Black holes in N = 2 supergravity

Harold Erbin

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Harold Erbin

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devant le jury composé de

Iosif Bena	Membre invité
Geoffrey Compère	Rapporteur
Atish Dabholkar	Examinateur
Nick Halmagyi	Directeur de thèse
Giuseppe Policastro	Examinateur
Henning Samtleben	Rapporteur
Antoine Van Proeyen	Examinateur

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Abstract

The most general black hole solution of Einstein–Maxwell theory has been discovered by Plebański and Demiański in 1976. This thesis provides several steps towards generalizing this solution by embedding it into N = 2 gauged supergravity. The (bosonic fields of the) latter consists in the metric together with gauge fields and two kinds of scalar fields (vector scalars and hyperscalars); as a consequence finding a general solution is involved and one needs to focus on specific subclasses of solutions or to rely on solution generating algorithms.

In the first part of the thesis we approach the problem using the first strategy: we restrict our attention to BPS solutions, relying on a symplectic covariant formalism. First we study the possible Abelian gaugings involving the hyperscalars in order to understand which are the necessary conditions for obtaining $N = 2$ adS_4 vacua and near-horizon geometries associated to the asymptotics of static black holes. A preliminary step is to obtain covariant expressions for the Killing vectors of symmetric special quaternionic-Kähler manifolds. Then we describe a general analytic solutions for 1/4-BPS (extremal) black holes with mass, NUT, dyonic charges and running scalars in $N = 2$ Fayet–Iliopoulos gauged supergravity with a symmetric very special Kähler manifold.

In the second part we provide an extension of the Janis–Newman algorithm to all bosonic fields with spin less than 2, to topological horizons and to other dimensions. This provides all the necessary tools for applying this solution generating algorithm to (un)gauged supergravity, and interesting connections with the $N = 2$ supergravity theory are unravelled.

Résumé

La solution des équations d'Einstein–Maxwell décrivant le trou noir le plus général a été découverte par Plebański et Demiański en 1976. Cette thèse accomplit plusieurs étapes en vue d'intégrer une généralisation de cette solution en supergravité jaugée $N = 2$. Le contenu bosonique de cette dernière comprend la métrique assortie de champs de jauge et de deux types de champs scalaires (appelés scalaires-vecteurs et hyperscalaires) ; cela implique qu'il est beaucoup plus compliqué de trouver une solution générale et l'on doit se restreindre à des classes particulières de solutions ou bien utiliser des algorithmes pour générer des solutions.

Dans la première partie de cette thèse nous approchons ce problème grâce à la première stratégie en nous restreignant aux solutions BPS. Dans un premier temps nous étudions les jaugeages abéliens qui impliquent les hyperscalaires afin de comprendre quelles sont les conditions nécessaires pour obtenir des vides $N = 2$ adS_4 ainsi que des géométries de proche-horizon associées à des trous noirs statiques. Par la suite nous décrivons une solution générale et analytique pour des trous noirs (extrémaux) 1/4-BPS qui possèdent une masse, une charge de NUT, des charges dyoniques et des champs scalaires non-triviaux dans le contexte de la supergravité $N = 2$ jaugée à la Fayet–Iliopoulos.

Dans la seconde partie nous obtenons une extension de l'algorithme de Janis-Newman afin de prendre en compte tous les champs bosoniques de spin inférieur à 2, les horizons topologiques et le cas des autres dimensions. Ainsi cela met à disposition tous les outils nécessaires pour appliquer cet algorithme à la supergravité (jaugée ou non).

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Introduction

0.1 Background

0.1.1 Quantum gravity and string theory

Finding a theory of quantum gravity is a major goal of theoretical physics. Indeed the 20th century has seen the discovery of two great theories – quantum field theory (QFT) and general relativity (GR) – that both work extremely well in their respective domains of application but which cannot be reconciled on the overlap. The main difficulty resides in the fact that QFT rely heavily on the concept of renormalization in order to obtain sensible results from the computations that would otherwise yield divergences. On the other hand GR is non-renormalizable and leads to incurable divergences.

A theory of quantum gravity is needed in order to answer some of the most important questions concerning our universe. In particular primordial cosmology and the origin of the universe can be properly address only within this context as they touch the very nature of spacetime and the latter require a complete theory of quantum gravity to be properly understood. Similarly black holes are objects formed by a huge concentration of matter and they cannot be properly described in general relativity. For the moment these problems get only partial answers by using semi-classical methods. Both cases are linked to the presence of *singularities* (the Big-Bang and the center of the black hole) that should be resolved by a proper quantum treatment of gravity.

Another interesting quest is the unification of the forces and the understanding of the very nature of interactions and matter. The current knowledge culminates in the standard model of particle physics which describe all matter and non-gravitational forces that have been measured. But this theory is still unsatisfactory for several reasons: there are many free parameters (19 plus 7-8 neutrino masses) that are lacking theoretical interpretation. Similarly the hierarchy problem states that the Higgs mass should be of the same order of the cut-off scale where new physics appear (or the Planck mass otherwise), and in the current framework this value can be understand only by a very fine-tuning of the parameters, which is not natural. Another problem is the prediction of a huge value for the cosmological constant. The two last points are related to the question of naturalness which asks that parameters have natural values (in the correct units). Finally the standard model does not explain why there are three generations of fermions, the mass of the neutrinos nor why the gauge group is

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1). \tag{0.1.1}$$

A satisfying theory should be able to provide the derivation of the parameters from more fundamental properties (for examples through the dynamics of background fields) and to explain why one observes this field content. A first possibility is to unify the gauge group into one unique group at higher energy which would reduce the number of gauge couplings and unify matter families (through the embedding into representations of this group).

String theory is a promising candidate for a consistent quantum gravity theory which provides a grand unification framework at the same time. In this theory the fundamental constituents are strings and the usual fields appear as excitation modes of these strings.

The interactions of the strings are non-local in spacetime and this smearing reduces the UV divergences as interactions cannot be concentrated at a point. The very existence of a fundamental string puts very stringent constraint on the structure of spacetime: supersymmetry is necessary for having a consistent theory, and spacetime should have 10 dimensions (for the five possible superstring theories). Hence one needs to hide these dimensions, either by compactification (with Kaluza–Klein dimensional reduction) or by using a braneworld scenario [1–3]. On the bright side string theory is unique and it describes quantum gravity unified to matter and interactions, and there are no free parameters (before compactification).

For decades the developments of string theory were limited to a perturbative analysis. Recently the understanding of string theory has been deepened by a series of discoveries concerning its non-perturbative structure: all five superstring theories (type II A and B, type I and two heterotic) are related by dualities to each other, and to an 11-dimensional theory called M-theory. The latter is unique and is believed to be the fundamental theory, but its definition is not known, and only some of its aspects are understood in some limits. Finally the previous analysis yielded the existence of branes which are extended objects generalizing particles and strings. They proved to be fundamental in the realization of black holes from string theory.

0.1.2 Supersymmetry and supergravity

In order to pursue the goal of unification one could ask if the internal gauge symmetry can be unified with spacetime symmetries. A no-go theorem from Coleman and Mandula [4] stated that it was impossible and the symmetry group is necessary a direct product

$$\text{conformal} \times \text{internal} \tag{0.1.2}$$

(in general one considers the Poincaré subgroup of the conformal group). But Haag, Łopuszański and Sohnius discovered a loophole in the argument [5]: the above group can be extended into the superconformal group (which includes the super-Poincaré group) by adding anticommuting generators. This group contains an automorphism subgroup called the R-symmetry group that acts both on the fermionic generators and as an internal symmetry.

Supersymmetry is generated by fermionic generators Q and it relates bosons to fermions, and conversely

$$Q\,|\text{fermion}\rangle = |\text{boson}\rangle\,, \qquad Q\,|\text{boson}\rangle = |\text{fermion}\rangle\,, \tag{0.1.3}$$

and the anticommutator of these generators is equivalent to a translation

$$\{Q, Q\} \sim P. \tag{0.1.4}$$

Fields of different spins are gathered into multiplets that transform irreducibly under super-Poincaré transformations. A theory with supersymmetry is characterized by the number N of fermionic generators; in $d = 4$ the condition that no spin higher than 2 are generated implies that $N \leq 8$ (when $N \geq 2$ one speaks about extended supersymmetry). This symmetry is very powerful and imposes constraints – the higher N is, the more severe they are – on the theory. For example $N = 1$ is already sufficient for curing some of the problems of the standard model (even if these extensions suffer from other problems): the Higgs mass is stabilized as it inherits the mass protection from its partner. For extended supersymmetry exact solutions could be derived, see for example the work of Seiberg and Witten on $N = 2$ [6, 7] and the integrability of $N = 4$ [8–10]. The reason is that the scalar fields ϕ^i parametrizes a non-linear sigma model

$$\mathcal{L} = -\frac{1}{2}\, g_{ij}(\phi)\, \partial\phi^i \partial\phi^j \tag{0.1.5}$$

whose target manifold with metric g_{ij} is very constrained by supersymmetry, and other fields of the multiplets inherits these properties. In particular the isometry group of this manifold translates (mostly) into the global symmetry of the Lagrangian.

Interestingly local supersymmetry includes general relativity: indeed the fact that the anticommutators of two supersymmetries close on the momentum implies that one cannot make local supersymmetry without making local the Poincaré group. This theory is called *supergravity*. In this context the R-symmetry group is made local and provides gauge interactions: this leads to a unification of spacetime and internal gauge symmetries!

As seen in the previous section, supersymmetry is necessary ingredient of string theory for including fermions in the spectrum and for removing inconsistencies (such as the tachyons). In this case supergravity corresponds to the low-energy approximation of superstring theories.

In this thesis we focus on $N = 2$ supergravity. The latter admits three main multiplets: the gravity multiplet (containing the metric and a vector field called the graviphoton), the vector multiplet (containing a vector field and a complex scalar field) and the hypermultiplet (containing four real scalar fields). This theory has more symmetries than $N = 1$ and the additional structures facilitate the computations, but it is also less constrained than higher N theories (such as the maximal $N = 8$ supergravity) and as a consequence it has a richer dynamics and admits more different models. The scalar manifold in $N = 1$ is only Kähler, while in $N = 2$ additional conditions imply that it is a direct product

$$\text{special Kähler} \times \text{quaternionic}, \tag{0.1.6}$$

and there is little freedom in their definition (for example a unique holomorphic function is sufficient to define a special Kähler manifold). Finally the scalar manifolds of $N > 2$ supergravity are all symmetric and fixed once the number of vector multiplets is given (hence the manifold is unique for $N > 4$). These spaces possess very interesting geometrical properties which all have an interpretation from supersymmetry.

Currently supersymmetry has not been found in nature, which means that it should be broken at an energy higher than those accessible in the current experiments. From the phenomenological point of view theories with a low number of supersymmetries ($N = 1, 2$) are preferable since they are closer to the standard model. Moreover $N = 2$ supergravity corresponds to the effective action of the low-energy limit of type II string theory compactified on a Calabi–Yau manifold. These models present some interest because they are very similar to the $N = 1$ theories resulting from the compactification of the heterotic string theory on a Calabi–Yau manifold [11–13].

The simplest version of these theories are called ungauged theories because the only local symmetry corresponds to the local super-Poincaré group. The $N = 2$ theory is quite simple in this case as some fields decouple from the others due to the absence of scalar potential (this also imply a vanishing cosmological constant). In order to get a richer dynamics one needs to deform the theory by using some of the vector fields as gauge fields for a local gauge symmetry – one then obtains a gauged supergravity. In the context of string compactification, this corresponds to some p-forms which are not vanishing along cycles of the internal manifold.

Finally supergravity is interesting by itself as a theory of quantum gravity: it is known that supersymmetry improves the ultraviolet behaviour of a theory. For example $N = 4$ super Yang–Mills is perturbatively finite. There is hope that a similar property is true for the maximal $N = 8$ supergravity: in particular recent studies have shown by explicit computations that expected loop divergences (from symmetry arguments) do not appear, for example at 3-loops in $N = 4$ (see for example [14–17]).

0.1.3 Black holes

General relativity is the theory of gravitational phenomena. It describes the dynamical evolution of spacetime through the Einstein–Hilbert action that leads to Einstein equations.

The latter are highly non-linear differential equations and finding exact solutions is a notoriously difficult problem. There are different types of solutions but this thesis will cover only black-hole-like solutions (type-D in the Petrov classification) which can be described as particle-like objects that carry some charges, such as a mass or an electric charge.

Black holes are very specific entities that put a lot of strain on theories of quantum gravity, and as such they are useful sandboxes where one can test the properties and the predictions of the theory. Rotating black holes are the most relevant subcases for astrophysics as it is believed that most astrophysical black holes are rotating. These solutions may also provide exterior metric for rotating stars.

They resemble a lot a particle in the sense that they do not seem to have a structure: they are defined by few parameters – such as the mass, the electric charge or the angular momentum –, and any perturbation of a black hole dies off quickly. The most general solution of this type in pure Einstein–Maxwell gravity is the Plebański–Demiański metric [18, 19]: it possesses six charges: mass m, NUT charge n, electric charge q, magnetic charge p, rotation j and acceleration a.

Classically a black hole is a region delimited by an horizon where the gravitational field is so strong that nothing can escape from it (not even light), and they can be formed from the gravitational collapse of a supermassive star. At the center of the black hole is a singularity where the curvature of spacetime becomes infinite. Such divergence indicates a breakdown of the theory: indeed gravitational effects are so important close to the origin that classical GR is not sufficient and one needs a full quantization of gravity in order to account for quantum effects.

Bekenstein and Hawking discovered that a black hole behaves like a thermodynamical system in the sense that it has a temperature T, an entropy S, and each charge is associated to a potential. A black hole emits a perfect black body radiation at the temperature T which is related to the gravity on the horizon (called the surface gravity). Then the entropy can be derived from the first law using the relation between the mass and the energy. This picture explains the apparent simplicity of black holes: a statistical ensemble made of a great number of particles moving in a box is determined only by few parameters (temperature, pressure. . .). Statistical physics teaches us that entropy is related to the number of microstates of a system, and it is very natural to ask from a theory of quantum gravity what are these states for the black holes. A specific subclass consists of extremal black holes which have a vanishing temperature.

Usual systems have accustomed us to think that the entropy of a system should be proportional to its volume. This is not the case in gravity where the entropy follows an area law

$$S = \frac{A k_b}{4 \ell_p^2}, \tag{0.1.7}$$

where A is the area of the horizon. This means that there is far less degrees of freedom than what one would think, and these would live on the horizon of the black hole. This suggests the existence of an holographic principle which states that (some) gravitational systems can be entirely described by data on their boundary. This principle has seen a nice realization within string theory under the adS/CFT correspondence.

Black holes are such special that it is always useful to classify all possible black hole solution that can be found in a given theory or in its low-energy limit. Hence studying black holes in supergravity gives indirect clues on the structure of string theory. In their seminal paper [20], Strominger and Vafa set up a framework where the microstates were identified with branes. The agreement between the microscopic counting and the macroscopic entropy computed in the corresponding supergravity have been shown to hold for many BPS or extremal black holes.

0.1.4 BPS solutions and adS black holes

A BPS solution of supergravity is a solution of the equations of motion which preserves some supersymmetry (indicated as a fraction), i.e. it is annihilated by the action of some supersymmetry generators and it defines a background with its own supersymmetry algebra. Extremal black holes form long BPS representations and the action of supersymmetry is well defined, which is not the case for finite temperature black holes [21, p. 8], and for this reason they share similar properties.[1] These solutions are very useful because some of their properties are protected by non-renormalization theorem due to supersymmetry, and this makes it possible to infer their behaviour at strong coupling. In particular this last property is essential for comparing the entropy with the microstate counting.

Extremal black holes can be seen as solitons, i.e. solutions interpolating between two vacua, one sitting at the radial infinity (called the UV), the other being the near-horizon geometry (the IR) – both are solutions of the BPS equations. They are subject to the so-called *attractor mechanism* [22–26]: the scalar fields take on the horizon constant values which depends only of the electromagnetic charges of the solution. This is as if the fields were forgetting everything about their radial evolution outside the black hole, and in particular the corresponding values do not depend on the values at infinity.

We will mainly focus on adS black holes which have a negative cosmological constant. The first motivation is to provide solutions that can be used in the context of the adS/CFT correspondence, and in particular for the application to condensed matter through adS/CMT [27–29]. Moreover solutions with a negative cosmological constant are more natural in the context of gauged supergravity and string theory. AdS black holes present a richer thermodynamics [30, 31] than their asymptotically flat cousins; this results from the cosmological constant which acts as a space cut-off, the black hole does not feel the entire spacetime and is more stable as a consequence. Another interesting property of adS space is that a field can have a negative mass without being unstable if it satisfies the BreitenLohner–Freedman (BF) bound [32, 33].

Strictly speaking adS black holes are not asymptotic to adS space: if magnetic charges are present then the asymptotic space is deformed to the so-called magnetic adS (madS). It can be shown that to each madS vacuum is associated an adS vacuum. 1/2-BPS black holes are asymptotically adS but they correspond typically to a naked singularity, and for this reason we will concentrate on 1/4-BPS black holes.

0.1.5 Taub–NUT spacetime

The Taub–NUT spacetime is very peculiar and Misner said it was “a counterexample to almost anything” believed in general relativity. For example it can be BPS without being extremal. This solution is characterized by the NUT charge n which plays the same role as the magnetic charge in electromagnetism (in this analogy the usual mass corresponds to the electric charge) and for this reason one also refers to it as a magnetic mass.

This spacetime is a solution of the vacuum Einstein equation with no cosmological constant. In this case the space is not asymptotically flat and it is characterized by the value of n, the off-diagonal component of the metric giving a vector potential

$$A_\phi \sim g_{t\phi} = 2n\cos\theta. \tag{0.1.8}$$

This is recognized as being the potential of a magnetic-like monopole. On the other hand the solution can also include a mass m which asymptotically gives the usual scalar potential

$$\phi \sim \frac{1}{2}\,(1 - g_{tt}) = -\frac{m}{r} \tag{0.1.9}$$

which is the potential of an electric-like point source. Then the Taub–NUT solution with mass is a gravitational dyon.

[1] Moreover a static BPS black hole is necessarily extremal.

The metric does not have any curvature singularity, in particular the space is regular at $r = 0$. But the metric suffers from a worse pathology which is the presence of Misner strings due to wire-like singularities (this is similar to the Dirac strings that one introduces with magnetic monopoles). These strings can be removed by using two patches of coordinates, but as a consequence closed timelike curves appear, with the periodicity of the time given by

$$\Delta t = 8\pi n. \tag{0.1.10}$$

Closed timelike curve may not appear for hyperbolic black holes if the NUT charge lies in some range [34, 35].

The solution is better behaved in Euclidean signature. There it corresponds to a gravitational instanton, which is a non-singular solution of the equations of motion with a finite action that contributes to the computation of the partition function in the saddle point approximation.

The NUT charge can be incorporated in more general solutions, for example in supergravity and with a non-vanishing cosmological constant.

0.2 Motivations

0.2.1 Supergravity

The last decades has seen a lot of works on $N = 2$ gauged supergravity for its applications on string phenomenology, holography and black holes. While many the ungauged theory has been deeply studied and understood, much less is known on the gauged version. For example a complete classification of BPS solutions exist [36–39], the attractor mechanism has received a lot of attention [40–42]), and fairly general non-extremal solutions have been found [43, 44].

The first step is to study the vacua that can be obtained in this theory. In particular the most natural one is the $N = 2$ adS_4 vacua which have been discussed in [45–49], while adS_4 vacua with less supersymmetries were found in [48, 50, 51]. Another important type of vacua consists in the near-horizon geometries $\mathrm{adS}_2 \times \Sigma_g$ where Σ_g is a Riemann surface of genus g, and it has also received attention recently [46, 49, 52, 53]. Some steps towards a classification of the BPS solutions have been taken in [54–57]. The equations for more specific ansatz have also been studied, for example static black holes [58–63] or maximally supersymmetric solutions [47]. The supersymmetry algebras associated to BPS solutions were worked out in [64, 65]. Finally the attractor mechanism also takes place in these theories [52, 58, 66–71].

As reviewed above the archetypal black hole of Einstein–Maxwell theory with cosmological constant is the Plebański–Demiański (PD) solution [18, 19] which contains six charges: mass m, NUT n, electric q and magnetic p charges, spin j and acceleration a. In the context of supergravity on adS space and of adS/CFT it is natural to consider topological horizons, which are not only spherical, but also flat or hyperbolic (or a compact Riemann surface obtained by quotienting with a discrete group) [72–74]; indeed the usual wisdom about horizon topology in asymptotically flat spaces does not hold for adS spaces [75]. The supersymmetry of the (topological) PD solution and its truncations has been studied in [31, 75–78] by embedding it into pure $N = 2$ gauged supergravity, which is equivalent to taking constant scalars. Non-BPS solutions with running scalars have been studied in the STU model (which includes three vector multiplets) and its truncations [79–83]. Constructing the general solution with non-constant scalars in general $N = 2$ gauged supergravity is an outstanding goal, and a first step is to look at the BPS subclass which is simpler to study.

In ungauged supergravity static black holes are 1/2-BPS. The corresponding solutions in gauged supergravity are naked singularity (but there are regular 1/2-BPS rotating black holes) and cannot have magnetic charges [60, 61, 63, 75]. A static 1/4-BPS black hole with constant scalars was found in [59] where it was put forward that the solution is regular only if the horizon is hyperbolic. An important step has been taken by Cacciatori and Klemm who found the first regular 1/4-BPS black holes with running scalars in the STU model [58], and it was generalized to any symmetric very special manifold in [46] in the case of vanishing axions. In particular it was shown in [52, 60] that spherical horizons are possible if the scalars are non-trivial. These solutions have no flat space limit and are thus very different from the 1/2-BPS solutions [60]; as explained above they have a madS vacua. Finally the general analytic 1/4-BPS solution of Fayet–Iliopoulos (FI) gauged supergravity with a symmetric scalar manifold (with an arbitrary number of vector multiplets, running scalars and dyonic charges) was built in [84] using a formalism developed in [85] which rely heavily on the properties of very special Kähler manifolds. A 1/4-BPS black hole with NUT and magnetic charges was constructed in the case of only one vector multiplet [86]. All the previous discussion apply to FI gauged supergravity, but very few solutions with hypermultiplets have been found: recently an analytic BPS solution have been described in [67], while some numerical 1/4-BPS solutions were built in [62] (1/2-BPS solutions with pathological behaviour have been discussed in [61]). Finally 1/8-BPS solutions were classified in [57].

Solutions with a NUT charge are interesting in the fluid/gravity correspondence where a NUT charge in spacetime translates to vorticity in the dual fluid [87–90]. Another interesting path is to perform a Wick rotation and to compare the free energy with the result in the dual CFT using localization. Indeed it was put in evidence in a series of papers by Martelli and collaborators on minimal $N = 2$ gauged supergravity that the NUT charge and the acceleration correspond to the two squashing parameters of the boundary S^3 [91–94].

0.2.2 Demiański–Janis–Newman algorithm

As the complexity of the equations of motion increase, it is harder to find exact analytical solutions, and one often consider specific types of solutions (extremal, BPS), truncations (some fields are constant, equal or vanishing) or solutions with restricted number of charges. Then it is interesting to find solution generating algorithms which are procedures which transform a seed configuration to another configuration with a greater complexity (for example with a higher number of charges).

An algorithm which is on-shell is very previous because one is sure to obtain a solution when starting with a seed configuration which solves the equations of motion. On the other hand off-shell algorithms do not necessarily preserve the equations of motion, but they are nonetheless very precious: they provide a motivated ansatz, and it is always easier to check if an ansatz satisfy the equations than solving them from scratch. Even if in practice this kind of solution generating technique does not provide so many new solutions, it can help to understand better the underlying theory (which can be general relativity, modified gravities or even supergravity) [95] and it may shed light on the structure of gravitational solutions.

Janis–Newman (JN) algorithm is one of these (off-shell) solution generating techniques, which – in its original formulation – could be used to generate rotating metrics from static ones. It was used by Janis and Newman to give another derivation of the Kerr metric [96], while shortly after it has been used again to discover the Kerr–Newman metric [97].

This algorithm provides a way to generate axisymmetric metrics from a spherically symmetric seed metric through a particular complexification of radial and (null) time coordinates, followed by a complex coordinate transformation. Often one performs a change of coordinates to write the result in Boyer–Lindquist coordinates. The original prescription uses the Newman–Penrose tetrad formalism, which appears to be very tedious since it requires to invert the metric, to find a null tetrad basis where the transformation can be applied, and lastly to invert again the metric. In [98] Giampieri introduced another formulation of the JN algorithm which avoids gymnastics with null tetrads and which appears to be very useful for extending the procedure to more complicated solutions (such as higher dimensional ones). However it has been so far totally ignored in the literature and the first published and widely accessible paper on this topic is [99]. We stress that *all* results are totally equivalent in both approaches, and every computation that can be done with Giampieri's prescription can be done with the other.

In order for the metric to be still real, the seed metric functions[2] must be transformed such that reality is preserved.[3] Despite that there is *no* rigorous statement concerning the possible complexification of these functions, some general features have been worked out in the last decades and a set of rules has been established. Note that this step is the same in both prescriptions. In particular these rules can be obtained by solving the equations of motion for some examples and by identifying the terms in the solution [100]. Another approach consists in expressing the metric functions in terms of the Boyer–Lindquist functions – that appear in the change of coordinates and which are real –, the latter being then determined from the equations of motion [101, 102].

It is widely believed that the JN algorithm is just a trick without any physical or mathematical basis, which is not accurate. Indeed it was proved by Talbot [103] shortly after its discovery why this transformation was well-defined, and he characterizes under which conditions the algorithm is on-shell for a subclass of Kerr–Schild (KS) metrics (see also [104]).[4] KS metrics admit a very natural formulation in terms of complex functions for which (some) complex change of coordinates can be defined. Note that KS metrics are physically interesting as they contain solutions of Petrov type II and D. Another way to understand this algorithm has been provided by Schiffer et al. [105] who showed that some KS metrics can be written in terms of a unique complex generating function, from which other solutions can be obtained through a complex change of coordinates. In various papers, Newman shows that the imaginary part of complex coordinates may be interpreted as an angular momentum, and there are similar correspondences for other charges (magnetic. . .) [106–108]. More recently Ferraro shed a new light on the JN algorithm using Cartan formalism [109]. A recent account on these points can be found in [110].

Other solution generating algorithm rely on a complex formulation of general relativity which allows complex changes of coordinates. This is the case of the Ernst potential formulation [111, 112] or of Quevedo's formalism who decomposes the Riemann tensor in irreducible representations of $\mathrm{SO}(3, \mathbb{C}) \sim \mathrm{SO}(3, 1)$ and then uses the symmetry group to generate new solutions [113, 114].

The JN algorithm has been used to find new solutions as well as to show that known solutions could be derived in this way. For instance it has been applied to dilatonic gravity [115], interior solutions [101, 102, 116–121] and other dimensions [122–124].[5] The list is short because the algorithm could be used only to derive the metric, and all other fields had to be found using equations of motion. Moreover many works [125–130, 131, sec. 5.4.2] (to cite only few) are wrong or not reliable because they do not check the equations of motion or they perform non-integrable Boyer–Lindquist changes of coordinates [99, 121, 132, 133].

The algorithm has later been extended to what we call the Demianski–Janis–Newman (DJN) algorithm, when Demiański (and partially Newman) showed that other parameters can be added [100, 134], even in the presence of a cosmological constant.[6]

More recently it has been investigated whether the JN algorithm can be applied in modified theories of gravity. Pirogov put forward that rotating metrics obtained from the JN algorithm in Brans–Dicke theory are not solutions if $\alpha \neq 1$ [138]. Similarly Hansen and

[2]We call a "seed/stationary metric function" a function that appears in the seed/stationary metric. The term "stationary" is used to describe the metric resulting from the DJN algorithm, which generically is non-static.

[3]For simplifying, we will say that we complexify the functions inside the metric when we perform this transformation, even if in practice we "realify" them.

[4]It has not been proved that the KS condition is necessary, but all known examples seem to fit in this category.

[5]A general strategy for interior solutions is the following: find the stationary metric, and then describe the fluid stress-energy tensor that allows to solve the equations of motion. Note that here the fluid is in general not present and the algorithm is just seen as a way to provide a motivated ansatz for the metric. As a consequence one can also add angular momentum with non-vanishing cosmological constant, despite the Demiański's result [100] (more details later).

[6]Demiański's metric has been generalized in [135–137].

Yunes have shown a similar result in quadratic modified gravity (which includes Gauss–Bonnet) [139].[7] These do not include Sen's dilaton–axion black hole for which $\alpha = 1$ (section 16.3.4), nor the BBMB black hole from conformal gravity (section 16.2.3). Finally it was proved in [140] that it does not work either for Einstein–Born–Infled theories. We note that all these no-go theorem have been found by assuming a transformation with only rotation.

Detailed reviews on generalizations and explanations of the JN algorithm can be found in [110, 142, chap. 19, 101, 131, sec. 5.4] (see also [143]).

0.3 Content

0.3.1 Supergravity

An important motivation of this work is to study black holes which can be embedded into M-theory, such as the STU model with a specific choice of gaugings which is a dimensional reduction of $d = 11$ supergravity on S^7. In presence of a NUT charge the holographic duals correspond to the ABJM theory on a curved manifold. In particular after the Euclidean continuation these contain Seifert spaces (given by a U(1) bundle over Σ_g), including the Lens spaces $S^3/\mathbb{Z}_n$, where supersymmetry has been preserved by twisting the theory with respect to a general $\mathrm{U}(1) \subset \mathrm{SU}(4)_R \times \mathrm{U}(1)_R$. From an $N = 2$ point of view this includes flavour as well as R-symmetries.

The goal of this work is to deepen the understanding of BPS solutions in (matter-coupled) $N = 2$ gauged supergravity with abelian gaugings. When there are no hypermultiplet this corresponds to Fayet–Iliopoulos (FI) gauging.

In the case where hypermultiplets are present, the hyperscalars are the only scalar fields to be charged. Fortunately the isometries of homogeneous (symmetric or not) special quaternionic manifolds have been classified by de Wit and van Proeyen [12, 144–146]. These manifolds are constructed as a fibration over a special Kähler manifold through the c-map, and some isometries of the latter can be lifted to the full quaternionic spaces. In this work we are building on these results to provide symplectic covariant expressions for the Killing vectors and prepotentials for symmetric spaces only. This helps to clarify a conceptual point on the so-called hidden Killing vectors: they must act symplectically on the coordinates of the base special Kähler space and this was not evident in the analysis of de Wit and van Proeyen. Symmetric manifolds are coset spaces for which all possible isometries are realized and form a semi-simple Lie algebra.

The holonomy group of quaternionic manifolds contains an SU(2) factor which corresponds to the SU(2) R-symmetry of the $N = 2$ super-Poincaré algebra. A Killing vector does not need to preserve the SU(2) connections and it can induce a rotation given by a 3-vector called the compensator. It was already known that a necessary condition for getting a $N = 2$ adS_4 vacua is that at least one isometry with a non-trivial compensator be gauged [48, 50]. In particular we list the isometries with such compensators, and all of them are model-dependent (the isometries of the Heisenberg algebra associated to the Ramond scalars).

We also analyse $\mathrm{adS}_2 \times \Sigma_g$ vacua. In the case of FI gaugings this was solved in [53]. Since the equations for the vector and hyperscalars are decoupled we find that the entropy is given by the same formulas in both cases, except for the replacement of the FI parameters by the Killing prepotentials.

The idea in these two cases is to first solve the problem in FI supergravity by treating the prepotentials as constants. This provides a solution for the vector scalars in terms of

[7] There are some errors in the introduction of [139]: they report incorrectly that the result from [138] implies that Sen's black hole cannot be derived from the JN algorithm, as was done by Yazadjiev [115]. But this black hole corresponds to $\alpha = 1$ and as reported above there is no problem in this case (see [141] for comparison). Moreover they argue that several works published before 2013 did not take into account the results of Pirogov [138], published in 2013. . .

the charges, gauging parameters and hyperscalars which can be fed into the other equations. We give examples for models which correspond to consistent truncations of M-theory.

Solutions with less charges are easier to find and we focus on NUT charged ones. The addition of this charge is very natural because it preserves the SU(2) isometry and the hope is that BPS equations are not much different from the static case. The simple adS–NUT Schwarzschild black hole can be obtained from a limit of the PD solution, and there are two BPS branches preserving a half and quarter of the supersymmetry. An intriguing property in the presence of a NUT charge is the existence of BPS solutions that are not extremal and without horizons. On the other hand if there is an horizon then the solution is necessarily extremal. We discuss the root structure of the metric functions in order to clarify the different possibilities.

Then we compute the 1/4-BPS equations for NUT black hole in FI gauged supergravity and we look for solutions by using the techniques of [84]. In the case of extremal black hole we arrive at an analytic solution with running scalars and dyonic charges which generalize the one of [84]. In particular the near-horizon geometry does not feel the NUT charge. We were not able to find the general solution in the case where the black hole is not extremal, and it is not known if there are solutions with different near-horizon geometries or if they would simply be without horizons. Nonetheless we construct the constant scalar solutions in this formalism.

Symmetric Kähler manifolds are endowed with a invariant symmetric 4-tensor because the isometry group are of type E_7 [147, 148]. This quartic invariant appears in the expressions of the Killing vectors of symmetric special quaternionic manifolds, of the black hole entropy and the radius of adS_4, of the BPS equations and of the analytic solutions for static and NUT-charged dyonic 1/4-BPS black holes [53, 84, 144, 149–151].

In conclusion the achievements of the current work are:

- symplectic covariant expressions for the quaternionic isometries;
- BPS equations with magnetic gaugings for matter-coupled $N = 2$ gauged supergravity;
- a framework for studying $N = 2$ adS_4 and $adS_2 \times \Sigma_g$ vacua with abelian gaugings;
- quite generic solution for 1/4-BPS black holes with FI gaugings;

As a future direction one can extend the analysis of the BPS black holes (both static and with a NUT charge) in order to include hypermultiplets. A simpler intermediate goal would be to find an analytic solution of the scalars in terms of the charges for the vacua. Another topic which has recently benefited from the study of quaternionic isometries is inflation in $N = 2$ supergravity where it was shown that at least one hidden isometry needs to be gauged in order to construct a physical model [152, 153].

Despite the fact that it would be very interesting to find the most general 1/4-BPS NUT solution when the horizon is not $adS_2 \times \Sigma_g$, it may be more important to look first to solutions with rotation and acceleration[8] or at 1/2-BPS NUT solutions with running scalars. With more supersymmetry it would be easier to compute the microstates of these black holes.

It is not clear how the solution of Chow and Compère [80] is related to the known 1/4-BPS solutions and this point calls for an explanation. Finally computing the holographic free energy of the NUT charged solution is an interesting problem.

In all cases keeping the symplectic covariance of the equations by considering the general case was a key step in order to build the solutions by exploiting the power of the special geometry, and in particular of the quartic invariant. In the same idea it would be useful to extend the symplectic covariance of the Killing vectors to the case of homogeneous spaces and for non-abelian gaugings.

[8] In particular solutions with acceleration has been discovered recently [154, 155], and the rotating black holes from [79, 80] may give some intuitions. Also in this case the near-horizon geometries will certainly be different and a first analysis would be to look at these solutions.

0.3.2 Demiański–Janis–Newman algorithm

As explained in the previous section, the JN algorithm was formulated only for the metric and all other fields had to be found using the equations of motion (with or without using an ansatz). For example neither the Kerr–Newman gauge field or its associated field strength could be derived in [97]. The solution to this problem is to perform a gauge transformation in order to remove the radial component of the gauge field in null coordinates [99]. It is then straightforward to apply the JN algorithm in either prescription.[9]

Another problem was exemplified by the derivation of Sen's axion–dilaton rotating black hole [157] by Yazadjiev [115], who could find the metric and the dilaton, but not the axion nor the gauge field. The reason is that while the JN algorithm applies directly to real scalar fields, it does not for complex scalar fields (or for a pair of real fields that can naturally be gathered into a complex scalar). Then it is necessary to consider the complex scalar as a unique object and to perform the transformation without trying to keep it real [158].

Hence this completes the JN algorithm for all bosonic fields with spin less than 2.

Demiański's analysis reveals itself to be very useful in order to find the most general transformation. We have extended its analysis to Einstein–Maxwell gravity and to topological horizons [159], fixing also some errors that appeared in his work due to an hidden hypothesis. This has also been the occasion to provide very generic formulas for the configurations obtained after performing the DJN algorithm. A long standing issue of this analysis was to find how one should complexify the metric function: the usual rules do not work in presence of a NUT charge, and if there is no way to obtain the function by complexification it would imply that the most general transformation are useless because they can not be used in other cases (except if one is willing to solve Einstein equations, which is not the goal of a solution generating technique). We have found that it is necessary to complexify also the mass and to consider the complex parameter $m + in$ [158, 159]. Similarly configurations with magnetic charges were out of reach, and we have shown that one needs to consider the complex charge $q + ip$ [158]. Such a complex combination is quite natural from the point of view of Plebański–Demiański solution [18, 19]. It is to notice, that the appearance of complex coordinate transformations mixed with complex parameter transformations was a feature of Quevedo's solution generating technique [113, 114]. Yet it is unclear what the link with our approach really is, despite the fact that it may probably provide some clues for generalizing further the DJN algorithm (higher dimensions, cosmological backgrounds…).

Hence the final metric may contain (for vanishing cosmological constant) five of the six Plebański–Demiański parameters [18, 19] along with Demiański's parameter. It is intriguing that one could get all Plebański–Demiański parameters but the acceleration, which appears in the combination $a + i\alpha$.

We also comment the group properties that some of the DJN transformations possess [159]. This observation can be useful for chaining several transformations or to add parameters to solutions that already contain some of the parameters (for example adding a rotation to a solution that already contains a NUT charge).

We also extended the algorithm to five dimensions [160], where the key idea is to perform the transformation only on the metric parts that describe the rotation plane that we are looking for. We also give a proposal for the metric in higher dimensions but we could not transform the function itself.

Finally a very general *Mathematica* package has been written for the DJN algorithm in Einstein –Maxwell theory and it is available on demand.

All these results provide a complete framework for most of the theories of gravity that are commonly used. A major playground for this modified Demiański–Janis–Newman (DJN) algorithm would be (gauged) supergravity where many interesting solutions remain to be discovered.

As a conclusion we summarize the features of our new results:

[9] Another solution has been proposed by Keane [156], but it is applicable only to the Newman–Penrose coefficients of the field strength.

- all bosonic fields with spin ≤ 2;
- topological horizons;
- charges m, n, q, p, a (with a only for $\Lambda = 0$);
- group properties;
- extend to $d = 3, 5$ dimensions (and proposal for higher).

Here is a list of new examples that have been derived using the previous results (all in $4d$ except when said explicitly):

- Kerr–Newman–NUT;
- dyonic Kerr–Newman;
- Yang–Mills Kerr–Newman black hole [161];
- adS–NUT Schwarzschild;
- ungauged $N = 2$ BPS solutions [36];
- non-extremal solution in T^3 model [157] (partly derived in [115]);
- SWIP solutions [162];
- charged Taub–NUT–BBMB with Λ [35];
- $5d$ Myers–Perry [163];
- $5d$ BMPV [164].

We also found [160] a more direct derivation of the rotating BTZ black hole (derived in another way by Kim [123, 124]). Moreover Klemm and Rabbiosi showed how to recover the NUT charged black hole in gauged $N = 2$ sugra with $F = -i\, X^0 X^1$ from [82].[10] Note that all these examples appear to be related to $N = 2$ supergravity.

Despite the fact that the JN algorithm is partly understood, a better understanding is called for. In particular it seems linked with ($N = 2$) supergravity and it is possible that a natural explanation could be found in supersymmetry. Another interesting application would be to derive generating functions (e.g. the fake superpotential in $N = 2$ supergravity) for rotating black holes from static ones. Moreover another question is to understand which 1/4-BPS static black holes from section 12.3 can be mapped to the solutions of section 13.4. Finally the question of acceleration remains open.

0.4 Structure

In part I we review the ungauged and gauged $N = 2$ supergravity: it describes the multiplets, the bosonic Lagrangian, the supersymmetry variations and the gauging procedure. These chapters are mostly self-contained and include a minimal description of the scalar manifolds. Next in part II we describe the properties of the scalar manifolds: this corresponds to a special Kähler manifold for the vector scalars, and to a quaternionic manifold for the hyperscalars. We describe the Riemannian properties of these manifolds and we build the isometries, focusing particularly on symmetric spaces. Then in part III we look at the BPS equations and their static and NUT charged solutions. We start this part with a chapter on the general properties of adS–NUT black holes. Finally part IV is devoted to the Demiański–Janis–Newman algorithm. We start by a simple presentation of the algorithm before giving general formulas for all fields with spins less than two. Conventions, background informations, long formulas and computations are relegated in appendix V.

[10] Private communication by D. Klemm.

0.5 Publications

The content of the thesis is based on the following papers:

- Chapters 7, 9, 11, 12, section 2.6 [149]
  H. Erbin and N. Halmagyi. "Abelian Hypermultiplet Gaugings and BPS Vacua in $N = 2$ Supergravity". *JHEP* 2015.5 (May 2015), arXiv: 1409.6310.

- Chapters 10, 13 [150]
  H. Erbin and N. Halmagyi. "Quarter-BPS Black Holes in AdS_4-NUT from $N = 2$ Gauged Supergravity". Accepted in *JHEP* (Mar. 2015), arXiv: 1503.04686.

- Chapter 14, sections 17.1, 17.2 [99]
  H. Erbin. "Janis-Newman algorithm: simplifications and gauge field transformation". *General Relativity and Gravitation* 47.3 (Mar. 2015), arXiv: 1410.2602.

- Chapter 15, section 17.3 [159]
  H. Erbin. "Deciphering and generalizing Demiański-Janis-Newman algorithm". Submitted to *Classical and Quantum Gravity* (Nov. 2014), arXiv: 1411.2909

- Chapter 16 [158]
  H. Erbin and L. Heurtier. "Supergravity, complex parameters and the Janis-Newman algorithm". *Classical and Quantum Gravity* 32.16 (July 2015), p. 165005, arXiv: 1501.02188.

- Chapter 18 [160]
  H. Erbin and L. Heurtier. "Five-dimensional Janis-Newman algorithm". *Classical and Quantum Gravity* 32.16 (July 2015), p. 165004, arXiv: 1411.2030.

Part I

$N = 2$ supergravity

Chapter 1

Introduction to $N = 2$ supergravity

Four-dimensional $N = 2$ supergravity can be obtained as the low-energy effective action of type II superstring theory compactified on Calabi–Yau 3-fold [165, sec. 21.4.3, 166, sec. 5] or on a $N = (2, 2)$ superconformal theory with $c = 9$ [12, 13, 167]. This case is interesting because heterotic string theory can be compactified on these manifolds and give rise to $N = 1$ supergravity in four dimensions, and some details of the resulting theory are independent of the number of supersymmetries [12, 13]. Finally $N = 2$ supergravity can also be found from M-theory on a 7-dimensional manifold with SU(3) structure [168, 169]. If fluxes are present then one gets gauged supergravity and we address this topic in the next chapter.

In this section we present the supersymmetry algebra and the corresponding multiplets. We then display the Lagrangian that describes the interaction of the hyper-, vector and gravity multiplets and we comment the electromagnetic duality of this theory. Finally we present the main details of the manifolds described by the scalar fields – the special Kähler and quaternionic geometries – which described in more details in later chapters.

General introductions can be found in the classical references [165, 170–172].[1] Several thesis have been written recently on the topic [173–175].

1.1 Algebra and multiplets

The $N = 2$ supersymmetry algebra corresponds to [165, app. 6A]

$$[J_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\nu, \tag{1.1.1a}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\nu\sigma} J_{\mu\rho}, \tag{1.1.1b}$$

$$\{Q_\alpha, \bar Q^\beta\} = -\frac{i}{2}\, \delta_\alpha{}^\beta\, P_L \gamma_\mu\, P^\mu, \qquad \{Q^\alpha, \bar Q_\beta\} = -\frac{i}{2}\, \delta^\alpha{}_\beta\, P_R \gamma_\mu\, P^\mu, \tag{1.1.1c}$$

$$\{Q_\alpha, \bar Q_\beta\} = 0, \qquad \{Q^\alpha, \bar Q^\beta\} = 0, \tag{1.1.1d}$$

$$[P_\mu, Q_\alpha] = 0, \qquad [P_\mu, Q^\alpha] = 0, \tag{1.1.1e}$$

$$[J_{\mu\nu}, Q_\alpha] = -\frac{i}{2}\, \gamma_{\mu\nu} Q_\alpha, \quad [J_{\mu\nu}, Q^\alpha] = -\frac{i}{2}\, \gamma_{\mu\nu} Q^\alpha, \tag{1.1.1f}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\, \varepsilon_{\alpha\beta} P_L\, Z, \qquad \{Q^\alpha, Q^\beta\} = -\frac{1}{2}\, \varepsilon^{\alpha\beta} P_R\, \bar Z, \tag{1.1.1g}$$

$$[R^A, Q_\alpha] = (U^A)_\alpha{}^\beta Q_\beta, \qquad [R^A, Q^\alpha] = (U^A)^\alpha{}_\beta Q^\beta, \tag{1.1.1h}$$

$$[T^a, T^b] = f^{ab}{}_c T^c, \tag{1.1.1i}$$

[1] In particular an summary of the historical works may be found in [170, sec. 4].

where P_μ and $J_{\mu\nu}$ generate translation and Lorentz transformations and form the Poincaré algebra, Q_α are the fermionic generator of supersymmetry, R^A are the generator of the $\mathrm{U}(2)_R$ R-symmetry represented by the matrices U^A, T^a are generators of the internal symmetry, and finally Z is the central charge. The index α corresponds to the fundamental representation of $\mathrm{U}(2)_R$.

Note that $J_{\mu\nu}$ and P_μ describe the Poincaré subalgebra. The commutators of $J_{\mu\nu}$ with respectively itself, P_μ and Q_α show that they behave as an antisymmetric 2-tensor, a vector and a spinor. Two supersymmetric transformations close on a translation: as a consequence if supersymmetry is made local, so are the translations and one cannot have local supersymmetry without gravity. The R-symmetry group corresponds to the automorphism group: this is the only internal group that does not commute with the supersymmetry generators.

The algebra is given in terms of Weyl spinors (Q_α, Q^α) where the position of the index gives the chirality (see appendix A.5)

$$Q_\alpha = P_L\, Q_\alpha, \qquad Q^\alpha = P_R\, Q^\alpha. \tag{1.1.2}$$

Poincaré fields are organized into multiplets in this extended algebra. One of the constraint for building these representations is that the highest spin should not exceed $s = 2$ as interacting higher-spin theories (with a finite number of fields) are not consistent. The different multiplets are summarized in table 1.1. Using the table A.2 one can see that the bosonic and fermionic on-shell degrees of freedom match in each multiplets.

There are additional multiplets that we will not discuss, the tensor (or hypertensor, scalar-tensor) multiplet [49, 176–182], the double tensor multiplet [178] and the vector-tensor multiplet [176, 177, 183, 184]. While it is possible to always dualize the tensor into scalars in ungauged supergravity (where the vector-tensor and (double) tensor multiplets give respectively the vector and hyper-multiplets), this is not the case in gauged supergravity where the coupling of the multiplets with and without tensors are different. For example the masses of the tensor multiplets give magnetic gaugings. These multiplets have their interest in the context of flux compactifications where p-forms naturally arise.

multiplet	$s_{\max}$	$s=2$	$s=3/2$	$s=1$	$s=1/2$	$s=0$
gravity	2	1	2	1		
	3/2		1	2	1	
vector	1			1	2	2
hyper	1/2				2	4

Table 1.1 – $N = 2$ supergravity multiplets and spin content.

We consider the following field content:

- Gravity multiplet
$$\{g_{\mu\nu}, \psi_{\alpha\mu}, \psi^\alpha_\mu, A^0_\mu\}. \tag{1.1.3}$$
- n_v vector multiplets
$$\{A^i_\mu, \lambda^{\alpha i}, \lambda^{\bar\imath}_\alpha, \tau^i\}, \tag{1.1.4}$$
with $\tau^i \in \mathbb{C}$.
- n_h hypermultiplets
$$\{\zeta^{\mathcal{A}}, \zeta_{\mathcal{A}}, q^u\}, \tag{1.1.5}$$
with $q^u \in \mathbb{R}$.

The fields $\psi_{\alpha\mu}$, $\lambda^{\alpha i}$ and $\zeta^{\mathcal{A}}$ are respectively called gravitini, gaugini and hyperini. The ranges of the indices are

$$\alpha = 1, 2, \qquad i = 1, \dots, n_v, \qquad u = 1, \dots, 4n_h, \qquad \mathcal{A} = 1, \dots, 2n_h. \tag{1.1.6}$$

The index α corresponds to the fundamental representation of $\mathrm{SU}(2) \sim \mathrm{Sp}(1)$ and $\mathcal{A}$ to the fundamental of $\mathrm{Sp}(n_h)$.

1.2 Lagrangian

It is natural to gather gauge fields into one vector of dimension $n_v + 1$

$$A^\Lambda = (A^0, A^i), \qquad \Lambda = 0, \dots, n_v. \tag{1.2.1}$$

The bosonic part of the Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\text{bos}} = \frac{R}{2} + \frac{1}{4}\, \operatorname{Im}\mathcal{N}(\tau)_{\Lambda\Sigma}\, F^\Lambda_{\mu\nu}\, F^{\Sigma\mu\nu} - \frac{1}{8}\, \operatorname{Re}\mathcal{N}(\tau)_{\Lambda\Sigma}\, \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\, F^\Lambda_{\mu\nu}\, F^\Sigma_{\rho\sigma} \\ - g_{i\bar\jmath}(\tau)\, \partial_\mu \tau^i \partial^\mu \bar\tau^{\bar\jmath} - \frac{1}{2}\, h_{uv}(q)\, \partial_\mu q^u \partial^\mu q^v\end{aligned} \tag{1.2.2}$$

where the field strengths are defined by

$$F^\Lambda = \mathrm{d}A^\Lambda. \tag{1.2.3}$$

All fields are minimally coupled to gravity (through the factor $\sqrt{-g}$ in the action). Both vector- and hyperscalars describe a non-linear sigma model since the coefficient of the kinetic term is field-dependent. Moreover the gauge fields are coupled to the vector scalars through the period matrix $\mathcal{N}$: the imaginary and real parts correspond respectively to a generalization of the gauge coupling and of the topological θ-term. Finally the hyperscalars do not interact with the gauge fields nor the vector scalars.

Supersymmetry dictates the form of the various functions that appear. In particular the period matrix $\mathcal{N}$ and the metric $g_{i\bar\jmath}$ can be derived from a unique holomorphic function F called the prepotential (see section 1.4).[2]

All the kinetic terms should be positive definite [185, sec. 2], and this imposes some restrictions on the scalar fields. The normalisation of the curvature term corresponds to a gauge choice.[3] Moreover the kinetic term for the gauge field has the correct signature because $\operatorname{Im}\mathcal{N}$ is negative definite (see section 4.4).

The Lagrangian is invariant under the local R-symmetry with gauge group $\mathrm{U}(2)_R$ for which there are two composite gauge fields $\mathcal{A}_\mu(\tau, \bar\tau)$ and $\mathcal{V}^x_\mu(q)$ with $x = 1, 2, 3$. Their origin can be seen most clearly from the superconformal tensor calculs. The scalar fields are neutral under this group.

We are not interested in the fermionic part of the Lagrangian but we will comment some of its properties. Fermions are coupled to the gauge fields through Pauli terms $F\psi\psi$ (and so on) which give rise to anomalous magnetic moments – in particular for the gaugini they are given by the quantity W_{ijk} (see section 4.5) [171, sec. 4.3]. Moreover the fermions are minimally coupled to the composite $\mathrm{U}(2)_R$ gauge fields. The Lagrangian includes four-fermion terms, but there are no mass terms.

The full Lagrangian is invariant under supersymmetry variations, we will give them only in the case of gauged supergravity (section 2.4).

1.3 Electromagnetic duality

Electromagnetic duality with and without scalars was studied in full generality by Gaillard and Zumino [187] (see also [21, sec. 3]). For a review of this topic see [170, sec. 2, 188, sec. 3, 189, 186, sec. 2, 165, sec. 4.2].

Recall that the field strength are determined from the gauge potential by

$$F^\Lambda = \mathrm{d}A^\Lambda. \tag{1.3.1}$$

[2] There are formulation of the theory without prepotential but we will not worry about this subtlety.

[3] In particular the term which appears before gauge fixing is $-i\left\langle \mathcal{V}, \bar{\mathcal{V}}\right\rangle R$, and we recover R by setting $\left\langle \mathcal{V}, \bar{\mathcal{V}}\right\rangle = i$ as in (4.2.25) [186, sec. 4].

Dual (magnetic) field strengths are given by

$$G_\Lambda = \star \left(\frac{\delta \mathcal{L}_{\text{bos}}}{\delta F^\Lambda} \right) = \operatorname{Re} \mathcal{N}_{\Lambda\Sigma}\, F^\Lambda + \operatorname{Im} \mathcal{N}_{\Lambda\Sigma} \star F^\Lambda. \tag{1.3.2}$$

It is also possible to introduce magnetic gauge potential A_Λ such that

$$G_\Lambda = \mathrm{d} A_\Lambda. \tag{1.3.3}$$

Both types of field strengths and gauge fields form together a symplectic vector

$$\mathcal{F} = \mathrm{d}A = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}, \qquad A = \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}. \tag{1.3.4}$$

The self-dual and anti-self-dual field strength is defined by

$$F^\pm = \frac{1}{2}(F \mp i \star F), \tag{1.3.5}$$

and similarly for $G^\pm$. Using equation (4.3.3) one finds

$$G^+ = \mathcal{N} F^+, \qquad G^- = \bar{\mathcal{N}} F^-. \tag{1.3.6}$$

Using these fields the kinetic term for the gauge fields can be rewritten as [170, p. 5, 165, p. 446]

$$\mathcal{L}_{\text{vec}} = \frac{1}{2} \operatorname{Im}(\mathcal{N}_{\Lambda\Sigma} F^{+\Lambda} F^{+\Sigma}) = -\frac{i}{4} \mathcal{N}_{\Lambda\Sigma} F^{+\Lambda} F^{+\Sigma} + \text{c.c.} = -\frac{i}{4}\, G^+_\Lambda F^{+\Lambda} + \text{c.c.} \tag{1.3.7}$$

This can be proven using the fact that

$$F^+_{\mu\nu} F^{+\mu\nu} = \frac{1}{2}(F_{\mu\nu} F^{\mu\nu} - i F_{\mu\nu} \star F^{\mu\nu}), \tag{1.3.8}$$

then one ends up with

$$\mathcal{L}_{\text{vec}} = -\frac{1}{4} \operatorname{Re} \Big(i \mathcal{N}_{\Lambda\Sigma} (F_{\mu\nu} F^{\mu\nu} - i F_{\mu\nu} \star F^{\mu\nu}) \Big). \tag{1.3.9}$$

Maxwell equations and Bianchi identities

$$\mathrm{d} F^\Lambda = 0, \qquad \mathrm{d} G_\Lambda = 0 \tag{1.3.10}$$

can be gathered as

$$\mathrm{d}\mathcal{F} = 0. \tag{1.3.11}$$

Note also that they can be traded for their dual

$$\mathrm{d} \star F^\Lambda = 0, \qquad \mathrm{d} \star G_\Lambda = 0 \Longrightarrow \mathrm{d} \star \mathcal{F} = 0. \tag{1.3.12}$$

They can also be rewritten as

$$\mathrm{d} \operatorname{Im} \mathcal{F}^\pm = 0. \tag{1.3.13}$$

These equations are invariant under linear transformations from $\mathrm{GL}(2n_v + 2, \mathbb{R})$, which reduces to symplectic transformations

$$\mathcal{F} \longrightarrow \mathcal{U}\mathcal{F}, \qquad \mathcal{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n_v + 2, \mathbb{R}) \tag{1.3.14}$$

if one wants to preserve the relation between F and G

$$G_\Lambda = \mathcal{N}_{\Lambda\Sigma} F^\Sigma \Longrightarrow G'_\Lambda = \mathcal{N}'_{\Lambda\Sigma} F'^\Sigma. \tag{1.3.15}$$

This is a consequence of the fact that a symplectic transformation of the various sections will induce a diffeomorphism of the scalar manifold, and the action will be of the same form only if both transformations are consistent together. The fact that both scalar and gauge fields transform can be seen as a consequence of supersymmetry which relates both fields: indeed if only the vector fields were transforming then the supersymmetry transformation would not be consistent anymore.

In presence of matter the dualities of the full equations of motion are restricted to a subgroup $G \subset \mathrm{Sp}(2n_v + 2, \mathbb{R})$, called the U-duality group, because the self-interaction terms are not invariant under the full symplectic group (see section 1.4).

It is important to note that the equations of motion – but not the action — are only covariant with respect to these symplectic transformations (called also duality-rotations or field-redefinitions), and as a consequence these are *not* symmetries of the action. [170, p. 7]. Symmetries of the equations of motion (and Bianchi identities) correspond to the subgroup of the symplectic transformation that leaves the equations invariant, and they are called duality transformations. We used this word duality because in general the action is not invariant, only the equations of motion are [165, p. 84].

The gauge field Lagrangian (1.3.7) transforms according to [170, p. 7, 186, p. 3]

$$\begin{aligned} 2\mathcal{L}_{\text{vec}} = \operatorname{Im}(G^+_\Lambda F^{+\Lambda}) \longrightarrow \operatorname{Im}(G'^+_\Lambda F'^{+\Lambda}) = \operatorname{Im}\big(&G^+_\Lambda F^{+\Lambda} + 2F^- C^t B G^- \\ &+ F^- C^t A F^- + G^- D B G^-\big). \end{aligned} \tag{1.3.16}$$

Then a symmetry of the Lagrangian is possible only if $B = 0$ since the last term was not present in the original Lagrangian – these symmetries are called electric. Moreover it seems that we would have to require also $C = 0$, this is not necessary if one asks only for a symmetry of the action: the term $(C^t A)_{\Lambda\Sigma} F^{-\Lambda} F^{-\Sigma}$, which corresponds to a constant shift of $\mathcal{N}$

$$\mathcal{N} \longrightarrow A^{t-1} \mathcal{N} A^{-1} + C A^{t-1}, \tag{1.3.17}$$

is a topological density since the coefficient is constant. Nonetheless this term would have a quantum effect as it modifies the θ-angle of the theory. In particular the path integral is invariant only if the coefficients are integer multiples of 2π, which restricts the U-duality group G to a discrete subgroup [170, p. 27]. In the case $C \neq 0$ the prepotential is shifted [165, sec. 21.1.2], from (5.1.22)

$$\delta F = \frac{1}{2} X S^t Q X. \tag{1.3.18}$$

The transformation for which $B \neq 0$ are non-perturbative because they mix the electric and magnetic field strengths into the Lagrangian which does not involve the latter. From the microscopic point of view this is equivalent to exchanging the electric and magnetic currents, and then the elementary states with the soliton states [170, p. 28].

The electric and magnetic charges q_Λ and p^Λ contained in a volume V with boundary Σ are defined by

$$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \frac{1}{\operatorname{Vol}(\Sigma)} \int_\Sigma \mathcal{F}. \tag{1.3.19}$$

The charges are defined as densities to avoid infinite charges in the case of non-compact surfaces. For compact horizons one takes

$$\operatorname{Vol}(\Sigma) = \operatorname{Vol}(S^2) = 4\pi. \tag{1.3.20}$$

Note also that the charges are a priori not constant. Since the charges $\mathcal{Q}$ are obtained by integrating the field strengths $\mathcal{F}$, they also transform under symplectic transformations [186, sec. 2]. Let us stress that identifying charges as being magnetic or electric is a frame-dependent question as a consequence of the previous point.

The graviphoton dressed field strength T and its (anti-)self-dual parts are defined by

$$T^+ = -\left\langle \bar{\mathcal{V}}, \mathcal{F}^+ \right\rangle, \qquad T^- = -\left\langle \mathcal{V}, \mathcal{F}^- \right\rangle \tag{1.3.21}$$

since [165, p. 478]

$$\langle \mathcal{V}, \mathcal{F}^+ \rangle = \langle \bar{\mathcal{V}}, \mathcal{F}^- \rangle = 0. \tag{1.3.22}$$

Similarly one defines the dressed field strengths T^i of the vector multiplet fields as

$$T_i^+ = -\langle U_j, \mathcal{F}^+ \rangle, \qquad T_{\bar{\imath}}^- = -\langle \bar{U}_{\bar{\imath}}, \mathcal{F}^- \rangle, \tag{1.3.23}$$

while the tensors with the upper index are $T^{\bar{\imath}+} = g^{\bar{\imath}j} T_j^+$ and $T^{i-} = g^{i\bar{\jmath}} T_{\bar{\jmath}}^-$.

Important quantities are the central and matter charges defined by

$$\mathcal{Z} = -\frac{1}{2} \int_\Sigma T^-, \qquad \mathcal{Z}_i = -\frac{1}{2} \int_\Sigma T_i^-. \tag{1.3.24}$$

If $\mathcal{V}$ does not depend on the coordinates on Σ, one can move $\mathcal{V}$ outside the integral in (1.3.24). Then the central and matter charges correspond to the components of $\mathcal{Q}$ along the basis $(\mathcal{V}, U_i)$ following (4.4.14)

$$\mathcal{Z} = \Gamma(\mathcal{Q}) = \langle \mathcal{V}, \mathcal{Q} \rangle, \qquad \mathcal{Z}_i = \mathrm{D}_i \mathcal{Z} = \langle U_i, \mathcal{Q} \rangle. \tag{1.3.25}$$

1.4 Scalar geometry

Scalar fields describe a non-linear sigma model with target space

$$\mathcal{M} = \mathcal{M}_v(\tau^i) \times \mathcal{M}_h(q^u) \tag{1.4.1}$$

where supergravity imposes constraints on the manifold holonomies which determine their types:[4]

- $\mathcal{M}_v$: special Kähler (SK) manifold (chapter 4), $\dim_\mathbb{R} = 2n_v$ [167];
- $\mathcal{M}_h$: quaternionic Kähler (QK) manifold (chapter 8), $\dim_\mathbb{R} = 4n_h$ [190].

The R-symmetry group of the supersymmetry algebra can be split as

$$\mathrm{U}(2)_R = \mathrm{SU}(2)_R \times \mathrm{U}(1)_R, \tag{1.4.2}$$

and this is mirrored in the structure of the multiplets: SK manifolds have a U(1) bundle while QK manifolds have an SU(2) bundle. In particular if the manifolds $\mathcal{M}_v$ and $\mathcal{M}_h$ are cosets G/H, then their maximal compact subgroup H contains respectively a factor U(1) or SU(2).

In considering the fields as coordinates for the non-linear sigma model all relevant formulas are obtained through a pull-back, in particular

$$\mathrm{d}\tau^i = \partial_\mu \tau^i \, \mathrm{d}x^\mu, \qquad \mathrm{d}q^u = \partial_\mu q^u \, \mathrm{d}x^\mu. \tag{1.4.3}$$

1.4.1 Isometries

The isometry group[5]

$$G \equiv \mathrm{ISO}(\mathcal{M}) \tag{1.4.4}$$

of this manifold translates into an invariance of the scalar kinetic term which is just the pullback of the metric on $\mathcal{M}$. On the other hand through its embedding into the symplectic group (as explained in section 1.3) it defines the global symmetry group of the equations of motion and it is called the U-duality group. A subgroup of G can be gauged in order to generate new interactions, and this is the topic of chapter 2.

According to the discussion of section 1.3, an isometry can be of one of the three following types [170, sec. 6, 189]:

[4] The manifold described by the scalars of n_t vector-tensor multiplets is real.

[5] We will also use the notations $G_v \equiv \mathrm{ISO}(\mathcal{M}_v)$ and $G_h \equiv \mathrm{ISO}(\mathcal{M}_h)$.

- Classical symmetries: the matrix $\mathcal{U}$ is block diagonal

$$\mathcal{U} = \begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}, \tag{1.4.5}$$

(where the lower component follows from the constraints (5.1.3)), and it is a true symmetry of the Lagrangian.

- Perturbative symmetries: the matrix $\mathcal{U}$ is lower triangular

$$\mathcal{U} = \begin{pmatrix} A & 0 \\ C & A^{t-1} \end{pmatrix}. \tag{1.4.6}$$

At the classical level the action is invariant, while at the quantum level only the path integral is invariant for a subgroup $G(\mathbb{Z}) \subset G(\mathbb{R})$.

- Non-perturbative symmetries: the matrix U has the general form (1.3.14)

$$\mathcal{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.4.7}$$

and they are symmetries of the quantum theory but they cannot be defined perturbatively.

Isometries of the scalar manifold extend to a symmetry of the Lagrangian if all couplings are diffeomorphism invariant, which means that they depend only on the metric, the curvature and Christoffel symbols [170, sec. 7.1].

In $d = 4$ all symmetries of the scalar manifold extend to symmetries of the full Lagrangian (as opposed to $d = 5$) [144, 191] and this is a consequence of supersymmetry.[6]

If one considers models obtained from compactification of type II, then the corresponding SK manifold $\mathcal{M}_v$ is symmetric and the QK is special, which means that it entirely specified by another SK manifold $\mathcal{M}_z$ which is also symmetric. Moreover the manifolds $\mathcal{M}_v$ and $\mathcal{M}_z$ are interchanged when compactifying type II A and B on the same manifold [144].

We review the main properties of these manifolds and we refer the reader to part II for more details.

1.4.2 Special Kähler manifolds

A special Kähler manifold is a Kähler manifold with a bundle with group $\mathrm{Sp}(2n_v + 2, \mathbb{R})$.

SK manifolds are better described in terms of projective coordinates X^Λ where

$$\tau^i = \frac{X^i}{X^0}. \tag{1.4.8}$$

Then the prepotential is a holomorphic function $F = F(X^\Lambda)$ of weight 2. The gradient of the prepotential gives a set of functions

$$F_\Lambda = \frac{\partial F}{\partial X^\Lambda} \tag{1.4.9}$$

that together with X^Λ form a section of the symplectic bundle

$$v = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}. \tag{1.4.10}$$

Then the Kähler potential reads

$$K = -\ln i\big(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda\big) \tag{1.4.11}$$

[6] This was proved only for cubic prepotentials, but no counter-example is known [144, p. 15].

from which derives the metric

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K. \tag{1.4.12}$$

It is always possible to describe the SK manifold in terms of a prepotential and we will focus on this case [186]. But this does not mean that symplectically rotated theories are equivalent (for example different theories with the same geometry may have different gauge groups, and partial symmetry breaking from $N = 2$ to $N = 1$ in FI gauged supergravity is impossible if a superpotential exists) [186, sec. 4.2].

The pull-back of the U(1) connection (3.2.46) is

$$\mathcal{A}_\mu = -\frac{i}{2}\big(K_i \partial_\mu \tau^i - K_{\bar{\imath}} \partial_\mu \bar{\tau}^{\bar{\imath}}\big). \tag{1.4.13}$$

1.4.3 Quaternionic manifolds

The quaternionic manifold with metric h_{uv} has a triplet of structures J^x satisfying the quaternionic algebran $\mathrm{SU}(2) \sim \mathrm{Sp}(1)$

$$J^x J^y = -\delta^{xy} + \varepsilon^{xyz} J^z \tag{1.4.14}$$

where $x = 1, 2, 3$ is the vector representation of $\mathrm{SO}(3) \sim \mathrm{SU}(2)$. They define a triplet of 2-forms

$$K^x = J^x_{uv}\, \mathrm{d}q^u \wedge \mathrm{d}q^v, \qquad J^x_{uv} = h_{uw} (J^x)_v{}^w. \tag{1.4.15}$$

The manifold has an SU(2) bundle with connection ω^x and a curvature proportional to the quaternionic 2-forms

$$\Omega^x = \nabla \omega^x = \lambda K^x \tag{1.4.16}$$

These forms are covariantly closed

$$\nabla \Omega^x = \nabla K^x = 0. \tag{1.4.17}$$

Finally one can introduce vielbeine

$$h_{uv} = \mathbb{C}_{\mathcal{A}\mathcal{B}} \varepsilon_{\alpha\beta} U_u^{\alpha\mathcal{A}} U_v^{\mathcal{B}\beta}, \tag{1.4.18}$$

where the indices $\mathcal{A}$ and α run respectively in the fundamental representations of $\mathrm{Sp}(n_h)$ and $\mathrm{Sp}(1)$, where the corresponding symplectic metrics are $\mathbb{C}$ and ε. This splitting of the indices is a consequence of the holonomy of the manifold.

