

A physical interpretation of the Newman Penrose formalism and its application to Bertrand Spacetime

II

Jeevitha T U, Sanjit Das

Division of Physics, School of Advanced Sciences, Vellore Institute of Technology, Chennai - 600127, India

E-mail: sanjitdas@vit.ac.in

Abstract. In general relativity, problems are usually approached with coordinate basis method, but there is an other efficient method for solving these problems, which is tetrad formalism. Newman Penrose formalism is an old and renowned approach in general relativity which served immense benefits in analytical and computational aspects. The basic ingredients on the NP equations are the null tetrad $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ here we construct a null tetrad for spherically symmetric static solutions that admit stable closed orbits called Bertrand Spacetime (BST) which has two type I & II, specially we concentrated on type-II. Along with this we studied Ricci spin coefficients, Ricci tensor and Weyl tensor.

1. Introduction

General theory of relativity published by Einstein in 1915 has revolutionized modern physics as never before. The Einstein field equation connects the Ricci tensor and the energy-momentum tensor deciding how one affects the curvature of the spacetime. The NPF is a priceless methodology and it has been utilized in many parts of analytic and computational relativity. It is furthermore a tetrad formalism with a remarkable choice of basis vectors. In 1954, A. Z. Petrov presented the Petrov classification, it is an invariant portrayal of the solutions of Einstein field equations [1, 2]. There are various kinds of Petrov types, in that Petrov type N is an answer of Einstein vacuum field equations are among the most captivating, however rather inconvenient also small-researched of all unfilled space matrices. From the actual perspective, they address spacetime finished off through and through with gravitational radiation, while numerically they structure a class of arrangements of Einstein equations which should be conceivable to be settled explicitly [3].

2. Exordium of BST

There are numerous solutions to Einstein's equations that were developed, one of the popular examples is Bertrand spacetime which is exposed by Perlick. He composed summed up the notable Bertrand theorem (BT) in classical mechanics to the general theory of relativity. The theorem says that inverse square law and Hooke's law are the only central force which gives rise to bounded closed orbits. From Bertrand theorem, Perlick has exposed his works and it's named Bertrand spacetime which states that all spherically symmetric static solutions of



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Einstein's equations, that admit stable closed orbits. Bertrand spacetime has been separated into two sections which is type-I and type-II [4]. In this exploration article, we study Newman Penrose's formalism for BST type-II.

$$ds^2 = -\frac{dt^2}{G + \sqrt{r^{-2} + K}} + \frac{dr^2}{\beta^2(1 + Kr^2)} + r^2 d\theta^2 + r^2 \sin \theta^2 d\theta^2 \quad (1)$$

So, we have

$$g_{\mu\nu} = \begin{pmatrix} -\frac{1}{G + \sqrt{r^{-2} + K}} & 0 & 0 & 0 \\ 0 & \frac{1}{\beta^2(1 + Kr^2)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (2)$$

3. Newman-Penrose Formalism

Over the past century, Black hole astronomy has seen plenty of impeccable improvements. In the before time of 1900s, they be just a theorem or just a mathematical object with the characteristics of blackhole singularity. In the early 1960s, Penrose analytically proved the existence of such possibility. Dr.Roger Penrose was granted the 2020 Nobel Prize in Physics for "the revelation that black hole formation is a vigorous expectation of the general theory of relativity. Tetrad formalism also called Spin-coefficient formalism along a specialized basis vector. It was first proposed in 1962 by Roger Penrose, he was certain the fundamental part of space-time is its light cone game plan which makes reachable the introduction of a spinor basis.

Newman Penrose (NP) approach broadly utilized in analytical as well as mathematical investigations of Einstein's equations. Throughout the previous forty years, the advancement made in the comprehension of Einstein's equation might be credited to NP formalism.

In simpler terms Newman-Penrose equation has is a well-established arrangement of the condition describe the spinor element of the Riemann curvature tensor with the spinor association element [5]. Also, this has been recognized and utilized in a continuous writing audit. NP formalism is significant for its enough as well as inbound versatility for deriving the exact solution of the Einstein field equations and for various observation [6].

