-m^2)} \\ &= -p^2\frac{g^2}{8\pi^2}\left(ln(\frac{\Lambda^2}{m^2}) + ln(\frac{\Lambda^2_{eff}}{m^2})\right) + \ldots . \end{aligned} \quad (2.24)$$

   - Auxiliary fields $F,\ G$

     For F field, at one loop there are two diagrams While, for G field, at one loop there is only one diagram. However, they have the same contribution

$$\begin{aligned}
&\Gamma^{(FF)}(p^2) = \Gamma^{(GG)}(p^2) \\
= \ & -4g^2 \int \frac{d^4k}{i(2\pi)^4} \cos^2(\frac{1}{2}p \times k) \frac{1}{(k^2 - m^2)((k+p)^2 - m^2)} \\
= \ & -\frac{g^2}{8\pi^2} \left(ln(\frac{\Lambda^2}{m^2}) + ln(\frac{\Lambda_{eff}^2}{m^2}) \right) + \ldots . \qquad (2.25)
\end{aligned}$$

- Mixed fields

$$\Gamma^{(FA)}(p^2) = \Gamma^{(GB)}(p^2) = 0. \qquad (2.26)$$

Again, all the integrations can be done directly by using the Schwinger parametrization technique [28, 31]. The divergent terms of the wave function renormalizations of all fields are the same, whereas the finite terms of $\Gamma^{(FF)}$ and $\Gamma^{(GG)}$ are different from those of the others. Note that in the NC Wess-Zumino model the planar and nonplanar contributions have the same multiplicative factor. Renormalization is cut by half compared to that of the ordinary case.

2. Mass renormalizations
   - Since, at one-loop $\Gamma^{(\bar{\Psi}\Psi)}(\not{p})$ is proportional to only $\not{p}$ and both Γ^{FA} and Γ^{GB} are zero, there is no mass renormalization.
3. Vertex corrections
   - FA^2, FB^2, ABG

     For each vertex, at one loop there are two diagrams and they add up to zero. So, there is no correction for each vertex.
   - $\bar{\Psi} \star A \star \Psi$, $\bar{\Psi} \star \gamma_5 B \star \Psi$

     Similarly, there is no correction for each of these two vertices. Since, at one loop there are two diagrams and they add up to finite values.

Just as in the Φ^4 theory, the UV/IR mixing also appears in the NCWZ theory, which is the general consequence of the uncertainty relations among

NC coordinates [10]. Renormalization in the NCWZ theory is very similar to the commutative one. Compared with the ordinary Wess-Zumino theory, the counter term for the wave function renormalization reduces by one-half, but the cancelations, in particular the absence of mass and vertex corrections, persist due to supergauge invariance. The renormalization of the wave function of the commutative theory can be recovered by setting $\Theta^{\mu\nu}$ equal to zero. Supergauge invariance sustains generally in NC field theories. In the next chapter we will study superPoincaré algebra in NC field theories and again verify the supergauge invariance from an algebraic point of view.

CHAPTER 3
DEFORMED SUPERPOINCARÉ ALGEBRA

In this chapter following an introduction on classification of representations of Poincaré and superPoincaré algebra, algebra of NCFT's are studied. Conserved currents are derived by Noether's procedure, then a representation of the generators of deformed Poincaré or superPoincaré algebra is suggested, and commutation relations are calculated explicitly. NC ϕ^4 and NCWZ theory are studied as the examples.

3.1 Unitary Representations of SuperPoincaré Algebra

Poincaré invariance is considered as a fundamental property of modern theory since the discovery of special relativity. In recent decades superPoincaré invariance, as an enlarged invariance, is also considered as a property of fundamental theory for theoretical concerns (see [35] for a review), although there is no experimental evidence clearly supporting the conjecture yet. The importance of the unitary representations of the Poincaré algebra and their classification is originally recognized by Wigner [36]. The following is a review about the theory of unitary Poincaré and superPoincaré representation in $3+1$ dimensional space.

Poincaré algebra includes Lorentz generators $M_{\mu\nu}$ and translation generators P_μ, satisfying commutation relations,

$$\begin{aligned}
[P^\mu, P^\nu] &= 0 \ , \\
[M^{\mu\nu}, P^\sigma] &= i(\eta^{\mu\sigma} P^\nu - \eta^{\nu\sigma} P^\mu) \ , \\
[M^{\mu\nu}, M^{\alpha\beta}] &= i(\eta^{\mu\alpha} M^{\nu\beta} + \eta^{\alpha\nu} M^{\beta\mu} + \eta^{\nu\beta} M^{\mu\alpha} + \eta^{\beta\mu} M^{\alpha\nu}) \ ,
\end{aligned}$$

where $\eta_{\mu\nu} = (-1, 1, 1, 1)$ and $\mu = 0, 1, 2, 3$. The representations are characterized by the values of the Casimir operators, $P_\mu P^\mu$, and the squares of the Pauli-Lubanski forms built out of the Levi-Civitá symbols. In $d = 4$ space-time dimensions, the Pauli-Lubanski vector is

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma \ . \tag{3.1}$$

Modern physics theory, built in the framework of quantum mechanics, assumes existence of a Hilbert space in which physical particles are described by quantum states. Poincaré invariance of the theory implies that elementary free particles can be classified in unitary representations of the Poincaré group.

Elements of the Poincaré group satisfy,

$$T(a)T(b) = T(a+b) \tag{3.2}$$

$$d(\Lambda)T(a) = T(\Lambda a)d(\Lambda) \tag{3.3}$$

$$d(\Lambda)d(I) = \pm d(\Lambda I) \ . \tag{3.4}$$

Here $T(a)$ represents the abelian translational group, and $d(\Lambda)$ represents the Lorentz group, where $\pm$ indicates $d(\Lambda)$ is a double valued representation.

Consider the wave function $\phi(p, \xi)$ parameterized by the momentum variables p_μ and the variable ξ labels an auxiliary space, so that translation elements are

$$T(a)\phi(p, \xi) = e^{ipa}\phi(p, \xi) \ . \tag{3.5}$$

Now define the operators

$$P(\Lambda)\phi(p, \xi) = \phi(\Lambda^{-1}p, \xi) \ . \tag{3.6}$$

It is easy to show

$$P(\Lambda)T(a) = T(\Lambda a)p(\Lambda) \ . \tag{3.7}$$

Considering both the above equation and Eqn. (3.3), an operator defined by

$$Q(\Lambda) \equiv d(\Lambda)P(\Lambda)^{-1} \tag{3.8}$$

can be shown to act on the parameter ξ alone, which can depend, however, on p,

$$Q(\Lambda)\phi(p,\xi) = \sum_{\eta} Q(p,\Lambda)_{\xi\eta}\phi(p,\eta) \ . \tag{3.9}$$

$Q(p,\Lambda)$ is actually an unitary representation of the little group of the Lorentz group $d(\Lambda)$. It suffices to consider $Q(p_{fix},\lambda)$ as a representation of $\Lambda p_{fix} = p_{fix}$ for a particular vector p_{fix}.

Representation of Poincaré algebra is also characterized by the particular value of the casimir operators, $P_\mu P^\mu$ and $W_\mu W^\mu$. In Eqn. (3.1), $M^{\mu\nu}$, acting on the wave function $\phi(p,\xi)$, is generically represented as

$$M^{\mu\nu} = -i\left(p^\mu \frac{\partial}{\partial p_\nu} - p^\nu \frac{\partial}{\partial p_\mu}\right) + S^{\mu\nu} \ , \tag{3.10}$$

where $S^{\mu\nu}$ is associated with ξ. Three classes of irreducible physical representations are found,

- $P_\mu P^\mu = m^2 > 0$

  Choose $P^\mu_{fix} = (m,0,0,0)$, then

  $$W_\mu W^\mu = m^2(S^{ij})^2 = m^2 s(s+1) \ , \tag{3.11}$$

  where $S^{ij}(i,j=1,2,3)$ is an irreducible representation of $SO(3)$, labeled by spin s. It is well known that s takes the value of zero or positive integer or half integer. This class of representation describes massive particles with spin s.

- $P_\mu P^\mu = 0, W_\mu W^\mu = 0$

  This class of representation is just the massless limit ($m \to 0$) of the massive representation and describes massless particles. Choose $P^\mu_{fix} = (E,0,0,E)$, or in light cone

$$P^+ = \sqrt{2}E, P^- = P^1 = P^2 = 0 \ . \qquad (3.12)$$

Since

$$W_\mu W^\mu = (W_1)^2 + (W_2)^2 = 0 \ , \qquad (3.13)$$

$W_1 = W_2 = 0$ or $S^{-1} = S^{-2} = 0$. Also it is easy to show that $W^- = 0$ and $W^+ = P^+ S^{12}$. Therefore $W^\mu = S^{12} P^\mu$, where S^{12} , helicity operator, is the generator of the little group $U(1)$. Single or double valueness of the Lorentz group demand the value of the helicity generator to be half integer or integer. One particular variant of this class of representation, obtained by taking infinite momentum limit of massive representations [37], can be used to represent irreducible degrees of freedom of strings. Its supersymmetric generalization will be able to represent superstrings of various flavors.

- $P_\mu P^\mu = 0, W_\mu W^\mu = \Xi^2$

  Again choose $P^\mu_{fix} = (E, 0, 0, E)$. $W_\mu W^\mu = 2E[(S^{-1})^2 + (S^{-2})^2]$ and $W^- = 0$, but W^1, W^2 and W^+ are nonzero. The little group which leaves P^μ invariant is $SE(2)$ with generators S^{-1}, S^{-2} and S^{12}. Only Ξ, the length of W^μ, is needed to label the representation. Two types of representation, single valued or double valued, belong to this class. This class of representation is originally called continuous spin representation by Wigner, due to the reason that for each representation, the states can be labeled by the value of usual helicity generator S^{12} which is all the integers or half integers.

The above representations should be able to represent all physical particles in $3 + 1$ dimension. There are good reasons to disregard higher spin representations [38, 39] in the first and second class. Naive quantization of the continuous spin representation leads to nonlocality or breakdown of causality [40, 41].

SuperPoincaré group is the extension of Poincaré group, including supercharge Q_A, which satisfies the commutation relations,

$$\begin{aligned} [Q_A \ , \ P^\mu] &= 0 \ , & (3.14) \\ [M^{\mu\nu} \ , \ Q_A] &= -\frac{1}{2}(\Sigma^{\mu\nu}Q)_A \ , & (3.15) \\ \{Q_A \ , \ Q_B^\dagger\} &= (\gamma^\mu P_\mu \gamma^0)_{AB} \ , & (3.16) \end{aligned}$$

where $A, B = 1, 2, \dot{1}, \dot{2}$ are spinor indices. In the above,

$$\Sigma^{\mu\nu} = -\frac{i}{2}[\gamma^\mu \ , \ \gamma^\nu] \ ,$$

where the γ matrices satisfy the anticommutation relation,

$$\{\gamma^\mu \ , \ \gamma^\nu\} \ = \ 2\eta^{\mu\nu} \ .$$

Massless supermultiplets are particularly important, which yield the basic physical spectra of the supersymmetric models. Spectra of Wess-Zumino model can be deduced in the following way [34]. The same arguments (Eqn. (3.5) and below) can be applied to superPoincaré group. The little group $Q(\Lambda)$ in this case includes supercharge generators. Since $[W^\mu \ , Q_A] \neq 0$, $W_\mu W^\mu$ is not casimir operator any more. For massless supermultiplet, still choose $P^\mu = (E, 0, 0, E)$, then use representation independent light cone projectors to split the supercharge generators,

$$Q = Q_+ + Q_- \ , \qquad Q_\pm \equiv \mathcal{P}_\pm Q \ . \tag{3.17}$$

Eqn. (3.16), in the Weyl representation, shows that commutators between supercharge generators generally vanish, except

$$\left\{Q_{+a} \ , \ Q_{+b}^\dagger\right\} = 2E\delta_{ab} \ , \tag{3.18}$$

where $a, b = 1, 2$. General arguments can show that Q_{+1} and Q_{+2} actually belong to two disconnected algebras. Therefore, considering just the minimal supersymmetric model, the little group contains only the generators, $Q_{+1}, Q^\dagger_{+1}$ and S^{12}, where S^{12} comes from the Poincaré algebra. These generators obey the commutation relations,

$$\left\{Q_{+1}\ ,\ Q^\dagger_{+1}\right\} \;=\; 2E \tag{3.19}$$

$$\{S_{12}\ ,\ Q_{+1}\} \;=\; -\frac{1}{2}Q_{+1} \tag{3.20}$$

Previously (Eqn. (3.12) and below) it is shown that the massless representation of Poincaré group includes states which are labeled by $|E,\lambda\rangle$, where E is proportional to P^+_{fix} and λ is the eigenvalue of the helicity generator S^{12}. Eqn. (3.20) indicates that the supercharge Q_{+1} or $Q^\dagger_{+1}$ are the lowering or raising operators for the helicity. Thus the state $|E,\lambda\rangle$ can be defined to be the lowest helicity state since $Q^2_{+1} = 0$. Therefore the minimum supermultiplet contains two states $|E,\lambda\rangle$ and $Q^\dagger_{+1}|E,\lambda\rangle$. The supersymmetric Wess-Zumino model describes the supermultiplet with helicity $\lambda = 0$.

Other supermultiplets with more spinor supercharges or with central charges in various dimensions have been constructed explicitly [34]. In higher $(d+1)$-dimension $(d > 3)$, little group of superPoincaré group is enlarged to contain $SO(d-1)$ and corresponding spinor supercharge generators. Massless supermultiplets in $(9+1)$- dimension correspond to various superstring theories. The little group $SO(8)$ has triality symmetry which leads to marvelous cancelations in quantum perturbation calculations ([42]). More recently, M-theory in $(10+1)$-dimension emerges as unification of superstring theories, whose low energy limit is suggested to be $(10+1)$- dimension supergravity theory. The little group $SO(9)$ is the maximum subgroup of the exceptional group $F4$. As a result, Euler triplets arises as solutions of Kostant equations [43]. The lowest level triplet is supermultiplet and corresponds to $(10+1)$-dimension supergravity theory. The higher level multiplets have accidental supersymmetry and maybe able to describe the zero-tension limits of string theory [4]. Chapter 5 shows the construction of these solutions at all levels. Continuous spin representation in $(d+1)$ dimension is the representation of little group $SE(d-1)$. Upon supersymmetrization, it has one to one correspondence with ordinary massless supermultiplet in d dimension [3]. The paper also shows that if light cone translations are represented by Grassman variables, nilpotent continuous spin representations lead to supermultiplets with central charges. Such analogy is already suggested in previous classification of $(3+1)$-dimension representation, where both the massless supermultiplet and the continuous spin representation contain states with connected helicities.

NC field theory, as a low energy limit of string theory, does not have Lorentz symmetry. In the following sections deformed superPoincaré algebra of NC field theories are studied in an intuitive way, which is expected to gain some basic understanding of underlying algebra and representation of NC field theories.

3.2 Deformed SuperPoincaré Algebra

3.2.1 Notations and Identities

To facilitate the calculations involving NC fields star product, we introduce the following notations and list the useful identities.

Define an operator Δ, which acts nontrivially on a scalar pair-product (f,g) as,

$$\begin{aligned}
\Delta(f,g) &\equiv \partial_\mu f \tilde{\partial}^\mu g, \\
\Delta^2(f,g) &= \partial_\mu \partial_\nu f \tilde{\partial}^\mu \tilde{\partial}^\nu g, \\
\vdots &= \vdots \\
\Delta^n(f,g) &= \underbrace{\partial_\mu \partial_\nu \cdots \partial_\rho}_{n} f \underbrace{\tilde{\partial}^\mu \tilde{\partial}^\nu \cdots \tilde{\partial}^\rho}_{n} g,
\end{aligned} \tag{3.21}$$

where $\tilde{\partial}^\mu \equiv \frac{i}{2}\Theta^{\mu\nu}\partial_\nu$.

With our definition, a star product between two scalar fields A and B can be written as

$$\begin{aligned} A \star B &= e^{\Delta}(A, B) \\ &= \left(1 + \Delta + \frac{\Delta^2}{2!} + \frac{\Delta^3}{3!} + \cdots \right)(A, B) \\ &= AB + \partial_\mu \left(E(\Delta)(A, \tilde{\partial}^\mu B) \right), \end{aligned} \tag{3.22}$$

where the operator $E(\Delta)$ is

$$E(\Delta) = \frac{e^{\Delta} - 1}{\Delta} = \sum_{n=0}^{\infty} \frac{\Delta^n}{(n+1)!}. \tag{3.23}$$

By using the above notations, we obtain some useful identities:

1. $B \star A = AB - \partial_\mu \left(E(-\Delta)(A, \tilde{\partial}^\mu B) \right)$.
2. $[A, B]_\star \equiv A \star B - B \star A = 2\partial_\mu \left(\frac{\sinh(\Delta)}{\Delta}(A, \tilde{\partial}^\mu B) \right)$.
3. $\{A, B\}_\star \equiv A \star B + B \star A = 2AB + 2\partial_\mu \left(\frac{\cosh(\Delta) - 1}{\Delta}(A, \tilde{\partial}^\mu B) \right)$.
4. $(x_\rho A) \star B = x_\rho (A \star B) + A \star \tilde{\partial}_\rho B$.
5. $B \star (x_\rho A) = x_\rho (B \star A) - \tilde{\partial}_\rho B \star A$.
6. $[(x_\rho A), B]_\star = x_\rho [A, B]_\star + \{A, \tilde{\partial}_\rho B\}_\star$.
7. $[B, (x_\rho A)]_\star = x_\rho [B, A]_\star - \{A, \tilde{\partial}_\rho B\}_\star$.
8. $\{(x_\rho A), B\}_\star = x_\rho \{A, B\}_\star + [A, \tilde{\partial}_\rho B]_\star$.

We assume $\theta^{0i} = 0$ from now on for causality and unitarity reasons [25]. The immediate consequence is that noncommutativity will not introduce higher order time derivatives of the fields in Lagrangian.

3.2.2 Φ^4 Theory

Now let us calculate the Noether currents of the NC Φ^4 theory following standard technique [44] . Varying the Lagrangian (2.7), and using the above identities and also the equation of motion, one gets

$$\delta \int d^4x\mathcal{L} = \int d^4x\partial_\mu \left(\frac{1}{2}\{\partial^\mu\Phi, \delta_0\Phi\}_\star + \delta x^\mu \mathcal{L} + \frac{\lambda}{12}\frac{\sinh(\Delta)}{\Delta}(\Phi\star\Phi \ , \ \tilde{\partial}^\mu[\Phi, \delta_0\Phi]_\star) \right). \tag{3.24}$$

Under an infinitesimal translation, $\delta x^\mu = g^{\mu\nu}\epsilon_\nu,\ \delta_0\Phi = -\epsilon_\nu\partial^\nu\Phi$, one yields the energy-momentum tensor,

$$T^{\mu\nu} = \frac{1}{2}\{\partial^\mu\Phi, \partial^\nu\Phi\}_\star - g^{\mu\nu}\mathcal{L} + \frac{\lambda}{12}\frac{\sinh(\Delta)}{\Delta}(\Phi\star\Phi \ , \ \tilde{\partial}^\mu[\Phi, \partial^\nu\Phi]_\star). \tag{3.25}$$

As explicitly seen, the energy-momentum tensor $T^{\mu\nu}$ is conserved since its divergence is zero.

Under the infinitesimal Lorentz transformation, $\delta x^\mu = \epsilon^{\mu\nu}x_\nu = -\frac{1}{2}\epsilon^{\rho\sigma}(x_\rho g^\mu_\sigma - x_\sigma g^\mu_\rho),\ \delta_0\Phi = \frac{1}{2}\epsilon^{\rho\sigma}(x_\rho\partial_\sigma\Phi - x_\sigma\partial_\rho\Phi)$, where $\epsilon^{\rho\sigma}$ is an anti-symmetric second rank tensor, one obtains a three-index current

$$\begin{aligned} j^\mu_{\rho\sigma} &= T^\mu_\rho x_\sigma + \frac{1}{2}[\partial_\rho\Phi, \tilde{\partial}_\sigma\partial^\mu\Phi]_\star + \frac{\lambda}{12}(\sinh(\Delta)/\Delta)'(\tilde{\partial}_\sigma(\Phi\star\Phi) \ , \ \tilde{\partial}^\mu[\Phi, \partial_\rho\Phi]_\star) \\ &\quad - \frac{\lambda}{12}\frac{\sinh(\Delta)}{\Delta}(\Phi\star\Phi \ , \ \tilde{g}^\mu_\sigma[\Phi, \partial_\rho\Phi]_\star + \tilde{\partial}^\mu\{\tilde{\partial}_\sigma\Phi, \partial_\rho\Phi\}_\star) - (\rho\leftrightarrow\sigma), \end{aligned} \tag{3.26}$$

where $(\sinh(\Delta)/\Delta)' = (\Delta\cosh(\Delta) - \sinh(\Delta))/\Delta^2$. The divergence of the three-index current is not equal to zero due to the presence of the terms proportional to the non-commutativity $\Theta^{\mu\nu}$. However, note that the Noether currents of the commutative scalar field theory can be obtained by setting $\Theta^{\mu\nu}$ equal to zero.

In the case of the commutative Φ^4 theory, one yields the momentum and Hamiltonian generators from the energy-momentum tensor, and the angular momentum and boost generators from the three-index current [44]. These generators form the Poincaré algebra. For the NC Φ^4 theory, one obtains its generators analogous to those of the commutative one,

$$P^i \equiv \int d^3x\mathcal{P}^i = \int d^3x(\partial^i\Phi)\dot{\Phi}, \tag{3.27}$$

$$P^0 \equiv \int d^3x\mathcal{P}^0 = \int d^3x\left(\frac{1}{2}(\dot{\Phi}^2 + (\vec{\partial}\Phi)^2 + m^2\Phi^2) + \frac{\lambda}{4!}\Phi^{\star 4}\right), \tag{3.28}$$

$$M^{0i} = \int d^3x(x^0\mathcal{P}^i - x^i\mathcal{P}^0), \tag{3.29}$$

$$M^{ij} = \int d^3x(x^i\mathcal{P}^j - x^j\mathcal{P}^i). \tag{3.30}$$

The surface terms of M^{0i} and M^{ij} are dropped out. These generators generate the translational, rotational and boost transformations on Φ.