In supergravity one has the restriction [171, p. 6, 192, p. 719]

$$\lambda = -1 \tag{1.4.19}$$

which implies that the quaternionic spaces have negative curvature

$$R = -8 n_h (n_h + 2). \tag{1.4.20}$$

The pull-back of the SU(2) connection corresponding to the composite $\mathrm{SU}(2)_R$ gauge field is

$$\mathcal{V}^x_\mu = -\omega^x_u\, \partial_\mu q^u. \tag{1.4.21}$$

In most of the cases that are of interest to us the quaternionic manifold is special (see chapter 8.5) and all its properties are given by a special Kähler manifold $\mathcal{M}_z$ of dimension $2(n_h - 1)$ with prepotential G. These manifolds are constructed from the c-map: $d = 4$ supergravity is reduced to $d = 3$ where all vectors can be dualized to scalar fields. Since there are only scalar fields (coming from the original vector and hypermultiplets, and from the reduction) the geometry can only be quaternionic. Then the manifold that are constructed in this way can be used for $\mathcal{M}_h$ in $d = 4$ [12, 13, 145]. The idea is that dualities of the $d = 4$

equations of motion will translate into invariance of the $d = 3$ Lagrangian since there are no more gauge fields [12, 144, sec. 2.3].

In this case the fields are denoted by $(\phi, \sigma, \xi^A, \tilde{\xi}_A)$ where $A = 0, \ldots, n_h - 1$. Physically ϕ is the dilaton (coming from the metric), σ is the axion (coming from dualization of the NS B-field) and the $(\xi^A, \tilde{\xi}_A)$ corresponds to the RR scalars (coming from the reduction of the RR forms) [50, p. 5].

Chapter 2

Gauged supergravity

A gauged supergravity is obtained from an ungauged theory by using some of the gauge fields in order to introduce a local gauge symmetry. In this chapter we describe the two main possibilities which consists in gauging a subgroup of the isometry group of the scalar manifolds or in introducing Fayet–Iliopoulos gaugings (both are not exclusive). The gauging procedure is described in [170, sec. 7, 165, chap. 21, 174, chap. 2, 175, chap. 1] (see also [57, 193]).

Gauged supergravities typically appear in flux compactifications which refers to compactifications where some p-form field of the higher-dimensional theory has a value along a (non-trivial) cycle of the internal manifold [166, sec. 5, 194, sec. 4].

In order to understand the details of the gauging one needs to understand the isometries of the SK and QK scalar manifolds, which are the topics of chapters 7 and 7. Our study of the BPS solutions will rely heavily on a symplectic covariant formalism: this requires us to introduce magnetic gaugings in order to treat equally electric and magnetic field strengths. Constructing a Lagrangian with magnetic gaugings is a difficult task and we will restrict ourselves to a simple case involving only the equations of motion/BPS.

2.1 Generalities

Since the Lagrangian (1.2.2) is invariant under the global isometry group G of the scalar manifold $\mathcal{M}$ (section 1.4) one can gauge a subgroup K of the global symmetry group G such that part of the symmetries are made local

$$K \subset G. \tag{2.1.1}$$

The group should be at most $n_v + 1$, which corresponds to the number of gauge fields

$$m = \dim K \leq n_v + 1. \tag{2.1.2}$$

This produces typically a non-abelian theory with gauge fields A^Λ in the adjoint representation, and by supersymmetry the fields X^Λ also sits in the adjoint representation. Vector scalar and hyperscalars are minimally coupled to the gauge fields through the Killing vectors of SK and QK geometries respectively, and they are in some representation of the gauge group. The fermions are coupled through the Killing prepotentials (or moment maps) acting as a deformation of the composite $\mathrm{U}(2)_R$ connections and derivatives of the SK/QK Killing vectors for the gaugini/hyperini. If the SK P^0_Λ and QK P^x_Λ moment maps are non-zero then the fermions are charged respectively under the $\mathrm{U}(1)_R$ and $\mathrm{SU}(2)_R$ factors of the R-symmetry which are gauged by physical gauge fields (in particular this is the only coupling for the gravitini), while only non-dynamical gauge fields were gauging it in ungauged supergravity [165, sec. 19.5, 193].

If only QK isometries are made local then the gauge group is necessarily abelian

$$K = \mathrm{U}(1)^m, \qquad m \leq \dim G_h. \tag{2.1.3}$$

Indeed since the fields X^Λ are in the adjoint representation, non-abelian gaugings are possible only if a subgroup of G_v is gauged.

If there are hypermultiplets then the quaternionic moment maps are fully determined from Killing vectors. On the other hand if $n_h = 0$ then the quaternionic moment maps can still be (non-vanishing) constants called Fayet–Iliopoulos parameters. They correspond to the coupling constants of the gravitini to the gauge fields using the R-symmetry group $\mathrm{SU}(2)_R$.[1] If one is not gauging a subgroup of G_v then the resulting group is abelian and for each gauge field this amounts to consider a U(1) inside the $\mathrm{SU}(2)_R$

$$\mathrm{U}(1) \subset \mathrm{SU}(2)_R. \tag{2.1.4}$$

Then one often considers the maximal case with

$$K = \mathrm{U}(1)^{n_v+1}, \tag{2.1.5}$$

(it is convenient to consider the diagonal U(1) inside $\mathrm{SU}(2)_R$), which is referred to as *Fayet–Iliopoulos gauging*. Minimal gauged supergravity is constructed in this way.

Gauging adds complexity to the theory and additional terms are generated in order to preserve supersymmetry:

- a scalar potential $V(\tau, q)$;
- (scalar-dependent) fermion masses;
- Chern–Simons terms for A^Λ.

The hypermultiplets are not spectators anymore and the dynamics is much richer. Moreover a non-trivial potential is necessary for obtaining AdS_4 vacua.

In section 1.4.1 we explained that the isometry group is embedded into the symplectic group, and that different types of symmetries can be distinguished. In particular within the current formalism it is possible to gauge only isometries which correspond to perturbative (or electric) symmetries, i.e. those which have a lower triangular embedding into the symplectic group; this issue will discussed further in section 2.5.

Hence the choice of the symplectic frame is important for determining the gauging. In particular it is always possible to find a frame where the gaugings are electric. On the other hand a prepotential may not exist in this frame, or it can be ugly, and there is a trade-of between having electric gaugings and the existence of a prepotential [165, sec. 21.2.2].

As soon as the theory is gauged, models related by symplectic transformations are not equivalent anymore because the gauging breaks the symplectic invariance. Indeed even if the bosonic part of the Lagrangian is invariant, minimal coupling of the gauge fields to the fermions breaks this duality invariance [195].

2.2 Gaugings

2.2.1 Isometries

Except in the FI case, the gauging is encoded by $n_v + 1$ Killing vectors

$$k_\Lambda = k_\Lambda^i(\tau)\,\partial_i + k_\Lambda^{\bar\imath}(\bar\tau)\,\partial_{\bar\imath} + k_\Lambda^u(q)\,\partial_u \tag{2.2.1}$$

[1] We stress that this is compatible with the previous option of gaugings a sugroup of G_v. This procedure amounts to gauge the R-symmetry by physical gauge fields furthermore with constant couplings.

which act on the fields as

$$\delta\tau^i = \alpha^\Lambda k_\Lambda^i(\tau), \qquad \delta q^u = \alpha^\Lambda k_\Lambda^u(q). \tag{2.2.2}$$

where α^Λ are the parameters of the gauge transformation. The vectors $\{k_\Lambda^i, k_\Lambda^{\bar{\imath}}, k_\Lambda^u\}$ correspond to linear combinations of the Killing vectors generating the isometries of $\mathcal{M}_v$ and $\mathcal{M}_h$

$$k_\Lambda = \theta_\Lambda^\alpha\, k_\alpha, \qquad \alpha = 1, \ldots, \dim G. \tag{2.2.3}$$

The coefficients θ_Λ^α of the linear combination are called the gauging parameters and the vectors k_α span the algebra of the full isometry group.

The Killing vectors form a Lie algebra

$$[k_\Lambda, k_\Sigma] = f_{\Lambda\Sigma}{}^\Omega\, k_\Omega \tag{2.2.4}$$

where $f_{\Lambda\Sigma}{}^\Omega$ are the structure constants. This provides constraints for the gauging parameters which are not all independent [194, sec. 3.1, 179, sec. 3]: the constraints can be worked out by using the explicit algebras $\mathfrak{g}_v$ and $\mathfrak{g}_h$ on the LHS and by identifying the coefficients with the RHS. In particular if no isometries of $\mathcal{M}_v$ are gauged then the Killing vector algebra is necessarily abelian (but this does not mean that the isometries of the manifolds are abelian: only their linear combination needs to be abelian, see section 2.6 for an example).

The isometry induces a symplectic $T = \alpha^\Lambda T_\Lambda$ and a Kähler $f = \alpha^\Lambda f_\Lambda$ transformation

$$\delta\mathcal{V} = T\mathcal{V} + f(\tau)\mathcal{V}, \tag{2.2.5}$$

where T_Λ is lower triangular

$$T_\Lambda = \begin{pmatrix} A_\Lambda & 0 \\ C_\Lambda & A_\Lambda^{t-1} \end{pmatrix}, \tag{2.2.6}$$

and C_Λ is symmetric. This transformation needs to be consistent with the transformation of the field strength F^Λ under a non-abelian gauge transformation [165, p. 474]

$$\delta F^\Lambda = \alpha^\Omega F^\Sigma f_{\Sigma\Omega}{}^\Lambda. \tag{2.2.7}$$

In particular this justifies the restriction to electric gaugings with $B_\Lambda = 0$, and this indicates that T_Λ should be

$$T_\Lambda = \begin{pmatrix} -f_\Lambda & 0 \\ C_\Lambda & f_\Lambda^t \end{pmatrix} = \begin{pmatrix} -f_{\Lambda\Sigma}{}^\Omega & 0 \\ C_{\Lambda\Sigma\Omega} & f_{\Lambda\Omega}{}^\Sigma \end{pmatrix}. \tag{2.2.8}$$

These generators satisfy the Lie algebra under the conditions

$$C_{(\Lambda\Sigma\Omega)} = 0, \tag{2.2.9a}$$

$$f_{\Xi\Omega}{}^\Gamma C_{\Gamma\Lambda\Sigma} = 2 f_{\Lambda[\Xi}{}^\Gamma C_{\Omega]\Sigma\Gamma} + 2 f_{\Sigma[\Xi}{}^\Gamma C_{\Omega]\Lambda\Gamma}. \tag{2.2.9b}$$

If the second term is present it induces a Kähler transformation

$$\delta K = \alpha^\Lambda (f_\Lambda + \bar{f}_\Lambda). \tag{2.2.10}$$

This implies the constraint

$$k_\Lambda^i \partial_i f_\Sigma - k_\Sigma^i \partial_i f_\Lambda = f_{\Lambda\Sigma}{}^\Omega f_\Omega. \tag{2.2.11}$$

In the kinetic term of the scalar fields the partial derivatives are modified to covariant derivatives through minimal coupling

$$\mathrm{D}_\mu = \partial_\mu - A_\mu^\Lambda k_\Lambda. \tag{2.2.12}$$

The fact that only the electric gauge field A^Λ are introduced implies that one breaks the symplectic covariance. Moreover the field strengths of the gauge fields are modified by a non-abelian piece

$$F^\Lambda_{\mu\nu} = \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu + f_{\Sigma\Omega}{}^\Lambda A^\Sigma_\mu A^\Omega_\nu. \tag{2.2.13}$$

Moment maps are real functions that can be built from special and quaternionic Killing vector

$$P^0_\Lambda = i\big(k^i_\Lambda \partial_i K - f_\Lambda\big), \qquad P^x_\Lambda = k^u_\Lambda i_k \omega^x_u + W^x_\Lambda \tag{2.2.14}$$

where f_Λ is the shift of the Kähler potential and W^x_Λ the SU(2) rotation of the triplet of hyperkähler structures induced by the isometry.

There are two important relations

$$k^i_\Lambda L^\Lambda = 0, \qquad P^0_\Lambda L^\Lambda = 0. \tag{2.2.15}$$

The Kähler U(1) connection (1.4.13) is modified to

$$\mathcal{A}_\mu = -\frac{i}{2}\big(K_i \mathrm{D}_\mu \tau^i - K_{\bar\imath} \mathrm{D}_\mu \bar\tau^{\bar\imath}\big) - \frac{1}{4}\, A^\Lambda_\mu (f_\Lambda - \bar f_\Lambda) \tag{2.2.16a}$$

$$= -\frac{i}{2}\big(K_i \partial_\mu \tau^i - K_{\bar\imath} \partial_\mu \bar\tau^{\bar\imath}\big) - \frac{i}{2}\, A^\Lambda_\mu P^0_\Lambda, \tag{2.2.16b}$$

while the SU(2) connection becomes

$$\mathcal{V}^x_\mu = -\omega^x_u\, \mathrm{D}_\mu q^u + \frac{1}{2}\, A^\Lambda_\mu W^x_\Lambda \tag{2.2.17a}$$

$$= -\omega^x_u\, \partial_\mu q^u - \frac{1}{2}\, A^\Lambda_\mu P^x_\Lambda. \tag{2.2.17b}$$

The fact that spinors are charged implies Dirac-like quantization conditions on the Killing prepotentials

$$p^\Lambda P^0_\Lambda \in \mathbb{Z}, \qquad p^\Lambda P^x_\Lambda \in \mathbb{Z}. \tag{2.2.18}$$

where p^Λ are the magnetic charges.

One defines the prepotential charges (also called the superpotential)

$$\mathcal{L}^x = -P^x_\Lambda L^\Lambda \tag{2.2.19}$$

(see (2.5.5) for a symplectic covariant definition).

2.2.2 Fayet–Iliopoulos gauging

A good reference is [83, sec. 2] (see also [165, sec. 21.5.1]).

In Fayet–Iliopoulos (FI) gauging the fermions become charged under a subgroup K_{FI} of the R-symmetry group

$$K_{\text{FI}} \subset \mathrm{SU}(2)_R \tag{2.2.20}$$

This corresponds to constant quaternionic moment maps ξ^x_Λ called the FI parameters

$$\xi^x_\Lambda \equiv P^x_\Lambda = \text{cst}, \tag{2.2.21}$$

which is possible only if $n_h = 0$ (otherwise they are determined by the quaternionic geometry and they are non-constant). These moment maps can be non-vanishing even if $n_h = 0$ because there is always a compensating hypermultiplet, which was fixed during the construction of the theory. If one gauges also a subgroup $K \subset G_v$, then a necessary condition is [174, p. 35]

$$K_{\text{FI}} \subset K. \tag{2.2.22}$$

If one considers abelian isometries, then the equivariance condition (8.3.26) reads

$$\varepsilon^{xyz} \xi^y_\Lambda \xi^z_\Sigma = 0. \tag{2.2.23}$$

As a consequence it is possible to choose a direction for the SU(2) vector

$$\xi^x_\Lambda = (0, 0, g_\Lambda) \tag{2.2.24}$$

which corresponds to

$$\mathrm{U}(1) \subset \mathrm{SU}(2)_R \tag{2.2.25}$$

(U(1) being the diagonal subgroup). The parameters g_Λ are the electric charges of the gravitini under this U(1) symmetry: the gauge fields are coupled to the gravitini through the linear combinations $g_\Lambda A^\Lambda$, and the two gravitini have opposite charges $\pm g_\Lambda$. Note that the vector scalars are neutral. In general speaking about FI gauging refers to this latter case.

Pure supergravity is a subcase of (abelian) FI gauged supergravity.

2.3 Lagrangian

2.3.1 General case

The bosonic part of the Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\text{bos}} = \frac{R}{2} &+ \frac{1}{4}\,\mathrm{Im}\,\mathcal{N}(\tau)_{\Lambda\Sigma}\,F^\Lambda_{\mu\nu}\,F^{\Sigma\mu\nu} - \frac{1}{8}\,\mathrm{Re}\,\mathcal{N}(\tau)_{\Lambda\Sigma}\,\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\,F^\Lambda_{\mu\nu}\,F^\Sigma_{\rho\sigma} \\ &- g_{i\bar{\jmath}}(\tau)\,\mathrm{D}_\mu\tau^i\mathrm{D}^\mu\bar{\tau}^{\bar{\jmath}} - \frac{1}{2}\,h_{uv}(q)\,\mathrm{D}_\mu q^u\mathrm{D}^\mu q^v \\ &+ \frac{2}{3}\,C_{\Lambda,\Sigma\Xi}\,\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\,A^\Lambda_\mu A^\Sigma_\nu\left(\partial_\rho A^\Xi_\sigma + \frac{3}{8}\,f_{\Omega\Gamma}{}^\Xi A^\Omega_\rho A^\Gamma_\sigma\right) - V(\tau,\bar{\tau},q).\end{aligned} \tag{2.3.1}$$

The term proportional to $C_{\Lambda\Sigma\Omega}$ is necessary to compensate the transformation of the matrix $\mathcal{N}$

$$\delta\mathcal{N}_{\Lambda\Sigma} = -\alpha^\Gamma(f_{\Gamma\Lambda}{}^\Omega\mathcal{N}_{\Sigma\Omega} + f_{\Gamma\Sigma}{}^\Omega\mathcal{N}_{\Lambda\Omega} + C_{\Gamma\Lambda\Sigma}). \tag{2.3.2}$$

under a gauge transformation.

The scalar potential reads

$$V = \big(f^\Lambda_i g^{i\bar{\jmath}} \bar{f}^\Sigma_{\bar{\jmath}} - 3L^\Lambda\bar{L}^\Sigma\big)P^x_\Lambda P^x_\Sigma + \bar{L}^\Lambda L^\Sigma\big(2\,h_{uv}k^u_\Lambda k^v_\Sigma + g_{i\bar{\jmath}}k^i_\Lambda k^{\bar{\jmath}}_\Sigma\big). \tag{2.3.3}$$

Note that there is only one negative term in the potential. Another expression for the potential is

$$V = \left(-\frac{1}{2}\,\mathrm{Im}\,\mathcal{N}^{\Lambda\Sigma} - 4L^\Lambda\bar{L}^\Sigma\right)P^x_\Lambda P^x_\Sigma + 2\bar{L}^\Lambda L^\Sigma\,h_{uv}k^u_\Lambda k^v_\Sigma + 2\,\mathrm{Im}\,F^{\Lambda\Sigma}P^0_\Lambda P^0_\Sigma \tag{2.3.4}$$

using (4.3.6) to rewrite the first term and writing the SK Killing vectors in terms of their prepotentials [165, p. 475].

We will not describe the full Lagrangian which is complicated and instead we refer the reader to [170, sec. 8, 165, sec. 21.3]. We are only interested in the mass terms of the fermions

$$\mathcal{L}_m = \frac{1}{2}\,S_{\alpha\beta}\,\bar{\psi}^\alpha_\mu\gamma^{\mu\nu}\psi^\beta_\nu - \frac{1}{2}\,m^{\alpha\beta}_{ij}\,\bar{\lambda}^i_\alpha\lambda^j_\beta - m^{\mathcal{A}}_{\alpha\bar{\imath}}\,\bar{\lambda}^{\alpha\bar{\imath}}\zeta_{\mathcal{A}} - \frac{1}{2}\,m_{\mathcal{A}\mathcal{B}}\,\bar{\zeta}^{\mathcal{A}}\zeta^{\mathcal{B}} - \bar{\psi}_{\mu\alpha}\gamma^\mu\chi^\alpha + \text{c.c.} \tag{2.3.5}$$

In the last term χ^α corresponds to the gravitini

$$\chi^\alpha = \frac{1}{2}\,W^{\alpha\beta}_i\,\lambda^i_\beta + 2\,N^\alpha_{\mathcal{A}}\zeta^{\mathcal{A}}. \tag{2.3.6}$$

The various mass matrices are given by

$$S_{\alpha\beta} = i\,\bar{L}^\Lambda P_\Lambda^x \sigma^x{}_\alpha{}^\gamma \varepsilon_{\gamma\beta}, \tag{2.3.7a}$$

$$m_{ij}^{\alpha\beta} = \frac{i}{2}\, C_{ijk} g^{k\bar{k}} \bar{f}_{\bar{k}}^\Lambda P_\Lambda^x\, \varepsilon^{\alpha\gamma} \sigma^x{}_\gamma{}^\beta + \varepsilon^{\alpha\beta} g_{j\bar{\imath}} k_\Lambda^{\bar{\imath}} f_i^\Lambda, \tag{2.3.7b}$$

$$m_{\alpha\bar{\imath}}^{\mathcal{A}} = 2i k_\Lambda^u\, \varepsilon_{\alpha\beta} U_u^{\beta\mathcal{A}} \bar{f}_{\bar{\imath}}^\Lambda, \tag{2.3.7c}$$

$$m_{\mathcal{A}\mathcal{B}} = -2L^\Lambda\, \varepsilon^{\alpha\beta} U_{\alpha\mathcal{A}}^v U_{\beta\mathcal{B}}^u \nabla_v k_{u\Lambda}, \tag{2.3.7d}$$

$$W_i^{\alpha\beta} = i\big(\varepsilon^{\alpha\beta} P_\Lambda^0 - P_\Lambda^x\, \varepsilon^{\alpha\gamma} \sigma^x{}_\gamma{}^\beta\big) f_i^\Lambda, \tag{2.3.7e}$$

$$N_{\mathcal{A}}^\alpha = -i\,\mathbb{C}_{\mathcal{A}\mathcal{B}}\, U_u^{\alpha\mathcal{B}} k_\Lambda^u L^\Lambda. \tag{2.3.7f}$$

Another expression for $W_i^{\alpha\beta}$ is

$$W_i^{\alpha\beta} = -\varepsilon^{\alpha\beta} g_{i\bar{\jmath}} k_\Lambda^{\bar{\jmath}} L^\Lambda - P_\Lambda^x\, \varepsilon^{\alpha\gamma} \sigma^x{}_\gamma{}^\beta f_i^\Lambda \tag{2.3.8}$$

These masses are related to the fermion shift that appears in the supersymmetric variations. Through Ward identities for supersymmetry the superpotential is also given by [170, sec. 9]

$$V\,\delta^\alpha{}_\beta = -3\, S^{\alpha\gamma} S_{\gamma\beta} + W_i^{\alpha\gamma} g^{i\bar{\jmath}} W_{\bar{\jmath}\beta\gamma} + 4\, N_{\mathcal{A}}^\alpha \bar{N}_\beta^{\mathcal{A}}. \tag{2.3.9}$$

2.3.2 Fayet-Iliopoulos gaugings

The scalar potential reads

$$V(\tau,\bar{\tau}) = \big(g^{i\bar{\jmath}} f_i^\Lambda \bar{f}_{\bar{\jmath}}^\Sigma - 3\bar{L}^\Lambda L^\Sigma\big)\, g_\Lambda g_\Sigma. \tag{2.3.10}$$

2.3.3 Minimal gauged sugra

Pure supergravity corresponds to $n_v = n_h = 0$. Its bosonic action is equivalent to Einstein–Maxwell theory. Its prepotential reads [165, ex. 21.3]

$$F = -\frac{i}{2}\,(X^0)^2. \tag{2.3.11}$$

Gauge fixing gives

$$X^0 = \frac{1}{\sqrt{2}} \tag{2.3.12}$$

which gives the value of $\mathcal{N}$

$$\mathcal{N} = -i, \tag{2.3.13}$$

which implies in particular

$$G = -\star F. \tag{2.3.14}$$

The $T_{\mu\nu}$ tensor equals simply the field strength up to a factor

$$T_{\mu\nu} = 2\sqrt{2}\, F_{\mu\nu}. \tag{2.3.15}$$

The scalar potential is constant

$$V = \Lambda = -6g^2 \tag{2.3.16}$$

with Λ the cosmological constant.

2.4 Supersymmetry variations

The bosonic part of the supersymmetry variations with parameter ε^α of the fermionic fields is given by

$$\delta\psi^\alpha_\mu = \hat{\mathrm{D}}_\mu \varepsilon^\alpha = \mathrm{D}_\mu \varepsilon^\alpha - \frac{i}{8}\, T^-_{ab}\, \gamma^{ab}\gamma_\mu \varepsilon^{\alpha\beta}\varepsilon_\beta + \frac{1}{2}\,\gamma_\mu S^{\alpha\beta}\varepsilon_\beta, \tag{2.4.1a}$$

$$\delta\lambda^i_\alpha = \mathrm{D}_\mu \tau^i\, \varepsilon_\alpha + \frac{1}{4}\, T^{-i}_{ab}\, \gamma^{ab}\varepsilon_{\alpha\beta}\varepsilon^\beta + g^{i\bar{\jmath}}\bar{W}_{\bar{\jmath}\beta\alpha}\varepsilon^\beta, \tag{2.4.1b}$$

$$\delta\zeta^{\mathcal{A}} = \frac{i}{2}\, U^{\alpha\mathcal{A}}_u\, \mathrm{D}_\mu q^u\, \varepsilon_\alpha + \bar{N}^{\mathcal{A}}_\alpha \varepsilon^\alpha. \tag{2.4.1c}$$

The additional terms are quadratic in the fermions and can be found in [170, sec. 8].

We denote by $\hat{\mathrm{D}}_\mu$ the supercovariant derivative. The gauge and spacetime covariant derivatives are

$$\mathrm{D}_\mu \varepsilon^\alpha = \nabla_\mu \varepsilon^\alpha + i\,\mathcal{V}^x_\mu \sigma^{x\alpha}{}_\beta \varepsilon^\beta, \tag{2.4.2a}$$

$$\nabla_\mu \varepsilon^\alpha = \left(\partial_\mu + \frac{1}{4}\,\omega_{\mu ab}\gamma^{ab} - i\mathcal{A}_\mu\right)\varepsilon^\alpha. \tag{2.4.2b}$$

The (bosonic part of) the anti-self-dual field strengths T_{ab} and T^i_{ab} were defined in (1.3.21)

$$T^- = -\left\langle \mathcal{V}, \mathcal{F}^- \right\rangle, \qquad T^-_i = -g^{i\bar{\jmath}} \left\langle \bar{U}_{\bar{\jmath}}, \mathcal{F}^- \right\rangle. \tag{2.4.3}$$

Finally the composite U(1) and SU(2) connections were given in (2.2.16) and (2.2.17).

A BPS solution is a field configuration that solves the equations of motion and which preserves some amount of supersymmetry, which is equivalent to the invariance of the configuration under supersymmetry variations. Moreover for classical solutions the fermionic fields typically vanish which ensures that the variations of the bosonic fields are zero. Then we just need to compute the variations of the fermionic fields (if they were not vanishing they would acquire a non-zero value after a supersymmetry transformation)

$$\delta\psi_{\alpha\mu} = \delta\lambda^{\alpha i} = \delta\zeta^{\mathcal{A}} = 0. \tag{2.4.4}$$

These equations will typically separate into matrix equations, which project out some components of the parameter ε_α, and scalar equations, which can be differential or algebraic. Together with Maxwell equations they provide a solution to the equations of motion.

The condition for ε^α to be a Killing spinor is equivalent to ε^α being covariantly constant with respect to the supercovariant derivative. In particular by taking the commutator of this equation one obtains the integrability condition

$$\left[\hat{\mathrm{D}}_\mu, \hat{\mathrm{D}}_\nu\right] \varepsilon^\alpha = \hat{R}_{\mu\nu}\varepsilon^\alpha = 0 \tag{2.4.5}$$

which is necessary but not sufficient. This equation is non-differential and gives constraints and projectors.

2.5 Magnetic gaugings

In order to obtain symplectic covariant expressions it is also possible to introduce magnetic gauging parameters such that the magnetic gauge fields A_Λ from (1.3.4) will be coupled to the scalars through the covariant derivatives. A Lagrangian description of this theory is quite involved as one needs to introduce new (tensor) fields and gauge invariances, and this is better formulated with the embedding tensor formalism [49, 179, 194]. When gaugings are abelian another possibility is to work directly with the BPS equations and the equations of motion since on-shell quantities are easier to deal with: these equations are completed such that they become symplectic covariant [52, 149]. For other works on magnetic gaugings, see also [176, 178, 180, 196].

2.5.1 Generalities

Introducing magnetic Killing vectors k^Λ that are paired with the electric ones k_Λ into a symplectic vector

$$\mathcal{K} = \begin{pmatrix} k^\Lambda \\ k_\Lambda \end{pmatrix}, \qquad \mathcal{K} = \mathcal{K}^i\,\partial_i + \mathcal{K}^{\bar\imath}\,\partial_{\bar\imath} + \mathcal{K}^u\,\partial_u, \tag{2.5.1}$$

the covariant derivative of the scalar fields becomes

$$\mathrm{D}_\mu = \partial_\mu - A_\mu \Omega \mathcal{K} = \partial_\mu - A_\mu^\Lambda k_\Lambda + A_{\Lambda\mu} k^\Lambda \tag{2.5.2}$$

in order to respect symplectic covariance [169, sec. 4.2, 196, sec. 3]. The Killing vectors can be expanded on the set of Killing vectors k_α generating the isometries of $\mathcal{M}$ (these are the same as the one appearing at the beginning of section 2.2.1)

$$\mathcal{K} = \Theta^\alpha\, k_\alpha, \qquad \Theta^\alpha = \begin{pmatrix} \theta^{\alpha\Lambda} \\ \theta^\alpha_\Lambda \end{pmatrix}. \tag{2.5.3}$$

Hence the coefficients of the linear combination are symplectic vectors, and $\theta^{\alpha\Lambda}$ and θ^α_Λ being respectively the magnetic and electric gauging parameters.

The Killing vectors satisfy constraints from closure of the algebra. There are three possibilities, depending if the vectors are both electric, both magnetic, or one electric and one magnetic.

The symplectic Killing prepotentials are given by

$$\mathcal{P}^x = \mathcal{K}^u \omega^x_u - \mathcal{W}^x, \tag{2.5.4a}$$

or in components

$$P^{x\Lambda} = k^{\Lambda u}\omega^x_u - W^{x\Lambda}, \qquad P^x_\Lambda = k^u_\Lambda \omega^x_u - W^x_\Lambda, \tag{2.5.4b}$$

One defines the prepotential charges (also called the superpotential)

$$\mathcal{L}^x = \langle \mathcal{V}, \mathcal{P}^x \rangle, \qquad \mathcal{L}^x_i = \langle U_i, \mathcal{P}^x \rangle. \tag{2.5.5}$$

In the case of FI gauging (section 2.2.2), one adds the constants g^Λ which correspond to the magnetic charges of the gravitini under the local U(1). The symplectic vector is denoted by

$$\mathcal{G} \equiv \mathcal{P}^3 = \begin{pmatrix} g^\Lambda \\ g_\Lambda \end{pmatrix}. \tag{2.5.6}$$

2.5.2 Constraints from locality

To ensure the existence of a Lagrangian and, more importantly, of an electric frame (since we derived the BPS equations from an electric frame, before doing a symplectic rotation), we must impose locality conditions on the parameters [179, sec. 3]. Then the locality constraints read [149, sec. 6.1, app. C] (see also [169, sec. 2])

$$\left\langle \Theta^\alpha, \Theta^\beta \right\rangle = 0. \tag{2.5.7}$$

It is necessary to impose this condition only when the gauge group is abelian, which is the case here [49, sec. 5]. This constraint is also a consequence of the Ward identity from which the scalar potential (2.3.9) is obtained [180].

The constraints imply that

$$\langle \mathcal{K}^u, \mathcal{P}^x \rangle = 0. \tag{2.5.8}$$

First we denote by k^u_α the generic set of Killing vectors such that

$$\mathcal{K}^u = \Theta^\alpha\, k^u_\alpha, \qquad \mathcal{W}^x = \Theta^\alpha\, w^x_\alpha, \tag{2.5.9}$$

then using the formula (2.5.4) for the prepotential we have

$$\begin{aligned}\langle \mathcal{K}^u, \mathcal{P}^x \rangle &= \langle \mathcal{K}^u, \mathcal{K}^v \omega_v^x - \mathcal{W}^x \rangle \\ &= k_\alpha^u (\omega_v^x k_\beta^v - w_\beta^x) \langle \Theta^\alpha, \Theta^\beta \rangle ,\end{aligned}$$

and this vanishes from the locality constraint.

2.6 Quaternionic gaugings

In this section we consider only abelian gaugings of the isometries of special quaternionic manifolds [149].

The Killing vector $k_\Lambda^u \partial_u$ can be expanded on the basis of Killing vectors on $\mathcal{M}_h$ (studied in section 9.1)

$$k_\alpha = \{k_{\mathbb{U}}, k_\xi, \widehat{k}_\xi, k_+, k_0, k_-\} \tag{2.6.1}$$

with the coefficients

$$\theta_\Lambda^\alpha = \{\mathbb{U}_\Lambda, \alpha_\Lambda, \widehat{\alpha}_\Lambda, \epsilon_{\Lambda+}, \epsilon_{\Lambda 0}, \epsilon_{\Lambda-}\}, \tag{2.6.2}$$

using the notations of [12, 144, 145] for the parameters. Note that α_Λ and $\widehat{\alpha}_\Lambda$ are symplectic vectors (of the base SK space $\mathcal{M}_z$) of dimensions $2n_h$

$$\alpha_\Lambda = \begin{pmatrix} \alpha_\Lambda^A \\ \alpha_{A\Lambda} \end{pmatrix}, \qquad \widehat{\alpha}_\Lambda = \left(\widehat{\alpha}_\Lambda^A \_ \widehat{\alpha}_{A\Lambda} \right). \tag{2.6.3}$$

Explicitly this reads

$$k_\Lambda = k_\Lambda^u \,\partial_u = k_{\mathbb{U}_\Lambda} + \alpha_\Lambda^t \mathbb{C}\, k_\xi + \widehat{\alpha}_\Lambda^t \mathbb{C}\, \widehat{k}_\xi + \epsilon_{+\Lambda} k_+ + \epsilon_{0\Lambda} k_0 + \epsilon_{-\Lambda} k_-. \tag{2.6.4}$$

Similarly the magnetic Killing vector is written $k^{u\Lambda}$ and all the magnetic parameters have the index Λ up.

All these parameters are not independent and consistency conditions impose relations between them (see also appendix E.2). The number of constraints can be much greater than the number of parameters, showing that some of these constraints are redundant.

The Killing algebra is abelian if the right hand side of (2.2.4) vanishes. From the algebra with electric/electric Killing vectors we derive the following constraints [149, sec. 6.1, app. C]

$$\begin{aligned}
0 &= \mathbb{T}(\alpha_\Lambda, \hat{\alpha}_\Sigma) - \mathbb{T}(\alpha_\Sigma, \hat{\alpha}_\Lambda), &(2.6.5a)\\
0 &= -(\mathbb{U}_\Lambda \alpha_\Sigma - \mathbb{U}_\Sigma \alpha_\Lambda) + (\epsilon_{0\Lambda}\alpha_\Sigma - \epsilon_{0\Sigma}\alpha_\Lambda) + (\epsilon_{+\Lambda}\widehat{\alpha}_\Sigma - \epsilon_{+\Sigma}\widehat{\alpha}_\Lambda), &(2.6.5b)\\
0 &= (\mathbb{U}_\Lambda \widehat{\alpha}_\Sigma - \mathbb{U}_\Sigma \widehat{\alpha}_\Lambda) + (\epsilon_{-\Lambda}\alpha_\Sigma - \epsilon_{-\Sigma}\alpha_\Lambda) + (\epsilon_{0\Lambda}\widehat{\alpha}_\Sigma - \epsilon_{0\Sigma}\widehat{\alpha}_\Lambda), &(2.6.5c)\\
0 &= \alpha_\Lambda^t \mathbb{C} \alpha_\Sigma + 2(\epsilon_{+\Sigma}\epsilon_{0\Lambda} - \epsilon_{+\Lambda}\epsilon_{0\Sigma}), &(2.6.5d)\\
0 &= (\widehat{\alpha}_\Lambda^t \mathbb{C}\alpha_\Sigma - \alpha_\Lambda^t \mathbb{C}\widehat{\alpha}_\Sigma) + 2(\epsilon_{+\Sigma}\epsilon_{-\Lambda} - \epsilon_{+\Lambda}\epsilon_{-\Sigma}), &(2.6.5e)\\
0 &= \widehat{\alpha}_\Lambda^t \mathbb{C}\widehat{\alpha}_\Sigma + 2(\epsilon_{0\Lambda}\epsilon_{-\Sigma} - \epsilon_{0\Sigma}\epsilon_{-\Lambda}). &(2.6.5f)
\end{aligned}$$

And we recall the definition of $\mathbb{T}_{\alpha,\hat{\alpha}}$ from (9.2.4a) We have defined

$$\mathbb{T}(\alpha_\Lambda, \hat{\alpha}_\Sigma) = (\alpha_\Lambda^t \mathbb{C} \partial_\xi)(\hat{\alpha}_\Sigma^t \mathbb{C}\partial_\xi)\, \mathbb{S}. \tag{2.6.6}$$

For the details of the computations, see appendix F.2. It is straightforward to obtain all the other constraints (electric/magnetic and magnetic/magnetic) from the electric/electric ones.

Without hidden vectors it reduces to

$$\begin{aligned}
0 &= \mathbb{U}_\Lambda \alpha_\Sigma - \mathbb{U}_\Sigma \alpha_\Lambda + \epsilon_{0\Lambda}\alpha_\Sigma - \epsilon_{0\Sigma}\alpha_\Lambda, &(2.6.7a)\\
0 &= \alpha_\Lambda^t \mathbb{C}\alpha_\Sigma + 2(\epsilon_{+\Sigma}\epsilon_{0\Lambda} - \epsilon_{+\Lambda}\epsilon_{0\Sigma}) &(2.6.7b)
\end{aligned}$$

and for $\epsilon_{0\Lambda} = 0$ furthermore to

$$0 = \mathbb{U}_\Lambda \alpha_\Sigma - \mathbb{U}_\Sigma \alpha_\Lambda, \tag{2.6.8a}$$

$$0 = \alpha_\Lambda^t \mathbb{C} \alpha_\Sigma, \tag{2.6.8b}$$

which can be found in [169, eq. (2.20)].

Part II

Kähler geometries

Chapter 3

Hermitian and Kähler manifolds

In $N = 2$ supergravity the manifold described by the vector scalars is special Kähler: hence we first start by describing separately the Kähler manifold and the more generic Hermitian and complex manifolds of which a Kähler manifold is a subcase. Then in chapter 4 we will explain what are the additional conditions for making a Kähler manifold special.

Great references for this section and the next one are [165, chap. 13, 197, chap. 8] (see also [198, sec. 9.A, 199]).

3.1 Hermitian manifold

3.1.1 Definition and properties

Consider a manifold $(\mathcal{M}, g)$ of (real) dimension $2n$ and with metric

$$\mathrm{d}s^2 = g_{ab}\, \mathrm{d}\phi^a \mathrm{d}\phi^b, \qquad a = 1, \dots, 2n, \tag{3.1.1}$$

endowed with a torsionless Levi–Civita covariant derivative, i.e.

$$\mathrm{D}_k g_{ij} = 0. \tag{3.1.2}$$

Definition 3.1 (Almost-complex manifold) The manifold $\mathcal{M}$ is almost-complex if it admits an almost-complex structure ${J_a}^b(\phi)$ which square to $-{\delta_a}^b$

$${J_a}^c {J_c}^b = -{\delta_a}^b. \tag{3.1.3}$$

An almost-complex manifold is necessarily even-dimensional (in fact it can be shown that any such manifold is almost-complex). The definition (3.1.3) implies that the eigenvalues of J are $\pm i$ (and of equal numbers).

From the almost-complex structure one defines the Nuijenhuis tensor

$${N_{ab}}^c = {J_a}^d \partial_{[c} {J_{b]}}^k - {J_b}^d \partial_{[c} {J_{a]}}^k. \tag{3.1.4}$$

The qualifier "almost" is used to indicate that J may not be defined globally.

Definition 3.2 (Complex manifold) An almost-complex manifold $(\mathcal{M}, J)$ is said to be complex if J is integrable, i.e. if it can be defined globally.

For a complex manifold the Nijenhuis tensor vanishes

$${N_{ab}}^c = 0. \tag{3.1.5}$$

Definition 3.3 (Hermitian manifold) A manifold $(\mathcal{M}, J)$ is said to be hermitian if J is compatible with the metric

$$J_a{}^c g_{cd} J_b{}^d = g_{ab} \Longleftrightarrow J\, g\, J^t = g. \tag{3.1.6}$$

Using the metric to lower an index produces the antisymmetric tensor

$$J_{ab} = J_a{}^c g_{cb}, \qquad J_{ab} = -J_{ba} \tag{3.1.7}$$

as can be seen by multiplying (3.1.6) by $J_e{}^b$

$$g_{ab} J_e{}^b = J_{ea},$$
$$J_a{}^c g_{cd} J_b{}^d J_e{}^b = -J_a{}^c g_{cd} \delta_e{}^d = g_{ab} J_e{}^b = -J_a{}^c g_{ce} = -J_{ae}$$

(in one word, hermicity implies antisymmetry). Thus it defines a 2-form called the fundamental form of $\mathcal{M}$, denoted by Ω

$$\Omega = -J_{ab}\, \mathrm{d}\phi^a \wedge \mathrm{d}\phi^b. \tag{3.1.8}$$

Note that Ω is real.

Since Ω^n is a $(2n)$-form nowhere vanishing it can serves as a volume element on the manifold [197, sec. 8.4.2].

3.1.2 Complex coordinates

Locally it is possible to introduce complex coordinates

$$\phi^a = (\tau^i, \bar\tau^{\bar\imath}), \qquad i, \bar\imath = 1, \ldots, n \tag{3.1.9}$$

such that the metric reads

$$\mathrm{d}s^2 = g_{i\bar\jmath}\, \mathrm{d}\tau^i \bar{\mathrm{d}}\tau^{\bar\jmath} + g_{\bar\imath j}\, \mathrm{d}\bar\tau^{\bar\imath} \mathrm{d}\tau^j = 2\, g_{i\bar\jmath}\, \mathrm{d}\tau^i \mathrm{d}\bar\tau^{\bar\jmath}. \tag{3.1.10}$$

Note that this metric is real since it was in the original coordinates, and as a consequence

$$g_{i\bar\jmath} = g^*_{j\bar\imath}. \tag{3.1.11}$$

A generic complex manifold that is not hermitian cannot be set in this form [165, sec. 13.1]. Conversely it can be shown that in coordinates where J is diagonal, the definition (3.1.6) implies that g_{ij} and its conjugate vanish. In matrix form one has

$$g_{ab} = \begin{pmatrix} 0 & g_{i\bar\jmath} \\ g_{j\bar\imath} & 0 \end{pmatrix}. \tag{3.1.12}$$

The index i and $\bar\imath$ are called holomorphic and antiholomorphic. The convention is to write the holomorphic index first. Moreover it is always possible to use the metric to convert a (anti)holomorphic index into its counterpart. For example one can use the metric on $A_i{}^{\bar\jmath}$ to get A_{ij}

$$A_{ij} = g_{j\bar\jmath} A_i{}^{\bar\jmath} \tag{3.1.13}$$

or $\partial_{\bar\imath}$ to ∂^i. Vectors of dimension n_v will sometimes be denoted in boldface, for example $\boldsymbol{\tau}$.

In these coordinates the almost-complex structure takes the diagonal form

$$J_a{}^b = i\, \mathrm{diag}(\delta_i{}^j, -\delta_{\bar\imath}{}^{\bar\jmath}). \tag{3.1.14}$$

Inserting this expression into (3.1.8), one obtains the fundamental form in complex coordinates

$$J_{i\bar\jmath} = -i\, g_{i\bar\jmath}, \tag{3.1.15a}$$
$$\Omega = 2i\, g_{i\bar\jmath}\, \mathrm{d}\tau^i \wedge \mathrm{d}\bar\tau^{\bar\jmath}. \tag{3.1.15b}$$

Due to the hermicity some Christoffel symbols vanish

$$\Gamma^{i}{}_{\bar{j}\bar{k}} = \Gamma^{\bar{i}}{}_{jk} = 0. \tag{3.1.16}$$

The Dobeault operators are defined by

$$\mathrm{d} = \partial + \bar{\partial}, \qquad \partial = \mathrm{d}\tau^{i}\,\partial_{i}, \qquad \bar{\partial} = \mathrm{d}\bar{\tau}^{\bar{i}}\,\partial_{\bar{i}}. \tag{3.1.17}$$

A useful relation is

$$\partial\bar{\partial} = -\frac{1}{2}\,\mathrm{d}(\partial - \bar{\partial}). \tag{3.1.18}$$

3.2 Kähler manifold

3.2.1 Definition

Definition 3.4 (Kähler manifold) A hermitian manifold $\mathcal{M}$ is said to be Kähler if the fundamental form Ω is closed

$$\mathrm{d}\Omega = 0. \tag{3.2.1}$$

In this case Ω is also called the Kähler 2-form.

This is equivalent to J being covariantly constant[1]

$$\mathrm{D}_{k} J_{ij} = 0. \tag{3.2.2}$$

A Kähler manifold has a holonomy group $\mathrm{U}(n)$. The Kähler form is a symplectic form, and as such Kähler manifolds also have a symplectic structure [199, p. 20].

Example 3.1 Examples of Kähler manifolds include:

- Calabi–Yau manifolds, for which the holonomy is restricted to $\mathrm{SU}(n)$. They have a vanishing first Chern class c_1 and admit a non-vanishing holomorphic n-form [199, sec. 5].
- All Hermitian manifolds of real dimension 2 due to the fact that any 2-form in 2 dimensions is closed [199, p. 20].
- The complex projective planes $\mathbb{C}P^{n}$.

In complex coordinates the condition (3.2.1) translates to

$$\mathrm{d}\Omega = -i(\partial_{i} g_{j\bar{k}} - \partial_{j} g_{i\bar{k}})\mathrm{d}\tau^{i} \wedge \mathrm{d}\tau^{j} \wedge \mathrm{d}\bar{\tau}^{\bar{k}} + \text{c.c.} = 0 \tag{3.2.3}$$

where the expression (3.1.15) of J_{ab} was used. Then the Kähler form is closed if

$$\partial_{i} g_{j\bar{k}} - \partial_{j} g_{i\bar{k}} = 0. \tag{3.2.4}$$

The latter implies the existence of a real function $K(\tau, \bar{\tau})$ called the Kähler potential that determines the metric

$$g_{i\bar{j}} = \partial_{i}\partial_{\bar{j}} K. \tag{3.2.5}$$

This presents a huge simplification since a single function gives the full metric. The *Kähler cone* is defined as the range of coordinates τ^{i} for which the metric is positive definite.

[1] Indeed if a form is closed, then one gets derivatives of the components which can be transformed to covariant ones since the Christoffel symbols will vanish by antisymmetry.

This function is not unique as shifts – called Kähler transformations – by holomorphic and antiholomorphic functions $f(\tau)$ and $\bar{f}(\bar{\tau})$

$$K(\tau, \bar{\tau}) \longrightarrow K(\tau, \bar{\tau}) + f(\tau) + \bar{f}(\bar{\tau}) \tag{3.2.6}$$

leave the metric invariant. Moreover K does not need to be defined globally, and the Kähler potentials on various patches are related by Kähler transformations

$$K_j(\tau, \bar{\tau}) = K_i(\tau, \bar{\tau}) + f_{ij}(\tau) + \bar{f}_{ij}(\bar{\tau}). \tag{3.2.7}$$

Using Dobeault operators (3.1.17) one can write the Kähler form as

$$\Omega = 2i\, \partial\bar{\partial} K. \tag{3.2.8}$$

3.2.2 Riemannian geometry

Recall that

$$\Gamma^i{}_{\bar{j}\bar{k}} = \Gamma^{\bar{\imath}}{}_{jk} = 0 \tag{3.2.9}$$

because the manifold is hermitian. Additional symbols vanish because of the Kähler condition

$$\Gamma^i{}_{j\bar{k}} = \Gamma^{\bar{\imath}}{}_{\bar{j}k} = 0. \tag{3.2.10}$$

Then the only non-vanishing symbols are

$$\Gamma^i{}_{jk} = g^{i\bar{\ell}} \partial_j g_{k\bar{\ell}} = g^{i\bar{\ell}} \partial_j \partial_k \partial_{\bar{\ell}} K \tag{3.2.11}$$

and their conjugates. The trace of the Christoffel is particularly simple

$$\Gamma^j{}_{ij} = \partial_i \ln \det g. \tag{3.2.12}$$

Similarly only the component $R_{i\bar{j}k\bar{\ell}}$ of the Riemann tensor and its permutations do not vanish

$$R^i{}_{jk\bar{\ell}} = -\partial_{\bar{\ell}} \Gamma^i{}_{jk}, \tag{3.2.13a}$$

$$R_{i\bar{j}k\bar{\ell}} = \partial_i \partial_{\bar{j}} g_{k\bar{\ell}} - g^{m\bar{n}} \partial_{\bar{j}} g_{m\bar{\ell}}\, \partial_i g_{k\bar{n}} \tag{3.2.13b}$$

$$= \partial_i \partial_{\bar{j}} \partial_k \partial_{\bar{\ell}} K - g^{m\bar{n}} (\partial_{\bar{j}} \partial_{\bar{\ell}} \partial_m K) \partial_i \partial_{\bar{n}} \partial_k K. \tag{3.2.13c}$$

The Ricci tensor

$$R_{i\bar{j}} = R^k{}_{ki\bar{j}} = -g^{k\bar{\ell}} R_{i\bar{\ell}k\bar{j}} \tag{3.2.14}$$

can be obtained directly from

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \ln \det g. \tag{3.2.15}$$

3.2.3 Symmetries

To each symmetry of the manifold preserving both structures g (in order to be an isometry) and J corresponds an holomorphic Killing vector k which generates infinitesimal transformations (or holomorphic isometries) through Lie derivative [170, sec. 7.1, 200, sec. 2]: its Lie derivative acting on g and J should vanish

$$\mathcal{L}_k g_{ij} = \nabla_a k_b + \nabla_b k_a = 0, \tag{3.2.16a}$$

$$\mathcal{L}_k J_a{}^b = J_c{}^b \nabla_a k^c - J_a{}^c \nabla_c k_a = 0. \tag{3.2.16b}$$

Together these implies the invariance of the Kähler form

$$\mathcal{L}_k \Omega = 0. \tag{3.2.17}$$

In fact the last requirement is more fundamental than the vanishing of $\mathcal{L}_k J_i{}^j$, since it means that the volume is invariant (the Lie derivative of the volume element Ω^n vanishes) and we will see that a condition similar to $\mathcal{L}_k \Omega = 0$ is the correct on in the case of quaternionic manifold.

Using the explicit formula (A.2.11) for $\mathcal{L}_k$ and the fact that $\mathrm{d}\Omega = 0$ gives

$$\mathrm{d} i_k \Omega = 0. \tag{3.2.18}$$

Then the Poincaré lemma states that it exists a (real) function P called the moment map (or Killing potential) such that

$$i_k \Omega = -2\, \mathrm{d} P_k. \tag{3.2.19}$$

P_k is not unique as it can be shifted by a constant (note that it depends on k)

$$P_k \longrightarrow P_k + \xi_k. \tag{3.2.20}$$

In the rest of this section we omit the index k.

In complex coordinates the condition (3.2.16b) gives the constraints

$$\partial_{\bar{\imath}} k^j = 0, \qquad \partial_i k^{\bar{\jmath}} = 0, \tag{3.2.21}$$

which mean that the Killing vector (with the index up) splits into a holomorphic and an antiholomorphic parts

$$k = k^a(\phi)\partial_a = k^i(\tau)\partial_i + k^{\bar{\imath}}(\bar{\tau})\partial_{\bar{\imath}}. \tag{3.2.22}$$

Then a variation of the coordinates with parameter θ reads

$$\delta\tau^i = \theta\, k^i(\tau), \qquad \delta\bar{\tau}^{\bar{\imath}} = \theta\, k^{\bar{\imath}}(\bar{\tau}) \tag{3.2.23}$$

and the transformation preserves the split in holomorphic and antiholomorphic coordinates. On the other hand the Killing equation (3.2.16a) gives two conditions

$$\nabla_i k_j + \nabla_j k_i = 0, \qquad \nabla_i k_{\bar{\jmath}} + \nabla_{\bar{\jmath}} k_i = 0. \tag{3.2.24}$$

The first equation is trivial since

$$\nabla_i k_j = g_{j\bar{k}} \nabla_i k^{\bar{k}} = g_{j\bar{k}} \partial_i k^{\bar{k}} = 0 \tag{3.2.25}$$

In coordinates the definition (3.2.19) of the moment map reads (from now on we remove the index k denoting the vector)

$$k_i = g_{i\bar{\jmath}} k^{\bar{\jmath}} = i\, \partial_i P, \qquad k_{\bar{\imath}} = -i\, \partial_{\bar{\imath}} P. \tag{3.2.26}$$

Then the second equation of (3.2.24) is immediately satisfied. An equation for P can be obtained from the first condition in (3.2.24)

$$\nabla_i \partial_j P = 0. \tag{3.2.27}$$

Kähler manifolds are simpler than arbitrary manifolds because a Killing vector is fully determined by one unique real function, mirroring the fact that the metric is given by the Kähler potential.

In general the Kähler potential is not invariant under Killing transformation which can induces a Kähler transformation

$$\mathcal{L}_k K = (k^i \partial_i + k^{\bar{\imath}} \partial_{\bar{\imath}}) K = f + \bar{f}, \tag{3.2.28}$$

which leaves the metric invariant. This makes possible to find an explicit expression for P. Indeed using the expression of the metric, (3.2.26) can be rewritten as

$$k_{\bar{\jmath}} = g_{i\bar{\jmath}} k^i = k^i \partial_i \partial_{\bar{\jmath}} K, \tag{3.2.29}$$

and comparing with (3.2.26) gives

$$P = i\big(k^i \partial_i K - r\big) \tag{3.2.30}$$

where $r = r(\tau)$. This last function can be identified by requiring the reality of P

$$P + \bar{P} = 2P \Longrightarrow (k^i \partial_i + k^{\bar{\imath}} \partial_{\bar{\imath}}) K = r + \bar{r}. \tag{3.2.31}$$

Then the equation (3.2.28) implies that $r = f$ and one obtains

$$P = i\big(k^i \partial_i K - f\big) = -i\big(k^{\bar{\imath}} \partial_{\bar{\imath}} K - \bar{f}\big). \tag{3.2.32}$$

In particular any constant shift ξ of the prepotential can be taken into account by shifting f to $f + i\xi$. There will be an ambiguity only for U(1) factors.

In general a metric admits several Killing vectors k_Λ that generate a non-abelian group with Lie algebra

$$[k_\Lambda, k_\Sigma] = f_{\Lambda\Sigma}{}^{\Omega} k_\Omega. \tag{3.2.33}$$

All quantities then get a Λ index. The bracket does not mix holomorphic and antiholomorphic vectors, and in components they read

$$k_\Lambda^j \partial_j k_\Sigma^i - k_\Sigma^j \partial_j k_\Lambda^i = f_{\Lambda\Sigma}{}^{\Omega} k_\Omega^i \tag{3.2.34}$$

with $\mathcal{L}_\Lambda \equiv \mathcal{L}_{k_\Lambda}$.

For a simple non-abelian group the moment map can be shifted by the constants such that they transform into the adjoint

$$\mathcal{L}_\Lambda P_\Sigma = (k_\Lambda^i \partial_i + k_\Lambda^{\bar{\imath}} \partial_{\bar{\imath}}) P_\Sigma = f_{\Lambda\Sigma}{}^{\Omega} P_\Omega. \tag{3.2.35}$$

This last condition, which is also called the equivariance condition, can be rewritten as

$$k_\Lambda^i g_{i\bar{\jmath}} k_\Sigma^{\bar{\jmath}} - k_\Sigma^i g_{i\bar{\jmath}} k_\Lambda^{\bar{\jmath}} = i f_{\Lambda\Sigma}{}^{\Omega} P_\Omega. \tag{3.2.36}$$

There are four families and two exceptional cases of symmetric Kähler space [165, p. 270]

$$\begin{gathered}\frac{\mathrm{SU}(p,q)}{\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)}, \qquad \frac{\mathrm{SO}^*(2n)}{\mathrm{U}(n)}, \qquad \frac{\mathrm{Sp}(2n)}{\mathrm{U}(n)}, \qquad \frac{\mathrm{SO}(n,2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}, \\ \frac{\mathrm{E}_{6,-14}}{\mathrm{SO}(10)\times\mathrm{U}(1)}, \qquad \frac{\mathrm{E}_{7,-25}}{\mathrm{E}_6\times\mathrm{U}(1)}.\end{gathered} \tag{3.2.37}$$

3.2.4 Kähler–Hodge manifold

Kähler–Hodge manifolds (or Kähler manifold of restricted type) are discussed in [171, sec. 2, 170, sec. 4.1, 4.2, 186, sec. 4.1, 165, sec. sec. 17.3.6, 17.5.1, app. 17A]. In the context of supergravity, the presence of fermions implies a Dirac-like quantization condition on the Kähler form and this is equivalent to the Hodge condition [186, sec. 4.1].

Definition 3.5 (Kähler–Hodge manifold) A Kähler–Hodge manifold $\mathcal{M}$ is a Kähler manifold for which it exists a line bundle $\mathcal{L} \to \mathcal{M}$ such that the first Chern class is equal to the (de Rham) cohomology class of the Kähler form

$$c_1(\mathcal{L}) = [\Omega]. \tag{3.2.38}$$

Given a metric $h(z^i, \bar{z}^{\bar{\imath}})$ on the fiber, the connection reads[2]

$$\theta = \partial \ln h = h^{-1} \partial h \tag{3.2.39}$$

and similarly for $\bar{\theta}$. Then the cohomology class is

$$c_1(\mathcal{L}) = 2i\,[\bar{\partial}\theta] = 2i\,[\bar{\partial}\partial \ln h]. \tag{3.2.40}$$

Recalling (3.2.8)

$$\Omega = 2i\,\partial\bar{\partial}K, \tag{3.2.41}$$

the definition implies that the metric is given by the exponential of the Kähler potential

$$h = \mathrm{e}^K \Longrightarrow \theta = \partial K. \tag{3.2.42}$$

Note that a Kähler transformation corresponds to a gauge transformation on θ

$$\theta \longrightarrow \theta + \partial f, \tag{3.2.43}$$

since the derivative of the Kähler potential transforms as

$$\partial_i K \longrightarrow \partial_i K + \partial_i f. \tag{3.2.44}$$

Then the transition function between two patches if given by e^f which corresponds to a Kähler transformation. A line bundle can be mapped to a U(1) bundle $\mathcal{U} \to \mathcal{M}$, and the corresponding transition function is $\exp(i \operatorname{Im} f)$. The connection on the line and on the U(1) bundles are related by

$$\mathcal{A} = \operatorname{Im}\theta = \frac{i}{2}\,(\theta - \bar{\theta}). \tag{3.2.45}$$

A way to motivate this result is that $\partial_i f = 2i\partial_i \operatorname{Im} f$, whereas taking the real part would give a total derivative and thus a vanishing curvature [165, p. 379]. Using the expression for θ, one obtains

$$\mathcal{A} = -\frac{i}{2}\big(\partial_i K\,\mathrm{d}\tau^i - \partial_{\bar{\imath}} K\,\mathrm{d}\bar{\tau}^{\bar{\imath}}\big). \tag{3.2.46}$$

In real coordinates this can be written

$$\mathcal{A}_a = -\frac{1}{2}\,J_a{}^b \partial_b K. \tag{3.2.47}$$

A field ψ^i (corresponding to a section of $\mathcal{U}$) is said to be of weight $(p, \bar{p})$ if it transforms as

$$\psi^i \longrightarrow \psi'^i = \mathrm{e}^{-\frac{1}{2}(pf + \bar{p}\bar{f})}\,\psi^i \tag{3.2.48}$$

under a Kähler transformation (3.2.6). Then the covariant derivative is

$$\mathrm{D}_i\psi^j = \partial_i\psi^j + \Gamma^j{}_{ik}\psi^k + \frac{p}{2}\,\partial_i K\,\psi^j, \qquad \mathrm{D}_{\bar{\imath}}\psi^j = \partial_{\bar{\imath}}\psi^j + \frac{\bar{p}}{2}\,\partial_{\bar{\imath}} K\,\psi^j. \tag{3.2.49}$$

Moreover the conjugate field $\bar{\psi}^{\bar{\imath}}$ has weight $(-p, -\bar{p})$. In general one has $\bar{p} = -p$ from the fact that the derivative of a section ϕ on $\mathcal{U}$ is

$$\mathrm{D}\phi = (\mathrm{d} + ip\,\mathcal{A})\phi. \tag{3.2.50}$$

Then one can map the sections of $\mathcal{U}$ into sections of $\mathcal{L}$ through

$$\Psi^i = \mathrm{e}^{-\frac{\bar{p}}{2}K}\,\psi^i, \tag{3.2.51}$$

such that the covariant derivatives are

$$\mathrm{D}_i\Psi^j = \partial_i\Psi^j + \Gamma^j{}_{ik}\Psi^k + p\,\partial_i K\,\Psi^j, \qquad \mathrm{D}_{\bar{\imath}}\psi^j = \partial_{\bar{\imath}}\Psi^j. \tag{3.2.52}$$

If ψ^i is holomorphic then the field Ψ^i is covariantly holomorphic

$$\partial_{\bar{\imath}}\psi^j = 0 \Longrightarrow \mathrm{D}_{\bar{\imath}}\Psi^j = 0. \tag{3.2.53}$$

Note also that

$$R_{i\bar{\jmath}} = [\mathrm{D}_i, \mathrm{D}_{\bar{\jmath}}] = i\,g_{i\bar{\jmath}} = -J_{i\bar{\jmath}} \tag{3.2.54}$$

meaning that the curvature of the bundle is the Kähler form.

[2] h is just a function since the line is 1-dimensional, such that $h^{-1} = 1/h$.

Chapter 4

Special Kähler geometry

Special Kähler (SK) manifolds appear as target spaces of non-linear sigma models of the vector scalars in $N = 2$ supergravity. These spaces correspond to Kähler–Hodge manifolds endowed with a symplectic bundle. The U(1) bundle associated to the Hodge condition has the interpretation of the $\mathrm{U}(1)_R$ R-symmetry of the supersymmetry algebra. The simplest formulation is using projective coordinates which are necessary for using a symplectic covariant formalism, which can then be used to formulate more efficiently the $N = 2$ theory. In particular many analytic results for BPS and non-BPS solutions rely heavily on this formulation, and additionally some quaternionic Kähler (QK) manifolds – and more specifically most of those of interest in $N = 2$ supergravity – can be described as a fibration over a SK manifold (see chapter 8.5). Finally both for SK and QK manifolds the isometries are more easily understood using symplectic covariant expressions. For these reasons we propose to review these manifolds in some details: we first start by defining the manifold, its projective parametrization and its Riemannian properties. Then in the following chapters we cover in details other important aspects such as the symplectic invariants, the classification of the homogeneous spaces and the most important models (called quadratic and cubic) and at the end the isometries.

The first axiomatic definition was given in [167], and it was refined in [186] (see also [201]). Major references on the topic are the book [165] and the papers [170, 188, 189].

4.1 Definition

Definition 4.1 (Special Kähler manifold) A special Kähler (SK) manifold $(\mathcal{M}_v, g)$ of real dimension $2n_v$ with complex (or special) coordinates $\{\tau^i, \bar\tau^{\bar\imath}\}$, $i = 1, \dots, n_v$, is a Kähler–Hodge manifold equipped with a (flat) holomorphic vector bundle with group $\mathrm{Sp}(2n_v + 2, \mathbb{R})$, and for which there exists a section v such that the exponential of the Kähler potential is given by

$$K = -\ln(-i\ \langle v, \bar v\rangle) \tag{4.1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the symplectic inner product [170, sec. 4, 186, sec. 4.2.2, 167, sec. 4]. An additional necessary property is[1]

$$\langle v, \partial_i v\rangle = 0. \tag{4.1.2}$$

Other equivalent definitions can be found in [186, sec. 4.2]. Since this manifold is Kähler–Hodge it satisfies all the properties from chapter 3.

The line and vector bundles are respectively denoted by $\mathcal{L} \to \mathcal{M}_v$ and $\mathcal{SV} \to \mathcal{M}_v$. The section v is an element of the tensor bundle $\mathcal{L} \otimes \mathcal{SV}$.

[1] This condition was missing in [167].

The metric is written

$$ds^2 = 2\, g_{i\bar{\jmath}}\, d\tau^i d\tau^{\bar{\jmath}}, \qquad i = 1, \dots, n_v \tag{4.1.3}$$

4.2 Homogeneous coordinates and symplectic structure

4.2.1 Vectors

Let's denote the components of the section v by

$$v = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}, \qquad \Lambda = 0, \dots, n_v. \tag{4.2.1}$$

The X^Λ are called homogeneous coordinates (or projective) coordinates and they provide a projective parametrization of the manifold such that

$$\tau^i = \frac{X^i}{X^0}. \tag{4.2.2}$$

The special coordinates are left unchanged by rescaling of the homogeneous coordinates X^Λ. As a consequence the section v are defined up to rescaling

$$v \longrightarrow e^{-f(\tau)} v. \tag{4.2.3}$$

A convenient gauge choice is[2]

$$X^0 = 1, \qquad X^i = \tau^i. \tag{4.2.4}$$

The transformation properties of this section will be addressed in more details in section 5.1.

We restrict ourselves to the case where the components F_Λ can be derived from a prepotential F which is an homogeneous (holomorphic) function of order 2 in the X^Λ

$$F(\lambda X) = \lambda^2 F(X). \tag{4.2.5}$$

Then one has

$$F_\Lambda = \frac{\partial F}{\partial X^\Lambda} \equiv \partial_\Lambda F. \tag{4.2.6}$$

One can write [170, sec. 4.5, 185, sec. 5]

$$F(X^0, \tau) = (X^0)^2 f(\tau) \tag{4.2.7}$$

where $f(\tau)$ is invariant under rescaling of the coordinates due to the property (4.2.5).

More generally a symplectic vector A of dimension $2(n_v + 1)$ is defined by

$$A = \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}, \tag{4.2.8}$$

where the upper and lower components are distinguished only by the positions of the index (and from the vector itself by the presence of the index).

The symplectic 2-form Ω reads explicitly

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.2.9}$$

It defines a scalar product

$$\langle A, B \rangle \equiv A^t \Omega B = A^\Lambda B_\Lambda - B^\Lambda A_\Lambda. \tag{4.2.10}$$

Sometimes we will need to write explicitly the symplectic indices

$$A^M = \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}, \qquad \Omega_{MN} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad M = 1, \dots, 2(n_v + 1). \tag{4.2.11}$$

With these notations the symplectic product is

$$\langle A, B \rangle = A^M \Omega_{MN} B^N. \tag{4.2.12}$$

[2] Sometimes the name "special coordinates" is used to designate explicitly this gauge choice.

4.2.2 Metric and Kähler potential

The Kähler potential is

$$K = -\ln(-i\,\langle v, \bar{v}\rangle) = -\ln i\big(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda\big). \tag{4.2.13}$$

This definition can be understood from the following fact: the inner product between v and its conjugate transforms as

$$\langle v, \bar{v}\rangle \longrightarrow \mathrm{e}^{-f-\bar{f}}\,\langle v, \bar{v}\rangle\,, \tag{4.2.14}$$

under rescaling of v (4.2.3), and one recognizes in the exponential a possible Kähler transformation [188, p. 4, 167, sec. 2].