NP condition is a vital basis for thinking about the asymptotic properties of the gravitational field. further, it is essential to analyze the working of the fields at infinity. Close by the chance of conformal infinity, engage Hawking strategy is to look at the gravitational field of a black hole.

3.1. The spin coefficients along with Complex Null tetrad

Computations in the Newman Penrose formalism of general theory of relativity ordinarily start with the development of a complex null tetrad $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$, where $\{l^\mu, n^\mu\}$ is a combine of real null vectors and $\{m^\mu, \bar{m}^\mu\}$ is a couple of complex null vectors. These tetrad vectors regard the accompanying standardization and metric conditions expecting the spacetime signature $(-, +, +, +)$: [7]

The null quality interprets as obeys

$$l.l = n.n = m.m = \bar{m}.\bar{m} = 0 \quad (3)$$

as well as the orthogonality shows as

$$l.m = l.\bar{m} = n.m = n.\bar{m} = 0. \quad (4)$$

Aside against this, the standardization follows

$$l.n = -1 \quad (5)$$

$$m.\bar{m} = 1 \quad (6)$$

$\eta_{(a)(b)}$ create the fundamental matrix, to frame a steady symmetric matrix [8].

$$g_{\mu\nu} = -l_\mu n_\nu - l_\nu n_\mu + m_\mu \bar{m}_\nu + m_\nu \bar{m}_\mu$$

$$\eta_{(\alpha)(\beta)} = \eta^{(\alpha)(\beta)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (7)$$

4. NP formalism with BST-II : Calculations

The orthonormal basis one-forms can act composed through watching the raised metric condition just as [8]:

$$\omega^{\hat{\theta}} = rd\theta$$

$$\omega^{\hat{\phi}} = rsin\theta d\phi$$

$$\omega^{\hat{r}} = \left(\frac{1}{\beta^2(1+kr^2)} \right)^{1/2} dr$$

$$\omega^{\hat{t}} = \left(\frac{1}{G+\sqrt{r^{-2}+K}} \right)^{1/2} dt$$

Now applying the conditions on null tetrad will have,

$$\begin{pmatrix} -n_\mu \\ -l_\mu \\ \bar{m}_\mu \\ m_\mu \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} l^\mu \\ n^\mu \\ m^\mu \\ m^{*\mu} \end{pmatrix} \quad (8)$$

we have the following relations:

$$l = \frac{1}{\sqrt{2}} \left[\left(\frac{1}{G+\sqrt{r^{-2}+K}} \right)^{1/2} dt + \left(\frac{1}{\beta^2(1+kr^2)} \right)^{1/2} dr \right]$$

$$n = \frac{1}{\sqrt{2}} \left[\left(\frac{1}{G+\sqrt{r^{-2}+K}} \right)^{1/2} dt - \left(\frac{1}{\beta^2(1+kr^2)} \right)^{1/2} dr \right]$$

$$m = \frac{1}{\sqrt{2}} (rd\theta + ir \sin\theta d\phi)$$

$$\bar{m} = \frac{1}{\sqrt{2}} (rd\theta - ir \sin\theta d\phi)$$

Using the notation $v^a = (v^t, v^r, v^\theta, v^\phi)$, we compose the parts of null tetrads just as:

$$\begin{aligned} l^\mu &= \frac{1}{\sqrt{2}} \left[\sqrt{\frac{1}{G + \sqrt{r^{-2} + K}}}, \sqrt{\frac{1}{\beta^2(1 + Kr^2)}}, 0, 0 \right] \\ n^\mu &= \frac{1}{\sqrt{2}} \left[\sqrt{\frac{1}{G + \sqrt{r^{-2} + K}}}, -\sqrt{\frac{1}{\beta^2(1 + Kr^2)}}, 0, 0 \right] \\ m^\mu &= \frac{1}{\sqrt{2}} [0, 0, r, ir \sin \theta] \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}} [0, 0, r, -ir \sin \theta] \end{aligned}$$