By using the quantization condition, $[\Phi(\vec{x}), \dot{\Phi}(\vec{y})] = i\delta^3(\vec{x} - \vec{y})$, one can easily obtain the following equal-time commutation relations:

$$[P^\mu, P^\nu] = 0, \tag{3.31}$$

$$[M^{ij}, M^{kl}] = i(\eta^{il}M^{jk} + \eta^{jk}M^{il} - \eta^{ik}M^{jl} - \eta^{jl}M^{ik}), \tag{3.32}$$

$$[M^{ij}, P^k] = i(\eta^{jk}P^i - \eta^{ik}P^j), \tag{3.33}$$

$$[M^{0i}, P^j] = i\eta^{ij}P^0. \tag{3.34}$$

The above commutation relations of the NC Φ^4 theory are the same as those of the commutative one. In particular, (3.31) verifies that the NC Φ^4 Lagrangian has translational invariance and the translation generator P^μ is conserved. But the following commutation relations have some additional terms proportional to $\Theta^{\mu\nu}$, due to the symmetry-breaking term $\frac{\lambda}{4!}\Phi^{*4}$,

$$
\begin{aligned}
[M^{0i}, P^0] &= -i\eta^{00}P^i - i\frac{\lambda}{4!}\int d^3x\{\dot{\Phi}, [\Phi^{\star 2}, \tilde{\partial}^i\Phi]_\star\}_\star, &(3.35)\\
[M^{ij}, P^0] &= -i\frac{\lambda}{3!}\int d^3x x^i(\partial^j\Phi)\Phi^{\star 3} + (i \leftrightarrow j), &(3.36)\\
[M^{0i}, M^{0j}] &= -i\eta^{00}M^{ij} + i\frac{\lambda}{4!}\int d^3x\left(x^j\{\dot{\Phi}, [\Phi^{\star 2}, \tilde{\partial}^i\Phi]_\star\}_\star - (i \leftrightarrow j)\right), &(3.37)\\
[M^{0i}, M^{jk}] &= i(\eta^{ij}M^{0k} - \eta^{ik}M^{0j}) - i\frac{\lambda}{4!}\int d^3x x^i\\
&\quad \times\Big([\partial^k\Phi, \tilde{\partial}^j\Phi^{\star 3}]_\star + \Phi^{\star 2}\star\partial^k\Phi\star\tilde{\partial}^j\Phi - \tilde{\partial}^j\Phi^{\star 2}\star\partial^k\Phi\star\Phi\\
&\quad +\Phi\star\partial^k\Phi\star\tilde{\partial}^j\Phi^{\star 2} - \tilde{\partial}^j\Phi\star\partial^k\Phi\star\phi^{\star 2} - (j \leftrightarrow k)\Big). &(3.38)
\end{aligned}
$$

The Eqn. (3.35) and (3.36) explicitly show that the Lorentz generators are not conserved in the theory, and all the deformation terms are directly proportional to $\Theta^{\mu\nu}$.

3.2.3 Wess-Zumino Model

For the NCWZ model, one start from an on-shell Lagrangian analogous to the commutative one [34],

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu A\partial^\mu A - m^2A^2) + \frac{1}{2}(\partial_\mu B\partial^\mu B - m^2B^2) + \frac{1}{2}(i\bar{\Psi}\ \partial\!\!\!/\Psi - m\bar{\Psi}\Psi)\\
&\quad -mgA(A^{\star 2} + B^{\star 2}) - mgB(A\star B + B\star A)\\
&\quad -g(A\bar{\Psi}\star\Psi - B\bar{\Psi}\star\gamma_5\Psi) - \frac{1}{2}g^2(A - iB)^{\star 2}(A + iB)^{\star 2} &(3.39)\\
&= \frac{1}{2}(\partial_\mu\phi\partial^\mu\bar{\phi} - m^2\phi\bar{\phi}) + \frac{1}{2}(i\psi\sigma^\mu\partial_\mu\bar{\psi} + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - m\bar{\psi}\bar{\psi} - m\psi\psi)\\
&\quad -\frac{1}{2}mg(\phi\bar{\phi}^{\star 2} + \bar{\phi}\phi^{\star 2}) - g(\phi\bar{\psi}\star\bar{\psi} + \bar{\phi}\psi\star\psi) - \frac{1}{2}g^2\phi^{\star 2}\bar{\phi}^{\star 2}. &(3.40)
\end{aligned}
$$

where $\phi \equiv A - iB, \bar{\phi} \equiv A + iB$, and $\psi, \bar{\psi}$ are the Weyl components of the Majorana field Ψ, following the notations and conventions by Bailin and Love [45].

Following the similar procedure as done in the Φ^4 theory, the variation of the Lagrangian under the infinitesimal Poincaré and supergauge transformations yields the generators as,

$$
\begin{aligned}
P^i &\equiv \int d^3x \mathcal{P}^i = \int d^3x \left(\frac{1}{2}\partial^i\phi\dot{\bar{\phi}} + \frac{1}{2}\dot{\phi}\partial^i\bar{\phi} + i\bar{\psi}\bar{\sigma}^0\partial^i\psi\right), && (3.41)\\
P^0 &\equiv \int d^3x \mathcal{P}^0 \\
&= \int d^3x \left(\frac{1}{2}(\dot{\phi}\dot{\bar{\phi}} + \partial^i\phi\partial^i\bar{\phi} + m^2\phi\bar{\phi}) + \frac{1}{2}(i\bar{\psi}\bar{\sigma}^i\partial^i\psi + i\psi\sigma^i\partial^i\bar{\psi} + m\psi\psi + m\bar{\psi}\bar{\psi})\right.\\
&\left. +\frac{1}{2}mg(\phi\bar{\phi}^{*2} + \bar{\phi}\phi^{*2}) + g(\phi\bar{\psi}\star\bar{\psi} + \bar{\phi}\psi\star\psi) + \frac{1}{2}g^2\phi^{*2}\bar{\phi}^{*2}\right), && (3.42)\\
M^{0i} &= \int d^3x \left(x^0\mathcal{P}^i - x^i\mathcal{P}^0\right), && (3.43)\\
M^{ij} &= \int d^3x \left(x^i\mathcal{P}^j - x^j\mathcal{P}^i\right), && (3.44)\\
\chi Q &= \chi\int d^3x \left(\dot{\phi}\psi - 2\partial_i\phi\sigma^{0i}\psi + im\phi\sigma^0\bar{\psi} + ig\phi^{\star 2}\sigma^0\bar{\psi}\right), && (3.45)\\
\bar{\chi}\bar{Q} &= \bar{\chi}\int d^3x \left(\dot{\bar{\phi}}\bar{\psi} - 2\partial_i\bar{\phi}\bar{\sigma}^{0i}\bar{\psi} + im\bar{\phi}\bar{\sigma}^0\psi + ig\bar{\phi}^{\star 2}\bar{\sigma}^0\psi\right) \;=\; (\chi Q)^\dagger, && (3.46)
\end{aligned}
$$

where χ is an arbitrary Majorana spinor parameter.

In the case of the commutative Wess-Zumino model, the analogs of the above generators are those of the Poincaré algebra and supercharge, which form the $N = 1$ super-Poincaré algebra. With the representations obtained here in the NCWZ model, one can calculate the commutation relations between those generators,

$$[P^\mu, P^\nu] = 0, \tag{3.47}$$

$$[M^{ij}, M^{kl}] = i(\eta^{il}M^{jk} + \eta^{jk}M^{il} - \eta^{ik}M^{jl} - \eta^{jl}M^{ik}), \tag{3.48}$$

$$[M^{ij}, P^k] = i(\eta^{jk}P^i - \eta^{ik}P^j), \tag{3.49}$$

$$[M^{0i}, P^j] = i\eta^{ij}P^0, \tag{3.50}$$

The above commutation relations are exactly the same as those obtained in the NC Φ^4 theory, which suggests the generality of such relations for all NCFT's. In particular, (3.47) verifies the translational invariance of the theory. Equation

(3.50) is a little surprising. The calculation of it in any way involves the NC interaction terms. Nevertheless it is true for both NCFT's.

Other commutation relations are

$$[\chi Q, \zeta Q] = [\bar{\chi}\bar{Q}, \bar{\zeta}\bar{Q}] = 0, \tag{3.51}$$

$$[\chi Q, \bar{\zeta}\bar{Q}] = 2\chi\sigma^{\mu}\bar{\zeta}P_{\mu}, \tag{3.52}$$

$$[P^{\mu}, \chi Q] = 0, \tag{3.53}$$

$$[M^{ij}, \chi Q] = -i\chi\sigma^{ij}Q, \tag{3.54}$$

$$[M^{ij}, \bar{\chi}\bar{Q}] = -i\bar{\chi}\bar{\sigma}^{ij}\bar{Q}. \tag{3.55}$$

All the above relations are exactly the same as those of the commutative Wess-Zumino model. In particular, one finds the supercharge generators, Q and $\bar{Q}$, and the translation generators P^{μ}'s form a close algebra, and the supercharge generators are conserved.

The rest commutation relations have additional terms proportional to $\Theta^{\mu\nu}$, including the similar ones as appears in the NC Φ^4 theory,

$$\begin{aligned}
&[M^{0i}, P^0] = -i\eta^{00}P^i \\
&-\int d^3x \left(\frac{i}{2}mg([\phi, \tilde{\partial}^i\bar{\phi}]_\star\dot{\bar{\phi}} + \dot{\phi}[\bar{\phi}, \tilde{\partial}^i\phi]_\star) + mg([\bar{\phi}, \tilde{\partial}^i\psi]_\star\sigma^0\bar{\psi} + [\phi, \tilde{\partial}^i\bar{\psi}]_\star\bar{\sigma}^0\psi)\right. \\
&-2ig([\bar{\phi}, \tilde{\partial}^i\psi]_\star\sigma^{0l}\partial_l\psi + [\phi, \tilde{\partial}^i\bar{\psi}]_\star\bar{\sigma}^{0l}\partial_l\bar{\psi}) + \frac{i}{2}g^2(\dot{\phi}[\bar{\phi}^{\star 2}, \tilde{\partial}^i\phi]_\star + [\phi^{\star 2}, \tilde{\partial}^i\bar{\phi}]_\star\dot{\bar{\phi}}) \\
&\left.+g^2([\phi, \tilde{\partial}^i\psi]_\star\{\bar{\phi}, \psi\}_\star - \{\phi, \bar{\psi}\}_\star[\bar{\phi}, \tilde{\partial}^i\psi]_\star)\right),
\end{aligned} \tag{3.56}$$

$$\begin{aligned}
&[M^{ij}, P^0] \\
= &\int d^3x \left(\frac{i}{2}mg([\partial^i\phi, \tilde{\partial}^j\phi]_\star\bar{\phi} + [\partial^i\bar{\phi}, \tilde{\partial}^j]_\star\phi) + ig([\partial^i\bar{\psi}, \tilde{\partial}^j\bar{\psi}]_\star\phi + [\partial^i\psi, \tilde{\partial}^j\psi]_\star\bar{\phi})\right. \\
&\left.+\frac{i}{2}g^2([\partial^i\phi, \tilde{\partial}^j\phi]_\star\bar{\phi}^{\star 2} + [\partial^i\bar{\phi}, \tilde{\partial}^j\bar{\phi}]_\star\phi^{\star 2})\right) - (i \leftrightarrow j),
\end{aligned} \tag{3.57}$$

$$[M^{0i}, M^{0j}] = -i\eta^{00}M^{ij} + \int d^3x \Big(mgx^i(\psi\sigma^0[\phi, \tilde{\partial}^j\bar{\psi}]_\star + [\bar{\phi}, \tilde{\partial}^j\psi]_\star\sigma^0\bar{\psi})$$
$$- \tfrac{i}{2}mgx^i([\phi, \tilde{\partial}^j\bar{\phi}]_\star\dot{\bar{\phi}} + \dot{\phi}[\bar{\phi}, \tilde{\partial}^j\phi]_\star) + 2igx^i([\phi, \tilde{\partial}^j\bar{\psi}]_\star\bar{\sigma}^{0l}\partial_l\bar{\psi} + [\bar{\phi}, \tilde{\partial}^j\psi]_\star\sigma^{0l}\partial_l\psi)$$
$$-\tfrac{i}{2}g^2x^i([\phi^{\star 2}, \tilde{\partial}^j\bar{\phi}]_\star\dot{\bar{\phi}} + \dot{\phi}[\bar{\phi}^{\star 2}, \tilde{\partial}^j\phi]_\star) + g^2\{x^i\phi, \bar{\psi}\}_\star\{\psi, x^j\bar{\phi}\}_\star - (i \leftrightarrow j)\Big)\,, \quad (3.58)$$

$$[M^{0i}, M^{jk}] = i(\eta^{ij}M^{0k} - \eta^{ik}M^{0j})$$
$$- \int d^3x \Big(\tfrac{i}{2}mgx^i(\bar{\phi}[\partial^k\phi, \tilde{\partial}^j\phi]_\star + \phi[\partial^k\bar{\phi}, \tilde{\partial}^j\bar{\phi}]_\star) + igx^i(\phi[\partial^k\bar{\psi}, \tilde{\partial}^j\bar{\psi}]_\star + \bar{\phi}[\partial^k\psi, \tilde{\partial}^j\psi]_\star)$$
$$+\tfrac{i}{2}g^2x^i([\partial^k\phi, \tilde{\partial}^j\phi]_\star\bar{\phi}^{\star 2} + [\partial^k\bar{\phi}, \tilde{\partial}^j\phi]_\star) - (j \leftrightarrow k)\Big)\,, \quad (3.59)$$

and also the transformations of the supercharge generators under the Lorentz boosts,

$$[M^{0i}, \chi Q] = -i\chi\sigma^{0i}Q + \int d^3x \Big(g[\dot{\phi}, \phi]_\star\chi\sigma^0\tilde{\partial}^i\bar{\psi} + g[\phi, \tilde{\partial}^i\partial_l(\phi)]\chi\sigma^l\bar{\psi}$$
$$-2ig\psi \star \psi\chi\psi + img[\phi, \bar{\phi}]_\star\chi\tilde{\partial}^i\psi + ig^2[\phi^{\star 2}, \bar{\phi}]_\star\chi\tilde{\partial}^i\psi\Big)\,, \quad (3.60)$$

$$[M^{0i}, \bar{\chi}\bar{Q}] = -i\bar{\chi}\bar{\sigma}^{0i}\bar{Q} + \int d^3x \Big(g[\dot{\bar{\phi}}, \bar{\phi}]_\star\bar{\chi}\bar{\sigma}^0\tilde{\partial}^i\psi + g[\bar{\phi}, \tilde{\partial}^i\partial_l\bar{\phi}]\bar{\chi}\bar{\sigma}^l\psi$$
$$-2ig\bar{\psi} \star \bar{\psi}\bar{\chi}\bar{\psi} + img[\bar{\phi}, \phi]_\star\bar{\chi}\tilde{\partial}^i\bar{\psi} + ig^2[\bar{\phi}^{\star 2}, \phi]_\star\bar{\chi}\tilde{\partial}^i\bar{\psi}\Big)\,. \quad (3.61)$$

To simplify the expression, we reorder the conjugate fields on the right hand side of the above equations, which induces extra infinite constant terms not explicitly shown here.

In summary, The commutation relations of the Lorentz rotation and boost generators generally have additional terms compared with those of the Poincaré or super-Poincaré algebras. Other commutation relations verify certain symmetries preserved by NCFT's, such as the translational and supergauge invariance. In the limit of $\Theta^{\mu\nu} \to 0$, the Poincaré or Super-Poincaré algebra is recovered.

3.3 Discussions

In this chapter we suggest a representation of the translation, Lorentz and supercharge generators. The commutation relations of those quantities are calculated directly based on this representation.

The NCFT has nonlocal interaction terms, which explicitly break the Lorentz invariance, but still preserve the translational and supergauge invariance. It is found that in the NCFT the translation and supercharge generators form the same algebra as in the commutative theory. But, the commutation relations of the Lorentz generators, or between the Lorentz generators and the translation or supercharge generators, generally have extra terms proportional to the non-commutativity $\Theta^{\mu\nu}$. In addition to that, there are also other interesting commutation relations, such as $[M^{0i}, P^j] = i\eta^{ij}P^0$, still hold true in the NC case.

Preservation of supersymmetry algebra suggests a supersymmetry generalization of NC geometry, which is not clearly understood yet. In that frame, fermions can better be defined as supersymmetry partners of bosons, instead of being a naive generalization of Lorentz algebra representation, since Lorentz invariance is broken. Indeed supersymmetry algebra has been represented on NC space and a divergence free supergravity model is expected to be constructed with this representation [46].

The superPoincaré algebra generators are "fundamental quantities", representation of which can be used to construct a theory of a dynamical system [47]. It remains a question whether the representation we obtained, Eqn. (3.41) to (3.46), could be used to construct a new theory on NC space consistent with the NC theory we start with.

CHAPTER 4
QUANTIZATION OF NONCOMMUTATIVE SOLITONS

4.1 Introduction

Solitons, known as "extended objects", exist in field theories with nonlinear interactions [48]. They are defined to be classical solutions to the equation of motion of a local field theory with the property that the energy density is, at all times, localized within a given region of space. A wave packet, which spreads as time evolves, in general is not this type. These objects have special features since at classical level they are already particle-like, and yet they possess an extended structure. Such objects have been studied extensively around mid-70's to early 80's of the last century. It was speculated hadrons may simply be described as quantized states of the extend objects, where the possibility of confining quarks exists, since even classically quarks are trapped to some extent [49, 50].

Solitons discussed in this chapter are also localized extended objects, but exist in NC field theories, called NC solitons. NC solitons was first discovered by Gopakumar, Minwalla and Strominger (GMS) [51] in NC scalar field theory with the potential having a local minimum besides a global minimum at the origin. Since then, the soliton solutions have been explicitly constructed in different NC gauge theories with or without matter [5, 6]. Monopole-like solutions in $(3 + 1)$ dimension with a string attached turns out to be realization of D3-D1 system, where D1 string ends on the D3 brane [52]. NC solitons can also be interpreted as lower dimensional D-branes in string field theory [53, 54].

As explained in chapter 1, perturbative dynamics of NC field theories reveals a very intriguing structure, i. e. UV/IR mixing, which suggests their analogy to string theories. Thus it becomes interesting to investigate the quantum behavior of nonperturbative structures, NC solitons, in NC field theory.

Next section is a review of the theory of NC solitons and their interpretation as D-branes, and also quantum theory of ordinary commutative solitons. In section three a new type of the NC soliton solutions, NC Q-balls, is investigated first and its difference from NC GMS solitons, existence at arbitrary small θ, is emphasized. Next canonical quantization of NC Q-balls at very small theta is discussed in detail. Quantum correction to the energy is calculated with phase shift summation method. The same method is further generalized to NC GMS soliton for smooth enough solutions. In both cases UV/IR mixing terms are present in the energy correction of NC solitons. The future direction could be quantization of NC solitons in gauge theories, which would increase the understanding of the UV/IR mixing terms.

4.2 Noncommutative Solitons and D-branes

4.2.1 Noncommutative Solitons in Scalar Field Theory

Classical NC solitons were first discovered in the scalar field theory with more than one space dimension [22]. Start with an action of scalar field theory in two NC space dimensions,

$$S = \int d^3x \frac{1}{2}(\partial\Phi)^2 - V(\Phi) \ , \tag{4.1}$$

where the fields are multiplied by NC star products (1.1) implicit here. In chapter 2 we discussed operator and function representations of NC algebra, and the isomorphism (Weyl transform) between them. The two descriptions are completely equivalent. It turns out classical solutions of NC field theory can be easily represented in operator formalism. Specifically, in two NC space dimensions, NC field theory are isomorphic to an algebra of operators defined on a one particle Hilbert space,

$$[\hat{x}^1 \ , \hat{x}^2] = i\theta \ . \tag{4.2}$$

Also star product of fields is just product of Weyl transform of the fields (1.1 and 2.5),

$$f(x) \star g(x) \longleftrightarrow \hat{f}(\hat{x})\hat{g}(\hat{x}) \ , \tag{4.3}$$

where $\hat{f}(\hat{x})$ and $f(x)$ or $\hat{g}(\hat{x})$ and $g(x)$ are Weyl transforms of each other. The integration over the space is actually equivalent of operator trace,

$$\int d^2x \leftrightarrow 2\pi\theta \mathrm{Tr} \ . \tag{4.4}$$

The derivative is equivalent to

$$\frac{\partial}{\partial x^i}\phi(x) \longleftrightarrow \frac{i}{\theta}\epsilon_{ij}[\hat{x}^j, \hat{\phi}(\hat{x})] \ . \tag{4.5}$$

Define creation and annihilation operators

$$a = \frac{1}{\sqrt{2\theta}}(\hat{x}^1 + i\hat{x}^2) \ , \qquad a^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}^1 - i\hat{x}^2) \ , \tag{4.6}$$

with $[a, a^\dagger] = 1$ as usual. The field $\Phi(x)$ or $\hat{\Phi}(\hat{x})$ can be expanded in the orthonormal basis $f_{nm}(x)$ or $|n\rangle\langle m|$. A systematic way of calculating $f_{nm}(x)$ can be obtained [5, 55]. In particular, Weyl transform of projection operator $P_n = |n\rangle\langle n|$, $f_{nn}(r)$, are just central functions, and can be expressed in Laguerre polynomials,

$$f_{nn}(r) = (-1)^n 2e^{-r^2} L_n(2r^2) \ . \tag{4.7}$$

In the operator formalism the action integral (4.1) becomes

$$S[\hat{\Phi}] = \int dt 2\pi\theta \mathrm{Tr}\left(\frac{1}{2}\dot{\hat{\Phi}}^2 + \frac{1}{\theta}([a, \hat{\Phi}][a^\dagger, \hat{\Phi}] - V(\hat{\Phi})\right) . \tag{4.8}$$

The equation of motion is

$$\partial_0^2\hat{\Phi} + \frac{2}{\theta}[a, [a^\dagger, \hat{\Phi}]] + V'(\hat{\Phi}) = 0 \ . \tag{4.9}$$

In commutative theory, it is well known that time independent scalar solitons do not exist in dimension more than one [56]. However, in NC scalar field theory, such solitons generally exist provided θ is larger enough. For radial symmetric solution,

$$\hat{\Phi} = \sum_n \lambda_n P_n = \sum_n \lambda_n |n\rangle\langle n| \ . \tag{4.10}$$

Put into Eqn. (4.9),

$$(n+1)(\lambda_{n+1} - \lambda_n) - n(\lambda_n - \lambda_{n-1}) = \frac{\theta}{2} V'(\lambda_n) \ . \tag{4.11}$$

If

$$V'(\Phi) = c\Phi(\Phi - \alpha_1) \cdots (\Phi - \alpha_l) \ , \tag{4.12}$$

at $\theta \to \infty$, the solutions are just

$$\Phi = \alpha_i P_n \qquad \text{or} \qquad \alpha_i(1 - P_n) \ . \tag{4.13}$$

4.2.2 Noncommutative Solitons in Gauge Theory

NC solitons are also widely found in gauge theories [6], and their interpretation as D-branes is precise [53, 54]. In the following we review the theory of solitons found in gauge theories. Start with the action,

$$S = 2\pi\theta \mathrm{Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D^\mu \phi D_\nu \phi - V(\phi) \right) \ . \tag{4.14}$$

Here

$$F_{oi} = \dot{A}_i - \partial_i A_0 \ , \tag{4.15}$$

and

$$F_{ij} = i[D_i, D_j] = \frac{1}{\theta}([C, \bar{C}] + 1)\epsilon^{ij} \ , \tag{4.16}$$

where

$$D_i \phi = \partial_i - i[A_i, \phi] \to U D_i \phi \bar{U} \tag{4.17}$$

is covariant derivative under U(1) transformation,

$$\phi \to U \phi \bar{U} \ , \tag{4.18}$$

and $C \equiv a^\dagger + i\sqrt{\theta}A$.