The metric is derived from the Kähler potential

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K. \tag{4.2.15}$$

An expression in homogeneous coordinates is given by [165, p. 445]

$$g_{i\bar{\jmath}} = 2\,\mathrm{Im}\,F_{\Lambda\Sigma}\,\partial_i X^\Lambda \partial_{\bar{\jmath}} X^\Sigma. \tag{4.2.16}$$

The metric is invariant under Kähler transformations

$$K \longrightarrow K' = K + f + \bar{f}. \tag{4.2.17}$$

Let's come back to the condition (4.1.2): despite that v is a section of the bundle, the covariant derivative is not necessary because

$$\langle v, \mathrm{D}_i v\rangle = \langle v, \partial_i v\rangle \tag{4.2.18}$$

since the symplectic product is antisymmetric [186, sec. 4.2.2].

4.2.3 Covariant holomorphic fields

The manifold is Kähler–Hodge which means that there is a U(1) bundle (see section 3.2.4 for more details). The section v has weight $p = 1$

$$\mathrm{D}_i v = \partial_i v + \frac{1}{2}\,\partial_i K\, v \tag{4.2.19}$$

and is holomorphic

$$\partial_{\bar{\imath}} v = 0, \tag{4.2.20}$$

such that one can define the holomorphic section

$$\mathcal{V} = \mathrm{e}^{\frac{K}{2}} v \equiv \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \tag{4.2.21}$$

and its covariant derivative

$$U_i = \mathrm{D}_i \mathcal{V} \equiv \begin{pmatrix} f^i_\Lambda \\ h_{i\Lambda} \end{pmatrix}. \tag{4.2.22}$$

One then has

$$\mathrm{D}_{\bar{\imath}} \mathcal{V} = 0. \tag{4.2.23}$$

Note that the coordinates τ^i can also be written as

$$\tau^i = \frac{L^i}{L^0}. \tag{4.2.24}$$

Moreover the section $\mathcal{V}$ is invariant under Kähler transformations by construction (see the previous section).

Taking the exponential of the Kähler potential (4.2.13) and using the expression of the sections (4.2.21) give the normalizations

$$\langle \mathcal{V}, \bar{\mathcal{V}} \rangle = i, \qquad \langle U_i, \bar{U}_{\bar{j}} \rangle = -i\, g_{i\bar{j}}. \tag{4.2.25}$$

The last relation can be used to obtain the metric if one knows U_i.

Decomposing $\mathcal{V}$ into its real and imaginary part, (4.2.25) implies that

$$\langle \operatorname{Re} \mathcal{V}, \operatorname{Im} \mathcal{V} \rangle = -\frac{1}{2}, \qquad \langle \operatorname{Re} U_i, \operatorname{Im} \bar{U}_{\bar{j}} \rangle = \frac{1}{2}\, g_{ij}. \tag{4.2.26}$$

The symplectic product of a vector A with $\mathcal{V}$ and U_i are defined by

$$\Gamma(A) = \langle \mathcal{V}, A \rangle\,, \qquad \Gamma_i(A) = \mathrm{D}_i \Gamma(A) = \langle U_i, A \rangle \tag{4.2.27}$$

and similarly for the complex conjugates $\bar{\Gamma}(A)$ and $\bar{\Gamma}_{\bar{i}}(A)$. Note that these operators are linear and $\Gamma_i(A)$ can be defined only if the vector A is independent of τ^i. In particular one has

$$\Gamma(\bar{\mathcal{V}}) = i, \qquad \Gamma(\operatorname{Re} \mathcal{V}) = \frac{i}{2}, \qquad \Gamma(\operatorname{Im} \mathcal{V}) = -\frac{1}{2}, \qquad \Gamma(U_i) = 0. \tag{4.2.28}$$

Note that as a consequence of the previous relations one has

$$\mathrm{D}_{\bar{\imath}} \Gamma(A) = 0, \qquad \mathrm{D}_{\bar{j}} \mathrm{D}_i \Gamma(A) = g_{i\bar{j}}\, \Gamma(A). \tag{4.2.29}$$

4.2.4 Prepotential properties

The nth derivative of the prepotential is

$$F_{\Lambda_1 \cdots \Lambda_n} \equiv \frac{\partial F}{\partial X^{\Lambda_1} \cdots \partial X^{\Lambda_n}}. \tag{4.2.30}$$

The homogeneity of the prepotential implies several identities for its derivatives [185, sec. 2, 165, p. 433]

$$X^{\Lambda_n}\, F_{\Lambda_1 \cdots \Lambda_n} = (3 - n)\, F_{\Lambda_1 \cdots \Lambda_{n-1}} \tag{4.2.31}$$

(for $n = 1$ we define $F_{\Lambda_1 \Lambda_0} \equiv F$) and in particular [12]

$$F = \frac{1}{2}\, F_\Lambda X^\Lambda, \qquad F_\Lambda = F_{\Lambda\Sigma} X^\Sigma, \qquad F_{\Lambda\Sigma\Delta} X^\Delta = 0. \tag{4.2.32}$$

The special case $n = 3$ implies the following relation

$$\mathrm{d}F_\Lambda = F_{\Lambda\Sigma} \mathrm{d}X^\Sigma \tag{4.2.33}$$

since

$$\mathrm{d}F_\Lambda = \mathrm{d}(F_{\Lambda\Sigma} X^\Sigma) = F_{\Lambda\Sigma} \mathrm{d}X^\Sigma + X^\Sigma \mathrm{d}F_{\Lambda\Sigma} = F_{\Lambda\Sigma} \mathrm{d}X^\Sigma + \cancel{X^\Sigma F_{\Lambda\Sigma\Xi} \mathrm{d}X^\Xi}. \tag{4.2.34}$$

Two prepotentials that differ by a quadratic polynomial in X^Λ with real coefficients are equivalent as they do not contribute to the Kähler potential [202, p. 5, 144, p. 5]. Moreover such terms can be removed/added by a symplectic transformation (see section 5.1).

4.3 Homogeneous matrices

4.3.1 Hessian matrix

The Hessian matrix $\mathcal{F}$ of the prepotential F is written

$$F_{\Lambda\Sigma} = \partial_\Lambda F_\Sigma = \partial_\Sigma F_\Lambda. \tag{4.3.1}$$

In section 4.4 we will prove that $\operatorname{Im} \mathcal{F}$ has n_v positive and one negative eigenvalues.

4.3.2 Period matrix

The period matrix[3] [165, p. 448]

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i \frac{\operatorname{Im} F_{\Lambda\Delta} \operatorname{Im} F_{\Sigma\Xi} \, X^\Delta X^\Xi}{\operatorname{Im} F_{\Delta\Xi} \, X^\Delta X^\Xi} \tag{4.3.2}$$

is symmetric and is an object that allows to lower the index of L^Λ as

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma. \tag{4.3.3}$$

On the other hand f^i_Λ and $h_{i\Lambda}$ are related by

$$h_{i\Lambda} = \bar{\mathcal{N}}_{\Lambda\Sigma} f^\Sigma_i. \tag{4.3.4}$$

This means that $\mathcal{N}$ is not a metric for Λ index. Note also that $\mathcal{I}$ is negative definite, which is a consequence of the positivity of the metric. The real and imaginary parts of this matrix are written as $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{R}_{\Lambda\Sigma} + i\, \mathcal{I}_{\Lambda\Sigma}. \tag{4.3.5}$$

The inverse of the matrices is denoted with upper indices.

There are some useful identities

$$L^\Lambda \mathcal{I}_{\Lambda\Sigma} \bar{L}^\Sigma = -\frac{1}{2}, \qquad f^\Lambda_i \mathcal{I}_{\Lambda\Sigma} \bar{f}^\Sigma_{\bar{\jmath}} = -\frac{1}{2}\, g_{i\bar{\jmath}}, \qquad L^\Lambda \mathcal{I}_{\Lambda\Sigma} f^\Sigma_i = 0, \tag{4.3.6a}$$

$$U^{\Lambda\Sigma} = f^\Lambda_i g^{i\bar{\jmath}} \bar{f}^\Sigma_{\bar{\jmath}} = -\frac{1}{2} \mathcal{I}^{\Lambda\Sigma} - \bar{L}^\Lambda L^\Sigma. \tag{4.3.6b}$$

4.4 Symplectic matrices

Let's denote by $\mathcal{T}_{\Lambda\Sigma}$ a symmetric matrix of dimension $(n_v + 1)$, and define its real and imaginary parts[4]

$$\mathcal{T} = \mathcal{R} + i\,\mathcal{I}. \tag{4.4.1}$$

Then the (real) symplectic matrix $\mathcal{M}(\mathcal{T})$ is defined by [203, sec. 3.2, 40, sec. 1] (see also [52, p. 5, 165, p. 514, 46, app. A, 53, app. A])

$$\mathcal{M}(\mathcal{T}) = \begin{pmatrix} 1 & -\mathcal{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathcal{R} & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}, \tag{4.4.2}$$

of dimension $2(n_v + 1)$, where 1 denotes the identity matrix of dimension $(n_v + 1)$. The matrix is symmetric since $\mathcal{R}$ and $\mathcal{I}$ are symmetric. It is also symplectic because it satisfies the relation

$$\mathcal{M}^t \Omega \mathcal{M} = \Omega \quad \Longrightarrow \quad \mathcal{M}\Omega\mathcal{M} = \Omega, \tag{4.4.3}$$

the second relation following from the symmetric shape of the matrix.

The product[5] of Ω with this matrix $\mathcal{M}$

$$\Omega\mathcal{M} = -\begin{pmatrix} \mathcal{I}^{-1}\mathcal{R} & -\mathcal{I}^{-1} \\ \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \end{pmatrix} \tag{4.4.4}$$

is also symplectic

$$(\Omega\mathcal{M})^t\, \Omega\, (\Omega\mathcal{M}) = \Omega. \tag{4.4.5}$$

Two matrices of this type are of interest

$$\mathcal{M}_+ \equiv \mathcal{M}(\mathcal{N}) \equiv \mathcal{M}, \qquad \mathcal{M}_- \equiv \mathcal{M}(\mathcal{F}), \tag{4.4.6}$$

[3] This expression could also be given in terms of L^Λ because it has weight 0.
[4] Later we will use normal letters instead of curly ones for the real and imaginary parts.
[5] Some authors call this product $\mathcal{M}$ [169, sec. 2.2].

where $\mathcal{F}$ and $\mathcal{N}$ are respectively the period (4.3.2) and Hessian (4.3.1) matrices. Similarly by convention $\mathcal{R}$ and $\mathcal{I}$ without further specifications are the real and imaginary parts of $\mathcal{N}$.

The product $\Omega\mathcal{M}$ defines a complex structure on the bundle [52, sec. 2.2]

$$\Omega\mathcal{M}\mathcal{V} = i\mathcal{V}, \qquad \Omega\mathcal{M}U_i = -iU_i. \tag{4.4.7}$$

For this reason eigenvalues of $\Omega\mathcal{M}$ are $\pm i$ ($n_v + 1$ of each). This matrix squares to -1

$$(\Omega\mathcal{M})^2 = -1. \tag{4.4.8}$$

This last expression gives the inverse of $\Omega\mathcal{M}$ as

$$(\Omega\mathcal{M})^{-1} = -\Omega\mathcal{M} \tag{4.4.9}$$

and this can be rewritten (4.4.5) as

$$(\Omega\mathcal{M})^t\Omega = -\Omega(\Omega\mathcal{M}). \tag{4.4.10}$$

Since $\mathcal{M}$ and $\Omega\mathcal{M}$ are symplectic they preserve the inner product and they can be moved inside

$$\langle \Omega\mathcal{M}A, \Omega\mathcal{M}B\rangle = \langle A, B\rangle\,, \qquad \langle \Omega\mathcal{M}A, B\rangle = \langle A, \Omega\mathcal{M}B\rangle\,. \tag{4.4.11}$$

Since the vectors $(\mathcal{V}, \bar{\mathcal{V}}, U_i, \bar{U}_i)$ form a complete basis of $\mathcal{M}_v$ [204, app. A], the identity and $\mathcal{M}$ can be expanded

$$1 = i\,\mathcal{V}\bar{\mathcal{V}}^t\Omega - i\,\bar{\mathcal{V}}\mathcal{V}^t\Omega - i\,g^{i\bar{\jmath}}\,U_i\bar{U}_{\bar{\jmath}}^t\Omega + i\,g^{i\bar{\jmath}}\,\bar{U}_{\bar{\jmath}}U_i^t\Omega, \tag{4.4.12a}$$

$$-\Omega\mathcal{M} = \mathcal{V}\bar{\mathcal{V}}^t\Omega + \bar{\mathcal{V}}\mathcal{V}^t\Omega + g^{i\bar{\jmath}}\,U_i\bar{U}_{\bar{\jmath}}^t\Omega + g^{i\bar{\jmath}}\,\bar{U}_{\bar{\jmath}}U_i^t\Omega. \tag{4.4.12b}$$

The decompositions of Ω and $\mathcal{M}$ are straightforward. These relations can be checked by multiplying them on the right by $\mathcal{V}$ and U_i and their conjugates before using the orthonormality (4.2.25) (implying that only one term of the sum contributes) and the properties of the complex structure (4.4.7); as an example multiply the second one by $\mathcal{V}$

$$\mathcal{M}\mathcal{V} = -i\mathcal{V} = \mathcal{V}\bar{\mathcal{V}}^t\Omega\mathcal{V}. \tag{4.4.13}$$

In particular any (real) vector A on can be expanded on the basis $(\mathcal{V}, \bar{\mathcal{V}}, U_i, \bar{U}_i)$ through (4.4.12a) [53, app. A]

$$A = i\,\langle\bar{\mathcal{V}}, A\rangle\,\mathcal{V} - i\,\langle\mathcal{V}, A\rangle\,\bar{\mathcal{V}} + i\,g^{i\bar{\jmath}}\,\langle U_i, A\rangle\,\bar{U}_{\bar{\jmath}} - i\,g^{\bar{\imath}j}\,\langle\bar{U}_{\bar{\imath}}, A\rangle\,U_j \tag{4.4.14a}$$

$$= i\,\bar{\Gamma}(A)\,\mathcal{V} - i\,\Gamma(A)\,\bar{\mathcal{V}} + i\,g^{i\bar{\jmath}}\,\Gamma_i(A)\,\bar{U}_{\bar{\jmath}} - i\,g^{\bar{\imath}j}\,\bar{\Gamma}_{\bar{\imath}}(A)\,U_j \tag{4.4.14b}$$

$$= 2\,\mathrm{Im}\,\big(\bar{\mathcal{V}}\,\langle A, \,\rangle\,\mathcal{V}\big) - 2g^{i\bar{\jmath}}\,\mathrm{Im}\,\big(\,\langle\bar{U}_{\bar{\jmath}}, A\rangle\,U_i\big) \tag{4.4.14c}$$

$$= 2\,\mathrm{Im}\,\big(\bar{\Gamma}(A)\,\mathcal{V}\big) - 2g^{i\bar{\jmath}}\,\mathrm{Im}\,\big(\bar{\Gamma}_{\bar{\jmath}}(A)\,U_i\big). \tag{4.4.14d}$$

From Ω and $\mathcal{M}$ another matrix can be defined [52, sec. 2.2]

$$\mathcal{C} = \frac{1}{2}(\mathcal{M} - \varepsilon_\Omega\, i\Omega) \tag{4.4.15}$$

This matrix is hermitian [205, sec. 3]

$$\mathcal{C}^\dagger = \mathcal{C}. \tag{4.4.16}$$

and from (4.4.7) it satisfies the twisted self-duality

$$\mathcal{C}\mathcal{V} = -\varepsilon_\Omega\, i\,\Omega\mathcal{V}. \tag{4.4.17}$$

Using equation (4.4.3) one can show that

$$\mathcal{C}\Omega\mathcal{C} = \varepsilon_\Omega\, i\,\mathcal{C}. \tag{4.4.18}$$

Taking a symplectic vector A, the decomposition (4.4.12b) implies the sum rule [203, sec. 3.2]

$$-\frac{1}{2}\, A^t \mathcal{M} A = |\Gamma(A)|^2 + |\Gamma_i(A)|^2 . \tag{4.4.19}$$

Hence $\mathcal{M}$ defines a quadratic form which is negative definite if the metric is positive definite [165, p. 448], which reflects the fact that $\operatorname{Im}\mathcal{N}$ is negative definite. This is a consequence of the fact that $\mathcal{R}$ does not play any role since, defining the vector

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ -\mathcal{R} & 1 \end{pmatrix} A, \tag{4.4.20}$$

one can rewrite the previous relation as

$$A^t \mathcal{M} A = \tilde{A}^t \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I}^{-1} \end{pmatrix} \tilde{A}. \tag{4.4.21}$$

Similarly $\mathcal{M}(\mathcal{F})$ defines a quadratic form through another sum rule

$$-\frac{1}{2}\, A^t \mathcal{M}(\mathcal{F}) A = |\Gamma(A)|^2 - |\Gamma_i(A)|^2 . \tag{4.4.22}$$

This shows that $\operatorname{Im}\mathcal{F}$ has one negative and n_v positive eigenvalues.

Note also the relation

$$-\frac{1}{2}\, A^t \mathcal{M}(\mathcal{F}) A = \frac{1}{2}\, A^t \mathcal{M} A + 2\, |\Gamma(A)|^2 . \tag{4.4.23}$$

4.5 Structure coefficients

For a summary of this section, see [170, sec. 4.3, 206, sec. 4] (and also [171, sec. 2, 167, sec. 2]).

The structure constant of the SK space is a symmetric 3-tensor defined by

$$C_{ijk} = \langle \mathrm{D}_i U_j, U_k \rangle \tag{4.5.1}$$

and it is covariantly holomorphic of weight 2

$$\mathrm{D}_{\bar m} C_{ijk} = 0. \tag{4.5.2}$$

(this covariant derivative does not involve Christoffel symbol). Notice that, as it is a 3-tensor, the covariant derivative reads explicitly

$$\mathrm{D}_i C_{jk\ell} = \partial_i C_{jk\ell} + (\partial_i K) C_{jk\ell} + \Gamma^m{}_{ij} C_{mk\ell} + \Gamma^m{}_{ik} C_{jm\ell} + \Gamma^m{}_{i\ell} C_{jkm} \tag{4.5.3}$$

(this expression is symmetric in ij), and we recall the expression of the Christoffel symbol

$$\Gamma^i{}_{jk} = -g^{i\bar\ell} \partial_j g_{k\bar\ell}. \tag{4.5.4}$$

From this tensor one defines the rescaled structure constant

$$W_{ijk} = \mathrm{e}^{-K} C_{ijk} \tag{4.5.5}$$

which satisfies

$$\partial_{\bar m} W_{ijk} = 0. \tag{4.5.6}$$

The complex conjugate is written $\bar C_{\bar\imath\bar\jmath\bar k}$, and the quantities with upper indices are obtained from

$$C^{\bar\imath\bar\jmath\bar k} = g^{i\bar\imath} g^{j\bar\jmath} g^{k\bar k} C_{ijk}, \qquad \bar C^{ijk} = g^{i\bar\imath} g^{j\bar\jmath} g^{k\bar k} \bar C_{\bar\imath\bar\jmath\bar k}. \tag{4.5.7}$$

The corresponding rescaled quantities are

$$\bar{W}^{ijk} = g^{i\bar{\imath}} g^{j\bar{\jmath}} g^{k\bar{k}} \bar{W}_{\bar{\imath}\bar{\jmath}\bar{k}} = \mathrm{e}^{-K} \bar{C}^{ijk}. \tag{4.5.8}$$

As a consequence one finds

$$\mathrm{D}_i U_j = i\, C_{ijk} g^{k\bar{k}} \bar{U}_{\bar{k}} \tag{4.5.9}$$

which implies

$$\mathrm{D}_i \mathrm{D}_j \Gamma(A) = i\, C_{ijk} g^{k\bar{k}} \bar{\Gamma}_{\bar{k}}(A). \tag{4.5.10}$$

Given a vector A the so-called cubic norm reads [207, sec. 2.1, 40, sec. 5]

$$N(A) = C_{ijk} \bar{\Gamma}^i(A) \bar{\Gamma}^j(A) \bar{\Gamma}^k(A), \qquad \bar{N}(A) = \bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}} \Gamma^{\bar{\imath}}(A) \Gamma^{\bar{\jmath}}(A) \Gamma^{\bar{k}}(A). \tag{4.5.11}$$

Note that

$$N(\mathcal{V}) = 0 \Longrightarrow N(\mathrm{Re}\, \mathcal{V}) = N(\mathrm{Im}\, \mathcal{V}) = 0 \tag{4.5.12}$$

because of the orthogonality conditions (4.2.25).

One defines finally the rank 5 E-tensor

$$E^m{}_{ijk\ell} = g^{m\bar{m}} E_{\bar{m}ijk\ell}, \qquad E_{\bar{m}ijk\ell} = \frac{1}{3} \bar{\mathrm{D}}_{\bar{m}} \mathrm{D}_i C_{jk\ell}. \tag{4.5.13}$$

It is symmetric in all covariant indices. An explicit expression can be computed

$$E^m{}_{ijk\ell} + \frac{4}{3} C_{(ijk} \delta_{\ell)}{}^m = g^{m\bar{m}} g^{n\bar{n}} g^{p\bar{p}}\, C_{n(ij} C_{k\ell)p} \bar{C}_{\bar{m}\bar{n}\bar{p}}. \tag{4.5.14}$$

4.6 Riemannian geometry

The Riemann geometry of SK manifolds is described in [202, 185, sec. 2, 167, sec. 2], and additional details on symmetric spaces are in [40, sec. 5, 144, 208].

4.6.1 General properties

Since the space is Kähler, the expressions from section 3.2.2 can be used. But the additional properties give alternative expressions.

The Riemann tensor read

$$R_{i\bar{\jmath}k\bar{\ell}} = g_{i\bar{\jmath}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{\jmath}} - g^{m\bar{n}} C_{ikm} \bar{C}_{\bar{\jmath}\bar{\ell}\bar{n}}, \tag{4.6.1}$$

the sign being chosen such that $R < 0$ [206, sec. 4]. In the rigid limit only the last term survives.

Contracting with the metric gives the Ricci tensor

$$R_{i\bar{\jmath}} = g^{k\bar{\ell}} R_{i\bar{\ell}k\bar{\jmath}} = -(n_v + 1) g_{i\bar{\jmath}} + g^{k\bar{\ell}} g^{m\bar{n}} C_{ikm} \bar{C}_{\bar{\jmath}\bar{\ell}\bar{n}}. \tag{4.6.2}$$

And finally one finds the curvature

$$R = g^{i\bar{\jmath}} R_{i\bar{\jmath}} = -n_v(n_v + 1) + g^{i\bar{\jmath}} g^{k\bar{\ell}} g^{m\bar{n}} C_{ikm} \bar{C}_{\bar{\jmath}\bar{\ell}\bar{n}}. \tag{4.6.3}$$

4.6.2 Symmetric space

The space $\mathcal{M}_v$ is symmetric if the Riemann tensor is covariantly constant

$$\mathrm{D}_m R_{i\bar{j}k\bar{\ell}} = 0. \tag{4.6.4}$$

This implies that[6]

$$\mathrm{D}_\ell C_{ijk} = \mathrm{D}_{(\ell} C_{i)jk} = 0, \tag{4.6.5}$$

and as a consequence the E-tensor (4.5.13) vanishes

$$E^m{}_{ijk\ell} = 0. \tag{4.6.6}$$

From (4.5.14) this implies the relation

$$\frac{4}{3}\, C_{(ijk} g_{\ell)\bar{m}} = g^{n\bar{n}} g^{p\bar{p}}\, C_{n(ij} C_{k\ell)p} \bar{C}_{\bar{m}\bar{n}\bar{p}}, \tag{4.6.7}$$

and thus

$$g^{n\bar{n}} R_{(i|\bar{m}|j|\bar{n}} C_{n|k\ell)} = -\frac{2}{3}\, g_{(i|\bar{m}} C_{|jk\ell)}. \tag{4.6.8}$$

[6] Note that the next two equations are necessary conditions for the manifold to be symmetric, but they are not sufficient [206, sec. 4].

Chapter 5

Symplectic transformations and invariants

The description of SK manifolds in terms of the section and its derivative is symplectic covariant and we are free to change the parametrization of the bundle section $\mathcal{V}$ by performing a $\mathrm{Sp}(2n_v+2,\mathbb{R})$ rotation. This means that the expressions are not *invariant* when written in coordinates (for example the prepotential changes) but they keep the same form when given in terms of symplectic vectors. This can be compared to general relativity where expressions are covariant/invariant with respect to diffeomorphisms/isometries. A given basis is called a (symplectic) *frame*.

The next question is to construct objects that are invariant under isometries. It appears that a quartic symmetric tensor exist for SK symmetric manifolds G/H since the group G is of type E_7. This invariant tensor plays an important role in many places, such as the definition of isometries of special quaternionic manifolds (see chapter 9), in the construction of analytic solutions to the BPS equations or in some important quantities defining the black holes, such as the area of the adS_4 radius. This structure is most clearly seen using a symplectic covariant formalism, which also simplifies the formulation of the equations and of the Lagrangian.

5.1 Symplectic transformations

References include [170, sec. 2, 188, 189, 186, sec. 2, app. A].

5.1.1 Holomorphic section

A matrix $\mathcal{U}$ is symplectic if

$$\mathcal{U}^t \Omega \mathcal{U} = \Omega. \tag{5.1.1}$$

Parametrizing the matrix as

$$\mathcal{U} = \begin{pmatrix} Q & R \\ S & T \end{pmatrix} = \begin{pmatrix} Q^\Lambda{}_\Sigma & R^{\Lambda\Sigma} \\ S_{\Lambda\Sigma} & T_\Lambda{}^\Sigma \end{pmatrix}. \tag{5.1.2}$$

this implies the following constraints

$$Q^t S - S^t Q = 0, \qquad R^t T - T^t R = 0, \qquad Q^t T - S^t R = 1. \tag{5.1.3}$$

From these one can determine the dimension of the group [165, p. 85]

$$\dim \mathrm{Sp}(2n_v+2,\mathbb{R}) = (n_v+1)(2n_v+3). \tag{5.1.4}$$

The matrix $\mathcal{U}$ acts on $\mathcal{V}$ as

$$\mathcal{V}' = \mathcal{U}\mathcal{V} \qquad \Longrightarrow \qquad \begin{cases} L'^\Lambda = Q^\Lambda{}_\Sigma L^\Sigma + R^{\Lambda\Sigma} M_\Sigma, \\ M'_\Lambda = S_{\Lambda\Sigma} L^\Sigma + T_\Lambda{}^\Sigma M_\Sigma. \end{cases} \tag{5.1.5}$$

Since the matrix is constant it acts in the same way on U_i

$$U'_i = \mathcal{U} U_i = \mathrm{D}_i(\mathcal{U}\mathcal{V}). \tag{5.1.6}$$

In order to preserve the relation (4.3.3) in the new frame

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma \qquad \Longrightarrow \qquad M'_\Lambda = \mathcal{N}'_{\Lambda\Sigma} L'^\Sigma \tag{5.1.7}$$

it is necessary for the matrix $\mathcal{N}$ to transform as

$$\mathcal{N}' = (S + T\mathcal{N})(Q + R\mathcal{N})^{-1}. \tag{5.1.8}$$

For this one needs to replace M_Λ in (5.1.5)

$$L' = (Q + R\mathcal{N})L, \qquad M' = (S + T\mathcal{N})M = \mathcal{N}' L'. \tag{5.1.9}$$

For some applications it is convenient to consider infinitesimal transformations

$$\mathcal{U} = \mathrm{e}^{\mathfrak{U}} \sim 1 + \mathfrak{U} \tag{5.1.10}$$

where $\mathfrak{U} \in \mathfrak{sp}(2n_v + 2, \mathbb{R})$ and one writes

$$\delta\mathcal{V} = \mathfrak{U}\,\mathcal{V}. \tag{5.1.11}$$

The condition (5.1.1) translates into

$$\mathfrak{U}^t \Omega + \Omega \mathfrak{U} = 0, \tag{5.1.12}$$

or as

$$t = -q^t, \qquad r = r^t, \qquad s = s^t \tag{5.1.13}$$

in terms of the parametrization

$$\mathfrak{U} = \begin{pmatrix} q & r \\ s & t \end{pmatrix}. \tag{5.1.14}$$

5.1.2 Section and coordinates

The variation of the homogeneous coordinates can be written as [185, sec. 6, 209]

$$\delta X^\Lambda = q^\Lambda{}_\Sigma X^\Sigma + r^{\Lambda\Sigma} F_\Sigma = \Big(q^\Lambda{}_\Sigma + r^{\Lambda\Xi} F_{\Xi\Sigma} \Big) X^\Sigma \tag{5.1.15}$$

using the homogeneity of F. One sees that $\delta X^0 \neq 0$ which implies that the two sets of special coordinates

$$\tau^i = \frac{X^i}{X^0}, \qquad \tau'^i = \frac{X'^i}{X'^0} \tag{5.1.16}$$

are not equivalent anymore, i.e. the transformation does not preserve the gauge choice imposed on X^0 for defining the special coordinates. For this reason one needs to rescale the coordinates X^Λ by multiplying by X'^0/X^0. Infinitesimally this implies

$$\delta\tau^i = \Big(q^i{}_\Sigma + r^{i\Xi} F_{\Xi\Sigma} \Big) \tau^\Sigma - \tau^i \Big(q^0{}_\Sigma + r^{0\Xi} F_{\Xi\Sigma} \Big) \tau^\Sigma \tag{5.1.17}$$

where $\tau^0 = 1$.

A first condition on these transformations is that [144, app. C]

$$\frac{\partial X'^\Lambda}{\partial X^\Sigma} \neq 0 \tag{5.1.18}$$

is non-singular, which means that the transformation of X'^Λ in terms of X^Λ (with F_Λ taken as a function of X^Λ) is invertible.

If one wants to keep the same class of Lagrangian – derivable from a prepotential – then one also needs that it exists a function F' such that

$$F'_\Lambda = \frac{\partial F'(X')}{\partial X'^\Lambda}. \tag{5.1.19}$$

This is the case when

$$F'_{\Lambda\Sigma} = \frac{\partial F'_\Lambda}{\partial X'^\Sigma} \tag{5.1.20}$$

is symmetric.

The new prepotential F' is obtained by using the relation

$$F' = \frac{1}{2}\, F'_\Lambda X'^\Lambda \tag{5.1.21}$$

and the explicit expression for F'_Λ and X'^Λ.

The expression for the new prepotential is

$$F'(X') = F(X) + X S^t R F + \frac{1}{2}\, X S^t Q X + \frac{1}{2}\, F T^t R F \tag{5.1.22}$$

where all F except in the first term are denoting the vector F_Λ.

It is always possible to find a frame where a prepotential exists [186, sec. 4.2].

5.2 Symplectic invariants

Any quantity made from symplectic products behaves as a scalar under symplectic transformations – and by an abuse of language we write sometimes "symplectic invariant". This corresponds to H-invariance [210, sec. 1]. In particular this is the case of the structure constant (4.5.1) since it is defined as a symplectic product, and – given a vector A – of the products $\Gamma(A)$ and $\Gamma_i(A)$, and of the cubic norm $N(A)$, given by (4.2.27) and (4.5.11).

If the manifold is a coset $\mathcal{M}_v \equiv G/H$, then symplectic scalars with no free (anti)holomorphic indices are only H-invariant if the coordinates are fixed. Conversely H-invariant expressions are also symplectic covariant.

In the following invariants associated to a vector A are built, and we write $\Gamma \equiv \Gamma(A)$, $\Gamma_i \equiv \Gamma_i(A)$ and $N(A) \equiv N$. The independent invariants were listed in [40, sec. 5] (see also [210, 42, sec. 2]). Two invariants are given by

$$I_\pm(A, \mathcal{V}) = -\frac{1}{2}\, A^t \mathcal{M}_\pm = |\Gamma|^2 \pm |\Gamma_i|^2\,. \tag{5.2.1}$$

They can be written in terms of the two invariants

$$i_1 = |\Gamma|^2\,, \tag{5.2.2a}$$

$$i_2 = |\Gamma_i|^2\,. \tag{5.2.2b}$$

Two others can be introduced

$$i_3 = \frac{1}{3!}\,(\Gamma\, N + \bar\Gamma\, \bar N), \tag{5.2.2c}$$

$$i_4 = \frac{i}{3!}\,(\Gamma\, N - \bar\Gamma\, \bar N), \tag{5.2.2d}$$

along with the Poisson bracket

$$i_5 = \left\{N, \bar N\right\} = g^{i\bar\imath}\, \frac{\partial N}{\partial \bar\Gamma^i}\, \frac{\partial \bar N}{\partial \Gamma^{\bar\imath}} = g^{i\bar\imath} C_{ijk} \bar C_{\bar\imath\bar\jmath\bar k}\, \bar\Gamma^j \bar\Gamma^k \Gamma^{\bar\jmath} \Gamma^{\bar k}. \tag{5.2.2e}$$

5.3 Duality invariants

In this section we assume that the SK space is symmetric

$$\mathcal{M}_v = \frac{G}{H}, \tag{5.3.1}$$

where G is called the duality group.

5.3.1 General definition

A duality invariant

$$I_n = I_n(A, \mathcal{V}) = I_n\big(\Gamma(A), \Gamma_i(A)\big) \tag{5.3.2}$$

(where A is any symplectic vector) is a homogeneous polynomial of order n which is invariant under G-transformations (i.e. under the isometries). One consequence is that it does not depend on the manifold coordinates [40, 210, footnote 1]

$$\partial_i I_n = 0 \Longleftrightarrow I_n = I_n(A). \tag{5.3.3}$$

In $d = 4$ duality invariants for all symmetric manifolds G/H are quartic,[1] i.e. $n = 4$. This is a consequence of the fact that the group G is always of type E_7 [148, 151, 211].

Definition 5.1 (E_7-type Lie group) A group of type E_7 is a Lie groups for which there exists a representation $\mathbf{R}$ such that ($A_i \in \mathbf{R}$ in the following) [147, sec. 4, 148, sec. 2.1]:

1. $\mathbf{R}$ is symplectic, which means that the singlet $\mathbf{1}$ sits into the antisymmetric product

   $$1 = (\mathbf{R} \times \mathbf{R})_a, \tag{5.3.4}$$

   and the associated invariant tensor $\mathbb{C}$ corresponds to the symplectic metric (skew-symmetric 2-form). The latter defines a symplectic product for vectors in $\mathbf{R}$

   $$\langle A_1, A_2 \rangle = \mathbb{C}_{MN} A_1^M A_2^N. \tag{5.3.5}$$

2. There exists a unique invariant symmetric 4-tensor t (called a primitive G-invariant structure)

   $$1 = (\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R})_s, \tag{5.3.6}$$

   and then one can define the map $I_4 : \mathbf{R}^4 \to \mathbb{R}$

   $$I_4(A_1, A_2, A_3, A_4) = t_{MNPQ} A_1^M A_2^N A_3^P A_4^Q. \tag{5.3.7}$$

3. The trilinear map $I_4' : \mathbf{R}^3 \to \mathbf{R}$ defined by

   $$\langle I_4'(A_1, A_2, A_3), A_4 \rangle = I_4(A_1, A_2, A_3, A_4), \tag{5.3.8}$$

   satisfies

   $$\langle I_4'(A_1, A_1, A_1), I_4'(A_2, A_2, A_2) \rangle = -2\, I_4(A_1, A_1, A_2, A_2)\ \langle A_1, A_2 \rangle\,. \tag{5.3.9}$$

These properties are linked to the connection between Jordan algebras (and Freudenthal triple system) and special Kähler manifolds. They imply various identities for the quartic invariant.

[1] As we will see later, the groups are *degenerate* for quadratic prepotential, and there is a quadratic invariant.

5.3.2 Quartic invariant

A quartic invariant can be defined for symmetric SK manifold [40, sec. 5] (for other references, see [207, sec. 2.1, 151, sec. 4, 212, sec. 4.3, 85, app. A])

$$I_4 = (i_1 - i_2)^2 - 4\, i_4 - i_5 \tag{5.3.10a}$$

or using explicit expression

$$I_4 = \big(|\Gamma|^2 - |\Gamma_i|^2 \big)^2 - \frac{2i}{3} \big(\Gamma\, N - \bar\Gamma\, \bar N\big) - g^{i\bar\imath} C_{ijk} \bar C_{\bar\imath\bar\jmath\bar k}\, \bar\Gamma^j \bar\Gamma^k \Gamma^{\bar\jmath} \Gamma^{\bar k}. \tag{5.3.10b}$$

This expression does not depend of the symplectic frame and is invariant under diffeomorphisms of $\mathcal{M}_v$ (detailed in section 7) [151, sec. 4].

The above general expression is sometimes said to be given in the complex basis [207] (as opposed to its expression for cubic prepotentials which is real). In [212, sec. 4.3] it is called the "entropy functional". This quartic invariant can be built directly from the generators of the group G [213, sec. 3, 211].

5.3.3 Invariant tensor

Then one can define a symmetric 4-tensor [151, sec. 4, 85, app. B]

$$t_{MNPQ} = \frac{\partial^4 I_4(A)}{\partial A^M \partial A^N \partial A^P \partial A^Q}. \tag{5.3.11}$$

Explicit expressions for this tensor can be found in [212, sec. 4.3].

Then one can define a function I_4 that takes four arguments

$$I_4(A, B, C, D) = t_{MNPQ} A^M B^N C^P D^Q \tag{5.3.12}$$

along with its gradient

$$I_4'(A, B, C)^M = \Omega^{MR}\, t_{RNPQ} A^N B^P C^Q \tag{5.3.13}$$

where Ω is used to get a vector and not a form.

Finally one defines the formulas for equal arguments

$$I_4(A) = \frac{1}{4!}\, I_4(A, A, A, A), \qquad I_4'(A) = \frac{1}{3!}\, I_4'(A, A, A). \tag{5.3.14}$$

Since I_4' defines a vector it is possible to nest expressions. These expressions can be simplified using identities for the product $t_{MNPQ} \Omega^{MR} t_{RSTU}$, which depend on the type of the manifold under consideration (magical, cubic non-magical and quadratic models) [148, sec. 2] (see also [147]).

By definition one has

$$\langle A_1, I_4'(A_2, A_3, A_4) \rangle = I_4(A_1, A_2, A_3, A_4). \tag{5.3.15}$$

From (4.2.25) one finds that

$$I_4(\operatorname{Re} \mathcal{V}) = I_4(\operatorname{Im} \mathcal{V}) = \frac{1}{16} \tag{5.3.16}$$

using (4.2.28) and the fact that all other terms vanish.

5.3.4 Freudenthal duality

The Freudenthal dual $\mathfrak{f}(A)$ of a vector A is defined by [151]

$$\mathfrak{f}(A)^M = \Omega^{MN} \frac{\partial \left| \sqrt{I_4(A)} \right|}{\partial A^N}. \tag{5.3.17}$$

This operator $\mathfrak{f}$ is an anti-involution and preserves the quartic invariant

$$\mathfrak{f}(\mathfrak{f}(A)) = -A, \qquad I_4(\mathfrak{f}(A)) = I_4(A). \tag{5.3.18}$$

Then $\mathfrak{f}$ is a complex structure.

5.4 Non-symmetric spaces

The function I_4 can be defined for non-symmetric spaces, but then it depends on the scalars and does not provide an invariant. Nonetheless it can be useful.

For an example with cubic prepotential, see section 6.3.4.

Chapter 6

Special Kähler classification. Quadratic and cubic prepotentials

We provide elements concerning the classification of homogeneous symmetric and non-symmetric spaces, and we give more details on quadratic and cubic models. Both these models appear frequently in $N = 2$ supergravity and they contain all the possible symmetric spaces: we will use them frequently in our study of BPS solutions and we will also classify the isometries in these two cases.

6.1 Classification of spaces

Spaces with cubic prepotentials are referred to as very special Kähler spaces. They are obtained from the dimensional reduction of $d = 5$ $N = 2$ supergravity for which the scalar manifold is real; this operation is called the r-map. As a consequence they have real structure constants.

The classification of symmetric spaces have been done in [208, 214], while homogeneous spaces were described in [13, 215] (see also [144–146]). Other useful references include [152, p. 78, tab. 2, 165, p. 443, tab. 20.5].

6.1.1 Symmetric spaces

For all symmetric SK spaces there exists a symplectic basis where the prepotential is quadratic or cubic [40, p. 29]. Properties of the Riemann tensor and the curvature of theses spaces are described in [208].

Spaces with quadratic prepotentials correspond to complex projective spaces (see section 6.2) [208]

$$\mathbb{C}P^{n_v} \equiv \frac{\mathrm{SU}(n_v, 1)}{\mathrm{SU}(n_v) \times \mathrm{U}(1)} \tag{6.1.1}$$

(for $n_v = 1$ there is only one U(1) in the denominator). They originally appeared in [216].

Günaydin, Sierra and Townsend obtained all symmetric spaces with cubic prepotentials by studying the link between Jordan algebra and symmetric real geometries in $d = 5$ $N = 2$ supergravity and reducing to $d = 4$ [214]. It was proven by Cremmer and van Proeyen that this list was indeed complete, using a classification of symmetric Kähler spaces (3.2.37) and imposing the "special" conditions [208] (see also [185, sec. 5, app.]).

There is an infinite family of cubic spaces (sometimes called the *generic Jordan sequence* [206])

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(n_v,2)}{\mathrm{SO}(n_v)\times \mathrm{SO}(2)} \tag{6.1.2}$$

(for $n_v = 1$ there is only the first factor), along with four exceptional cases (sometimes called magical models) [208, 214, sec. 5]

$$\frac{\mathrm{Sp}(6)}{\mathrm{U}(3)}, \quad \frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3)\times \mathrm{SU}(3)\times \mathrm{U}(1)}, \quad \frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6)\times \mathrm{U}(1)}, \quad \frac{\mathrm{E}_{7,-25}}{\mathrm{E}_6\times \mathrm{U}(1)} \tag{6.1.3}$$

for $n_v = 6, 9, 15, 27$ respectively (related to the magic square – they are linked with the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O}$). An interesting point of the generic sequence is that they are the only SK spaces with a direct product structure [170, p. 11].

Note that

$$\mathrm{SU}(1,1) \sim \mathrm{SL}(2,\mathbb{R}). \tag{6.1.4}$$

The cubic case $n_v = 3$ (called the STU model) is very special because [165, p. 452]

$$\mathcal{M}_v = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,2)}{\mathrm{SO}(2)\times \mathrm{SO}(2)} \sim \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)^3. \tag{6.1.5}$$

This implies that the geometry will factorize and this manifold exhibits very interesting properties.

In the case $n_v = 1$, the manifolds are $\mathrm{SU}(1,1)/\mathrm{U}(1)$ for both the quadratic and cubic prepotentials, but they are different since they have different curvature [208, p. 451]

$$R_{\text{quad}} = -2, \qquad R_{\text{cubic}} = -\frac{2}{3}. \tag{6.1.6}$$

Symmetric spaces are also Einstein

$$R_{i\bar{j}} = \Lambda g_{i\bar{j}}, \qquad \Lambda = \frac{R}{n_v}, \tag{6.1.7}$$

where [206, sec. 5]

$$\Lambda_{\text{quad}} = -(n_v+1), \qquad \Lambda_{\text{cubic}} = -\frac{n_v^2 - 2n_v + 3}{n_v}, \qquad \Lambda_{\text{magic}} = -\frac{2}{3}\, n_v. \tag{6.1.8}$$

6.1.2 Homogeneous spaces

The classification of homogeneous SK spaces with cubic prepotential was started by Cecotti [215] and completed by de Wit and van Proeyen [13]. As reviewed in section 8.5, QK manifolds can be obtained from SK manifolds through the c-map. Homogeneous quaternionic spaces were classified by Alekseevskii and Cecotti used this fact to obtain homogeneous SK manifolds as the inverse of the c-map. In their paper de Wit and van Proeyen discovered new SK spaces, showing that Alekseevskii's classification was incomplete (since new QK manifolds could be derived from the c-map).

De Wit and van Proeyen found interesting links with Clifford algebras, while Cecotti showed that these spaces were related to T-algebras, which are a generalization of Jordan algebras.

6.2 Quadratic prepotential

For references see [212, sec. 4.2, 165, sec. 13.3].

Quadratic prepotentials

$$F = \frac{i}{2}\,\eta_{\Lambda\Sigma}\,X^\Lambda X^\Sigma \tag{6.2.1}$$

correspond to the complex projective spaces $\mathbb{C}P^{n_v}$

$$\mathcal{M}_v = \frac{\mathrm{SU}(n_v,1)}{\mathrm{SU}(n_v)\times\mathrm{U}(1)} \tag{6.2.2}$$

which are maximally symmetric. The flat metric on this space is given by

$$\eta_{\Lambda\Sigma} = \mathrm{diag}(-1,1,\ldots,1). \tag{6.2.3}$$

The coefficients of F are imaginary because real quadratic terms are irrelevant as seen in section 4.2.4.

Because the isotropy group is $\mathrm{SU}(n_v)\times\mathrm{U}(1)$ there is a natural split between the timelike direction $\Lambda = 0$ and the spacelike ones $\Lambda = i$.

6.2.1 General formulas

In special coordinates [149, app. A.1]

$$X^\Lambda = \begin{pmatrix} 1 \\ \tau^i \end{pmatrix} \tag{6.2.4}$$

the F_Λ are given by

$$F_\Lambda = i\,\eta_{\Lambda\Sigma}X^\Sigma = i\begin{pmatrix} -1 \\ \tau^i \end{pmatrix} \tag{6.2.5}$$

The "spatial" indices are raised and lowered with $\delta_{i\bar{\jmath}}$ and $\delta^{i\bar{\jmath}}$.

The Kähler potential is given by

$$\mathrm{e}^{-K} = 2(|\boldsymbol{\tau}|^2 - 1) \tag{6.2.6}$$

where $\boldsymbol{\tau}$ is the vector with components τ^i. The metric reads

$$g_{i\bar{\jmath}} = \frac{\delta_{i\bar{\jmath}}}{1-|\boldsymbol{\tau}|^2} + \frac{\bar{\tau}_i\tau_{\bar{\jmath}}}{(1-|\boldsymbol{\tau}|^2)^2}. \tag{6.2.7}$$

The structure constants vanish

$$C_{ijk} = 0 \tag{6.2.8}$$

and for this reason these models in supergravity are called minimally coupled. This implies that three invariants from (5.2.2) are zero [148, sec. 8.4]

$$i_3 = i_4 = i_5 = 0. \tag{6.2.9}$$

The curvature of these spaces is read from (4.6.3)

$$R = -n_v(n_v+1). \tag{6.2.10}$$

Quadratic spaces can be obtained as a truncation from symmetric cubic spaces since [148, sec. 8.3]

$$\frac{\mathrm{SU}(n_v,1)}{\mathrm{SU}(n_v)\times\mathrm{U}(1)} \subset \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2n_v,2)}{\mathrm{SO}(2n_v)\times\mathrm{SO}(2)} \tag{6.2.11}$$

6.2.2 Quartic and quadratic invariants

The groups $\mathrm{SU}(n_v, 1)$ are degenerate groups of type E_7, as is seen in the vanishing of the structure constants [148]. As a consequence the quartic invariant I_4 becomes the square of a quadratic invariant I_2 [12, p. 227, 149, sec. 3.2]

$$I_4(A) = I_2(A)^2. \tag{6.2.12}$$

The quadratic invariant reads [148, sec. 8.4]

$$I_2 = i_1 - i_2 \tag{6.2.13}$$

and one denotes by θ_{MN} the associated tensor

$$I_2(A_1, A_2) = \theta_{MN} A_1^M A_2^N, \qquad \theta_{MN} = \frac{1}{2} \frac{\partial^2 I_2(A)}{\partial A^M \partial A^N}. \tag{6.2.14}$$

The quartic tensor (6.2.12) can be derived from this 2-tensor

$$t_{MNPQ} = 4!\, \theta_{MN} \theta_{PQ}. \tag{6.2.15}$$

Using (4.4.22) I_2 can also be written

$$I_2(A) = -\frac{1}{2}\, A^t \mathcal{M}(\mathcal{F}) A, \tag{6.2.16}$$

where $\mathcal{M}(F)$ was defined in section 4.4.

Writing explicitly the components with $\mathcal{Q} = (p^\Lambda q_\Lambda)$, the quadratic invariant is [40, sec. 5, 149, sec. 3.2, 212, sec. 1]

$$I_2(\mathcal{Q}) = \frac{i}{2}\, p^\Lambda \eta_{\Lambda\Sigma}\, p^\Sigma + \frac{i}{2}\, q_\Lambda \eta^{\Lambda\Sigma} q_\Sigma. \tag{6.2.17}$$

Note that I_2 can be rewritten as

$$I_2(\mathcal{Q}) = \frac{1}{2}\, T^{\Lambda\Sigma} T^{\Delta\Xi} \eta_{\Lambda\Delta} \eta_{\Sigma\Xi}, \qquad T_{\Lambda\Sigma} = p^\Lambda q^\Sigma - p^\Sigma q^\Lambda. \tag{6.2.18}$$

This implies

$$\theta = \frac{i}{2} \begin{pmatrix} \eta_{\Lambda\Sigma} & 0_\Lambda{}^\Sigma \\ 0^\Lambda{}_\Sigma & \eta^{\Lambda\Sigma} \end{pmatrix}. \tag{6.2.19}$$

The gradient defines a new vector

$$I_2'(A)^M = \Omega^{MN} \theta_{NP} A^P. \tag{6.2.20}$$

Because of the existence of I_2, the Freudenthal operator (see section 5.3.4) becomes [148, sec. 10]

$$\mathfrak{f}(A)^M = \Omega^{MN} \frac{\partial I_2(A)}{\partial A^N} \tag{6.2.21}$$

while using the definition of the gradient gives

$$\mathfrak{f}(A) = \frac{1}{2}\, I_2'(A). \tag{6.2.22}$$

It preserves the quadratic invariant

$$I_2(\mathfrak{f}(A)) = I_2(A). \tag{6.2.23}$$

In this context the operator I_2' also defines a complex structure (up to a normalization) since we have seen that $\mathfrak{f}$ defines one.

6.3 Cubic prepotential

6.3.1 General case

Manifolds with cubic prepotential are called very special Kähler manifolds or d-geometries. These manifolds can be obtained by reducing $d = 5$ supergravity to $d = 4$ through the r-map.

For details see [206, 185, sec. 5, 144, p. 7, sec. 4, 149, sec. 3.1, app. A].

For space with cubic prepotential there is a frame where F can be put in the form[1]

$$F = -D_{ijk} \frac{X^i X^j X^k}{X^0} \tag{6.3.1}$$

where D_{ijk} is a symmetric 3-tensor. The associated f function is

$$f(\tau) = -D_{ijk}\tau^i\tau^j\tau^k. \tag{6.3.2}$$

We will use the abbreviations

$$D_\tau = D_{\tau\tau\tau} = D_{ijk}\,\tau^i\tau^j\tau^k, \qquad D_{\tau,i} = D_{ijk}\,\tau^j\tau^k \tag{6.3.3}$$

and similarly for other quantities like D_y (y being the imaginary part of τ).

The (rescaled) structure constant are given in terms of the D-tensor

$$W_{ijk} = D_{ijk}, \tag{6.3.4}$$

and it is convenient to define the tensor $\widehat{D}^{ijk}$ [149, sec. 3.1]

$$\widehat{D}^{ijk} = \frac{1}{D_y^2}\, g^{i\ell} g^{jm} g^{kn}\, D_{\ell mn} \tag{6.3.5}$$

which corresponds to $\mathrm{e}^{2K}\bar{W}^{ijk}$ up to a normalization. The tensor D_{ijk} (and hence W_{ijk}) is always constant, but this is not necessarily the case for $\widehat{D}^{ijk}$ [50, app. D]. Since $\widehat{D}^{ijk}$ is real we use also holomorphic indices.

In special coordinates, the conjugates are

$$F_\Lambda = \begin{pmatrix} D_\tau \\ -3\,D_{\tau,i} \end{pmatrix}. \tag{6.3.6}$$

The Kähler potential is

$$\mathrm{e}^{-K} = 2\big(\operatorname{Im} f + 2i \operatorname{Im}\tau^i \operatorname{Re}(\partial_i f)\big) = 8\,D_y \tag{6.3.7}$$

since

$$\begin{aligned} \mathrm{e}^{-K} &= -i\big(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda\big) = i(D_\tau - D_{\bar\tau}) - 3i(D_{\tau\tau\bar\tau} - D_{\bar\tau\bar\tau\tau}) \\ &= -2\operatorname{Im} D_\tau + 6\operatorname{Im} D_{\tau\tau\bar\tau} = 2(D_y - 3\,D_{xxy}) + 6(D_{xxy} + D_y). \end{aligned}$$

The metric is [84, app A.1]

$$g_{ij} = -\frac{3}{2}\frac{D_{y,ij}}{D_y} + \frac{9}{4}\frac{D_{y,i}D_{y,j}}{D_y^2} \tag{6.3.8}$$

The Riemann tensor is

$$R^i{}_{jk}{}^\ell = \delta^i_j\delta^\ell_k + \delta^i_k\delta^\ell_j - \frac{9}{16}\,\widehat{D}^{i\ell m} D_{mjk}. \tag{6.3.9}$$

[1] The minus sign is conventional, other factors can be found in the literature, such as $\pm 1, \pm i$, along with some different normalization, for example 1/3! [42, 185].

The E-tensor (4.5.13) reads [149, sec. 3.1]

$$E^i{}_{jk\ell m} = \widehat{D}^{ijk} D_{j(\ell m} D_{np)k} - \frac{64}{27}\, \delta^i{}_{(m} D_{np\ell)}. \qquad (6.3.10)$$

If $\mathcal{M}_v$ is symmetric, then $\widehat{D}^{ijk}$ entries are constant and they satisfy

$$\widehat{D}^{ijk} D_{j\ell(m} D_{np)k} = \frac{16}{27}\left(\delta^i_\ell D_{mnp} + 3\,\delta^i_{(m} D_{np)\ell}\right) \qquad (6.3.11a)$$

$$\widehat{D}^{ijk} D_{j(\ell m} D_{np)k} = \frac{64}{27}\,\delta^i_{(\ell} D_{mnp)}. \qquad (6.3.11b)$$

6.3.2 Generic symmetric models

As explained in section 6.1.1, the generic cubic symmetric models are the manifolds

$$\mathcal{M}_v = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(n_v,2)}{\mathrm{SO}(n_v)\times \mathrm{SO}(2)}. \qquad (6.3.12)$$

In this case again there is a natural split between the timelike direction $\Lambda = 0$ and spacelike ones $\Lambda = i$ because the isotropy group is $\mathrm{SO}(n_v)\times\mathrm{SO}(2)$.

6.3.3 Jordan algebras and quartic invariant

The existence and the form of the quartic invariant for symmetric very special Kähler manifolds is related to Freudenthal triple systems and the associated Jordan algebra; good references includes [148, 207] (for a mathematical paper, see [147]).

For symmetric cubic spaces the quartic invariant is given by [207, sec. 2.1, 40, sec. 5, 149, sec. 3.1, 144, p. 26] (see also [147, sec. 3])

$$I_4(\mathcal{Q}) = -(q_\Lambda p^\Lambda)^2 + \frac{1}{16}\, p^0\, \widehat{D}^{ijk} q_i q_j q_k - 4\, q_0\, D_{ijk} p^i p^j p^k + \frac{9}{16}\, \widehat{D}^{ijk} D_{k\ell m} q_i q_j\, p^\ell p^m \qquad (6.3.13)$$

with $\mathcal{Q} = (p^\Lambda, q_\Lambda)$.

The explicit components of the tensor t_{MNPQ} are [50, app. D, 84, app. A.3]

$$\begin{gathered} t_{00}{}^{00} = -4, \qquad t_{0i}{}^{0j} = -2\delta_i{}^j, \qquad t_{ij}{}^{k\ell} = -4\delta_i{}^{(k}\delta_k{}^{\ell)} + \frac{9}{4}\, D_{ijm}\widehat{D}^{k\ell m}, \\ t_0{}^{ijk} = -\frac{3}{8}\,\widehat{D}^{ijk}, \qquad t_{ijk}{}^0 = 24\, D_{ijk}. \end{gathered} \qquad (6.3.14)$$

A fundamental identity is [207, sec. 2.1, 85, app. B]

$$I_4'(I_4'(A), A, A) = -8\, A\, I_4(A) \qquad (6.3.15)$$

which is called the Freudenthal identity and is a consequence of the Jordan algebra structure of the space. Some identities that are satisfied by combinations of the invariant evaluated with two vectors are given in the appendix E.1.

One of the most useful identity is [85, app. B]

$$I_4'(A, \mathrm{Im}\,\mathcal{V}, \mathrm{Im}\,\mathcal{V}) = -4\,\langle \mathrm{Im}\,\mathcal{V}, A\rangle\, \mathrm{Im}\,\mathcal{V} - 8\,\langle \mathrm{Re}\,\mathcal{V}, A\rangle\, \mathrm{Re}\,\mathcal{V} - \Omega\mathcal{M}A. \qquad (6.3.16)$$

ans from it one deduces the relation

$$\mathrm{Re}\,\mathcal{V} = -2\, I_4'(\mathrm{Im}\,\mathcal{V}) = -\frac{I_4'(\mathrm{Im}\,\mathcal{V})}{2\sqrt{I_4(\mathrm{Im}\,V)}}. \qquad (6.3.17)$$

It is remarkable that none of these identities changes when $\mathcal{V}$ is multiplied by a phase.

6.3.4 Non-symmetric spaces

As shown in [202, 206], spaces with a cubic prepotential have a least the isometry group

$$G = \mathrm{SO}(1,1) \times \mathbb{R}^{n_v} \tag{6.3.18}$$

where the first factor is related to overall rescaling while the second corresponds to n_v shifts of the axions $\mathrm{Re}\,\tau^i$. As we will see in the section 7.2, these isometries correspond to the universal transformations associated to parameters $\{\beta, b^i\}$. As a consequence the quartic function can depend only on the dilatons $\mathrm{Im}\,\tau^i$, and the terms that are scalar-dependent will be proportional to the E-tensor (4.5.13)

$$I_4(A, \tau^i) \sim I_4(A) + (D_y)^{5/3} E^m{}_{ijk\ell}\, p^j p^k p^\ell q_m q_n \frac{\partial D_p}{\partial p^i \partial p^n} \tag{6.3.19}$$

with $I_4(A)$ is the quartic invariant (6.3.13).

Chapter 7

Special Kähler isometries

The main motivation of this chapter is to understand the isometries of the quadratic and cubic models. This is an important step in order to construct gauged supergravities based on these models as one needs to know the correspoding Killing vectors that appear in the covariant derivatives. Moreover some isometries of the QK manifolds are inherited from its base SK space.

7.1 General case

Special Kähler isometries were worked out in [185, sec. 6, 12, p. 222, 146] (see also [149, sec. 3]).

Isometries (also called duality transformations) on special Kähler manifolds are given by symplectic transformations (see section 5.1) that are consistent with the symplectic vectors [165, p. 450, 209]. In particular this means that the duality transformation of F_Λ agrees with the transformation induced by the fact that F_Λ is a function of X^Λ [12, p. 222]. For homogeneous spaces some isometries are constrained while other are universal and their existence is always guaranteed. In the case of symmetric spaces all isometries are realized [12, p. 222]. These isometries are generated by holomorphic Killing vectors since the manifold is Kähler, and all the properties described in section 3.2.3 also apply.

The isometry group is denoted by

$$G_v = \mathrm{ISO}(\mathcal{M}_v). \tag{7.1.1}$$

The variation of the section is

$$\delta v = \mathfrak{U}\, v \tag{7.1.2}$$

with

$$\mathfrak{U} = \begin{pmatrix} q & r \\ s & t \end{pmatrix} \in \mathfrak{sp}(2n_v + 2) \tag{7.1.3}$$

and the constraints

$$t = -q^t, \qquad r = r^t, \qquad s = s^t. \tag{7.1.4}$$

Consistency of the transformation of the vector v with the expression $F_\Lambda(X)$ implies that the prepotential keeps the same functional form [144, p. 6, app. C]

$$F'(X') = F(X'). \tag{7.1.5}$$

In supergravity this condition implies that the Lagrangian is invariant. Note that this does not mean that the function itself is invariant, and one finds that [185, sec. 6]

$$\delta F(X) = F(X') - F(X) = i\left(XsX - \frac{1}{4}\,FrF\right). \tag{7.1.6}$$

As said in section 4.2.4 pure imaginary quadratic terms have no effect.

This is equivalent to the chain rule

$$\delta F_\Lambda = \frac{\partial F_\Lambda}{\partial X^\Sigma}\, \delta X^\Sigma = F_{\Lambda\Sigma} \delta X^\Sigma. \tag{7.1.7}$$

Contracting this equation with X^Λ and using the homogeneity of F gives

$$X^\Lambda \delta F_\Lambda = F_\Lambda \delta X^\Lambda. \tag{7.1.8}$$

This last condition is sufficient to classify all the isometries and it reads explicitly [185, sec. 6, 12, p. 223]

$$X^\Lambda s_{\Lambda\Sigma} X^\Sigma - 2 X^\Lambda (q^t)_\Lambda{}^\Sigma F_\Sigma - F_\Lambda r^{\Lambda\Sigma} F_\Sigma = 0. \tag{7.1.9}$$

From the relation

$$F = \frac{1}{2}\, F_\Lambda X^\Lambda \tag{7.1.10}$$

one obtains the variation

$$\delta F = \frac{1}{2}\big(\delta F_\Lambda X^\Lambda + F_\Lambda \delta X^\Lambda\big) = \delta F_\Lambda X^\Lambda = F_\Lambda \delta X^\Lambda, \tag{7.1.11}$$

the last two equalities coming from (7.1.8).

The number of isometries is given by the number of independent parameters ω^m in the matrix $\mathfrak{U}$ and they can be found by expanding (7.1.9) in τ^i. Then the Killing vectors and the symplectic matrix can be written as linear combinations

$$k^i = \omega^m\, k^i_m, \qquad \mathfrak{U} = \omega^m\, \mathfrak{U}_m \tag{7.1.12}$$

where each k^i_m and $\mathfrak{U}_m$ generates an independent isometry.

Also the Kähler potential (4.2.13)

$$\mathrm{e}^{-K} = -i\, \langle v, \bar v \rangle \tag{7.1.13}$$

is obviously invariant under isometries since it is written only in terms of symplectic invariant quantities, but this does not need to be the case in special coordinates: there may be a compensating Kähler transformation

$$\mathcal{L}_k K = 2\, \mathrm{Re}\, f_k \tag{7.1.14}$$

associated to the transformation with Killing vector k. The reason is that a transformation may change $X^0 = 1$ to another value $X'^0 \neq 1$, and one needs to perform a compensating Kähler transformation in order to set $X'^0 = 1$ [186].

7.2 Cubic prepotential

Let's consider the cubic prepotential

$$F = -D_{ijk}\, \frac{X^i X^j X^k}{X^0}. \tag{7.2.1}$$

The isometries were studied in [185, sec. 6, 13, 144] (see also [149, sec. 3.1]).

7.2.1 Parameters

The matrix $\mathfrak{U}$ is parametrized as [149, sec. 3.1, 144, p. 7, sec. 4.2]

$$\begin{gathered} q^{\Lambda}{}_{\Sigma} = -(t^t)^{\Lambda}{}_{\Sigma} = \begin{pmatrix} \beta & a_j \\ b^i & B^i{}_j + \frac{1}{3}\,\beta\,\delta^i{}_j \end{pmatrix}, \\ s_{\Lambda\Sigma} = \begin{pmatrix} 0 & 0 \\ 0 & -6D_{ijk}b^k \end{pmatrix}, \qquad r^{\Lambda\Sigma} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{3}{32}\widehat{D}^{ijk}a_k \end{pmatrix}. \end{gathered} \tag{7.2.2}$$

In special coordinates the variation of τ^i is given by

$$\delta\tau^i = b^i - \frac{2}{3}\,\beta\,\tau^i + B^i{}_j\tau^j - \frac{1}{2}\,R^i{}_{jk}{}^{\ell}\,\tau^j\tau^k a_\ell \tag{7.2.3}$$

and the Killing vector is

$$k = k^i\partial_i = k_\beta + b^i\,k_{b,i} + a_i\,k_a^i + B^i{}_j\,(k_B)_i{}^j. \tag{7.2.4}$$

The unconstrained symmetries associated to β and b^i generate respectively a rescaling and a shift of the axions.

The other rescaling symmetries associated to $B^i{}_j$ are constrained by[1]

$$B_{(i}{}^{\ell}D_{jk)\ell} = 0. \tag{7.2.5}$$

Finally the non-linear symmetries must satisfy

$$a_i E^i{}_{jk\ell m} = 0 \tag{7.2.6}$$

where the E-tensor is given by (4.5.14) or (6.3.10). This condition is necessary and sufficient for having $\widehat{D}^{ijk}a_k = \text{cst}$ (which is needed because the matrix $\mathfrak{U}$ is constant) [144, sec. 4.2].

If $\mathcal{M}_v$ is symmetric, then $\widehat{D}^{ijk}$ is constant and $E^i{}_{jk\ell m} = 0$ such that a_i is unconstrained. Then the symmetry group will be a simple Lie algebra, with b^i and a_i being associated to lowering and raising operators, while $(\beta, B^i{}_j)$ are associated to Cartan elements.

7.2.2 Lie derivative

Transformation associated to β and a_i induce a Kähler transformation of the potential with [149, sec. 3.3, app. A.1]

$$f = \beta + a_i\tau^i. \tag{7.2.7}$$

7.2.3 Algebra

The algebra can be found in [185, sec. 6, 144, sec. 4.2]

$$[k_\beta, k_{b,i}] = \frac{2}{3}\,k_{b,i}, \qquad [k_\beta, k_a^i] = -\frac{2}{3}\,k_a^i, \qquad [k_{b,i}, k_a^j] = \delta^i{}_j\,k_\beta + \tilde{R}^i{}_{jk}{}^{\ell}(k_B)_\ell{}^k, \tag{7.2.8a}$$

$$\Big[(k_B)_i{}^j, k_{b,k}\Big] = \tilde{R}^i{}_{jk}{}^{\ell}k_{b,\ell}, \qquad \Big[(k_B)_i{}^j, k_a^k\Big] = -\tilde{R}^i{}_{j\ell}{}^k k_a^\ell \tag{7.2.8b}$$

where

$$\tilde{R}^i{}_{jk}{}^{\ell} = R^i{}_{jk}{}^{\ell} + \frac{2}{3}\,\delta^i{}_j\delta_k{}^{\ell}. \tag{7.2.9}$$

Due to the form of the algebra the existence of a transformation with parameter a_i imply one of the form $B_i{}^j$.

[1] This constraint is discussed more deeply in [13, 144, sec. 5] in which the authors study which d_{ijk} satisfy it, and this has some link with Clifford algebra.

The algebra $\mathfrak{g}_v$ of G_v can be decomposed in eigenspaces associated to the symmetry β [144, sec. 2.2]

$$\mathfrak{g}_v = \mathfrak{g}_{-2/3} + \mathfrak{g}_0 + \mathfrak{g}_{2/3} \tag{7.2.10}$$

where

$$[k_\beta, \mathfrak{g}_a] = a\, \mathfrak{g}_a. \tag{7.2.11}$$

The space $\mathfrak{g}_0$ contains β and $B^i{}_j$ while $\mathfrak{g}_{2/3}$ contains b^i, and as a result

$$\dim \mathfrak{g}_{2/3} = n_v. \tag{7.2.12}$$

Hidden symmetries a_i are in $\mathfrak{g}_{-2/3}$ and the associated roots are located on the left of the root diagram, while the dimension of the space

$$\dim \mathfrak{g}_{-2/3} \leq n_v \tag{7.2.13}$$

According to the denomination of [144, sec. 2.2], symmetries associated to a_i are hidden ones. This bound is saturated – meaning that a_i exist – for symmetric spaces, in which case the curvature and $\widehat{D}^{ijk}$ are constant, or equivalently when $E^i{}_{jk\ell m} = 0$. Otherwise the Lie algebra is not semisimple.