4.1. Ricci rotation coefficients

The 12 complex functions, called as Spin-coefficients with respect to the complex null tetrad, act determine over the ricci rotation coefficients [5] as

$$\begin{aligned} -k &= \gamma_{(2)(0)(0)} = e^i {}_{(2)} e_{(0)i;j} e_{(0)}{}^j = l_{i;j} m^i l^j \\ -\rho &= \gamma_{(2)(0)(3)} = l_{i;j} m^i \bar{m}^j \\ -\sigma &= \gamma_{(2)(0)(2)} = l_{i;j} m^i m^j \\ -\tau &= \gamma_{(2)(0)(1)} = l_{i;j} m^i n^j \\ \nu &= \gamma_{(1)(3)(1)} = n_{i;j} \bar{m}^i n^j \\ \mu &= \gamma_{(1)(3)(2)} = n_{i;j} \bar{m}^i m^j \\ \lambda &= \gamma_{(1)(3)(3)} = n_{i;j} \bar{m}^i \bar{m}^j \\ \pi &= \gamma_{(1)(3)(0)} = n_{i;j} \bar{m}^i l^j \\ -\epsilon &= \frac{1}{2} (\gamma_{(1)(0)(0)} - \gamma_{(3)(2)(0)}) = \frac{1}{2} (l_{i;j} n^i l^j - m_{i;j} \bar{m}^i l^j) \\ -\beta &= \frac{1}{2} (\gamma_{(1)(0)(2)} - \gamma_{(2)(3)(2)}) = \frac{1}{2} (l_{i;j} n^i m^j - m_{i;j} \bar{m}^i m^j) \\ -\gamma &= \frac{1}{2} (\gamma_{(2)(3)(1)} - \gamma_{(0)(1)(1)}) = \frac{1}{2} (l_{i;j} n^i n^j - m_{i;j} \bar{m}^i n^j) \\ -\alpha &= \frac{1}{2} (\gamma_{(2)(3)(3)} - \gamma_{(0)(1)(3)}) = \frac{1}{2} (l_{i;j} n^i \bar{m}^j - m_{i;j} \bar{m}^i \bar{m}^j) \end{aligned}$$

The general standards may be seen that the complex conjugate form of any sum can be gotten by supplanting the index 3, any spot it occurs, by index 4, and conversely[7].

With respect to complex null tetrad the 12 complex functions, known as spin-coefficients, are described through the Ricci rotation coefficients as

$$\begin{aligned}\pi &= 0; \quad \nu = 0; \quad \lambda = 0; \\ \kappa &= 0; \quad \tau = 0; \quad \sigma = 0; \\ \mu = \rho &= -\frac{1}{\sqrt{2}r\sqrt{\frac{1}{\beta^2(Kr^2+1)}}}; \\ \epsilon &= \frac{\beta^2\sqrt{K+\frac{1}{r^2}}\sqrt{\frac{1}{\beta^2(Kr^2+1)}}}{\sqrt{2}\left(4Gr+4r\sqrt{K+\frac{1}{r^2}}\right)}; \\ \gamma &= \frac{G^2r^2\sqrt{K+\frac{1}{r^2}}+2G(Kr^2+1)+(Kr^2+1)\sqrt{K+\frac{1}{r^2}}}{4\sqrt{2}r^3\sqrt{K+\frac{1}{r^2}}\left(G+\sqrt{K+\frac{1}{r^2}}\right)^2\left(Gr^2\sqrt{K+\frac{1}{r^2}}+Kr^2+1\right)\sqrt{\frac{1}{\beta^2(Kr^2+1)}}}; \\ \alpha = -\beta &= -\frac{\cot\theta}{2\sqrt{2}r}.\end{aligned}$$

4.2. Weyl tensor

The Ricci tensor is found by the contraction [9]

$$g^{il}R_{ijkl} = R_{ik}$$

The Weyl tensor is currently identified with the Riemann and the Ricci tensors by

$$C_{ijkl} = R_{ijkl} - \frac{1}{2}(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk}) + \frac{1}{6}(g_{ik}g_{jl} - g_{jk}g_{il})R$$