Suppose the potential is $V(\phi - \phi_*)$, which has a local minimum at $\phi = \phi_*$ and a local maximum at $\phi = 0$, and $V(0) = 0$. Static solutions locally minimize the Hamiltonian

$$H = 2\pi\theta \mathrm{Tr}\left(\frac{1}{4}F_{ij}^2 + \frac{1}{\theta}[C,\phi][\bar{C},\phi] + V(\phi - \phi_*)\right) \ . \tag{4.19}$$

The lowest energy solution (vacuum) can be easily found,

$$\phi = \phi_* I \ , \qquad C = a^\dagger \ . \tag{4.20}$$

A class of solutions with nonzero energy can be constructed through a transformation with a nonunitary isometry operator , shift operator S ,

$$S \ : \ |n\rangle \rightarrow |n+1\rangle \ , \qquad S = \sum_{n=0}^{\infty} |n+1\rangle\langle n| \ , \tag{4.21}$$

where $\bar{S}S = 1$, but $S\bar{S} = 1 - P_0 \equiv 1 - |0\rangle\langle 0|$. The n^{th} solution is

$$\begin{aligned} \phi &= S^n \phi_* I \bar{S}^n = \phi_*(I - P_n) \ , \\ C &= S^n \bar{a} \bar{S}^n \ , \qquad \bar{C} = S^n a \bar{S}^n \ , \end{aligned} \tag{4.22}$$

with the field strength

$$F = \frac{1}{2}\epsilon^{ij}F^{ij} = \frac{1}{\theta}([C,\bar{C}] + 1) = \frac{P_n}{\theta} \tag{4.23}$$

and the energy

$$E = 2\pi\theta n \left(\frac{1}{2\theta^2} + V(-\phi_*)\right) \ . \tag{4.24}$$

The NC solitons constructed above can in fact be interpreted as Bosonic D-branes. As explained in the introduction section, the effective field theory of the tachyon and gauge field degrees of freedom of the open strings on the D-brane will be a NC field theory in terms of the effective field theory at $B = 0$, but with the open string effective metric G_{ij} and the noncommutativity θ_{ij} determined by (1.11) and (1.13).

Consider a Bosonic open string with an unstable space-filling D25-brane. The leading terms in the effective action for constant tachyon and gauge field strength have the Born-Infeld form [57–59]. Integrating out the massive string degrees of freedom leads to an effective action of the form (for $B_{ij} = 0$),

$$\begin{aligned} S_{eff} &= \frac{c}{g_s} \int d^{26}x \sqrt{\det g} \left[-\frac{1}{4} h(\phi - 1) F^{\mu\nu} F_{\mu\nu} \right. \\ &\qquad \left. + \cdots + \frac{1}{2} f(\phi - 1) \partial^\mu \phi \partial_\mu \phi + \cdots - V(\phi - 1) \right] , \end{aligned} \tag{4.25}$$

where $c = T_{25} g_s$ is independent of g_s with T_{25} the D25-brane tension. The potential $V(\phi - 1)$ has a local maximum at $\phi = 0$ with $V(-1) = 1$ representing the unstable D25 brane configuration, and a local minimum at $\phi = 1$ with $V(0) = 0$ representing the closed string vacuum, according to the conjecture of Sen [60, 61].

Now turn on the background B field with only $B_{24,25} = b < 0$, then the action becomes NC in $\mathbf{R^2}$ including 24 and 25 direction. In chapter 1, explicit form of the NC gauge field term is given in (1.19) analogous to (1.22). Therefore the NC action analogous to (4.25) is

$$S = \frac{2\pi\theta c}{G_s} \int d^{24}x \mathcal{L}_{nc} , \tag{4.26}$$

with

$$\begin{aligned} \mathcal{L}_{nc} &= \sqrt{\det G} \mathrm{Tr} \left[-\frac{1}{4} h(\phi - 1)(F^{\mu\nu} + \Phi^{\mu\nu})(F_{\mu\nu} + \Phi_{\mu\nu}) \right. \\ &\qquad \left. + \cdots + \frac{1}{2} f(\phi - 1) D^\mu \phi D_\mu \phi + \cdots - V(\phi - 1) \right] . \end{aligned} \tag{4.27}$$

Focusing on only the NC directions 24, 25, a special choice [10]

$$\theta = \frac{1}{B} = \frac{1}{|b|} \qquad \Phi = -B = |b| \tag{4.28}$$

in Eqn. (1.20) gives the value of the open string metric (1.11) and the string coupling (1.24) in the zero coupling limit,

$$G = -(2\pi\alpha' b)^2 \ , \qquad G_s = g_s(2\pi\alpha'|b|) \ . \tag{4.29}$$

In evaluating energy of the NC soliton solution (4.22), remember the soliton can be obtained by the transformation with the shifting operator S^n from the closed string vacuum configuration. All covariant derivatives of ϕ or $F_{\mu\nu}$ vanish as they do in the closed string vacuum, so are the gauge fields A_μ vertical to the brane. Also

$$h(\phi-1)(F+\Phi)^2 = \frac{1}{\theta^2}h(-P_n)(1-P_n) = \frac{1}{\theta^2}h(-1)P_n(1-P_n) = 0 \ , \tag{4.30}$$

and

$$V(\phi-1) = V(-P_n) = V(-1)P_n = P_n \ . \tag{4.31}$$

Therefore the energy of the soliton has only the potential term, derived from Eqn. (4.27),

$$E = \frac{2\pi\theta c\sqrt{\det G}}{G_s}\int d^{24}x \mathrm{Tr} P_n = \frac{(2\pi)^2\alpha' nc}{g_s} \ , \tag{4.32}$$

which identifies the tension as

$$T = \frac{(2\pi)^2\alpha' nc}{g_s} = (2\pi)^2\alpha' n T_{25} = nT_{23} \ . \tag{4.33}$$

In the above construction configuration of n Dp-branes arises as the NC soliton solution shifted from the closed string vacuum. This construction is exact for any value of B or finite θ, which reduces to that of [62] in the limit of large B-field.

4.3 Classical Noncommutative Q-ball Solution

GMS solitons, which exist in $(2+1)$ dimensional NC scalar field theory, while classically stable, cease to exist at sufficiently small NC parameter θ, due to the nonexistence theorem of Derrick [56] in the commutative limit($\theta \to 0$). In this commutative limit, however, time dependent nontopological solitons, or Q-balls exist in all space dimensions [48, 63].

4.3.1 Hamiltonian and Equation of Motion

In this section we derive the equation of motion for NC Q-ball solutions, following brief introduction of NC scalar field theory. The form of the solution has already been given [64]. We discuss the existence and stability of the solutions, and show that in the commutative limit NC Q-balls just reduce to the commutative Q-balls.

Consider a NC scalar field theory action with global $U(1)$ phase invariance,

$$S = -\int dtd^2x \left[\partial_\mu\bar{\phi}\partial^\mu\phi + V(\frac{1}{2}\{\bar{\phi},\phi\})\right] \ , \tag{4.34}$$

where the space-time metric is $(-,+,+)$, and the fields are multiplied by NC star product, generally made implicit in this paper, and $\{A\ ,B\} \equiv A\star B + B\star A$. The potential V has a global minimum at the origin, with the scaling property

$$V(g,\bar{\phi}\phi) = g^{-2}V(g^2\bar{\phi}\phi) \ . \tag{4.35}$$

g is then the coupling constant assumed to be small. The commutative limit of this action is where ordinary Q-balls have already been constructed [48, 63]. The NC star product is defined to be,

$$(\phi\star\psi)(x) \equiv \exp(i\frac{\theta}{2}\epsilon_{jk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial y_k})\phi(x)\psi(y)\Big|_{y=x} \ , \tag{4.36}$$

where $j,k = 1,2$.

In the operator formalism the action integral (4.34) becomes

$$S[\hat{\phi},\hat{\bar{\phi}}] = \int dt 2\pi\theta \mathrm{Tr}\left(\partial_0\hat{\bar{\phi}}\partial_0\hat{\phi} + \frac{1}{\theta}([a,\hat{\bar{\phi}}][a^\dagger,\hat{\phi}] + [a^\dagger,\hat{\bar{\phi}}][a,\hat{\phi}]) - V(\hat{\bar{\phi}}\hat{\phi})\right) . \tag{4.37}$$

The equation of motion is

$$\partial_0^2\hat{\phi} + \frac{2}{\theta}[a,[a^\dagger,\hat{\phi}]] + \hat{\phi}V'(\hat{\bar{\phi}}\hat{\phi}) = 0 \ . \tag{4.38}$$

The action has global $U(1)$ phase invariance, which yields a conserved charge

$$Q[\hat{\phi}, \hat{\bar{\phi}}] = \int d^2x j^0 = i2\pi\theta \mathrm{Tr}(\hat{\bar{\phi}}\partial_0\hat{\phi} - \partial_0\hat{\bar{\phi}}\hat{\phi}) \ . \tag{4.39}$$

Q is interpreted as particle number in the physical system. A particular system always exists with fixed particle number $N = Q[\hat{\phi}, \hat{\bar{\phi}}]$. To find nondissipative soliton solutions [48, 63] under this constraint, we write Hamiltonian

$$H = 2\pi\theta \mathrm{Tr}\left(\partial_0\hat{\bar{\phi}}\partial_0\hat{\phi} - \frac{1}{\theta}([a, \hat{\bar{\phi}}][a^\dagger, \hat{\phi}] + [a^\dagger, \hat{\bar{\phi}}][a, \hat{\phi}]) + V(\hat{\bar{\phi}}\hat{\phi})\right) + \omega(N - Q[\hat{\phi}, \hat{\bar{\phi}}]) \ , \tag{4.40}$$

with the constraint applied before the Poisson bracket is worked out [47]. The minimum energy solution occurs at

$$\left.\frac{\delta H}{\delta(\partial_0\hat{\bar{\phi}})}\right|_N = \partial_0\hat{\phi} + i\omega\hat{\phi} = 0 \ , \tag{4.41}$$

which means

$$\hat{\phi} = \frac{1}{\sqrt{2}}\hat{\sigma}(\hat{x})e^{-i\omega t}. \tag{4.42}$$

Assuming hermitian $\hat{\sigma}(\hat{x})$ or real $\sigma(x)$, H becomes

$$H = 2\pi\theta \mathrm{Tr}\left(-\frac{1}{2}\omega^2\hat{\sigma}^2 - \frac{1}{\theta}[a, \hat{\sigma}][a^\dagger, \hat{\sigma}] + V(\frac{1}{2}\hat{\sigma}^2)\right) + \omega N \ , \tag{4.43}$$

with the particle number

$$N = 2\pi\theta\omega \mathrm{Tr}(\hat{\sigma}^2) \ . \tag{4.44}$$

and the equation of motion (4.38)

$$\frac{2}{\theta}[a, [a^\dagger, \hat{\sigma}]] - \omega^2\hat{\sigma} + \hat{\sigma}V'(\frac{1}{2}\hat{\sigma}^2) = 0 \ . \tag{4.45}$$

Note the equation of motion (4.45) also follows from $(\delta H/\delta\hat{\sigma})|_N = 0$, which means that the solution $\hat{\sigma}$ has the same form as the static GMS soliton solution in the potential

$$U(\hat{\sigma}) = V(\frac{1}{2}\hat{\sigma}^2) - \frac{1}{2}\omega^2\hat{\sigma}^2 \ . \tag{4.46}$$

Consider spherically symmetric solution [51] expanded in terms of the projection operators,

$$\hat{\sigma}(\hat{x}) = \sum_{n=0}^{\infty} \lambda_n P_n \ , \tag{4.47}$$

where $P_n \equiv |n><n|$. Replace $\hat{\sigma}$ in (4.43), (4.44), and the equation of motion (4.45),

$$H = 2\pi \sum_n [(n+1)(\lambda_{n+1} - \lambda_n)^2 + \theta U(\lambda_n)] + \omega N \ , \tag{4.48}$$

$$N = 2\pi\theta\omega \sum_n \lambda_n^2 \ , \tag{4.49}$$

$$(n+1)(\lambda_{n+1} - \lambda_n) - n(\lambda_n - \lambda_{n-1}) = \frac{\theta}{2} U(\lambda_n) \ . \tag{4.50}$$

Sum the equation of motion from $n = 0$ to $n = K$

$$\lambda_{K+1} - \lambda_K = \frac{\theta}{2(K+1)} \sum_{n=0}^{K} U'(\lambda_n) \ , \tag{4.51}$$

where $K \geq 0$ is an arbitrary integer. A particular set of λ_n's defines a solution. Many properties of the solution can still be derived from Eqn. (4.48-4.51) though a closed form has not been constructed. For example, because of the finiteness of both the energy H and the particle number N, we have,

$$\lambda_{n+1} = \lambda_n \ , \qquad \lambda_n = 0 \ , \quad \text{for} \ \ n \to \infty \ , \tag{4.52}$$

4.3.2 Q-ball Solutions

In static GMS soliton theory, the global minimum of the potential is generally assumed to be at the origin, and the core of the soliton is localized at the local minimum of the potential (false bubble solution). It is the noncommutativity that forbids the classical decay of the solitons. The corresponding commutative potential does not have nontrivial topological structures, and hence yields no soliton solutions. Therefore NC GMS solitons are genuine NC effects and they disappear at small enough θ, where the commutative limit is approached.

This is not the case with Q-ball solutions. The existence and stability of Q-ball solution rely on the conservation of the charge Q as the consequence of the global symmetry. The potential for Q-ball solutions does not have nontrivial topological structure. Therefore NC Q-balls are expected to exist even for very small θ . We will show that such NC Q-ball solutions would smoothly reduce to the Q-ball solution in the commutative limit.

In the following we discuss the existence of NC Q-ball solutions in a typical potential form,

$$U(\sigma) = V(\sigma^2) - \frac{1}{2}\omega^2\sigma^2 = a\sigma^2 - b\sigma^4 + c\sigma^6 \ , \tag{4.53}$$

where the coefficients b and c are larger than zero, and $a = \frac{1}{2}(m^2 - \omega^2)$.

$U(\sigma)$ varies for different ω. If $\omega^2 > m^2$ or $a < 0$, $U(\sigma)$ has a local maximum at the origin. In the commutative limit there is only a plane wave solution. Here similar plane wave solution in NC limit can also be constructed. Since for a stable soliton solution λ_n would have to take values between s and the origin and monotonically decrease in n [65], a simple argument can show that solitons cannot exist. There is a constraint that $\sum_{n=0}^{K} U'(\lambda_n)$ converges to zero as K goes to infinity, which cannot be satisfied in this case. To prove this constraint, suppose that

$$\sum_{n=0}^{\infty} U'(\lambda_n) \sim v \neq 0 \ , \tag{4.54}$$

Sum Eqn. (4.51) from a particular $K = q$ sufficiently large to a point p close to infinity,

$$\lambda_p - \lambda_q \sim \sum_{K=q}^{p} \frac{v}{K} \ . \tag{4.55}$$

It is then easy to see λ_p will not converge to zero as p goes to infinity.

When $\omega^2 < \nu^2$, $\nu^2 = m^2 - b^2/2c$, $U(\sigma)$ has only a global minimum at the origin. Even though in the commutative theory no soliton solutions exist, for NC theory at sufficiently large θ, there are GMS type solitons exist. It has been shown that there is a critical lower bound on θ for the existence of NC soliton [66]. Similar bounds would be expected to exist for NC Q-ball for $0 < \omega < \nu$ as well.

As $\nu^2 < \omega^2 < m^2$, $U(\sigma)$ has a local minimum at the origin, a global minimum at s ($U(s) < 0$) and a zero $w = [(b-\sqrt{b^2 - 4ac})/2a]^{1/2}$ between 0 and s ($0 < w < s$). In the commutative case such potential form enables the existence of Q- ball solutions. In the NC case it is expected that NC-Q ball solutions exist even for small θ. In the following we take the continuum limit of Eqn. (4.51) for very small θ , and show that all the solitons sP_K exist at $\theta \to \infty$ converges to the commutative Q-ball solution as $\theta \to 0$.

For very small θ, all λ_n's can be considered as sufficiently close. Therefore λ_n can be approximated by a continuous function $\lambda(u)$ [1] . Let $u = K\theta$, and $\lambda_K = \lambda(K\theta) = \lambda(u)$. Eqn. (4.51) becomes

$$\lambda'(u) = \frac{1}{2(u+\theta)} \int_0^u U'[\lambda(s)]ds + \mathcal{O}(\theta) \ . \tag{4.56}$$

Ignore $\mathcal{O}(\theta)$ term, we have

$$\frac{d\lambda}{du} + u\frac{d^2\lambda}{du^2} = \frac{1}{2}\frac{dU}{d\lambda} \ . \tag{4.57}$$

Let $u = \frac{1}{2}v^2$,

$$\frac{d^2\lambda}{dv^2} + \frac{1}{v}\frac{d\lambda}{dv} = \frac{dU(\lambda)}{d\lambda} \ . \tag{4.58}$$

This is exactly the equation of motion for the commutative Q-ball solution $\lambda(v)$, with v identified as radius r . This can be explained as follows: The Weyl transform of $\frac{1}{2}r^2$ is $a^\dagger a$, and $a^\dagger a$ has the eigenvalue $n\theta$ on the state $|n>$. As θ gets smaller, the eigenvalues $n\theta$ gets closer, and eventually becomes continuous as $\frac{1}{2}r^2$ in the commutative limit. The coefficient λ_n just becomes the field $\lambda(r)$ in

[1] Thanks to Dr. Shabanov for helping on this point

this limit. In this description the commutative Q-ball can be considered as the analytical continuation of the NC Q-balls in θ .

The formula for the energy and particle number in the commutative limit can also be recovered by taking the continuum limit of Eqn. (4.48) and (4.49),

$$H = 2\pi \int_0^\infty v dv \left[\frac{1}{2} (\frac{d\lambda}{dv})^2 + U(\lambda) + \omega N \right] , \tag{4.59}$$

$$N = 2\pi\omega \int_0^\infty du \lambda^2(u) = 2\pi\omega \int_0^\infty v dv \lambda^2(v) . \tag{4.60}$$

The existence of the commutative Q-ball solution are proved by considering an analogous problem in which a classical particle moves in the one-dimensional potential $-U(\lambda)$ [48]. The field configuration $\lambda(v)$ of the Q-ball starts from a unique value $\rho = \lambda(0)$ between the zero w and the global minimum s, then monotonically decreases in v , and approaches 0 when $v \to \infty$. This property of $\lambda(v)$ is consistent with those of the general stable NC soliton solutions λ_n at finite θ . It is found [65] that there exist smooth θ families of spherically symmetric solutions in which λ_n is monotonically decreasing in n . In the infinite θ limit such solution is just sP_K . As θ decrease, $\lambda_0, \cdots, \lambda_K$ decrease from s , while other $\lambda_n (n > K)$ starts to move away from the origin towards s , but the whole λ_n series remain monotonically decrease in n . Since in the commutative limit λ_n just becomes $\lambda(v)$, one can conclude that as θ decreases from ∞ to zero, λ_0 will decrease from s and eventually to ρ at the commutative limit.

4.3.3 Virial Relation

The Hamiltonian (4.43) in the function formalism is

$$H[\sigma] = 2\pi \int r dr \left(\frac{1}{2} (\partial_i \sigma)^2 + U(\theta , \sigma) + \omega N \right) , \tag{4.61}$$

where the potential U has explicit dependence on θ through star product. Suppose $\sigma(x)$ is the Q-ball solution, $H[\sigma(x/a)]$ must be stationary at $a = 1$. A change of the integration variables shows that

$$H[\sigma(x/a)] = 2\pi \int rdr \left(\frac{1}{2}(\partial_i \sigma(x))^2 + a^2 U(\frac{\theta}{a^2}\ , \sigma(x)) + \omega N \right)\ , \tag{4.62}$$

and

$$\frac{d}{da} H[\sigma(x/a)] \Big|_{a=1} = 2\pi \int rdr \left(2U(\theta\ , \sigma) - 2\frac{\partial}{\partial\theta} U(\theta\ , \sigma) \right) = 0\ . \tag{4.63}$$

Unlike the Virial theorem for $d = 1$ space dimension, here the kinetic energy is scale invariant. Scaling dependence of the energy includes two separate terms from the potential and from its dependence on θ through the star products. The significance of Eqn. (4.63) is more explicit in GMS soliton case, where the potential energy

$$2\pi \int rdr U(\sigma) = \theta \sum_{n=0}^{\infty} U(\lambda_n) > 0\ . \tag{4.64}$$

The scaling variable a can be thought of as the size of the NC soliton. While the positive potential energy favors shrinking of the soliton, but the NC star products keep it from decay.

4.4 Quantization of Noncommutative Q-ball

Solitons are extended objects exist in field theory, the properties of which receive quantum corrections as the fields are quantized. In this section we follow very closely to the canonical quantization procedure [49, 63]. Then we evaluate the ultraviolet divergences in the quantum corrections to the soliton energy at very small θ .

4.4.1 Canonical Quantization

The general procedure to investigate the properties of the solitons is to expand the fields around the classical solution. Because the momentum and particle number are conserved in the system, we will have to impose the corresponding constraints to erase the zero-frequency modes in the expansion.

We start by making a point canonical transformation of $\phi(x)$,

$$\phi = \frac{1}{\sqrt{2}} e^{-i\beta(t)} [\sigma(x - X(t)) + \chi(x - X(t), t)] \ , \tag{4.65}$$

$$\chi(x - X(t), t) \equiv \chi_R(x - X(t), t) + i\chi_I(x - X(t), t) \ , \tag{4.66}$$

where $\beta(t)$ and $X^i(t)$ are the collective coordinates represent the over-all phase and the center of mass position. Impose the constraints on χ to ensure the above transformation is a canonical transformation with equal number of degrees of freedom before and after,

$$\int \sigma\chi_I = 0 \ , \qquad \int \chi_R \partial_i \sigma = 0 \ , \tag{4.67}$$

where $i = 1, 2$. The integral sign denotes two dimensional integrations over x. The star product is suppressed. Unless indicated otherwise, from now on the differential d^2x and the NC star product are implied wherever applicable. The above constraints also remove the perturbative zero mode solutions in the meson field χ. Let

$$\chi_R(x, t) = \sum_{a=3}^{\infty} q_{Ra}(t) f_a(x) \ , \tag{4.68}$$

$$\chi_I(x, t) = \sum_{\dot{a}=2}^{\infty} q_{I\dot{a}}(t) g_{\dot{a}}(x) \ , \tag{4.69}$$

where $f_a(x)$ and $g_{\dot{a}}(x)$ are the real normal functions satisfy

$$\int f_a f_b = \delta_{ab} \ , \tag{4.70}$$

$$\int g_{\dot{a}} g_{\dot{b}} = \delta_{\dot{a}\dot{b}} \ , \tag{4.71}$$

$$\tag{4.72}$$

and under the constraints,

$$\int \partial_i \sigma f_a = 0 \,, \tag{4.73}$$

$$\int \sigma g_{\dot{a}} = 0 \,, \tag{4.74}$$

where $a = 3, 4, \ldots$ and $\dot{a} = 2, 3, \ldots$ always in this paper.