7.3 Quadratic prepotential

Now one considers quadratic prepotentials

$$F = \frac{i}{2}\, \eta_{\Lambda\Sigma}\, X^\Lambda X^\Sigma. \tag{7.3.1}$$

7.3.1 Parameters

The solution to the constraints (7.1.9) is given by [149, sec. 3.2, app. A.1]

$$s_{\Lambda\Sigma} = -\eta_{\Lambda\Xi}\, r^{\Xi\Upsilon}\, \eta_{\Upsilon\Sigma}, \qquad \eta_{\Lambda(\Sigma} q^\Lambda{}_{\Xi)} = 0 \tag{7.3.2}$$

where there is no sum on Λ in the last constraint (i.e. all diagonal elements are vanishing). The second constraint is equivalent to

$$q^0{}_i = q^i{}_0, \qquad q^i{}_j = -q^j{}_i, \qquad q^\Lambda{}_\Lambda = 0. \tag{7.3.3}$$

The variations of the coordinates is given by

$$\delta\tau^i = A^i{}_0 + (A^i{}_j - A^0{}_0 \delta^i{}_j)\tau^j - A^0{}_j \tau^j \tau^i \tag{7.3.4}$$

where

$$A = q + i\, r\, \eta. \tag{7.3.5}$$

Looking at the variation of τ^i, the trace of A and $A^0{}_0$ have the same action and one should be removed, and this is equivalent to removing one of them for r. The number of parameters contained in each matrices is

$$r: \quad \frac{1}{2}(n_v+1)(n_v+2) - 1, \qquad q: \quad \frac{1}{2}\, n_v(n_v-1) + n_v, \tag{7.3.6}$$

giving a total number of $n_v(n_v+2)$ which agrees with the number of Killing vectors on $\mathbb{C}P^{n_v}$.

7.3.2 Lie derivative

A Kähler transformation is induced for some of the isometries [149, sec. 3.3, app. A.1]

$$f = 2\, \bar{A}^0{}_i\, \tau^i. \tag{7.3.7}$$

Chapter 8

Quaternionic geometry

Quaternionic Kähler manifold (QK) manifolds form the target manifold of hypermultiplets in $N = 2$ supergravity. These manifolds possess a SU(2) bundle which correspond to the $\mathrm{SU}(2)_R$ symmetry of the supersymmetry algebra, and as a consequence there is a triplet of complex structures that obey the quaternionic algebra. After giving the definition of these manifolds we describe their geometrical properties followed by a general description of isometries. In particular we describe the SU(2) compensator which is interpreted as a rotation of the complex structures under a transformation, and it will be an important ingredient in the construction of BPS vacua. Finally we describe the special quaternionic manifolds that are constructed as a fibration over a SK manifolds and which are simpler than generic QK spaces, and in the following chapter we build the isometries of these spaces.

General references include [170, sec. 5, 171, 165, chap. 13 and 20, 217, sec. 2] (see also [218, 199, sec. 5]). Some historical and mathematical references are [200, 219–224].

8.1 Definitions

Definition 8.1 (Quaternionic manifold) A quaternionic Kähler (QK) manifold $(\mathcal{M}_h, h)$ is a $4n_h$-dimensional real manifold with metric

$$\mathrm{d}s^2 = h_{uv}\, \mathrm{d}q^u \mathrm{d}q^v, \qquad u = 1, \dots, 4n_h \tag{8.1.1}$$

endowed with three (almost-)complex structures J^x, $x = 1, 2, 3$, satisfying the quaternionic algebra

$$J^x J^y = -\delta^{xy} + \varepsilon^{xyz} J^z. \tag{8.1.2}$$

Alternatively a QK manifold is characterized by its holonomy group [220, 221]

$$\mathrm{Hol}(\mathcal{M}_h) = \mathcal{H} \cdot \mathrm{Sp}(1) \equiv \mathcal{H} \times \mathrm{Sp}(1)/\mathbb{Z}_2, \qquad \mathcal{H} \subset \mathrm{Sp}(n_h). \tag{8.1.3}$$

Locally the coordinates q^u can be gathered into quaternions, but in general this is not possible globally [200, p. 126–127]. Similarly these spaces are not Kähler strictly speaking in general and this is an abuse of language.

We note that $\mathrm{Sp}(n_h) \subset \mathrm{SO}(4n_h)$ and it is the subgroup that leaves invariant the J^x. $\mathrm{Sp}(n_h) \cdot \mathrm{Sp}(1)$ is a maximal subgroup of $\mathrm{SO}(4n)$ [220]. We recall that $\mathrm{Sp}(1) \sim \mathrm{SU}(2)$.

The connection 1-form of the SU(2) factor is denoted by

$$\omega^x = (\omega^x)_u\, \mathrm{d}q^u, \tag{8.1.4}$$

and the associated curvature is

$$\Omega^x = \nabla \omega^x = \mathrm{d}\omega^x + \frac{1}{2}\, \varepsilon^{xyz} \omega^y \wedge \omega^z. \tag{8.1.5}$$

Moreover the metric must be hermitian with respect to the three J^x (denoted collectively as $\boldsymbol{J}$), i.e.

$$\forall x: \qquad J^x\, h\, (J^x)^t = h \tag{8.1.6}$$

(no sum over x) and they should be covariantly constant

$$\nabla_w \boldsymbol{J}_u{}^v = \mathrm{D}_w \boldsymbol{J}_u{}^v + 2\boldsymbol{\omega}_w \times \boldsymbol{J}_u{}^v = 0 \tag{8.1.7a}$$

$$\nabla_w (J^x)_u{}^v = \mathrm{D}_w (J^x)_u{}^v + \varepsilon^{xyz} (\omega^y)_w (J^z)_u{}^v = 0, \tag{8.1.7b}$$

where D_u is the covariant derivative associated to h_{uv}. This relation means that the J^x are covariantly constant with respect to D_u up to an SU(2) rotation with vector $(\omega^x)_u(q)$.

The triplet of hyperkähler 2-forms

$$K^x = J^x_{uv}\, \mathrm{d}q^u \wedge \mathrm{d}q^v, \qquad J^x_{uv} = h_{uw} (J^x)_v{}^w. \tag{8.1.8}$$

have to be closed with respect to Sp(1) connection

$$\nabla K^x = \mathrm{d}K^x + \varepsilon^{xyz} \omega^y \wedge K^z = 0. \tag{8.1.9}$$

For a quaternionic manifold the SU(2) curvature 2-form needs to be proportional to the hyperkähler 2-form

$$\Omega^x = \lambda K^x. \tag{8.1.10}$$

In supergravity $\lambda = -1$ [171, p. 6], but we will keep it general for two reasons:

- some authors use different normalizations;
- the limit $\lambda = 0$ corresponds to hyperkähler manifolds and rigid supersymmetry.

Because of the connection the covariant exterior derivative does not square to zero but to [200, sec. 4, 224, sec. 4]

$$\nabla^2 f^x = \varepsilon^{xyz} \Omega^y f^z \tag{8.1.11}$$

for any p-form f^x.

The fundamental (quaternionic) 4-form is defined as [219, 221, 224]

$$\mathit{\Omega} = K^x \wedge K^x = \frac{1}{\lambda^2}\, \Omega^x \wedge \Omega^x, \tag{8.1.12}$$

it is globally defined, non-vanishing and covariantly closed (i.e. parallel)

$$\nabla \mathit{\Omega} = 0 \tag{8.1.13}$$

since it is invariant under $\mathrm{Sp}(n_h) \cdot \mathrm{Sp}(1)$ [219, 222] (or in the opposite sense, a manifold is quaternionic if $\mathit{\Omega}$ is covariantly closed). This implies that $\mathit{\Omega}$ is closed and harmonic (this is equivalent to $K^x = \lambda \Omega^x$) [200, sec. 4]

$$\mathrm{d}\mathit{\Omega} = 0, \qquad \Delta \mathit{\Omega} = 0. \tag{8.1.14}$$

This is automatic for $n_h = 1$ since $\mathit{\Omega} = 3\varepsilon$ (ε being the volume form of the space, not to be counfounded with ε^{xyz}) [224, sec. 2]. Recall that the laplacian on forms is defined by

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d} \tag{8.1.15}$$

where δ is the codifferential.

We want to prove that $\mathit{\Omega}$ is closed. Using the definition (8.1.5) of Ω^x we have

$$\begin{aligned} \lambda^2\, \mathit{\Omega} &= \left(\mathrm{d}\omega^x + \frac{1}{2}\, \varepsilon^{xyz} \omega^y \wedge \omega^z\right) \wedge \left(\mathrm{d}\omega^x + \frac{1}{2}\, \varepsilon^{xuv} \omega^u \wedge \omega^v\right) \\ &= \mathrm{d}\omega^x \wedge \mathrm{d}\omega^x + \varepsilon^{xyz} \mathrm{d}\omega^x \wedge \omega^y \wedge \omega^z + \varepsilon^{xuv} \varepsilon^{xyz} \omega^u \wedge \omega^v \wedge \omega^y \wedge \omega^z. \end{aligned}$$

The last term vanishes because the ε will give a symmetric factor, so we have [225, sec. 3]

$$\lambda^2\, \Omega = \mathrm{d}\left(\omega^x \wedge \mathrm{d}\omega^x + \frac{1}{3}\,\varepsilon^{xyz}\omega^x \wedge \omega^y \wedge \omega^z\right). \tag{8.1.16}$$

This implies that Ω is closed as announced. For $n_h > 2$ this is a necessary and sufficient condition for the manifold to be quaternionic and $\mathrm{d}\Omega$ determines entirely $\nabla\Omega$, while for $n_h = 2$ we need to take some care [224, sec. 2, app. A].

The volume element on $\mathcal{M}_h$ is given by Ω^{n_h}.

Closely related to the quaternionic manifolds are the hyperkähler ones, for which the SU(2) bundle is trivial, and the holonomy group is contained in $\mathrm{Sp}(n_h)$.

8.2 Geometry

8.2.1 Vielbein

Let's introduce the vielbein 1-form $U^{\alpha\mathcal{A}}$

$$U^{\alpha\mathcal{A}} = U^{\alpha\mathcal{A}}_u\, \mathrm{d}q^u \tag{8.2.1}$$

such that

$$h_{uv} = \mathbb{C}_{\mathcal{A}\mathcal{B}}\varepsilon_{\alpha\beta}U^{\alpha\mathcal{A}}_u U^{\mathcal{B}\beta}_v. \tag{8.2.2}$$

The flat coordinates have been split in two indices due to the fact that the holonomy group is $\mathrm{Sp}(n_h)\cdot\mathrm{Sp}(1)$: $\mathcal{A}$ and α runs respectively in the fundamental representations of $\mathrm{Sp}(n_h)$ and $\mathrm{Sp}(1)$

$$\alpha = 1, 2, \qquad \mathcal{A} = 1, \dots, 2n_h, \tag{8.2.3}$$

where the corresponding symplectic flat metrics are $\mathbb{C}$ and ε (see the appendix A.3 for conventions)

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \qquad \mathbb{C}_{\mathcal{A}\mathcal{B}} = -\mathbb{C}_{\mathcal{B}\mathcal{A}}. \tag{8.2.4}$$

The inverse vielbein $U^u_{\alpha\mathcal{A}}$ is defined such that

$$U^{\alpha\mathcal{A}}_u U^v_{\alpha\mathcal{A}} = \delta_u{}^v, \qquad U^{\alpha\mathcal{A}}_u U^u_{\beta\mathcal{B}} = \delta_\alpha{}^\beta \delta_\mathcal{A}{}^\mathcal{B} \tag{8.2.5}$$

and it obeys the reality condition

$$(U^{\alpha\mathcal{A}})^* = U_{\alpha\mathcal{A}} = \mathbb{C}_{\mathcal{A}\mathcal{B}}\varepsilon_{\alpha\beta}U^{\mathcal{B}\beta}_v. \tag{8.2.6}$$

These conditions imply

$$2\, U^{\alpha\mathcal{A}}_u U^v_{\beta\mathcal{A}} = \delta_u{}^v \delta_\alpha{}^\beta + i\, \sigma^x{}_\beta{}^\alpha (J^x)_u{}^v, \tag{8.2.7a}$$

$$(J^x)_u{}^v = -i\, \sigma^x{}_\alpha{}^\beta\, U^{\alpha\mathcal{A}}_u U^v_{\beta\mathcal{A}}. \tag{8.2.7b}$$

Other relations are satisfied, such as

$$\mathbb{C}_{\mathcal{A}\mathcal{B}}(U^{\alpha\mathcal{A}}_u U^{\beta\mathcal{B}}_v + U^{\alpha\mathcal{A}}_v U^{\beta\mathcal{B}}_u) = \varepsilon^{\alpha\beta}\, h_{uv}, \tag{8.2.8a}$$

$$\varepsilon_{\alpha\beta}(U^{\alpha\mathcal{A}}_u U^{\beta\mathcal{B}}_v + U^{\alpha\mathcal{A}}_v U^{\beta\mathcal{B}}_u) = \frac{1}{n_h}\, \mathbb{C}^{\mathcal{A}\mathcal{B}}\, h_{uv}. \tag{8.2.8b}$$

The vielbein is covariantly constant

$$\nabla_v U^{\alpha\mathcal{A}}_u = \partial_v U^{\alpha\mathcal{A}}_u + \omega_{v\beta}{}^\alpha\, U^{\beta\mathcal{A}}_u + \Delta_{v\mathcal{B}}{}^\mathcal{A}\, U^{\alpha\mathcal{B}}_u - \Gamma^w{}_{vu}\, U^{\alpha\mathcal{A}}_w = 0, \tag{8.2.9}$$

where ω and Δ are the SU(2) and $\mathrm{Sp}(n_h)$ (Lie algebra valued) connections

$$\omega_\beta{}^\alpha = \omega_{u\beta}{}^\alpha\, \mathrm{d}q^u = i\, \omega^x\, \sigma^x{}_\beta{}^\alpha, \qquad \Delta_\mathcal{B}{}^\mathcal{A} = \Delta_{v\mathcal{B}}{}^\mathcal{A}\mathrm{d}q^u, \tag{8.2.10}$$

and ω^x is the connection (8.1.4).

8.2.2 Curvature

Due to the holonomy of the manifold the Riemann tensor factorizes. Its precise form can be found from (8.2.9) and it reads

$$R_{uv}{}^{w}{}_{s} = U^{w}_{\alpha\mathcal{A}} U^{\alpha\mathcal{B}}_{s} \mathcal{R}_{uv\mathcal{A}}{}^{\mathcal{B}} - \boldsymbol{J}_{s}{}^{w} \cdot \boldsymbol{\Omega}_{uv}, \tag{8.2.11a}$$

$$\mathcal{R}_{uv\mathcal{A}}{}^{\mathcal{B}} = 2\, \partial_{[u} \Delta_{v]\mathcal{B}}{}^{\mathcal{A}} + 2\, \Delta_{[u|\mathcal{C}|}{}^{\mathcal{A}} \Delta_{v]\mathcal{B}}{}^{\mathcal{C}}, \tag{8.2.11b}$$

$$\boldsymbol{\Omega}_{uv} = 2\, \partial_{[u} \boldsymbol{\omega}_{v]} + 2\, \boldsymbol{\omega}_{u} \times \boldsymbol{\omega}_{v} \tag{8.2.11c}$$

where Ω^x is the SU(2) curvature (8.1.5), and we recall that it is proportional to the hyperkähler 2-form (8.1.10).

Quaternionic manifolds are Einstein [224]

$$R_{uv} = \frac{R}{4n_h}\, h_{uv} \tag{8.2.12}$$

and thus have constant curvature. Moreover the latter is related to the coefficient of proportionality between Ω^x and K^x

$$\lambda = \frac{R}{8n_h(n_h+2)}. \tag{8.2.13}$$

Even stronger one can prove that the Riemann tensor decomposes as (we omit the indices) [165, p. 455, 200, sec. 4]

$$R = 2\lambda\, R_{\mathbb{H}P} + R_0 \tag{8.2.14}$$

where $R_{\mathbb{H}P}$ is the curvature on quaternionic projective space, and R_0 is the Ricci-flat curvature part (related to the Weyl tensor) of $\mathrm{Sp}(n_h)$ (it behaves as a curvature tensor for a Riemannian manifold whose holonomy is a subgroup of $\mathrm{Sp}(n_h)$).

8.3 Symmetries

As for the case of Kähler manifold a Killing vector k acts with a Lie derivative to generate isometries. It should preserve the metric h_{uv} and the fundamental 4-form $\mathit{\Omega}$ [200, sec. 4], that is

$$\mathcal{L}_k h_{uv} = \mathcal{L}_k \mathit{\Omega} = 0. \tag{8.3.1}$$

We have proved that $\mathrm{d}\mathit{\Omega} = 0$ so we have

$$\mathcal{L}_k \mathit{\Omega} = \mathrm{d} i_k \mathit{\Omega} = 0. \tag{8.3.2}$$

Invoking the Poincaré lemma, it exists a 2-form P_k such that [224, sec. 4]

$$i_k \mathit{\Omega} = \mathrm{d} P_k \tag{8.3.3}$$

generalizing the moment map from the Kähler manifolds. We can decompose it (locally) as

$$P_k = P^x_k \Omega^x. \tag{8.3.4}$$

Instead of continuing on this path, we introduce the definitions as in [170, sec. 7.3]. We assume that the action of the Lie group generates triholomorphic isometries, which means that $\mathcal{L}_k$ acts on Ω^x and ω^x [192]

$$\mathcal{L}_k \Omega^x = \varepsilon^{xyz} W^y_k \Omega^z, \qquad \mathcal{L}_k \omega^x = \nabla W^x_k \tag{8.3.5}$$

where W^x_k is an SU(2) compensator.[1] The reason is that the Sp(1) curvature being nonzero, we cannot trivialize the Sp(1) bundle: then all quantities that transform under this group

[1] With respect to [170, 192] we have $W \to -W$ since they define it by $\mathcal{L}_k \Omega^x = \varepsilon^{xyz} \Omega^y W^z_k$.

(such as K^x) are defined on this bundle, and not just on the quaternionic base space, and thus they are subject to local Sp(1) gauge transformations [226, sec. 1] or, said another way, they must transform covariantly.

In the same way we associated a prepotential to a Killing vector of Kähler manifolds, we would like to introduce triholomorphic prepotentials (or moment maps) P^x_k satisfying [224, sec. 4]

$$i_k K^x = \nabla P^x_k. \tag{8.3.6}$$

We can express them in terms of the hyperkähler forms (under certain conditions of regularity) [200, sec. 4]. Introduce first the 1-form

$$\beta^x = i_k K^x = \frac{1}{\lambda}\, i_k \Omega^x = \nabla P^x_k, \tag{8.3.7}$$

and take its covariant derivative

$$\nabla \beta^x = \nabla^2 P^x_k \Longrightarrow \mathrm{d}\beta^x + \varepsilon^{xyz}\omega^y \wedge \beta^z = \varepsilon^{xyz}\Omega^y P^z_k \tag{8.3.8}$$

using (8.1.11). Applying i_k and noting that $i_k\beta^x = 0$ since $i_k^2 = 0$ (and $i_k f = 0$ for f a 0-form) we get

$$i_k \mathrm{d}\beta^x + \varepsilon^{xyz} i_k\omega^y\, \beta^z = \varepsilon^{xyz} i_k \Omega^y\, P^z_k. \tag{8.3.9}$$

We can introduce the Lie derivative in the first term since

$$i_k \mathrm{d}\beta^x = i_k\, \mathrm{d} i_k \Omega^x = i_k \mathcal{L}_k \Omega^x \tag{8.3.10}$$

again because $i_k^2 = 0$. The we use (8.3.5) to replace the Lie derivative

$$i_k \mathrm{d}\beta^x = \varepsilon^{xyz} W^y_k\, i_k \Omega^z = \varepsilon^{xyz} W^y_k\, i_k \beta^z. \tag{8.3.11}$$

Replacing $i_k\Omega^y = \lambda\beta^y$ in the last term and switching y and z, we finally find

$$\varepsilon^{xyz}(W^y_k + i_k\omega^y\, \beta^z + \lambda P^y_k)\beta^z = 0. \tag{8.3.12}$$

Under certain condition on $i_k\Omega^x$ [200, sec. 4] this implies

$$P^x_k = \frac{1}{\lambda}\Big(- i_k\omega^x - W^x_k \Big). \tag{8.3.13}$$

We deduce that any isometry is associated to a triplet of moment maps, and moreover we can rewrite (8.3.5) as [226, sec. 2]

$$\mathcal{L}_k \Omega^x = \varepsilon^{xyz}(i_k\omega^x - \lambda P^x_k)\Omega^z, \tag{8.3.14}$$

In terms of the triplet of complex structures this gives

$$\mathcal{L}_k \boldsymbol{J} = 2\lambda\, \boldsymbol{J} \times \boldsymbol{P}_k. \tag{8.3.15}$$

The statement (8.3.5) that a Killing vector is triholomorphic means that its covariant derivative commutes with all three complex structures (we omit the index k in the rest of the section)

$$\nabla_u k^w\, \boldsymbol{J}_w{}^v = \boldsymbol{J}_u{}^w\, \nabla_v k^w \tag{8.3.16}$$

In coordinates equation (8.3.6) reads

$$\lambda\, \nabla_u \boldsymbol{P}^x = k^v\, \boldsymbol{\Omega}_{uv}. \tag{8.3.17}$$

The moment map can also be found from

$$4\lambda n_h\, \boldsymbol{P} = \boldsymbol{J}_u{}^v\, \nabla_v k^u. \tag{8.3.18}$$

From Killing equation

$$\nabla_u k_v + \nabla_v k_u = 0, \tag{8.3.19}$$

using the commutator

$$[\nabla_u, \nabla_v]\, k^w = R_{uv}{}^w{}_s k^s \tag{8.3.20}$$

and the explicit value of the Ricci, one finds that k^u satisfy a Poisson equation [218, app. A]

$$\nabla_v \nabla^v k^u + 2\lambda(n_h + 2) k^u = 0. \tag{8.3.21}$$

Then using the relation with the prepotentials implies that the latter also satisfy a Poisson equation (but with different eigenvalues) The prepotentials are harmonic functions

$$\nabla_u \nabla^u P^x + 4 n_h \lambda\, P^x = 0. \tag{8.3.22}$$

Note that the commutator on P^x yields

$$[\nabla_u, \nabla_v]\, P^x = 2\varepsilon^{xyz} \Omega^y_{uv} P^z. \tag{8.3.23}$$

Then the Poisson equation can be used to find a direct expression for the Killing vector

$$k^u = -\frac{1}{6\lambda^2}\, h^{uv} \Omega^x_{vw} \nabla^w P^x. \tag{8.3.24}$$

Let's denote by $\{k_\Lambda\}$ the set of Killing vectors generating the isometries on $\mathcal{M}_h$ (we will use an index Λ as a shortcut for k_Λ in the compensator, etc.). Then one has the cocycle identity

$$\mathcal{L}_\Lambda W^x_\Sigma - \mathcal{L}_\Sigma W^x_\Lambda + \varepsilon^{xyz} W^y_\Lambda W^z_\Sigma = f_{\Lambda\Sigma}{}^\Xi W^x_\Xi \tag{8.3.25}$$

where $f_{\Lambda\Sigma}{}^\Xi$ are the structure constants of the algebra. There is also an equivariance condition

$$J^x_{uv} k^u_\Lambda k^v_\Sigma = \frac{1}{2}\, f_{\Lambda\Sigma}{}^\Omega P^x_\Omega + \frac{\lambda}{2}\, \varepsilon^{xyz} P^x_\Lambda P^y_\Sigma. \tag{8.3.26}$$

8.4 Classification of spaces

Homogeneous QK manifolds have been classified by Alekseevsky [227], but it was shown by de Wit and van Proeyen that it was incomplete [13, 228]. The symmetric manifolds (called Wolf spaces) were given by Wolf [229] (see also [190, 222]). Useful references include [170, p. 77, tab. 2, 152, p. 78, tab. 2, 165, p. 443, tab. 20.5].

The symmetric spaces that are special (i.e. which can be obtained from the c-map, see chapter 8.5) consist in two families

$$\frac{\mathrm{SU}(n_h, 2)}{\mathrm{SU}(n_h) \times \mathrm{SU}(2) \times \mathrm{U}(1)}, \qquad \frac{\mathrm{SO}(n_h, 4)}{\mathrm{SO}(n_h) \times \mathrm{SO}(4)}, \tag{8.4.1}$$

(when $n_h = 1$ the factor $\mathrm{SU}(n_h)$ is not present) given respectively by the quadratic and cubic models (section 6.1.1), and five exceptional cases

$$\begin{gathered} \frac{\mathrm{G}_{2,2}}{\mathrm{SO}(4) \times \mathrm{SO}(2)}, \quad \frac{\mathrm{F}_{4,4}}{\mathrm{USp}(6) \times \mathrm{SU}(2)}, \quad \frac{\mathrm{E}_{6,2}}{\mathrm{SU}(6) \times \mathrm{SU}(2)}, \\ \frac{\mathrm{E}_{7,-5}}{\mathrm{SO}(12) \times \mathrm{SU}(2)}, \quad \frac{\mathrm{E}_{8,-24}}{\mathrm{E}_7 \times \mathrm{SU}(2)} \end{gathered} \tag{8.4.2}$$

for $n_h = 7, 10, 16, 28$ respectively. The first of these exceptional spaces corresponds to the c-map with a cubic model since the spaces of the two families are isomorphic for $n_h = 2$ and it is given by a quadratic model [169, p. 5, tab. 2]. Note that $\mathrm{SU}(2) \subset \mathrm{SO}(4)$.

Finally the only symmetric spaces that cannot be obtained from the c-map are the projective quaternionic manifolds

$$\mathbb{H}P^{n_h} \equiv \frac{\mathrm{Sp}(n_h, 1)}{\mathrm{Sp}(n_h) \times \mathrm{Sp}(1)}, \tag{8.4.3}$$

and recall that $\mathrm{Sp}(1) \sim \mathrm{SU}(2)$.

8.5 Special quaternionic manifolds

Special (or dual) quaternionic manifolds $\mathcal{M}_h$ are a subclass of quaternionic manifolds which fully specified by a special Kähler manifold $\mathcal{M}_z$ [11, 13, 144, 165]. The map $\mathcal{M}_z \to \mathcal{M}_h$ is called the c-map. The latter is useful for determining the isometries of the QK manifold; in particular if $\mathcal{M}_z$ is symmetric then $\mathcal{M}_h$ is also symmetric [12, pp. 222, 224].

8.5.1 Quaternionic metric from the c-map

We recall that $\dim \mathcal{M}_h = 4n_h$. A special quaternionic manifold is made of a base special Kähler manifold $\mathcal{M}_z$ of dimension $2(n_h - 1)$ with a fibration. Homogeneous coordinates on $\mathcal{M}_z$ are denoted by Z^A, and the fibers are $(\phi, \sigma, \xi^A, \tilde{\xi}_A)$ where

$$A = 0, \dots, n_h - 1. \tag{8.5.1}$$

Physically ϕ is the dilaton (coming from the metric), σ is the axion (coming from dualization of the B-field) and the $(\xi^A, \tilde{\xi}_A)$ corresponds to the NS scalars (coming from the reduction of the NS forms).

The explicit construction can be found in [11, 152, sec. 4].

Sometimes we will parametrize the dilaton as

$$\rho = \mathrm{e}^{-2\phi}. \tag{8.5.2}$$

The special coordinates are

$$z^a = \frac{Z^a}{Z^0}, \qquad a = 1, \dots, n_h - 1. \tag{8.5.3}$$

Finally we group the Ramond coordinates into a symplectic vector

$$\xi = \begin{pmatrix} \xi^A \\ \tilde{\xi}_A \end{pmatrix} \tag{8.5.4}$$

Before describing the metric and other geometrical objects we set up the notation for the base special Kähler manifold.

8.5.2 Base special Kähler manifold

The properties of this embedded manifold are exactly the same as the ones described in chapter 4. In this section we are just recalling the main quantities and defining the notations: instead of curly letters $\mathcal{A}$ we will use blackboard bold letter $\mathbb{A}$.

The prepotential is denoted by G and its derivatives together with Z^A form the symplectic vector

$$Z = \begin{pmatrix} Z^A \\ G_A \end{pmatrix}. \tag{8.5.5}$$

The symplectic metric is $\mathbb{C}$.

We obtain the Kähler potential from

$$K_z = -\ln\big(-i\bar{Z}^t \mathbb{C} Z\big) = -\ln i\big(\bar{Z}^A G_A - Z^A \bar{G}_A\big) \tag{8.5.6}$$

from which we obtain the metric

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K_z. \tag{8.5.7}$$

We obtain the period matrix

$$\mathbb{N}_{AB} = \bar{G}_{AB} + 2i\, \frac{\operatorname{Im} G_{AC} \operatorname{Im} G_{BD}\, Z^C Z^D}{\operatorname{Im} G_{CD}\, Z^C Z^D}. \tag{8.5.8}$$

and the complex structure

$$\mathbb{M} = \begin{pmatrix} \operatorname{Im}\mathbb{N} + \operatorname{Re}\mathbb{N}(\operatorname{Im}\mathbb{N})^{-1}\operatorname{Re}\mathbb{N} & -\operatorname{Re}\mathbb{N}(\operatorname{Im}\mathbb{N})^{-1} \\ -(\operatorname{Im}\mathbb{N})^{-1}\operatorname{Re}\mathbb{N} & (\operatorname{Im}\mathbb{N})^{-1} \end{pmatrix}. \tag{8.5.9}$$

Cubic prepotentials will be written as

$$G = -d_{abc}\,\frac{Z^a Z^b Z^c}{Z^0}. \tag{8.5.10}$$

The associated manifolds are called very special quaternionic.

8.5.3 Geometrical structures

The metric $\mathcal{M}_h$ is given by

$$ds_h^2 = d\phi^2 + g_{a\bar{b}}\,dz^a d\bar{z}^{\bar{b}} + \frac{1}{4}\,e^{4\phi}\left(d\sigma + \frac{1}{2}\,\xi^t \mathbb{C} d\xi\right)^2 - \frac{1}{4}\,e^{2\phi}\,d\xi^t \mathbb{M} d\xi. \tag{8.5.11}$$

Note that the second term in parenthesis can be rewritten as

$$\xi^t \mathbb{C} d\xi = \xi^A d\tilde{\xi}_A - \tilde{\xi}_A d\xi^A. \tag{8.5.12}$$

The spin connection ω_u^x is given[2] by [169, sec. 4.2, 62, sec. 3.1, 51, sec. 4]

$$\begin{aligned} \omega^+ &= \sqrt{2}\,e^{\phi + K_z/2}\,Z^t \mathbb{C} d\xi, \\ \omega^3 &= \frac{e^{2\phi}}{2}\left(da + \frac{1}{2}\,\xi^t \mathbb{C} d\xi\right) - 2\,e^{K_z}\operatorname{Im}\left(Z^A \operatorname{Im} G_{AB} d\bar{Z}^B\right). \end{aligned} \tag{8.5.13}$$

where we defined

$$\omega^{\pm} = \omega^1 \pm i\,\omega^2 \tag{8.5.14}$$

which are complex conjugate. These expressions are not invariant under SU(2) transformations.

We can also rewrite [149, app. B]

$$\operatorname{Im}\left(Z^A \operatorname{Im} G_{AB} d\bar{Z}^B\right) = \frac{1}{4}\,Z \mathbb{C} d\bar{Z} + \text{c.c.} \tag{8.5.15}$$

since

$$\begin{aligned} \operatorname{Im}\left(Z^A \operatorname{Im} G_{AB} d\bar{Z}^B\right) &= \operatorname{Im}\left(\frac{1}{2i}\,Z^A (G_{AB} - \bar{G}_{AB}) d\bar{Z}^B\right) = -\frac{1}{2}\operatorname{Re}\left(G_A d\bar{Z}^A - Z^A d\bar{G}_A\right) \\ &= -\frac{1}{4}\left(G_A d\bar{Z}^A - Z^A d\bar{Z}_A + \bar{G}_A dZ^A - \bar{Z}^A dG_A\right) \end{aligned}$$

where we used the homogeneity of G (4.2.31)

$$G_{AB} Z^B = G_A, \qquad G_{AB} dZ^B = dG_A. \tag{8.5.16}$$

[2] Note that this involves a choice of SU(2) basis. Other possibilities are also fine.

Chapter 9

Quaternionic isometries

In this chapter we focus on the isometries of special quaternionic manifolds. As reviewed in the chapter 7 on SK isometries, knowing the Killing vectors of the target space of the non-linear sigma models involved in the $N = 2$ supergravity is necessary in order to write the gauged theory. Since there is a base SK space we are able to use symplectic covariant expressions which simplify the construction of the Killing vectors and which provide a nice interpretation of them.

The isometries of special quaternionic manifolds were classified by de Wit and Van Proeyen [12, 144–146]. There are three kinds of isometries [144, 149]:

- duality symmetries, inherited from the base special Kähler manifolds;
- extra symmetries, whose origin is seen directly from the gauge transformations;
- hidden symmetries, which are not generic and whose existence depends on specific properties of the manifold.

9.1 Killing vectors

We will denote the isometry group by

$$G_h = \mathrm{ISO}(\mathcal{M}_h). \tag{9.1.1}$$

In order to simplify the notation, we define

$$\partial_\xi = \begin{pmatrix} \partial_A \\ \partial^A \end{pmatrix}, \qquad \partial_A = \partial_{\xi^A} = \frac{\partial}{\partial \xi^A}, \qquad \partial^A = \partial_{\tilde{\xi}_A} = \frac{\partial}{\partial \tilde{\xi}_A}. \tag{9.1.2}$$

We will also make use of

$$\mathbb{C}\partial_\xi = \begin{pmatrix} \partial^A \\ -\partial_A \end{pmatrix}. \tag{9.1.3}$$

Similarly we write

$$\partial_Z = \begin{pmatrix} \partial_{Z^A} \\ \partial_{G_A} \end{pmatrix}, \qquad \partial_{Z^A} = \frac{\partial}{\partial Z^A}, \qquad \partial_{G_A} = \frac{\partial}{\partial G_A}. \tag{9.1.4}$$

9.1.1 Duality symmetries

Isometries of the base SK space (described in section 7) can be lifted to the full quaternionic space by adding a transformation of the fibers [12, p. 223]. They consist in symplectic (infinitesimal) transformations $\mathbb{U} \in \mathfrak{sp}(2n_H, \mathbb{R})$ that leave invariant the prepotential. Since the metric is made only of symplectic products, it is easy to see that the Killing vector on the full space is [169, sec. 4.2]

$$k_{\mathbb{U}} = (\mathbb{U}Z)^t\, \partial_Z + (\mathbb{U}\bar{Z})^t\, \partial_{\bar{Z}} + (\mathbb{U}\xi)^t \partial_\xi. \tag{9.1.5}$$

Writing explicitly the product gives

$$k_{\mathbb{U}} = (\mathbb{U}Z)^A \partial_{Z^A} + (\mathbb{U}Z)_A \partial_{G_A} + (\mathbb{U}\xi)^A \partial_A + (\mathbb{U}\xi)_A \partial^A + \text{c.c.} \tag{9.1.6}$$

In order to use conventions similar to the other Killing vectors we should write this vector as a linear combination of each Killing vector associated to independent parameters, but this is not the usual approach taken in the literature.

The matrix $\mathbb{U}$ is parametrized by (see section 7)

$$\mathbb{U} = \begin{pmatrix} v^A{}_B & t^{AB} \\ s_{AB} & u_A{}^B \end{pmatrix}, \qquad t^{AB} = t^{BA}, \quad s_{AB} = s_{BA}, \quad v^A{}_B = -u_B{}^A \tag{9.1.7}$$

where the constraint are equivalent to

$$\mathbb{U}^t \mathbb{C} + \mathbb{C}\mathbb{U} = 0. \tag{9.1.8}$$

We refer to section 7 for more details on the classification of duality isometries. Since the parameters are subject to the constraints not all these symmetries are universal.

9.1.2 Extra symmetries

These symmetries act on the Heisenberg fiber: they originate from the gauge symmetry of gauge fields that have been dualized to scalar fields [12, p. 223]. Only the derivative of the scalar fields that have been dualized from vector fields appear, and shift symmetries result from this.

The first symmetry is a translation of the axion [169, sec. 4.2]

$$k_+ = \partial_\sigma. \tag{9.1.9}$$

In general nothing depends on the axion and everything is invariant under shift of this field.

Then there is a scaling symmetry of all the fields

$$k_0 = \partial_\phi - 2\sigma\partial_\sigma - \xi^t \partial_\xi. \tag{9.1.10}$$

Expanding the product gives explicitly

$$k_0 = \partial_\phi - 2\sigma\partial_\sigma - \tilde{\xi}_A \partial^A - \xi^A \partial_A. \tag{9.1.11}$$

Finally there are $2n_h$ translations of the Ramond fields ξ accompanied by a transformation of σ [169, sec. 4.2] (this is really a $2n_h$-dimensional vector)

$$k_\xi = \mathbb{C}\partial_\xi + \frac{1}{2}\, \xi\, \partial_\sigma \tag{9.1.12}$$

or more explicitly[1]

$$k^A = \partial^A + \frac{1}{2}\, \xi^A \partial_\sigma, \tag{9.1.13a}$$

$$k_A = -\partial_A + \frac{1}{2}\, \tilde{\xi}_A \partial_\sigma. \tag{9.1.13b}$$

The shift of the fibers can be written

$$k_\xi = \mathbb{C}\partial_\xi + \frac{1}{2}\, \xi\, \partial_\sigma. \tag{9.1.14}$$

All these symmetries are universal and do not depend on the model.

[1] Note that k_A gets a minus sign with respect to the definition in [169, sec. 4.2].

9.1.3 Hidden vectors

There are several hidden symmetries [12, 144, 145, sec. 3]. In [149, sec. 4] these vectors have been expressed in a symplectic covariant form.[2]

Since the quaternionic metric does not contain linear term in $\mathrm{d}z^a$, any isometry of the full space needs to be an isometry of the base SK space when the vector is restricted to the latter

$$\mathcal{L}_k h_{uv} = 0 \Longrightarrow \mathcal{L}_k|_{\mathcal{M}_z} g_{a\bar{b}} = 0. \tag{9.1.15}$$

In particular this implies that the transformation of the homogeneous SK coordinates are of the form

$$\delta Z = \mathbb{S} Z \tag{9.1.16}$$

where $\mathbb{S} \in \mathfrak{sp}(2n_h)$ and it satisfies the equivalent of (7.1.9). In particular this matrix can depend on all the fields of the fiber

$$\mathbb{S} = \mathbb{S}(\phi, \sigma, \xi^A, \tilde{\xi}_A) \tag{9.1.17}$$

as they are just constant from the point of view of the base SK space, but it appears that $\mathbb{S}$ depends only on ξ.

The first vector is given by

$$k_- = -\sigma\, \partial_\phi + (\sigma^2 - \mathrm{e}^{-4\phi} - W)\partial_\sigma + (\sigma\xi - \mathbb{C}\partial_\xi W)^t \partial_\xi - (\mathbb{S}Z)^t\partial_Z + \text{c.c.} \tag{9.1.18a}$$

Then there are $2n_h$ vectors

$$\begin{aligned} \widehat{k}_\xi = &-\frac{1}{2}\,\xi\,\partial_\phi + \left(\frac{\sigma}{2}\,\xi - \frac{1}{2}\,\mathbb{C}\partial_\xi W\right)\partial_\sigma + \sigma\,\mathbb{C}\partial_\xi + \left(\frac{1}{2}\,\xi^t\xi - \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t\right)\partial_\xi \\ &- (\mathbb{C}\partial_\xi \mathbb{S}\, Z)^t \partial_Z + \text{c.c.} \end{aligned} \tag{9.1.18b}$$

Explicitly they are given by

$$\begin{aligned} \widehat{k}^A = &-\frac{1}{2}\,\xi^A\,\partial_\phi + \left(\frac{\sigma}{2}\,\xi^A - \frac{1}{2}\,\partial^A W\right)\partial_\sigma + \sigma\,\partial^A + \left(\frac{1}{2}\,\xi^A\,\xi - \mathbb{C}\partial_\xi\partial^A W\right)^t\partial_\xi \\ &- (\partial^A \mathbb{S}\, Z)^t\partial_Z + \text{c.c.} \end{aligned} \tag{9.1.18c}$$

$$\begin{aligned} \widehat{k}_A = &-\frac{1}{2}\,\tilde{\xi}_A\,\partial_\phi + \left(\frac{\sigma}{2}\,\tilde{\xi}_A + \frac{1}{2}\,\partial_A W\right)\partial_\sigma - \sigma\,\partial_A + \left(\frac{1}{2}\,\tilde{\xi}_A\,\xi + \mathbb{C}\partial_\xi\partial_A W\right)^t\partial_\xi \\ &+ (\partial_A \mathbb{S}\, Z)^t\partial_Z + \text{c.c.} \end{aligned} \tag{9.1.18d}$$

We have used several quantities

$$W = \frac{1}{4}\,h(\xi) - \frac{1}{2}\,\mathrm{e}^{-2\phi}\,\xi^t\mathbb{C}\mathbb{M}\xi, \tag{9.1.19a}$$

$$\mathbb{S} = \frac{1}{2}\left(\xi\xi^t + \frac{1}{2}\,H\right)\mathbb{C}, \tag{9.1.19b}$$

$$H = \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h)^t = \begin{pmatrix} \partial^A\partial^B h & -\partial^A\partial_B h \\ -\partial_A\partial^B h & \partial_A\partial_B h \end{pmatrix}. \tag{9.1.19c}$$

h is a homogeneous quartic polynomial constructed from the quartic invariant [151, sec. 4], while $\mathbb{S}$ is a symplectic matrix

$$\mathbb{S}^t = \mathbb{C}\mathbb{S}\mathbb{C}. \tag{9.1.20}$$

H is a symmetric matrix.

Some of the quantities involved are homogeneous in ξ:

[2] Also this paper provides corrections to the expression from [144] that were incorrect.

- h: order 4;
- $\mathbb{S}, H$: order 2.

This means that

$$\xi^t \partial_\xi h = 4h, \qquad \xi^t \partial_\xi H = 2H, \qquad \xi^t \partial_\xi \mathbb{S} = 2\mathbb{S}. \tag{9.1.21}$$

When the space is symmetric the quartic invariant h is independent of the fields z^i [144, pp. 13, 17]. In particular it is possible to obtain conditions by taking derivatives. If h depends on z^i then some symmetries of $\mathfrak{g}_{-1/2}$ can still exist if some linear combinations of $\partial_A h$ and $\partial^A h$ are independent of z^i. For this last reason it may be interesting to keep parameters in Killing vectors since the Killing vectors $\widehat{k}^A$ and $\widehat{k}_A$ may not exist by themselves, but only linear combinations.

Some interesting results on possible hidden vectors are proved in [144, sec. 4.3] for $\mathcal{M}_z$ with cubic prepotential. For example $\widehat{\alpha}_0$ always exists, whereas $\widehat{\alpha}^0$ exists only for symmetric spaces, and the others exist if

$$E^a{}_{bcde}\,\widehat{\alpha}^e = 0, \qquad E^a{}_{bcde}\,\widehat{\alpha}_a = 0. \tag{9.1.22}$$

Note that the second constraint coincides with the one for the existence of a_a, such that if the later exist, then there also exist symmetries such that $\widehat{\alpha}_a \propto a_a$.

Cubic prepotential

For cubic prepotential the quartic invariant is given by (6.3.13)

$$h(\xi, \tilde{\xi}) = -(\tilde{\xi}_A \xi^A)^2 + \frac{1}{16}\, \xi^0\, \widehat{d}^{abc} \tilde{\xi}_a \tilde{\xi}_b \tilde{\xi}_c - 4\, \tilde{\xi}_0\, d_{abc} \xi^a \xi^b \xi^c + \frac{9}{16}\, \widehat{d}^{abc} d_{cde} \tilde{\xi}_a \tilde{\xi}_b\, \xi^d \xi^e. \tag{9.1.23}$$

The parameters of the matrix $\mathbb{S}$ as written in section 7.2 are

$$\beta = -\frac{1}{2} \left(3\, \tilde{\xi}_0 \xi^0 + \tilde{\xi}_a \xi^a\right), \tag{9.1.24a}$$

$$b^a = -\frac{1}{2} \left(2\, \tilde{\xi}_0 \xi^a - \frac{3}{32}\, \widehat{d}^{abc} \tilde{\xi}_b \tilde{\xi}_c\right), \tag{9.1.24b}$$

$$a_a = -\frac{1}{2} \left(2\, \xi^0 \xi_a + 6\, d_{abc} \xi^b \xi^c\right), \tag{9.1.24c}$$

$$B^a{}_b = -\frac{1}{2} \left(\frac{2}{3}\, \delta^a{}_b \tilde{\xi}_c \xi^c - \frac{9}{8}\, \widehat{d}^{acd} d_{bde} \tilde{\xi}_d \xi^e\right). \tag{9.1.24d}$$

Quadratic prepotential

For quadratic prepotential the quartic invariant is given by (6.2.12) and (6.2.17)

$$h(\xi, \tilde{\xi}) = I_2(\xi, \tilde{\xi})^2, \qquad I_2(\xi, \tilde{\xi}) = \frac{i}{2}\, \xi^A \eta_{AB}\, \xi^B + \frac{i}{2}\, \tilde{\xi}_A \eta^{AB} \tilde{\xi}_B. \tag{9.1.25}$$

The parameters of the matrix $\mathbb{S}$ as written in section 7.3 are

$$r^{AB} = -\frac{1}{2} \left(-\xi^A \xi^B - i\, I_2(\xi, \tilde{\xi})\, \eta^{AB} + (\eta^{-1}\tilde{\xi})^A (\eta^{-1}\tilde{\xi})^B\right), \tag{9.1.26a}$$

$$s_{AB} = -\frac{1}{2} \left(\tilde{\xi}^A \tilde{\xi}^B + i\, I_2(\xi, \tilde{\xi})\, \eta_{AB} - (\eta\xi)_A (\eta\xi)_B\right), \tag{9.1.26b}$$

$$q^A{}_B = -\frac{1}{2} \left(-\xi^A \tilde{\xi}_B - (\eta^{-1}\tilde{\xi})^A (\eta\xi)_B\right). \tag{9.1.26c}$$

Some relations

For later computations we look at various expressions involving the previous objects.

The ϕ derivative of W is equal to

$$\partial_\phi W = \mathrm{e}^{-2\phi}\, \xi^t \mathbb{C}\mathbb{M}\xi. \tag{9.1.27}$$

W is not homogeneous (since it has quadratic and quartic pieces) but using the last equation we have

$$(\xi^t \partial_\xi - \partial_\phi) W = 4W, \tag{9.1.28}$$

or written in various other ways

$$\xi^t \partial_\xi W = 2W + \frac{1}{2}\, h = 4W + \mathrm{e}^{-2\phi}\, \xi^t \mathbb{C}\mathbb{M}\xi = 4W + \partial_\phi W. \tag{9.1.29}$$

Similarly for the derivative of W we get

$$(\xi^t \partial_\xi - \partial_\phi)\partial_\xi W = 3\, \partial_\xi W \tag{9.1.30}$$

or differently

$$(\xi^t \partial_\xi)\partial_\xi W = W + \frac{1}{2}\, \partial_\xi h = 3\partial_\xi W + \frac{1}{2}\, \mathrm{e}^{-2\phi}\, \partial_\xi(\xi^t \mathbb{C}\mathbb{M}\xi) = 3\partial_\xi W + \mathrm{e}^{-2\phi}\, \mathbb{C}\mathbb{M}\xi \tag{9.1.31}$$

using the relation (9.1.32) proved below.

The derivative with respect to ξ of the second term in W reads

$$\mathrm{e}^{2\phi}\, \partial_\xi(\partial_\phi W) = \partial_\xi(\xi^t \mathbb{C}\mathbb{M}\xi) = 2\mathbb{C}\mathbb{M}\xi \tag{9.1.32}$$

since

$$\partial_\xi(\xi^t \mathbb{C}\mathbb{M}\xi) = \mathbb{C}\mathbb{M}\xi + \xi^t \mathbb{C}\mathbb{M} = \mathbb{C}\mathbb{M}\xi - \mathbb{M}^t \mathbb{C}\xi = 2\mathbb{C}\mathbb{M}\xi.$$

Equivalently

$$(\mathbb{C}\partial_\xi)(\xi^t \mathbb{C}\mathbb{M}\xi) = -2\mathbb{M}\xi. \tag{9.1.33}$$

Taking the derivative a second time gives

$$\partial_\xi[\partial_\xi(\xi^t \mathbb{C}\mathbb{M}\xi)]^t = 2\mathbb{C}\mathbb{M}, \qquad \mathbb{C}\partial_\xi[\mathbb{C}\partial_\xi(\xi^t \mathbb{C}\mathbb{M}\xi)]^t = -2\mathbb{C}\mathbb{M}, \tag{9.1.34}$$

On the other hand we defined

$$H = \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h)^t \tag{9.1.35}$$

so we get that

$$\mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t = H - 2\,\mathrm{e}^{-2\phi}\mathbb{C}\mathbb{M} = -2\xi\xi^t - 4\mathbb{S}\mathbb{C} - 2\,\mathrm{e}^{-2\phi}\mathbb{C}\mathbb{M}. \tag{9.1.36}$$

9.1.4 Summary

As a summary, the list of all the Killing vectors is

$$k_{\mathbb{U}} = (\mathbb{U}Z)^t\, \partial_Z + (\mathbb{U}\bar{Z})^t\, \partial_{\bar{Z}} + (\mathbb{U}\xi)^t \partial_\xi, \tag{9.1.37a}$$

$$k_\xi = \mathbb{C}\partial_\xi + \frac{1}{2}\, \xi\, \partial_\sigma, \tag{9.1.37b}$$

$$k_0 = \partial_\phi - 2\sigma\, \partial_\sigma - \xi^t \partial_\xi, \tag{9.1.37c}$$

$$k_+ = \partial_\sigma, \tag{9.1.37d}$$

for the normal symmetries and

$$
\begin{aligned}
k_- &= -\sigma\partial_\phi + (\sigma^2 - \mathrm{e}^{-4\phi} - W)\partial_\sigma + (\sigma\xi - \mathbb{C}\partial_\xi W)^t\partial_\xi - (\mathbb{S}Z)^t\partial_Z + \text{c.c.}, && (9.1.37\text{e}) \\
\widehat{k}_\xi &= -\frac{1}{2}\,\xi\,\partial_\phi + \left(\frac{\sigma}{2}\,\xi - \frac{1}{2}\,\mathbb{C}\partial_\xi W\right)\partial_\sigma + \sigma\,\mathbb{C}\partial_\xi + \left(\frac{1}{2}\,\xi\xi^t - \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t\right)\partial_\xi && (9.1.37\text{f}) \\
&\quad - (\mathbb{C}\partial_\xi\mathbb{S}\,Z)^t\partial_Z + \text{c.c.}
\end{aligned}
$$

for the hidden symmetries.

We have used several quantities

$$
\begin{aligned}
W &= \frac{1}{4}\,h(\xi^A,\tilde{\xi}_A) - \frac{1}{2}\,\mathrm{e}^{-2\phi}\,\xi^t\mathbb{C}\mathbb{M}\xi, && (9.1.38\text{a}) \\
\mathbb{S} &= \frac{1}{2}\left(\xi\xi^t + \frac{1}{2}\,H\right)\mathbb{C}, && (9.1.38\text{b}) \\
H &= \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h)^t = \begin{pmatrix} \partial^A\partial^B h & -\partial^A\partial_B h \\ -\partial_A\partial^B h & \partial_A\partial_B h \end{pmatrix}. && (9.1.38\text{c})
\end{aligned}
$$

9.2 Algebra

We define the commutator of two vectors of Killing vectors k_1 and k_2 as

$$
\left[k_1, k_2^t\right] = k_1 k_2^t - (k_1 k_2^t)^t. \tag{9.2.1}
$$

Another possibility is to introduce one parameter for each Killing vector which turns the previous matrix commutator into a normal scalar commutator

$$
\left[\epsilon_1^t k_1, \epsilon_2^t k_2\right] = \epsilon_1^t k_1 k_2^t \epsilon_2 - \epsilon_2^t (k_1 k_2^t)^t \epsilon_1 \tag{9.2.2}
$$

and specific commutators can be extracted by taking all parameters to zeros except those we are interested in which are set to one.[3]

The non-vanishing commutators of the algebra are [149, sec. 4.3, 144, sec. 3]

$$
\begin{gathered}
[k_0, k_+] = 2k_+, \qquad [k_0, k_\xi] = k_\xi, \qquad \left[k_\xi, k_\xi^t\right] = \mathbb{C}\,k_+, \qquad [k_\mathbb{U}, k_\xi] = \mathbb{U}\,k_\xi, \\
[k_0, k_-] = -2\,k_-, \qquad \left[k_0, \widehat{k}_\xi\right] = -\widehat{k}_\xi, \qquad [k_-, k_\xi] = -\widehat{k}_\xi, \\
[k_+, k_-] = -k_0, \qquad \left[k_+, \widehat{k}_\xi\right] = k_\xi, \qquad \left[k_\mathbb{U}, \widehat{k}_\xi\right] = \mathbb{U}\,\widehat{k}_\xi, \\
\left[\widehat{k}_\xi, \widehat{k}_\xi^t\right] = \mathbb{C}\,k_-, \qquad \left[\hat{\alpha}^t\widehat{k}_\xi, \alpha^t k_\xi\right] = \frac{1}{2}\,\hat{\alpha}\mathbb{C}\alpha\,k_0 + k_{\mathbb{T}_{\alpha,\hat{\alpha}}}
\end{gathered} \tag{9.2.3}
$$

with

$$
\begin{aligned}
\mathbb{T}_{\alpha,\hat{\alpha}} &= (\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\mathbb{C}\partial_\xi)\,\mathbb{S} = -\frac{1}{2}\,\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t) + \frac{1}{4}\,H''_{\alpha,\hat{\alpha}}\mathbb{C}, && (9.2.4\text{a}) \\
H''_{\alpha,\hat{\alpha}} &= \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h''_{\alpha,\hat{\alpha}})^t = (\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\mathbb{C}\partial_\xi)H, && (9.2.4\text{b}) \\
h''_{\alpha,\hat{\alpha}} &= (\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\mathbb{C}\partial_\xi)h. && (9.2.4\text{c})
\end{aligned}
$$

Some commutators are computed in appendix F.1, others have been checked with Mathematica.

We see that there are two Heisenberg subalgebra, one generated by $\{k_\xi, k_+\}$ [169, sec. 4], the other by $\{\widehat{k}_\xi, k_-\}$.

[3] The same idea is used for supersymmetry where ϵQ can be used to turn anticommutators into commutators.

The algebra $\mathfrak{g}_h$ corresponding to these Killing vectors can be decomposed into eigenspaces of k_0 [12, pp. 222–223, 144, sec. 2.3]

$$\mathfrak{g}_h = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1 \tag{9.2.5}$$

where the Killing vectors contained in $\mathfrak{g}_a$ satisfy

$$[k_0, \mathfrak{g}_a] = a\, \mathfrak{g}_a. \tag{9.2.6}$$

We note that the dimensions of extra symmetry subspaces are

$$\dim \mathfrak{g}_{1/2} = 2n_h, \qquad \dim \mathfrak{g}_1 = 1 \tag{9.2.7}$$

while for hidden symmetries the dimensions are

$$\text{symmetric } \mathcal{M}_h\text{:} \qquad \dim \mathfrak{g}_{-1} = 1, \qquad \dim \mathfrak{g}_{-1/2} = 2n_h, \tag{9.2.8a}$$

$$\text{otherwise:} \qquad \dim \mathfrak{g}_{-1} = 0, \qquad \dim \mathfrak{g}_{-1/2} \leq n_h. \tag{9.2.8b}$$

Note that the algebra of $\mathcal{M}_z$ is contained in $\mathfrak{g}_0$. As a conclusion very special quaternionic manifolds have at least $2n_h + 2$ isometries (k_0, k_ξ and k_+) [11]

$$\dim \mathfrak{g} \geq 2n_h + 2. \tag{9.2.9}$$

Using the algebra we can obtain some information about the number of symmetries that will be realized. For example if for a given A the symmetries $\widehat{k}_A$ and $\widehat{K}^A$ exist, then from the algebra we deduce that k_- exists also and the space is symmetric [12, p. 228]. Similarly the bound on the dimension of $\mathfrak{g}_{-1/2}$ is obtained from the commutators with $k_{\mathbb{U}}$, so if we have one symmetry of this subspace we can build other by taking the commutator.

Projective quaternionic space

$$\mathcal{M}_h = \frac{\mathrm{Sp}(n_h, 1)}{\mathrm{Sp}(n_h) \times \mathrm{Sp}(1)} \tag{9.2.10}$$

are associated to the algebra C_1^1 are not in the image of the c-map since

$$\dim \mathfrak{g}_1 = 3 \tag{9.2.11}$$

which is in contradiction with what we have seen above [144, p. 12].

9.3 Compensators

The expressions for the compensators are not invariant under SU(2) transformations, and they depend on the choice of the spin connection.

We recall that the compensators are defined by

$$\mathcal{L}_k \omega^+ = \mathrm{d}W_k^+ - i\, \omega^+ W_k^3 + i\, \omega^3 W_k^-, \tag{9.3.1}$$

and also

$$\omega^+ = \sqrt{2}\, \mathrm{e}^{\phi + K_z/2}\, Z^t \mathbb{C} \mathrm{d}\xi. \tag{9.3.2}$$

In homogeneous coordinates, ω_u^x is explicitly invariant and the compensator vanishes

$$W^x = 0. \tag{9.3.3}$$

Then for getting their expressions one needs to compute the Lie derivative in special coordinates.

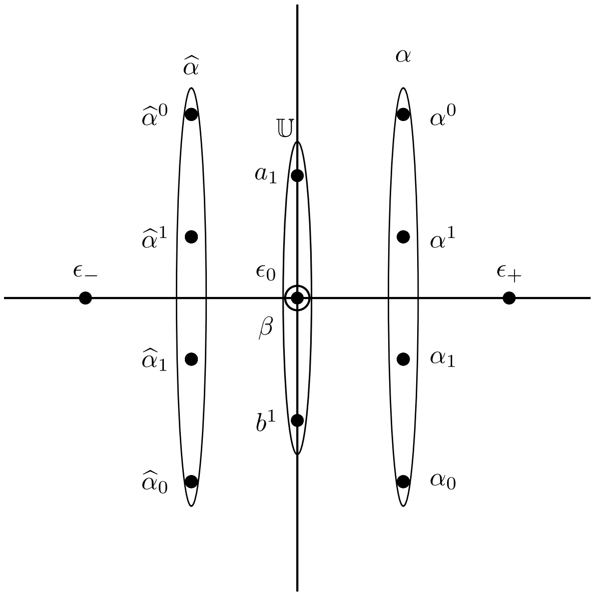

Figure 9.1 – G_2 root diagram [144, sec. 2.3], see [230, sec. 3.1] for the construction. This corresponds to $n_h = 2$, and in this case $B^1{}_1 = 0$.

9.3.1 Duality symmetries

Cubic prepotential

The only non-zero compensator is [149, sec. 5.1.1, app. B.3.1]

$$W^3_{\mathbb{U}} = a_c \operatorname{Im} z^c. \tag{9.3.4}$$

from

$$\mathcal{L}_{\mathbb{U}} \omega^+ = -i\, a_c \operatorname{Im} z^c \,\omega^+. \tag{9.3.5}$$

Quadratic prepotential

The only non-zero compensator is [149, sec. 5.1.1]

$$W^3_{\mathbb{U}} = \operatorname{Im}(A^a{}_0 z^a) = q^a{}_0 \operatorname{Im} z^a + r^{a0} \operatorname{Re} z^a \tag{9.3.6}$$

from

$$\mathcal{L}_{\mathbb{U}} \omega^+ = -i \left(q^a{}_0 \operatorname{Im} z^a + r^{a0} \operatorname{Re} z^a\right) \omega^+. \tag{9.3.7}$$

9.3.2 Hidden symmetries

Compensators for hidden symmetries are [149, sec. 5.1.2, app. B.3.2]

$$W^+_- = 2i\sqrt{2}\, \mathrm{e}^{K_z - \phi}\, Z\mathbb{C}\xi, \tag{9.3.8a}$$

$$W^3_- = -W^3_{\mathbb{S}} - \mathrm{e}^{-2\phi}, \tag{9.3.8b}$$

$$\widehat{W}^+_\xi = -\mathbb{C}\partial_\xi W^+_-, \tag{9.3.8c}$$

$$\widehat{W}^3_\xi = -2\,\mathbb{C}\partial_\xi W^3_-. \tag{9.3.8d}$$

9.4 Prepotentials

The expressions for the prepotentials are not invariant under SU(2) transformations, and they depend on the choice of the spin connection.

We recall that Killing prepotentials are given by

$$P_\Lambda^x = k_\Lambda^u \omega_u^x - W_\Lambda^x \tag{9.4.1}$$

and they are real. We will sometimes use

$$P^\pm = P^1 \pm i\, P^2. \tag{9.4.2}$$

The prepotentials for the universal symmetries are

$$P_+^+ = 0, \qquad P_+^3 = \frac{1}{2}\,\mathrm{e}^{2\phi}, \tag{9.4.3a}$$

$$P_0^+ = \sqrt{2}\,\mathrm{e}^{K_z/2+\phi}\,Z\mathbb{C}\xi, \qquad P_0^3 = -\sigma\,\mathrm{e}^{2\phi}, \tag{9.4.3b}$$

$$P_\xi^+ = \sqrt{2}\,\mathrm{e}^{K_z/2+\phi}\,Z, \qquad P_\xi^3 = \frac{1}{2}\,\mathrm{e}^{2\phi}\,Z, \tag{9.4.3c}$$

while those for the base SK isometries are

$$P_{\mathbb{U}}^+ = \frac{1}{2}\,\mathrm{e}^{K_z/2+\phi}\,\xi\mathbb{C}\mathbb{U}Z, \qquad P_{\mathbb{U}}^3 = \frac{1}{4}\,\mathrm{e}^{2\phi}\,\xi\mathbb{C}\mathbb{U}\xi + \frac{1}{2}\,\mathrm{e}^{K_z}\,Z\mathbb{C}\mathbb{U}\bar{Z}, \tag{9.4.3d}$$

and those for the hidden isometries are

$$P_-^+ = -\frac{1}{2}\,\mathrm{e}^{-2\phi} + \frac{\sigma^2}{2}\,\mathrm{e}^{2\phi}(2W - \xi\partial_\xi W) - \frac{1}{2}\,\mathrm{e}^{K_z}\,\bar{Z}\mathbb{C}\mathbb{S}Z, \tag{9.4.3e}$$

$$P_-^3 = \sqrt{2}\,\mathrm{e}^{K_z/2+\phi}(\sigma\,Z\mathbb{C}\xi + Z\partial_\xi W), \tag{9.4.3f}$$

$$\widehat{P}_\xi^+ = \frac{1}{\sqrt{2}}\,\mathrm{e}^{K_z/2+\phi}\,(Z\mathbb{C}\xi)\,\xi + \mathbb{C}\partial_\xi P_-^+, \tag{9.4.3g}$$

$$\widehat{P}_\xi^3 = \frac{\sigma}{2}\,\mathrm{e}^{2\phi}\,\xi + \mathbb{C}\partial_\xi P_-^3. \tag{9.4.3h}$$

Part III

BPS equations for black holes

Chapter 10

Generalities on AdS–NUT black holes

10.1 Ansatz

In this section we consider asymptotically adS and adS–NUT black holes. The goal is to provide an overview of the structure of these solutions [150].

We take the following ansatz for the metric and the gauge fields

$$\mathrm{d}s^2 = -\,\mathrm{e}^{2U}\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big)^2 + \mathrm{e}^{-2U}\mathrm{d}r^2 + \mathrm{e}^{2(V-U)}\,\mathrm{d}\Sigma_g^2, \tag{10.1.1a}$$

$$A^\Lambda = \tilde q^\Lambda\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big) + \tilde p^\Lambda\,H(\theta)\,\mathrm{d}\phi. \tag{10.1.1b}$$

The functions $U, V, \tilde q$ and $\tilde p$ depend only on r, and n is the NUT charge. The space Σ_g is defined in section A.7

$$\mathrm{d}\Sigma_g^2 = \mathrm{d}\theta^2 + H'(\theta)^2\,\mathrm{d}\phi^2, \qquad H(\theta) = \begin{cases} -\cos\theta & \kappa = 1,\\ \theta & \kappa = 0,\\ \cosh\theta & \kappa = -1. \end{cases} \tag{10.1.2}$$

We mainly work with $\kappa = \pm 1$, but one can check that key equations are also valid for $\kappa = 0$, possibly with a rescaling of the Maxwell and NUT charges.

10.2 Motivation: constant scalar black holes

10.2.1 Solution

In order to motivate our general analysis let us start with the adS–NUT charged black hole in Einstein–Maxwell theory with a cosmological constant $\Lambda = -3g^2$, which corresponds to minimal gauged supergravity with coupling g ($n_v = n_h = 0$), following [150, sec. 2].

The metric and the gauge field read [76]

$$\mathrm{d}s^2 = -\frac{\mathrm{e}^{2V}}{r^2+n^2}\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big)^2 + \frac{r^2+n^2}{\mathrm{e}^{2V}}\,\mathrm{d}r^2 + (r^2+n^2)\,\mathrm{d}\Sigma_g^2, \tag{10.2.1a}$$

$$A = \frac{Qr - nP}{r^2+n^2}\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big) + P\,H(\theta)\,\mathrm{d}\phi. \tag{10.2.1b}$$

using the functions

$$e^{2V} = g^2(r^2+n^2)^2 + (\kappa + 4g^2n^2)(r^2-n^2) - 2mr + P^2 + Q^2, \tag{10.2.2a}$$

$$e^{2(V-U)} = r^2 + n^2, \tag{10.2.2b}$$

$$\tilde{q} = \frac{Qr - nP}{r^2+n^2}, \tag{10.2.2c}$$

$$\tilde{p} = P. \tag{10.2.2d}$$

The ϕ-component of the gauge field reads

$$A_\phi = \frac{P(r^2-n^2) + 2nQr}{r^2+n^2}\, H(\theta). \tag{10.2.3}$$

The parameters P and Q are the magnetic and electric charges, and m is the mass. The ADM mass and charges depend on the genus of the surface [75, p. 5].

It it well-known that Taub–NUT spacetimes have closed timelike curve (which are present in order to avoid Misner strings), and the periodicity is related to the NUT charge [231, 39, chap. 9]. The only exception to the previous statement is for $\kappa = -1$ where there is a range for n where the solution is free of closed timelike curves [34]

$$0 \le 2g^2n^2 \le 1. \tag{10.2.4}$$

When the NUT charge is set to zero the solution corresponds to the adS Reissner–Nordström.

10.2.2 Root structure and supersymmetry

The supersymmetric properties of adS black holes ($n = 0$) were first studied by Romans in its seminal paper [31]. He found two classes of BPS solutions

$$\frac{1}{2}\text{-BPS}: \qquad m = |Q|, \qquad P = 0, \tag{10.2.5a}$$

$$\frac{1}{4}\text{-BPS}: \qquad m = 0, \qquad P = \pm\frac{1}{2g}, \tag{10.2.5b}$$

and only Q is not constrained. The 1/2-BPS solution has a naked singularity for any κ, while the 1/4-BPS solution also has a naked singularity, except for $\kappa = -1$ and $Q = 0$, in which case it has a horizon $\mathrm{adS}_2 \times H^2$.