The Weyl tensor is the trace-free part of the Riemann tensors, and its tetrad components are exposed by

$$R_{ijkl} = C_{ijkl} + \frac{1}{2}(g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk} + g_{jl}R_{ik}) - \frac{1}{6}(g_{ik}g_{jl} - g_{il}g_{jk})R$$

In the NP formalism, from the Weyl tensor, the ten independent components are shown by the five complex scalars [10, 11],

$$\begin{aligned}\psi_0 &= C_{0202} = C_{abcd}l^a m^b l^c m^d \\ \psi_1 &= C_{0102} = C_{abcd}l^a n^b l^c m^d \\ \psi_2 &= C_{0231} = C_{abcd}l^a m^b \bar{m}^c n^d \\ \psi_3 &= C_{0131} = C_{abcd}l^a n^b \bar{m}^c n^d \\ \psi_4 &= C_{1313} = C_{abcd}n^a \bar{m}^b n^c \bar{m}^d\end{aligned}$$

while it is sure about general grounds that the Weyl tensor is totally indicated by the five complex scalars ψ_0, \dots, ψ_4 , it will be helpful to have an overall equation which communicates the various parts of the Weyl tensor expressly regarding the five scalars [12, 13, 14].

The tetrad components of Weyl tensor have the mathematical properties given in Table.1 [6, 15]:

Table 1. Petrov type classification

Petrov type	ψ_0	ψ_1	ψ_2	ψ_3	ψ_4	Propagation vector
N	P_t	0	0	0	0	n_i
III	0	P_t	0	0	0	n_i
D	0	0	P_t	0	0	l_i, n_i
III	0	0	0	P_t	0	l_i
N	0	0	0	0	P_t	l_i

Where P_t is a non zero weyl tensor.

NP formalism assigns the ten independent Weyl tensor components to five complex scalar.

$$\psi_0 = 0$$

$$\psi_1 = 0$$

$$\begin{aligned} & 3Gr^2 \sqrt{K + \frac{1}{r^2}} (Kr^2 + 1) (-\beta^2 + 4Kr^2 + 4) \\ & -6(\beta^2 - 2) G^2 r^2 (Kr^2 + 1) (Kr^2 + 1) \\ & -4(\beta^2 - 1) G^3 r^4 ((Kr^2 + 1)) \sqrt{K + \frac{1}{r^2}} \\ & + (Kr^2 + 1) (Kr^2 + 1) (-\beta^2 + 2(\beta^2 + 2) Kr^2 + 4) \\ \psi_2 = & \frac{24r^4 \sqrt{K + \frac{1}{r^2}} (G + \sqrt{K + \frac{1}{r^2}}) (Gr^2 \sqrt{K + \frac{1}{r^2}} + Kr^2 + 1)^2}{+ (Kr^2 + 1) (Kr^2 + 1) (-\beta^2 + 2(\beta^2 + 2) Kr^2 + 4)} \end{aligned}$$

$$\psi_3 = 0$$

$$\psi_4 = 0$$

In Weyl tensor, out of five tetrad components the BST-II ψ_2 is non zero other all the components ψ_0 , ψ_1 , ψ_3 , and ψ_4 are zero. This suggests that BST-II are Petrov type D from gravitational field, which has l^i and n^i as the propagation vectors.

4.3. Ricci tensor

The ten Ricci tensor components are denoted from three complex and four real scalars are given underneath [16, 17]

$$\phi_{00} = \frac{1}{2} R_{ab} l^a l^b; \quad \phi_{11} = \frac{1}{4} R_{ab} (l^a n^b + m^a m^{*b}); \quad \phi_{22} = \frac{1}{2} R_{ab} n^a n^b$$

$$\phi_{01} = \frac{1}{2} R_{ab} l^a m^b; \quad \phi_{02} = \frac{1}{2} R_{ab} m^a m^b; \quad \phi_{12} = \frac{1}{2} R_{ab} m^a n^b; \quad \Lambda = \frac{R}{24}$$