Rewrite the Lagrangian (4.34) with (4.65),

$$L = \frac{1}{2}\dot{q}^T \mathcal{M} \dot{q} + \mathcal{V}(q), \tag{4.75}$$

where $q^T = (X_1 \,, X_2 \,, \beta \,, q_{R3} \,, \ldots \,, q_{I2} \,, \ldots)$ and T denotes matrix transpose, and

$$\mathcal{V}(q) \equiv \int (\partial_i \bar{\phi} \partial_i \phi + V(\frac{1}{2}\{\bar{\phi} \,, \phi\})) \tag{4.76}$$

The matrix elements of the symmetric $\mathcal{M}$ are

$$\mathcal{M}_{ij} = M_0 \delta_{ij} + \int (2\partial_i \sigma \partial_j \chi_R + \partial_i \chi_R \partial_j \chi_R + \partial_i \chi_I \partial_j \chi_I) \,, \tag{4.77}$$

$$\mathcal{M}_{\beta\beta} = I + \int (2\sigma \chi_R + \chi_R^2 + \chi_I^2) \,, \tag{4.78}$$

$$\mathcal{M}_{\beta i} = \int (-2\partial_i \sigma \chi_I + \chi_R \partial_i \chi_I - \chi_I \partial_i \chi_R) \,, \tag{4.79}$$

$$\mathcal{M}_{ia} = -\int f_a \partial_i \chi_R \,, \tag{4.80}$$

$$\mathcal{M}_{i\dot{a}} = -\int g_{\dot{a}} \partial_i \chi_I \,, \tag{4.81}$$

$$\mathcal{M}_{\beta a} = \int f_a \chi_I \,, \tag{4.82}$$

$$\mathcal{M}_{\beta \dot{a}} = -\int g_{\dot{a}} \chi_R \,, \tag{4.83}$$

$$\mathcal{M}_{ab} = \delta_{ab} \,, \tag{4.84}$$

$$\mathcal{M}_{\dot{a}\dot{b}} = \delta_{\dot{a}\dot{b}} \,. \tag{4.85}$$

where $M_0 \equiv \frac{1}{2}\int (\partial_i \sigma)^2$ and $I \equiv \int \sigma^2$. The conjugate momentum of q is

$$p = M\dot{q} \equiv (P_1 \,, P_2 \,, N \,, p_{R3} \,, \ldots \,, p_{I2} \,, \ldots)^T \,. \tag{4.86}$$

The particle number N and the total momentum P_i are conserved since the Lagrangian (4.75) is independent of the collective coordinates β and X^i. Quantize the new canonical coordinates,

$$[X^i\ , P^j] = i\delta^{ij}\ , \tag{4.87}$$

$$[\beta\ , N] = i\ , \tag{4.88}$$

$$[q_{Ra}\ , p_{Rb}] = i\delta_{ab}\ , \tag{4.89}$$

$$[q_{I\dot{a}}\ , p_{I\dot{b}}] = i\delta_{\dot{a}\dot{b}}\ . \tag{4.90}$$

The Hamiltonian

$$H = \frac{1}{2}J^{-1}p^T\mathcal{M}^{-1}Jp + \mathcal{V}(q)\ , \tag{4.91}$$

where $J = \sqrt{\det\mathcal{M}}$ because the operator ordering in H is unambiguously determined by the ordinary quantized Hamiltonian with the coordinates ϕ [49].

The quantum states can be labeled as $|P^1\ , P^2\ , N\ , q_{Ra}\ , q_{I\dot{a}}\rangle$. One can solve the Schrödinger Equation perturbatively around the one soliton ground state $|P^1 = P^2 = 0\ , N = I\omega\ , 0\rangle$. In this state P^i and N are the momentum and particle number of the classical solution σ, which can be obtained by letting $\phi = \sigma$ in Eqn. (4.49) and $P^i = \int \dot{\bar{\phi}}\partial^i\phi + \partial^i\bar{\phi}\dot{\phi}$ [1]. 0 labels the lowest energy state with the given P^i and N value.

We can then treat χ_R and χ_I as perturbative degrees of freedom, and expand the Hamiltonian perturbatively around the one soliton ground state order by order in the weak coupling constant g , defined in Eqn. (4.35).

σ is at the order of g^{-1} as a soliton solution. M_0 and I are g^{-2} order. Then P_i and N are at the g^{-2} order, while p_{Ra} and $p_{I\dot{a}}$ at g^0 order. Since J commute with P_i and N , and at the leading g^{-3} order, $J = M_0\sqrt{I}$ is a constant or $[p\ , J] = 0$, one can check that J would not be a factor in the Hamiltonian up to the order g^0.

The Hamiltonian can be expanded order by order, $H = H_0 + H_1 + H_2$, with the expansion relation,

$$\mathcal{M}^{-1} = \mathcal{M}_0^{-1} + \mathcal{M}_0^{-1}\Delta\mathcal{M}_0^{-1} + \mathcal{M}_0^{-1}\Delta\mathcal{M}_0^{-1}\Delta\mathcal{M}_0^{-1} + \cdots \ , \tag{4.92}$$

where $\mathcal{M} = \mathcal{M}_0 + \Delta$, and $\mathcal{M}_0$ has only nonzero diagonal elements, $\mathcal{M}_0^{qq} = (M_0 \ , M_0 \ , I \ , 1 \ , 1)$.

H_0, equal to the energy of the classical solution, is at the order of g^{-2},

$$H_0 = M_0 + \frac{1}{2}I\omega^2 + V(\frac{1}{2}\sigma^2) \ , \tag{4.93}$$

H_1, linear in χ, vanishes due to the fixed N and P_i, which ensures that χ_R and χ_I are at the order of g^0.

The term quadratic in χ is at the g^0 order,

$$H_2 = \frac{1}{2}(p_{Ra} - \omega \int f_a\chi_I)^2 + \frac{1}{2}(p_{I\dot{a}} + \omega \int g_{\dot{a}}\chi_R)^2 + 2\frac{\omega^2}{M_0}(\int \partial_i\sigma\chi_I)^2 + 2\frac{\omega^2}{I}(\int \sigma\chi_R)^2 + \mathcal{V}_2(q) \ , \tag{4.94}$$

where

$$\begin{aligned} \mathcal{V}_2(q) \ &= \ \int \Big\{ \frac{1}{2}[(\partial_i\chi_R)^2 + (\partial_i\chi_I)^2] - \frac{1}{2}\omega^2(\chi_R^2 + \chi_I^2) \\ &\quad + \frac{1}{2}(\chi_R^2 + \chi_I^2)V'(1/2\sigma^2) + V\left(\frac{1}{2}\sigma^2 \ , \frac{1}{2}\{\chi_R, \sigma\} \ , \frac{1}{2}\{\chi_R, \sigma\}\right)\Big\} \ , \end{aligned} \tag{4.95}$$

where $V\left(\frac{1}{2}\sigma^2 \ , \frac{1}{2}\{\chi_R, \sigma\} \ , \frac{1}{2}\{\chi_R, \sigma\}\right)$ represents the terms from the expansion of the potential V quadratic in χ_R.

4.4.2 Energy Corrections at Very Small θ

The Hamiltonian is separated into two parts, described by the baryon degrees of freedom $(P^i \ , N)$ and meson degrees of freedom $(\chi_R \ , \chi_I)$ respectively. The sum of the frequencies of the meson excitations is the zero-point energy of H_2,

$$\langle P^1 = P^2 = 0 \ , N = I\omega \ , 0| \ H_2 \ |P^1 = P^2 = 0 \ , N = I\omega \ , 0\rangle \ , \tag{4.96}$$

which, subtracted by the vacuum energy $E_{\text{vac}} = \int d^2k/(2\pi)^2\sqrt{k^2+m^2}$, gives the quantum corrections to the soliton energy.

$\mathcal{V}_2(q)$ is the perturbative expansion of the effective potential $\mathcal{V}(q) - \omega Q[\phi, \bar{\phi}]$, Eqn. (4.76) and (4.39), around the solution σ . It is easy to check that $\chi_R = \partial_i\sigma$ and $\chi_I = \sigma$ are the eigenmodes of $\mathcal{V}_2(q)$ with eigenfrequency 0, due to the translational and rotational invariance of the potential. Therefore we can define the normal functions f_a and $g_{\dot{a}}$ to be the eigenmodes of $\mathcal{V}_2$, or

$$\mathcal{V}_2(q) = \frac{1}{2}\Omega^2_{Ra}q^2_{Ra} + \frac{1}{2}\Omega^2_{I\dot{a}}q^2_{I\dot{a}} \ , \tag{4.97}$$

with the frequencies Ω_{Ra} and $\Omega_{I\dot{a}}$. The potential $\mathcal{V}_2$ are highly nonlocal since the fields are multiplied by NC star product. In the commutative Q-ball case, $\mathcal{V}_2$ has been shown to have only one s-wave eigenmode in χ_R sector with imaginary frequency Ω_{R3} [63]. In last section, we have shown that as NC parameter θ is taken to be small enough, the NC soliton solution will reduce arbitrary close to its commutative analog. Therefore close to the commutative limit $\mathcal{V}_2$ is expected to have the similar eigenvalues and eigenmodes as its commutative analog. NC Q-ball is also expected to be stable as its commutative analog. We will assume θ is chosen to be such a small value in evaluating the quantum effects of the noncommutativity.

Define $f_i = 1/\sqrt{M_0}\partial_i\sigma$ and $g_1 = 1/\sqrt{I}\sigma$, Rewrite the Hamiltonian H_2 , (4.94), in the matrix form,

$$H_2 = \frac{1}{2}(\mathcal{P}^T - \omega\mathcal{Q}^T\Upsilon^T)(\mathcal{P} - \omega\Upsilon\mathcal{Q}) + 2\omega^2\mathcal{Q}^T\Xi\mathcal{Q} + \frac{1}{2}\mathcal{Q}^T\Omega^2\mathcal{Q} \ , \tag{4.98}$$

where the matrices are defined as follows:

$$\mathcal{P}^T \equiv (p_{Ra} \ , p_{I\dot{a}}) \ , \qquad \mathcal{Q}^T \equiv (q_{Ra} \ , q_{I\dot{a}}) \ , \tag{4.99}$$

$$\Upsilon \equiv \begin{pmatrix} 0 & \Gamma_{a\dot{a}} \\ & \\ -\Gamma^T_{\dot{a}a} & 0 \end{pmatrix} , \qquad \Xi \equiv \begin{pmatrix} \mathcal{F}_{ab} & 0 \\ & \\ 0 & \mathcal{G}_{\dot{a}\dot{b}} \end{pmatrix} , \qquad \Omega \equiv \begin{pmatrix} \Omega_{Ra} & 0 \\ & \\ 0 & \Omega_{I\dot{a}} \end{pmatrix} , \tag{4.100}$$

where

$$\Gamma_{a\dot{a}} \equiv \int f_a g_{\dot{a}} \ , \qquad \mathcal{F}_{ab} \equiv \int g_1 f_a \int g_1 f_b \ , \qquad \mathcal{G}_{\dot{a}\dot{b}} \equiv \int f_i g_{\dot{a}} \int f_i g_{\dot{b}} \ . \tag{4.101}$$

The equation of motion,

$$\dot{\mathcal{Q}} = \frac{\partial H_2}{\partial \mathcal{P}} \ , \qquad \dot{\mathcal{P}} = -\frac{\partial H_2}{\partial \mathcal{Q}} \ , \tag{4.102}$$

give

$$\dot{\mathcal{Q}} \ = \ \mathcal{P} - \omega \Upsilon \mathcal{Q} \ , \tag{4.103}$$

$$\dot{\mathcal{P}} \ = \ \omega \Upsilon^T (\mathcal{P} - \Upsilon \mathcal{Q}) - 4\omega^2 \Xi \mathcal{Q} - \Omega^2 \mathcal{Q} \ . \tag{4.104}$$

Therefore,

$$\ddot{\mathcal{Q}} + 2\omega \Upsilon \dot{\mathcal{Q}} + 4\omega^2 \Xi \mathcal{Q} + \Omega^2 \mathcal{Q} = 0 \ . \tag{4.105}$$

Let the real normal eigenmodes of $\mathcal{Q}$ be

$$\mathcal{Q}^A = (\xi^A_{Ra} \ , \xi^A_{I\dot{a}})^T, \tag{4.106}$$

where $\mathcal{Q}^{A^T} \mathcal{Q}^B = \delta^{AB}$. Replace $\mathcal{Q} = \mathcal{Q}^A \exp(-i\Lambda_A t)$ (Index A is not summed over) in the above equation. Since $\mathcal{Q}^{A^T} \Upsilon \mathcal{Q}^A = 0$,

$$\Lambda_A = \sqrt{\mathcal{Q}^{A^T} (4\omega^2 \Xi + \Omega^2) \mathcal{Q}^A} \ . \tag{4.107}$$

Introduce creation and annihilation operators, $[\mathcal{C}_A \ , \mathcal{C}^\dagger_B] = \delta_{AB}$, $\mathcal{Q}$ can then be quantized as,

$$\mathcal{Q} = \sum_A \frac{\mathcal{Q}^A}{\sqrt{2\Lambda_A}} (\mathcal{C}_A e^{-i\Lambda_A t} + \mathcal{C}^\dagger_A e^{i\Lambda_A t}) \ . \tag{4.108}$$

Use this equation and Eqn. (4.104) and (4.98), one can define the one soliton ground state,

$$\mathcal{C}_A|P^1 = P^2 = 0 \ , N = I\omega \ , 0\rangle = 0 \ , \tag{4.109}$$

then the zero-point energy of H_2 (4.96) is

$$\frac{1}{2}\sum_A \Lambda_A = \frac{1}{2}\mathrm{Tr}\{\Lambda\} \ , \tag{4.110}$$

where the matrix Λ is diagonal with the eigenvalues Λ_A.

In the commutative theory the zero-point energy contains the divergences even after subtraction of the vacuum energy. The finiteness of the soliton energy is recovered by starting from the renormalized form of the action (4.34), which induces the counter terms also contain the divergences [67].

Work in the specific form of the ϕ^6 potential (4.53),

$$V(\frac{1}{2}\{\bar{\phi},\phi\}) = m^2(\frac{1}{2}\{\bar{\phi},\phi\}) - bm^2g^2(\frac{1}{2}\{\bar{\phi},\phi\})^2 + cm^2g^4(\frac{1}{2}\{\bar{\phi},\phi\})^3 \ . \tag{4.111}$$

At the g^0 order, or the one-loop order, the general formula for the soliton energy is

$$E_{\text{soliton}} \equiv \langle P^1 = P^2 = 0 \ , N = I\omega \ , 0| \ H \ |P^1 = P^2 = 0 \ , N = I\omega \ , 0\rangle - E_{\text{vac}} \tag{4.112}$$

$$= H_0 + \frac{1}{2}\mathrm{Tr}\{\Lambda\} - E_{\text{vac}} + \frac{1}{2}\delta m^2 \int \sigma^2 - bm^2\delta g^2_{(4)} \int \sigma^4 \ , \tag{4.113}$$

where δm^2 and $\delta g^2_{(4)}$ are the counter terms for the mass and the ϕ^4 coupling respectively. The ϕ^6 coupling does not receive loop corrections. The ϕ^4 coupling terms yield the right coefficients and can be renormalized [32].

The loop integration in the NC field theory generally contains phase factors which yield the interesting UV-IR phenomenon upon renormalization [22]. In the following we evaluate the quantum correction from the zero-point energy of H_2 in Eqn. (4.110) and show that it contains the same phase factors as those appear in the counter terms δm^2 and $\delta g^2_{(4)}$.

We start by arguing that only $1/2\mathrm{Tr}\{\Omega\}$ is needed in evaluating the leading divergence. In Eqn. (4.107), it is easy to see ${\mathcal{Q}^A}^T \Xi \mathcal{Q}^A$ is finite,

$$\begin{aligned} {\mathcal{Q}^A}^T \Xi \mathcal{Q}^A &= (\int g_1 f_a \xi_{Ra})^2 + (\int f_i g_{\dot{a}} \xi_{I\dot{a}})^2 \\ &\leq [\int g_1^2 + \int (f_a \xi_{Ra})^2]^2 + [\int f_1^2 + \int (g_{\dot{a}} \xi_{I\dot{a}})^2]^2 + [\int f_2^2 + \int (g_{\dot{a}} \xi_{I\dot{a}})^2]^2 \\ &= (1 + \xi_{Ra}^2)^2 + 2(1 + \xi_{I\dot{a}}^2)^2 \ \leq \ 12 \ . \end{aligned} \tag{4.114}$$

As we will see that the eigenvalues Ω_{Ra} and $\Omega_{I\dot{a}}$ behave like $\sqrt{k^2 - \omega^2 + m^2}$ at very large k. The leading divergence of $\mathrm{Tr}\{\Lambda\}$ will be determined by $\mathrm{Tr}\{\Omega\}$.

Ω_{Ra} and $\Omega_{I\dot{a}}$, eigenfrequencies of $\mathcal{V}_2(q)$ in Eqn. (4.95), satisfy the linear equations,

$$(-\partial_i^2 - \omega^2 + m^2)\chi_I - \frac{1}{2} b m^2 g^2 \{\sigma^2, \chi_I\} + \frac{3}{8} c m^2 g^4 \{\sigma^4, \chi_I\} = \Omega_{I\dot{a}}^2 \chi_I \ , \tag{4.115}$$

$$\begin{aligned} &(-\partial_i^2 - \omega^2 + m^2)\chi_R - b m^2 g^2 (\{\sigma^2, \chi_R\} + \sigma \chi_R \sigma) + \\ &\qquad \frac{3}{4} c m^2 g^4 (\{\sigma^4, \chi_R\} + \sigma^2 \chi_R \sigma^2 + \sigma\{\sigma, \chi_R\}\sigma) = \Omega_{Ra}^2 \chi_R \ . \end{aligned} \tag{4.116}$$

The above equations are just time independent Schrödinger equations. In particular phase shifts from the central potential have been used in calculating the soliton energy correction [68]. The basic idea is that in the central potential for each partial wave, the difference of the density of the states between the scattered wave and the free wave is related to the derivative of the phase shift,

$$\rho_l(k) - \rho_0(k) = \frac{1}{\pi} \frac{d\delta_l(k)}{dk} \ , \tag{4.117}$$

where l goes from $-\infty$ to ∞. The finiteness of the particle number, $N = \omega \int \sigma^2$, determines that $\sigma \to o$ as $r \to \infty$. Therefore the NC potential in Eqn. (4.115) and (4.116) is radial symmetric and vanishes at ∞. For the most general potential term $\mathcal{W}_F(r) \star \chi \star \mathcal{W}_B(r)$,

$$[\mathcal{W}_F(r) \star \chi \star \mathcal{W}_B(r) \ , L] = \mathcal{W}_F(r) \star [\chi \ , L] \star \mathcal{W}_B(r) \ , \tag{4.118}$$

where $L = -i\epsilon^{ij}x^i\partial^j$ is the angular momentum. The star product is made explicit here and in the rest of the section. This formula can be easily proved in the Weyl transforms of the fields. Going to the momentum space, one can generalize the result in [15] and show that

$$\mathcal{W}_F(x) \star \chi(x) \star \mathcal{W}_B(x) = \int \frac{d^2p_f}{(2\pi)^2}\frac{d^2p_b}{(2\pi)^2}\widetilde{\mathcal{W}}_F(p_f)\widetilde{\mathcal{W}}_B(p_b)e^{i\mathbf{p}_f(\mathbf{x}+\frac{i\theta}{2}\tilde{\partial})}e^{i\mathbf{p}_b(\mathbf{x}-\frac{i\theta}{2}\tilde{\partial})}\chi(x) \ , \tag{4.119}$$

where $\tilde{\partial}^i \equiv \epsilon^{ij}\partial^j$.

Using Eqn. (4.117), consider only the leading divergence, we have [69],

$$\begin{aligned} \frac{1}{2}\mathrm{Tr}\{\Lambda\} - E_{\mathrm{vac}} &\sim \frac{1}{2}\mathrm{Tr}\{\Omega\} - E_{\mathrm{vac}} \\ &\sim \frac{1}{2\pi}\int d\sqrt{k^2+m^2}\sum_l [\delta_{Il}(k) + \delta_{Rl}(k)] \ , \end{aligned} \tag{4.120}$$

where $\delta_{Il}(k)$ and $\delta_{Rl}(k)$ are the phase shifts for χ_I and χ_R. The sum of the phase shifts can be evaluated through Born approximation. In the commutative case, this leads to the cancelation of the tadpole diagram [68].

Eqn. (4.115) and (4.116) have the Jost solution form for the lth partial wave at large r ,

$$\chi \sim h_l^*(kr) + e^{2i\delta_l}h_l(kr) \ . \tag{4.121}$$

Considering the asymptotic ($r \to \infty$) behavior of the solution, the standard procedure [70] leads to the scattering amplitude,

$$f(\mathbf{k}', \mathbf{k}) = f(\phi) = \sum_l f_l(k)e^{il\phi} = \frac{1}{\sqrt{k}}\sum_l e^{i\delta_l}\sin\delta_l e^{il\phi} \ , \tag{4.122}$$

where $k' = k$ and ϕ is the angle between $\mathbf{k}'$ and $\mathbf{k}$. At large l, or $\delta_l \approx 0$, we can see

$$\sum_l \delta_l \approx \sqrt{k}f(\phi = 0) \tag{4.123}$$

$f(\mathbf{k}', \mathbf{k})$ can also be calculated through Born approximation, replacing χ by $e^{-i\mathbf{kx}}$ in the potential form (4.119),

$$\begin{aligned} f(\mathbf{k}', \mathbf{k}) &= -\frac{1}{4\sqrt{k}} \int d^2x e^{-i\mathbf{k}'\mathbf{x}} \sum_{\mathrm{i}} \mathcal{W}_F^{(\mathrm{i})} \star e^{-i\mathbf{kx}} \star \mathcal{W}_B^{(\mathrm{i})} \\ &= -\frac{1}{4\sqrt{k}} \int d^2x e^{-i(\mathbf{k}'-\mathbf{k})\mathbf{x}} \sum_{\mathrm{i}} \mathcal{W}_F^{(\mathrm{i})}(x - \frac{\theta}{2}\tilde{k}) \mathcal{W}_B^{(\mathrm{i})}(x + \frac{\theta}{2}\tilde{k}) \end{aligned} \quad (4.124)$$

where i labels the potential terms in Eqn. (4.115) and (4.116), and $\tilde{k}^i \equiv \epsilon^{ij}k^j$. Therefore

$$\begin{aligned} \sum_l \delta_l &= -\frac{1}{4} \int d^2x \sum_{\mathrm{i}} \mathcal{W}_F^{(\mathrm{i})}(x - \frac{\theta}{2}\tilde{k}) \mathcal{W}_B^{(\mathrm{i})}(x + \frac{\theta}{2}\tilde{k}) \\ &= -\frac{1}{4} \int \frac{d^2p}{(2\pi)^2} \sum_{\mathrm{i}} \widetilde{\mathcal{W}}_F^{(\mathrm{i})}(p) \widetilde{\mathcal{W}}_B^{(\mathrm{i})}(-p) e^{-i\theta \mathbf{p}\tilde{\mathbf{k}}} \ . \end{aligned} \quad (4.125)$$

The right hand side only depends on the magnitude k due to the central potential $\mathcal{W}(r)$.