This has been generalized in [76] which found again two classes

$$\frac{1}{2}\text{-BPS}: \qquad m = |Q|\sqrt{\kappa + 4g^2n^2}, \qquad P = \pm n\sqrt{\kappa+4g^2n^2}, \tag{10.2.6a}$$

$$\frac{1}{4}\text{-BPS}: \qquad m = |2gnQ|, \qquad P = \pm\frac{\kappa+4g^2n^2}{2g}, \tag{10.2.6b}$$

where q and n are not constrained.

On these two BPS branches the root structure corresponds to

$$e^{2V} = g^2\,(r-r_1^+)(r-r_1^-)(r-r_2^+)(r-r_2^-), \tag{10.2.7}$$

where

$$\frac{1}{2}\text{-BPS}: \qquad r_1^\pm = \frac{i}{2g}\left(\sqrt{\kappa+4g^2n^2} \pm \sqrt{\kappa + 8g^2n^2 + 4igQ}\right), \tag{10.2.8a}$$

$$\frac{1}{4}\text{-BPS}: \qquad r_1^\pm = i\left(n \pm \frac{1}{\sqrt{2}g}\sqrt{\kappa+4g^2n^2+2igQ}\right), \tag{10.2.8b}$$

and in both cases one has $r_2^{\pm}(Q) = -r_1^{\pm}(-Q)$.

The 1/4-BPS branch has a real root only if

$$Q^2 = -2n^2(\kappa + 2g^2n^2), \tag{10.2.9}$$

which requires $\kappa = -1$. Then the solution possesses an extremal horizon located at

$$r_1^- = r_2^- = \frac{\sqrt{1-\kappa-4g^2n^2}}{2\sqrt{2}g} > 0. \tag{10.2.10}$$

Note that the squareroot is well defined only if n is situated in the range (10.2.4) where there is no closed timelike curve according to [34]. One can see that if one of the root is real, then another root is automatically real and the black hole is extremal.

On the other hand for the 1/2-BPS solution a real root exists if

$$Q^2 = -n^2(\kappa + 4g^2n^2) \tag{10.2.11}$$

but this is in contraction with the requirement that the magnetic charge is real

$$\kappa + 4g^2n^2 > 0. \tag{10.2.12}$$

In this case the spacetime can reach negative r and there is no horizon. This should be contrasted with the Euclidean analysis where the associated solutions have a single root (corresponding to a bolt). This quantitative difference is due to the fact that one continues also the NUT charge when performing the Wick rotation from Lorentzian to Euclidean signatures.

10.3 Root structure and IR geometry

In general e^{2V} could be any function; nonetheless from known examples it seems that the most general form is a quartic polynomial [150, sec. 4] (see for example [18, 19, 76])

$$e^{2V} = \sum_{p=0}^{4} v_p\, r^p. \tag{10.3.1}$$

The root structure of this functions is particularly important as it determines the existence and the location of horizons, along which other properties such as extremality. Before proceeding remember that it is possible to shift the radial coordinates. Finally the temperature of the black hole is proportional to $(e^V)'$.

The various possibilities are:

- Naked singularity: pair of complex conjugate roots, $v_3 = 0$.

  The solution has no horizon.

- Black hole: two real roots, $v_0 = 0$.

  There is at least one horizon and the black hole has a finite temperature.

- Extremal black hole: real double root, $v_0 = v_1 = 0$.

  Two horizons of the previous case coincide, which implies that the first derivative vanishes, and the temperature is zero. We also recall that *static* BPS black holes are extremal.

- Double extremal black hole [58]: pair of real double roots, $v_0 = v_1 = 0$ and $v_3 = \sqrt{v_2 v_4}$.

- Ultracold black hole [31, sec. 3.1]: real triple root, $v_0 = v_1 = v_2 = 0$.

It is implicit that the other roots are different, and they may be real (giving additional horizons) or in complex conjugate pairs. Shifting r has been used to set $v_0 = 0$ – which is equivalent to move one of the root to $r = 0$ – when at least one root is real, or to set $v_3 = 0$. It is possible that for some special values of the v_i the class of a black hole changes, as we have seen in the previous section.

Extremal black holes which have

$$v_0 = v_1 = 0, \qquad v_2 \neq 0 \tag{10.3.2}$$

possess a near-horizon geometry of the form $\mathrm{adS}_2 \times \Sigma_g$ with respective radii R_1 and R_2. They are related to the metric functions by

$$\mathrm{e}^{2V} \sim_0 v_2\, r^2, \qquad \mathrm{e}^{2(V-U)} \sim_0 R_2^2, \qquad v_2 = \frac{R_2}{R_1}. \tag{10.3.3}$$

Plugging these functions into (10.1.1a) gives

$$\mathrm{d}s^2 = -\frac{r^2}{R_1^2}\big(\mathrm{d}t + 2n\, H(\theta)\, \mathrm{d}\phi\big)^2 + \frac{R_1^2}{r^2}\, \mathrm{d}r^2 + R_2^2\, \mathrm{d}\Sigma_g^2 \tag{10.3.4}$$

which approaches $\mathrm{adS}_2 \times \Sigma_g$ after the rescaling

$$r \longrightarrow \epsilon r, \qquad t \longrightarrow t/\epsilon, \tag{10.3.5}$$

followed by $\epsilon \to 0$.

In order to find BPS solutions without NUT charge, Cacciatori and Klemm used an ansatz with two double roots [58]

$$\mathrm{e}^{V} = \frac{r^2}{R} - v \tag{10.3.6}$$

where R is the radius of the asymptotic adS_4 vacua, and $v > 0$ is fixed by the near-horizon geometry [84]. Hence the function V is completely fixed by the boundary conditions in the IR and in the UV. Solutions in this category include [46, 58]; in the symplectic frame where the gaugings are electric they have magnetic charges.

Chapter 11

Static BPS equations

We are looking for static $\frac{1}{4}$-BPS solutions of $N = 2$ matter-coupled gauged supergravity. As it is well known [57, 69], BPS equations imply the equations of motion for the metric and for the scalar fields, but not Maxwell equations which need to be solved separately.[1]

11.1 Ansatz

The ansatz for the metric and for the gauge fields are

$$\mathrm{d}s^2 = -\,\mathrm{e}^{2U}\mathrm{d}t^2 + \mathrm{e}^{-2U}\mathrm{d}r^2 + \mathrm{e}^{2(V-U)}\,\mathrm{d}\Sigma_g^2, \tag{11.1.1a}$$

$$A^\Lambda = \tilde{q}^\Lambda\,\mathrm{d}t - p^\Lambda F'(\theta)\mathrm{d}\phi. \tag{11.1.1b}$$

The functions $U, V, \tilde{q}$ and p depend only on r, while Σ_g is a Riemann surface of genus g (see appendix A.7) with metric

$$\mathrm{d}\Sigma_g^2 = \mathrm{d}\theta^2 + H'(\theta)^2\,\mathrm{d}\phi^2, \qquad H'(\theta) = \begin{cases} \sin\theta & \kappa = 1, \\ 1 & \kappa = 0, \\ \sinh\theta & \kappa = -1. \end{cases} \tag{11.1.2}$$

All scalars are function only on r

$$\tau^i = \tau^i(r), \qquad q^u = q^u(r). \tag{11.1.3}$$

We consider only abelian gaugings.

The magnetic field strength reads

$$G_\Lambda = R_{\Lambda\Sigma}F^\Sigma - I_{\Lambda\Sigma}\star F^\Sigma. \tag{11.1.4}$$

The electric and magnetic charges are given explicitly by

$$p^\Lambda = \frac{1}{4\pi}\int_{\Sigma_g} F^\Lambda, \tag{11.1.5a}$$

$$q_\Lambda = \frac{1}{4\pi}\int_{\Sigma_g} G_\Lambda = -\,\mathrm{e}^{2(V-U)}\,I_{\Lambda\Sigma}\tilde{q}'^\Sigma + \kappa\,R_{\Lambda\Sigma}p^\Sigma. \tag{11.1.5b}$$

The latter can be used for deriving an expression for $\tilde{q}'^\Lambda$

$$\tilde{q}'^\Lambda = \mathrm{e}^{2(U-V)}\,I^{\Lambda\Sigma}\big(R_{\Sigma\Delta}p^\Delta - q_\Sigma\big). \tag{11.1.6}$$

[1] In this section we follow the conventions of [62, 149].

The central and matter charges are[2]

$$\mathcal{Z} = \langle \mathcal{Q}, \mathcal{V} \rangle = p^\Lambda M_\Lambda - q_\Lambda L^\Lambda, \qquad \mathcal{Z}_i = \langle \mathcal{Q}, U_i \rangle \,. \tag{11.1.7}$$

Similarly one defines the prepotential charges

$$\mathcal{L}^x = \langle \mathcal{P}^x, \mathcal{V} \rangle = -P^x_\Lambda L^\Lambda, \qquad \mathcal{L}^x_i = \langle \mathcal{P}^x, U_i \rangle \,. \tag{11.1.8}$$

Another expression for the central charge is

$$\mathcal{Z} = L^\Lambda I_{\Lambda\Sigma} \big(\mathrm{e}^{2(V-U)} \tilde{q}^\Sigma + i\, p^\Sigma \big). \tag{11.1.9}$$

11.2 Equations

BPS equations for $N = 2$ matter-coupled gauged supergravity have been derived in [62, sec. 2.2, app. B] (see also [149, app. D]).

For deriving the equations one choose a frame where the gaugings are purely electric

$$P^{x\Lambda} = 0 \tag{11.2.1}$$

such that

$$\mathcal{L}^x = -P^x_\Lambda L^\Lambda. \tag{11.2.2}$$

The Killing spinor reads

$$\varepsilon_\alpha = \mathrm{e}^{U/2}\, \mathrm{e}^{i\psi/2}\, \varepsilon_{0\alpha} \tag{11.2.3a}$$

where $\varepsilon_{0\alpha}$ is a constant spinor satisfying the two projection conditions

$$\begin{aligned} \varepsilon_{0\alpha} &= i\, \gamma^0 \varepsilon_{\alpha\beta} \varepsilon_0^\beta, && (11.2.3b) \\ \varepsilon_{0\alpha} &= -p^\Lambda P^x_\Lambda\, \gamma^{01} \sigma^x{}_\alpha{}^\beta \varepsilon_{0\beta}. && (11.2.3c) \end{aligned}$$

Each projection halves the number of independent components. If $p^\Lambda = 0$ then the second projection is removed and one obtains 1/2-BPS solutions.

There are algebraic equations

$$\begin{aligned} (p^\Lambda P^x_\Lambda)^2 &= \kappa^2, && (11.2.4a) \\ p^\Lambda k^u_\Lambda &= 0, && (11.2.4b) \\ \mathrm{Re}(\, \mathrm{e}^{-i\psi} \mathcal{L}^x)\, p^\Lambda P^x_\Lambda &= -\, \mathrm{e}^{2(U-V)}\, \mathrm{Im}(\, \mathrm{e}^{-i\psi} \mathcal{Z}) && (11.2.4c) \end{aligned}$$

and differential equations

$$\begin{aligned} p'^\Lambda &= 0, && (11.2.4d) \\ \psi' &= -\mathcal{A}_r + 2\, p^\Lambda P^x_\Lambda\, \mathrm{e}^{-U}\, \mathrm{Re}(\, \mathrm{e}^{-i\psi} \mathcal{L}^x), && (11.2.4e) \\ (\, \mathrm{e}^U)' &= -p^\Lambda P^x_\Lambda\, \mathrm{Im}(\, \mathrm{e}^{-i\psi} \mathcal{L}^x) + \mathrm{e}^{2(U-V)}\, \mathrm{Re}(\, \mathrm{e}^{-i\psi} \mathcal{Z}), && (11.2.4f) \\ (\, \mathrm{e}^V)' &= -2\, \mathrm{e}^{V-U} p^\Lambda P^x_\Lambda\, \mathrm{Im}(\, \mathrm{e}^{-i\psi} \mathcal{L}^x), && (11.2.4g) \\ \tau'^i &= \mathrm{e}^{-U}\, \mathrm{e}^{i\psi} g^{i\bar{\jmath}} \big(\, \mathrm{e}^{2(U-V)} \mathrm{D}_{\bar{\jmath}} \mathcal{Z} - i\, p^\Lambda P^x_\Lambda\, \mathrm{D}_{\bar{\jmath}} \mathcal{L}^x \big), && (11.2.4h) \\ q'^u &= -2\, \mathrm{e}^{-U}\, h^{uv} \partial_v \big(p^\Lambda P^x_\Lambda\, \mathrm{Im}(\, \mathrm{e}^{-i\psi} \mathcal{L}^x) \big), && (11.2.4i) \\ q'_\Lambda &= 2\, \mathrm{e}^{-U}\, \mathrm{e}^{2(V-U)}\, h_{uv} k^u_\Lambda k^v_\Sigma\, \mathrm{Re}(\, \mathrm{e}^{-i\psi} L^\Sigma), && (11.2.4j) \end{aligned}$$

the primes denoting the radial derivative, and $\mathcal{A}_r$ is the composite U(1) connection. The equation (11.2.4a) corresponds to Dirac quantization condition (2.2.18) for the particular cases where the integer of the RHS is ± 1. The last equation (11.2.4j) corresponds to Maxwell

[2] There is a minus sign with respect to the notations of appendix A.6.

equation: the fact that its RHS is non-trivial implies that some electric charges will not be conserved (they correspond to massive vector fields).

The equations for the vector scalars can also be written in terms of $\mathcal{L}_i^x$ and $\mathcal{Z}_i$.

Combining the equations n_v (complex) equations for τ^i, the one for U and the one for ψ, one can obtain n_v+1 complex equations for the sections [84]

$$\begin{aligned} 2\,\mathrm{e}^{2U}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}L^\Lambda)\big) &= -\,\mathrm{e}^{2(U-V)}p^\Lambda + p^\Delta P^x_\Delta\, I^{\Lambda\Sigma}P^x_\Sigma \\ &\quad - 8\,p^\Sigma P^x_\Sigma\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}^x)\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Lambda), \end{aligned} \tag{11.2.5a}$$

$$2\,\partial_r\big(\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Lambda)\big) = \mathrm{e}^{2(U-V)}I^{\Lambda\Sigma}R_{\Sigma\Delta}p^\Delta - I^{\Lambda\Sigma}q_\Sigma. \tag{11.2.5b}$$

One can also derive equations for M_Λ

$$\begin{aligned} 2\,\mathrm{e}^{2U}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}M_\Lambda)\big) &= -\,\mathrm{e}^{2(U-V)}q_\Lambda + p^\Delta P^x_\Delta\, R_{\Lambda\Sigma}I^{\Sigma\Xi}P^x_\Xi \\ &\quad - 8\,p^\Sigma P^x_\Sigma\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}^x)\,\mathrm{Re}(\mathrm{e}^{-i\psi}M_\Lambda), \end{aligned} \tag{11.2.6a}$$

$$2\,\partial_r\big(\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}M_\Lambda)\big) = \mathrm{e}^{2(U-V)}\Big(R_{\Lambda\Sigma}I^{\Sigma\Delta}\big(R_{\Delta\Xi}p^\Xi - q_\Sigma\big) + I_{\Lambda\Sigma}p^\Sigma\Big) + p^\Sigma P^x_\Sigma\, P^x_\Lambda \tag{11.2.6b}$$

which are not independent.

One finds that

$$\tilde{q}^\Lambda = 2\,\mathrm{e}^U\,\mathrm{Re}(\mathrm{e}^{-i\psi}L^\Lambda). \tag{11.2.7}$$

Let's define

$$P^x_p = p^\Lambda P^x_\Lambda. \tag{11.2.8}$$

Then if $p^\Lambda \neq 0$ one can use a local SU(2) transformation in order to set [149, app. D]

$$P^1_p = P^2_p = 0, \tag{11.2.9}$$

which is a weaker condition than setting $P^1_\Lambda = P^2_\Lambda = 0$ as was done in [62]. This is possible only because p^Λ is constant. Then all remaining P^1_Λ and P^2_Λ in the BPS equations disappear, and the above equations can be rewritten uniquely in terms of $P^\Lambda \equiv P^3_\Lambda$ (this should not be confound with the momentum map of the SK gauged symmetries), and similarly we write $\mathcal{L} \equiv \mathcal{L}^3$.

Then the Dirac condition can be rewritten as

$$p^\Lambda P_\Lambda = \epsilon_D\,\kappa \tag{11.2.10}$$

with $\epsilon_D = \pm 1$ (a common choice is $\epsilon_D = -1$ [52, 149]). Replacing this in all equations one sees that κ only appears in the Dirac condition, meaning that solutions are independent of the curvature of the horizon, but regularity does depend on it [46, p. 6].

If $p^\Lambda = 0$ then the Dirac condition should not be imposed.

11.3 Symplectic extension

In this section we introduce magnetic gaugings by performing a symplectic transformation (see section 2.5). Most parts of the equations (11.2.4) are already written in a symplectic form.

One can see that $\tilde{q}'^\Lambda$ from (11.1.6) corresponds to the first row of the $-\Omega\mathcal{M}\mathcal{Q}$, where $\mathcal{M}$ was defined in (4.4.2). Then symplectic equations can be obtained from the replacement

$$\tilde{q}'^\Lambda = -\,\mathrm{e}^{2(U-V)}(\Omega\mathcal{M}\mathcal{Q})^\Lambda. \tag{11.3.1}$$

Similarly terms involving the electromagnetic charges and the gaugings, such as $I^{-1}\mathcal{P}$, can be replaced (missing terms due to the fact we had $P^\Lambda = 0$ can be guessed).

We now list the symplectic algebraic equations

$$\langle \mathcal{Q}, \mathcal{P}\rangle = \epsilon_D\,\kappa, \tag{11.3.2a}$$

$$\langle \mathcal{Q}, \mathcal{K}^u\rangle = 0, \tag{11.3.2b}$$

$$\epsilon_D\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}) = -\,\mathrm{e}^{2(U-V)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{Z}) \tag{11.3.2c}$$

and differential equations

$$(\,\mathrm{e}^U)' = -\epsilon_D\ \mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}) + \,\mathrm{e}^{2(U-V)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{Z}), \tag{11.3.2d}$$

$$(\,\mathrm{e}^V)' = -2\epsilon_D\ \mathrm{e}^{V-U}\ \mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}), \tag{11.3.2e}$$

$$\tau'^i = \,\mathrm{e}^{-U}\,\mathrm{e}^{i\psi} g^{i\bar{\jmath}}\big(\,\mathrm{e}^{2(U-V)}\mathrm{D}_{\bar{\jmath}}\mathcal{Z} - i\,\epsilon_D\,\mathrm{D}_{\bar{\jmath}}\mathcal{L}\big), \tag{11.3.2f}$$

$$q'^u = -2\epsilon_D\ \mathrm{e}^{-U}\ h^{uv}\partial_v\big(\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L})\big), \tag{11.3.2g}$$

$$\mathcal{Q}' = 2\,\mathrm{e}^{-U}\,\mathrm{e}^{2(V-U)}\,h_{uv}\mathcal{K}^u\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\,\langle\mathcal{V},\mathcal{K}^u\rangle), \tag{11.3.2h}$$

$$\psi' = -\mathcal{A}_r + 2\epsilon_D\ \mathrm{e}^{-U}\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}). \tag{11.3.2i}$$

We note that the symplectic Maxwell equations correctly reduce to (11.2.4d) and (11.2.4j) in a symplectic frame since $k^{u\Lambda} = 0$.

Instead of working with the real and imaginary parts of $\mathrm{e}^{-i\psi}\,\mathrm{e}^{-U}\mathcal{V}$ as independent equations as in (11.2.5), one can combine (11.2.5a) and (11.2.6a) in the symplectic equation

$$2\,\mathrm{e}^{2U}\partial_r\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{-U}\mathcal{V}) = -\,\mathrm{e}^{2(U-V)}\mathcal{Q} + \epsilon_D\,\Omega\mathcal{M}\mathcal{P} - 8\,\epsilon_D\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}). \tag{11.3.3a}$$

We stress that this equation is totally equivalent to (11.3.2d), (11.3.2f) and (11.3.2i). Then the remaining equations are combined as

$$2\,\partial_r\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{U}\mathcal{V}) = -\,\mathrm{e}^{2(U-V)}\,\Omega\mathcal{M}\mathcal{Q} + \epsilon_D\,\mathcal{P} \tag{11.3.3b}$$

and they are redundant since $\mathrm{Im}\,\mathcal{V}$ already exhausts the $2n_v + 2$ variables τ^i, ψ and U. Here it is useful to have the equations (11.2.6b) for M_Λ because the second term is not visible in (11.2.5b).

For a future purpose we want to obtain another form of (11.3.3a). Multiplying by $\mathrm{e}^{2(V-U)}$, we want to rewrite the LHS with a factor e^V inside the derivative

$$\begin{aligned}\mathrm{e}^{2V}\partial_r\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{-U}\mathcal{V}) &= \mathrm{e}^V\partial_r\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{V-U}\mathcal{V}) - \,\mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\,\partial_r\,\mathrm{e}^V \\ &= \mathrm{e}^V\partial_r\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{V-U}\mathcal{V}) + 2\,\epsilon_D\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V}),\end{aligned}$$

and this combines with the RHS as

$$\begin{aligned}2\,\mathrm{e}^V\partial_r\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\,\mathrm{e}^{V-U}\mathcal{V}) = -\mathcal{Q} + \epsilon_D\ \mathrm{e}^{2(V-U)}\Big(&\Omega\mathcal{M}\mathcal{P} - 8\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ &- 4\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\ \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V})\Big).\end{aligned} \tag{11.3.4}$$

Equation (11.3.2c) can be directly integrated to get the phase in terms of $\mathcal{L}$ and $\mathcal{Z}$ [52, eq. (2.39)]

$$\mathrm{e}^{2i\psi} = \frac{\mathcal{Z} - i\,\epsilon_D\ \mathrm{e}^{2(V-U)}\mathcal{L}}{\bar{\mathcal{Z}} + i\,\epsilon_D\ \mathrm{e}^{2(V-U)}\bar{\mathcal{L}}}. \tag{11.3.5}$$

This is obtained by writing explicitly the real and imaginary parts in order to get a second order equation for $\mathrm{e}^{i\psi}$, which then can be solved.

11.4 Symmetric $\mathcal{M}_v$ with FI gaugings

In this section we consider only FI gaugings such that $\mathcal{P} = \mathrm{cst}$. A seminal approach developed in [85] allows to greatly simplify the equations and this lead to complete analytical solution of a full 1/4-BPS black hole in [84]. The idea is to rewrite the equations in terms of the quartic invariant (and its gradient) and to exploit the power of special geometry.

First let's define a rescaled section

$$\widetilde{\mathcal{V}} = \mathrm{e}^{V-U}\,\mathrm{e}^{-i\psi}\,\mathcal{V}. \tag{11.4.1}$$

The equation (11.3.4) can be simplified using relation (E.1.2c)

$$2\,\mathrm{e}^{V}\partial_r\,\mathrm{Im}\,\widetilde{\mathcal{V}} = -\mathcal{Q} + \epsilon_D\,I_4'(\mathcal{P}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}). \tag{11.4.2}$$

In these terms equation (11.3.2e) reads

$$(\,\mathrm{e}^{V})' = -2\epsilon_D \left\langle \widetilde{\mathcal{V}}, \mathcal{P} \right\rangle, \tag{11.4.3}$$

while the constraint (11.3.2c) becomes

$$2\epsilon_D\,I_4(\mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{P}) = \left\langle \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{Q} \right\rangle \tag{11.4.4}$$

using (E.1.2b) to replace $\mathrm{Re}\,\widetilde{\mathcal{V}}$

$$\mathrm{Re}\,\widetilde{\mathcal{V}} = 2\,\mathrm{e}^{2(U-V)}\,I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}). \tag{11.4.5}$$

A more convenient form for this equation can be achieved by writing

$$I_4(\mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{P}) = \left\langle \mathrm{Im}\,\widetilde{\mathcal{V}}, I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{P}) \right\rangle \tag{11.4.6}$$

and by inserting (11.4.2)

$$\mathrm{e}^{V} \left\langle \mathrm{Im}\,\widetilde{\mathcal{V}}, \partial_r\,\mathrm{Im}\,\widetilde{\mathcal{V}} \right\rangle = \left\langle \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{Q} \right\rangle. \tag{11.4.7}$$

Let's summarize the equations that have been obtained

$$2\,\mathrm{e}^{V}\partial_r\,\mathrm{Im}\,\widetilde{\mathcal{V}} = -\mathcal{Q} + \epsilon_D\,I_4'(\mathcal{P}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}), \tag{11.4.8a}$$

$$(\,\mathrm{e}^{V})' = -2\epsilon_D \left\langle \widetilde{\mathcal{V}}, \mathcal{P} \right\rangle, \tag{11.4.8b}$$

$$\mathrm{e}^{V} \left\langle \mathrm{Im}\,\widetilde{\mathcal{V}}, \partial_r\,\mathrm{Im}\,\widetilde{\mathcal{V}} \right\rangle = \left\langle \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{Q} \right\rangle, \tag{11.4.8c}$$

$$\langle \mathcal{Q}, \mathcal{P} \rangle = \epsilon_D\,\kappa. \tag{11.4.8d}$$

The main advantage of these equations is that they do not involve $\mathrm{Re}\,\widetilde{\mathcal{V}}$, U or ψ, they only contain $\mathrm{Im}\,\widetilde{\mathcal{V}}$ and V (as dynamical objects). Another useful point is the removal of the matrix $\mathcal{M}$ whose explicit form is involved in the general case. All other objects can be deduced from them, for example one can obtain $\mathrm{Re}\,\widetilde{\mathcal{V}}$ from (11.4.5).

Chapter 12

Static BPS solutions

We will focus on solutions that are black holes interpolating between a (magnetic) adS_4 (of radius R) for $r \to \infty$ and a topological horizon of Bertotti–Robinson type $\mathrm{adS}_2 \times \Sigma_g$ (with respective radius R_1 and R_2) for $r \to 0$. Both these spacetimes are also BPS solutions and can be studied separately, and for this reason the full black hole can be seen as a soliton (or a domain wall) [84].

12.1 $N = 2$ adS_4

An anti-de Sitter vacua is characterized by constant scalars and vanishing charges

$$\tau^i(r) = \tau^i_0, \qquad q^u(r) = q^u_0, \qquad \mathcal{Q} = 0, \tag{12.1.1}$$

which implies in particular $\mathcal{Z} = 0$. The metric functions are

$$\mathrm{e}^U = \frac{r}{R}, \qquad \mathrm{e}^V = \frac{r^2}{R} \tag{12.1.2}$$

giving the metric

$$\mathrm{d}s^2 = -\frac{r^2}{R^2}\,\mathrm{d}t^2 + \frac{R_1^2}{r^2}\,\mathrm{d}r^2 + \frac{r^2}{R^2}\,\mathrm{d}\Sigma_g^2 \tag{12.1.3}$$

As discussed in the previous section vanishing charges imply that the solution is 1/2-BPS. Moreover in the case of adS_4 vacua there is a special enhancement of supersymmetry which increases it to a full BPS solution. Moreover one cannot use the trick of the SU(2) rotation to set $\mathcal{P}^1 = \mathcal{P}^2 = 0$.

Typically the asymptotic geometry of a 1/4-BPS black hole will be a madS vacua. There is a one-to-one relationship between adS and madS vacua.

From (11.3.2f) one gets the equation

$$\mathcal{L}_i^x = \langle U_i, \mathcal{P}^x \rangle = 0. \tag{12.1.4}$$

In a frame where the gaugings are purely electric, this equation is equivalent to

$$P^x_\Lambda f_i^\Lambda = 0. \tag{12.1.5}$$

In the space spanned by the $n_v + 1$ directions of Λ, f_i^Λ represents n_v vectors indexed by i. Then the previous equation implies that, for fixed x, P^x_Λ is orthogonal to these n_v vectors and thus

$$P^x_\Lambda = c^x(q^u)\, P_\Lambda. \tag{12.1.6}$$

Then a local SU(2) rotation can be used to set

$$c^1 = c^2 = 0. \tag{12.1.7}$$

Note that the latter equations must be enforced as they are not a generic consequence of the theory. We then denote $\mathcal{P} \equiv \mathcal{P}^3$ and $\mathcal{L} \equiv \mathcal{L}^3$ as usual.

The BPS equations are

$$\operatorname{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) = 0, \tag{12.1.8a}$$

$$\operatorname{Im}(\mathrm{e}^{-i\psi}\mathcal{L}) = \frac{1}{R}, \tag{12.1.8b}$$

$$\mathcal{L}_i = 0, \tag{12.1.8c}$$

$$\psi' = 0, \tag{12.1.8d}$$

$$\langle \mathcal{V}, \mathcal{K}^u \rangle = 0. \tag{12.1.8e}$$

From (11.3.2h) one obtains

$$\operatorname{Re}(\mathrm{e}^{-i\psi}\langle \mathcal{V}, \mathcal{K}^u \rangle) = 0, \tag{12.1.9}$$

while the derivative in (11.3.2g) can be used to replace the prepotential by the Killing vector

$$\operatorname{Im}(\mathrm{e}^{-i\psi}\langle \mathcal{V}, \mathcal{K}^u \rangle) = 0. \tag{12.1.10}$$

Combining both equations gives (12.1.8e).

The equations for the sections are

$$2\operatorname{Re}(\mathrm{e}^{-i\psi}\mathcal{V}) = R\,\mathcal{P}, \qquad 2\operatorname{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) = R\,\Omega\mathcal{M}\mathcal{P}. \tag{12.1.11}$$

Using the matrix $\mathcal{C}$ defined in (4.4.15) this can be rewritten as

$$\mathrm{e}^{-i\psi}\mathcal{V} = i\,R\,\Omega\mathcal{C}\mathcal{P}. \tag{12.1.12}$$

All the equations but the last one in (12.1.8) do not involve the Killing vectors. Hence a strategy to solve these equations is to consider $\mathcal{P}$ as a constant (which is the case for the FI gaugings $\mathcal{P} \to \mathcal{G}$ and $n_h = 0$) and to solve for the vector scalars in terms of $\mathcal{P}$. Then the remaining equation (12.1.8e) can be used to solve for the hyperscalars which can be replaced at the end in the vector scalars.

Following this strategy we first analyse the equations for the vector scalar sector [46, sec. 3]. Equation (12.1.8d) means that the phase is constant

$$\psi(r) = \psi_0. \tag{12.1.13}$$

We rewrite (12.1.8b) as

$$\mathcal{L} = \frac{i}{R}\,\mathrm{e}^{i\psi_0}. \tag{12.1.14}$$

Because of (12.1.8c) the prepotentials have components only in the direction of $\mathcal{V}$ and its conjugate

$$\mathcal{P} = -2\operatorname{Im}(\bar{\mathcal{L}}\mathcal{V}). \tag{12.1.15}$$

Note that these equations are identical to those of the $\mathrm{adS}_2 \times S^2$ near-horizon in ungauged $N = 2$ supergravity, with the replacement $\mathcal{P} \to \mathcal{Q}$, which can be solved explicitly in some cases (such as symmetric cubic $\mathcal{M}_v$) [232]. The value for the phase is taken to be

$$\psi_0 = -\frac{\pi}{2} \tag{12.1.16}$$

which implies

$$\mathcal{L} = \frac{1}{R}, \tag{12.1.17}$$

and also

$$\mathcal{P} = -\frac{2}{R}\operatorname{Im}\mathcal{V}. \tag{12.1.18}$$

These equations are consistent with (12.1.11).

Let's turn to the last equation (12.1.8e)

$$\langle \mathcal{V}, \mathcal{K}^u \rangle = 0 \tag{12.1.19}$$

following the analysis of [149, sec. 2.2].

First we want to clarify this equation. Using the results of section 8.3, the spin connection ω^x is invariant under symmetry transformation generated by k only up to an SU(2) transformation (we consider only the electric frame here)

$$\mathcal{L}_k \omega^x = \nabla W_k^x \tag{12.1.20}$$

where W_k^x is an SU(2) vector called the compensator. This allows to relate directly the Killing vector and prepotential

$$P^x = k^u \omega_u^x + W^x. \tag{12.1.21}$$

Contracting (12.1.8e) with ω_u^x and plugging this last result gives

$$\mathrm{e}^{-i\psi}\mathcal{L} - \mathrm{e}^{-i\psi} \langle \mathcal{V}, \mathcal{W} \rangle = 0. \tag{12.1.22}$$

If the compensator vanishes $\mathcal{W} = 0$ one obtains a singular solution since $\mathcal{L} = 0$ implies $R \to \infty$. Then a necessary condition for having a $N = 2$ adS_4 vacua is that at least one isometry with a non-trivial compensator is gauged [48, 50]. In the case of special quaternionic manifold, isometries with compensators are not generic as only the isometries inherited from the base special Kähler space and the hidden symmetries have compensators (see section 9).

It may seem that (12.1.8e) are too many equations since there are $2n_h$ equations ($\mathcal{V}$ being complex) for the n_h variables q^u. But in fact the imaginary part is already implied by (12.1.18)

$$\langle \mathrm{Im}\, \mathcal{V}, \mathcal{K}^u \rangle \sim \langle \mathcal{P}, \mathcal{K}^u \rangle = 0 \tag{12.1.23}$$

where the last equality follows from the locality constraints (2.5.7). Then the only equations that we need to solve are

$$\langle \mathrm{Re}\, \mathcal{V}, \mathcal{K}^u \rangle = 0. \tag{12.1.24}$$

We restrict ourselves to the case of symmetric very special Kähler manifold (section 6.3). Using the relation (6.3.17)

$$\mathrm{Re}\, \mathcal{V} = -\frac{1}{8} I_4'(\mathrm{Im}\, \mathcal{V}) \tag{12.1.25}$$

the previous equation can be rewritten as

$$\langle I_4'(\mathrm{Im}\, \mathcal{V}), \mathcal{K}^u \rangle = I_4(\mathcal{K}^u, \mathrm{Im}\, \mathcal{V}, \mathrm{Im}\, \mathcal{V}, \mathrm{Im}\, \mathcal{V}) = 0 \tag{12.1.26}$$

and then as

$$I_4(\mathcal{K}^u, \mathcal{P}, \mathcal{P}, \mathcal{P}) \sim \nabla^u I_4(\mathcal{P}) = 0 \tag{12.1.27}$$

thanks to (12.1.15).

As a summary the equations to solve for are

$$\mathcal{P} = -2\, \mathrm{Im}(\bar{\mathcal{L}}\mathcal{V}), \tag{12.1.28a}$$

$$\mathcal{L} = \frac{1}{R}, \tag{12.1.28b}$$

$$0 = \nabla^u I_4(\mathcal{P}). \tag{12.1.28c}$$

The first two equations in the case of FI gaugings were explicitly solved in some cases in [46].

12.2 Near-horizon $\mathrm{adS}_2 \times \Sigma_g$

These equations have been studied with $n_h = 0$ and FI gaugings in [58, sec. 4, 52, sec. 3], and further in [53] (see also [46, sec. 5]). For $n_h \neq 0$ they were studied in the electric frame in [62, sec. 2.3] and in general in [149, sec. 2.3].

There is a supersymmetry enhancement at the horizon because there are two extra superconformal charges [149, p. 6].

Denoting the horizon radius by r_h and by r_Λ the radius where the scalars τ^i vanish, the solution is regular only if $r_h > r_\Lambda$ for all Λ [46, p. 15].

Scalars and charges are constant for near-horizon geometries

$$\tau^i(r) = \tau^i_0, \qquad q^u(r) = q^u_0, \qquad \mathcal{Q} = \mathrm{cst}. \tag{12.2.1}$$

The metric functions are

$$\mathrm{e}^U = \frac{r}{R_1}, \qquad \mathrm{e}^V = \frac{R_2}{R_1}\, r \tag{12.2.2}$$

giving the metric

$$\mathrm{d}s^2 = -\frac{r^2}{R_1^2}\,\mathrm{d}t^2 + \frac{R_1^2}{r^2}\,\mathrm{d}r^2 + R_2^2\,\mathrm{d}\Sigma_g^2 \tag{12.2.3}$$

The BPS equations are

$$\langle \mathcal{Q}, \mathcal{P} \rangle = \epsilon_D\, \kappa, \tag{12.2.4a}$$
$$\mathrm{Im}(\,\mathrm{e}^{-i\psi} \mathcal{Z}) = \epsilon_D\, R_2^2\, \mathrm{Re}(\,\mathrm{e}^{-i\psi} \mathcal{L}), \tag{12.2.4b}$$
$$\mathrm{Re}(\,\mathrm{e}^{-i\psi} \mathcal{Z}) = \frac{R_2^2}{2R_1}, \tag{12.2.4c}$$
$$\epsilon_D\, \mathrm{Im}(\,\mathrm{e}^{-i\psi} \mathcal{L}) = -\frac{1}{2R_1}, \tag{12.2.4d}$$
$$\mathcal{Z}_i = i\, \epsilon_D\, R_2^2\, \mathcal{L}_i, \tag{12.2.4e}$$
$$\psi' = 2\epsilon_D\, \frac{R_1}{r}\, \mathrm{Re}(\,\mathrm{e}^{-i\psi} \mathcal{L}), \tag{12.2.4f}$$
$$\langle \mathcal{Q}, \mathcal{K}^u \rangle = 0, \tag{12.2.4g}$$
$$\langle \mathcal{V}, \mathcal{K}^u \rangle = 0. \tag{12.2.4h}$$

We can adopt the same strategy as in the previous section: all equations except the last two do not contain the Killing vectors, such that they can be solved as if $\mathcal{P}$ was constant, giving a solution for the vector scalars in terms of the charges, the gaugings and the hyperscalars

$$\tau^i = \tau^i(\mathcal{P}, \mathcal{Q}, q^u). \tag{12.2.5}$$

Then the remaining equations can be used to solve for the hyperscalars in terms of the charges and the gaugings

$$q^u = q^u(\mathcal{P}, \mathcal{Q}), \quad \Longrightarrow \quad \tau^i = \tau^i(\mathcal{P}, \mathcal{Q}). \tag{12.2.6}$$

From the equations one can also write

$$\mathrm{Re}(\,\mathrm{e}^{-i\psi} \mathcal{Z}) = -\epsilon_D\, R_2^2\, \mathrm{Im}(\,\mathrm{e}^{-i\psi} \mathcal{L}). \tag{12.2.7}$$

Combining this with (12.2.4b) gives

$$\mathcal{Z} = i\, \epsilon_D\, R_2^2\, \mathcal{L}. \tag{12.2.8}$$

Since R_2^2 is real this means that the phases of $\mathcal{Z}$ and $\mathcal{L}$ differ by $\pi/2$ [52, p. 12]. Plugging the relation (12.2.8) into (11.3.5) implies that ψ is a multiple of π

$$\psi(r) = \pi. \tag{12.2.9}$$

Another way to see this is by taking the imaginary part of (12.2.8): this is consistent with (12.2.4b) only if $\psi = \pi$. Then inserting this result into (12.2.4f) gives

$$\operatorname{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) = 0 \Longrightarrow \operatorname{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) = 0, \tag{12.2.10}$$

and as a consequence

$$\mathcal{Z} = \frac{R_2^2}{2R_1}, \qquad \mathcal{L} = -\epsilon_D\,\frac{i}{2R_1}. \tag{12.2.11}$$

Instead of working with (12.2.4e) it is easier to work with the sections. Using the previous elements one has

$$\frac{2R_2^2}{R_1}\operatorname{Im}\mathcal{V} = \mathcal{Q} - \epsilon_D\,R_2^2\,\Omega\mathcal{M}\mathcal{P}, \tag{12.2.12a}$$

$$\frac{2R_2^2}{R_1}\operatorname{Re}\mathcal{V} = \Omega\mathcal{M}\mathcal{Q} + \epsilon_D\,R_2^2\,\mathcal{P}. \tag{12.2.12b}$$

Adding the two equations gives

$$\mathcal{V} = i\,\frac{R_1}{2R_2^2}\,\Omega\mathcal{C}(\mathcal{Q} + \epsilon_D\,R_2^2\,\Omega\mathcal{M}\mathcal{P}) \tag{12.2.13}$$

where $\mathcal{C}$ was defined in (4.4.15). Note the similarity with (12.1.12).

Another way to derive the equation for the section is to contract (12.2.4e) with $\Omega\mathcal{M}$. Using the relation (4.4.7)

$$\Omega\mathcal{M}U_i = -iU_i \tag{12.2.14}$$

one obtains

$$\begin{aligned} 0 &= \langle U_i, \mathcal{Q}\rangle - i\,\epsilon_D\,R_2^2\,\langle U_i, \mathcal{P}\rangle = \langle U_i, \mathcal{Q}\rangle + \epsilon_D\,R_2^2\,\langle \Omega\mathcal{M}U_i, \mathcal{P}\rangle \\ &= \langle U_i, \mathcal{Q}\rangle + \epsilon_D\,R_2^2\,\langle U_i, \Omega\mathcal{M}\mathcal{P}\rangle = \big\langle U_i, \mathcal{Q} + \epsilon_D\,R_2^2\,\Omega\mathcal{M}\mathcal{P}\big\rangle \end{aligned}$$

because of (4.4.11). As a consequence the quantity $\mathcal{Q} + \epsilon_D\,R_2^2\,\Omega\mathcal{M}\mathcal{P}$ has no components along the direction U_i in the basis $(\mathcal{V}, U_i)$ such that

$$\mathcal{Q} + \epsilon_D\,R_2^2\,\Omega\mathcal{M}\mathcal{P} = -2\operatorname{Im}(\big\langle\bar{\mathcal{V}}, \mathcal{Q} + \epsilon_D\,R_2^2\,\Omega\mathcal{M}\mathcal{P}\big\rangle\,\mathcal{V}). \tag{12.2.15}$$

Now we can introduce the central charge and after using the relation (12.2.8) one obtains

$$\mathcal{Q} + \epsilon_D\,R_2^2\,\Omega\mathcal{M}\mathcal{P} = -4\operatorname{Im}(\bar{\mathcal{Z}}\,\mathcal{V}). \tag{12.2.16}$$

This is equivalent to (12.2.12a) once $\mathcal{Z}$ is replaced by its value.

Contracting (12.2.16) with $\mathcal{P}$ gives

$$\langle\mathcal{Q}, \mathcal{P}\rangle + \epsilon_D\,R_2^2\,\langle\Omega\mathcal{M}\mathcal{P}, \mathcal{P}\rangle = -4\operatorname{Im}(\bar{\mathcal{Z}}\,\mathcal{L}), \tag{12.2.17}$$

while with $\mathcal{Q}$ one gets

$$\langle\Omega\mathcal{M}\mathcal{P}, \mathcal{Q}\rangle = 0. \tag{12.2.18}$$

Then using the relation (12.2.8) modifies the first equation to

$$\langle\mathcal{Q}, \mathcal{P}\rangle - \epsilon_D\,R_2^2\,\mathcal{P}\mathcal{M}\mathcal{P} = 4\epsilon_D\,R_2^2\,|\mathcal{L}|^2\,, \tag{12.2.19}$$

and using (4.4.23) one obtains [52, p. 13]

$$\frac{\epsilon_D}{R_2^2}\,\langle\mathcal{Q}, \mathcal{P}\rangle = -\mathcal{P}\mathcal{M}(\mathcal{F})\mathcal{P} = 2(|\mathcal{L}|^2 - |\mathcal{L}_i|^2). \tag{12.2.20}$$

A similar relation for $\mathcal{Z}$ follows directly

$$\epsilon_D\,R_2^2\,\langle\mathcal{Q}, \mathcal{P}\rangle = -\mathcal{Q}\mathcal{M}(\mathcal{F})\mathcal{Q} = 2(|\mathcal{Z}|^2 - |\mathcal{Z}_i|^2). \tag{12.2.21}$$

These formulas are helpful for understanding why it is not possible to find asymptotically adS_4 solutions with spherical horizon and constant scalars: the adS_4 vacua has $\mathcal{L}_i = 0$ from (12.1.8b), and the previous equations give

$$R_2^2 = -\frac{\epsilon_D}{2\,|\mathcal{L}|^2}\,\langle \mathcal{Q}, \mathcal{P} \rangle = -\frac{\kappa}{2\,|\mathcal{L}|^2}. \tag{12.2.22}$$

The latter is positive only for $\kappa = -1$.

As a summary the equations to solve are

$$\mathcal{Q} + i\epsilon_D\, R_2^2\, \Omega \mathcal{M} \mathcal{P} - = -4\,\mathrm{Im}(\bar{\mathcal{Z}}\,\mathcal{V}), \tag{12.2.23a}$$

$$\mathcal{Z} = \frac{R_2^2}{2R_1}, \tag{12.2.23b}$$

$$\langle \mathcal{Q}, \mathcal{P} \rangle = \epsilon_D\,\kappa, \tag{12.2.23c}$$

$$\langle \mathcal{Q}, \mathcal{K}^u \rangle = 0, \tag{12.2.23d}$$

$$\langle \mathcal{V}, \mathcal{K}^u \rangle = 0. \tag{12.2.23e}$$

The first two equations were solved for FI gaugings with cubic $\mathcal{M}_v$ explicitly in the case of symmetric spaces and implicitly otherwise in [62]. Note that for $\mathcal{P} = 0$ it reduces to the attractor equations of ungauged supergravity.

After some work one can see that the vector scalar equations imply [149, p. 7, 53]

$$I_4(\mathcal{Q} - iR_2^2\,\mathcal{P}) = 0. \tag{12.2.24}$$

In particular this gives the radius of Σ_g (and hence the entropy)

$$R_2^4 = I_4(\mathrm{Im}\,\widetilde{\mathcal{V}}) = \frac{1}{I_4(\mathcal{P})}\left(I_4(\mathcal{Q},\mathcal{Q},\mathcal{P},\mathcal{P}) \pm \sqrt{I_4(\mathcal{Q},\mathcal{Q},\mathcal{P},\mathcal{P})^2 - I_4(\mathcal{Q}) I_4(\mathcal{P})} \right). \tag{12.2.25}$$

At this point $\mathcal{P}$ depends on q^u, which needs to be solved for using the other equations.

The entropy is

$$S = \pi R_2^2 = \pi \sqrt{I_4(\mathrm{Im}\,\widetilde{\mathcal{V}})}. \tag{12.2.26}$$

One finds also the constraint

$$\begin{aligned} 0 = 4\, I_4(\mathcal{P}) I_4(\mathcal{P},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2 + 4\, I_4(\mathcal{Q}) I_4(\mathcal{Q},\mathcal{P},\mathcal{P},\mathcal{P})^2 \\ - I_4(\mathcal{P},\mathcal{Q},\mathcal{Q},\mathcal{Q}) I_4(\mathcal{P},\mathcal{P},\mathcal{Q},\mathcal{Q}) I_4(\mathcal{Q},\mathcal{P},\mathcal{P},\mathcal{P}). \end{aligned} \tag{12.2.27}$$

12.3 General solution

A general solution to the set of BPS equations for FI gauged supergravity (11.4.8) was provided in [84]. We will only give the most important details of the analysis.

As explained in section 10.3, BPS static black holes are extremal and we are considering near-horizon geometry $\mathrm{adS}_2 \times \Sigma_g$. As a consequence the ansatz for e^V is

$$\mathrm{e}^{2V} = r^2 (v_4 r^2 + v_3 r + v_2). \tag{12.3.1}$$

This root structure and the degenerate double extremal case are the only ones allowed for this type of black holes [150, p. 11].

The ansatz for $\mathrm{Im}\,\widetilde{\mathcal{V}}$ is more involved

$$\mathrm{Im}\,\widetilde{\mathcal{V}} = \mathrm{e}^{-V}(A_3 r^3 + A_2 r^2 + A_1 r) \tag{12.3.2}$$

where the A_i are symplectic vectors.

The next steps is to expand each of the equations (11.4.8) in powers of r and to identify the coefficients. In principle one should be able to find the constraint (12.2.27) from the analysis, but this did not appear feasible, and for this reason it is used as an input for simplifying the equations, using it for replacing $I_4(\mathcal{P}, \mathcal{P}, \mathcal{Q}, \mathcal{Q})$.

Note also that the system contains much more equations than variables, and there is a lot of redundancy. In particular (11.4.8b) implies the following relations

$$v_{i+1} = \frac{4}{i+1} \langle \mathcal{P}, A_i \rangle \,. \tag{12.3.3}$$

The UV boundary condition can be read from (11.4.8a) and gives

$$A_3 = \frac{I'_4(\mathcal{P})}{4\sqrt{I_4(\mathcal{P})}}, \qquad v_4 = \frac{1}{R^2_{\text{adS}}} = \sqrt{I_4(\mathcal{P})}. \tag{12.3.4}$$

The overall normalization was not fixed and it was determined by comparison with [46].

The solution for A_2 and A_3 is found by expanding these vectors on the basis (E.1.1), and it can be found that only third order terms are non-vanishing

$$A_i = a_{i1}\, I'_4(\mathcal{P}) + a_{i2}\, I'_4(\mathcal{P}, \mathcal{P}, \mathcal{Q}) + a_{i3}\, I'_4(\mathcal{P}, \mathcal{Q}, \mathcal{Q}) + a_{i4}\, I'_4(\mathcal{Q}). \tag{12.3.5}$$

Explicit formulas can be found in [84, sec. 3], and one needs to use the identities of appendix E.1.

The real part of $\widetilde{\mathcal{V}}$ can be found from

$$\operatorname{Re} \widetilde{\mathcal{V}} = 2\, \mathrm{e}^{2(U-V)}\, I'_4(\operatorname{Im} \widetilde{\mathcal{V}}), \tag{12.3.6}$$

then the function U from

$$I_4(\operatorname{Im} \widetilde{\mathcal{V}}) = \frac{1}{16}\, \mathrm{e}^{4(V-U)}, \tag{12.3.7}$$

and finally the physical scalars from

$$\tau^i = \frac{\widetilde{L}^i}{\widetilde{L}^0} \tag{12.3.8}$$

(the overall rescaling are cancelling).

The solution has $2n_v$ charges since $\mathcal{Q}$ has $2n_v + 2$ components and there are two constraints, the Dirac condition (11.4.8d) and the constraint (12.2.27). This is the maximum number from the near-horizon analysis from [53].

As a conclusion, it is much easier to find a general solution using a symplectic formalism where the underlying structure simplifies the computations rather than choosing a particular model with electric gaugings.

12.4 Examples

In this section we work through two examples of gauged supergravity theories which arise from M-theory and which have $\mathcal{M}_h = \mathrm{G}_{2(2)}/\mathrm{SO}(4)$, reproducing the $N = 2$ adS_4 vacuum and then look at black hole horizons. It is well known that when a FI-gauged supergravity theory (i.e. with $n_h = 0$ and $\mathrm{U}(1)_R$ gauging) admits an $N = 2$ adS_4 vacuum it also admits a constant scalar flow to $\mathrm{adS}_2 \times H^2/\tilde{\Gamma}$ (one can find a very general proof of this in [52]). With the addition of hypermultiplets, one can set them also constant and then the only additional constraints are $\langle \mathcal{K}^u, \mathcal{Q} \rangle = 0$. Subject to this condition being solved, the hypermultiplets decouple and the constant scalar flow is also a solution of the theory with hypermultiplets. We demonstrate this in our two examples.

Our first example was obtained in [169] corresponding to the invariant dimensional reduction of M-theory on $V_{5,2}$. Our second example comes from [233] and corresponds to a consistent truncation of the dimensional reduction of maximal gauged supergravity on the Einstein three-manifold $M_3 \in \{H^3/\Gamma, T^3, S^3\}$, where Γ is a discrete subgroup of $SL(2, \mathbb{C})$.

12.4.1 $V_{5,2}$

The invariant reduction of M-theory on seven-dimensional cosets was performed in [169] where in addition the general reduction on $SU(3)$-structure manifolds was found. All the resulting four dimensional gauged supergravity models found in that work fall into the class studied here, namely the hypermultiplet scalar manifold is a symmetric space which lies in the image of a c-map. Black hole solutions in many of these models were studied in [62]. We restrict ourselves to the example where $\mathcal{M}_h = /\mathrm{SO}(4)$ corresponding to the reduction on $V_{5,2}$.

The following data specifies the four dimensional supergravity theory [169]

$$n_v = 1, \qquad \mathcal{M}_v = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}, \qquad F = -\frac{(X^1)^3}{X^0}, \qquad X^\Lambda = \begin{pmatrix} 1 \\ \tau \end{pmatrix} \tag{12.4.1a}$$

$$n_h = 2, \qquad \mathcal{M}_h = \frac{\mathrm{G}_{2(2)}}{\mathrm{SO}(4)}, \qquad \mathcal{M}_z = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}, \qquad G = -\frac{(Z^1)^3}{Z^0}, \qquad Z^A = \begin{pmatrix} 1 \\ z \end{pmatrix}. \tag{12.4.1b}$$

The non-vanishing electric gaugings are given by

$$b^1_\Lambda = \frac{4}{\sqrt{3}}\,\delta_{\Lambda 0}, \qquad a_{1,\Lambda} = -\frac{4}{\sqrt{3}}\,\delta_{\Lambda 0}, \qquad \varepsilon_{+\Lambda} = -e_0\,\delta_{\Lambda 0}. \tag{12.4.2}$$

The non-vanishing magnetic gauging is given by

$$\varepsilon^\Lambda_+ = -2\delta^{\Lambda 1}. \tag{12.4.3}$$

The constant e_0 has its origin in the M-theory three-form with legs in the external four dimensional spacetime which has been dualized to a constant [169].

We note that the gaugings which specify this model were incorrectly reported in [169] to have vanishing compensator W^x_Λ. This of course is incompatible with the existence of a supersymmetric adS$_4$ vacuum. The solution is that the Killing vectors $k_{\mathbb{U}}$ with $a_i \neq 0$ have non-trivial compensators and we now see this is nontrivially gauged. In fact this is the only gauging with a non-trivial compensator in this reduction.

AdS$_4$ vacua

The Killing prepotentials $P^\pm_\Lambda$ are set to vanish by the condition

$$\xi^A = \tilde{\xi}_A = 0 \tag{12.4.4}$$

Then from $\langle \mathcal{K}^A, \mathrm{Im}\,\mathcal{V}\rangle = 0$ (in the direction of $\mathcal{M}_z$) we get

$$\mathcal{K}^A = 0 \Longrightarrow z = i\sqrt{3}, \tag{12.4.5}$$

and from $\langle \mathcal{K}^\sigma, \mathrm{Im}\,\mathcal{V}\rangle = 0$ (in the direction of the axion σ) we get

$$e^\phi = \sqrt{\frac{6}{e_0}} \tag{12.4.6}$$

while the axion is unfixed. As a result we have the Killing prepotentials

$$P^3_\Lambda = (1,0), \qquad \tilde{P}^{3\Lambda} = (0, -6/e_0). \tag{12.4.7}$$

The vector multiplet scalars are then given by

$$x = 0, \qquad y = \sqrt{\frac{e_0}{6}} \tag{12.4.8}$$

and the adS$_4$ radius is given by

$$R^2 = \frac{12\sqrt{6}}{e_0^{3/2}}. \tag{12.4.9}$$

$\mathrm{AdS}_2 \times \Sigma_g$ vacua

There is a related $\mathrm{adS}_2 \times H^2/\tilde{\Gamma}$ vacuum at the same point on the scalar moduli spaces $\mathcal{M}_v \times \mathcal{M}_h$. The charges are

$$\mathcal{Q} = (1/4, 0, 0, e_0/8) \tag{12.4.10}$$

and the radii are

$$R_1 = \frac{e_0^{3/4}}{8\ 2^{1/4}\ 3^{3/4}}, \qquad R_2 = \frac{e_0^{3/4}}{4\ 2^{1/4}\ 3^{3/4}}. \tag{12.4.11}$$

12.4.2 $\mathrm{SO}(5)$ gauged supergravity on M_3

The maximal gauged supergravity in seven dimensions has been dimensionally reduced on three-dimensional constant curvature Einstein manifold and consistently truncated to a four dimensional gauged supergravity theory in [233]. The resulting theory is given by the following data

$$n_v = 1, \qquad \mathcal{M}_v = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}, \qquad F = -4\,\frac{(X^1)^3}{X^0}, \qquad X^\Lambda = \begin{pmatrix} 1 \\ \tau \end{pmatrix} \tag{12.4.12a}$$

$$n_h = 2, \qquad \mathcal{M}_h = \frac{\mathrm{G}_{2(2)}}{\mathrm{SO}(4)}, \qquad \mathcal{M}_z = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}, \qquad G = -\frac{(Z^1)^3}{Z^0}, \qquad Z^A = \begin{pmatrix} 1 \\ z \end{pmatrix}. \tag{12.4.12b}$$

We have computed the gaugings in our terminology by careful comparison with [233].

We find that $k_1 = 0$ and the non-vanishing electric components are in k_0

$$\alpha^0{}_0 = \frac{1}{2}, \qquad \hat{\alpha}_{0,0} = 3^{3/4}, \qquad \alpha_{1,0} = \frac{3^{3/4}\ell}{4}. \tag{12.4.13}$$

Likewise we find that $k^0 = 0$ and the non-vanishing magnetic components are in k^1

$$\alpha_1{}^1 = -\frac{1}{2\sqrt{3}}. \tag{12.4.14}$$

The integer $\ell = \{-1, 0, 1\}$ corresponds to the reduction on $M_3 = \{H^3/\Gamma, T^3, S^3\}$ respectively. The gauging from $\hat{\alpha}_{0,0}$ provides the non-trivial compensator required to have a supersymmetric adS_4 vacuum.

We write

$$z = \chi + i\,\mathrm{e}^{-2\varphi}. \tag{12.4.15}$$

This yields the magnetic Killing prepotentials

$$P^{x,0} = 0, \qquad P^{1,1} = \frac{3^{1/4}}{2}\,\mathrm{e}^{\phi+3\varphi}\chi, \qquad P^{2,1} = \frac{3^{1/4}}{2}\,\mathrm{e}^{\phi+\varphi}, \qquad P^{3,1} = \frac{3^{1/4}}{2}\,\mathrm{e}^{2\phi}\xi^1 \tag{12.4.16}$$

and the electric Killing prepotentials

$$P_0^1 = \frac{1}{4\,3^{3/4}}\Big[-9\,\mathrm{e}^{4\varphi}\chi\ell + 2\chi(\,\mathrm{e}^{4\varphi}\chi^2-3) + 3^{3/2}\Big(6\xi^0(\xi^1-\chi\xi^0) + \mathrm{e}^{4\varphi}\big(-2\sigma+\xi^0(\tilde\xi_0+2\chi^3\xi^0)+\tilde\xi_1\xi^1-6\chi^2\xi^0\xi^1+6\chi(\xi^1)^2\big)\Big)\Big] \tag{12.4.17a}$$

$$P_0^2 = \frac{1}{4\,3^{3/4}}\Big[-9\,\mathrm{e}^{\phi+\varphi}\ell + 2\,\mathrm{e}^{-\phi-3\varphi}\Big(\,\mathrm{e}^{2\phi}(3\,\mathrm{e}^{4\varphi}\chi^2-1) + 3^{3/2}\big(\,\mathrm{e}^{2\phi}(3\,\mathrm{e}^{4\varphi}(-\chi\xi^0+\xi^1)^2)-(\xi^0)^2-\,\mathrm{e}^{6\varphi}\big)\Big)\Big] \tag{12.4.17b}$$

$$P_0^3 = \frac{1}{4\,3^{3/4}}\Big[18\sqrt{3}\,\mathrm{e}^{2\varphi}(\chi\xi^0-\xi^1) + \,\mathrm{e}^{2\phi}\Big(\tilde\xi_0(2+3^{3/2}(\xi^0)^2)-9\ell\xi^1 + 3^{3/2}\big(\tilde\xi_1\xi^0\xi^1+2(\xi^1)^3-2\sigma\xi^0\big)\Big)\Big] \tag{12.4.17c}$$

$$P_1^x = 0. \tag{12.4.17d}$$

AdS_4 **vacua**

The supersymmetric adS_4 vacuum is at

$$\xi^A = \tilde\xi_A = \chi = \sigma = \phi = 0, \qquad \mathrm{e}^{\varphi} = \frac{1}{3^{1/4}}, \qquad \tau = \frac{i}{2\sqrt{2}} \tag{12.4.18}$$

and in particular requires $\ell = -1$, corresponding to a reduction on H^3/Γ. The adS_4 radius is

$$R = \frac{1}{\sqrt{2}}. \tag{12.4.19}$$

Evaluated at this vacuum the Killing prepotentials become

$$P_\Lambda^1 = P_\Lambda^3 = P^{1,\Lambda} = P^{3,\Lambda} = 0, \qquad P_0^2 = -\frac{1}{4}, \qquad P^{2,2} = \frac{1}{2}. \tag{12.4.20}$$

$\mathrm{AdS}_2 \times \Sigma_g$ **vacua**

The $\mathrm{adS}_2 \times \Sigma_g$ vacuum is located at the same point on the scalar manifold. The charges are given by

$$p^0 = -1, \qquad p^1 = 0, \qquad q_0 = 0, \qquad q_1 = -\frac{3}{2}. \tag{12.4.21}$$

The radii are given by

$$R_1 = \frac{1}{2^{3/4}}, \qquad R_2 = \frac{1}{2^{1/4}}. \tag{12.4.22}$$

When lifted to M-theory this is a solution of the form

$$\mathrm{adS}_2 \times H^2/\tilde\Gamma \times (H^3 \times_w S^4) \tag{12.4.23}$$

where the S^4 is fibered non-trivially over H^3. It arises as the IR of a domain wall $\mathrm{adS}_4 \to \mathrm{adS}_2 \times H^2$ where the scalar fields take constant values along the whole flow.

Chapter 13

BPS AdS–NUT black holes

We focus on 1/4-BPS adS–NUT black holes. BPS equations for $N = 2$ FI gauged supergravity and several classes of analytical solutions were derived in [150].[1]

13.1 Ansatz

We consider the ansatz from section 10 where the metric and for the gauge fields are

$$\mathrm{d}s^2 = -\,\mathrm{e}^{2U}\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big)^2 + \mathrm{e}^{-2U}\mathrm{d}r^2 + \mathrm{e}^{2(V-U)}\,\mathrm{d}\Sigma_g^2, \tag{13.1.1a}$$

$$A^\Lambda = \tilde{q}^\Lambda\big(\mathrm{d}t + 2n\,H(\theta)\,\mathrm{d}\phi\big) + \tilde{p}^\Lambda\,H(\theta)\,\mathrm{d}\phi. \tag{13.1.1b}$$

The functions $U, V, \tilde{q}$ and $\tilde{p}$ depend only on r, while Σ_g is a Riemann surface of genus g (see appendix A.7) with metric

$$\mathrm{d}\Sigma_g^2 = \mathrm{d}\theta^2 + H'(\theta)^2\,\mathrm{d}\phi^2, \qquad H'(\theta) = \begin{cases} \sin\theta & \kappa = 1, \\ 1 & \kappa = 0, \\ \sinh\theta & \kappa = -1. \end{cases} \tag{13.1.2}$$

All scalars are function only on r

$$\tau^i = \tau^i(r), \qquad q^u = q^u(r). \tag{13.1.3}$$

We consider only abelian gaugings.

The magnetic field strength reads

$$G_\Lambda = R_{\Lambda\Sigma}F^\Sigma - I_{\Lambda\Sigma}\star F^\Sigma. \tag{13.1.4}$$

The electric and magnetic charges are given explicitly by

$$p^\Lambda = \frac{1}{4\pi}\int_{\Sigma_g} F^\Lambda = \tilde{p}^\Lambda - 2n\tilde{q}^\Lambda, \tag{13.1.5a}$$

$$q_\Lambda = \frac{1}{4\pi}\int_{\Sigma_g} G_\Lambda = -\,\mathrm{e}^{2(V-U)}\,I_{\Lambda\Sigma}\tilde{q}'^\Sigma + \kappa\,R_{\Lambda\Sigma}p^\Sigma. \tag{13.1.5b}$$

Using these expressions one can rewrite the gauge field as

$$A^\Lambda = \tilde{q}^\Lambda\,\mathrm{d}t + p^\Lambda\,H(\theta)\,\mathrm{d}\phi, \tag{13.1.6}$$

and finds again an expression for $\tilde{q}'^\Lambda$

$$\tilde{q}'^\Lambda = \mathrm{e}^{2(U-V)}\,I^{\Lambda\Sigma}\big(R_{\Sigma\Delta}p^\Delta - q_\Sigma\big). \tag{13.1.7}$$

[1] In this section we follow the conventions of [150]. The main difference is the replacement of Ω by $-\Omega$.

The central and matter charges are[2]

$$\mathcal{Z} = \langle \mathcal{Q}, \mathcal{V} \rangle \,, \qquad \mathcal{Z}_i = \langle \mathcal{Q}, U_i \rangle \,. \tag{13.1.8}$$

Similarly one defines the prepotential charges

$$\mathcal{L}^x = \langle \mathcal{P}^x, \mathcal{V} \rangle \,, \qquad \mathcal{L}^x_i = \langle \mathcal{P}^x, U_i \rangle \,. \tag{13.1.9}$$

13.2 BPS equations

For the following we consider FI gaugings and $n_h = 0$.

The Killing spinor has the same form (11.2.3) as for $n = 0$

$$\begin{aligned} \varepsilon_\alpha &= \mathrm{e}^{U/2}\,\mathrm{e}^{i\psi/2}\,\varepsilon_{0\alpha}, & (13.2.1a)\\ \varepsilon_{0\alpha} &= i\,\gamma^0 \varepsilon_{\alpha\beta}\varepsilon_0^\beta, & (13.2.1b)\\ \varepsilon_{0\alpha} &= -p^\Lambda P^x_\Lambda\,\gamma^{01}\sigma^x{}_\alpha{}^\beta \varepsilon_{0\beta}, & (13.2.1c) \end{aligned}$$

$\varepsilon_{0\alpha}$ being a constant spinor.

The symplectic covariant equations are

$$\begin{aligned} \langle \mathcal{Q}, \mathcal{G} \rangle + 4n\,\mathrm{e}^U\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}) &= \varepsilon_D\,\kappa, & (13.2.2a)\\ \varepsilon_D\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}) &= \mathrm{e}^{2(U-V)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{3U-2V} & (13.2.2b)\\ 2\,\mathrm{e}^{2V}\partial_r\big(\,\mathrm{e}^{-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) &= \big(4n\,\mathrm{e}^U - 8\varepsilon_D\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\big)\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) & \\ &\quad - \mathcal{Q} - \varepsilon_D\,\mathrm{e}^{2(V-U)}\Omega\mathcal{M}\mathcal{G}, & (13.2.2c)\\ (\,\mathrm{e}^V)' &= -2\varepsilon_D\,\mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}), & (13.2.2d)\\ \mathcal{Q}' &= -2n\,\mathrm{e}^{2(U-V)}\,\Omega\mathcal{M}\mathcal{Q}. & (13.2.2e) \end{aligned}$$

At the end one finds Maxwell equations, while the first one is a generalization of the Dirac condition.

We also have the equation for the real part of $\mathcal{V}$

$$2\,\partial_r\big(\,\mathrm{e}^U\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = -\mathcal{G} - \mathrm{e}^{2(U-V)}\Omega\mathcal{M}\mathcal{Q}. \tag{13.2.3}$$

Finally we recall the equations for ψ', U' and z'^i

$$\begin{aligned} \psi' &= -A_r - 2\,\mathrm{e}^{-U}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}) - n\,\mathrm{e}^{2(U-V)}, & (13.2.4a)\\ (\,\mathrm{e}^U)' &= -\varepsilon_D\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{Z}), & (13.2.4b)\\ (z^i)' &= \mathrm{e}^{-U}\,\mathrm{e}^{i\psi}g^{i\bar{\jmath}}\big(\,\mathrm{e}^{2(U-V)}\,\mathrm{D}_{\bar{\jmath}}\mathcal{Z} + i\,\mathrm{D}_{\bar{\jmath}}\mathcal{L}\big). & (13.2.4c) \end{aligned}$$

The equation (13.2.2c) can be modified using (F.3.30e) to include one factor e^V inside the derivative

$$\begin{aligned} 2\,\mathrm{e}^V\partial_r\big(\,\mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = 4\,&\big(n\,\mathrm{e}^U - 2\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\big)\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V})\\ &- 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\\ &- \mathcal{Q} - \mathrm{e}^{2(V-U)}\Omega\mathcal{M}\mathcal{G}. \end{aligned} \tag{13.2.5}$$

One can also use Maxwell equation (13.2.2e) to rewrite (13.2.3) as

$$2\,\partial_r\big(\,\mathrm{e}^U\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = \frac{1}{2n}\,\mathcal{Q}' - \mathcal{G}. \tag{13.2.6}$$

[2]There is a minus sign with respect to the notations of appendix A.6.