The Ricci tensor of non-zero component of the tetrad components are

$$\begin{aligned} \Phi_{00} = - & \frac{2\beta^2 G^3 Kr^6 (Kr^2 + 1) \sqrt{K + \frac{1}{r^2}} + \beta^2 (Kr^2 + 1) (Kr^2 + 1)^2 (2Kr^2 - 1)}{4r^4 \sqrt{K + \frac{1}{r^2}} (G + \sqrt{K + \frac{1}{r^2}}) (Gr^2 \sqrt{K + \frac{1}{r^2}} + Kr^2 + 1)^2} \\ & + 2\beta^2 Gr^2 + (Kr^2 + 1) \sqrt{K + \frac{1}{r^2}} (3K^2 r^4 + 2Kr^2 - 1) \\ & + \beta^2 G^2 r^2 + (Kr^2 + 1) (6K^2 r^4 + 5Kr^2 - 1) \end{aligned}$$

$$\begin{aligned}
\Phi_{11} = & -\frac{4G^3r^4(Kr^2+1)\sqrt{K+\frac{1}{r^2}}(\beta^2(Kr^2+1)-1)}{16r^4\sqrt{K+\frac{1}{r^2}}\left(G+\sqrt{K+\frac{1}{r^2}}\right)\left(Gr^2\sqrt{K+\frac{1}{r^2}}+Kr^2+1\right)^2} \\
& + (4G^2r^2(Kr^2+1)(Kr^2+1)(\beta^2(3Kr^2+4)-3)) \\
& + \left((Kr^2+1)^2(5\beta^2+4\beta^2K^2r^4+4(3\beta^2-1)Kr^2-4)\right) \\
& + \left(Gr^2(Kr^2+1)\sqrt{K+\frac{1}{r^2}}(17\beta^2+12\beta^2K^2r^4+4(8\beta^2-3)Kr^2-12)\right) \\
\Phi_{22} = & -\frac{2\beta^2G^3Kr^6(Kr^2+1)\sqrt{K+\frac{1}{r^2}}+\beta^2(Kr^2+1)^3(2Kr^2-1)}{4r^4\sqrt{K+\frac{1}{r^2}}\left(G+\sqrt{K+\frac{1}{r^2}}\right)\left(Gr^2\sqrt{K+\frac{1}{r^2}}+Kr^2+1\right)^2} \\
& + 2\beta^2Gr^2(Kr^2+1)\sqrt{K+\frac{1}{r^2}}(3K^2r^4+2Kr^2-1) \\
& + \beta^2G^2r^2+(Kr^2+1)(6K^2r^4+5Kr^2-1) \\
\Lambda = & -\frac{4G^3r^4(Kr^2+1)\sqrt{K+\frac{1}{r^2}}(\beta^2(3Kr^2+1)-1)}{48r^4\sqrt{K+\frac{1}{r^2}}\left(G+\sqrt{K+\frac{1}{r^2}}\right)\left(Gr^2\sqrt{K+\frac{1}{r^2}}+Kr^2+1\right)^2} \\
& + 12G^2r^2+(Kr^2+1)(Kr^2+1)(\beta^2(3Kr^2+1)-1) \\
& + 3Gr^2(Kr^2+1)\sqrt{K+\frac{1}{r^2}}(5\beta^2+12\beta^2K^2r^4+4(4\beta^2-1)Kr^2-4) \\
& + (Kr^2+1)^2(7\beta^2+12\beta^2K^2r^4+4(4\beta^2-1)Kr^2-4)
\end{aligned}$$

5. Conclusion

In Einstein's theory of relativity, the Newman Penrose formalism assumes a significant job, we examined Petrov classification in this article particularly by utilizing a consolidated spin coefficient for BST-II, we get Petrov type D. It commonly speaks for field built by timelike cosmic items. Additionally, the BST-II likewise treated to build up a similar type of field. Similarly, the BST-II also treated to develop the same type of field. Using Newman Penrose formalism we indicate the tetrad components of Ricci tensor for BST-II as we expected the complex components have been disappeared. NP formalism has demonstrated numerous application to this type of circumstance, one of that has been introduced in this article. Experiencing comparative computations done in this article, for the overall case, likewise we check spacetime that Petrov type D are all the spherically symmetric static spacetimes. While computing various Ricci coefficients we have several time cross checked it using the computer algebra Mathematica.

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