Now we are ready to evaluate Eqn. (4.120),

$$\begin{aligned} & \frac{1}{2\pi} \int d\sqrt{k^2 + m^2} \sum_l [\delta_{Il}(k) + \delta_{Rl}(k)] \\ = & -\frac{1}{8\pi} \int d\sqrt{k^2 + m^2} \int \frac{d^2p}{(2\pi)^2} \sum_{\mathrm{i}} \widetilde{\mathcal{W}}_F^{(\mathrm{i})}(p) \widetilde{\mathcal{W}}_B^{(\mathrm{i})}(-p) e^{-i\theta \mathbf{p}\tilde{\mathbf{k}}} \\ = & -\frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{e^{-i\theta \mathbf{p}\tilde{\mathbf{k}}}}{2\sqrt{k^2 + m^2}} \sum_{\mathrm{i}} \widetilde{\mathcal{W}}_F^{(\mathrm{i})}(p) \widetilde{\mathcal{W}}_B^{(\mathrm{i})}(-p) \ . \end{aligned} \quad (4.126)$$

The integration over k is exactly part of the tadpole diagram belongs to δm^2 [32], and it contains the UV/IR divergence ($\Lambda \to \infty$ and $p \to 0$) evaluated with the cutoff Λ [22],

$$\int \frac{d^2k}{(2\pi)^2} \frac{e^{-i\theta \mathbf{p}\tilde{\mathbf{k}}}}{2\sqrt{k^2 + m^2}} = \frac{2}{(4\pi)^{3/2}} (m\Lambda_{eff})^{1/2} K_{\frac{1}{2}} \left(\frac{2m}{\Lambda_{eff}} \right) = \frac{\Lambda_{eff}}{8\pi} + \mathcal{O}(1) \ , \quad (4.127)$$

where $\Lambda_{eff} \equiv (\theta^2 p^2/4 + 1/\Lambda^2)^{-1/2}$. Notice that the above UV/IR divergence from Eqn. (4.126) occurs only when both $\mathcal{W}_F$ and $\mathcal{W}_B$ exist. In other words, only the terms $\sigma\chi_R\sigma$, $\sigma^2\chi_R\sigma^2$ and $\sigma\{\sigma, \chi_R\}\sigma$ in Eqn. (4.115) and (4.116) yield UV/IR divergence. All other potential terms only give the normal UV divergence where the phase factor is absent. Since the counter term δm^2 and δg^2 do not include UV/IR divergence, we are certain that the Q-ball energy correction includes UV/IR divergence. Cancelation of the UV divergences is not obvious because the exact value of the eigenfrequencies Λ_A is unknown.

4.5 Finite θ and Noncommutative GMS Solitons

The above calculation assumes that the NC parameter θ is sufficiently small so that the NC potential will generate the Jost solution form as in the commutative case. Let us consider the effects of the NC potential (4.119) in the case that θ is not small.

Since

$$[x^i \pm \frac{i\theta}{2}\tilde{\partial}^i, x^j \pm \frac{i\theta}{2}\tilde{\partial}^j] = \pm i\theta\epsilon^{ij} \ , \tag{4.128}$$

the effective scattering potential for the NC interaction $\mathcal{W}_F(x) \star \chi \star \mathcal{W}_B(x)$ (4.119) is just

$$\widehat{\mathcal{W}}_F(x + \frac{i\theta}{2}\tilde{\partial})\widehat{\mathcal{W}}_B(x - \frac{i\theta}{2}\tilde{\partial}) \ , \tag{4.129}$$

multiplication of the Weyl transforms of $\mathcal{W}_F(x)$ and $\mathcal{W}_B(x)$. Notice $\widehat{\mathcal{W}}_F$ and $\widehat{\mathcal{W}}_B$ commute since $[x^i + i\theta/2\tilde{\partial}^i, x^j - i\theta/2\tilde{\partial}^j] = 0$.

Now considering

$$\widehat{\mathcal{W}}_F(x + \frac{i\theta}{2}\tilde{\partial}) = \int \frac{d^2p_f}{(2\pi)^2}\widetilde{\mathcal{W}}_F(p_f)e^{i\mathbf{p}_f(\mathbf{x}+\frac{i\theta}{2}\tilde{\partial})} \ , \tag{4.130}$$

the noncommutativity,

$$[p_f^1(x^1 + \frac{i\theta}{2}\tilde{\partial}^1) \ , p_f^2(x^2 + \frac{i\theta}{2}\tilde{\partial}^2)] = i\theta p_f^1 p_f^2 \ , \tag{4.131}$$

can be suppressed even if θ is not small, as long as $\mathcal{W}(x)$ is smooth enough or $\widetilde{\mathcal{W}}(p_f) \to 0$ at large p_f. Notice small p_f is also the IR limit we discussed in the last section. Under this assumption we can write

$$\widehat{\mathcal{W}}_F(x + \frac{i\theta}{2}\tilde{\partial}) \approx \mathcal{W}_F(x + \frac{i\theta}{2}\tilde{\partial}) = \mathcal{W}_F(|\mathbf{x} + \frac{i\theta}{2}\tilde{\partial}|^2) = \mathcal{W}_F(r^2 - \frac{\theta}{2}L + \partial_i^2) \ . \quad (4.132)$$

Acting on the field $\chi(x) = u_l(kr)e^{il\phi}$, the effective potential becomes $\mathcal{W}_F(r^2 - \theta l/2 + \partial_i^2)$. Similar calculation applied to $\widehat{\mathcal{W}}_B(x)$ yields $\mathcal{W}_B(r^2 + \theta l/2 + \partial_i^2)$. Therefore at large k and large r, we can treat the scattering potential perturbatively as in the commutative case. The phase shift evaluation of the energy of the soliton could still apply provided that $\mathcal{W}$ or the soliton solution σ are smooth enough.

NC GMS soliton only exists at finite θ. Based on the above arguments, we can still evaluate its quantum corrections with the phase shift method in the last section.

Quantization of GMS soliton and Q-ball share a lot of similarities. To get the GMS soliton theory, we make the replacements ($\phi\ ,\bar{\phi} \rightarrow 1/\sqrt{2}\Phi$) in the previous complex scalar field theory (4.34). With the potential (4.111), the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\Phi)^2 - V(\Phi) = -\frac{1}{2}(\partial_\mu\Phi)^2 - \frac{1}{2}m^2\Phi^2 + \frac{1}{4}bm^2g^2\Phi^4 - \frac{1}{8}cm^2g^4\Phi^6 \ , \quad (4.133)$$

where Φ is multiplied by the star product. The renormalizability of the theory has been proved [30]. Let $\omega = 0$ because there is no conserved charge Q (4.39) in the theory. When $b^2 - 4c < 0$, the potential in (4.133) has a local minimum at $\sqrt{b}g/2$ besides the global minimum at the origin, and the GMS soliton solution σ exists. Replace the expansion (4.65) by $\Phi = \sigma + \chi$, where $\chi = \chi_R$ is real. As a result, the meson degrees of freedom are only χ_R or χ. Upon quantization, the soliton energy is still given by Eqn. (4.113). Since $\omega = 0$ and $\Lambda = \Omega$ are the exact eigenfrequencies of H_2 (4.94), we have

$$(-\partial_i^2 + m^2)\chi_R - bm^2g^2(\{\sigma^2, \chi_R\} + \sigma\chi_R\sigma) \\ + \frac{3}{4}cm^2g^4(\{\sigma^4, \chi_R\} + \sigma^2\chi_R\sigma^2 + \sigma\{\sigma, \chi_R\}\sigma) = \Lambda_a^2\chi_R \ . \quad (4.134)$$

Eqn. (4.120) describes the exact ultraviolet divergences in $1/2\mathrm{Tr}\{\Lambda\} - E_{\mathrm{vac}}$. Therefore we are able to check the cancelation of the divergences in Eqn. (4.113). As mentioned in the previous section, in the above equation, the terms $\sigma\chi_R\sigma$, $\sigma^2\chi_R\sigma^2$ and $\sigma\{\sigma, \chi_R\}\sigma$ yield UV/IR divergences, while the rest terms yield UV divergence. A critical observation is that those terms yield UV/IR divergence have one to one correspondence with the contractions of the fields yield Nonplanar Feynman diagrams [22], and those terms yield UV divergence correspond exactly to the planar diagrams. We can just spare the details of counting the divergences. Since the counter terms δm^2 and $\delta g^2_{(4)}$ cancel exactly the UV divergence part, we conclude that the soliton energy (4.113) is UV finite, but includes all the UV/IR divergences.

4.6 Conclusion and Discussion

In this chapter we discussed the quantization of NC solitons in (2 + 1) dimensional scalar field theory. In particular, classical solutions and quantization of the NC Q-balls at very small θ are investigated in detail. Classically NC Q-balls reduce to the commutative Q-ball as θ goes to zero. Quantum mechanically, because loop integrations in the NC field theory have different ultraviolet structure from those in the commutative theory, i. e. UV/IR mixing, quantum corrections to the NC soliton energy necessarily include the UV/IR divergent terms which cannot be renormalized away. The existence of such terms in the energy is demonstrated through the phase shift summation. The same method is further generalized to NC GMS solitons which exist only at finite θ. In the small momentum limit, or for the sufficiently smooth soliton solutions, divergence structure of the soliton energy can be calculated exactly. In this case the energy is found to contain no UV divergence but all the UV/IR divergences. Quantum corrections to the NC soliton energy have also been calculated but at very large θ [7], where no UV/IR divergence is found. We believe that is because at large θ, the noncommutativity (4.131) is not small and cannot be ignored, and the potential term is the dominant term instead of a perturbative one. In this case the phase shift sum is not a good approximation to the energy correction.

An interpretation to the UV/IR divergence [23] is that because new light degrees of freedom are introduced in the Wilsonian effective action. UV/IR divergence can be reproduced by integrating out those new degrees of freedom, which are then interpreted as closed string modes with channel duality. Future research direction is to consider the NC solitons in the gauge theory, where they are interpreted as D-branes [53, 54] and D-brane action is properly recovered. One expects to gain better understanding of interactions between D-brane and closed strings through quantization of NC solitons.

CHAPTER 5
SOLUTION OF KOSTANT EQUATION

In chapter 3 theory of superPoincaré algebra and its representations is reviewed. Classification of irreducible unitary representations of Poincaré and superPoincaré algebra reveals physical spectrum of the corresponding theory. M-theory, as unification of string theory, is conjectured to be 11-dimensional theory [12, 13], whose low energy limit is 11-dimensional supergravity. Little group of 10 dimensional Poincaré group is $SO(8)$, which classifies the spectrum of 10 dimensional superstring theory. Triality symmetry of $SO(8)$ leads to marvelous cancelations between Bosonic and Fermionic contributions, which renders the theory to be UV finite. Little group of 11-dimensional superPoincaré algebra, $SO(9)$ does not have such symmetry, and $SO(9)$ is nonrenormalizable in high loop order [71]. It is found that some irreps of $SO(9)$ naturally group together into an infinite tower of triplets [72], the lowest of which includes the spectrum of 11 dimensional supergravity. This suggests that the tower of triplets might be able to describe certain limit of M-theory. A possible candidate is the infinite Regge slope limit, or zero tension limit of string theory, where one expects all states to become massless, with an infinite number of states for each spin.

A mathematical understanding of the triplets has been given [73]. They arise for embeddings where both group and subgroup have the same rank. $SO(9)$ is a subgroup of $F4$ with the same rank, and the quotient space $F4/SO(9)$ has Euler number three giving a triplet of $SO(9)$ to every irrep of $F4$. There exist other cases the triplets arise the lowest of which describe $N = 8$ supergravity, $N = 4$ Yang-Mills and $N = 2$ hypermultiplet [74–76]. All these Euler triplets arise as solutions of Kostant's equation [43], which is a Dirac-like equation on the coset.

This chapter focus on differential form of the representation of Kostant's equation and its solutions, which is the beginning step for the Lagrangian construction. After a review of Euler triplet in a toy coset space, coset $SU(3)/SU(2)\times U(1)$, Euler triplet solutions in coset $F4/SO(9)$ are discussed in detail. In particular, variable representations of 11-dimensional superPoincaré algebra, and $F4$ and $SO(9)$ algebra, are worked out explicitly, as well as representations of Kostant equation and its triplet solutions.

5.1 Euler Triplet for $SU(3)/SU(2)\times U(1)$

As a learning example start with a detailed analysis of the Euler triplets associated with the coset $SU(3)/SU(2)\times U(1)$. There is an infinity of Euler triplets which are solutions of Kostant's equation associated with this coset. The most trivial solution describes the light-cone degrees of freedom of the $N=2$ in four dimensions, when the $U(1)$ is interpreted as helicity.

5.1.1 The $N=2$ Hypermultiplet in 4 Dimensions

The massless $N=2$ scalar hypermultiplet contains two Weyl spinors and two complex scalar fields, on which the $N=2$ SuperPoincaré algebra is realized. Introduce the light-cone Hamiltonian

$$P^- = \frac{p\overline{p}}{p^+}\ , \tag{5.1}$$

where $p=\frac{1}{\sqrt{2}}(p^1+ip^2)$. The front-form supersymmetry generators satisfy the anticommutation relations

$$\begin{aligned}
\{\mathcal{Q}^m_+,\overline{\mathcal{Q}}^n_+\} &= -2\delta^{mn}p^+\ ,\\
\{\mathcal{Q}^m_-,\overline{\mathcal{Q}}^n_-\} &= -2\delta^{mn}\frac{p\overline{p}}{p^+}\ , \qquad m,n=1,2\ ,\\
\{\mathcal{Q}^m_+,\overline{\mathcal{Q}}^n_-\} &= -2p\delta^{mn}\ .
\end{aligned} \tag{5.2}$$

The kinematic supersymmetries are expressed as

$$\mathcal{Q}_+^m = -\frac{\partial}{\partial\overline{\theta}^m} - \theta_m p^+ \ , \qquad \overline{\mathcal{Q}}_+^m = \frac{\partial}{\partial\theta^m} + \overline{\theta}_m p^+ \ , \tag{5.3}$$

while the kinematic Lorentz generators are given by

$$\begin{aligned} M^{12} &= i(x\overline{p} - \overline{x}p) + \frac{1}{2}\theta_m\frac{\partial}{\partial\theta_m} - \frac{1}{2}\overline{\theta}^m\frac{\partial}{\partial\overline{\theta}^m} \ , \\ M^{+-} &= -x^-p^+ - \frac{i}{2}\theta_m\frac{\partial}{\partial\theta_m} - \frac{i}{2}\overline{\theta}^m\frac{\partial}{\partial\overline{\theta}^m} \ , \\ M^+ &\equiv \frac{1}{\sqrt{2}}(M^{+1} + iM^{+2}) = -xp^+ \ , \qquad \overline{M}^+ = -\overline{x}p^+ \ , \end{aligned} \tag{5.4}$$

where $x = \frac{1}{\sqrt{2}}(x^1 + ix^2)$, and where the two complex Grassman variables satisfy the anticommutation relations

$$\begin{aligned} \{\theta_m, \frac{\partial}{\partial\theta_n}\} &= \{\overline{\theta}^m, \frac{\partial}{\partial\overline{\theta}^n}\} = \delta^{mn} \ , \\ \{\theta_m, \frac{\partial}{\partial\overline{\theta}^n}\} &= \{\overline{\theta}^m, \frac{\partial}{\partial\theta_n}\} = 0 \ . \end{aligned}$$

The (free) Hamiltonian-like supersymmetry generators are simply

$$\mathcal{Q}_-^m = \frac{\overline{p}}{p^+}\mathcal{Q}_+^m \ , \qquad \overline{\mathcal{Q}}_-^m = \frac{p}{p^+}\overline{\mathcal{Q}}_+^m \ , \tag{5.5}$$

and the light-cone boosts are given by

$$\begin{aligned} M^- &= x^-p - \frac{1}{2}\{x, P^-\} + i\frac{p}{p^+}\theta_m\frac{\partial}{\partial\theta_m} \ , \\ \overline{M}^- &= x^-\overline{p} - \frac{1}{2}\{\overline{x}, P^-\} + i\frac{\overline{p}}{p^+}\overline{\theta}^m\frac{\partial}{\partial\overline{\theta}^m} \ . \end{aligned} \tag{5.6}$$

This representation of the superPoincaré algebra is reducible, as it can be seen to act on reducible superfields $\Phi(x^-, x^i, \theta_m, \overline{\theta}^m)$, because the operators

$$\mathcal{D}_+^m = \frac{\partial}{\partial\overline{\theta}^m} - \theta_m p^+ \ , \tag{5.7}$$

anticommute with the supersymmetry generators. As a result, one can achieve irreducibility by acting on superfields for which

$$\mathcal{D}_+^m\ \Phi = [\frac{\partial}{\partial\overline{\theta}^m} - \theta_m p^+]\Phi = 0\ , \tag{5.8}$$

solved by the chiral superfield

$$\Phi(y^-, x^i, \theta_m) = \psi_0(y^-, x^i) + \theta_m\psi^m(y^-, x^i) + \theta_1\theta_2\psi^{12}(y^-, x^i)\ . \tag{5.9}$$

The field entries of the scalar hypermultiplet now depend on the combination

$$y^- = x^- - i\theta_m\overline{\theta}^m\ , \tag{5.10}$$

and the transverse variables. Acting on this chiral superfield, the constraint is equivalent to requiring that

$$\mathcal{Q}_+^m \approx -2p^+\theta_m\ , \qquad \overline{\mathcal{Q}}_+^m \approx \frac{\partial}{\partial\theta_m}\ , \tag{5.11}$$

where the derivative is meant to act only on the naked θ_m's, not on those hiding in y^-. This light-cone representation is well-known, but we repeat it here to set our conventions and notations.

5.1.2 Coset Construction

Let T^A , $A = 1, 2, \ldots 8$, denote the $SU(3)$ generators. Its $SU(2) \times U(1)$ subalgebra is generated by T^i, $i = 1, 2, 3$, and T^8. Introduce Dirac matrices over the coset

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}\ ,$$

for $a, b = 4, 5, 6, 7$, to define the Kostant equation over the coset $SU(3)/SU(2) \times U(1)$ as

$$\not{\mathcal{K}}\,\Psi\ =\ \sum_{a=4,5,6,7}\gamma^a T_a\,\Psi\ =\ 0\ .$$

The Kostant operator commutes with the $SU(2) \times U(1)$ generators

$$L_i = T_i + S_i \ , \ i = 1, 2, 3 \ ; \qquad L_8 = T_8 + S_8 \ , \tag{5.12}$$

sums of the $SU(3)$ generators and of the "spin" part, expressed in terms of the γ matrices as

$$S_j = -\frac{i}{4} f_{jab} \gamma^{ab} \ , \qquad S_8 = -\frac{i}{4} f_{8ab} \gamma^{ab} \ , \tag{5.13}$$

where $\gamma^{ab} = \gamma^a \gamma^b$, $a \neq b$, and f_{jab} , f_{8ab} are structure functions of $SU(3)$.

The Kostant equation has an infinite number of solutions which come in groups of three representations of $SU(2) \times U(1)$, called Euler triplets. For each representation of $SU(3)$, there is a unique Euler triplet, each given by three representations

$$\{a_1, a_2\} \ \equiv \ [a_2]_{-\frac{2a_1+a_2+3}{6}} \oplus [a_1 + a_2 + 1]_{\frac{a_1-a_2}{6}} \oplus [a_1]_{\frac{2a_2+a_1+3}{6}} \ ,$$

where a_1, a_2 are the Dynkin labels of the associated $SU(3)$ representation. Here, $[a]$ stands for the $a = 2j$ representation of $SU(2)$, and the subscript denotes the $U(1)$ charge. The Euler triplet corresponding to $a_1 = a_2 = 0$,

$$\{0, 0\} \ = \ [0]_{-\frac{1}{2}} \oplus [1]_0 \oplus [0]_{\frac{1}{2}} \ ,$$

describes the degrees of freedom of the $N = 2$ supermultiplet, where the properly normalized $U(1)$ is interpreted as the helicity of the four-dimensional Poincaré algebra.

5.1.3 Grassmann Numbers and Dirac Matrices

In order to use the superfield technique we will identify the spin part of the $U(1)$ generator S_8 with the spin part in Eqn. (5.4) taking the condition (5.8) into account. This will mean that we write also S_i in terms of the $\theta's$. An appropriate representation is then

$$\gamma^4 + i\gamma^5 \;=\; i\sqrt{\frac{2}{p^+}}\mathcal{Q}^1_+ \;, \qquad \gamma^4 - i\gamma^5 = i\sqrt{\frac{2}{p^+}}\overline{\mathcal{Q}}^1_+ \tag{5.14}$$

$$\gamma^6 + i\gamma^7 \;=\; i\sqrt{\frac{2}{p^+}}\mathcal{Q}^2_+ \;, \qquad \gamma^6 - i\gamma^7 = i\sqrt{\frac{2}{p^+}}\overline{\mathcal{Q}}^2_+ \;, \tag{5.15}$$

in terms of the kinematic $N = 2$ light-cone supersymmetry generators defined in the previous section. We can check that S_8 indeed agrees with the spin part of Eqn. (5.4) (after proper normalization). As the Kostant operator anticommutes with the constraint operators

$$\{\; \not{\mathcal{K}},\; \mathcal{D}^m_+\} \;=\; 0 \;, \tag{5.16}$$

its solutions can be written as chiral superfields, on which the γ's become

$$\gamma^4 + i\gamma^5 \;=\; -2i\sqrt{2p^+}\;\theta_1 \;, \qquad \gamma^4 - i\gamma^5 = i\sqrt{\frac{2}{p^+}}\;\frac{\partial}{\partial\theta_1} \tag{5.17}$$

$$\gamma^6 + i\gamma^7 \;=\; -2i\sqrt{2p^+}\;\theta_2 \;, \qquad \gamma^6 - i\gamma^7 = i\sqrt{\frac{2}{p^+}}\;\frac{\partial}{\partial\theta_2} \;, \tag{5.18}$$

The complete "spin" parts of the $SU(2) \times U(1)$ generators, expressed in terms of Grassmann variables, do not depend on p^+,

$$\begin{aligned} S_1 &= \frac{1}{2}(\theta_1\frac{\partial}{\partial\theta_2} + \theta_2\frac{\partial}{\partial\theta_1}) \;, \qquad S_2 = -\frac{i}{2}(\theta_1\frac{\partial}{\partial\theta_2} - \theta_2\frac{\partial}{\partial\theta_1}) \\ S_3 &= \frac{1}{2}(\theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2}) \;, \qquad S_8 = \frac{\sqrt{3}}{2}(\theta_1\frac{\partial}{\partial\theta_1} + \theta_2\frac{\partial}{\partial\theta_2} - 1) \;. \end{aligned} \tag{5.19}$$

Using Grassmann properties, the $SU(2)$ Casimir operator can be written as

$$\vec{S}^2 = \frac{3}{4}(\theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2})^2 \;; \tag{5.20}$$

it has only two eigenvalues, 3/4 and zero. These $SU(2)$ generators obey a simple algebra

$$S_i\,S_j = \frac{1}{3}\vec{S}\cdot\vec{S}\,\delta_{ij} + \frac{i}{2}\epsilon_{ijk}S_k\ . \tag{5.21}$$

The helicity, identified with S_8 up to a normalizing factor of $\sqrt{3}$, leads to half-integer helicity values on the Grassmann-odd components of the (constant) superfield representing the hypermultiplet.