It is then straightforward to integrate this equation

$$4n\,\mathrm{e}^{U}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) = \mathcal{Q} - 2n\,\mathcal{G}\,r - \widehat{\mathcal{Q}} \tag{13.2.7}$$

where $\widehat{\mathcal{Q}}$ is the integration constant

$$\widehat{\mathcal{Q}} = \begin{pmatrix} P^\Lambda \\ Q_\Lambda \end{pmatrix}. \tag{13.2.8}$$

In turn one can use this to get the expression for $\mathcal{Q}$ if one knows the other quantities. Moreover plugging this result into Dirac quantization equation (F.3.30a) gives

$$\left\langle \widehat{\mathcal{Q}}, \mathcal{G} \right\rangle = \varepsilon_D\,\kappa \tag{13.2.9}$$

where the LHS is constant and $\widehat{\mathcal{Q}}$ corresponds to the conserved charges.

Finally one can use this expression for $\mathcal{Q}$ in order to rewrite the equations for $\mathrm{Im}\,\mathcal{V}$ (13.2.2c)

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\,\mathrm{e}^{-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = 8\big(n\,\,\mathrm{e}^{U} - \varepsilon_D\,\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\big)\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ - 2n\,\mathcal{G}\,r - \widehat{\mathcal{Q}} - \varepsilon_D\,\,\mathrm{e}^{2(V-U)}\Omega\mathcal{M}\mathcal{G}. \end{aligned} \tag{13.2.10}$$

and (13.2.5)

$$\begin{aligned} 2\,\mathrm{e}^{V}\partial_r\big(\,\mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = 8\,\big(n\,\,\mathrm{e}^{U} -\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\big)\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ - 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{V}) \\ - 2n\,\mathcal{G}\,r - \widehat{\mathcal{Q}} -\,\mathrm{e}^{2(V-U)}\Omega\mathcal{M}\mathcal{G}. \end{aligned} \tag{13.2.11}$$

The main advantage is that $\mathcal{Q}$ has been replaced by the constant $\widehat{\mathcal{Q}}$, while the extra term $\mathcal{G}\,r$ is not a big problem.

Note that we can use (13.2.2b) in order to get an expression for $\mathrm{e}^{i\psi}$. This last expression will not help to solve the equation since it is complicated, but it means that we can always integrate the differential equation for the phase (13.2.4a), and we can obtain the expression if we know all other quantities. The result is[3]

$$\mathrm{e}^{i\psi} = -\frac{n\,\mathrm{e}^{3U-2V}}{\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\,\bar{\mathcal{Z}}} \pm 2\sqrt{\left(\frac{n\,\mathrm{e}^{3U-2V}}{\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\,\bar{\mathcal{Z}}}\right)^2 - \frac{\mathcal{L} + i\,\mathrm{e}^{2(U-V)}\,\mathcal{Z}}{\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\,\bar{\mathcal{Z}}}}. \tag{13.2.12}$$

which is a consequence of the second order equation

$$\mathrm{e}^{2i\psi}\big(\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\,\bar{\mathcal{Z}}\big) - 2n\,\mathrm{e}^{3U-2V}\,\,\mathrm{e}^{i\psi} + \big(\mathcal{L} + i\,\mathrm{e}^{2(U-V)}\,\mathcal{Z}\big) = 0 \tag{13.2.13}$$

obtained by writing explicitly the real and imaginary parts. For $n = 0$ it reduces to (11.3.5).

13.3 Symmetric $\mathcal{M}_v$ with FI gaugings

Using techniques similar to section 11.4 one obtains the following equations for symmetric cubic $\mathcal{M}_v$

$$2\,\mathrm{e}^{V}\partial_r\,\mathrm{Im}\,\widetilde{\mathcal{V}} = -\widehat{\mathcal{Q}} + \epsilon_D\,I_4'(\mathcal{P}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}) + 2n\mathcal{P}r, \tag{13.3.1a}$$

$$(\,\mathrm{e}^{V})' = -2\epsilon_D\left\langle \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{P} \right\rangle, \tag{13.3.1b}$$

$$\mathrm{e}^{V}\left\langle \mathrm{Im}\,\widetilde{\mathcal{V}}, \partial_r\,\mathrm{Im}\,\widetilde{\mathcal{V}} \right\rangle = \left\langle \mathrm{Im}\,\widetilde{\mathcal{V}}, \widehat{\mathcal{Q}} \right\rangle + 3n\,\mathrm{e}^{V} + 4nr\left\langle \mathcal{P}, \mathrm{Im}\,\widetilde{\mathcal{V}} \right\rangle, \tag{13.3.1c}$$

$$\left\langle \widehat{\mathcal{Q}}, \mathcal{P} \right\rangle = \varepsilon_D\,\kappa \tag{13.3.1d}$$

where we defined

$$\widetilde{\mathcal{V}} = \,\mathrm{e}^{V-U}\,\,\mathrm{e}^{-i\psi}\,\mathcal{V}. \tag{13.3.2}$$

[3] To lighten notations we take $g_\Lambda \tilde{p}^\Lambda = \kappa$.

13.4 Solutions

In this section we are looking for solutions of the previous equations. Following section 10.3 and the example of section 10.2, we will consider first extremal black holes (of general and CK types), and then solutions with complex roots. Indeed other cases do not seem to appear.

The derivation uses techniques that are similar to those described in section 12.3. In particular one imposes the near-horizon constraint (12.2.27), and the identities from appendix E.1 are used.

13.4.1 Pair of double roots

When there is a pair of double roots our ansatz is:

$$\mathrm{e}^{2V} = r^2(v_4 r^2 + 2\sqrt{v_2 v_4}\, r + v_2), \tag{13.4.1}$$

$$\operatorname{Im}\widetilde{\mathcal{V}} = \frac{1}{\epsilon\sqrt{2\,\langle\mathcal{G}, A_1\rangle}}\, A_1 + A_3\, r \tag{13.4.2}$$

where (A_1, A_3) are symplectic vectors which we must determine and we include a sign $\epsilon = \pm 1$ to keep track of both branches of the square root. We have introduced this particular normalization of A_1 to make contact with expressions elsewhere. The IR and UV asymptotics completely fix the solution, the BPS equations then over-constrain this ansatz and for a solution to exist there must be significant cancellations.

We first solve the second equation of (13.3.1b) to get

$$\sqrt{v_2} = \epsilon\sqrt{2\,\langle\mathcal{G}, A_1\rangle}, \qquad \sqrt{v_4} = \langle\mathcal{G}, A_3\rangle\,, \tag{13.4.3}$$

and then expand the BPS equations (13.3.1a) in r to get

$$0 = I_4'(\mathcal{G}, A_3, A_3) - 2\,\langle\mathcal{G}, A_3\rangle\, A_3, \tag{13.4.4a}$$

$$0 = I_4'(\mathcal{G}, A_1, A_3) - 2\,\langle\mathcal{G}, A_1\rangle\, A_3 + n\kappa\epsilon\sqrt{2\,\langle\mathcal{G}, A_1\rangle}\,\mathcal{G}, \tag{13.4.4b}$$

$$0 = I_4'(\mathcal{G}, A_1, A_1) - 2\,\langle\mathcal{G}, A_1\rangle\,\mathcal{Q}. \tag{13.4.4c}$$

The constraint (13.3.1c) is also expanded and we get

$$0 = \sqrt{2}\,\langle A_1, A_3\rangle - n\kappa\epsilon\sqrt{\langle\mathcal{G}, A_1\rangle}, \tag{13.4.5a}$$

$$0 = \langle\mathcal{Q}, A_1\rangle + 2\,\langle A_1, A_3\rangle\,, \tag{13.4.5b}$$

$$0 = \sqrt{2}n\kappa\epsilon\,\langle\mathcal{G}, A_1\rangle^{3/2} + \langle\mathcal{G}, A_3\rangle\,\langle\mathcal{Q}, A_1\rangle + 2\,\langle\mathcal{G}, A_1\rangle\,\big(\langle\mathcal{Q}, A_3\rangle + \langle A_1, A_3\rangle\big), \tag{13.4.5c}$$

$$0 = \langle\mathcal{Q}, A_1\rangle\,. \tag{13.4.5d}$$

All the free parameters are fixed by the UV and IR asymptotics. From the UV we get

$$A_3 = \frac{I_4'(\mathcal{G})}{4 I_4(\mathcal{G})^{1/4}}, \qquad v_4 = \sqrt{I_4(\mathcal{G})} \tag{13.4.6}$$

where we have appealed to [53] to fix the normalization of A_3. The solution for A_1, found from the IR equation (13.4.4c), is the same as in [84]

$$A_1 = a_1\, I_4'(\mathcal{G}, \mathcal{G}, \mathcal{G}) + a_2\, I_4'(\mathcal{G}, \mathcal{G}, \mathcal{Q}) + a_3\, I_4'(\mathcal{G}, \mathcal{Q}, \mathcal{Q}) + a_4\, I_4'(\mathcal{Q}, \mathcal{Q}, \mathcal{Q}) \tag{13.4.7}$$

with

$$a_1 = -\frac{a_3}{3}\,\frac{I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})}{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})}, \tag{13.4.8a}$$

$$a_2 = \frac{a_3}{6}\,\frac{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2}{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 I_4(\mathcal{Q}) - I_4(\mathcal{G})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2}, \tag{13.4.8b}$$

$$a_3 = \frac{9\big(I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})I_4(\mathcal{G}) - I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})I_4(\mathcal{Q})\big)}{I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})(\langle I_4'(\mathcal{G},\mathcal{G},\mathcal{G}), I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q})\rangle + \kappa I_4(\mathcal{G},\mathcal{G},\mathcal{Q},\mathcal{Q}))}, \tag{13.4.8c}$$

$$a_4 = -\frac{a_2}{3}\,\frac{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})}{I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})}. \tag{13.4.8d}$$

The effect of the NUT charge is through (13.4.4b) as well as the constraints (13.4.5a) and (13.4.5c). We find that these three equations are redundant and there is a single non-trivial constraint on the system

$$n\kappa\epsilon = -\frac{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})}{144\sqrt{2}\,I_4(\mathcal{G})^{1/4}}\times \\ \times\sqrt{\frac{18\,\langle\mathcal{G},\mathcal{Q}\rangle\,I_4(\mathcal{G},\mathcal{G},\mathcal{Q},\mathcal{Q}) - \langle I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q}), I_4'(\mathcal{G},\mathcal{G},\mathcal{G})\rangle}{\big(I_4(\mathcal{G})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2 - I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 I_4(\mathcal{Q})\big)^2 + 16 I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^3 I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^3}}. \tag{13.4.9}$$

When $n = 0$ then (13.4.9) is solved by $I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q}) = I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) = 0$ and the solutions reduce to those in [46, 58, 85].

13.4.2 Single double root

Only a single double root is required in e^{2V} in order to have an $\mathrm{adS}_2 \times \Sigma_g$ vacuum in the IR but this more general solution is somewhat more complicated. We found that in order to have a pair of double roots, there is a relation between the NUT charge and the electromagnetic charges (13.4.9), whereas there is no such constraint when requiring a single double root. The only constraint is that for $\mathrm{adS}_2 \times \Sigma_g$ vacua (12.2.27).

We take the same ansatz as in section 12.3

$$\mathrm{e}^{2V} = r^2(v_2 + v_3 r + v_4 r^2) \tag{13.4.10}$$

$$\mathrm{Im}\,\widetilde{\mathcal{V}} = \mathrm{e}^{-V}\widehat{A} \tag{13.4.11}$$

$$\widehat{A} = A_1 r + A_2 r^2 + A_3 r^3 \tag{13.4.12}$$

where A_i are constant symplectic vectors whose dependence on $\mathcal{G}$ and $\mathcal{Q}$ we seek to determine.

We first solve (13.3.1b) with

$$v_{i+1} = \frac{4}{i+1}\,\langle\mathcal{G}, A_i\rangle\,, \qquad i = 2,3,4. \tag{13.4.13}$$

The symplectic vector of BPS equations (13.3.1a) is then

$$2\,\mathrm{e}^{2V}\widehat{A}' - (\,\mathrm{e}^{2V})'\widehat{A} = I_4'(\mathcal{G},\widehat{A},\widehat{A}) + \mathrm{e}^{2V}(2n\mathcal{G}r - \mathcal{Q}) \tag{13.4.14}$$

which breaks up into five components from different powers of r

$$0 = I_4'(\mathcal{G}, A_3, A_3) - 2\langle \mathcal{G}, A_3\rangle A_3, \tag{13.4.15}$$
$$0 = I_4'(\mathcal{G}, A_2, A_3) + n\kappa \langle \mathcal{G}, A_3\rangle \mathcal{G} - 2\langle G, A_2\rangle A_3, \tag{13.4.16}$$
$$0 = 2I_4'(\mathcal{G}, A_1, A_3) + I_4'(\mathcal{G}, A_2, A_2) - 8\langle \mathcal{G}, A_1\rangle A_3 - \langle \mathcal{G}, A_3\rangle \mathcal{Q} + 2\langle \mathcal{G}, A_3\rangle A_1 + \frac{4}{3}\langle \mathcal{G}, A_2\rangle \big(2\mathcal{G} - A_2\big), \tag{13.4.17}$$
$$0 = I_4'(\mathcal{G}, A_1, A_2) + 2\langle \mathcal{G}, A_1\rangle \big(n\kappa\mathcal{G} - A_2\big) + \langle \mathcal{G}, A_2\rangle \big(A_1 - \mathcal{Q}\big), \tag{13.4.18}$$
$$0 = I_4'(\mathcal{G}, A_1, A_1) - 2\langle \mathcal{G}, A_1\rangle \mathcal{Q}. \tag{13.4.19}$$

We also need to the expansion of the single real constraint (13.3.1c)

$$O(r^4): \quad 0 = 2\langle A_2, A_3\rangle - n\kappa \langle \mathcal{G}, A_3\rangle, \tag{13.4.20}$$
$$O(r^3): \quad 0 = 2\langle A_1, A_3\rangle + \langle \mathcal{Q}, A_3\rangle, \tag{13.4.21}$$
$$O(r^2): \quad 0 = \langle A_1, A_2\rangle + n\kappa \langle \mathcal{G}, A_1\rangle + \langle \mathcal{Q}, A_2\rangle, \tag{13.4.22}$$
$$O(r^1): \quad 0 = 2\langle \mathcal{Q}, A_1\rangle. \tag{13.4.23}$$

Note that once again, the highest order in r components of (13.4.14) and (13.3.1c) are independent of the NUT charge and therefore the solution for A_3 can be taken from [84]

$$A_3 = \frac{1}{4}\frac{I_4'(\mathcal{G})}{\sqrt{I_4(\mathcal{G})}}, \qquad v_4 = \sqrt{I_4(\mathcal{G})}. \tag{13.4.24}$$

We solve these equations with the ansatz

$$A_1 = a_1\, I_4'(\mathcal{G},\mathcal{G},\mathcal{G}) + a_2\, I_4'(\mathcal{G},\mathcal{Q},\mathcal{Q}) + a_3\, I_4'(\mathcal{G},\mathcal{Q},\mathcal{Q}) + a_4\, I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q}), \tag{13.4.25}$$
$$A_2 = b_1\, I_4'(\mathcal{G},\mathcal{G},\mathcal{G}) + b_2\, I_4'(\mathcal{G},\mathcal{Q},\mathcal{Q}) + b_3\, I_4'(\mathcal{G},\mathcal{Q},\mathcal{Q}) + b_4\, I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q}), \tag{13.4.26}$$

where $\{a_i, b_j\}$ are real constants with a non-trivial dependence on $(\mathcal{G},\mathcal{Q})$. The IR conditions which give a_i in terms of $(\mathcal{G},\mathcal{Q})$ are the same we obtained for the case when e^{2V} had a pair of double roots and are thus given by (13.4.8a)-(13.4.8d).

Then from (13.4.18) we find the solution for $\{b_1, b_2, b_4\}$ in terms of b_3

$$b_1 = \frac{b_3 I_4(\mathcal{Q}) I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})}{3I_4(\mathcal{G}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})} - \frac{2b_3 I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})}{3I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})} + \frac{n\kappa I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2}{18\Pi_3} + \frac{b_3\kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2}{54 I_4(\mathcal{G})\Pi_3}, \tag{13.4.27a}$$

$$b_2 = \frac{I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})\big(6nI_4(\mathcal{G})I_4(\mathcal{Q}) - b_3\Pi_2\big)}{6I_4(\mathcal{G})\Pi_3}, \tag{13.4.27b}$$

$$b_4 = -\frac{I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\big(3nI_4(\mathcal{G}) + b_3 I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})\kappa\big)}{9\Pi_3}. \tag{13.4.27c}$$

Finally from (13.4.17) we solve for b_3 and find the rather lengthy expression

$$b_3 = \frac{b_n}{b_d}$$

where the numerator and denominator are given by

$$\begin{aligned} b_n = {}& 6n\kappa I_4(\mathcal{G}) I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2 \langle I_4'(\mathcal{G},\mathcal{G},\mathcal{G}), I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q})\rangle\, \Pi_7 \\ &+ 3\Big[-I_4(\mathcal{G})^{3/2} I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\Pi_3^2\Pi_8\Big[-18 I_4(\mathcal{G})\Pi_3^2 \\ &\quad + (\kappa + 4n^2 I_4(\mathcal{G})^{1/2}) I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^{1/2} I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\Pi_5 \\ &\quad - 8n^2 I_4(\mathcal{G})^{3/2}\big[144\kappa I_4(\mathcal{Q})^2 I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 - \kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^3 \\ &\quad + 72 I_4(\mathcal{Q}) I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\Pi_6\big]\Big]\Big]^{1/2} \end{aligned} \tag{13.4.28}$$

and

$$
\begin{aligned}
b_d = 8I_4(\mathcal{G})\Big[&I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})\Big[2\kappa I_4(\mathcal{Q})I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2(144I_4(\mathcal{Q})^2I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})\\
&-I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^3)+I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})(288I_4(\mathcal{Q})^2I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})\\
&-I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^3)\left\langle I_4'(\mathcal{G},\mathcal{G},\mathcal{G}),I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q})\right\rangle\\
&+90\kappa I_4(\mathcal{Q})I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2\left\langle I_4'(\mathcal{G},\mathcal{G},\mathcal{G}),I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q})\right\rangle^2\\
&+9I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^3\left\langle I_4'(\mathcal{G},\mathcal{G},\mathcal{G}),I_4'(\mathcal{Q},\mathcal{Q},\mathcal{Q})\right\rangle^3\Big]+18\kappa I_4(\mathcal{G})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2\Pi_3\Big]\\
&-4\kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^3I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\Pi_5.
\end{aligned}
\tag{13.4.29}
$$

We have used the notation

$$\Pi_1 = I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\left\langle I_4'(\mathcal{G}),I_4'(\mathcal{Q})\right\rangle + 2\kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})I_4(\mathcal{Q}), \tag{13.4.30a}$$
$$\Pi_2 = I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})\left\langle I_4'(\mathcal{G}),I_4'(\mathcal{Q})\right\rangle + 2\kappa I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})I_4(\mathcal{Q}), \tag{13.4.30b}$$
$$\Pi_3 = I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\left\langle I_4'(\mathcal{G}),I_4'(\mathcal{Q})\right\rangle + 4\kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})I_4(\mathcal{Q}), \tag{13.4.30c}$$
$$\Pi_4 = 2\kappa I_4(\mathcal{Q})I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 + I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\Pi_1, \tag{13.4.30d}$$
$$\Pi_5 = I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\left\langle I_4'(\mathcal{G}),I_4'(\mathcal{Q})\right\rangle + 2\kappa I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})I_4(\mathcal{Q}), \tag{13.4.30e}$$
$$\Pi_6 = I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})\left\langle I_4'(\mathcal{G}),I_4'(\mathcal{Q})\right\rangle + 2\kappa I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})I_4(\mathcal{G}), \tag{13.4.30f}$$
$$\Pi_7 = 2\kappa I_4(\mathcal{G})I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})^2 + I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})\Pi_5, \tag{13.4.30g}$$
$$\Pi_8 = 2\kappa I_4(\mathcal{G})I_4(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{Q})^2 + I_4(\mathcal{G},\mathcal{Q},\mathcal{Q},\mathcal{Q})\Pi_6. \tag{13.4.30h}$$

These expression are fairly lengthy but in fact their derivation in Mathematica starting from (13.4.15)-(13.4.23) is quite straightforward when using the identities in appendix E.1. The $n \to 0$ limit of these expressions agrees with those found in [84].

13.4.3 Four independent roots

While extremal black holes necessarily have a double real root in e^{2V}, more general configurations are possible. For example we could have one or two pairs of complex conjugate roots. A natural ansatz for such solutions is

$$\mathrm{e}^{2V} = v_0 + v_1 r + v_2 r^2 + v_4 r^4, \tag{13.4.31a}$$
$$\mathrm{Im}\,\widetilde{\mathcal{V}} = \mathrm{e}^{-V}\widehat{A}, \tag{13.4.31b}$$
$$\widehat{A} = A_0 + A_1 r + A_2 r^2 + A_3 r^3. \tag{13.4.31c}$$

We have used a shift symmetry in r to set $v_3 = 0$ but one cannot in general use a real shift in r to set $v_0 = 0$.

An example of such solutions is the constant scalar asymptotically adS_4 solution of section 10, corresponding to the STU-model with

$$P^0 = Q_i = P, \qquad Q_0 = -P^i = Q. \tag{13.4.32}$$

In our formalism we find this constant scalar example to be given by the following data

$$A_0 = \frac{n\kappa(P-1)}{2g}\,\mathcal{G} + \frac{n\kappa}{8g^3}\,I_4'(\mathcal{G}), \tag{13.4.33a}$$
$$A_1 = \frac{Q}{2g}\,\mathcal{G} + \frac{P-3gn^2}{8g^3}\,I_4'(\mathcal{G}), \tag{13.4.33b}$$
$$A_2 = \frac{n\kappa}{2}\,\mathcal{G}, \tag{13.4.33c}$$
$$A_3 = \frac{I_4'(\mathcal{G})}{4\sqrt{I_4(\mathcal{G})}}, \tag{13.4.33d}$$

and the metric is given by

$$e^{2(V-U)} = r^2 + n^2 \tag{13.4.34a}$$

$$e^{2V} = 2\Big(P^2 + Q^2 + g^2 n^4 - 2gn^2 P + 4gn\kappa Q r + 2(3gn^2 - gP)r^2 + gr^4\Big). \tag{13.4.34b}$$

The phase of the spinor is given by

$$\sin\psi = e^{U-2V}\big(gr^3 + (-P + 3gn^2)r + n\kappa Q\big). \tag{13.4.35}$$

We have tried to obtain generalizations of this solution using the ansatz (13.4.31a)-(13.4.31c) but have not managed to decouple the set of algebraic equations. However this should not be seen as evidence that such solutions do not exist. Such solutions would not necessarily correspond to black holes since that requires the existence of a horizon. Since we expect BPS black holes to have extremal horizons, these solutions are covered by our analysis in section 13.4.2. Nonetheless looking ahead to possible extensions to Euclidean solutions, it is of some interest to have more general solutions with single real roots of e^{2V}.

13.5 Examples

13.5.1 T^3 model

We now write down a non-trivial example by restricting to the T^3 model and allowing for dyonic charges. One might first try to find the solution with the same charges $(p^1, q_0) = (0, 0)$ as Cacciatori–Klemm solution [58] but we find quite straight-forwardly that this requires $n = 0$ and thus does not admit a generalization with NUT charge.

For simplicity, such that the resulting expressions are not too cumbersome, we set $p^1 = 0$. We can solve the constraint (12.2.27) with

$$q_0 = \frac{(p^0 - q_1)^{3/2}}{\sqrt{2p^0}}, \tag{13.5.1}$$

then the imaginary parts of the sections are given by

$$\operatorname{Im}\widetilde{\mathcal{V}}^0 = \frac{\epsilon p^0}{\sqrt{g}\sqrt{5p^0 + 3q_1}} + \frac{g^2 r}{\sqrt{2}}, \qquad \operatorname{Im}\widetilde{\mathcal{V}}^i = \frac{\epsilon\sqrt{p^0}\sqrt{p^0 - q_1}}{\sqrt{8g}\sqrt{5p^0 + 3q_1}}, \tag{13.5.2}$$

$$\operatorname{Im}\widetilde{\mathcal{V}}_0 = \frac{\epsilon(2p^0 + q_1)\sqrt{p^0 - q_1}}{\sqrt{8g}\sqrt{p^0}\sqrt{5p^0 + 3q_1}}, \qquad \operatorname{Im}\widetilde{\mathcal{V}}_i = \frac{\epsilon(p^0 + q_1)}{2\sqrt{g}\sqrt{5p^0 + 3q_1}} + \frac{g^2 r}{\sqrt{2}}, \tag{13.5.3}$$

and the metric components are given by

$$e^{2V} = r^2\big[2\sqrt{2}\,g^3\,r + \epsilon\sqrt{g}\sqrt{5p^0 + 3q_1}\big]^2. \tag{13.5.4}$$

The NUT charge is given by the relation

$$n\kappa\epsilon = \frac{g^{3/2}}{2\sqrt{p^0}}\frac{(p^0 - q_1)^{3/2}}{\sqrt{5p^0 + 3q_1}} \tag{13.5.5}$$

and the BPS Dirac quantization condition is

$$-\kappa = g(p^0 + 3q_1). \tag{13.5.6}$$

When $\epsilon = +1$, the horizon is at $r = 0$ and we find that regular solutions exist for both $\kappa = \pm 1$. When $\epsilon = -1$ the horizon is at

$$r = \frac{\sqrt{5p^0 + 3q_1}}{g^{5/2}\sqrt{8}} \tag{13.5.7}$$

and for the absence of zeros in $\operatorname{Im}\widetilde{\mathcal{V}}$ we need

$$g(p^0+3q_1)>0 \tag{13.5.8}$$

which implies $\kappa=-1$.

13.5.2 Constant scalar solution

One can observe the limit $p^1=q_0$ which gives the constant scalar solution. The combination of constant scalar fields and a pair of double roots in e^V forces $n=0$ and as is well-known we have a hyperbolic horizon $\kappa=-1$. The solution data is given by

$$\operatorname{Im}\widetilde{\mathcal{V}}^0=\operatorname{Im}\widetilde{\mathcal{V}}_i=\frac{1}{2\sqrt{2}\,g}\big(2g^3r+\sqrt{p^0g}\big),\qquad \operatorname{Im}\widetilde{\mathcal{V}}_0=\operatorname{Im}\widetilde{\mathcal{V}}_i=0, \tag{13.5.9}$$

and the metric components are

$$\mathrm{e}^{2V}=2\sqrt{2}r^2\big(g^3\,r+\sqrt{p^0g}\big)^2,\qquad \mathrm{e}^{2(V-U)}=\frac{1}{g}\big(2g^3r+\sqrt{p^0g}\big)^2. \tag{13.5.10}$$

13.5.3 $F=-X^0X^1$

We can write quite explicitly the solution when

$$p^0=-q_1,\quad p^1=q_0,\quad p^3=p^2,\quad q_3=q_2 \tag{13.5.11}$$

which is equivalent to considering the prepotential

$$F=-X^0X^1 \tag{13.5.12}$$

and allowing for four arbitrary charges. The solution to the constraint (12.2.27) is taken to be

$$p^0=\frac{p^2q_0}{q_2} \tag{13.5.13}$$

and we then find the following data:

$$\begin{aligned}(A_0)^0=-(A_0)_1&=-\frac{q_2(p^2-q_0)^2}{2((p^2)^2+q_2^2)(p^2q_0+q_2^2)(q_0^2+q_2^2)}\times \\ &\quad\times\big[(q_0^2-q_2^2)^2q_2^2+(p^2)^2(q_0^2+4q_0q_2+q_2^2)+2q_2p^2(2q_2^2-q_0q_2)\big],\end{aligned} \tag{13.5.14}$$

$$(A_0)^2=-(A_0)^3=\frac{p^2(p^2-q_0)^2q_0q_2}{((p^2)^2+q_2^2)(q_0^2+q_2^2)}, \tag{13.5.15}$$

$$(A_0)_0=-(A_0)^1=\frac{(p^2-q_0)q_2^3}{((p^2)^2+q_2^2)(q_0^2+q_2^2)}, \tag{13.5.16}$$

$$\begin{aligned}(A_0)_2=-(A_0)_3&=-\frac{q_2(p^2-q_0)^2}{2((p^2)^2+q_2^2)(p^2q_0+q_2^2)(q_0^2+q_2^2)}\times \\ &\quad\times\big[(q_0^2+q_2^2)^2(p^2)^2+2p^2q_2(q_2^2-2q_0^2)+q_2^2(q_0^2-4q_0q_2+q_2^2)\big].\end{aligned} \tag{13.5.17}$$

The NUT charge is given by

$$n\kappa=-\frac{g^{3/2}}{q_2(q_2-q_0)+p^2(q_0+q_2)}\sqrt{-\frac{((p^2)^2+q_2^2)(p^2q_0+q_2^2)(q_0^2+q_2^2)}{2q_2}} \tag{13.5.18}$$

and the metric components can be obtained from

$$v_2=(q_0-p^2)\sqrt{\frac{-2gq_2\big[(p^2)^2q_0^2+4p^2(p^2-q_0)q_0q_2+((p^2)^2+q_2^2)q_2^2+4(p^2-q_0)q_2^3+q_2^4\big]}{(p^2)^2+q_2^2)(p^2q_0+q_2^2)(q_0^2+q_2^2)}} \tag{13.5.19}$$

and

$$v_4=\sqrt{8}g^3. \tag{13.5.20}$$

Part IV

Demiański–Janis–Newman algorithm

Chapter 14

Janis–Newman algorithm

In this chapter we recall the original Janis–Newman algorithm, followed by Giampieri's prescription [99]. We stress that both prescriptions are perfectly equivalent and each step can be matched; in particular the only arbitrary point – present in both approach – is the complexification of the metric function.

Then we describe the complexification of the gauge field in both prescriptions [99], showing that a simple gauge transformation brings the field in a form compatible with the algotihm. In this context the transformation cannot be performed directly on the field strength.

For flat space the JN algorithm reduces to a change of coordinates, from spherical to oblate ones. Finally we review the transformation from Reissner–Nordström to Kerr–Newman.

14.1 Original prescription

In their original paper [96], Janis and Newman demonstrated how to recover the Kerr metric from the Schwarzschild one, and they extended it to discover the Kerr–Newman metric in [97].

In this section we outline the procedure with the seed metric

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \qquad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2. \tag{14.1.1}$$

This simple model is sufficient to illustrate the main features of the algorithm, while more general transformations, metrics and matter contents will be the topic of the chapters 15 and 16. This approach relies on the Newman–Penrose null tetrads formalism and more details can be found in [96, 97, 101, 110, 131].

The algorithm proceeds as follows (explicit formulas are given in the next section):

1. Introduce the null coordinate
$$\mathrm{d}u = \mathrm{d}t - f^{-1}\,\mathrm{d}r. \tag{14.1.2}$$
The metric becomes
$$\mathrm{d}s^2 = -f\,\mathrm{d}u^2 - 2\,\mathrm{d}u\mathrm{d}r + r^2\,\mathrm{d}\Omega^2. \tag{14.1.3}$$

2. Find the contravariant form of the metric, introduce a set of null tetrads
$$Z_a^\mu = \{\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu\} \tag{14.1.4}$$
with expressions
$$\ell^\mu = \delta_r^\mu, \qquad n^\mu = \delta_u^\mu - \frac{f}{2}\,\delta_r^\mu, \qquad m^\mu = \frac{1}{\sqrt{2}\bar{r}}\left(\delta_\theta^\mu + \frac{i}{\sin\theta}\,\delta_\phi^\mu\right), \tag{14.1.5}$$

and rewrite the inverse metric under the form

$$g^{\mu\nu} = \eta^{ab} Z_a^\mu Z_b^\nu = -\ell^\mu n^\nu - \ell^\nu n^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu, \tag{14.1.6}$$

with the flat metric

$$\eta^{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{14.1.7}$$

At this point r is real such that $\bar{r} = r$, the latter is introduced in view of the next steps.

3. Allow the coordinates u and r to take complex values together with the conditions:
   - ℓ^μ and n^μ must be kept real;
   - m^μ and $\bar{m}^\mu$ must still be complex conjugated to each other;
   - one should recover the previous basis for $u, r \in \mathbb{R}$.

   The previous conditions imply that the function $f(r)$ should be replaced by a new function $\tilde{f}(r, \bar{r}) \in \mathbb{R}$ such that $\tilde{f}(r, r) = f(r)$. This step is the hardest to perform because there is no *a priori* rule to choose any particular complexification and one needs to check systematically if Einstein equations are satisfied. Examples have provided a set of rules that can be used [96, 97, 99, 101]

$$r \longrightarrow \frac{1}{2}(r + \bar{r}) = \operatorname{Re} r, \tag{14.1.8a}$$

$$\frac{1}{r} \longrightarrow \frac{1}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) = \frac{\operatorname{Re} r}{|r|^2}, \tag{14.1.8b}$$

$$r^2 \longrightarrow |r|^2. \tag{14.1.8c}$$

   All other functions can be reduced to a combination of them. For example $1/r^2$ is complexified as $1/|r|^2$.

4. Carry out a complex change of coordinates

$$u = u' + ia\cos\theta, \qquad r = r' - ia\cos\theta, \qquad \theta' = \theta, \qquad \phi' = \phi, \tag{14.1.9}$$

   a being a parameter (with the interpretation of angular momentum per unit of mass), with the restriction that $r', u' \in \mathbb{R}$. The tetrads transform as vectors

$$Z_a'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} Z_a^\nu, \tag{14.1.10}$$

   and now $\tilde{f} = \tilde{f}(r', \theta')$ (but note that the θ' dependence is not arbitrary and comes solely from $\operatorname{Im} r$).

   Explicitly one gets (forgetting the primes on the coordinates for convenience)

$$\begin{gathered} \ell'^\mu = \delta_r^\mu, \qquad n'^\mu = \delta_u^\mu - \frac{\tilde{f}}{2}\,\delta_r^\mu, \\ m'^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(\delta_\theta^\mu + \frac{i}{\sin\theta}\,\delta_\phi^\mu - ia\sin\theta\,(\delta_u^\mu - \delta_r^\mu)\right). \end{gathered} \tag{14.1.11}$$

5. Construct the metric $g^{\mu\nu}$ from the new set of tetrads and obtain its covariant expression $g_{\mu\nu}$ by inverting it.

6. Eventually change the coordinates into any other preferred system, e.g. Boyer–Lindquist. If the transformation is infinitesimal then one should check that it is a valid diffeomorphism, i.e. that it is integrable.

The two last steps are common with Giampieri's prescription and will be detailed in the next section.

14.2 Giampieri's prescription

In the former approach it is very tedious to invert twice the metric and find out the right tetrad basis. In an essay submitted only to the *Gravity Research Foundation* [98], Giampieri proposed a simplification to this algorithm: the complexification of u and r and the change of coordinates are done directly in the metric. Then all complex i factors are removed using a specific ansatz for the coordinate transformation. It is important that both approaches are equivalent since the ansatz can be recovered by direct comparison with the tetrad computations [98, 99].

Giampieri applied his method only to the Schwarzschild metric, thus it is worth to detail it in the more general context of (14.1.1) with arbitrary f. The procedure is the following:

1. Introduce the null coordinate u

$$\mathrm{d}s^2 = -f\,\mathrm{d}u^2 - 2\,\mathrm{d}u\mathrm{d}r + r^2\,\mathrm{d}\Omega^2. \tag{14.2.1}$$

2. Allow the coordinates u and r to take complex values and complexify the metric (14.2.1) to

$$\mathrm{d}s'^2 = -\tilde{f}\,\mathrm{d}u^2 - 2\,\mathrm{d}u\mathrm{d}r + |r|^2\,\mathrm{d}\Omega^2, \tag{14.2.2}$$

using the rules (C.1.1c) for the coefficient of $\mathrm{d}\Omega^2$ and where again $\tilde{f} = \tilde{f}(r,\bar{r})$ is the real-valued function which is replacing f. At this step the metric continues being real.

3. Apply the change of coordinates (14.1.9)

$$u = u' + ia\cos\psi, \qquad r = r' - ia\cos\psi, \qquad \theta' = \theta, \qquad \phi' = \phi, \tag{14.2.3}$$

where a new angle ψ is introduced. This amounts to embedding the spacetime in a 5-dimensional complex spacetime and the final metric will correspond to a 4-dimensional real slice. The differentials read

$$\mathrm{d}u = \mathrm{d}u' - ia\sin\psi\,\mathrm{d}\psi, \qquad \mathrm{d}r = \mathrm{d}r' + ia\sin\psi\,\mathrm{d}\psi, \tag{14.2.4}$$

and one gets the metric

$$\begin{aligned}\mathrm{d}s'^2 = &-\tilde{f}(\mathrm{d}u - ia\sin\psi\,\mathrm{d}\psi)^2 - 2\,(\mathrm{d}u - ia\sin\psi\,\mathrm{d}\psi)(\mathrm{d}r + ia\sin\psi\,\mathrm{d}\psi)\\ &+ (r^2 + a^2\cos^2\theta)\,\mathrm{d}\Omega^2.\end{aligned} \tag{14.2.5}$$

4. As one can easily notice, this metric cannot be correct because it has to be real. Giampieri found that this metric reduces to the result from the original formulation if one uses the ansatz

$$i\,\mathrm{d}\psi = \sin\psi\,\mathrm{d}\phi \tag{14.2.6a}$$

followed by the replacement

$$\psi = \theta. \tag{14.2.6b}$$

Deleting all the primes, the metric obtained in the Kerr coordinates [96] is

$$\mathrm{d}s^2 = -\tilde{f}\,(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi)^2 - 2\,(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi)(\mathrm{d}r + a\sin^2\theta\,\mathrm{d}\phi) + \rho^2\mathrm{d}\Omega^2 \tag{14.2.7}$$

where we have introduced

$$\rho^2 = r^2 + a^2\cos^2\theta. \tag{14.2.8}$$

5. Finally one can go to Boyer–Lindquist coordinates with

$$\mathrm{d}u = \mathrm{d}t' - g(r)\mathrm{d}r, \qquad \mathrm{d}\phi = \mathrm{d}\phi' - h(r)\mathrm{d}r. \tag{14.2.9}$$

The conditions $g_{tr} = g_{r\phi'} = 0$ are solved for

$$g = \frac{r^2 + a^2}{\Delta}, \qquad h = \frac{a}{\Delta} \tag{14.2.10}$$

where we have defined

$$\Delta = \tilde{f}\rho^2 + a^2 \sin^2\theta. \tag{14.2.11}$$

As indicated by the r-dependence this change of variable is integrable provided that g and h are functions of r only. However Δ as given in (14.2.11) could in principle contain a dependence on θ, thus it is absolutely essential that one checks that this is not the case.

Given this condition one gets the metric (deleting the prime) [234, p. 14]

$$\mathrm{d}s^2 = -\tilde{f}\,\mathrm{d}t^2 + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\phi^2 + 2a(\tilde{f}-1)\sin^2\theta\,\mathrm{d}t\mathrm{d}\phi \tag{14.2.12}$$

with

$$\frac{\Sigma^2}{\rho^2} = r^2 + a^2 + a g_{t\phi}. \tag{14.2.13}$$

The rr-term has been computed from

$$g - a\sin^2\theta\,h = \frac{\rho^2}{\Delta}. \tag{14.2.14}$$

We stress that the order of the steps should be respected, otherwise the ansatz (14.2.6) cannot be consistently applied. The second important point is that JN and Giampieri's prescriptions differ only in the computation of the metric since the rules (C.1.1) are identical in both cases. Therefore this new approach is not adding nor removing any of the ambiguity that is already present and well-known in JN algorithm. In particular the ansatz (14.2.6) is a direct consequence of the fact that the 2-dimensional slice (θ, ϕ) is given by

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2, \tag{14.2.15}$$

the function in the RHS of (14.2.6) corresponding to $\sqrt{g^{\Omega}_{\phi\phi}}$ (where g is the static metric) as can be seen by doing the computation with $i\,\mathrm{d}\psi = H(\psi)\mathrm{d}\phi$ and identifying H at the end (in particular see section 15.2.1).

Another peculiar feature of this approach is that one should consider the complexification of the differentials and the complexification of the metric functions as two different processes: one can derive general formula as we did by taking f arbitrary while the differentials are transformed. From this point of view the r^2 factor in front of $\mathrm{d}\Omega^2$ can also be considered as a function with its own complexification.

Comparing (14.2.1) and (14.2.7) makes clear that the effect of the ansatz (14.2.6) can be reduced to modifying the formula (14.2.4) into

$$\mathrm{d}u = \mathrm{d}u' - a\sin^2\theta\,\mathrm{d}\phi, \qquad \mathrm{d}r = \mathrm{d}r' + a\sin^2\theta\,\mathrm{d}\phi. \tag{14.2.16}$$

Using directly these expressions allows to avoid introducing the angle ψ altogether. Although some authors [109, 119] mentioned the equivalence of these formulae and the result from the tetrads as a curiosity, it is surprising that this direction has not been followed further.

14.3 Validity

It is not necessary to specify the action for performing the algorithm as one needs only the expressions of the various seed fields, but one must check that the result is a solution of the equations of motion. Indeed it is not fully understood under which conditions the algorithm will send a solution to another solution since the complex transformation does not preserve Einstein equations in general.

Another important point is to check that the Boyer–Lindquist transformation (14.2.10) is integrable, i.e. that the function g and h depends only on r.

14.4 Gauge field

As already mentioned in the introduction, the authors of [97] face serious difficulties while trying to derive the field strength of the Kerr–Newman black hole from the Reissner–Nordström one. Indeed, in the null tetrad formalism, the field strength is given in terms of Newman–Penrose coefficients and problems arise when trying to generate the rotating solution since one of the coefficients, vanishing in the case of Reissner–Nordström, is non-zero for Kerr–Newman.

Three different prescriptions have been proposed recently: two works in the Newman–Penrose formalism – one with the field strength [156] and one with the gauge field [99] – while the third extends Giampieri's approach to the gauge field [99].

Our formulation is much more natural because it is more convenient to work with the gauge field rather than using the field strength or its Newman–Penrose coefficients (for example in view of matter coupling). Moreover it is also closer to the original spirit of the algorithm as one works with contravariant components (written with tetrads) for both the metric and the gauge field, and the transformation follows the same pattern.

Let's consider the simple gauge field

$$A = f_A(r)\,\mathrm{d}t, \tag{14.4.1}$$

the most general case being discussed in chapter 15 and in section 16.2.

14.4.1 Giampieri's prescription

We show that using Giampieri's prescription allows to circumvent the problem in a very simple way.

Expressing the gauge field (14.4.1) in terms of the (u, r) coordinates gives

$$A = f_A\,(\mathrm{d}u + f^{-1}\,\mathrm{d}r). \tag{14.4.2}$$

The second term actually does not contribute to the field strength since $A_r = A_r(r)$ and one can remove it by a gauge transformation, getting

$$A = f_A\,\mathrm{d}u. \tag{14.4.3}$$

Applying the transformations (14.2.16) gives

$$A' = f_A\,(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi). \tag{14.4.4}$$

Going to Boyer–Lindquist coordinates, using (14.2.10), provides

$$A' = \tilde{f}_A\left(\mathrm{d}t - \frac{\rho^2}{\Delta}\,\mathrm{d}r - a\sin^2\theta\,\mathrm{d}\phi\right) \tag{14.4.5}$$

where the relation (14.2.14) has been used. Generically the only θ-dependence of the function $\tilde{f}_A$ is in a factor $1/\rho^2$ which cancels the one in front of dr. Then we are left with $A'_r = A'_r(r)$, and it can again be removed by a gauge transformation, leaving (omitting the prime)

$$A = \tilde{f}_A \, (\mathrm{d}t - a \sin^2\theta \, \mathrm{d}\phi). \tag{14.4.6}$$

Notice that before the transformation $A_r/A_t = g_{rr} = f^{-1}$, while after the transformation $A'_r/A'_t = g'_{rr} = \rho^2/\Delta$.

14.4.2 Newman's prescription

Expression (14.4.3) for the static gauge potential – after the gauge transformation – can be rewritten as

$$A_\mu = f_A \, \delta^u_\mu. \tag{14.4.7}$$

Using the inverse of the metric (14.2.1) with function (14.5.3) one obtains the contravariant expression

$$A^\mu = -f_A \, \delta^\mu_r = -f_A \, \ell^\mu \tag{14.4.8}$$

where $\ell^\mu = \delta^\mu_r$, see (14.1.5).

The JN transformation applied to the previous expression yields

$$A'^\mu = -\tilde{f}_A \, \ell'^\mu = -\tilde{f}_A \, \delta^\mu_r \tag{14.4.9}$$

with $\ell'^\mu = \ell^\mu$ is defined in (14.1.11). Finally the 1-form

$$A' = \tilde{f}_A \, (\mathrm{d}u - a \sin^2\theta \, \mathrm{d}\phi) \tag{14.4.10}$$

is retrieved using the metric (14.2.7) with the function (14.5.5).

The result is identical to the one derived with Giampieri's formalism, showing again that the two approaches are totally equivalent, and that it was not necessary to use the null Lorentz rotation from [156]. It is possible to check that the transformation cannot be performed without first removing the r-component with the gauge transformation.

14.4.3 Keane's prescription

It is worth mentioning that another solution was recently proposed in [156], where a null Lorentz transformation on the tetrads is used to obtain the correct Newman–Penrose coefficients for the field strength.

14.5 Examples

14.5.1 Flat space

It is straightforward to check that the algorithm applied to the flat Minkowski metric – which has $f = 1$ – in spherical coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \big(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\big) \tag{14.5.1}$$

gives again the Minkowski metric but in spheroidal coordinates (after a Boyer–Lindquist transformation) (B.2.9)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\rho^2}{r^2 + a^2} \, \mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 + (r^2 + a^2) \sin^2\theta \, \mathrm{d}\phi^2, \tag{14.5.2}$$

recalling that $\rho^2 = r^2 + a^2 \cos^2\theta$. The metric is exactly diagonal because $g_{t\phi} = 0$ for $\tilde{f} = 1$ from (14.2.12). Hence for flat space the JN algorithm reduces to a change of coordinates, from spherical to spheroidal coordinates (the 2-spheres foliating the space in the radial direction are deformed to ellipses).

This fact is an important consistency check that will be useful when extending the algorithm to higher dimensions (chapter 18) or to other coordinate systems (such as one with direction cosines). Moreover in this case one can forget about the time direction and consider only the transformation of the radial coordinate.

14.5.2 Kerr–Newman black hole

In this section we apply the formalism to the Reissner–Nordström black hole in order to get the Kerr–Newman rotating black hole [97, 156], both of which are solutions of Einstein–Maxwell theory.

The seed solution corresponds to the metric

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2 d\Omega^2, \qquad f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \tag{14.5.3}$$

and to the gauge field

$$A = \frac{q}{r}\,dt \tag{14.5.4}$$

where the parameters m and q correspond respectively to the mass and to the electric charge.

Metric

Using the rules (C.1.1b) and (C.1.1c) for the second and third terms respectively, the function f can be complexified as

$$\tilde{f}(r,\theta) = 1 + \frac{q^2 - 2mr}{\rho^2} \tag{14.5.5}$$

where we recall that $\rho^2 = |r|^2 = r^2 + a^2\cos^2\theta$.

As already described in [97, 101], plugging this function into (14.2.12) gives the well-known Kerr–Newman metric

$$ds^2 = -\tilde{f}\,dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,d\phi^2 + 2a(\tilde{f}-1)\sin^2\theta\,dt d\phi, \tag{14.5.6}$$

where functions Δ and Σ are given by

$$\frac{\Sigma^2}{\rho^2} = r^2 + a^2 - \frac{q^2 - 2mr}{\rho^2}\,a^2\sin^2\theta, \tag{14.5.7a}$$

$$\Delta = r^2 - 2mr + a^2 + q^2, \tag{14.5.7b}$$

and it is to point out that Δ depends only on r so that the transformation (14.2.10) to Boyer–Lindquist coordinates is well defined.

Gauge field

Applying the recipe of section 14.4, the potential (14.5.4) of the Reissner–Nordström black hole leads directly to

$$A = \frac{qr}{\rho^2}\left(dt - \frac{\rho^2}{\Delta}\,dr - a\sin^2\theta\,d\phi\right), \tag{14.5.8}$$

where as usual $\rho^2 = r^2 + a^2\cos^2\theta$. The prefactor has been transformed using the rule (C.1.1b). Finally, the factor ρ^2 in front of dr cancels with the prefactor, and we are left with

$$A_r = \frac{qr}{\Delta} \tag{14.5.9}$$

which depends only on r. After a gauge transformation one obtains the traditional form of the electromagnetic gauge field for the Kerr–Newman black hole

$$A = \frac{qr}{\rho^2}\,(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi). \tag{14.5.10}$$

Chapter 15

Extended algorithm

In the previous chapter we chose a very specific complex change of coordinates (14.1.9). A natural question is to ask whether one can perform other changes of coordinates, and to find how to interpret them. Demiański gave an answer by considering a transformation with two unknown θ-dependent functions and by solving the equations of motion in a simple case [100] – we will call this version the Demiański–Janis–Newman (DJN) algorithm. Then one can hope that these transformations will be the most general ones (under the assumptions that are made), and one can use these transformations in other cases without having to solve the equations. The latter claim can be justified by looking at the equations of motions for more complex examples: even if one cannot find directly a solution, one finds that the same structure persists [159] (this is also motivated by the solutions in [136, 137]). Another strength of this approach is to remove the ambiguity of the algorithm since the functions are found from the equations of motion, and this may help when one does not know how to perform precisely the algorithm (for example in higher dimensions, see chapter 18).

In his analysis, Demiański finds that very few transformations can be done: they contain three parameters (rotation a, NUT charge n and c) when the cosmological constant is zero, and only one for non-vanishing cosmological constant (NUT charge n). At this point a new problem arises: when the transformation implies the NUT charge the usual rules (C.1.1) are not sufficient to transform the seed function. This lack would seriously reduce the utility of this improved algorithm because one cannot use it to discover new solutions without solving the equations of motion, which is not the goal of the algorithm. We demonstrate in section 15.4 that the transformation can be achieved by a complexification of the mass together with a shift of the horizon curvature [159]

$$m = m' + i\kappa\, n, \qquad \kappa \longrightarrow \kappa - \frac{4\Lambda}{3}\, n^2, \tag{15.0.1}$$

establishing that Demiański's transformations can be interpreted as an extension of the usual JN algorithm.

Demiański's paper [100] is short and results are extremely condensed and we explain in more details his approach. In particular we uncover an hidden assumption on the form of the metric function which explains the error in his formula (14) [113, 159]. A generalization of this hypothesis leads to other equations that we could not solve analytically, but this would lead to another solution. A result from Demiański's analysis is the impossibility to find Kerr–AdS from the DJN algorithm and it is often quoted as a no-go theorem. But this outcome relies on the assumption that no parameter already present in the static metric is complexified, which may not be justified.

One of the obvious generalization is the inclusion of a gauge field which is needed to obtain (electrically) charged solutions [159]. It appears that the analysis is left unchanged, the Maxwell equations being also integrable within Demiański's ansatz. This solution was already found in [136] but we demonstrate how to perform the full computation using the DJN algorithm, having in mind the possible generalizations to other cases.

Another improvement of the DJN algorithm that results from our analysis is the generalization of all formula to topological horizons [159]. In particular all existing formula can be straightforwardly generalized to the case of hyperbolic horizons,[1] and we prove all formula by solving explicitly Einstein equations. Topological horizons are of particular interest in supergravity models since asymptotically AdS black holes can possess non-spherical horizons.

We end the introduction by describing our ansatz. We consider the most general seed metric for which (θ, ϕ)-section are 2-dimensional maximally symmetric spaces (it can be the sphere S^2 or the hyperboloid H^2). Similarly the gauge field contains only one unknown radial function and it is purely electric. The DJN algorithm generates a stationary metric coupled to a gauge field for a total of six unknown functions (with only five being independent). We provide several formula in (u, r) and (t, r) coordinates that should be suitable for any application of the DJN algorithm.[2] Similar formula for subcases have been obtained in [101, 102, 121, 131]. All these computations are gathered in a Mathematica file (available on demand) which includes the computations of Einstein–Maxwell equations. We insist on the fact that all these results can also be derived from the tetrad formalism.

15.1 Setting up the ansatz

Einstein–Maxwell gravity with cosmological constant Λ reads [165, chap. 22]

$$S = \int \mathrm{d}^4x \, \sqrt{-g} \left(\frac{1}{2\varkappa^2} (R - 2\Lambda) - \frac{1}{4} F^2 \right), \tag{15.1.1}$$

where $\varkappa^2 = 8\pi G$ is the Einstein coupling constant, g is the metric with Ricci scalar R and $F = \mathrm{d}A$ is the field strength of the Maxwell field. In our conventions the spacetime signature is mostly plus and in the following we set κ to 1.

The associated equations of motion are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 2\, T_{\mu\nu}, \qquad \nabla_\mu F^{\mu\nu} = 0, \tag{15.1.2}$$

where the stress–energy tensor for the electromagnetic field is

$$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4}\, g_{\mu\nu} F^2. \tag{15.1.3}$$

The static electromagnetic one-form is taken to be

$$A(r) = f_A(r)\, \mathrm{d}t. \tag{15.1.4}$$

This ansatz is purely electric since only the time component is non-zero.

The static metric ansatz in coordinates (t, r, θ, ϕ) reads

$$\mathrm{d}s^2 = -f_t(r)\, \mathrm{d}t^2 + f_r(r)\, \mathrm{d}r^2 + f_\Omega(r)\, \mathrm{d}\Omega^2. \tag{15.1.5}$$

One of the functions is redundant since we are free to redefine the radial coordinate.

The (θ, ϕ) sections correspond to 2-dimensional maximally symmetric spaces, which are the sphere S^2, the euclidean plane $\mathbb{R}^2$ and the hyperboloid H^2 respectively for positive, vanishing and negative curvature [76].[3] Defining κ as the sign of the surface curvature, the uniform metric $\mathrm{d}\Omega^2$ is given by

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + H(\theta)^2\, \mathrm{d}\phi^2 \tag{15.1.6}$$

[1] We do not treat the case of flat horizon but this could be obtained from some easy reparametrization.

[2] We stress that at this stage these formula do not satisfy Einstein equations, they are just proxy to simplify later computations.

[3] The convention are slightly different from the one in the appendix A.7. One needs to make the replacement $(H, H') \to (-\kappa H', H)$.

with

$$H(\theta) = \begin{cases} \sin\theta & \kappa = 1, \\ \sinh\theta & \kappa = -1. \end{cases} \tag{15.1.7}$$

We focus on $\kappa = \pm 1$, the case $\kappa = 0$ can be deduced easily.

Introducing the null coordinates u through the change of coordinates

$$\mathrm{d}t = \mathrm{d}u + \sqrt{\frac{f_r}{f_t}}\,\mathrm{d}r, \tag{15.1.8}$$

the static metric (15.1.5) becomes

$$\mathrm{d}s^2 = -f_t\,\mathrm{d}u^2 - 2\sqrt{f_t f_r}\,\mathrm{d}u\mathrm{d}r + f_\Omega\left(\mathrm{d}\theta^2 + H^2\,\mathrm{d}\phi^2\right), \tag{15.1.9}$$

while the gauge field (15.1.4) is found to be

$$A = f_A\left(\mathrm{d}u + \sqrt{\frac{f_r}{f_t}}\,\mathrm{d}r\right). \tag{15.1.10}$$

Since the component A_r depends only on r it can be removed by a gauge transformation [99] such that

$$A = f_A\,\mathrm{d}u. \tag{15.1.11}$$

This step is primordial for having a consistent DJN transformation.

15.2 Demiański–Janis–Newman algorithm

In this section we apply the Janis–Newman algorithm to the ansatz of the previous section. Using arbitrary functions for the complex transformation and for the functions inside the metric, we obtain a very general ansatz; then we will solve Einstein–Maxwell equations in the next section in order to find their forms. We will directly use Giampieri's prescription [98, 99] in order to avoid the introduction of tetrads and the computation of the contravariant components of the metric and of the gauge field.

15.2.1 Janis–Newman transformation

The Janis–Newman algorithm can be summarized as the following sequence of steps:

1. Start with a seed metric in (u, r) coordinates.
2. Let the coordinates u and r become complex.
3. Replace the functions inside the metric by other functions depending on r and its conjugate.
4. Make a change of coordinates $(r, u) \to (r', u')$, the new coordinates being real.
5. Apply Giampieri's ansatz to recover a real metric.

The complex change of coordinates is given by[4] [100]

$$r = r' + i\,F(\theta), \qquad u = u' + i\,G(\theta), \tag{15.2.1}$$

[4] Similar transformations have been studied by Talbot [103].

where $u', r' \in \mathbb{R}$, and $F(\theta)$ and $G(\theta)$ are two arbitrary functions.[5] Usually these functions are taken to be

$$F(\theta) = -a\cos\theta, \qquad G(\theta) = a\cos\theta, \tag{15.2.2}$$

but here they are kept general and the most general transformation will be determined by Einstein equations.

As given by

$$\mathrm{d}r = \mathrm{d}r' + i\,F'(\theta)\,\mathrm{d}\theta, \qquad \mathrm{d}u = \mathrm{d}u' + i\,G'(\theta)\,\mathrm{d}\theta, \tag{15.2.3}$$

(the prime on F and G denoting the differentiation with respect to θ), the differentials of the coordinates are complex which is not coherent with having a complex metric. The (generalized) Giampieri's ansatz consists in the replacement

$$i\,\mathrm{d}\theta = \sqrt{g^{\Omega}_{\phi\phi}}\,\mathrm{d}\phi = H(\theta)\,\mathrm{d}\phi, \tag{15.2.4}$$

where the RHS is given by comparison of the final result with the tetrad formalism [98, 99, 109]. As a consequence the transformation of the differentials are

$$\mathrm{d}r = \mathrm{d}r' + F'(\theta)H(\theta)\,\mathrm{d}\phi, \qquad \mathrm{d}u = \mathrm{d}u' + G'(\theta)H(\theta)\,\mathrm{d}\phi. \tag{15.2.5}$$

Finally the four functions

$$f_i(r) = \{f_t, f_r, f_\Omega, f_A\} \tag{15.2.6}$$

are transformed to

$$\tilde{f}_i(r,\bar{r}) = \{\tilde{f}_t, \tilde{f}_r, \tilde{f}_\Omega, \tilde{f}_A\}. \tag{15.2.7}$$

There are only two conditions that we impose on these functions

$$\tilde{f}_i = \tilde{f}_i(r,\bar{r}) = \tilde{f}_i\big(r', F(\theta)\big) \in \mathbb{R}, \qquad \tilde{f}_i(r', 0) = f_i(r'). \tag{15.2.8}$$

The first relation means that the dependence in θ is solely contained in the functional dependence of $F(\theta)$.[6] On the other hand we do not try to get the functions $\tilde{f}_i$ from the complexification of the static functions [100]; this is the topic of section 15.4.

As a consequence the θ-derivative of $\tilde{f}_i$ reads

$$\partial_\theta \tilde{f}_i = F'\,\partial_F \tilde{f}_i \tag{15.2.9}$$

such that it is sufficient to obtain the dependence of $\tilde{f}_i$ in term of F.

Note that general conditions that need to be satisfied by F and G can be found in [110, sec. 2.3, 103].

15.2.2 Metric

Applying the transformations (15.2.1) and (15.2.5) and replacing the functions, the resulting stationary metric in Eddington–Finkelstein coordinates is

$$\mathrm{d}s^2 = -\tilde{f}_t(\mathrm{d}u + \alpha\,\mathrm{d}r + \omega H\,\mathrm{d}\phi)^2 + 2\beta\,\mathrm{d}r\mathrm{d}\phi + \tilde{f}_\Omega\big(\mathrm{d}\theta^2 + \sigma^2 H^2 \mathrm{d}\phi^2\big) \tag{15.2.10}$$

where we defined the quantities

$$\omega = G' + \sqrt{\frac{\tilde{f}_r}{\tilde{f}_t}}\,F', \qquad \sigma^2 = 1 + \frac{\tilde{f}_r}{\tilde{f}_\Omega}\,F'^2, \qquad \alpha = \sqrt{\frac{\tilde{f}_r}{\tilde{f}_t}}, \qquad \beta = \tilde{f}_r\,F'H. \tag{15.2.11}$$

The transformation

$$\mathrm{d}u = \mathrm{d}t - g(r)\mathrm{d}r, \qquad \mathrm{d}\phi = \mathrm{d}\phi' - h(r)\mathrm{d}r \tag{15.2.12}$$

[5] In his paper [100] Demiański considers functions that depend on θ and ϕ, but he drops the ϕ-dependence at an intermediate step. In our case we want to keep the U(1) isometry so we do not consider this case.

[6] This assumption is not explicit in Demiański's paper [100].

can be used to set the coefficient g_{ur} and $g_{r\phi}$ to zero and to cast the metric in Boyer–Lindquist (BL) coordinates. The solution to these two conditions is

$$g(r) = \frac{\sqrt{\left(\tilde{f}_t \tilde{f}_r\right)^{-1}}\, \tilde{f}_\Omega - F' G'}{\Delta}, \qquad h(r) = \frac{F'}{H(\theta)\Delta} \tag{15.2.13}$$

with

$$\Delta = \frac{\tilde{f}_\Omega}{\tilde{f}_r} + F'^2 = \frac{\tilde{f}_\Omega}{\tilde{f}_r}\, \sigma^2. \tag{15.2.14}$$

We stress that the functions g and h cannot depend on θ, otherwise the change of variables (15.2.12) is not integrable. It is thus necessary to check for given functions $\tilde{f}_i$, F and G that all the θ-dependence cancels.

Finally the metric in (t, r) coordinates can be written (removing the prime on ϕ)

$$\mathrm{d}s^2 = -\tilde{f}_t \big(\mathrm{d}t + \omega H\, \mathrm{d}\phi\big)^2 + \frac{\tilde{f}_\Omega}{\Delta}\, \mathrm{d}r^2 + \tilde{f}_\Omega \big(\mathrm{d}\theta^2 + \sigma^2 H^2 \mathrm{d}\phi^2\big). \tag{15.2.15}$$

15.2.3 Gauge field

Applying the DJN transformations (15.2.5) to the gauge field (15.1.11)

$$A = f_A\, \mathrm{d}u \tag{15.2.16}$$

gives[7]

$$A = \tilde{f}_A\, (\mathrm{d}u + G' H\, \mathrm{d}\phi). \tag{15.2.17}$$

Using the explicit formula (15.2.13), the previous expression becomes in Boyer–Lindquist coordinates

$$A = \tilde{f}_A \left(\mathrm{d}t - \frac{\tilde{f}_\Omega}{\sqrt{\tilde{f}_t \tilde{f}_r}\, \Delta}\, \mathrm{d}r + G' H\, \mathrm{d}\phi \right). \tag{15.2.18}$$

Here the function

$$A_r = -\frac{\tilde{f}_A \tilde{f}_\Omega}{\sqrt{\tilde{f}_t \tilde{f}_r}\, \Delta} = -\frac{g_{rr}}{\sqrt{\tilde{f}_t \tilde{f}_r}}\, A_t \tag{15.2.19}$$

may depend on θ in which case it would not be possible to remove it by a gauge transformation.[8]

15.3 Charged topological solution

In this section we solve Einstein–Maxwell equations (15.1.2) for the system

$$f_t = f, \qquad f_r = f^{-1}, \qquad f_\Omega = r^2. \tag{15.3.1}$$

First the static solution is recalled for later comparison – it corresponds to the static limit of the stationary solution. The stationary solution is derived in (u, r) coordinates in order to avoid the question of the validity of the Boyer–Lindquist transformation and because the metric looks simpler.

[7] This may also be derived from the tetrad formalism [99, 110, 156].

[8] In several examples where BL coordinates exist, A_r depends only on r. This seems to be the generic case.

15.3.1 Static case

Consider the static metric (15.1.5) and gauge field (15.1.4).

Only the (t) component of Maxwell equations is non trivial

$$2f_A' + r f_A'' = 0, \tag{15.3.2}$$

the prime being a derivative with respect to r, and its solution is

$$f_A(r) = \frac{q}{r} \tag{15.3.3}$$

where q is a constant of integration that is interpreted as the charge (we set the additional constant to zero since it can be removed by a gauge transformation).

The only relevant Einstein equation is

$$\frac{q^2}{r^2} - \kappa + r^2\Lambda + f + r f' = 0 \tag{15.3.4}$$

whose solution reads

$$f(r) = \kappa - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{\Lambda}{3}\, r^2, \tag{15.3.5}$$

m being a constant of integration that is identified to the mass.

We stress that we are just looking to solutions of Einstein equations and we are not concerned with regularity (in particular it is well-known that only $\kappa = 1$ is well-defined for $\Lambda = 0$).

15.3.2 Simplifying the equations

The component $(r\theta)$ gives the equation

$$F'\left(G'' + \frac{H'}{H}\, G'\right) = 2FF' \tag{15.3.6}$$

which depends only on θ and it allows to solve for G in terms of F. If $F' \neq 0$ it implies the equation (rr) which is

$$G'' + \frac{H'}{H}\, G' = \pm 2F. \tag{15.3.7}$$

If $F' = 0$ this last equation should be used instead and the sign can be absorbed into F since it is an arbitrary constant. As a result the equation in both cases is

$$G'' + \frac{H'}{H}\, G' = 2F. \tag{15.3.8}$$

The r-component of the Maxwell equation can be integrated to

$$\tilde{f}_A = \frac{q\, r}{r^2 + F^2} + \alpha\, \frac{r^2 - F^2}{r^2 + F^2}. \tag{15.3.9}$$

We can remove the constant α by matching with the static case in the limit $F \to 0$, but we can also get this result from the θ-equation

$$\alpha\, F' = 0. \tag{15.3.10}$$

The (tr) equation contains only r-derivative of $\tilde{f}$ and it can be integrated to[9]

$$\tilde{f} = \kappa - \frac{2mr - q^2 + 2F(\kappa\, F + K)}{r^2 + F^2} - \frac{\Lambda}{3}\,(r^2 + F^2) - \frac{4\Lambda}{3}\, F^2 + \frac{8\Lambda}{3}\, \frac{F^4}{r^2 + F^2} \tag{15.3.11}$$

[9]In [100] the last term of $\tilde{f}$ is missing [113], as can be compared with other references on (A)dS–Taub–NUT, see for example [76].

where again m is a constant of integration interpreted as the mass. The function K is defined by

$$2K = F'' + \frac{H'}{H} F'. \tag{15.3.12}$$

This implies the equations $(r\phi)$ and $(\theta\theta)$.

As explained in section 15.2.1 the θ-dependence should be contain in $F(\theta)$ only. The second term of the function $\tilde{f}$ contains some lonely θ from the $H(\theta)$ in the function K: this means that they should be compensated by the F, and we therefore ask that the sum $\kappa F + K$ be constant[10]

$$\kappa F' + K' = 0 \Longrightarrow \kappa F + K = \kappa n. \tag{15.3.13}$$

The parameter n is interpreted as the NUT charge.

The components $(t\theta)$ and $(\theta\phi)$ give the same equation

$$\Lambda F' = 0. \tag{15.3.14}$$

Finally one can check that the last three equations $(tt), (t\phi)$ and $(\phi\phi)$ are satisfied.

Let's summarize the equations

$$2F = G'' + \frac{H'}{H} G', \tag{15.3.15a}$$

$$\kappa n = \kappa F + K, \tag{15.3.15b}$$

$$0 = \Lambda F' \tag{15.3.15c}$$

and the function $\tilde{f}$

$$\tilde{f} = \kappa - \frac{2mr - q^2 + 2F(\kappa F + K)}{r^2 + F^2} - \frac{\Lambda}{3}(r^2 + F^2) - \frac{4\Lambda}{3} F^2 + \frac{8\Lambda}{3} \frac{F^4}{r^2 + F^2}. \tag{15.3.15d}$$

We also defined

$$2K = F'' + \frac{H'}{H} F'. \tag{15.3.15e}$$

As explained in the introduction, a major issue of Demiański's approach is the impossibility to obtain – at least in a direct manner – the stationary $\tilde{f}$ function (15.3.15d) as a complexification of the static f function (15.3.5). Not being able to reproduce the stationary function from the static one is equivalent to a failure because it would not be possible to apply the algorithm to other cases. This is one of the reason explaining why applications of the JN algorithm have been limited to adding a rotation parameter. We address this question in section 15.4 and show how to recover $\tilde{f}$ from f.

In the next sections we solve explicitly the equations (15.3.15), and because the case $\Lambda = 0$ and $\Lambda \neq 0$ are really different we consider them separately.

15.3.3 Solution for $\Lambda \neq 0$

Equation (15.3.15c) implies that $F' = 0$ and then

$$F(\theta) = n \tag{15.3.16}$$

by compatibility with (15.3.15b) and since $K(\theta) = 0$.

Solution to (15.3.15a) is

$$G(\theta) = c_1 - 2\kappa n \ln H(\theta) + c_2 \ln \frac{H(\theta/2)}{H'(\theta/2)} \tag{15.3.17}$$

[10] In section 15.3.5 we relax this last assumption by allowing non-constant $\kappa F + K$. In this context the equations and the function $\tilde{f}$ are modified and this provides an explanation for the error in $\tilde{f}$ of Demiański's paper [100].

where c_1 and c_2 are two constants of integration. Since only G' appears in the metric we can set $c_1 = 0$. On the other hand the constant c_2 can be removed by the transformation

$$\mathrm{d}u = \mathrm{d}u' - c_2\,\mathrm{d}\phi. \tag{15.3.18}$$

We summarize the solution to the system (15.3.15)

$$F(\theta) = n, \qquad G(\theta) = -2\kappa\, n \ln H(\theta). \tag{15.3.19}$$

The function $\tilde{f}$ then takes the form

$$\tilde{f} = \kappa - \frac{2mr - q^2 + 2\kappa n^2}{r^2 + n^2} - \frac{\Lambda}{3}(r^2 + 5n^2) + \frac{8\Lambda}{3}\,\frac{n^4}{r^2 + n^2} \tag{15.3.20a}$$

$$= \kappa - \frac{2mr - q^2 + 2\kappa n^2}{r^2 + n^2} - \frac{\Lambda}{3}\,\frac{r^4 + 6n^2 - 3n^4}{r^2 + n^2}. \tag{15.3.20b}$$

The transformation to BL coordinates is well defined (and $h = 0$)

$$g = \frac{r^2 + n^2}{\Delta}, \qquad \Delta = \kappa r^2 - 2mr + q^2 + \Lambda n^4 - \frac{\Lambda}{3}\, r^4 - n^2(\kappa + 2\Lambda r^2). \tag{15.3.21}$$

As noted by Demiański the only parameters that appear are the mass and the NUT charge, and it is not possible to add an angular momentum for non-vanishing cosmological constant.[11] As a consequence the JN algorithm cannot provide a derivation of (A)dS–Kerr–Newman.

15.3.4 Solution for $\Lambda = 0$

The solution to the differential equation (15.3.15b) is

$$F(\theta) = n - a\, H'(\theta) + \kappa\, c \left(1 + H'(\theta)\, \ln \frac{H(\theta/2)}{H'(\theta/2)}\right) \tag{15.3.22}$$

where a and c denote two constants of integration.

We solve the equation (15.3.15a) for G

$$\begin{aligned} G(\theta) = c_1 + \kappa\, a\, H'(\theta) - c\, H'(\theta)\, \ln \frac{H(\theta/2)}{H'(\theta/2)} - 2\kappa\, n \ln H(\theta) \\ + (a + c_2) \ln \frac{H(\theta/2)}{H'(\theta/2)} \end{aligned} \tag{15.3.23}$$

and c_1, c_2 are constants of integration. Again since only G' appears in the metric we can set $c_1 = 0$. We can also remove the last term with the transformation

$$\mathrm{d}u = \mathrm{d}u' - (c_2 + a)\mathrm{d}\phi. \tag{15.3.24}$$

We arrive at

$$F(\theta) = n - a\, H'(\theta) + \kappa\, c \left(1 + H'(\theta)\, \ln \frac{H(\theta/2)}{H'(\theta/2)}\right), \tag{15.3.25a}$$

$$G(\theta) = \kappa\, a\, H'(\theta) - \kappa\, c\, H'(\theta)\, \ln \frac{H(\theta/2)}{H'(\theta/2)} - 2n \ln H(\theta). \tag{15.3.25b}$$

The Boyer–Lindquist transformation is well defined only for $c = 0$, in which case

$$g = \frac{r^2 + a^2 + n^2}{\Delta}, \qquad h = \frac{\kappa a}{\Delta}, \qquad \Delta = \kappa r^2 - 2mr + q^2 - \kappa n^2 + \kappa a^2. \tag{15.3.26}$$

[11] In [95] Leigh et al. generalized Geroch's solution generating technique and also found that only the mass and the NUT charge appear when $\Lambda \neq 0$. We would like to thank D. Klemm for this remark.

The function $\tilde{f}$ reads [76, sec. 2.2]

$$\tilde{f} = \kappa - \frac{2mr - q^2}{\rho^2} + \frac{\kappa\, n(n - aH')}{\rho^2}, \qquad \rho^2 = r^2 + (n - a\, H')^2. \tag{15.3.27}$$

The constant a corresponds to the angular momentum (and one recognizes the usual JN algorithm). The interpretation is difficult because there is a wire-like singularity that extends to infinity [110, sec. 2.3, 235, sec. 5.3]. The spacetime is of type II if $c \neq 0$, otherwise it is of type D.

This solution was already found in [136] for the case $\kappa = 1$ by solving directly Einstein–Maxwell equations, starting with a metric ansatz of Demiański's form. In our case we wish to show that the same solution can be obtained by applying Demiański's method on all the quantities, including the gauge field.

15.3.5 Hidden assumptions in Demiański's paper

Demiański's paper is short and results are extremely condensed. In particular we uncover a hidden assumption on the form of the metric function which explains the error in his formula (14) [113].

In section 15.3.2 we obtained the equation (15.3.15b)

$$\kappa\, F + K = \kappa\, n, \qquad 2K = F'' + \frac{H'}{H}\, F' \tag{15.3.28}$$

by asking that the function (15.3.15d)

$$\tilde{f} = \kappa - \frac{2mr - q^2 + 2F(\kappa\, F + K)}{r^2 + F^2} - \frac{\Lambda}{3}\,(r^2 + F^2) - \frac{4\Lambda}{3}\, F^2 + \frac{8\Lambda}{3}\, \frac{F^4}{r^2 + F^2} \tag{15.3.29}$$

depends on θ only through $F(\theta)$.

A more general assumption would be that $\kappa F + K$ is some function $\chi = \chi(F)$

$$\kappa\, F + K = \kappa\, \chi(F). \tag{15.3.30}$$

The $(t\theta)$-component gives the equation

$$4\Lambda\, F^2 F' = F'\, \partial_F \chi. \tag{15.3.31}$$

If $F' = 0$ or $\Lambda = 0$ we found that

$$\partial_F \chi = 0 \Longrightarrow \chi = n \tag{15.3.32}$$

which reduces to the case studied in section 15.3.2.

On the other hand if $F' \neq 0$ then the previous equation becomes

$$\partial_F \chi = 4\Lambda\, F^2 \tag{15.3.33}$$

which can be integrated to

$$\chi(F) = n + \frac{4}{3}\, \Lambda F^3 \tag{15.3.34}$$

(notice that the limit $\Lambda \to 0$ is coherent). Plugging this function into equation (15.3.30) one obtains

$$\kappa\, F + K = \kappa \left(n + \frac{4}{3}\, \Lambda F^3 \right). \tag{15.3.35}$$

This differential equation is non-linear and we were not able to find an analytical solution.

Nonetheless by inserting the expression of χ in $\tilde{f}$ we see that the last term is killed

$$\tilde{f} = \kappa - \frac{2mr - q^2 + 2\kappa\, n\, F}{r^2 + F^2} - \frac{\Lambda}{3}\,(r^2 + F^2) - \frac{4\Lambda}{3}\, F^2. \tag{15.3.36}$$

One can recognize the function given by Demiański [100]. Then this function is valid at the condition that equation (15.3.15b) is modified to (15.3.35), but in this case the solution is not the general (A)dS–Taub–NUT anymore.

15.4 Finding the complexification

At the end of section 15.3.2, we mentioned the issue of finding the complexification of the stationary function from the static one. The rules (C.1.1) continue to apply with the parameter c, but they are not sufficient when one is considering the NUT charge n. Indeed the last case also requires the complexification of the mass parameter. In what follows we ignore the electric charge since it does not modify the discussion.

15.4.1 $\Lambda = 0$

The static Schwarzschild function (15.3.5)

$$f = \kappa - \frac{2m}{r} \tag{15.4.1}$$

is complexified as

$$\tilde{f} = \kappa - \left(\frac{m}{r} + \frac{\bar{m}}{\bar{r}} \right) = \kappa - \frac{2 \operatorname{Re}(m\bar{r})}{|r|^2}. \tag{15.4.2}$$

Performing the transformation

$$m = m' + i\kappa n, \qquad r = r' + iF \tag{15.4.3}$$

gives

$$\tilde{f} = \kappa - \frac{2mr + 2\kappa n F}{r^2 + F^2} \tag{15.4.4}$$

which corresponds to the correct function (15.3.15d).

15.4.2 $\Lambda \neq 0$

The procedure is less straightforward in this case and we only give some preliminary steps towards the solution.

The static Schwarzschild function (15.3.5)

$$f = \kappa - \frac{2m}{r} - \frac{\Lambda}{3}\, r^2 \tag{15.4.5}$$

is complexified as

$$\tilde{f} = \kappa - \frac{2 \operatorname{Re}(m\bar{r})}{|r|^2} - \frac{\Lambda}{3}\, |r|^2 . \tag{15.4.6}$$

The complexification of the mass parameter is[12]

$$m = m' + in \left(\kappa - \frac{4\Lambda}{3}\, n^2 \right), \qquad r = r' + in. \tag{15.4.7}$$

Moreover comparing the imaginary part of m with the previous case (15.4.3) suggests the replacement of the curvature sign[13] (only in the one appearing in f, not the one in (15.4.7))

$$\kappa \longrightarrow \kappa - \frac{4\Lambda}{3}\, n^2. \tag{15.4.8}$$

Note that the κ which appears in F and G are not shifted.

[12] The imaginary part of the new mass term appears in other contexts [18, 114, 236, 237]. In particular this corresponds to a condition of regularity in Euclidean signature.

[13] Notice that AdS–Taub–NUT (for $\kappa = -1$, $m = 0$) is supersymmetric for $n = \pm 1/(2g)$ where $g^2 = -\Lambda/3$ [76, tab. 1].

Presented in another way, the algorithm is to first perform the transformation (15.4.3) followed by the above replacement for κ everywhere

$$m = m' + i\kappa\, n, \qquad \kappa \longrightarrow \kappa - \frac{4\Lambda}{3}\, n^2. \tag{15.4.9}$$

One can notice that the limit $\Lambda \to 0$ agrees with the previous section (upon replacing n by F).

Inserting these transformations into $\tilde{f}$ gives the result

$$\tilde{f} = \kappa - \frac{2mr + 2\kappa n^2}{r^2 + n^2} - \frac{\Lambda}{3}\,(r^2 + 5n^2) + \frac{8\Lambda}{3}\,\frac{n^4}{r^2 + n^2} \tag{15.4.10}$$

and we retrieve (15.3.20).

Chapter 16

Algorithm with matter fields

Supergravity rotating solutions is an intense field of research, and it is surprising that the (D)JN algorithm has almost never been applied in this context (with the exception of [115]). One explanation is that such theories present a number of gauge fields and complex scalar fields that could not be transformed in the original formulation of the DJN algorithm. For instance, Yazadjiev [115] showed that it was possible to obtain the metric and the dilaton of Sen's dilaton–axion charged rotating black hole [157] (non-extremal solution of the T^3 model), but did not succeed in finding the axion nor the gauge field.

Each of these problems possess a different explanation. First of all, it was not known how to perform the transformation on the gauge field until recently, where two different prescriptions have been proposed [99, 110, 156].

The second problem is that you cannot transform independently the dilaton and the axion because they are naturally gathered into a complex scalar field. In particular the axion is vanishing for the static configuration, while it is non-zero for the rotating black hole. Moreover the usual transformation rules cannot be applied to complex scalar fields because they include a reality condition which is a too strong requirement for transforming complex fields, and one of our goal is to show how to modify the original prescription to accommodate this new fact [158]. We will illustrate this proposal on several examples, all taken from $N = 2$ ungauged supergravity, completing Yazadjiev's analysis [115] of Sen's rotating black hole, and showing how some BPS rotating black holes from [36] can be obtained (which include solutions from pure supergravity and from the STU model).

Another issue arises when one considers the NUT charge n. Indeed the usual rules (C.1.1) do not hold and it was shown in [159] that one needs to complexify the mass as $m = m' + in$ (see section 15.4).

A related case concerns dyonic solutions with electric and magnetic charges q and p, which can be used as a seed metric. It is necessary to follow the recipe of the previous examples, since the original JN rules are failing again. This is related to the fact that the electric and magnetic charges are naturally associated into the (complex) central charge $Z = q + ip$. In this way we succeed in performing the JN algorithm to a solution with magnetic charges.

First we describe explicitly the Kerr–Newman–Taub–NUT solution to recall the methods of the previous section, and then we turn to the more interesting dyonic Kerr–Newman–Taub–NUT and charged Taub–NUT–BBMB with Λ [35].

Finally let's note that Kerr–Newman solution and its extensions can be embedded into $N = 2$ supergravity [76, 77]. Another interesting point is that most of the examples presented in this chapter are truncations of the Chow–Compère black hole [44], and it would be useful to understand in which cases the DJN algorithm can be applied to this solution.

Moreover we describe two results which did not appear elsewhere before: the discussion of the Yang–Mills Kerr–Newman black hole [161] and Taub–NUT–BBMB solution of section 16.2.3.

16.1 Real scalar fields

Given a set of real scalar fields $\chi_a(r)$, they are complexified and transformed exactly as a metric function, see section 15.2.1

$$\chi_a(r) \longrightarrow \tilde{\chi}_a(r, \theta). \tag{16.1.1}$$

16.2 Gauge fields

16.2.1 Kerr–Newman–Taub–NUT black hole

A long-standing difficulty of Demiański's extension of the JN algorithm [100] was the impossibility to find the complexification of the metric function that was leading from Schwarzschild to Kerr–Taub–NUT. In this section we recall the solution to this problem that we gave in a previous paper [99], where we extended Demiański's result to Kerr–Newman–Taub–NUT.

Reissner–Nordström metric is given by

$$ds^2 = -f(r)\, dt^2 + f(r)^{-1}\, dr^2 + r^2 d\Omega^2, \qquad f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \tag{16.2.1a}$$

m and q being the mass and the electric charge, and the electromagnetic gauge field reads

$$A = \frac{q}{r}\, dt. \tag{16.2.1b}$$

As explained in section 15.4 it is necessary to complexify the mass. In this case the function f is complexified as

$$\tilde{f} = 1 - \left(\frac{m}{r} + \frac{\bar{m}}{\bar{r}}\right) + \frac{q^2}{|r|^2} = 1 - \frac{2\,\mathrm{Re}(m\bar{r}) + q^2}{|r|^2}, \tag{16.2.2}$$

and performing the transformation

$$m = m' + in, \qquad r = r' + iF \tag{16.2.3}$$

gives (omitting the primes)

$$\tilde{f} = 1 - \frac{2mr + 2nF}{\rho^2}, \qquad \rho^2 = r^2 + F^2. \tag{16.2.4}$$

Considering the transformations (16.2.3) leads to

$$\tilde{f} = 1 - \frac{2mr - q^2 + n(n - a\cos\theta)}{\rho^2}, \qquad \rho^2 = r^2 + (n - a\cos\theta)^2. \tag{16.2.5}$$

The metric and the gauge fields in BL coordinates can be read from (C.3.3) to be

$$ds^2 = -\tilde{f}\,(dt + \Omega\, d\phi)^2 + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2(d\theta^2 + \sigma^2 H^2 d\phi^2), \tag{16.2.6a}$$

$$A = \frac{q}{\rho^2}\Big(dt - (a\sin^2\theta + 2n\cos\theta)d\phi\Big) + A_r\, dr. \tag{16.2.6b}$$

One can check that A_r is a function of r only

$$A_r = -\frac{q}{\Delta} \tag{16.2.7}$$

and it can be removed by a gauge transformation. The various quantities that appear are given by

$$\Omega = -2n\cos\theta - (1 - \tilde{f}^{-1})\, a\sin^2\theta, \qquad \sigma^2 = \frac{\Delta}{\tilde{f}\rho^2}, \qquad \Delta = \tilde{f}\rho^2 + a^2\sin^2\theta. \tag{16.2.8}$$

This corresponds to the Kerr–Newman–Taub–NUT solution [76].

16.2.2 Dyonic Kerr–Newman black hole

The dyonic Reissner–Nordström metric is obtained from the electric one (16.2.1) by the replacement [238, sec. 6.6]

$$q^2 \longrightarrow |Z|^2 = q^2 + p^2 \tag{16.2.9}$$

where Z corresponds to the central charge [76]

$$Z = q + ip. \tag{16.2.10}$$

This is particularly useful when looking at the dyonic RN as a solution of pure $N = 2$ ungauged supergravity. Then the metric function reads

$$f(r) = 1 - \frac{2m}{r} + \frac{|Z|^2}{r^2}. \tag{16.2.11}$$

On the other hand the gauge field receives a new ϕ-component [76]

$$A = \frac{q}{r}\,\mathrm{d}t + p\cos\theta\,\mathrm{d}\phi = \frac{q}{r}\,\mathrm{d}u + p\cos\theta\,\mathrm{d}\phi \tag{16.2.12}$$

(the last equality being valid after a gauge transformation).

For simplifying the computations we only consider the case $n = 0$ with

$$F = -a\,\cos\theta, \qquad G = a\,\cos\theta, \tag{16.2.13}$$

but the general case $n \neq 0$ follows directly. The transformation of the metric is totally identical to the previous case (section 16.2.1) and one needs only to focus on the gauge field.

One has to rewrite first the gauge field as

$$A = \mathrm{Re}\left(\frac{Z}{r}\right)\,\mathrm{d}t + p\cos\theta\,\mathrm{d}\phi \tag{16.2.14}$$

before performing the JN transformation. The first term is complexified as

$$\mathrm{Re}\left(\frac{Z}{r}\right) = \frac{\mathrm{Re}(Z\bar{r})}{|r|^2} \tag{16.2.15}$$

and inserting the above transformation gives

$$A = \frac{qr - pa\cos\theta}{\rho^2}\,(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi) + p\cos\theta\,\mathrm{d}\phi. \tag{16.2.16}$$

After changing coordinates into the BL system, the A_r term is

$$\Delta\,A_r = -\frac{qr - pa\cos\theta}{\rho^2}\,\rho^2 - pa\cos\theta = -qr \tag{16.2.17}$$

($\Delta(r)$ is the denominator of the BL functions, not the Laplacian). Since $A_r = A_r(r)$ one can remove it and obtains finally

$$A = \frac{qr - pa\cos\theta}{\rho^2}\,(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi) + p\cos\theta\,\mathrm{d}\phi. \tag{16.2.18}$$

Using the fact that

$$a^2\sin^2\theta = r^2 + a^2 - \rho^2 \tag{16.2.19}$$

we rewrite it as

$$A = \frac{qr - pa\cos\theta}{\rho^2}\,\mathrm{d}t + \left(-\frac{qr}{\rho^2}\,a\sin^2\theta + \frac{p(r^2 + a^2)}{\rho^2}\,\cos\theta\right)\mathrm{d}\phi \tag{16.2.20a}$$

$$= \frac{qr}{\rho^2}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi) + \frac{p\cos\theta}{\rho^2}\,\big(a\,\mathrm{d}t + (r^2 + a^2)\,\mathrm{d}\phi\big) \tag{16.2.20b}$$

as it is presented in [76, 238, sec. 6.6].

The Yang–Mills Kerr–Newman black hole found by Perry [161] can also be derived in this way.

16.2.3 Charged BBMB–NUT black hole with cosmological constant

We consider Einstein–Maxwell theory with cosmological constant conformally coupled to a scalar field [35]

$$S = \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g} \left(R - 2\Lambda - \frac{1}{6} R\phi^2 - (\partial\phi)^2 - 2\alpha\phi^4 - F^2 \right), \tag{16.2.21}$$

where α is a coupling constant, and we have set $8\pi G = 1$.

For $F, \alpha, \Lambda = 0$, the Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) solution [239, 240] is static and spherically symmetric – it can be seen as the equivalent of the Schwarzschild black hole in conformal gravity.

The general charged solution with cosmological constant and quartic coupling reads

$$\mathrm{d}s^2 = -f\, \mathrm{d}t^2 + f^{-1}\, \mathrm{d}r^2 + r^2\, \mathrm{d}\Omega^2, \tag{16.2.22a}$$

$$A = \frac{q}{r}\, \mathrm{d}t, \qquad \phi = \sqrt{-\frac{\Lambda}{6\alpha}}\, \frac{m}{r-m}, \tag{16.2.22b}$$

$$f = -\frac{\Lambda}{3}\, r^2 + \kappa\, \frac{(r-m)^2}{r^2}, \tag{16.2.22c}$$

where $\mathrm{d}\Omega^2$ is S^2 or H^2 (see section 15.1). There is one constraint on the parameters

$$q^2 = \kappa m^2 \left(1 + \frac{\Lambda}{36\alpha} \right) \tag{16.2.23}$$

and one has $\alpha\Lambda < 0$ in order for ϕ to be real.

In order to add a NUT charge one performs the DJN transformation[1]

$$u = u' - 2n \ln H(\theta), \qquad r = r' + in, \qquad m = m' + in \tag{16.2.24}$$

together with the shift (15.4.8)

$$\kappa \longrightarrow \kappa - \frac{4\Lambda}{3}\, n^2. \tag{16.2.25}$$

Using the result (C.3.5) one obtains the metric (omitting the primes)

$$\mathrm{d}s^2 = -\tilde{f} \big(\mathrm{d}t - 2nH'\, \mathrm{d}\phi \big)^2 + \tilde{f}^{-1}\, \mathrm{d}r^2 + (r^2 + n^2)\, \mathrm{d}\Omega^2 \tag{16.2.26}$$

where the function $\tilde{f}$ is

$$\tilde{f} = -\frac{\Lambda}{3}\, (r^2 + n^2) + \left(\kappa - \frac{4\Lambda}{3}\, n^2 \right) \frac{(r-m)^2}{r^2 + n^2}, \tag{16.2.27}$$

where the term $r - m$ is invariant. Similarly one obtains the scalar field

$$\phi = \sqrt{-\frac{\Lambda}{6\alpha}}\, \frac{\sqrt{m^2 + n^2}}{r - m} \tag{16.2.28}$$

where the m in the numerator as been complexified as $|m|$.

Finally it is trivial to find the gauge field using the formula (C.3.3b)

$$A = \frac{q}{r^2 + n^2} \big(\mathrm{d}t - 2n \cos\theta\, \mathrm{d}\phi \big) \tag{16.2.29}$$

and the constraint (16.2.23) becomes

$$q^2 = \left(\kappa - \frac{4\Lambda}{3}\, n^2 \right) (m^2 + n^2) \left(1 + \frac{\Lambda}{36\alpha} \right). \tag{16.2.30}$$

[1] Due to the convention of [35] there is no κ in the transformations.

An interesting point is that the radial coordinate is redfined in [35] when obtaining the stationary solution from the static one.

Note that the BBMB solution and its NUT version are obtained from the limit

$$\Lambda, \alpha \longrightarrow 0, \qquad \text{with} \qquad -\frac{\Lambda}{36\alpha} \longrightarrow 1, \tag{16.2.31}$$

which also implies $q = 0$ from the constraint (16.2.23). Since no other modifications are needed, the derivation from the DJN algorithm also holds.

16.3 Complex scalar fields: rotation

In this section we expose the main ingredient for applying the JN transformation with $a \neq 0$ (but $n = 0$) on complex scalar fields: one needs to transform together the real and imaginary parts without enforcing any reality condition. Solutions with $n \neq 0$ require a more careful treatment and are studied in appendix 16.4.

We will give examples from ungauged $N = 2$ supergravity coupled to $n_v = 0, 1, 3$ vector multiplets (pure supergravity, STU model and T^3 model). Our aim is not to give a detailed account of supergravity, and more details can be found in the usual references [165, 170, 171].

16.3.1 Rule for complex fields

Let's consider a complex scalar field χ such that

$$\chi(r) = 1 + \frac{R}{r} \tag{16.3.1}$$

for the static configuration, R being a parameter. This is a very typical behaviour, where the imaginary part vanishes and the real part is harmonic with respect to the 3-dimensional spatial metric.

The first step of the JN algorithm is to complexify all the fields, using only the fact that r is complex. Namely, performing the JN transformation

$$r = r' - ia\cos\theta \tag{16.3.2}$$

gives

$$\tilde{\chi} = 1 + \frac{R}{r' - ia\cos\theta} = 1 + \frac{R\,(r' + ia\cos\theta)}{\rho^2}, \tag{16.3.3}$$

where as usual $\rho^2 = r'^2 + a^2\cos^2\theta$.

The imaginary part is thus proportional to the angular momentum a. Consequently it is impossible to generate the latter only from the static imaginary part since the traditional JN algorithm cannot generate a non-zero rotating field from a null static one. The main argument for this new rule is that one should not enforce any reality condition on the real or imaginary parts because they naturally form a pair. In other words, imaginary and real parts of the scalar fields naturally form a pair which cannot be reduced by any reality condition. Splitting a complex fields into its real and imaginary parts may hence obscure its structure and leads to a failure of the transformation (as it shows up in [115]). Note also that $\tilde{\chi}$ is now a complex harmonic function.

16.3.2 Review of $N = 2$ ungauged supergravity

In order for this chapter to be self-contained we recall the basic elements of $N = 2$ (ungauged) supergravity. The gravity multiplet contains the metric and the graviphoton

$$\{g_{\mu\nu}, A^0\} \tag{16.3.4}$$

while each of the vector multiplets contains a gauge field and a complex scalar field

$$\{A^i, z^i\}, \qquad i = 1, \ldots, n_v. \tag{16.3.5}$$

The scalar fields z^i (we denote the conjugate fields by $\bar{z}^i = z^{\bar{\imath}}$) parametrize a special Kähler manifold with metric $g_{i\bar{\jmath}}$. This manifold is uniquely determined by an holomorphic function called the prepotential F. The latter is better defined using the homogeneous (or projective) coordinates X^Λ such that

$$z^i = \frac{X^i}{X^0}. \tag{16.3.6}$$

The first derivative of the prepotential with respect to X^Λ is denoted by

$$F_\Lambda = \frac{\partial F}{\partial X^\Lambda}. \tag{16.3.7}$$

Finally it makes sense to regroup the gauge fields into one single vector

$$A^\Lambda = (A^0, A^i). \tag{16.3.8}$$

One needs to introduce two more quantities, respectively the Kähler potential and the Kähler connection

$$K = -\ln i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}^\Lambda), \qquad \mathcal{A}_\mu = -\frac{i}{2}(\partial_i K\, \partial_\mu z^i - \partial_{\bar{\imath}} K\, \partial_\mu z^{\bar{\imath}}). \tag{16.3.9}$$

The Lagrangian of this theory is given by

$$\mathcal{L} = -\frac{R}{2} + g_{i\bar{\jmath}}(z, \bar{z})\, z\partial_\mu z^i \partial^\nu z^{\bar{\jmath}} + \mathcal{R}_{\Lambda\Sigma}(z, \bar{z})\, F^\Lambda_{\mu\nu} F^{\Sigma\,\mu\nu} - \mathcal{I}_{\Lambda\Sigma}(z, \bar{z})\, F^\Lambda_{\mu\nu} \star F^{\Sigma\,\mu\nu} \tag{16.3.10}$$

where R is the Ricci scalar and $\star F^\Lambda$ is the Hodge dual of F^Λ. The matrix

$$\mathcal{N} = \mathcal{R} + i\,\mathcal{I} \tag{16.3.11}$$

can be expressed in terms of F. From this Lagrangian one can introduce the symplectic dual of F^Λ

$$G_\Lambda = \frac{\delta \mathcal{L}}{\delta F^\Lambda} = \mathcal{R}_{\Lambda\Sigma} F^\Sigma - \mathcal{I}_{\Lambda\Sigma} \star F^\Sigma. \tag{16.3.12}$$

16.3.3 BPS solutions

A BPS solution is a classical solution which preserves a part of the supersymmetry. The BPS equations are obtained by setting to zero the variations of the fermionic partners under a supersymmetric transformation. These equations are first order and under some conditions their solutions also solve the equations of motion.

In [36, sec. 3.1] (see also [61, sec. 2.2] for a summary), Behrndt, Lüst and Sabra obtained the most general stationary BPS solution for $N = 2$ ungauged supergravity. The metric for this class of solutions reads

$$\mathrm{d}s^2 = f^{-1}(\mathrm{d}t + \omega\, \mathrm{d}\phi)^2 + f\, \mathrm{d}\Sigma^2, \tag{16.3.13}$$

with the 3-dimensional spatial metric given in spherical or spheroidal coordinates

$$\mathrm{d}\Sigma^2 = h_{ij}\, \mathrm{d}x^i \mathrm{d}x^j = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) \tag{16.3.14a}$$

$$= \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 = \frac{\rho^2}{r^2 + a^2}\, \mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\, \mathrm{d}\phi^2, \tag{16.3.14b}$$

where i, j, k are flat spatial indices (which should not be confused with the indices of the scalar fields). The functions f and ω depend on r and θ only.

Then the solution is entirely given in terms of two sets of (real) harmonic functions[2] $\{H^\Lambda, H_\Lambda\}$

$$f = \mathrm{e}^{-K} = i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda), \tag{16.3.15a}$$

$$\varepsilon_{ijk}\partial_j\omega_k = 2\,\mathrm{e}^{-K}\mathcal{A}_i = (H_\Lambda \partial_i H^\Lambda - H^\Lambda \partial_i H_\Lambda), \tag{16.3.15b}$$

$$F^\Lambda_{ij} = \frac{1}{2}\,\varepsilon_{ijk}\partial_k H^\Lambda, \qquad G_{\Lambda\, ij} = \frac{1}{2}\,\varepsilon_{ijk}\partial_k H_\Lambda, \tag{16.3.15c}$$

$$i(X^\Lambda - \bar{X}^\Lambda) = H^\Lambda, \qquad i(F_\Lambda - \bar{F}_\Lambda) = H_\Lambda \tag{16.3.15d}$$

The only non-vanishing component of ω_i is $\omega \equiv \omega_\phi$.

Starting from the metric (16.3.13) in spherical coordinates with $\omega = 0$, one can use the JN algorithm of section 15.2 with

$$f_t = f^{-1}, \qquad f_r = f, \qquad f_\Omega = r^2 f \tag{16.3.16}$$

in order to obtain the metric (16.3.13) in spheroidal coordinates with $\omega \neq 0$ given by

$$\omega = a(1 - \tilde{f}) \sin^2\theta. \tag{16.3.17}$$

Then one needs only to find the complexification of f and to check that it gives the correct ω, as would be found from the equations (16.3.15). However it appears that one cannot complexify directly f. Therefore one needs to complexify first the harmonic functions H_Λ and H^Λ (or equivalently X^Λ), and then to reconstruct the other quantities. Nonetheless, equations (16.3.15) ensure that finding the correct harmonic functions gives a solution, thus it is not necessary to check these equations for all the other quantities.

In the next subsections we provide two examples,[3] one for pure supergravity as an appetizer, and then one with $n_v = 3$ multiplets (STU model).

Pure supergravity

As a first example we consider pure (or minimal) supergravity, i.e. $n_v = 0$ [36, sec. 4.2]. The prepotential reads

$$F = -\frac{i}{4}\,(X^0)^2. \tag{16.3.18}$$

The function H_0 and H^0 are related to the real and imaginary parts of the scalar X^0

$$H_0 = \frac{1}{2}(X^0 + \bar{X}^0) = \operatorname{Re} X^0, \qquad \bar{H}^0 = i(X^0 - \bar{X}^0) = -2 \operatorname{Im} X^0, \tag{16.3.19}$$

while the Kähler potential is given by

$$f = \mathrm{e}^{-K} = X^0 \bar{X}^0. \tag{16.3.20}$$

The static solution corresponds to [36, sec. 4.2]

$$H_0 = X^0 = 1 + \frac{m}{r} \tag{16.3.21}$$

Performing the JN transformation with the rule (16.3.3) gives

$$\tilde{X}^0 = 1 + \frac{m(r + ia\cos\theta)}{\rho^2}. \tag{16.3.22}$$

This corresponds to the second solution of [36, sec. 4.2] which is stationary with

$$\omega = \frac{m(2r + m)}{\rho^2}\, a \sin^2\theta. \tag{16.3.23}$$

[2] We omit the tilde that is present in [36] to avoid the confusion with the quantities that are transformed by the JNA. No confusion is possible since the index position will always indicate which function we are using.

[3] They correspond to singular solutions, but we are not concerned with regularity here.

STU model

We now consider the STU model $n_v = 3$ with prepotential [36, sec. 3]

$$F = -\frac{X^1 X^2 X^3}{X^0}. \tag{16.3.24}$$

The expressions for the Kähler potential and the scalar fields in terms of the harmonic functions are complicated and will not be needed (see [36, sec. 3] for the expressions). Various choices for the functions will give different solutions.

A class of static black hole-like solutions are given by the harmonic functions [36, sec. 4.4]

$$H_0 = h_0 + \frac{q_0}{r}, \qquad H^i = h^i + \frac{p^i}{r}, \qquad H^0 = H_i = 0. \tag{16.3.25}$$

These solutions carry three magnetic p^i and one electric q_0 charges.

Let's form the complex harmonic functions

$$\mathcal{H}_0 = H_0 + i\, H^0, \qquad \mathcal{H}_i = H^i + i\, H_i. \tag{16.3.26}$$

Then the rule (16.3.3) leads to

$$\mathcal{H}_0 = h_0 + \frac{q_0(r + ia\cos\theta)}{\rho^2}, \qquad \mathcal{H}_i = h^i + \frac{p^i(r + ia\cos\theta)}{\rho^2}, \tag{16.3.27}$$

for which the various harmonic functions read explicitly

$$H_0 = h_0 + \frac{q_0 r}{\rho^2}, \qquad H^i = h^i + \frac{p^i r}{\rho^2}, \qquad H^0 = \frac{q_0 a\cos\theta}{\rho^2}, \qquad H_i = \frac{p^i a\cos\theta}{\rho^2}. \tag{16.3.28}$$

This set of functions corresponds to the stationary solution of [36, sec. 4.4] where the magnetic and electric dipole momenta are not independent parameters but obtained from the magnetic and electric charges instead.

16.3.4 Dilaton–axion black hole – T^3 model

Sen derived his solution using the fact that Einstein–Maxwell gravity coupled to an axion σ and a dilaton ϕ (for a specific value of dilaton coupling constant) can be embedded in heterotic string theory. This model can also be embedded in $N = 2$ ungauged supergravity with $n_v = 1$, equal gauge fields $A \equiv A^0 = A^1$ and prepotential[4]

$$F = -i\, X^0 X^1, \tag{16.3.29}$$

The dilaton and the axion corresponds to the complex scalar field

$$z = \mathrm{e}^{-2\phi} + i\,\sigma. \tag{16.3.30}$$

The static metric, gauge field and the complex field read respectively

$$\mathrm{d}s^2 = -\frac{f_1}{f_2}\,\mathrm{d}t^2 + f_2\Big(f_1^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\Big), \tag{16.3.31a}$$

$$A = \frac{f_A}{f_2}\,\mathrm{d}t, \tag{16.3.31b}$$

$$z = \mathrm{e}^{-2\phi} = f_2 \tag{16.3.31c}$$

[4] This model can be obtained from the STU model by setting the sections pairwise equal $X^2 = X^0$ and $X^3 = X^1$ [44]. It is also a truncation of pure $N = 4$ supergravity.

where

$$f_1 = 1 - \frac{r_1}{r}, \qquad f_2 = 1 + \frac{r_2}{r}, \qquad f_A = \frac{q}{r}. \tag{16.3.32}$$

The radii r_1 and r_2 are related to the mass and the charge by

$$r_1 + r_2 = 2m, \qquad r_2 = \frac{q^2}{m}. \tag{16.3.33}$$

Applying now the Janis–Newman algorithm, the two functions f_1 and f_2 are complexified with the usual rules (C.1.1b)

$$\tilde{f}_1 = 1 - \frac{r_1 r}{\rho^2}, \qquad \tilde{f}_2 = 1 + \frac{r_2 r}{\rho^2}. \tag{16.3.34}$$

The final metric in BL coordinates is given by

$$\mathrm{d}s^2 = -\frac{\tilde{f}_1}{\tilde{f}_2}\left[\mathrm{d}t - a\left(1 - \frac{\tilde{f}_2}{\tilde{f}_1}\right)\sin^2\theta\,\mathrm{d}\phi\right]^2 + \tilde{f}_2\left(\frac{\rho^2\mathrm{d}r^2}{\Delta} + \rho^2\mathrm{d}\theta^2 + \frac{\Delta}{\tilde{f}_1}\sin^2\theta\,\mathrm{d}\phi^2\right) \tag{16.3.35}$$

for which the BL functions (C.3.4) are

$$g(r) = \frac{\hat{\Delta}}{\Delta}, \qquad h(r) = \frac{a}{\Delta} \tag{16.3.36}$$

with

$$\Delta = \tilde{f}_1\rho^2 + a^2\sin^2\theta, \qquad \hat{\Delta} = \tilde{f}_2\rho^2 + a^2\sin^2\theta. \tag{16.3.37}$$

Once f_A has been complexified as

$$\tilde{f}_A = \frac{qr}{\rho^2} \tag{16.3.38}$$

the transformation of the gauge field is straightforward

$$A = \frac{\tilde{f}_A}{\tilde{f}_2}\,(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi) - \frac{qr}{\Delta}\,\mathrm{d}r. \tag{16.3.39}$$

The A_r depending solely on r can again be removed thanks to a gauge transformation.

One cannot complexify the scalar z using the previous function $\tilde{f}_2$ since the latter is real and not complex. Instead one needs to follow the rule (16.3.3) a new time in order to obtain

$$z = 1 + \frac{r_2}{r} = 1 + \frac{r_2\bar{r}}{\rho^2}. \tag{16.3.40}$$

The explicit values for the dilaton and axion are then

$$\mathrm{e}^{-2\phi} = \tilde{f}_2, \qquad \sigma = \frac{r_2 a\cos\theta}{\rho^2}. \tag{16.3.41}$$

We have been able to find the full Sen's solution, completing the computations from [115]. It is interesting to note that for another value of the dilaton coupling we cannot use the transformation [138, 141].[5] Finally the truncation $\sigma = 0$ is also a solution of dilatonic gravity [141], but the JN algorithm generates directly the axion–dilaton metric such that we cannot recover the vanishing axion case [115].

[5] The authors of [139] report incorrectly that [138] is excluding all dilatonic solutions.

16.4 Complex scalar fields: NUT charge

16.4.1 Pure supergravity

In [36, sec. 4.2] a solution of pure supergravity (see 16.3.3 for the notations) with a NUT charge is presented. In this case the solution reads

$$X^0 = 1 + \frac{m + in}{r}, \qquad \omega = 2n \cos\theta. \tag{16.4.1}$$

The question is whether this configuration can be obtained from the $n = 0$ solution (16.3.21)

$$X^0 = 1 + \frac{m}{r} \tag{16.4.2}$$

from the transformation (15.4.3)

$$m = m' + in, \qquad r = r' + in. \tag{16.4.3}$$

It is straightforward to check that the full metric (16.3.13) is recovered, while the field X^0 in (16.4.1) follows from the rule (C.1.1a)

$$r \longrightarrow \frac{1}{2}\,(r + \bar{r}) = \operatorname{Re} r = r' \tag{16.4.4}$$

applied in the denominator. Hence a DJN transformation with the NUT charge does not act in the same way as a transformation with an angular momentum, since the transformation rule is different from (16.3.3).

16.4.2 SWIP solutions

Let's consider the action [162, 39, sec. 12.2]

$$S = \frac{1}{16\pi} \int \mathrm{d}^4x\, \sqrt{|g|}\, \left(R - 2(\partial\phi)^2 - \frac{1}{2}\, \mathrm{e}^{4\phi}\, (\partial\sigma)^2 - \mathrm{e}^{-2\phi} F^i_{\mu\nu} F^{i\mu\nu} + \sigma\, F^i_{\mu\nu} \tilde{F}^{i\mu\nu} \right) \tag{16.4.5}$$

where $i = 1, \ldots, M$. When $M = 2$ and $M = 6$ this action corresponds respectively to $N = 2$ supergravity with one vector multiplet and to $N = 4$ pure supergravity, but we keep M arbitrary. The axion σ and the dilaton ϕ are naturally paired into a complex scalar

$$z = \sigma + i\, \mathrm{e}^{-2\phi}. \tag{16.4.6}$$

In order to avoid redundancy we first provide the general metric with $a, n \neq 0$, and we explain how to find it from the restricted case $a = n = 0$.

Stationary Israel–Wilson–Perjés (SWIP) solutions correspond to

$$\mathrm{d}s^2 = -\,\mathrm{e}^{2U} W (\mathrm{d}t + A_\phi\, \mathrm{d}\phi)^2 + \mathrm{e}^{-2U} W^{-1} \mathrm{d}\Sigma^2, \tag{16.4.7a}$$

$$A^i_t = 2\,\mathrm{e}^{2U} \operatorname{Re}(k^i H_2), \qquad \tilde{A}^i_t = 2\,\mathrm{e}^{2U} \operatorname{Re}(k^i H_1), \qquad z = \frac{H_1}{H_2}, \tag{16.4.7b}$$

$$A_\phi = 2n \cos\theta - a \sin^2\theta (\,\mathrm{e}^{-2U} W^{-1} - 1), \tag{16.4.7c}$$

$$\mathrm{e}^{-2U} = 2 \operatorname{Im}(H_1 \bar{H}_2), \qquad W = 1 - \frac{r_0^2}{\rho^2}. \tag{16.4.7d}$$

This solution is entirely determined by the two harmonic functions

$$H_1 = \frac{1}{\sqrt{2}}\, \mathrm{e}^{\phi_0} \left(z_0 + \frac{z_0 \mathcal{M} + \bar{z}_0 \Upsilon}{r - ia\cos\theta} \right), \qquad H_2 = \frac{1}{\sqrt{2}}\, \mathrm{e}^{\phi_0} \left(1 + \frac{\mathcal{M} + \Upsilon}{r - ia\cos\theta} \right). \tag{16.4.8}$$

The spatial 3-dimensional metric $\mathrm{d}\Sigma^2$ reads

$$\mathrm{d}\Sigma^2 = h_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = \frac{\rho^2 - r_0^2}{r^2 + a^2 - r_0^2}\,\mathrm{d}r^2 + (\rho^2 - r_0^2)\mathrm{d}\theta^2 + (r^2 + a^2 - r_0^2)\sin^2\theta\,\mathrm{d}\phi^2. \quad (16.4.9)$$

Finally, r_0 corresponds to

$$r_0^2 = |\mathcal{M}|^2 + |\Upsilon|^2 - \sum_i \left|\Gamma^i\right|^2 \quad (16.4.10)$$

where the complex parameters are

$$\mathcal{M} = m + in, \qquad \Gamma^i = q^i + ip^i, \quad (16.4.11)$$

m being the mass, n the NUT charge, q^i the electric charges and p^i the magnetic charges, while the axion/dilaton charge Υ takes the form

$$\Upsilon = -\frac{1}{2}\sum_i \frac{(\bar{\Gamma}^i)^2}{\mathcal{M}}. \quad (16.4.12)$$

The latter together with the asymptotic values z_0 comes from

$$z \sim z_0 - i\,\mathrm{e}^{-2\phi_0}\frac{2\Upsilon}{r}. \quad (16.4.13)$$

The complex constant k^i are determined by

$$k^i = -\frac{1}{\sqrt{2}}\,\frac{\mathcal{M}\Gamma^i + \bar{\Upsilon}\bar{\Gamma}^i}{|\mathcal{M}|^2 - |\Upsilon|^2}. \quad (16.4.14)$$

As discussed in the previous section, the transformation of scalar fields is different whether one is turning on a NUT charge or an angular momentum. For this reason, starting from the case $a = n = 0$, one needs to perform the two successive transformations

$$u = u' - 2in\ln\sin\theta, \qquad r = r' + in, \qquad m = m' + in, \quad (16.4.15a)$$

$$u = u' + ia\cos\theta, \qquad r = r' - ia\cos\theta, \quad (16.4.15b)$$

the order being irrelevant (for definiteness we choose to add the NUT charge first). As explained in section 17.3, group properties of the DJN algorithm ensure that the metric will be transformed as if only one transformation was performed, and one can use the formula of section 15.2. Then the formulas (C.3.3) for the metric and for the gauge field directly apply, which ensures that the general form of the solution (16.4.7) is correct.[6] Since all the functions and the parameters depend only on $\mathcal{M}$, H_1 and H_2, it is sufficient to explain their complexification.

The function W is easily transformed, whereas H_1 and H_2 are more subtle since they are complex harmonic functions. Let's consider first the NUT charge with the transformation (16.4.15a). According to the previous appendix, the r in the denominator of both functions is transformed according to (C.1.1a)

$$r \longrightarrow \frac{1}{2}\,(r + \bar{r}) = \operatorname{Re} r = r'. \quad (16.4.16)$$

Then one can perform the second transformation (16.4.15b) in order to add the angular momentum. Using the recipe from section 16.3.1, one obtain the correct result (16.4.8) by just replacing r with (16.4.15b).

Finally let's note that it seems possible to also start from $p^i = 0$ and to turn them on using the transformation

$$q^i = q'^i = q^i + ip^i, \quad (16.4.17)$$

using different rules for complexifying the various terms (depending whether one is dealing with a real or a complex function/parameter).

[6]For that one needs to shift r^2 by r_0^2 in order to bring the metric (16.4.9) to the form (16.3.14). This modifies the function but one does not need this fact to obtain the general form. Then one can shift by $-r_0^2$ before dealing with the complexification of the functions.

Chapter 17

Technical properties

In this chapter we describe few technical properties of the algorithm. In particular some DJN transformations have an interesting group structure that allows to chain several transformations [159]. Another useful property of Giampieri's prescription is to allow to chain all coordinate transformation, making computations easier [99]. Then finally we discuss the fact that not all the rules (C.1.1) are independent and several choices of complexification are equivalent [99], contrary to what is widely believed.

17.1 Chaining transformations

The JN algorithm is summarized by the following table

$$\begin{array}{ccccccccc} t & \to & u & \to & u \in \mathbb{C} & \to & u' & \to & t' \\ r & & & \to & r \in \mathbb{C} & \to & r' & & \\ \phi & & & & & & & \to & \phi' \\ f & & & \to & \tilde{f} & & & & \\ g_{\mu\nu} & & & \to & g'_{\mu\nu} & & & & \end{array} \tag{17.1.1}$$

where the arrows correspond respectively to the steps 1, 2, 4 and 5 of section 14.2 (and 1, 3, 4 and 6 of section 14.1).

A major advantage of Giampieri's prescription is that one can chain all these transformations since it involves only substitutions and no tensor operations. For this reason it is much easier to implement on a computer algebra system such as Mathematica. It is then possible to perform a unique change of variables that leads directly from the static metric to the rotating metric in any system defined by the function (g, h)

$$\mathrm{d}t = \mathrm{d}t' + \big(ah \sin^2\theta\, (1 - \tilde{f}^{-1}) - g + \tilde{f}^{-1}\big)\, \mathrm{d}r' + a \sin^2\theta\, (\tilde{f}^{-1} - 1)\, \mathrm{d}\phi', \tag{17.1.2a}$$

$$\mathrm{d}r = (1 - ah \sin^2\theta)\, \mathrm{d}r' + a \sin^2\theta\, \mathrm{d}\phi', \tag{17.1.2b}$$

$$\mathrm{d}\phi = \mathrm{d}\phi' - h\, \mathrm{d}r', \tag{17.1.2c}$$

where the complexification of the metric function f can be made at the end. It is impressive that steps 1 to 5 from section 14.2 can be written in such a compact way.

17.2 Arbitrariness of the transformation

We provide a short comment on the arbitrariness of the complexification rules. In particular let's consider the functions

$$f_1(r) = \frac{1}{r}, \qquad f_2(r) = \frac{1}{r^2}. \tag{17.2.1}$$

The usual rule is to complexify these two functions as

$$\tilde{f}_1(r) = \frac{\operatorname{Re} r}{|r|^2}, \qquad \tilde{f}_2(r) = \frac{1}{|r|^2} \tag{17.2.2}$$

using respectively the rules (C.1.1b) and (C.1.1c) (in the denominator).

But it is possible to arrive at the same result with a different combinations of rules. In fact the functions can be rewritten as

$$f_1(r) = \frac{r}{r^2}, \qquad f_2(r) = \frac{1}{r}\frac{1}{r}. \tag{17.2.3}$$

The following set of rules results again in (17.2.2):

- f_1: (C.1.1a) (numerator) and (C.1.1c) (denominator);
- f_2: (C.1.1a) (first fraction) and (C.1.1b) (second fraction).

17.3 Group properties

In this section we want to show that (some of) DJN transformations form a group.

After a first transformation

$$r = r' + i\,F_1, \qquad u = u' + i\,G_1 \tag{17.3.1}$$

one obtains the metric

$$\begin{aligned} ds^2 = &- \tilde{f}_t^{\{1\}}(du + HG_1'\,d\phi)^2 + \tilde{f}_\Omega^{\{1\}}(d\theta^2 + H^2 d\phi^2) \\ &- 2\sqrt{\tilde{f}_t^{\{1\}}\tilde{f}_r^{\{1\}}}(du + G_1' H\,d\phi)(dr + F_1' H\,d\phi) \end{aligned} \tag{17.3.2}$$

where

$$\tilde{f}_i^{\{1\}} = \tilde{f}_i^{\{1\}}(r, F_1). \tag{17.3.3}$$

Applying a second transformation

$$r = r' + i\,F_2, \qquad u = u' + i\,G_2 \tag{17.3.4}$$

the previous metric becomes

$$\begin{aligned} ds^2 = &- \tilde{f}_t^{\{1,2\}}\big(du + H(G_1' + G_2')\,d\phi\big)^2 + \tilde{f}_\Omega^{\{1,2\}}(d\theta^2 + H^2 d\phi^2) \\ &- 2\sqrt{\tilde{f}_t^{\{1,2\}}\tilde{f}_r^{\{1,2\}}}\big(du + (G_1' + G_2')H\,d\phi\big)\big(dr + (F_1' + F_2')H\,d\phi\big) \end{aligned} \tag{17.3.5}$$

where this time

$$\tilde{f}_i^{\{1,2\}} = \tilde{f}_i^{\{1,2\}}(r, F_1, F_2). \tag{17.3.6}$$

As for the first transformation we only ask for the following conditions

$$\tilde{f}_i^{\{1,2\}}(r, F_1, 0) = \tilde{f}_i^{\{1\}}(r, F_1), \qquad \tilde{f}_i^{\{1,2\}}(r, F_1, F_2) = \tilde{f}_i^{\{2,1\}}(r, F_2, F_1). \tag{17.3.7}$$

In one word a zero transformation should just give back the old metric, and the two transformations should commute.

Looking at the expression of the metric, it is obvious that the DJN transformations which are such that

$$\tilde{f}_i^{\{1,2\}}(r, F_1, F_2) = \tilde{f}_i^{\{1\}}(r, F_1 + F_2) \tag{17.3.8}$$

form an (Abelian) group if the functions F and G are linear in the parameters (i. e. the group is additive). This last condition means that we can decompose them on a basis of generators $\{F_A(\theta)\}$ and $\{G_M(\theta)\}$, where A and M are (different) indices, such that

$$F(\theta) = f^A F_A(\theta), \qquad G(\theta) = g^M G_M(\theta), \tag{17.3.9}$$

f^A and g^M being the parameters of the transformations. It is possible that $f^A = g^M$ and $F_A \propto G_M$ for some A and M (as we obtained in section 15.3) which means that the corresponding parameters f^A and g^M are not independent.

These transformations form a group because composing two transformations (F_1, G_1) and (F_2, G_2) gives a third transformation (F_3, G_3) according to

$$F_3 = F_1 + F_2, \qquad G_3 = G_1 + G_2 \tag{17.3.10}$$

with the parameters combining linearly. Moreover there an identity $(0, 0)$ and also an inverse $(-F, -G)$.

All this structure implies that we can first add one parameter, and later another (say first the NUT charge, and then an angular momentum). Said another way this group preserves Einstein equations when the seed metric is a known (stationary) solution. But note that it may be very difficult to do it as soon as one begins to replace the F in the functions by their expression, because it obscures the original function – in one word we cannot find $\tilde{f}_i(r, F)$ from $\tilde{f}_i(r, \theta)$.

Another point worth mentioning is that not all DJN transformation are in this group since it may happen that the condition (17.3.8) is not satisfied. Such an example is provided in $5d$ where the function $f_\Omega(r) = r^2$ is successively transformed as [160]

$$r^2 \longrightarrow |r|^2 = r^2 + a^2 \cos^2\theta \longrightarrow |r|^2 + a^2 \cos^2\theta = r^2 + a^2 \cos^2\theta + b^2 \sin^2\theta, \tag{17.3.11}$$

where the two transformations were

$$F_1 = a\cos\theta, \qquad F_2 = b\sin\theta, \tag{17.3.12}$$

and

$$\tilde{f}_\Omega^{\{1,2\}} = r^2 + F_1^2 + F_2^2. \tag{17.3.13}$$

The condition (17.3.8) is clearly not satisfied. These group properties may explain why the JN algorithm is not working for $d > 5$, or maybe give a clue to solve this problem.

Chapter 18

Other dimensions

While in four dimensions we have at our disposal many theorems on the classification of solutions, this is not the case for higher dimensions and the bestiary for solutions is much wider and less understood [110, 241]. In particular important solutions have not yet been discovered, such as charged rotating black holes with several angular momenta (in pure Einstein–Maxwell gravity).

Generalizing the (D)JN algorithm in other dimensions is challenging and only small steps have been taken in this direction. For instance Xu recovered Myers–Perry solution [163] with one angular momentum from the Schwarzschild–Tangherlini solution [122] (see also [242]),[1] and Kim showed how the rotating BTZ black hole [243] can be obtained from its static limit [123, 124].

We first analyse the case $d = 5$ and we show how to generate solutions with two angular momenta from a static solution in the case of two examples [160]: the Myers–Perry black hole [163] and the Breckenridge–Myers–Peet–Vafa (BMPV) extremal black hole [164].

Parametrizing the metric on the sphere by direction cosines is a key step in order to generalize the transformation to any dimension since these coordinates are totally symmetric under interchange of angular momenta (at the opposite of the spherical coordinates). Despite the fact that it is possible to obtain the correct structure of the metric (for Myers–Perry-like metrics), it is very challenging to determine the functions inside the metric. Nonetheless this provides a unified view of the JN algorithm for $d = 3, 4, 5$. Indeed our formalism can be used to recover the rotating BTZ black hole more directly.

Here Giampieri's prescription simplifies greatly the computations as the tetrad formalism would imply working with matrices of size d. Note also that we could not reproduce the derivation using the tetrad formalism as some terms do not seem to cancel in this case.

A major application of our work would be to find the charged solution with two angular momenta of the $5d$ Einstein–Maxwell. This problem is highly non-trivial and there is few chances that this technique would work directly [242], but one can imagine that a generalization of Demiański's approach [100] (see chapter 15) could lead to new interesting solutions in five dimensions. An intermediate step is represented by the CCLP metric [244] which is a solution of Einstein–Maxwell together with a Chern–Simons term, but it cannot be obtained from the JN algorithm. Moreover it would be very desirable to derive the general d-dimensional Myers–Perry solution [163], or at least to understand why only the metric can be found, and not the function inside. Slowly rotating metrics could in principle be derived easily [242, sec. 4] using our prescriptions and could be a nice playground to understand better higher dimensional solution with $d \geq 6$. Finally one can ask whether the algorithm can be used to derive black rings [241, 245].

[1] Note that [127, 129] obtain higher dimensional metric with one angular momentum, but they are not solutions of the equations of motion.

18.1 Five-dimensional applications

We first look at the simple case of five dimensions, and later we generalize to any dimension.

18.1.1 Myers–Perry black hole

In this section we show how to recover Myers–Perry black hole in five dimensions through Giampieri's prescription. This is a solution of 5-dimensional pure Einstein theory which possesses two angular momenta and it generalizes the Kerr black hole. The importance of this solution lies in the fact that it can be constructed in any dimension.

Let us start with the five-dimensional Schwarzschild–Tangherlini metric

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_3^2 \tag{18.1.1}$$

where $\mathrm{d}\Omega_3^2$ is the metric on S^3, which can be expressed in Hopf coordinates (see section B.3.2)

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \cos^2\theta\,\mathrm{d}\psi^2, \tag{18.1.2}$$

and the function $f(r)$ is given by

$$f(r) = 1 - \frac{m}{r^2}. \tag{18.1.3}$$

An important feature of the JN algorithm is the fact that a given set of transformations in the (r,ϕ)-plane generates rotation in the latter. Generating several angular momenta in different 2-planes would then require successive applications of the JN algorithm on different hypersurfaces. In order to do so, one has to identify what are the 2-planes which will be submitted to the algorithm. In five dimensions, the two different planes that can be made rotating are the planes (r,ϕ) and (r,ψ). We claim that it is necessary to dissociate the radii of these 2-planes in order to apply separately the JN algorithm on each plane and hence to generate two distinct angular momenta. In order to dissociate the parts of the metric that correspond to the rotating and non-rotating 2-planes, one can protect the function r^2 to be transformed under complex transformations in the part of the metric defining the plane which will stay static. We thus introduce the function

$$R(r) = \mathrm{Re}(r) \tag{18.1.4}$$

such that the metric in null coordinates reads

$$\mathrm{d}s^2 = -\mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r) + (1-f)\,\mathrm{d}u^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + R^2\cos^2\theta\,\mathrm{d}\psi^2. \tag{18.1.5}$$

The first transformation – hence concerning the (r,ϕ)-plane – is

$$\begin{aligned} u &= u' + ia\cos\chi_1, & r &= r' - ia\cos\chi_1, \\ i\,\mathrm{d}\chi_1 &= \sin\chi_1\ \mathrm{d}\phi, & &\text{with}\ \ \chi_1 = \theta, \\ \mathrm{d}u &= \mathrm{d}u' - a\sin^2\theta\,\mathrm{d}\phi, & \mathrm{d}r &= \mathrm{d}r' + a\sin^2\theta\,\mathrm{d}\phi, \end{aligned} \tag{18.1.6}$$

and f is replaced by $\tilde{f}^{\{1\}} = \tilde{f}^{\{1\}}(r,\theta)$. Indeed we need to keep track of the order of the transformation, since the function f will be complexified twice consecutively. On the other hand $R(r) = \mathrm{Re}(r)$ transforms into $R(r) = r'$ and we find (omitting the primes)

$$\begin{aligned} \mathrm{d}s^2 = &-\mathrm{d}u^2 - 2\,\mathrm{d}u\mathrm{d}r + \big(1 - \tilde{f}^{\{1\}}\big)(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi)^2 + 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi \\ &+ (r^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2 + r^2\cos^2\theta\,\mathrm{d}\psi^2. \end{aligned} \tag{18.1.7}$$

The function $\tilde{f}^{\{1\}}$ is

$$\tilde{f}^{\{1\}} = 1 - \frac{m}{|r|^2} = 1 - \frac{m}{r^2 + a^2\cos^2\theta}. \tag{18.1.8}$$

There is a cancellation between the (u, r) and the (θ, ϕ) parts of the metric

$$\mathrm{d}s^2_{u,r} = (1 - \tilde{f}^{\{1\}})\,(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi)^2 - \mathrm{d}u(\mathrm{d}u + 2\mathrm{d}r) + 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi + a^2\sin^4\theta\,\mathrm{d}\phi^2, \tag{18.1.9a}$$

$$\mathrm{d}s^2_{\theta,\phi} = (r^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 + \big(r^2 + a^2(1 - \sin^2\theta)\big)\sin^2\theta\,\mathrm{d}\phi^2. \tag{18.1.9b}$$

In addition to the terms present in (18.1.5) we obtain new components corresponding to the rotation of the first plane (r, ϕ). We find the same terms as in (18.1.5) plus other terms that corresponds to the rotation in the first plane. Transforming now the second one – (r, ψ) – the transformation is[2]

$$\begin{aligned} u &= u' + ib\,\sin\chi_2, & r &= r' - ib\,\sin\chi_2, \\ i\,\mathrm{d}\chi_2 &= -\cos\chi_2\,\mathrm{d}\psi, & &\text{with}\ \ \chi_2 = \theta, \\ \mathrm{d}u &= \mathrm{d}u' - b\cos^2\theta\,\mathrm{d}\psi, & \mathrm{d}r &= \mathrm{d}r' + b\cos^2\theta\,\mathrm{d}\psi, \end{aligned} \tag{18.1.10}$$

can be applied directly to the metric

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}u^2 - 2\,\mathrm{d}u\mathrm{d}r &+ \big(1 - \tilde{f}^{\{1\}}\big)(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi)^2 + 2a\sin^2\theta\,\mathrm{d}R\mathrm{d}\phi \\ &+ \rho^2\mathrm{d}\theta^2 + (R^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2 + r^2\cos^2\theta\,\mathrm{d}\psi^2 \end{aligned} \tag{18.1.11}$$

where we introduced once again the function $R(r) = \mathrm{Re}(r)$ to protect the geometry of the first plane to be transformed under complex transformations.

The final result (using again $R = r'$ and omitting the primes) becomes

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}u^2 - 2\,\mathrm{d}u\mathrm{d}r &+ \big(1 - \tilde{f}^{\{1,2\}}\big)(\mathrm{d}u - a\sin^2\theta\,\mathrm{d}\phi - b\cos^2\theta\,\mathrm{d}\psi)^2 \\ &+ 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi + 2b\cos^2\theta\,\mathrm{d}r\mathrm{d}\psi \\ &+ \rho^2\mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2 + (r^2 + b^2)\cos^2\theta\,\mathrm{d}\psi^2 \end{aligned} \tag{18.1.12}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta. \tag{18.1.13}$$

Furthermore, the function $\tilde{f}^{\{1\}}$ has been complexified as

$$\tilde{f}^{\{1,2\}} = 1 - \frac{m}{|r|^2 + a^2\cos^2\theta} = 1 - \frac{m}{r'^2 + a^2\cos^2\theta + b^2\sin^2\theta} = 1 - \frac{m}{\rho^2}. \tag{18.1.14}$$

The metric can then be transformed into the Boyer–Lindquist (BL) using

$$\mathrm{d}u = \mathrm{d}t - g(r)\,\mathrm{d}r, \qquad \mathrm{d}\phi = \mathrm{d}\phi' - h_\phi(r)\,\mathrm{d}r, \qquad \mathrm{d}\psi = \mathrm{d}\psi' - h_\psi(r)\,\mathrm{d}r. \tag{18.1.15}$$

Defining the parameters[3]

$$\Pi = (r^2 + a^2)(r^2 + b^2), \qquad \Delta = r^4 + r^2(a^2 + b^2 - m) + a^2b^2, \tag{18.1.16}$$

the functions can be written

$$g(r) = \frac{\Pi}{\Delta}, \qquad h_\phi(r) = \frac{\Pi}{\Delta}\,\frac{a}{r^2 + a^2}, \qquad h_\psi(r) = \frac{\Pi}{\Delta}\,\frac{b}{r^2 + b^2}. \tag{18.1.17}$$

We get the final metric

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}t^2 &+ \big(1 - \tilde{f}^{\{1,2\}}\big)(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi - b\cos^2\theta\,\mathrm{d}\psi)^2 + \frac{r^2\rho^2}{\Delta}\,\mathrm{d}r^2 \\ &+ \rho^2\mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2 + (r^2 + b^2)\cos^2\theta\,\mathrm{d}\psi^2. \end{aligned} \tag{18.1.18}$$

[2]The easiest justification for choosing the sinus here is by looking at the transformation in terms of direction cosines, see section 18.3.3. Otherwise this term can be guessed by looking at Myers–Perry non-diagonal terms.

[3]See (18.2.17) for a definition of Δ in terms of $\tilde{f}$.

One recovers here the five dimensional Myers–Perry black hole with two angular momenta [163].

It is important to mention that following the same prescription in dimensions higher than five does not lead as nicely as we did in five dimensions to the exact Myers–Perry solution. Indeed we show in section 18.2 that the transformation of the metric can be done along the same line but that the only – major – obstacle comes from the function f that cannot be complexified as expected. Finding the correct complexification seems very challenging and it may be necessary to use a different complex coordinate transformation in order to perform a completely general transformation in any dimension. It might be possible to gain insight into this problem by computing the transformation within the framework of the tetrad formalism.

One may think that a possible solution would be to replace complex numbers by quaternions, assigning one angular momentum to each complex direction but it is straightforward to check that this approach is not working.

18.1.2 BMPV black hole

Few properties and seed metric

In this section we focus on another example in five dimensions, which is the BMPV black hole [164, 246]. This solution possesses many interesting properties, in particular it can be proven that it is the only rotating BPS asymptotically flat black hole in five dimensions with the corresponding near-horizon geometry [241, sec. 7.2.2, 8.5, 247].[4] It is interesting to notice that even if this extremal solution is a slowly rotating metric, it is an exact solution (whereas Einstein equations need to be truncated for consistency of usual slow rotation).

For a rotating black hole the BPS and extremal limits do not coincide [241, sec. 7.2, 246, sec. 1]: the first implies that the mass is related to the electric charge,[5] while extremality[6] implies that one linear combination of the angular momenta vanishes, and for this reason we set $a = b$ from the beginning.[7] We are thus left with two parameters that we take to be the mass and one angular momentum.

In the non-rotating limit BMPV black hole reduces to the charged extremal Schwarzschild–Tangherlini (with equal mass and charge) written in isotropic coordinates. For non-rotating black hole the extremal and BPS limit are equivalent.

Both the charged extremal Schwarzschild–Tangherlini and BMPV black holes are solutions of minimal ($N = 2$) $d = 5$ supergravity (Einstein–Maxwell plus Chern–Simons) whose action is [246, sec. 1, 248, sec. 2, 249, sec. 2]

$$S = -\frac{1}{16\pi G} \int \left(R \star 1 + F \wedge \star F + \frac{2\lambda}{3\sqrt{3}}\, F \wedge F \wedge A \right), \qquad (18.1.19)$$

where supersymmetry imposes $\lambda = 1$.

Since extremal limits are different for static and rotating black holes we can guess that the black hole we will obtain from the algorithm will not be a solution of the equations of motion and we will need to take some limit.