5.1.4 Solutions of Kostant's Equation

Consider now Kostant's equation over $SU(3)/SU(2)\times U(1)$. It is given by

$$\not{\mathcal{K}}\,\Psi \;=\; \sum_{a=4,5,6,7}\gamma^a T_a\,\Psi \;=\; 0\ . \tag{5.22}$$

Schwinger's celebrated representation of $SU(2)$ generators of in terms of one doublet of harmonic oscillators has been extended to other Lie algebras [77]. The generalization involves several sets of harmonic oscillators, each spanning the fundamental representations. Thus $SU(3)$ is generated by two sets of triplet harmonic oscillators, one transforming as a triplet the other as an antitriplet. Its generators are given by

$$T_1 + iT_2 = z_1\partial_2 - \overline{z}_2\partial_1\ , \qquad T_1 - iT_2 = z_2\partial_1 - \overline{z}_1\partial_2\ ,$$

$$T_4 + iT_5 = z_1\partial_3 - \overline{z}_3\partial_1\ , \qquad T_4 - iT_5 = z_3\partial_1 - \overline{z}_1\partial_3\ ,$$

$$T_6 + iT_7 = z_2\partial_3 - \overline{z}_3\partial_2\ , \qquad T_6 - iT_7 = z_3\partial_2 - \overline{z}_2\partial_3\ ,$$

and

$$T_3 = \frac{1}{2}(z_1\partial_1 - z_2\partial_2 - \overline{z}_1\partial_1 + \overline{z}_2\partial_2)\ ,$$

$$T_8 = \frac{1}{2\sqrt{3}}(z_1\partial_1 + z_2\partial_2 - \overline{z}_1\partial_1 - \overline{z}_2\partial_2 - 2z_3\partial_3 + 2\overline{z}_3\partial_3)\ ,$$

where we have defined

$$\partial_1 \equiv \frac{\partial}{\partial z_1}\ , \qquad \partial_1 \equiv \frac{\partial}{\partial \overline{z}_1}\ ,\ \text{etc.}\ .$$

The highest-weight states of each $SU(3)$ representation are holomorphic polynomials of the form

$$z_1^{a_1}\overline{z}_3^{a_2} \ ,$$

where a_1, a_2 are its Dynkin indices: all representations of $SU(3)$ are homogeneous holomorphic polynomials.

Now expand the Kostant equation (5.22) with the Dirac matrices in terms of Grassmann variables yields two independent pairs of equations

$$(T_4 + iT_5)\psi_1 + (T_6 + iT_7)\psi_2 \ = \ 0 \ ; \qquad (T_4 - iT_5)\psi_2 - (T_6 - iT_7)\psi_1 \ = \ 0 \ ,$$

and

$$(T_4 - iT_5)\psi_0 - (T_6 + iT_7)\psi_{12} \ = \ 0 \ ; \qquad (T_6 - iT_7)\psi_0 + (T_4 + iT_5)\psi_{12} \ = \ 0 \ ,$$

that is

$$(z_1\partial_3 - \overline{z}_3\partial_1)\psi_1 + (z_2\partial_3 - \overline{z}_3\partial_2)\psi_2 \ = \ 0 \ ; \qquad (z_3\partial_1 - \overline{z}_1\partial_3)\psi_2 - (z_3\partial_2 - \overline{z}_2\partial_3)\psi_1 \ = \ 0 \ ,$$

$$(z_3\partial_1 - \overline{z}_1\partial_3)\psi_0 - (z_2\partial_3 - \overline{z}_3\partial_2)\psi_{12} \ = \ 0 \ ; \qquad (z_3\partial_2 - \overline{z}_2\partial_3)\psi_0 + (z_1\partial_3 - \overline{z}_3\partial_1)\psi_{12} \ = \ 0 \ .$$

The homogeneity operators

$$D = z_1\partial_1 + z_2\partial_2 + z_3\partial_3 \ , \qquad \overline{D} = \overline{z}_1\partial_1 + \overline{z}_2\partial_2 + \overline{z}_3\partial_3$$

commute with $\not{K}$, allowing the solutions of Kostant equation to be arranged in terms of homogeneous polynomials, on which a_1 is the eigenvalue of D and a_2 that of $\overline{D}$. The solutions can also be labeled in terms of the $SU(2) \times U(1)$ generated by the operators

$$L_i = T_i + S_i \ , \ i = 1, 2, 3 \ ; \qquad L_8 = T_8 + S_8 \ .$$

The solutions for each triplet, are easily written for the highest weight states of each representation,

$$\begin{aligned}\Phi &= z_3^{a_1}\ \overline{z}_2^{a_2} && \text{labels}\ \ [a_2]_{-\frac{2a_1+a_2+3}{6}}\ ,\\ &+\ \theta_1\ z_1^{a_1}\ \overline{z}_2^{a_2} && \text{labels}\ \ [a_1+a_2+1]_{\frac{a_1-a_2}{6}}\ ,\\ &+\ \theta_1\theta_2\ z_1^{a_1}\ \overline{z}_3^{a_2}\ , && \text{labels}\ \ [a_1]_{\frac{2a_2+a_1+3}{6}}\ ,\end{aligned} \tag{5.23}$$

where $[\dots]$ are the $SU(2)$ Dynkin labels. All other states are obtained by repeated action of the lowering operator

$$L_1 - iL_2 = \theta_2\frac{\partial}{\partial\theta_1} + (z_2\partial_1 - \overline{z}_1\partial_2)\ ,$$

giving us all the states within each the Euler triplet.

5.2 Supergravity in Eleven Dimensions

The ultimate field theory without gravity is the finite $N = 4$ Super Yang-Mills theory in four dimensions. Eleven dimensional $N = 1$ Supergravity [78], the ultimate field theory with gravity, is not renormalizable; it does not stand on its own as a physical theory. However, the eleven-dimensional theory has been recently revived as the infrared limit of the presumably finite M-theory.

5.2.1 Superalgebra

$N = 1$ supergravity in eleven dimension is a local field theory that contains three massless fields, the familiar symmetric second-rank tensor, $h_{\mu\nu}$ which represents gravity, a three-form field $A_{\mu\nu\rho}$, and the Rarita-Schwinger spinor $\Psi_{\mu\,\alpha}$. From its Lagrangian, one can derive the expression for the super Poincaré algebra, which in the unitary transverse gauge assumes the particularly simple form in terms of the nine (16×16) γ_i matrices which form the Clifford algebra

$$\{\,\gamma^i, \gamma^j\,\}\ =\ 2\delta^{ij}\ , \qquad i, j\ = 1, \dots, 9\ .$$

Supersymmetry is generated by the sixteen real supercharges

$$\mathcal{Q}^a_\pm = \mathcal{Q}^{a\,*}_\pm \ ,$$

which satisfy

$$\{\mathcal{Q}^a_+, \mathcal{Q}^b_+\} \ = \ \sqrt{2}\,p^+\delta^{ab} \ , \qquad \{\mathcal{Q}^a_-, \mathcal{Q}^b_-\} \ = \ \frac{\vec{p}\cdot\vec{p}}{\sqrt{2}\,p^+}\delta^{ab} \ , \qquad \{\mathcal{Q}^a_+, \mathcal{Q}^b_-\} \ = \ -(\gamma_i)^{ab}p^i \ ,$$

and transform as Lorentz spinors

$$[M^{ij}, \mathcal{Q}^a_\pm] \ = \ \frac{i}{2}(\gamma^{ij}\mathcal{Q}_\pm)^a \ , \qquad [M^{+-}, \mathcal{Q}^a_\pm] \ = \ \pm\,\frac{i}{2}\mathcal{Q}^a_\pm \ , \tag{5.24}$$

$$[M^{\pm i}, \mathcal{Q}^a_\pm] \ = \ 0 \ , \qquad [M^{\pm i}, \mathcal{Q}^a_\mp] \ = \ \pm\,\frac{i}{\sqrt{2}}(\gamma^i\mathcal{Q}_\pm)^a \ . \tag{5.25}$$

A very simple representation of the 11-dimensional super-Poincaré generators can be constructed, in terms of sixteen anticommuting real χ's and their derivatives, which transform as the spinor of $SO(9)$, as

$$\mathcal{Q}^a_+ \ = \ \partial_{\chi^a} + \frac{1}{\sqrt{2}}p^+\chi^a \ , \qquad \mathcal{Q}^a_- \ = \ -\,\frac{p^i}{p^+}\,\left(\gamma^i\,\mathcal{Q}_+\right)^a \ ,$$

$$M^{ij} \ = \ x^ip^j - x^jp^i - \frac{i}{2}\chi\,\gamma^{ij}\partial_\chi \ , \tag{5.26}$$

$$M^{+-} \ = \ -x^-p^+ - \frac{i}{2}\chi\,\partial_\chi \ , \tag{5.27}$$

$$M^{+i} \ = \ -x^ip^+ \ , \tag{5.28}$$

$$M^{-i} \ = \ x^-p^i - \frac{1}{2}\{x^i, P^-\} + \frac{ip^j}{2p^+}\chi\gamma^i\gamma^j\partial_\chi \ . \tag{5.29}$$

The light-cone little group transformations are generated by

$$S^{ij} \ = \ -\,\frac{i}{2}\chi\,\gamma^{ij}\,\partial_\chi \ ,$$

which satisfy the $SO(9)$ Lie algebra. To construct its spectrum, we write the supercharges in terms of eight complex Grassmann variables

$$\theta^\alpha \ \equiv \ \frac{1}{\sqrt{2}}\left(\chi^\alpha + i\chi^{\alpha+8}\right) \ , \qquad \overline{\theta}^\alpha \ \equiv \ \frac{1}{\sqrt{2}}\left(\chi^\alpha - i\chi^{\alpha+8}\right) \ ,$$

and

$$\frac{\partial}{\partial\theta^\alpha} \;\equiv\; \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\chi^\alpha} - i\frac{\partial}{\partial\chi^{\alpha+8}}\right) , \qquad \frac{\partial}{\partial\overline{\theta}^\alpha} \;\equiv\; \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\chi^\alpha} + i\frac{\partial}{\partial\chi^{\alpha+8}}\right) ,$$

where $\alpha = 0, 1, 2, \ldots, 7$. The eight complex θ transform as the $(\mathbf{4}\,,\,\mathbf{2})$, and $\overline{\theta}$ as the $(\overline{\mathbf{4}}\,,\,\mathbf{2})$ of the $SU(4)\times SU(2)$ subgroup of $SO(9)$. The eight complex supercharges

$$\mathbf{Q}^\alpha_+ \;\equiv\; \frac{1}{\sqrt{2}}\left(\mathcal{Q}^\alpha_+ + i\mathcal{Q}^{\alpha+8}_+\right) \;=\; \frac{\partial}{\partial\overline{\theta}^\alpha} + \frac{1}{\sqrt{2}}p^+\theta^\alpha \;, \tag{5.30}$$

$$\mathbf{Q}^{\alpha\,\dagger}_+ \;\equiv\; \frac{1}{\sqrt{2}}\left(\mathcal{Q}^\alpha_+ - i\mathcal{Q}^{\alpha+8}_+\right) \;=\; \frac{\partial}{\partial\theta^\alpha} + \frac{1}{\sqrt{2}}p^+\overline{\theta}^\alpha \;, \tag{5.31}$$

satisfy

$$\{\,\mathbf{Q}^\alpha_+\,,\,\mathbf{Q}^{\beta\,\dagger}_+\,\} \;=\; \sqrt{2}\,p^+\,\delta^{\alpha\beta}\;.$$

To reduce the number of the grassmann variables, the usual way is to impose covariant derivatives as constraints,

$$\mathcal{D}^\alpha = \frac{\partial}{\partial\bar{\theta}^a} - \frac{1}{\sqrt{2}}p^+\theta^\alpha = 0, \tag{5.32}$$

since $\{\mathcal{D}^\alpha, \mathcal{Q}^a_+\} = 0$. It follows that $\partial/\partial\bar{\theta}^a$ can be replaced by $1/\sqrt{2}p^+\theta^\alpha$, when acting on the constraint fields depends only on θ^α . $\bar{\theta}$ can also be taken to be zero. Therefore,

$$\mathcal{Q}^\alpha_+ - i\mathcal{Q}^{\alpha+8}_+ \;=\; \sqrt{2}\frac{\partial}{\partial\theta^\alpha}, \tag{5.33}$$

$$\mathcal{Q}^\alpha_+ + i\mathcal{Q}^{\alpha+8}_+ \;=\; 2p^+\theta^\alpha \;. \tag{5.34}$$

This gives,

$$\mathcal{Q}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} \partial_\alpha + \sqrt{2}\theta^\alpha p^+ \\ i(\partial_\alpha - \sqrt{2}\theta^\alpha p^+) \end{pmatrix}, \tag{5.35}$$

where $\partial_\alpha = \partial/\partial\theta^\alpha$, and $\gamma_{ij} = \frac{1}{2}[\gamma_i, \gamma_j]$. They act irreducibly on chiral superfields which are annihilated by the covariant derivatives

$$\left(\frac{\partial}{\partial\overline{\theta}^\alpha} - \frac{1}{\sqrt{2}}p^+\theta^\alpha\right)\,\Phi(y^-,\theta) \;=\; 0\;,$$

where

$$y^- = x^- - \frac{i\theta\overline{\theta}}{\sqrt{2}} .$$

5.2.2 Representations of Grassman Variables

$SO(2n+1)$ representation in Dynkin basis$[H_{IJ}, E_{(I-J)}, E_{(I)}]$ can be constructed from its standard form$[M_{ij}]$ [79]. Here

$$[M_{ij}, M_{kl}] = i(\delta_{ik}M_{jl} + \delta_{jl}M_{ki} - \delta_{il}M_{jk} - \delta_{jk}M_{il}) \ , \tag{5.36}$$

and

$$H_I = M_{2I,2I}, \tag{5.37}$$

$$E_{(I)} = M_{2I-1,2n+1} + i\eta_{(I)}M_{2I,2n+1}, \tag{5.38}$$

$$E_{(I-J)} = \frac{i}{2}\eta_{(I+J)}(M_{2I-1,2J-1} + i\eta_{(I)}M_{2I,2J-1} - i\eta_{(J)}M_{2I-1,2J} + \eta_{(I)}\eta_{(J)}M_{2I,2J}), \tag{5.39}$$

Or conversely,

$$\begin{aligned}
M_{2I-1,2I} &= H_I, & (5.40)\\
M_{2I-1,2n+1} &= \frac{1}{2}(E_{(I)}+E_{-(I)}), & (5.41)\\
M_{2I,2n+1} &= -\frac{i}{2}\eta_{(I)}(E_{(I)}-E_{-(I)}), & (5.42)\\
M_{2I-1,2J-1} &= -\frac{i}{2}(\eta_{(I+J)}E_{(I-J)}-\eta_{(I+J)}E_{-(I-J)} & \\
&\quad +\eta_{(I-J)}E_{(I+J)}-\eta_{(I-J)}E_{-(I+J)}), & (5.43)\\
M_{2I,2J} &= -\frac{i}{2}\eta_{(I)}\eta_{(J)}(\eta_{(I+J)}E_{(I-J)}-\eta_{(I+J)}E_{-(I-J)} & \\
&\quad -\eta_{(I-J)}E_{(I+J)}+\eta_{(I-J)}E_{-(I+J)}), & (5.44)\\
M_{2I,2J-1} &= -\frac{1}{2}\eta_{(I)}(\eta_{(I+J)}E_{(I-J)}+\eta_{(I+J)}E_{-(I-J)} & \\
&\quad +\eta_{(I-J)}E_{(I+J)}+\eta_{(I-J)}E_{-(I+J)}), & (5.45)\\
M_{2I-1,2J} &= \frac{1}{2}\eta_{(J)}(\eta_{(I+J)}E_{(I-J)}+\eta_{(I+J)}E_{-(I-J)} & \\
&\quad -\eta_{(I-J)}E_{(I+J)}-\eta_{(I-J)}E_{-(I+J)})\ , & (5.46)
\end{aligned}$$

where $I, J = 1, \cdots, n$ and $i = 1, \cdots, 2n+1$ ($n = 4$) for $SO(9)$.

Consider spinor representation (**16**) of $SO(9)$,

$$M_{ij}^{(a),(b)} = -\frac{i}{2}f_{ij}^{(a),(b)} \equiv -\frac{i}{2}\gamma_i\gamma_j\ . \tag{5.47}$$

Here (a) represent 16 indices, are labeled by four $+$ or $-$ signs, and $-(a)$ means that all signs are flipped. They can all be switched to indices $0, \ldots, 15$ through binary counting. For example, if $(a) = (+++-) = (0001) = 1$, then $-(a) = (---+) = (1110) = 14$. γs are antisymmetric real matrices. A (**16**) spinor representation (5.47) can be naturally expressed in 16 Fermionic oscillators $\mathcal{Q}_+$, or 8 complex θ (5.35),

$$S_{ij} = -\frac{i}{4\sqrt{2p^+}}\mathcal{Q}_+^T\gamma^{ij}\mathcal{Q}_+\ . \tag{5.48}$$

A special choice of γ's, for the reason which will become clear soon, is

$$\begin{aligned}
\gamma_1 &= \sigma_3 \times \sigma_1 \times \mathbf{1} \times \sigma_3; & \gamma_2 &= \sigma_1 \times \sigma_1 \times \sigma_3 \times \mathbf{1};\\
\gamma_3 &= \sigma_3 \times \mathbf{1} \times \sigma_3 \times \sigma_1; & \gamma_4 &= \sigma_1 \times \sigma_3 \times \mathbf{1} \times \sigma_1;\\
\gamma_5 &= \sigma_3 \times \sigma_3 \times \sigma_1 \times \mathbf{1}; & \gamma_6 &= \sigma_1 \times \mathbf{1} \times \sigma_1 \times \sigma_3;\\
\gamma_7 &= \sigma_2 \times \sigma_2 \times \sigma_2 \times \sigma_2; & \gamma_8 &= \mathbf{1} \times \sigma_1 \times \sigma_1 \times \sigma_1;\\
\gamma_9 &= -\mathbf{1} \times \sigma_3 \times \sigma_3 \times \sigma_3;
\end{aligned} \tag{5.49}$$

Now explicit forms of Cartan generators and raising and lowering operators can be derived from Eqn. (5.48) (Eqn. (5.36) and below). Matrix elements of γ_{ij} are calculated with a C++ program displayed in appendix.

The cartan generators are

$$S_{12} = -\frac{1}{2}(-\theta^0\partial_0 + \theta^1\partial_1 + \theta^2\partial_2 - \theta^3\partial_3 - \theta^4\partial_4 + \theta^5\partial_5 + \theta^6\partial_6 - \theta^7\partial_7) \tag{5.50}$$
$$S_{34} = -\frac{1}{2}(-\theta^0\partial_0 - \theta^1\partial_1 + \theta^2\partial_2 + \theta^3\partial_3 + \theta^4\partial_4 + \theta^5\partial_5 - \theta^6\partial_6 - \theta^7\partial_7) \tag{5.51}$$
$$S_{56} = -\frac{1}{2}(-\theta^0\partial_0 + \theta^1\partial_1 - \theta^2\partial_2 + \theta^3\partial_3 + \theta^4\partial_4 - \theta^5\partial_5 + \theta^6\partial_6 - \theta^7\partial_7) \tag{5.52}$$
$$S_{78} = -\frac{1}{2}(-\theta^0\partial_0 + \theta^1\partial_1 + \theta^2\partial_2 - \theta^3\partial_3 + \theta^4\partial_4 - \theta^5\partial_5 - \theta^6\partial_6 + \theta^7\partial_7)\ . \tag{5.53}$$

The raising operators corresponding to the simple roots are

$$S_{(1-2)} = \theta^3\partial_6 + \theta^4\partial_1, \quad S_{(2-3)} = \theta^1\partial_2 + \theta^6\partial_5, \quad S_{(2+3)} = -(\theta^0\partial_3 + \theta^7\partial_4),$$

$$S_{(4)} = \theta^0\partial_7 + \theta^3\partial_4 + \theta^5\partial_2 + \theta^6\partial_1, \quad S_{(3-4)} = i(\frac{1}{\sqrt{2p^+}}\partial_3\partial_6 + \sqrt{2}p^+\theta^2\theta^7). \tag{5.54}$$

where $S_{12}, S_{34}, S_{56}, S_{(1-2)}, S_{(2-3)}$ and $S_{(2+3)}$ belong to $SU(4)$, S_{78} and $S_{(4)}$ belong to $SU(2)$, and $S_{(3-4)}$ mix $SU(4)$ and $SU(2)$ representations. $S_{(2+3)}, S_{(1-2)}, S_{(2-3)}$ correspond to the simple roots of $SU(4)$, $S_{(4)}$ corresponds to the simple root of $SU(2)$, and $S_{(1-2)}, S_{(2-3)}, S_{(3-4)}, S_{(4)}$ correspond to the simple roots of $SO(9)$. Also the lowering operators are

$$S_{-(1-2)} = \theta^6\partial_3 + \theta^1\partial_4, \quad S_{-(2-3)} = \theta^2\partial_1 + \theta^5\partial_6, \quad S_{-(2+3)} = -(\theta^3\partial_0 + \theta^4\partial_7),$$

$$S_{-(4)} = \theta^7\partial_0 + \theta^4\partial_3 + \theta^2\partial_5 + \theta^1\partial_6, \quad S_{-(3-4)} = i(\frac{1}{\sqrt{2}p^+}\partial_2\partial_7 + \sqrt{2}p^+\theta^3\theta^6). \tag{5.55}$$

Please note that each θ^α is an eigenstate of the cartan generators S_{12}, S_{34}, S_{56} and S_{78}. Take the highest state to be θ^0, then acting the lowering operators on it, θ^α's could be easily identified with the states of $(\mathbf{4}, \mathbf{2})$ in $SU(4) \times SU(2)$ with dynkin labels, represented in $(a_1, a_2, a_3) \times a_4$, where (a_1, a_2, a_3) and a_4 are dynkin labels for $SU(4)$ and $SU(2)$ respectively.

$$\theta^0 \sim (1,0,0) \times 1, \quad \theta^7 \sim (1,0,0) \times -1, \tag{5.56}$$

$$\theta^3 \sim (-1,1,0) \times 1, \quad \theta^4 \sim (-1,1,0) \times -1, \tag{5.57}$$

$$\theta^6 \sim (0,-1,1) \times 1, \quad \theta^1 \sim (0,-1,1) \times -1, \tag{5.58}$$

$$\theta^5 \sim (0,0,-1) \times 1, \quad \theta^2 \sim (0,0,-1) \times -1, \tag{5.59}$$

Alternatively, we can use the weight space representation for θ^α, and for the raising and lowering operators, expressed in eigenvalues of $S_{12}, S_{34}, S_{56}, S_{78}$.

$$\theta^0 \sim \tfrac{1}{2}(e_1 + e_2 + e_3 + e_4), \quad \theta^7 \sim \frac{1}{2}(e_1 + e_2 + e_3 - e_4), \tag{5.60}$$

$$\theta^3 \sim \tfrac{1}{2}(e_1 - e_2 - e_3 + e_4), \quad \theta^4 \sim \frac{1}{2}(e_1 - e_2 - e_3 - e_4), \tag{5.61}$$

$$\theta^6 \sim \tfrac{1}{2}(-e_1 + e_2 - e_3 + e_4), \quad \theta^1 \sim \frac{1}{2}(-e_1 + e_2 - e_3 - e_4), \tag{5.62}$$

$$\theta^5 \sim \tfrac{1}{2}(-e_1 - e_2 + e_3 + e_4), \quad \theta^2 \sim \frac{1}{2}(-e_1 - e_2 + e_3 - e_4), \tag{5.63}$$

$$S_{(2+3)} \sim (2,-1,0)\times 0 \sim (e_2+e_3), \tag{5.64}$$

$$S_{(1-2)} \sim (-1,2,-1)\times 0 \sim (e_1-e_2), \tag{5.65}$$

$$S_{(2-3)} \sim (0,-1,2)\times 0 \sim (e_2-e_3), \tag{5.66}$$

$$S_{(4)} \sim (0,0,0)\times 2 \sim (e_4) \tag{5.67}$$

To calculate the above formulas, first identify θ^0 with the highest weight state, $(1,0,0)\times 1$ in dynkin labels, i.e. $(1,0,0)$ in $SO(6)$, and 1 in $SO(3)$. since θ^0 state has the highest eigenvalues in terms of S_{12} , S_{34} , S_{56} and S_{78}, then by acting the lowering operators of $SO(6)$ and $SO(3)$,($S_{-(2+3)}, S_{-(1-2)}, S_{-(2-3)}$ for $SO(6)$ and $S_{-(4)}$ for $SO(3)$) , other θ's can also be identified with the dynkin labeled states.

Expansion of the superfield in powers of the eight complex θ's yields 256 components, with the following $SU(4)\times SU(2)$ properties

$$1 \sim (\mathbf{1},\mathbf{1}) \;, \tag{5.68}$$

$$\theta \sim (\mathbf{4},\mathbf{2}) \;, \tag{5.69}$$

$$\theta\theta \sim (\mathbf{6},\mathbf{3})\oplus(\mathbf{10},\mathbf{1}) \;, \tag{5.70}$$

$$\theta\theta\theta \sim (\overline{\mathbf{20}},\mathbf{2})\oplus(\overline{\mathbf{4}},\mathbf{4}) \;, \tag{5.71}$$

$$\theta\theta\theta\theta \sim (\mathbf{15},\mathbf{3})\oplus(\mathbf{1},\mathbf{5})\oplus(\mathbf{20'},\mathbf{1}) \;, \tag{5.72}$$

and the higher powers yield the conjugate representations by duality. These make up the three $SO(9)$ representations of $N=1$ supergravity

$$\mathbf{44} = (\mathbf{1},\mathbf{5})\oplus(\mathbf{6},\mathbf{3})\oplus(\mathbf{20'},\mathbf{1})\oplus(\mathbf{1},\mathbf{1}) \;, \tag{5.73}$$

$$\mathbf{84} = (\mathbf{15},\mathbf{3})\oplus(\overline{\mathbf{10}},\mathbf{1})\oplus(\mathbf{10},\mathbf{1})\oplus(\mathbf{6},\mathbf{3})\oplus(\mathbf{1},\mathbf{1}) \;, \tag{5.74}$$

$$\mathbf{128} = (\mathbf{20},\mathbf{2})\oplus(\overline{\mathbf{20}},\mathbf{2})\oplus(\mathbf{4},\mathbf{4})\oplus(\overline{\mathbf{4}},\mathbf{4})\oplus(\mathbf{4},\mathbf{2})\oplus(\overline{\mathbf{4}},\mathbf{2}) \;, \tag{5.75}$$

with the highest weights

$$\mathbf{44} \quad : \quad \theta^0\theta^3\theta^4\theta^7 \; = \; (0,2,0)\times 0 \quad \sim \quad (\mathbf{20}'\ \mathbf{1}) \tag{5.76}$$

$$\mathbf{84} \quad : \quad \theta^0\theta^7 \; = \; (2,0,0)\times 0 \quad \sim \quad (\mathbf{10}\,,\mathbf{1}\,) \tag{5.77}$$

$$\mathbf{128} \quad : \quad \theta^0\theta^3\theta^7 \; = \; (1,1,0)\times 1 \quad \sim \quad (\mathbf{20}\,,\mathbf{2}\,)\ , \tag{5.78}$$

together with their $SU(4) \times SU(2)$ properties. All other states are generated by acting on these highest weight states with the lowering operators. The highest weight chiral superfield that describes $N = 1$ supergravity in eleven dimensions is simply

$$\Phi \; = \; \theta^0\theta^7\, h(y^-,\vec{x}) \; + \; \theta^0\theta^3\theta^7\, \psi(y^-,\vec{x}) \; + \; \theta^0\theta^3\theta^4\theta^7\, A(y^-,\vec{x})\ ,$$

which summarizes the spectrum of the super-Poincaré algebra in eleven dimensions of either a free field theory or a free superparticle. All other states are obtained by applying the $SO(9)$ lowering operators.

5.2.3 $F4/SO(9)$ Oscillator and Differential form Representations

It turns out that all representations of the exceptional group F_4 are generated by three (not four [77]) sets of oscillators transforming as **26**.

Label each copy of 26 oscillators as $A_0^{[\kappa]}$, $A_i^{[\kappa]}$, $i = 1,\cdots,9$, $B_a^{[\kappa]}$, $a = 0,\cdots,15$, and their hermitian conjugates, and where $\kappa = 1,2,3$. Under $SO(9)$, the $A_i^{[\kappa]}$ transform as **9**, $B_a^{[\kappa]}$ transform as **16**, and $A_0^{[\kappa]}$ is a scalar. They satisfy the commutation relations of ordinary harmonic oscillators

$$[\, A_i^{[\kappa]}\,,\, A_j^{[\kappa']\,\dagger}\,] \; = \; \delta_{ij}\,\delta^{[\kappa]\,[\kappa']}\ , \qquad [\, A_0^{[\kappa]}\,,\, A_0^{[\kappa']\,\dagger}\,] \; = \; \delta^{[\kappa\,\kappa']}\ .$$

Note that the $SO(9)$ spinor operators satisfy Bose-like commutation relations

$$[\, B_a^{[\kappa]}\,,\, B_b^{[\kappa']\,\dagger}\,] \; = \; \delta_{ab}\,\delta^{[\kappa]\,[\kappa']}\ .$$

The generators T_{ij} and T_a

$$T_{ij} = -i \sum_{\kappa=1}^{4} \left\{ \left(A_i^{[\kappa]\dagger} A_j^{[\kappa]} - A_j^{[\kappa]\dagger]} A_i^{[\kappa]} \right) + \frac{1}{2}\, B^{[\kappa]\dagger}\, \gamma_{ij} B^{[\kappa]} \right\} \ , \tag{5.79}$$

$$\begin{aligned} T_a = -\frac{i}{2} \sum_{\kappa=1}^{4} \Big\{ (\gamma_i)^{ab} \left(A_i^{[\kappa]\dagger} B_b^{[\kappa]} - B_b^{[\kappa]\dagger} A_i^{[\kappa]} \right) \\ -\sqrt{3} \left(B_a^{[\kappa]\dagger} A_0^{[\kappa]} - A_0^{[\kappa]\dagger} B_a^{[\kappa]} \right) \Big\} \end{aligned} \tag{5.80}$$

satisfy the F_4 algebra,

$$[\, T_{ij}\, , T_{kl}\,] = -i\, (\delta_{jk}\, T_{il} + \delta_{il}\, T_{jk} - \delta_{ik}\, T_{jl} - \delta_{jl}\, T_{ik}) \ , \tag{5.81}$$

$$[\, T_{ij}\, , T_a\,] = \frac{i}{2}\, (\gamma_{ij})_{ab}\, T_b \ , \tag{5.82}$$

$$[\, T_a\, , T_b\,] = \frac{i}{2}\, (\gamma_{ij})_{ab}\, T_{ij} \ , \tag{5.83}$$

so that the structure constants are given by

$$f_{ij\,ab} = f_{ab\,ij} = \frac{1}{2}\, (\gamma_{ij})_{ab} \ .$$

The last commutator requires the Fierz-derived identity

$$\frac{1}{4}\, \theta\, \gamma^{ij}\, \theta\, \chi\, \gamma^{ij}\, \chi = 3\, \theta\, \chi\, \chi\, \theta + \theta\, \gamma^i\, \chi\, \chi\, \gamma^i \theta \ ,$$

from which we deduce

$$3\, \delta^{ac} \delta^{db} + (\gamma^i)^{ac}\, (\gamma^i)^{db} - (a \leftrightarrow b) = \frac{1}{4}\, (\gamma^{ij})^{ab}\, (\gamma^{ij})^{cd} \ .$$

To satisfy these commutation relations, we have required both A_0 and B_a to obey Bose commutation relations (Curiously, if both are anticommuting, the F_4 algebra is still satisfied). One can just as easily use a coordinate representation of the oscillators by introducing real coordinates x_i which transform as transverse space vectors, x_0 as scalars, and y_a as space spinors which satisfy Bose commutation rules

$$A_i = \frac{1}{\sqrt{2}}(x_i + \partial_{x_i}) , \qquad A_i^\dagger = \frac{1}{\sqrt{2}}(x_i - \partial_{x_i}) , \tag{5.84}$$

$$B_a = \frac{1}{\sqrt{2}}(y_a + \partial_{y_a}) , \qquad B_a^\dagger = \frac{1}{\sqrt{2}}(y_a - \partial_{y_a}) , \tag{5.85}$$

$$A_0 = \frac{1}{\sqrt{2}}(x_0 + \partial_{x_0}) , \qquad A_0^\dagger = \frac{1}{\sqrt{2}}(x_0 - \partial_{x_0}) . \tag{5.86}$$

From now on, let us use square brackets $[\cdots]$ to represent the dynkin label of $F4$, and round brackets $(\cdots)$ to represent the dynkin label of $SO(9)$, In the weight spaces of the cartan generators, (eigenvalues of $T_{12}, T_{34}, T_{56}, T_{78}$), the raising operators correspond to the simple roots of $F4$ are

$$T_{(2-3)} \sim [2-100] \sim (e_2 - e_3), \tag{5.87}$$

$$T_{(3-4)} \sim [-12-20] \sim (e_3 - e_4), \tag{5.88}$$

$$T_{(4)} \sim [0-12-1] \sim (e_4), \tag{5.89}$$

$$T_\eta \equiv \frac{1}{\sqrt{2}}(T_4 + iT_{12}) \sim [00-12] \sim \frac{1}{2}(e_1 - e_2 - e_3 - e_4). \tag{5.90}$$

where $T_{(2-3)}, T_{(3-4)}, T_{(4)}$ are defined by Eqn.(5.39) for $SO(9)$ generators as usual, and T_η represents the $F4$ simple root raising operator transform as a spinor under the $SO(9)$ subgroup, and also $T_{-\eta} \equiv \frac{1}{\sqrt{2}}(T_4 - iT_{12})$ will be used to represent the lowering operator. Also in the same space, the raising operators correspond to the simple roots of $SO(9)$ are

$$T_{(1-2)} \sim (2-100) \sim (e_1 - e_2), \tag{5.91}$$

$$T_{(2-3)} \sim (-12-10) \sim (e_2 - e_3), \tag{5.92}$$

$$T_{(3-4)} \sim (0-12-2) \sim (e_3 - e_4), \tag{5.93}$$

$$T_{(4)} \sim (00-12) \sim (e_4). \tag{5.94}$$

The weight states are

$$x_1 + ix_2 \sim [0001] \sim (1000) \sim (e_1), \tag{5.95}$$

$$x_3 + ix_4 \sim [100-1] \sim (-1100) \sim (e_2), \tag{5.96}$$

$$y_0 + iy_8 \sim [001-1] \sim (0001) \sim \frac{1}{2}(e_1 + e_2 + e_3 + e_4), \tag{5.97}$$

$$y_7 + iy_{15} \sim [01-10] \sim (001-1) \sim \frac{1}{2}(e_1 + e_2 + e_3 - e_4), \tag{5.98}$$

$$y_2 - iy_{10} \sim [1-110] \sim (01-11) \sim \frac{1}{2}(e_1 + e_2 - e_3 + e_4), \tag{5.99}$$

$$y_5 - iy_{13} \sim [10-11] \sim (010-1) \sim \frac{1}{2}(e_1 + e_2 - e_3 - e_4), \tag{5.100}$$

5.2.4 Solution of Kostant Equation in $F4/SO(9)$

Define Cliford algebra over 16-dimensional coset $F4/SO(9)$,

$$\{ \Gamma^a , \Gamma^b \} = 2\,\delta^{ab} , \quad a, b = 0, 1, \ldots, 15 , \tag{5.101}$$

generated by (256×256) matrices. The Kostant equation is defined as

$$\not{K}\, \Psi = \sum_{a=1}^{16} \Gamma^a\, T^a\, \Psi = 0 , \tag{5.102}$$

where T_a are F_4 generators not in $SO(9)$, with commutation relations

$$[\, T^a , T^b \,] = i\, f^{\,ab\, ij}\, T^{ij} . \tag{5.103}$$

Although it is taken over a compact manifold, it has non-trivial solutions. To see this, we rewrite its square as the difference of positive definite quantities,

$$\not{K}\not{K} = C^2_{F_4} - C^2_{SO(9)} + 72 , \tag{5.104}$$

where

$$C^2_{F_4} = \frac{1}{2} T^{ij}\, T^{ij} + T^a\, T^a , \tag{5.105}$$

is the F_4 quadratic Casimir operator, and

$$C^2_{SO(9)} \;=\; \frac{1}{2}\left(T^{ij} - i f^{ab\,ij}\,\widetilde{\Gamma}^{ab}\right)^2 \;, \tag{5.106}$$

is the quadratic Casimir for the sum

$$L^{ij} \;\equiv\; T^{ij} + S^{ij} \;, \tag{5.107}$$

where

$$S^{ij} = -\frac{i}{8}(\gamma_{ij})^{ab}\Gamma^a\Gamma^b \;, \tag{5.108}$$

is $SO(9)$ generator (5.48), which acts on the supergravity fields. The quadratic Casimir on the spinor representation is

$$\frac{1}{2}\,S^{ij}\,S^{ij} \;=\; 72 \;, \tag{5.109}$$

Kostant's operator commutes with the sum of the generators,

$$[\,\not{K},\, L^{ij}\,] \;=\; 0 \;, \tag{5.110}$$

allowing its solutions to be labeled by $SO(9)$ quantum numbers.

Since the little group generators S_{ij} act on a 256-dimensional space, they can be expressed in terms of sixteen (256×256) matrices, Γ^a, which satisfy the Dirac algebra

$$\{\,\Gamma^a\,,\,\Gamma^b\,\} \;=\; 2\delta^{ab} \;.$$

This leads to an elegant representation of the $SO(9)$ generators

$$S^{ij} = -\frac{i}{4}(\gamma^{ij})^{ab}\,\Gamma^a\,\Gamma^b \;\equiv\; -\frac{i}{2}f^{\,ij\,a\,b}\Gamma^a\,\Gamma^b \;.$$

, which can be identified with Eqn. (5.48), considering the replacement

$$\Gamma^a \leftrightarrow \sqrt{\frac{\sqrt{2}}{p^+}}Q_a \;. \tag{5.111}$$

The coefficients

$$f^{\,ij\,a\,b} \;\equiv\; \frac{1}{2}(\gamma^{ij})^{ab} \ ,$$

naturally appear in the commutator between the generators of $SO(9)$ and any spinor operator T^a, as

$$[\,T^{ij}\,,\,T^{a}\,] \;=\; \frac{i}{2}\left(\gamma^{ij}\,T\right)^{a} \;=\; if^{\,ij\,a\,b}\,T^{b}\ .$$

But there is more to it, the $(\gamma^{ij})^{ab}$ can also be viewed as structure constants of a Lie algebra. Manifestly antisymmetric under $a \leftrightarrow b$, they can appear in the commutator of two spinors into the $SO(9)$ generators

$$[\,T^{a}\,,\,T^{b}\,] \;=\; \frac{i}{2}(\gamma^{ij})^{ab}\,T^{ij} \;=\; f^{\,a\,b\,ij}\,T^{ij}\ ,$$

and one easily checks that they satisfy the Jacobi identities. Remarkably, the 52 operators T^{ij} and T^{a} generate the exceptional Lie algebra F_4, showing explicitly how an exceptional Lie algebra appears in the light-cone formulation of supergravity in eleven dimensions.

For Kostant solution $\Psi = \Theta(\theta)f(x,y)$

$$\Gamma^a T^a \Psi = (\Gamma^a \Theta(\theta))(T^a f(x,y)) = 0, \tag{5.112}$$

Therefore

$$\Gamma^a \Theta(\theta) = 0, \quad T^a f(x,y) = 0. \tag{5.113}$$

since $[L_{ij}, \Gamma^a T^a] = 0$, $\Theta(\theta)$ and $f(x,y)$ are both the highest weight states of the $SO(9)$ algebra, S_{ij} and T_{ij}, and

$$S_{ij}\Theta(\theta) = T_{ij}f(x,y) = 0 \tag{5.114}$$

To find the solution for the Kostant eqn., we have to choose the specific representation, (5.101) and (5.79),(5.80), for the generators, and therefore the solution is formed by the states of this representation.

Let us first show some useful relations based on the Dynkin diagram of 26 in $F4$, shown at the end of the chapter, where the numbers $1,2,3,4$ near the arrow represent the lowering operators $T_{-(2-3)}, T_{-(3-4)}, T_{-(4)}, T_{-\eta}$ respectively. Using the explicit formula, Eqn.(5.79), (5.80), (5.39) and (5.90), for those lowering operators, the calculation shows,

$$\begin{aligned}
T_{-\eta}(x_1+ix_2) &= \frac{i}{2}\left((\gamma_1)^{4b}+i(\gamma_2)^{4b}-i(\gamma_1)^{12b}+(\gamma_2)^{12b}\right)y_b = i(y_0+iy_8),\\
T_{-(4)}i(y_0+iy_8) &= \frac{1}{2}\left((\gamma_{79})^{a0}+i(\gamma_{79})^{a8}+i(\gamma_{89})^{a0}-(\gamma_{89})^{a8}\right)y_a = i(y_7+iy_{15})\\
T_{-(3-4)}i(y_7+iy_{15}) &= -\frac{i}{4}\left((\gamma_{57})^{a7}+i(\gamma_{57})^{a15}-i(\gamma_{67})^{a7}+(\gamma_{67})^{a15}+\right.\\
&\qquad \left. i(\gamma_{58})^{a7}-(\gamma_{58})^{a15}+(\gamma_{68})^{a7}+i(\gamma_{68})^{a15}\right)y_a = y_2-iy_{10}\\
T_{-(4)}(y_2-iy_{10}) &= -\frac{i}{2}\left((\gamma_{79})^{a2}-i(\gamma_{79})^{a10}-i(\gamma_{89})^{a2}-(\gamma_{89})^{a10}\right)y_a = -(y_5-iy_{13})\\
T_{-\eta}(-y_5+iy_{13}) &= \frac{i}{2}\left((\gamma_i)^{45}-i(\gamma_i)^{4,13}-i(\gamma_i)^{12,5}-(\gamma_i)^{12,13}\right)x_i = i(x_3+ix_4)
\end{aligned}$$

where we keep the coefficient of the states for the later antisymmetric construction of the highest weights.

To verify the solutions for the Kostant equation, we need to identify the generators,

$$\begin{aligned}
T_4+iT_{12} &= \sqrt{2}T_\eta, & T_4-iT_{12} &= \sqrt{2}T_{-\eta},\\
T_3+iT_{11} &= \sqrt{2}[T_{(4)},T_\eta], & T_3-iT_{11} &= -\sqrt{2}[T_{-(4)},T_{-\eta}].
\end{aligned}$$

and also from eqn. (5.101), Kostant operator can be rewrite as,

$$\Gamma_a T_a = i\left[\partial_\alpha(T_\alpha+iT_{\alpha+8})-\theta^\alpha(T_\alpha-iT_{\alpha+8})\right]\ . \tag{5.115}$$

The above explicit form shows that Kostant operators are just composed of raising and lowering operators constructed by T_a. Since we only need to verify the highest weight solutions, only the lowering operators are needed to be taken into consideration. Explicit calculation shows that when acting the Kostant operator on the highest weight solution, most of the terms will vanish due to ∂_α and θ^α, and only few lowering operators ($\frac{1}{\sqrt{2}}(T_3 - iT_{11})$, $\frac{1}{\sqrt{2}}(T_4 - iT_{12})$) need to be considered.

The solutions of kostant's equation form $SO(9)$ triplets. For every representation of $F4$, in dynkin label, $[a_1, a_2, a_3, a_4]$, there is a $SO(9)$ triplet solution associated [72],

$$(2{+}a_2{+}a_3{+}a_4, a_1, a_2, a_3)\oplus(a_2, a_1, 1{+}a_2{+}a_3, a_4)\oplus(1{+}a_2{+}a_3, a_1, a_2, 1{+}a_3{+}a_4) \quad (5.116)$$

Now let us parameterize the Γ^a by θ^a, and T^a by x, y, with Eqn.(5.101) and (5.79). Kostant equation $\Gamma^a T^a \Phi = 0$ will have the solution in the form $\Phi(\theta, x, y) = \Theta(\theta) f(x, y)$.

The solution in the first level is when $a_1 = a_2 = a_3 = a_4 = 0, (2000) \oplus (0010) \oplus (1001)$ or $(44) \oplus (84) \oplus (128)$. The highest weight solution is $\theta^0\theta^3\theta^4\theta^7, \theta^0\theta^7$ and $\theta^0\theta^3\theta^7$ found before.

To find solutions in the higher level, notice two things,

1. The solutions are in the form of $\Theta(\theta) f(x, y)$, where $\Theta(\theta) = \theta^0\theta^3\theta^4\theta^7, \theta^0\theta^7$ or $\theta^0\theta^3\theta^7$. $\Theta(\theta)$ and $f(x, y)$ are both the highest weights of the $SO(9)$ subgroup formed by generators, $L_{ij} = S_{ij} + T_{ij}$, eqn(5.2.4) and (5.79).
2. For the fundamental representation of $F4$(in dynkin label, $a_1, \ldots, a_4$ are all zero except one $a_i = 1$), suppose the associated solution is $\Theta(\theta) f(x, y)$, then $f(x, y)$ will be formed by the states of the $F4$ fundamental representation. Further more, since T_{ij} representations are homogeneous polynomial of x and y, $\Theta(\theta) f(x, y)^{a_i}$ is the solution for higher level($a_i > 1$).

Using the generalized form of the triplet solutions(5.116), the highest weight solutions correspond to each fundamental representation of $F4$ are constructed as follows:

1. $a_1 = a_2 = a_3 = 0, a_4 = 1$

$F4$ states are $\kappa = 1$ copy of 26 states. The highest weight solutions are

$$\theta^0\theta^3\theta^4\theta^7(x_1 + ix_2), \tag{5.117}$$

$$\theta^0\theta^7(y_0 + iy_8), \tag{5.118}$$

$$\theta^0\theta^3\theta^7(y_0 + iy_8), \tag{5.119}$$

where $(x_1 + ix_2)$ and $(y_0 + iy_8)$ are the highest weight states of $SO(9)$ representations (1000) and (0001) respectively, and they are also the states belong to the **26** of $F4$. Direct counting of the Dynkin label shows the above solution is consistent with the general form (5.116). To verify the solution, one need to use the properties of lowering simple root generators to traverse through weight states,

$$\begin{aligned} & \Gamma_a T_a \left(\theta^0\theta^3\theta^4\theta^7(x_1 + ix_2)\right) \\ = \; & i\left(\partial_\alpha(T_\alpha + iT_{\alpha+8}) - \theta^\alpha(T_\alpha - iT_{\alpha+8})\right)\left(\theta^0\theta^3\theta^4\theta^7(x_1 + ix_2)\right) = 0 \end{aligned} \tag{5.120}$$

$$\begin{aligned} & \Gamma_a T_a \left(\theta^0\theta^7(y_0 + iy_8)\right) \\ = \; & -i\left(\theta^3(T_3 - iT_{11}) + \theta^4(T_4 - iT_{12})\right)\left(\theta^0\theta^7(y_0 + iy_8)\right) \\ = \; & -i\left(\theta^3(-\sqrt{2})[T_{-(4)}, T_{-\eta}] + \theta^4\sqrt{2}T_{-\eta}\right)\left(\theta^0\theta^7(y_0 + iy_8)\right) = 0 \; . \end{aligned} \tag{5.121}$$

2. $a_2 = a_3 = a_4 = 0, a_1 = 1$

   $F4$ states are represented by antisymmetric products of $\kappa = 2$ copies of 26 states. From the general form of the solution, (5.116), we need to represent the $SO(9)$ highest weight state (0100) of **36** by the antisymmetric products of the two copies of **26** of $F4$. This state is also the highest weight state of **52** of $F4$. Since $(\mathbf{26} \times \mathbf{26})_\mathbf{a} = \mathbf{52} + \mathbf{273}$, To form this state, use the sixth highest weight states of **26**, antisymmetrize the first and the sixth, the second and the fourth, the third and the fourth, then choose proper coefficients to combine them. This highest weight state of $SO(9)$ is found is to be $([x_1+ix_2, x_3+ix_4]+[y_0+iy_8, y_5-iy_{13}]+[y_7+iy_{15}, y_2-iy_{10}])$. It is annihilated by all the simple roots raising operators of $SO(9)$ and $F4$. The highest weight solutions are

$$\begin{aligned}
&\theta^0\theta^3\theta^4\theta^7([x_1+ix_2, x_3+ix_4]+[y_0+iy_8, y_5-iy_{13}]+[y_7+iy_{15}, y_2-iy_{10}]),\\
&\theta^0\theta^7([x_1+ix_2, x_3+ix_4]+[y_0+iy_8, y_5-iy_{13}]+[y_7+iy_{15}, y_2-iy_{10}]),\\
&\theta^0\theta^3\theta^7([x_1+ix_2, x_3+ix_4]+[y_0+iy_8, y_5-iy_{13}]+[y_7+iy_{15}, y_2-iy_{10}]),
\end{aligned} \tag{5.122}$$

where $([x_1+ix_2, x_3+ix_4]+[y_0+iy_8, y_5-iy_{13}]+[y_7+iy_{15}, y_2-iy_{10}])$ is the highest weight of the $SO(9)$ representation (0100), and here we denote $[a,b]=a^{[1]}b^{[2]}-a^{[2]}b^{[1]}$, antisymmetric product of 2 copies of a and b states. The verification of this solution is similar to the previous case. For example,

$$\begin{aligned}
& \Gamma_a T_a \left(\theta^0\theta^7([x_1+ix_2, x_3+ix_4]+[y_0+iy_8, y_5-iy_{13}]+[y_7+iy_{15}, y_2-iy_{10}]\right)\\
= & -i\left(\theta^3(T_3-iT_{11})+\theta^4(T_4-iT_{12})\right)\\
& \quad \left(\theta^0\theta^7([x_1+ix_2, x_3+ix_4]+[y_0+iy_8, y_5-iy_{13}]+[y_7+iy_{15}, y_2-iy_{10}]\right)\\
= & -i\left(\theta^3(-\sqrt{2})[T_{-(4)}, T_{-\eta}]+\theta^4\sqrt{2}T_{-\eta}\right)\\
& \quad \left(\theta^0\theta^7([x_1+ix_2, x_3+ix_4]+[y_0+iy_8, y_5-iy_{13}]+[y_7+iy_{15}, y_2-iy_{10}]\right)\\
= & -i\left(-\sqrt{2}\theta^3\theta^0\theta^7([i(y_7+iy_{15}, x_3+ix_4]+[y_7+iy_{15}, -i(x_3+ix_4)])\right.\\
& \left. +\sqrt{2}\theta^4\theta^0\theta^7([i(y_0+iy_8), x_3+ix_4]+[y_0+iy_8, -i(x_3+ix_4)])\right) .
\end{aligned} \tag{5.123}$$

3. $a_1=a_2=a_4=0, a_3=1$

$F4$ states, (**273**), are also represented by antisymmetric products of $\kappa = 2$ copies of 26 states. The highest weight solutions are

$$\theta^0\theta^3\theta^4\theta^7[x_1 + ix_2, y_0 + iy_8], \tag{5.124}$$

$$\theta^0\theta^7[y_0 + iy_8, y_7 + iy_{15}], \tag{5.125}$$

$$\theta^0\theta^3\theta^7[x_1 + ix_2, y_0 + iy_8], \tag{5.126}$$

where $[x_1 + ix_2, y_0 + iy_8]$ and $[y_0 + iy_8, y_7 + iy_{15}]$ are the highest weights of the $SO(9)$ representations $(1001) \subset (\mathbf{16} \times \mathbf{9})$ and $(0010) \subset (\mathbf{16} \times \mathbf{16})_{\mathbf{a}}$ respectively. $[x_1 + ix_2, y_0 + iy_8]$ is also the highest weight of **273** of $F4$.

4. $a_1 = a_3 = a_4 = 0, a_2 = 1$

   $F4$ representation (**1274**) can be represented by Kronecker products of $\kappa = 3$ copies of 26 states. The highest weight state is simply the total antisymmetrization of the highest three states in 26. The highest weight solutions are

$$\theta^0\theta^3\theta^4\theta^7[x_1 + ix_2, y_0 + iy_8, y_7 + iy_{15}], \tag{5.127}$$

$$\theta^0\theta^7[x_1 + ix_2, y_0 + iy_8, y_7 + iy_{15}], \tag{5.128}$$

$$\theta^0\theta^3\theta^7[x_1 + ix_2, y_0 + iy_8, y_7 + iy_{15}], \tag{5.129}$$

where $[x_1 + ix_2, y_0 + iy_8, y_7 + iy_{15}]$ is the highest weight of the $SO(9)$ representation 1010, and $[a, b, c]$ is the antisymmetric products of 3 copies of a, b and c states. It is also the highest weight state of **1274** of $F4$.

CHAPTER 6
SUMMARY

This dissertation includes two loosely connected parts. The main focus is quantum aspects of NC field theories, including both perturbative and nonperturbative structures. In particular, perturbative behavior of NC supersymmetric Wess-Zumino model is discussed in detail. It is shown that NCWZ model has only wave function renormalization and UV finite as its commutative analog. It is suggested supersymmetric invariance of NCWZ model again leads to cancelation which renders mass and vertex corrections UV finite. UV/IR mixing terms, as a result of phase factors induced in the vertex, generally exist in all quantum perturbation calculations. NC solitons, nonperturbative structure in NC field theory, are interpreted as low energy manifestation of lower dimensional D-branes in string theory. Through quantization of NC solitons, corrections to the energy are calculated in detail. Energy of NC GMS solitons is found to be UV finite, and also includes UV/IR mixing terms, which need not be surprising considering their general existence in perturbation theory. UV/IR mixing terms in perturbative theory are suggested as results of particles traveling in extra dimensions, which in the context of string theory, are interpreted as low energy closed string modes dual to high energy open string modes living on the brane. Existence of UV/IR mixing terms in NC soliton energy suggests these modes also interact with lower dimensional D-branes. Properties of NC scalar solitons already make the UV/IR terms an intriguing subject. Some NC solitons (GMS solitons) exist only at large enough θ, and quantum corrections of which do not include UV/IR terms near infinite θ limit. There are many questions ready to be answered. How do UV/IR terms affect the stability of NC solitons? Is there any interpretation from string theory that such terms, which reveal the structure of string diagrams, cease to exist at very large θ? What new insight can those UV/IR terms give with regard to the interaction between D-brane and closed strings?

SuperPoincaré symmetry is considered as basic space-time symmetry of fundamental theory. NC field theories, as low energy limit of string theory, explicitly violate Lorentz symmetry. Thus it becomes important to understand the space-time symmetries on which NC field theories are constructed. A representation of deformed superPoincaré algebra is obtained and commutation relations are calculated in an intuitive way. Preservative of supersymmetry supports the attempt to construct supersymmetric NC field theory directly from supersymmetric generalization of NC space. The presence of the B field on the boundary of D-brane enables decoupling of low energy modes of string theory in certain limit, but also yields noncommutativity explicitly broken Lorentz symmetry. If field theories are fundamentally NC, there will be a very small upper bound of the NC parameter, since in our space field theory seems to be Lorentz invariant in high precision. An alternative explanation is to take into consideration the existence of a B field. Indeed covariance of the theory is easily justified if the NC parameters are taken to be $\theta^{\mu\nu}$, where the indices transform accordingly under Lorentz rotation. Symmetry, nonlocality, causality and unitarity will continue to be important issues in identifying NC field theories as realistic theories.

Solution of Kostant equation, as well as supersymmetry algebra representation in 11 dimension, is considered as an attempt to construct zero slope limits of string theory, if we believe they obey superPoincaré symmetry and reduce to 11 dimension supergravity in low energy limit. The appearance of an infinite tower of triplets is interesting and expected but construction of Lagrangian and interactions for those multiplets still needs further work.

APPENDIX
COMPUTER CODE

This program is designed to output matrix elements of γ^i and γ^{ij} (5.49).

```
//Define a complex number class and a sigma class,

//and a gamma class and a F_bar class.


#include<iostream>

using namespace std;


#ifdef _MSC_VER

class F_bar;

ostream& operator <<(ostream &, const F_bar &);

#endif


#ifdef _MSC_VER

class sComplex;

ostream& operator <<(ostream &, const sComplex &);

#endif


//define a class sComplex which represents the complex numbers either

//imaginary or real.

class sComplex

{

public:

int value;
```

```
bool real;
sComplex(){}
sComplex(int);
sComplex(int,bool);
sComplex operator+(sComplex);
sComplex operator*(sComplex);
friend ostream& operator <<(ostream &, const sComplex &);
};

//define a Sigma class to represent Dirac Sigma matrices or the
//unit matrix.

class Sigma
{
public:
int index;
sComplex sign;
sComplex ele[2][2];
Sigma(){}
Sigma(int);
Sigma(int,sComplex);
// ~Sigma();
Sigma operator*(Sigma);

protected:
void SigmaInit(int);
void EleInit(sComplex a[2][2]);
```

```
};

//define a Gamma class for gamma matrix, which includes an overall
//sign and 4 sigma matrices. Mathematically it is the direct
//product of four sigma matrices.

class Gamma
{
public:
Sigma ele[4];
sComplex sign;
Gamma(){}
Gamma(sComplex,int,int,int,int);
Gamma operator*(Gamma&);
};

//a function returns the antisymmetry or symmetry property of
//sigma matrices.

bool transSigma(Sigma& a)
{
if (a.index==0 || a.index==3 || a.index==1) return true;
    if (a.index==2) return false;
return true;
}
```

```
//a function returns the antisymmetry or symmetry property of
//gamma matrices.

bool transGamma(Gamma& a)
{
bool y=true;
for (int i=0;i<4;i++)
{
if (y==true) y=transSigma(a.ele[i]);
else y=!transSigma(a.ele[i]);
}
return y;
}

//operator== overloading for gamma matrices
bool operator==(Gamma& a,Gamma& b)
{
for (int i=0;i<4;i++)
{
if (a.ele[i].index!=b.ele[i].index) return false;
}
return true;
}

//output gamma matrices as the direct product of 4 sigma matrices.
ostream& operator <<(ostream& cout, const Gamma& gamma)
{
```

```
cout<<gamma.sign;
for (int i=0;i<4;i++)
{
cout<<gamma.ele[i].index<<"*";
}
return cout;
}

//F_bar class is a (16,16) array holding the elements of the
//16 times 16 dimension matrix, constructed by multiplication of
//2 gamma matrices. These elements are the structure functions of F4.
class F_bar
{
public:
sComplex ele[16][16];
F_bar(){}
F_bar(Gamma,Gamma);
friend ostream& operator <<(ostream &, const F_bar &);
};

//F_bar.cpp define the classes and overload operators

#include "F_bar.h" //Source file for class definition.
#include<iostream>
//#include<complex>
using namespace std;
```

```
//sComplex class constructor with an integer.
sComplex::sComplex(int value)
{
this->value=value;
this->real=true;
}

//sComplex class constructor with an integer and a boolean for real
//or imaginary

sComplex::sComplex(int value,bool real)
{
this->value=value;
this->real=real;
}

//operator+ overloading for the sComplex class
sComplex sComplex::operator+(sComplex a)
{
  if (this->real && a.real) return sComplex(this->value+a.value);
  else if (this->real || a.real) cout<<"mistake";
  else return sComplex(this->value+a.value,false);
}

//operator* overloading for the sComplex class. sComplex
//objects will multiply each other like complex numbers
sComplex sComplex::operator*(sComplex x)
```

```
{
int val=this->value*x.value;
if (this->real==true)
{
if (x.real==true) return sComplex(val);
else if (x.real==false) return sComplex(val,false);
}
else if (this->real==false)
{
if (x.real==true) return sComplex(val,false);
else if (x.real==false) return sComplex(-val,true);
}
}

//output sComplex objects like complex numbers.
ostream& operator<<(ostream& cout,const sComplex& x)
{
if (x.real==true || x.value==0) cout<<x.value;
else if (x.real==false && x.value!=0) cout<<x.value<<"i";
return cout;
}


//define two sComplex objects(imaginary i and -i) used later
sComplex ai=sComplex(1,false);
sComplex mAi=sComplex(-1,false);
```

```
//Sigma class constructor with index. The constructed Sigma
//objects will be Dirac sigma matrices, and the unit matrix.
Sigma::Sigma(int index)
{
SigmaInit(index);
this->sign=sComplex(1);

}

//Enable the Sigma objects to have complex coefficients.
Sigma::Sigma(int index,sComplex sign)
{
SigmaInit(index);
this->sign=sign;
}

void Sigma::SigmaInit(int index)
{
this->index=index;
if (index==0)
{
sComplex a[2][2]={{1,0},{0,1}};
EleInit(a);
}
else if (index==1)
{
sComplex a[2][2]={{0,1},{1,0}};
```

```
EleInit(a);
}
else if (index==2)
{
sComplex a[2][2]={{0,mAi},{ai,0}};
EleInit(a);
}
else if (index==3)
{
sComplex a[2][2]={{1,0},{0,-1}};
EleInit(a);
}
}

void Sigma::EleInit(sComplex a[2][2])
{
for (int i=0;i<2;i++)
{
for (int j=0;j<2;j++)
{
ele[i][j]=a[i][j];
}
}
}
```

```
/*Sigma::~Sigma()
{
for (int i=0;i<2;i++)
delete[] (ele[i]);
}*/

//operator* overloading for Sigma objects, which multiply
//each other like Dirac sigma matrices
Sigma Sigma::operator*(Sigma x)
{
sComplex thisSign=this->sign*x.sign;
if (this->index==0) return Sigma(x.index,thisSign);
if (x.index==0) return Sigma(this->index,thisSign);
if (this->index==1)
{
if (x.index==1) return Sigma(0,thisSign);
if (x.index==2) return Sigma(3,ai*thisSign);
if (x.index==3) return Sigma(2,mAi*thisSign);
}
if (this->index==2)
{
if (x.index==1) return Sigma(3,mAi*thisSign);
if (x.index==2) return Sigma(0,thisSign);
if (x.index==3) return Sigma(1,ai*thisSign);
}
if (this->index==3)
{
```

```
if (x.index==1) return Sigma(2,ai*thisSign);
if (x.index==2) return Sigma(1,mAi*thisSign);
if (x.index==3) return Sigma(0,thisSign);
}
}

//define Dirac sigma matrices and the unit matrics.
Sigma sigma_0=Sigma(0);
Sigma sigma_1=Sigma(1);
Sigma sigma_2=Sigma(2);
Sigma sigma_3=Sigma(3);

//each gamma matrix is constructed by 4 sigma matrices and a sign.
//mathematically it is the direct product of the 4 sigma matrices.
Gamma::Gamma(sComplex sign,int a,int b,int c,int d)
{
this->sign=sign;
ele[0]=a;
ele[1]=b;
ele[2]=c;
ele[3]=d;
}

//operator* overloading for the gamma matrices
Gamma Gamma::operator*(Gamma& x)
{
Sigma a[4];
```

```
for (int i=0;i<4;i++)
{
a[i]=this->ele[i]*x.ele[i];
}

sComplex thisSign=this->sign*x.sign;
for (int j=0;j<4;j++)
{
thisSign=thisSign*(a[j].sign);
}
return Gamma(thisSign,a[0].index,a[1].index,a[2].index,
a[3].index);
}

//calculate the matrix elements of the muliplication of 2 gamma
//matrices, and put them in the (16,16) array inside F_bar object.
F_bar::F_bar(Gamma x,Gamma y)
{
  Gamma z=x*y;
  for (int i1=0;i1<2;i1++)
    {
      for (int i2=0;i2<2;i2++)
{
  for (int i3=0;i3<2;i3++)
    {
      for (int i4=0;i4<2;i4++)
{
```

```
  for (int j1=0;j1<2;j1++)
    {
      for (int j2=0;j2<2;j2++)
{
  for (int j3=0;j3<2;j3++)
    {
      for (int j4=0;j4<2;j4++)
{
  ele[i1*8+i2*4+i3*2+i4*1][j1*8+j2*4+j3*2+j4*1]=
    z.sign*z.ele[0].ele[i1][j1]*z.ele[1].ele[i2][j2]*
    z.ele[2].ele[i3][j3]*z.ele[3].ele[i4][j4];
}
    }
}
    }
}
    }
}
    }
}

//output each nonzero element in the F_bar array.
ostream& operator <<(ostream & cout, const F_bar & f_bar)
{

/*offdiagonal elements,half of the matrix elements*/
for (int a=0;a<16;a++)
```

```
{
for (int b=a+1;b<16;b++)
{
if (f_bar.ele[a][b].value!=0)
{
cout<<a<<","<<b<<"   ";
cout<<f_bar.ele[a][b];
cout<<endl;

}
}
}

/*diagonal elements*/
for ( a=0;a<16;a++)
{
{
if (f_bar.ele[a][a].value!=0)
{
cout<<a<<","<<a<<"   ";
cout<<f_bar.ele[a][a];
cout<<endl;

}
}
}
return cout;
```

```
}


/*The purpose of this program is to calculate the structure functions
for the Exceptional Lie algebra F4. F4 has 36 generators forming a
subalgebra SO(9), and 16 generators transforming in a spinor
representation of the SO(9). The spinor representation of the SO(9)
is well known to be constructed by 9 gamma matrices. This program
will output all the nonzero matrix elements of the spinor
representation, given the input of the gamma matrices. */



//#include "F_bar.h"
#include "F_bar.cpp" //include class files.

int main()
{
Gamma* gamma=new Gamma[10];    //initialize
gamma[0]=Gamma(1,0,0,0,0);     //a unit matrix for special purpose
gamma[9]=Gamma(-1,0,3,3,3); //construct 9 gamma matrices
gamma[1]=Gamma(1,3,1,0,3);
gamma[2]=Gamma(1,1,1,3,0);
gamma[3]=Gamma(1,3,0,3,1);
gamma[4]=Gamma(1,1,3,0,1);
gamma[5]=Gamma(1,3,3,1,0);
gamma[6]=Gamma(1,1,0,1,3);
gamma[7]=Gamma(1,2,2,2,2);
gamma[8]=Gamma(1,0,1,1,1);
```

```
//output the F_bar objects for each i,j. The output is the nonzero elements
//of the matrix elements.


for (int i=1;i<10;i++)
{
for (int j=i+1;j<10;j++)
{
cout<<i<<","<<j<<"\n\n";
cout<<F_bar(gamma[i],gamma[j])<<endl;

}
}
return 0;
}
```

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BIOGRAPHICAL SKETCH

Xiao-Zhen Xiong was born on July 30th, 1972, in Xiu-Shui, a village town in Jian-Xi, a province in the south of China. He is the only son of Ya-te Xiong and Qi-fang Zhou, and the little brother of Xiao-lin Xiong.

Xiao-Zhen stayed in Xiu-Shui for roughly two years before he moved to Nan-Chang, capital city of Jiang-Xi, according to his mother. Xiao-Zhen only has flashes of memories before his fourth year. He started to show talent in math at five years of age, when he was able to multiply numbers without using a pencil. Also he was eager to learn more math and other things and entered the elementary school at six years old. His mother, a teacher in physics, deserves most of the credit for Xiao-Zhen's early education and interest in math and physics.

Xiao-Zhen stayed in Nan-Chang for 15 years until he finished high school, just like most people at his age in China did. For some reason, although Xiao-zhen has always been regarded as a person with very good potential, he has never been able to display that truly. According to his sister, although he is a very happy person living in a nice family, he seems to live in some sort of unconscious way. Somehow he is a little separate from the surrounding world. It is certain that he would face a lot of changes willingly or unwillingly later in his life.

The situation started to change a little bit after he left his parents for a college education thousands of miles away in the north of China. It was a small college in Chang-Chun, Ji-Lin, a very cold place compared to Nan-Chang. There he had very good classmates and friends. He started to grow stronger and play basketball and soccer. The physical life was not so easy. He remembers sitting in the crowded train for thousands of miles going to and from the school twice a year. He also remembers the hardness of study in the cold classroom and sleep in the cold dorm for the poor heating heating system. But it was not all bad. During that time, he learned how to try his best to achieve his goal, although he attempted but failed to enter graduate school to study high energy theory.

After Xiao-Zhen's graduation from college, he went on to work in a small company in Xia-Men, Fu-Jian, as an engineer. After a few months, he realized the simple work life was not his destiny. In his heart he still wanted more challenges and opportunities.

The dream all came true when he was accepted by the Physics Department ath the University of Florida in fall 1996. He started to enjoy everything, from the lectures by the best professors to fantastic football and NBA by the best players. The freedom in the USA always brought a variety of choices, and he was not really sure which to choose. After some tries, he finally settled to study with Prof. Pierre Ramond on high energy theory partly because he had always dreamed of being a pure theoretical physicist.

Since then for him it has been another life-changing 4 years. Xiao-Zhen had met a lot of frustrations from academic research to personal emotions. He was very fortunate to have Pierre as his advisor, from whom he learned the essence of research progress, as well as mentally being ready for the challenges. His research area has covered superPoincaré algebra and noncommutative field theories and solitons, from which he published 4 papers. He has also taught hundreds of students in the physics lab. By the time of his graduation, he would say he had mastered the ability of learning and research and established confidence to deal with the challenges in any theoretical science area.

After his graduation, he hopes to switch to a research area which has more practical applications. Currently he is switching to the computational biology area, and hopefully in this area he can exploit his theoretical background and research ability.