The charged extremal Schwarzschild–Tangherlini black hole is taken as a seed metric [249, sec. 3.2, 250, sec. 4, 251, sec. 1.3.1, 252, sec. 3]

$$\mathrm{d}s^2 = -H^{-2}\,\mathrm{d}t^2 + H\,(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_3^2) \qquad (18.1.20)$$

[4] Other possible near-horizon geometries are $S^1 \times S^2$ (for black rings) and T^3, even if the latter does not seem really physical. BMPV horizon corresponds to the squashed S^3.

[5] It is a consequence from the BPS bound $m \geq \sqrt{3}/2\ |q|$.

[6] Regularity is given by a bound, which is saturated for extremal black holes.

[7] If we had kept $a \neq b$ we would have discovered later that one cannot transform the metric to Boyer–Lindquist coordinates without setting $a = b$.

where $\mathrm{d}\Omega_3^2$ is the metric of the 3-sphere written in (18.1.2). The function H is harmonic

$$H(r) = 1 + \frac{m}{r^2}, \tag{18.1.21}$$

and the electromagnetic field reads

$$A = \frac{\sqrt{3}}{2\lambda}\,\frac{m}{r^2}\,\mathrm{d}t = (H-1)\,\mathrm{d}t. \tag{18.1.22}$$

In the next subsections we apply successively the transformations (18.1.6) and (18.1.10) with $a = b$ in the case $\lambda = 1$ because we are searching a supersymmetric solution.

Transforming the metric

The transformation to (u, r) coordinates of the seed metric (18.1.20)

$$\mathrm{d}t = \mathrm{d}u + H^{3/2}\,\mathrm{d}r \tag{18.1.23}$$

gives

$$\mathrm{d}s^2 = -H^{-2}\,\mathrm{d}u^2 - 2H^{-1/2}\,\mathrm{d}u\mathrm{d}r + Hr^2\,\mathrm{d}\Omega_3^2 \tag{18.1.24a}$$

$$= -H^{-2}\left(\mathrm{d}u - 2H^{3/2}\,\mathrm{d}r\right)\mathrm{d}u + Hr^2\,\mathrm{d}\Omega_3^2. \tag{18.1.24b}$$

For transforming the above metric one should follow the recipe of the previous section: transformations (18.1.6)

$$u = u' + ia\cos\theta, \qquad \mathrm{d}u = \mathrm{d}u' - a\sin^2\theta\,\mathrm{d}\phi, \tag{18.1.25}$$

and (18.1.10)

$$u = u' + ia\,\sin\theta, \qquad \mathrm{d}u = \mathrm{d}u' - a\cos^2\theta\,\mathrm{d}\psi \tag{18.1.26}$$

are performed one after another, transforming each time only the terms that pertain to the corresponding rotation plane.[8] In order to preserve the isotropic form of the metric the function H is complexified everywhere (even when it multiplies terms that belong to the other plane).

Since the procedure is exactly similar to the Myers–Perry case we give only the final result in (u, r) coordinates

$$\begin{aligned}
\mathrm{d}s^2 = &-\tilde{H}^{-2}\big(\mathrm{d}u - a(1-\tilde{H}^{3/2})(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\big)^2 \\
&- 2\tilde{H}^{-1/2}\big(\mathrm{d}u - a(1-\tilde{H}^{3/2})\,(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\big)\,\mathrm{d}r \\
&+ 2a\tilde{H}\,(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\,\mathrm{d}r - 2a^2\tilde{H}\cos^2\theta\sin^2\theta\,\mathrm{d}\phi\mathrm{d}\psi \\
&+ \tilde{H}\,\Big((r^2+a^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \cos^2\theta\,\mathrm{d}\psi^2) + a^2(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)^2\Big).
\end{aligned} \tag{18.1.27}$$

After both transformations the resulting function $\tilde{H}$ is

$$\tilde{H} = 1 + \frac{m}{r^2 + a^2\cos^2\theta + a^2\sin^2\theta} = 1 + \frac{m}{r^2+a^2} \tag{18.1.28}$$

which does not depend on θ.

It is easy to check that the Boyer–Lindquist transformation (18.1.15)

$$\mathrm{d}u = \mathrm{d}t - g(r)\,\mathrm{d}r, \qquad \mathrm{d}\phi = \mathrm{d}\phi' - h_\phi(r)\,\mathrm{d}r, \qquad \mathrm{d}\psi = \mathrm{d}\psi' - h_\psi(r)\,\mathrm{d}r \tag{18.1.29}$$

is ill-defined because the functions depend on θ. The way out is to take the extremal limit alluded above.

[8] For another approach see section 18.1.3.

Following the prescription of [164, 246] and taking the extremal limit

$$a, m \longrightarrow 0, \qquad \text{imposing} \qquad \frac{m}{a^2} = \text{cst}, \tag{18.1.30}$$

one gets at leading order

$$\tilde{H}(r) = 1 + \frac{m}{r^2} = H(r), \qquad a\,(1 - \tilde{H}^{3/2}) = -\frac{3\,ma}{2\,r^2} \tag{18.1.31}$$

which translate into the metric

$$\begin{aligned} \mathrm{d}s^2 = -H^{-2} &\left(\mathrm{d}u + \frac{3\,ma}{2\,r^2}\,(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\right)^2 \\ &- 2H^{-1/2}\left(\mathrm{d}u + \frac{3\,ma}{2\,r^2}\,(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\right)\mathrm{d}r \\ &+ H\,r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \cos^2\theta\,\mathrm{d}\psi^2). \end{aligned} \tag{18.1.32}$$

Then Boyer–Lindquist functions are

$$g(r) = H(r)^{3/2}, \qquad h_\phi(r) = h_\psi(r) = 0 \tag{18.1.33}$$

and one gets the metric in (t, r) coordinates

$$\begin{aligned} \mathrm{d}s^2 = &-\tilde{H}^{-2}\left(\mathrm{d}t + \frac{3\,ma}{2\,r^2}\,(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\right)^2 \\ &+ \tilde{H}\left(\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \cos^2\theta\,\mathrm{d}\psi^2\big)\right). \end{aligned} \tag{18.1.34}$$

We recognize here the BMPV solution [164, p. 4, 246, p. 16]. The fact that this solution has only one rotation parameter can be seen more easily in Euler angle coordinates [246, sec. 3, 253, sec. 2] or by looking at the conserved charges in the ϕ- and ψ-planes [164, sec. 3].

Transforming the Maxwell potential

Following the procedure described in [99] and recalled in section 14.5.2, one can also derive the gauge field in the rotating framework from the original static one (18.1.22). The latter can be written in the (u, r) coordinates

$$A = \frac{\sqrt{3}}{2}\,(H - 1)\,\mathrm{d}u, \tag{18.1.35}$$

since we can remove the $A_r(r)$ component by a gauge transformation. One can apply the two JN transformations (18.1.6) and (18.1.10) with $b = a$ to obtain

$$A = \frac{\sqrt{3}}{2}\,(\tilde{H} - 1)\Big(\mathrm{d}u - a\,(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\Big). \tag{18.1.36}$$

Then going into BL coordinates with (18.1.15) and (18.1.33) provides

$$A = \frac{\sqrt{3}}{2}\,(\tilde{H} - 1)\Big(\mathrm{d}t - a\,(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\Big) + A_r(r)\,\mathrm{d}r. \tag{18.1.37}$$

Again A_r depends only on r and can be removed by a gauge transformation. Applying the extremal limit (18.1.30) finally gives

$$A = \frac{\sqrt{3}}{2}\,\frac{m}{r^2}\Big(\mathrm{d}t - a\,(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\Big), \tag{18.1.38}$$

which is again the result presented in [164, p. 5].

Despite the fact that the seed metric (18.1.20) together with the gauge field (18.1.22) solves the equations of motion for any value of λ, the resulting rotating metric solves the equations only for $\lambda = 1$ (see [246, sec. 7] for a discussion). An explanation in this reduction can be found in the limit (18.1.30) that was needed for transforming the metric to Boyer–Lindquist coordinates and which gives a supersymmetric black hole – which necessarily has $\lambda = 1$.

18.1.3 Another approach to BMPV

In section 18.1.2 we applied the same recipe given in section 18.1.1 which, according to our claim, is the standard procedure in five dimensions.

There is another way to derive BMPV black hole. Indeed, by considering that terms quadratic in the angular momentum do not survive in the extremal limit, they can be added the metric without modifying the final result. Hence we can decide to transform all the terms of the metric[9] since the additional terms will be subleading. As a result the BL transformation is directly well defined and overall formulas are simpler, but we need to take the extremal limit before the end (this could be done either in (u, r) or (t, r) coordinates). This section shows that both approaches give the same result.

Applying the two transformations

$$u = u' + ia\cos\theta, \qquad \mathrm{d}u = \mathrm{d}u' - a\sin^2\theta\,\mathrm{d}\phi, \tag{18.1.39a}$$
$$u = u' + ia\sin\theta, \qquad \mathrm{d}u = \mathrm{d}u' - a\cos^2\theta\,\mathrm{d}\psi \tag{18.1.39b}$$

successively on all the terms one obtains the metric

$$\begin{aligned} \mathrm{d}s^2 = &- \tilde{H}^{-2}\big(\mathrm{d}u - a(1-\tilde{H}^{3/2})(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\big)^2 \\ &- 2\tilde{H}^{-1/2}\big(\mathrm{d}u - a(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\big)\,\mathrm{d}r \\ &+ \tilde{H}\left((r^2+a^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \cos^2\theta\,\mathrm{d}\psi^2) + a^2(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)^2\right), \end{aligned} \tag{18.1.40}$$

where again $\tilde{H}$ is given by (18.1.28)

$$\tilde{H} = 1 + \frac{m}{r^2 + a^2}. \tag{18.1.41}$$

Only one term is different when comparing with (18.1.27).

The BL transformation (18.1.15) is well-defined and the corresponding functions are

$$g(r) = \frac{a^2 + (r^2+a^2)\tilde{H}(r)}{r^2 + 2a^2}, \qquad h_\phi(r) = h_\psi(r) = \frac{a}{r^2 + 2a^2} \tag{18.1.42}$$

which do not depend on θ. They lead to the metric

$$\begin{aligned} \mathrm{d}s^2 = &- \tilde{H}^{-2}\big(\mathrm{d}t - a(1-\tilde{H}^{3/2})(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)\big)^2 \\ &+ \tilde{H}\left[(r^2+a^2)\left(\frac{\mathrm{d}r^2}{r^2+2a^2} + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \cos^2\theta\,\mathrm{d}\psi^2\right)\right. \\ &\left. + a^2(\sin^2\theta\,\mathrm{d}\phi + \cos^2\theta\,\mathrm{d}\psi)^2\right]. \end{aligned} \tag{18.1.43}$$

At this point it is straightforward to check that this solution does not satisfy Einstein equations and we need to take the extremal limit (18.1.30)

$$a, m \longrightarrow 0, \qquad \text{imposing} \qquad \frac{m}{a^2} = \text{cst} \tag{18.1.44}$$

[9] In opposition to our initial recipe, but this is done in a controlled way.

in order to get the BMPV solution (18.1.34)

$$\begin{aligned} \mathrm{d}s^2 = &- \tilde{H}^{-2} \left(\mathrm{d}t + \frac{3\,ma}{2\,r^2} \, (\sin^2\theta \, \mathrm{d}\phi + \cos^2\theta \, \mathrm{d}\psi) \right)^2 \\ &+ \tilde{H} \left(\mathrm{d}r^2 + r^2 \big(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 + \cos^2\theta \, \mathrm{d}\psi^2 \big) \right). \end{aligned} \tag{18.1.45}$$

It is surprising that the BL transformation is simpler in this case. Another point that is worth stressing is that we did not need to take the extremal limit in this computation, whereas in section 18.1.2 we had to in order to get a well-defined BL transformation.

18.1.4 CCLP black hole

It would be very interesting to find the CCLP black hole [244] (see also [248, sec. 2]), which is the corresponding non-extremal solution with four independent charges: two angular momenta a and b, an electric charge q and the mass m. This black hole is also a solution of $d = 5$ minimal supergravity (18.1.19).

The solution reads

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}t^2 &+ (1 - \tilde{f})(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi - b\cos^2\theta\,\mathrm{d}\psi)^2 + \frac{r^2\rho^2}{\Delta_r}\,\mathrm{d}r^2 \\ &+ \rho^2\mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2 + (r^2 + b^2)\cos^2\theta\,\mathrm{d}\psi^2 \\ &- \frac{2q}{\rho^2}\,(b\sin^2\theta\,\mathrm{d}\phi + a\cos^2\theta\,\mathrm{d}\psi)(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi - b\cos^2\theta\,\mathrm{d}\psi), \end{aligned} \tag{18.1.46a}$$

$$A = \frac{\sqrt{3}}{2}\,\frac{q}{\rho^2}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi - b\cos^2\theta\,\mathrm{d}\psi), \tag{18.1.46b}$$

where the function are given by

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \tag{18.1.47a}$$

$$\tilde{f} = 1 - \frac{2m}{\rho^2} + \frac{q^2}{\rho^4}, \tag{18.1.47b}$$

$$\Delta_r = \Pi + 2abq + q^2 - 2mr^2. \tag{18.1.47c}$$

Yet, using our prescription, it appears that the metric of this black hole cannot entirely be recovered. Indeed all the terms but one are generated by our algorithm, which also provides the correct gauge field. The missing term (corresponding to the last one in (18.1.46a)) is proportional to the electric charge and the current prescription cannot generate it.

This issue may be related to the fact that the CCLP solution cannot be written as a Kerr–Schild metric but as an extended Kerr–Schild one [254–256], which includes an additional term proportional to a spacelike vector. It appears that the missing term corresponds precisely to this additional term in the extended Kerr–Schild metric, and it is well-known that the JN algorithm works mostly for Kerr–Schild metrics. Moreover the Δ computed from (18.2.17) depends on θ and the BL transformation would not be well-defined if the additional term is not present to modify Δ to Δ_r.

18.2 Transformation in any dimension

In this appendix we consider the JN algorithm applied to a general static d-dimension metric. As we argued in a previous section it is important to consider separately the transformation of the metric and the complexification of the functions inside. Hence we are able to derive the general form of a rotating metric with the maximal number of angular momenta it can have in d dimensions, but we are not able to apply this result to any specific example for

$d \geq 6$, except if all momenta but one are vanishing [122]. Despite this last problem, this computation provides a unified framework for $d = 3, 4, 5$ (see section 18.3.4 for the BTZ black hole).

In the following the dimension is taken to be odd in order to simplify the computations, but the final result holds also for d even.

18.2.1 Metric transformation

Seed metric and discussion

Consider the d-dimensional static metric (notations are defined in appendix B.1)

$$ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2\,d\Omega_{d-2}^2 \tag{18.2.1}$$

where $d\Omega_{d-2}^2$ is the metric on S^{d-2}

$$d\Omega_{d-2}^2 = d\theta_{d-2} + \sin^2\theta_{d-2}\,d\Omega_{d-3}^2 = \sum_{i=1}^{n}\left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right). \tag{18.2.2}$$

The number $n = (d-1)/2$ denotes the number of independent 2-spheres.

In Eddington–Finkelstein coordinates the metric reads

$$ds^2 = (1-f)\,du^2 - du\,(du + 2dr) + r^2 \sum_i \Big(d\mu_i^2 + \mu_i^2\,d\phi_i^2\Big). \tag{18.2.3}$$

The metric looks like a 2-dimensional space (t, r) with a certain number of additional 2-spheres (μ_i, ϕ_i) which are independent from one another. Then we can consider only the piece (u, r, μ_i, ϕ_i) (for fixed i) which will transform like a 4-dimensional spacetime, while the other part of the metric (μ_j, ϕ_j) for all $j \neq i$ will be unchanged. After the first transformation we can move to another 2-sphere. We can thus imagine to put in rotation only one of these spheres. Then we will apply again and again the algorithm until all the spheres have angular momentum: the whole complexification will thus be a n-steps process. Moreover if these 2-spheres are taken to be independent this implies that we should not complexify the functions that are not associated with the plane we are putting in rotation.

To match these demands the metric is rewritten as

$$ds^2 = (1-f)\,du^2 - du\,(du + 2dr_{i_1}) + r_{i_1}^2\,(d\mu_{i_1}^2 + \mu_{i_1}^2 d\phi_{i_1}^2) + \sum_{i\neq i_1}\Big(r_{i_1}^2\,d\mu_i^2 + R^2\mu_i^2\,d\phi_i^2\Big). \tag{18.2.4}$$

where we introduced the following two functions of r

$$r_{i_1}(r) = r, \qquad R(r) = r. \tag{18.2.5}$$

This allows to choose different complexifications for the different terms in the metric. It may be surprising to note that the factors in front of $d\mu_i^2$ have been chosen to be $r_{i_1}^2$ and not R^2, but the reason is that the μ_i are all linked by the constraint

$$\sum_i \mu_i^2 = 1 \tag{18.2.6}$$

and the transformation of one i_1-th 2-sphere will change the corresponding μ_{i_1}, but also all the others, as it is clear from the formula (B.1.14) with all the a_i vanishing but one (this can also be observed in $5d$ where both μ_i are gathered into θ).

First transformation

The transformation is chosen to be

$$r_{i_1} = r'_{i_1} - i\, a_{i_1} \sqrt{1 - \mu_{i_1}^2}, \qquad u = u' + i\, a_{i_1} \sqrt{1 - \mu_{i_1}^2} \tag{18.2.7a}$$

which, together with the ansatz

$$i\, \frac{\mathrm{d}\mu_{i_1}}{\sqrt{1 - \mu_{i_1}^2}} = \mu_{i_1}\, \mathrm{d}\phi_{i_1}, \tag{18.2.7b}$$

gives the differentials

$$\mathrm{d}r_{i_1} = \mathrm{d}r'_{i_1} + a_{i_1} \mu_{i_1}^2\, \mathrm{d}\phi_{i_1}, \qquad \mathrm{d}u = \mathrm{d}u' - a_{i_1} \mu_{i_1}^2\, \mathrm{d}\phi_{i_1}. \tag{18.2.7c}$$

It is easy to check that this transformation reproduces the one given in four and five dimensions.

The complexified version of f is written as $\tilde{f}^{\{i_1\}}$: we need to keep track of the order in which we gave angular momentum since the function $\tilde{f}$ will be transformed at each step.

We consider separately the transformation of the (u, r) and $\{\mu_i, \phi_i\}$ parts. Inserting the transformations (18.2.7) in (18.2.3) results in

$$\begin{aligned}
\mathrm{d}s_{u,r}^2 &= (1 - \tilde{f}^{\{i_1\}}) \Big(\mathrm{d}u - a_{i_1} \mu_{i_1}^2\, \mathrm{d}\phi_{i_1}\Big)^2 - \mathrm{d}u\, (\mathrm{d}u + 2\mathrm{d}r_{i_1}) + 2 a_{i_1} \mu_{i_1}^2\, \mathrm{d}r_{i_1} \mathrm{d}\phi_{i_1} + a_{i_1}^2 \mu_{i_1}^4\, \mathrm{d}\phi_{i_1}^2, \\
\mathrm{d}s_{\mu,\phi}^2 &= \big(r_{i_1}^2 + a_{i_1}^2\big)(\mathrm{d}\mu_{i_1}^2 + \mu_{i_1}^2 \mathrm{d}\phi_{i_1}^2) + \sum_{i \neq i_1} \big(r_{i_1}^2 \mathrm{d}\mu_i^2 + R^2 \mu_i^2\, \mathrm{d}\phi_i^2\big) - a_{i_1}^2 \mu_{i_1}^4\, \mathrm{d}\phi_{i_1}^2 \\
&\quad + a_{i_1}^2 \bigg[- \mu_{i_1}^2 \mathrm{d}\mu_{i_1}^2 + (1 - \mu_{i_1}^2) \sum_{i \neq i_1} \mathrm{d}\mu_i^2 \bigg].
\end{aligned}$$

The term in the last bracket vanishes as can be seen by using the differential of the constraint

$$\sum_i \mu_i^2 = 1 \Longrightarrow \sum_i \mu_i \mathrm{d}\mu_i = 0. \tag{18.2.9}$$

Since this step is very important and non-trivial we expose the details

$$\begin{aligned}
[\cdots] &= \mu_{i_1}^2 \mathrm{d}\mu_{i_1}^2 - (1 - \mu_{i_1}^2) \sum_{i \neq i_1} \mathrm{d}\mu_i^2 = \Bigg(\sum_{i \neq i_1} \mu_i \mathrm{d}\mu_i \Bigg)^2 - \sum_{j \neq i_1} \mu_j^2 \sum_{i \neq i_1} \mathrm{d}\mu_i^2 \\
&= \sum_{i,j \neq i_1} \big(\mu_i \mu_j \mathrm{d}\mu_i \mathrm{d}\mu_j - \mu_j^2 \mathrm{d}\mu_i^2\big) = \sum_{i,j \neq i_1} \mu_j \big(\mu_i \mathrm{d}\mu_j - \mu_j \mathrm{d}\mu_i\big) \mathrm{d}\mu_i = 0
\end{aligned}$$

by antisymmetry.

Setting $r_{i_1} = R = r$ one obtains the metric

$$\begin{aligned}
\mathrm{d}s^2 &= (1 - \tilde{f}^{\{i_1\}}) \Big(\mathrm{d}u - a_{i_1} \mu_{i_1}^2\, \mathrm{d}\phi_{i_1}\Big)^2 - \mathrm{d}u\, (\mathrm{d}u + 2\mathrm{d}r) + 2 a_{i_1} \mu_{i_1}^2\, \mathrm{d}r \mathrm{d}\phi_{i_1} \\
&\quad + \big(r^2 + a_{i_1}^2\big)(\mathrm{d}\mu_{i_1}^2 + \mu_{i_1}^2 \mathrm{d}\phi_{i_1}^2) + \sum_{i \neq i_1} r^2 \big(\mathrm{d}\mu_i^2 + \mu_i^2\, \mathrm{d}\phi_i^2\big).
\end{aligned} \tag{18.2.10}$$

It corresponds to Myers–Perry metric in d dimensions with one non-vanishing angular momentum. We recover the same structure as in (18.2.4) with some extra terms that are specific to the i_1-th 2-sphere.

Iteration and final result

We should now split again r in functions (r_{i_2}, R). Very similarly to the first time we have

$$\begin{aligned} \mathrm{d}s^2 = (1-\tilde{f}^{\{i_1\}})\left(\mathrm{d}u - a_{i_1}\mu_{i_1}^2\,\mathrm{d}\phi_{i_1}\right)^2 - \mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r_{i_2}) + 2a_{i_1}\mu_{i_1}^2\,\mathrm{d}R\mathrm{d}\phi_{i_1} \\ + (r_{i_2}^2 + a_{i_1}^2)\mathrm{d}\mu_{i_1}^2 + \left(R^2 + a_{i_1}^2\right)\mu_{i_1}^2\mathrm{d}\phi_{i_1}^2 + r_{i_2}^2(\mathrm{d}\mu_{i_2}^2 + \mu_{i_2}^2\mathrm{d}\phi_{i_2}^2) \\ + \sum_{i\neq i_1,i_2}\left(r_{i_2}^2\mathrm{d}\mu_i^2 + R^2\mu_i^2\,\mathrm{d}\phi_i^2\right). \end{aligned} \tag{18.2.11}$$

We can now complexify as

$$r_{i_2} = r'_{i_2} - i a_{i_2}\sqrt{1-\mu_{i_2}^2}, \qquad u = u' + i\,a_{i_1}\sqrt{1-\mu_{i_2}^2}. \tag{18.2.12}$$

The steps are exactly the same as before, except that we have some inert terms. The complexified functions is now $\tilde{f}^{\{i_1,i_2\}}$.

Repeating the procedure n times we arrive at

$$\begin{aligned} \mathrm{d}s^2 =& -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \sum_i (r^2 + a_i^2)(\mathrm{d}\mu_i^2 + \mu_i^2\mathrm{d}\phi_i^2) - 2\sum_i a_i\mu_i^2\,\mathrm{d}r\mathrm{d}\phi_i \\ & + \left(1 - \tilde{f}^{\{i_1,\ldots,i_n\}}\right)\left(\mathrm{d}u + \sum_i a_i\mu_i^2\mathrm{d}\phi_i\right)^2. \end{aligned} \tag{18.2.13}$$

One recognizes the general form of the d-dimensional metric with n angular momenta [163].

Let's quote the metric in Boyer–Lindquist coordinates (omitting the indices on $\tilde{f}$) [163]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (1-\tilde{f})\left(\mathrm{d}t - \sum_i a_i\mu_i^2\mathrm{d}\phi_i\right)^2 + \frac{r^2\rho^2}{\Delta}\,\mathrm{d}r^2 + \sum_i (r^2 + a_i^2)\left(\mathrm{d}\mu_i^2 + \mu_i^2\,\mathrm{d}\phi_i^2\right) \tag{18.2.14}$$

which is obtained from the transformation

$$\mathrm{d}u = \mathrm{d}t - g\,\mathrm{d}r, \qquad \mathrm{d}\phi_i = \mathrm{d}\phi'_i - h_i\,\mathrm{d}r \tag{18.2.15}$$

with functions

$$g = \frac{\Pi}{\Delta} = \frac{1}{1 - F(1-\tilde{f})}, \qquad h_i = \frac{\Pi}{\Delta}\,\frac{a_i}{r^2 + a_i^2}, \tag{18.2.16}$$

and where the various quantities involved are (see appendix B.1.4)

$$\begin{gathered} \Pi = \prod_i (r^2 + a_i^2), \qquad F = 1 - \sum_i \frac{a_i^2\mu_i^2}{r^2 + a_i^2} = r^2\sum_i \frac{\mu_i^2}{r^2 + a_i^2}, \\ r^2\rho^2 = \Pi F, \qquad \Delta = \tilde{f}\,r^2\rho^2 + \Pi(1-F). \end{gathered} \tag{18.2.17}$$

Before ending this section, we comment the case of even dimensions: the term $\varepsilon'\,r^2\mathrm{d}\alpha^2$ is complexified as $\varepsilon'\,r_{i_1}^2\mathrm{d}\alpha^2$, since it contributes to the sum

$$\sum_i \mu_i^2 + \alpha^2 = 1. \tag{18.2.18}$$

This can be seen more clearly by defining $\mu_{n+1} = \alpha$ (we can also define $\phi_{n+1} = 0$), in which case the index i runs from 1 to $n + \varepsilon$, and all the previous computations are still valid.

18.3 Examples in various dimensions

18.3.1 Flat space

A first and trivial example is to take $f = 1$. In this case one recovers Minkowski metric in spheroidal coordinates with direction cosines (appendix B.1.4)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + F\,\mathrm{d}\bar{r}^2 + \sum_i (\bar{r}^2 + a_i^2)\Big(\mathrm{d}\bar{\mu}_i^2 + \bar{\mu}_i^2\,\mathrm{d}\bar{\phi}_i^2\Big) + \varepsilon'\,r^2\mathrm{d}\alpha^2. \tag{18.3.1}$$

In this case the JN algorithm is equivalent to a (true) change of coordinates [109] and there is no intrinsic rotation. The presence of a non-trivial function f then deforms the algorithm.

18.3.2 Myers–Perry black hole with one angular momentum

The derivation of the Myers–Perry metric with one non-vanishing angular momentum has been found by Xu [122].

The transformation is taken to be in the first plane

$$r = r' - ia\sqrt{1-\mu^2} \tag{18.3.2}$$

where $\mu \equiv \mu_1$. The transformation to the mixed spherical–spheroidal system (appendix B.1.5 is obtained by setting

$$\mu = \sin\theta, \qquad \phi_1 = \phi. \tag{18.3.3}$$

In these coordinates the transformation reads

$$r = r' - ia\cos\theta. \tag{18.3.4}$$

We will use the quantity

$$\rho^2 = r^2 + a^2(1-\mu^2) = r^2 + a^2\cos^2\theta. \tag{18.3.5}$$

The Schwarzschild–Tangherlini metric is [252]

$$\mathrm{d}s^2 = -f\,\mathrm{d}t^2 + f^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{d-2}^2, \qquad f = 1 - \frac{m}{r^{d-3}}. \tag{18.3.6}$$

Applying the previous transformation results in

$$\begin{aligned}\mathrm{d}s^2 = (1-\tilde{f})\left(\mathrm{d}u - a\mu^2\,\mathrm{d}\phi\right)^2 - \mathrm{d}u\,(\mathrm{d}u + 2\mathrm{d}r) + 2a\mu^2\,\mathrm{d}r\mathrm{d}\phi \\ + \big(r^2 + a^2\big)(\mathrm{d}\mu^2 + \mu^2\mathrm{d}\phi^2) + \sum_{i\neq 1} r^2\big(\mathrm{d}\mu_i^2 + \mu_i^2\,\mathrm{d}\phi_i^2\big).\end{aligned} \tag{18.3.7}$$

where f has been complexified as

$$\tilde{f} = 1 - \frac{m}{\rho^2 r^{d-5}}. \tag{18.3.8}$$

In the mixed coordinate system one has [122, 242]

$$\begin{aligned}\mathrm{d}s^2 = -\,\tilde{f}\,\mathrm{d}t^2 + 2a(1-\tilde{f})\sin^2\theta\,\mathrm{d}t\mathrm{d}\phi + \frac{r^{d-3}\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 \\ + \frac{\Sigma^2}{\rho^2}\,\sin^2\theta\,\mathrm{d}\phi^2 + r^2\cos^2\theta^2\,\mathrm{d}\Omega_{d-4}^2.\end{aligned} \tag{18.3.9}$$

where we defined as usual

$$\Delta = \tilde{f}\rho^2 + a^2\sin^2\theta, \qquad \frac{\Sigma^2}{\rho^2} = r^2 + a^2 + a g_{t\phi}. \tag{18.3.10}$$

This last expression explains why the transformation is straightforward with one angular momentum: the transformation is exactly the one for $d = 4$ and the extraneous dimensions are just spectators.

We have not been able to generalize this result for several non-vanishing momenta for $d \geq 6$, even for the case with equal momenta .

18.3.3 Five-dimensional Myers–Perry

We take a new look at the five-dimensional Myers–Perry solution in order to derive it in spheroidal coordinates because it is instructive.

The function

$$1 - f = \frac{m}{r^2} \tag{18.3.11}$$

is first complexified as

$$1 - \tilde{f}^{\{1\}} = \frac{m}{|r_1|^2} = \frac{m}{r^2 + a^2(1 - \mu^2)} \tag{18.3.12}$$

and then as

$$1 - \tilde{f}^{\{1,2\}} = \frac{m}{|r_2|^2 + a^2(1 - \mu^2)} = \frac{m}{r^2 + a^2(1 - \mu^2) + b^2(1 - \nu^2)}. \tag{18.3.13}$$

after the two transformations

$$r_1 = r_1' - ia\sqrt{1 - \mu^2}, \qquad r_2 = r_2' - ib\sqrt{1 - \nu^2}. \tag{18.3.14}$$

For $\mu = \sin\theta$ and $\nu = \cos\theta$ one recovers the transformations from sections 18.1.1 and 18.1.2.

Let's denote the denominator by ρ^2 and compute

$$\begin{aligned}
\frac{\rho^2}{r^2} &= r^2 + a^2(1 - \mu^2) + b^2(1 - \nu^2) = (\mu^2 + \nu^2)r^2 + \nu^2 a^2 + \mu^2 b^2 \\
&= \mu^2(r^2 + b^2) + \nu^2(r^2 + a^2) = (r^2 + b^2)(r^2 + a^2)\left(\frac{\mu^2}{r^2 + a^2} + \frac{\nu^2}{r^2 + b^2}\right).
\end{aligned}$$

and thus

$$r^2\rho^2 = \Pi F. \tag{18.3.15}$$

Plugging this into $\tilde{f}^{\{1,2\}}$ we have [163]

$$1 - \tilde{f}^{\{1,2\}} = \frac{mr^2}{\Pi F}. \tag{18.3.16}$$

18.3.4 Three dimensions: BTZ black hole

As another application we show how to derive the $d = 3$ rotating BTZ black hole from its static version [243]

$$ds^2 = -f\, dt^2 + f^{-1}\, dr^2 + r^2 d\phi^2, \qquad f(r) = -M + \frac{r^2}{\ell^2}. \tag{18.3.17}$$

In three dimensions the metric on S^1 in spherical coordinates is given by

$$d\Omega_1^2 = d\phi^2. \tag{18.3.18}$$

Introducing the coordinate μ we can write it in oblate spheroidal coordinates

$$d\Omega_1^2 = d\mu^2 + \mu^2 d\phi^2 \tag{18.3.19}$$

with the constraint

$$\mu^2 = 1. \tag{18.3.20}$$

Application of the transformation

$$u = u' + ia\sqrt{1-\mu^2}, \qquad r = r' - ia\sqrt{1-\mu^2} \tag{18.3.21}$$

gives from (18.2.13)

$$\begin{aligned} ds^2 = &- du^2 - 2du dr + (r^2 + a^2)(d\mu^2 + \mu^2 d\phi^2) - 2a\mu^2\, dr d\phi \\ &+ (1-\tilde{f})(du + a\mu^2 d\phi)^2. \end{aligned} \tag{18.3.22}$$

We still need to give the complexification of f which is

$$\tilde{f} = -m + \frac{\rho^2}{\ell^2}, \qquad \rho^2 = r^2 + a^2(1-\mu^2). \tag{18.3.23}$$

The transformation (18.2.16)

$$g = \frac{\rho^2(1-\tilde{f})}{\Delta}, \qquad h = \frac{a}{\Delta}, \qquad \Delta = r^2 + a^2 + (\tilde{f}-1)\rho^2 \tag{18.3.24}$$

to Boyer–Lindquist coordinates leads to the metric (18.2.14)

$$ds^2 = -dt^2 + (1-\tilde{f})(dt + a\mu^2 d\phi)^2 + \frac{\rho^2}{\Delta}\, dr^2 + (r^2 + a^2)(d\mu^2 + \mu^2\, d\phi^2). \tag{18.3.25}$$

Finally we can use the constraint $\mu^2 = 1$ to remove the μ. In this case we have

$$\rho^2 = r^2, \qquad \Delta = a^2 + \tilde{f} r^2 \tag{18.3.26}$$

and the metric simplifies to

$$ds^2 = -dt^2 + (1-\tilde{f})(dt + a d\phi)^2 + \frac{r^2}{a^2 + r^2\tilde{f}}\, dr^2 + (r^2 + a^2) d\phi^2. \tag{18.3.27}$$

We define the function

$$N^2 = \tilde{f} + \frac{a^2}{r^2} = -M + \frac{r^2}{\ell^2} + \frac{a^2}{r^2}. \tag{18.3.28}$$

Then redefining the time variable as [123, 124]

$$t = t' - a\phi \tag{18.3.29}$$

we get (omitting the prime)

$$ds^2 = -N^2 dt^2 + N^{-2}\, dr^2 + r^2(N^\phi dt + d\phi)^2 \tag{18.3.30}$$

with the angular shift

$$N^\phi(r) = \frac{a}{r^2}. \tag{18.3.31}$$

This is the solution given in [243] with $J = -2a$.

This has already been done by Kim [123, 124] in a different settings: he views the $d = 3$ solution as the slice $\theta = \pi/2$ of the $d = 4$ solution. Obviously this is equivalent to our approach: we have seen that $\mu = \sin\theta$ in $d = 4$ (appendix B.2), and the constraint $\mu^2 = 1$ is solved by $\theta = \pi/2$. Nonetheless our approach is more direct since the result just follows from a suitable choice of coordinates and there are no need for advanced justification.

Starting from the charged BTZ black hole

$$f(r) = -M + \frac{r^2}{\ell^2} - Q^2 \ln r^2, \qquad A = -\frac{Q}{2}\ln r^2, \tag{18.3.32}$$

it is not possible to find the charged rotating BTZ black hole from [257, sec. 4.2]: the solution solves Einstein equations, but not the Maxwell ones. This has been already remarked using another technique in [258, app. B]. It may be possible that a more general ansatz is necessary, following chapter 15 but in $3d$.

Part V

Appendices

Appendix A

Conventions

A.1 Generalities

We mostly follow the conventions of [165] (see also [170, app. C]).

Greek indices are curved, roman indices are flat (local Lorentz). Specific names for curved indices are given, such as (t, r, θ, ϕ), and numbers are reserved for flat indices, such as $(0, 1, 2, 3)$. In most of the text we use Planck units

$$8\pi G = \hbar = c = k = 1. \tag{A.1.1}$$

The signature of spacetime metric

$$\eta_{ab} = \epsilon_\eta \ \text{diag}(-1, 1, 1, 1) \tag{A.1.2}$$

is taken to be mostly plus $\epsilon_\eta = 1$. The Levi–Civita symbol ε_{abcd} (in flat indices) is

$$\varepsilon_{0123} = \epsilon_\varepsilon, \qquad \varepsilon^{0123} = -\epsilon_\varepsilon \tag{A.1.3}$$

and we will use $\epsilon_\varepsilon = 1$.

Given a Lagrangian $\mathcal{L}$ the action reads

$$S = \int \mathrm{d}^d x \, \sqrt{-g}\, \mathcal{L}. \tag{A.1.4}$$

Partial derivatives are abbreviated as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{A.1.5}$$

The (anti)symmetrization is done with unit weight

$$A_{[ab]} = \frac{1}{2}\,(A_{ab} - A_{ba}), \qquad A_{(ab)} = \frac{1}{2}\,(A_{ab} + A_{ba}). \tag{A.1.6}$$

We summarize the number of degrees of freedom in tables A.1 and A.2.

field	spin	off-shell	on-shell
ϕ	0	1	1
λ	1/2	$2^{\lfloor d/2 \rfloor}$	$2^{\lfloor d/2 \rfloor - 1}$
A_μ	1	$d - 1$	$d - 2$
ψ_μ	3/2	$(d-1)\, 2^{\lfloor d/2 \rfloor}$	$(d-3)\, 2^{\lfloor d/2 \rfloor - 1}$
$g_{\mu\nu}$	2	$\frac{1}{2}\, d(d-1)$	$\frac{1}{2}\, d(d-3)$

Table A.1 – Degrees of freedom off-shell and on-shell for the fields with spin ≤ 2 [165, tab. 6.2].

field	spin	off-shell	on-shell
ϕ	0	1	1
λ	1/2	4	2
A_μ	1	3	2
ψ_μ	3/2	12	2
$g_{\mu\nu}$	2	6	2

Table A.2 – Degrees of freedom off-shell and on-shell for the fields with spin ≤ 2 for $d = 4$.

fields	$\psi^\alpha_\mu, \lambda^{\alpha i}$	X^Λ, A^Λ_μ	$A^i_\mu, \lambda^{\alpha i}, \tau^i$	$\zeta^{\mathcal{A}}$	q^u	Z^A, ξ^A	z^a	σ^x
here	α	Λ	i	$\mathcal{A}$	u	A	a	x
[165]	i	I	α	A	u			
[170, 171]	A	Λ	i	α	u			
[50]		A	a			I	i	
range	$1, 2$	$0, \dots, n_v$	$1, \dots, n_v$	$1, \dots, 2n_h$	$1, \dots, 4n_h$	$0, \dots, n_h$	$1, \dots, n_h$	$1, 2, 3$

Table A.3 – Indices of the $N = 2$ fields in various conventions. n_v and n_h are the numbers of vector and hypermultiplets. The last column x corresponds to SU(2) index (σ^x are the Pauli matrices).

signs	ϵ_η	ϵ_ε	ϵ_Ω	$\epsilon_\mathbb{C}$
here	$+1$	$+1$	$+1$	$+1$
[165]	$+1$	$+1$	$+1$	
[170, 171]	-1			
[50]			$+1$	$+1$

Table A.4 – Sign conventions. For other comparisons of conventions see [259, problem C.1, p. 449–453, 165, app. A].

A.2 Differential geometry

Given a metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu, \tag{A.2.1}$$

the Christoffel symbol and the Riemann tensor are

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2}\, g^{\mu\sigma}\big(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\big), \tag{A.2.2}$$

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\rho\tau}\Gamma^\tau{}_{\nu\sigma} - \Gamma^\mu{}_{\sigma\tau}\Gamma^\tau{}_{\nu\rho}. \tag{A.2.3}$$

The Ricci tensor and the curvature are

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}. \tag{A.2.4}$$

A manifold is said to be Einstein if

$$R_{\mu\nu} = \Lambda\, g_{\mu\nu}, \qquad \Lambda = \frac{R}{d}, \tag{A.2.5}$$

d being the spacetime dimension. In the case $\Lambda = 0$ it is said to be Ricci flat

$$R_{\mu\nu} = 0. \tag{A.2.6}$$

A Killing vector k_μ generates an isometry of the corresponding manifold and it is defined by the equation

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0. \tag{A.2.7}$$

A p-form A_p with components $A_{\mu_1\cdots\mu_p}$ is defined by

$$A_p = \frac{1}{p!} A_{\mu_1\cdots\mu_p} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{A.2.8}$$

The exterior derivative d is nilpotent and maps a p-form into a $(p+1)$-form (example with a 1-form)

$$F = \mathrm{d}A = \partial_\mu A_\nu \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{A.2.9a}$$
$$F_{\mu\nu} = 2\,\partial_{[\mu} A_{\nu]}. \tag{A.2.9b}$$

The interior derivative i_k by a vector k maps a p-form into a $(p-1)$-form (example with a 1-form)

$$i_k A = k \lrcorner A = k^\mu A_\mu. \tag{A.2.10}$$

The Lie derivative $\mathcal{L}_k$ acting on forms is defined as

$$\mathcal{L}_k = i_k \mathrm{d} + \mathrm{d} i_k \tag{A.2.11}$$

and it commutes with the differential [260, sec. 4.21]

$$[\mathcal{L}_k, \mathrm{d}] = 0. \tag{A.2.12}$$

The integration of a d-form A reads

$$\int A = \frac{1}{d!} \int A_{\mu_1\cdots\mu_d} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_d} = \int A_{0\cdots D-1} \, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{d-1}. \tag{A.2.13}$$

Levi–Civita tensor is given in curved coordinates by

$$\varepsilon_{\mu_1\cdots\mu_d} = e^{-1} \, e^{a_1}_{\mu_1} \cdots e^{a_d}_{\mu_d} \, \varepsilon_{a_1\cdots a_d}, \qquad \varepsilon^{\mu_1\cdots\mu_d} = e \, e^{\mu_1}_{a_1} \cdots e^{\mu_d}_{a_d} \, \varepsilon^{a_1\cdots a_d}, \tag{A.2.14}$$

where e^a_μ is the vielbein. Contraction of two symbols is

$$\varepsilon_{\mu_1\cdots\mu_n\nu_1\cdots\nu_p} \varepsilon^{\mu_1\cdots\mu_n\rho_1\cdots\rho_p} = -n!p! \, \delta_{\nu_1}{}^{[\rho_1} \cdots \delta_{\nu_p}{}^{\rho_p]}. \tag{A.2.15}$$

Using this tensor one can define the Hodge operation

$$\star(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}) = \frac{\sqrt{-g}}{(d-p)!} \, \varepsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_d} \, \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_d}, \tag{A.2.16a}$$
$$\star(e^{a_1} \wedge \cdots \wedge e^{a_p}) = \frac{1}{(d-p)!} \, \varepsilon^{a_1\cdots a_p}{}_{a_{p+1}\cdots a_d} \, e^{a_{p+1}} \wedge \cdots \wedge e^{a_d}, \tag{A.2.16b}$$

and the dual of a p-form will produce a $(d-p)$-form. This operation squares to -1

$$\star\star F = -F. \tag{A.2.17}$$

One has the formula

$$\int \star F^{(p)} \wedge F^{(p)} = \frac{1}{p!} \int \mathrm{d}^d x \, \sqrt{-g} \, F^{\mu_1\cdots\mu_p} F_{\mu_1\cdots\mu_p}. \tag{A.2.18}$$

In particular the dual of a 2-form for $d = 4$ is denoted by [165, sec. 4.2.1]

$$\tilde{H}^{ab} = -\frac{i}{2} \, \varepsilon^{abcd} H_{cd} = -i \star F^{\mu\nu}. \tag{A.2.19}$$

Them one can define the self-dual and anti-self-dual of this tensor as

$$H^\pm_{ab} = \frac{1}{2}(H_{ab} \pm \tilde{H}_{ab}) \tag{A.2.20}$$

with the properties

$$H^{\pm}_{ab} = \pm \tilde{H}^{\pm}_{ab}, \qquad H^{\pm}_{ab} = (H^{\mp}_{ab})^*. \tag{A.2.21}$$

Moreover the dual operation is an involution (thanks to the factor i). In the curved frame one has

$$\star F_{\mu\nu} = \frac{1}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \qquad \star F^{\mu\nu} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{A.2.22}$$

Given two tensors F and G one has the following identities

$$\tilde{F}^{\mu\nu}\tilde{G}_{\mu\nu} = F_{\mu\nu}G_{\mu\nu}, \qquad \tilde{F}^{\mu\nu}G_{\mu\nu} = F_{\mu\nu}\tilde{G}_{\mu\nu}, \tag{A.2.23a}$$

$$F^{+}_{\mu\nu}G^{-\mu\nu} = 0, \qquad g^{\mu\nu}F^{+}_{\mu[\rho}G^{-}_{\sigma]\nu} = 0, \qquad g^{\mu\nu}F^{+}_{\mu(\rho}G^{+}_{\sigma)\nu} = -\frac{1}{4}\,g_{\rho\sigma}\,F^{+}_{\mu\nu}G^{+\mu\nu}. \tag{A.2.23b}$$

A.3 Symplectic geometry

Let's consider a space of dimension $2n$. We use indices $M, N = 1, \dots, 2n$.

Define the 2-dimensional antisymmetric matrix

$$\varepsilon = \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.3.1}$$

where $\epsilon = \pm 1$. Then the (flat) symplectic metric is defined by

$$\omega = \varepsilon \otimes 1_n = \epsilon \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \tag{A.3.2}$$

1_n denoting the n-dimensional identity. An alternative representation is the block-diagonal form

$$\omega' = 1_n \otimes \varepsilon = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \tag{A.3.3}$$

The symplectic metric squares to -1

$$\omega^2 = -1 \tag{A.3.4}$$

and the inverse is simply $-\omega$

$$\omega^{-1} = -\omega. \tag{A.3.5}$$

Let's consider a vector with contravariant components A^M. We are using the NW–SE convention [170, app. C, 165, p. 421, 471]

$$\omega_{MN}\omega^{NP} = -\delta_M{}^P, \qquad A_M = -\epsilon\,\omega_{MN}A^N, \qquad A^M = \epsilon\,\omega^{MN}A_N. \tag{A.3.6}$$

This implies that $\omega^{MN} = \omega_{MN}$ (in components) and ω^{MN} does *not* correspond to the components of ω^{-1}. In particular with $\epsilon = 1$ this implies

$$A_M = A^N\omega_{NM}, \qquad A^M = \omega^{MN}A_N. \tag{A.3.7}$$

and the symplectic inner product of two vectors A and B is

$$\langle A, B\rangle = A^M\omega_{MN}B^N = A_M B^M. \tag{A.3.8}$$

In the course of this thesis we will have several different symplectic spaces: Ω, $\mathbb{C}$, ε. Each will have a different sign ϵ_Ω, $\epsilon_\mathbb{C}$, etc. We choose $\epsilon_\Omega = \epsilon_\mathbb{C} = 1$. The sign is reversed with respect to [46, 53, 85, 150, 170, 171], but the same as in [62, 149, 165, 169].

A.4 Gamma matrices

Gamma matrices form a Clifford algebra

$$[\gamma_\mu, \gamma_\nu] = 2\, g_{\mu\nu}, \qquad [\gamma_a, \gamma_b] = 2\, \eta_{ab}. \tag{A.4.1}$$

The Hermitian conjugate of γ^μ is

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0. \tag{A.4.2}$$

Antisymmetric products are denoted by

$$\gamma_{a_1 \cdots a_n} = \gamma_{[a_1} \cdots \gamma_{a_n]}. \tag{A.4.3}$$

Finally in four dimensions one defines

$$\gamma_5 = i\, \gamma_0 \gamma_1 \gamma_2 \gamma_3, \qquad \varepsilon_{abcd} \gamma^d = i\, \gamma_5 \gamma_{abc}. \tag{A.4.4}$$

The left and right projectors are defined by

$$P_L = \frac{1 + \gamma_5}{2}, \qquad P_R = \frac{1 - \gamma_5}{2}. \tag{A.4.5}$$

A.5 Spinors

Given a Majorana spinor ϵ^α, the chiral left and right Weyl spinors are denoted by [65, sec. 2.1]

$$\varepsilon_\alpha = P_L\, \epsilon^\alpha, \qquad \varepsilon^\alpha = P_R\, \epsilon^\alpha. \tag{A.5.1}$$

The Majorana and Dirac conjugates are

$$\bar{\lambda} = \lambda^t C, \qquad \bar{\lambda} = i\, \lambda^\dagger \gamma^0. \tag{A.5.2}$$

The charge conjugation is

$$\lambda^C = B^{-1} \lambda^*, \qquad B = i C \gamma^0. \tag{A.5.3}$$

The matrix C satisfy

$$C^2 = -1, \qquad C^t = -C, \qquad (C\gamma^\mu)^t = C\gamma^\mu. \tag{A.5.4}$$

A.6 Supergravity

Given a Lagrangian $\mathcal{L}$ the dual of the field strength F^Λ is defined by

$$G_\Lambda = \star \left(\frac{\delta \mathcal{L}}{\delta F^\Lambda} \right). \tag{A.6.1}$$

The electric and magnetic charges q_Λ and p^Λ contained in a volume V with boundary Σ are defined by

$$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \frac{1}{\mathrm{Vol}(\Sigma)} \int_\Sigma \mathcal{F} \tag{A.6.2}$$

where $\mathcal{F} = (F^\Lambda, G_\Lambda)$ are the field strengths. The charges are defined as densities to avoid infinite charges in the case of non-compact surfaces. For compact horizons one takes

$$\mathrm{Vol}(\Sigma) = \mathrm{Vol}(S^2) = 4\pi. \tag{A.6.3}$$

The central charge is defined by

$$\mathcal{Z} = \epsilon_\Omega\, \Gamma(\mathcal{Q}) = \epsilon_\Omega\, \langle \mathcal{V}, \mathcal{Q} \rangle \tag{A.6.4a}$$

$$= L^\Lambda q_\Lambda - M_\Lambda p^\Lambda = \mathrm{e}^{K/2}\,(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda). \tag{A.6.4b}$$

Note that there is a factor 2 in [165, p. 480].[1]

Similarly one defines

$$\mathcal{L}^x = \epsilon_\Omega\, \Gamma(\mathcal{P}^x) = \epsilon_\Omega\, \langle \mathcal{V}, \mathcal{P}^x \rangle \tag{A.6.5a}$$

$$= L^\Lambda P^x_\Lambda - M_\Lambda \widetilde{P}^{x\Lambda}. \tag{A.6.5b}$$

A.7 Topological horizons

Black hole horizons correspond to 2-dimensional (θ, ϕ) sections Σ with spherical, planar or hyperbolic topology [75, 76]. The sign of the curvature is denoted by κ and correspond to

$$\kappa = \begin{cases} +1 & \text{spherical,} \\ 0 & \text{planar,} \\ -1 & \text{hyperbolic.} \end{cases} \tag{A.7.1}$$

In the case $\kappa = 0, -1$ the horizon is non-compact and the full solution describes a black membrane [75].

For a static spacetime the 2-dimensional section is maximally symmetric. The corresponding spaces are the sphere S^2, the euclidean plane $\mathbb{R}^2$ and the hyperboloid H^2 respectively for positive, vanishing and negative curvature (see table A.5). In these cases the uniform metric on Σ reads

$$\mathrm{d}\Sigma^2 = \mathrm{d}\theta^2 + H'(\theta)^2\, \mathrm{d}\phi^2 \tag{A.7.2}$$

where

$$H(\theta) = \begin{cases} -\cos\theta & \kappa = 1, \\ \theta & \kappa = 0, \\ \cosh\theta & \kappa = -1, \end{cases} \qquad H'(\theta) = \begin{cases} \sin\theta & \kappa = 1, \\ 1 & \kappa = 0, \\ \sinh\theta & \kappa = -1. \end{cases} \tag{A.7.3}$$

The function $H(\theta)$ may be defined by the differential equation

$$H'' + \kappa H = 0, \qquad H(0) = 0, \quad H'(0) = 1. \tag{A.7.4}$$

topology	Σ	κ	$\mathrm{ISO}(\Sigma)$
spherical	S^2	$+1$	$\mathrm{SO}(3)$
planar	$\mathbb{R}^2$	0	$\mathbb{R}^2$
cylindrical	$\mathbb{R} \times S^1$	0	$\mathbb{R} \times \mathrm{SO}(2)$
toroidal	T^2	0	$\mathrm{SO}(2)^2$
hyperbolic	H^2	-1	$\mathrm{SO}(2,1)$
Riemann surface ($g > 1$)	Σ_g	-1	$\mathrm{SO}(2,1)/\Gamma$

Table A.5 – Horizon topology for static spacetime. The last row corresponds to hyperbolic Riemann surfaces; non-hyperbolic surfaces are the sphere S^2 for $g = 0$ and the torus T^2 for $g = 1$.

By definition black holes have a compact (orientable) horizon. These can be obtained by taking the quotient of the isometry group $\mathrm{ISO}(\Sigma)$ by a discrete subgroup Γ. In this case taking the quotient is a global effect and does not affect the fields, and in particular one

[1] For $\epsilon_\Omega = -1$ one writes $\mathcal{Z} = \langle \mathcal{Q}, \mathcal{V} \rangle$.

can work with the above metric. An intermediate case corresponds to a cylindrical black hole with horizon $\mathbb{R} \times S^1$ when the direction ϕ is made compact using the quotient $\mathbb{R}/\mathbb{Z}$. Compact horizons are Riemann surfaces Σ_g where $g \in \mathbb{N}$ denotes the genus. The sphere $g = 0$ is already compact so we do not need to take a quotient. The surface $g = 1$ corresponds to the 2-torus $T^2 \sim S^1 \times S^1$ obtained by the quotient $(\mathbb{R}/\mathbb{Z})^2$, while higher genus surfaces $g > 1$ are obtained by taking the quotient of H^2 by a Fuchsian group Γ, which is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ (see table A.5). The sign of the curvature reads

$$\kappa = \mathrm{sign}(1 - g). \tag{A.7.5}$$

If the black hole is spinning then Σ is deformed as the isometry group is reduced. For example in the case of spherical topology one obtains a spheroid and the isometry is only SO(2) (corresponding to the Killing vector ∂_ϕ).

Appendix B

Coordinate systems

This appendix is partly based on [163, 261]. We present formula for any dimension before summarizing them for 4 and 5 dimensions.

B.1 d-dimensional

Let's consider $d = N + 1$ dimensional Minkowski space whose metric is denoted by

$$\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu, \qquad \mu = 0, \dots, N. \tag{B.1.1}$$

In all the following coordinates systems the time direction can separated from the spatial (positive definite) metric as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\Sigma^2, \qquad \mathrm{d}\Sigma^2 = \gamma_{ab}\, \mathrm{d}x^a \mathrm{d}x^b, \qquad a = 1, \dots, N, \tag{B.1.2}$$

where $x^0 = t$.

We also define

$$n = \left\lfloor \frac{N}{2} \right\rfloor \tag{B.1.3}$$

such that

$$d + \varepsilon = 2n + 2, \qquad N + \varepsilon = 2n + 1, \qquad \varepsilon' = 1 - \varepsilon \tag{B.1.4}$$

where

$$\varepsilon = \frac{1}{2}(1 - (-1)^d) = \begin{cases} 0 & d \text{ even (or } N \text{ odd)} \\ 1 & d \text{ odd (or } N \text{ even),} \end{cases} \tag{B.1.5}$$

and conversely for ε'.

B.1.1 Cartesian system

The usual Cartesian metric is

$$\mathrm{d}\Sigma^2 = \delta_{ab} \mathrm{d}x^a \mathrm{d}x^b = \mathrm{d}x^a \mathrm{d}x^a = \mathrm{d}\boldsymbol{x}^2. \tag{B.1.6}$$

B.1.2 Spherical

Introducing a radial coordinate r, the flat space metric can be written as a $(N-1)$-sphere of radius r [252]

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{N-1}. \tag{B.1.7}$$

The term $\mathrm{d}\Omega^2_{N-1}$ corresponds the metric on the unit $(N-1)$-sphere S^{N-1}, which is parametrized by $(N-1)$ angles θ_i and is defined recursively as

$$\mathrm{d}\Omega^2_{N-1} = \mathrm{d}\theta^2_{N-1} + \sin^2\theta_{N-1}\,\mathrm{d}\Omega^2_{N-2}. \tag{B.1.8}$$

This surface can be embedded in N-dimensional flat space with coordinates X^i constrained by

$$X^a X^a = 1. \tag{B.1.9}$$

B.1.3 Spherical with direction cosines

In d-dimensions there are n orthogonal 2-planes,[1] thus we can pair $2n$ of the embedding coordinates X^a (B.1.9) as (X_i, Y_i) which are parametrized as

$$X_i + iY_i = \mu_i\,\mathrm{e}^{i\phi_i}, \qquad a = 1, \ldots n. \tag{B.1.10}$$

For d even there is an extra unpaired coordinate that is taken to be

$$X^N = \alpha. \tag{B.1.11}$$

Each pair parametrizes a 2-sphere of radius μ_i. The μ_i are called the *direction cosines* and satisfy

$$\sum_i \mu_i^2 + \varepsilon'\alpha^2 = 1 \tag{B.1.12}$$

since there is one superfluous coordinate from the embedding.

Finally the metric is

$$\mathrm{d}\Omega^2_{N-1} = \sum_i \left(\mathrm{d}\mu_i^2 + \mu_i^2\,\mathrm{d}\phi_i^2\right) + \varepsilon'\,\mathrm{d}\alpha^2. \tag{B.1.13}$$

The interest of these coordinates is that all rotational directions are symmetric.

B.1.4 Spheroidal with direction cosines

From the previous system we can define the spheroidal $(\bar r, \bar\mu_i, \bar\phi_i)$ system – adapted when some of the 2-spheres are deformed to ellipses – by introducing parameters a_i such that (for d odd)

$$r^2\mu_i^2 = (\bar r^2 + a_i^2)\bar\mu_i^2, \qquad \sum_i \bar\mu_i^2 = 1. \tag{B.1.14}$$

This last condition implies that

$$r^2 = \sum_i (\bar r^2 + a_i^2)\bar\mu_i^2 = \bar r^2 + \sum_i a_i^2\bar\mu_i^2. \tag{B.1.15}$$

In these coordinates the metric reads

$$\mathrm{d}\Sigma^2 = F\,\mathrm{d}\bar r^2 + \sum_i (\bar r^2 + a_i^2)\Big(\mathrm{d}\bar\mu_i^2 + \bar\mu_i^2\,\mathrm{d}\bar\phi_i^2\Big) + \varepsilon'\,r^2\mathrm{d}\alpha^2 \tag{B.1.16}$$

and we defined

$$F = 1 - \sum_i \frac{a_i^2\bar\mu_i^2}{\bar r^2 + a_i^2} = \sum_i \frac{\bar r^2\bar\mu_i^2}{\bar r^2 + a_i^2}. \tag{B.1.17}$$

Here the a_i are just introduced as parameters in the transformation, but in the main text they are interpreted as "true" rotation parameters, i.e. angular momenta (per unit of mass) of a black hole. They all appear on the same footing.

Another quantity of interest is

$$\Pi = \prod_i (\bar r^2 + a_i^2). \tag{B.1.18}$$

[1] Note that this is linked to the fact that the little group of massive representation in D dimension is $\mathrm{SO}(N)$, which possess n Casimir invariants [163].

B.1.5 Mixed spherical–spheroidal

We consider the deformation of the spherical metric where one of the 2-sphere is replaced by an ellipse [242, sec. 3].

To shorten the notation let's define

$$\theta = \theta_{N-1}, \qquad \phi = \theta_{N-2}. \tag{B.1.19}$$

Doing the change of coordinates

$$\sin^2\theta \sin^2\phi = \cos^2\theta. \tag{B.1.20}$$

the metric becomes

$$\mathrm{d}\Sigma^2 = \frac{\rho^2}{r^2 + a^2}\,\mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2 + r^2\cos^2\theta^2\,\mathrm{d}\Omega^2_{d-4} \tag{B.1.21}$$

where as usual

$$\rho^2 = r^2 + a^2\cos^2\theta. \tag{B.1.22}$$

Except for the last term one recognize 4-dimensional oblate spheroidal coordinates (B.2.9).

B.2 4-dimensional

In this section one considers

$$d = 4, \quad N = 3, \quad n = 1. \tag{B.2.1}$$

B.2.1 Cartesian system

$$\mathrm{d}\Sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{B.2.2}$$

B.2.2 Spherical

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{B.2.3a}$$

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2, \tag{B.2.3b}$$

where $\mathrm{d}\Omega^2 \equiv \mathrm{d}\Omega_2^2$.

B.2.3 Spherical with direction cosines

$$\mathrm{d}\Omega^2 = \mathrm{d}\mu^2 + \mu^2\,\mathrm{d}\phi^2 + \mathrm{d}\alpha^2, \tag{B.2.4a}$$

$$\mu^2 + \alpha^2 = 1, \tag{B.2.4b}$$

where

$$x + iy = r\mu\,\mathrm{e}^{i\phi}, \qquad z = r\alpha, \tag{B.2.5}$$

Using the constraint one can rewrite

$$\mathrm{d}\Omega^2 = \frac{1}{1-\mu^2}\,\mathrm{d}\mu^2 + \mu^2\,\mathrm{d}\phi^2. \tag{B.2.6}$$

Finally the change of coordinates

$$\alpha = \cos\theta, \qquad \mu = \sin\theta. \tag{B.2.7}$$

solves the constraint and gives back the spherical coordinates.

B.2.4 Spheroidal with direction cosines

The oblate spheroidal coordinates from the Cartesian ones are [234, p. 15]

$$x + iy = \sqrt{r^2 + a^2}\, \sin\theta\, \mathrm{e}^{i\phi}, \qquad z = r\cos\theta, \tag{B.2.8}$$

and the metric is

$$\mathrm{d}\Sigma^2 = \frac{\rho^2}{r^2 + a^2}\, \mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 + (r^2 + a^2) \sin^2\theta\, \mathrm{d}\phi^2, \qquad \rho^2 = r^2 + a^2 \cos^2\theta. \tag{B.2.9}$$

In terms of direction cosines one has

$$\mathrm{d}\Sigma^2 = \left(1 - \frac{r^2\mu^2}{r^2 + a^2}\right) \mathrm{d}r^2 + (r^2 + a^2)\Big(\mathrm{d}\mu^2 + \mu^2\, \mathrm{d}\phi^2\Big) + r^2 \mathrm{d}\alpha^2. \tag{B.2.10}$$

B.3 5-dimensional

In this section one consider

$$d = 4, \quad N = 3, \quad n = 1. \tag{B.3.1}$$

B.3.1 Spherical with direction cosines

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\mu^2 + \mu^2\, \mathrm{d}\phi^2 + \mathrm{d}\nu^2 + \nu^2\, \mathrm{d}\psi^2, \qquad \mu^2 + \nu^2 = 1 \tag{B.3.2}$$

where for simplicity

$$\mu = \mu_1, \qquad \mu = \mu_2, \qquad \phi = \phi_1, \qquad \psi = \phi_2. \tag{B.3.3}$$

B.3.2 Hopf coordinates

The constraint (B.3.2) can be solved by

$$\mu = \sin\theta, \qquad \nu = \cos\theta \tag{B.3.4}$$

and this gives the metric in Hopf coordinates

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2 + \cos^2\theta\, \mathrm{d}\psi^2. \tag{B.3.5}$$

Appendix C

DJN algorithm: summary

C.1 Complexification

$$r \longrightarrow \frac{1}{2}(r + \bar{r}) = \operatorname{Re} r\,, \tag{C.1.1a}$$

$$\frac{1}{r} \longrightarrow \frac{1}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) = \frac{\operatorname{Re} r}{|r|^2}\,, \tag{C.1.1b}$$

$$r^2 \longrightarrow |r|^2\,. \tag{C.1.1c}$$

C.2 Transformations

Coordinates:

$$r = r' + i\,F(\theta), \qquad u = u' + i\,G(\theta), \tag{C.2.1a}$$

$$\mathrm{d}r = \mathrm{d}r' + F'(\theta)H(\theta)\,\mathrm{d}\phi, \qquad \mathrm{d}u = \mathrm{d}u' + G'(\theta)H(\theta)\,\mathrm{d}\phi. \tag{C.2.1b}$$

Mass and horizon curvature:

$$m = m' + i\kappa\, n, \qquad \kappa \longrightarrow \kappa - \frac{4\Lambda}{3}\, n^2, \tag{C.2.2}$$

Functions:

- $\Lambda \neq 0$

$$F(\theta) = n, \qquad G(\theta) = -2\kappa\, n \ln H(\theta). \tag{C.2.3}$$

- $\Lambda = 0$

$$F(\theta) = n - a\, H'(\theta) + \kappa\, c \left(1 + H'(\theta)\, \ln \frac{H(\theta/2)}{H'(\theta/2)}\right), \tag{C.2.4a}$$

$$G(\theta) = \kappa\, a\, H'(\theta) - \kappa\, c\, H'(\theta)\, \ln \frac{H(\theta/2)}{H'(\theta/2)} - 2n \ln H(\theta). \tag{C.2.4b}$$

C.3 Metric and gauge field

Static:

$$\mathrm{d}s^2 = -f_t(r)\,\mathrm{d}t^2 + f_r(r)\,\mathrm{d}r^2 + f_\Omega(r)\,\mathrm{d}\Omega^2, \tag{C.3.1a}$$

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + H(\theta)^2\,\mathrm{d}\phi^2, \qquad H(\theta) = \begin{cases} \sin\theta & \kappa = 1, \\ \sinh\theta & \kappa = -1, \end{cases} \tag{C.3.1b}$$

$$A = f_A\,\mathrm{d}t. \tag{C.3.1c}$$

Non-static (null coordinates):

$$\mathrm{d}s^2 = -\tilde{f}_t(\mathrm{d}u + \alpha\,\mathrm{d}r + \omega H\,\mathrm{d}\phi)^2 + 2\beta\,\mathrm{d}r\mathrm{d}\phi + \tilde{f}_\Omega\big(\mathrm{d}\theta^2 + \sigma^2 H^2 \mathrm{d}\phi^2\big), \tag{C.3.2a}$$

$$A = \tilde{f}_A\,(\mathrm{d}u + G' H\,\mathrm{d}\phi), \tag{C.3.2b}$$

$$\omega = G' + \sqrt{\frac{\tilde{f}_r}{\tilde{f}_t}}\,F', \qquad \sigma^2 = 1 + \frac{\tilde{f}_r}{\tilde{f}_\Omega}\,F'^2, \qquad \alpha = \sqrt{\frac{\tilde{f}_r}{\tilde{f}_t}}, \qquad \beta = \tilde{f}_r\,F' H. \tag{C.3.2c}$$

Non-static (Boyer–Lindquist):

$$\mathrm{d}s^2 = -\tilde{f}_t\big(\mathrm{d}t + \omega H\,\mathrm{d}\phi\big)^2 + \frac{\tilde{f}_\Omega}{\Delta}\,\mathrm{d}r^2 + \tilde{f}_\Omega\big(\mathrm{d}\theta^2 + \sigma^2 H^2 \mathrm{d}\phi^2\big), \tag{C.3.3a}$$

$$A = \tilde{f}_A\left(\mathrm{d}t - \frac{\tilde{f}_\Omega}{\sqrt{\tilde{f}_t\tilde{f}_r}\,\Delta}\,\mathrm{d}r + G' H\,\mathrm{d}\phi\right), \tag{C.3.3b}$$

$$\omega = G' + \sqrt{\frac{\tilde{f}_r}{\tilde{f}_t}}\,F', \qquad \sigma^2 = 1 + \frac{\tilde{f}_r}{\tilde{f}_\Omega}\,F'^2, \qquad \Delta = \frac{\tilde{f}_\Omega}{\tilde{f}_r}\,\sigma^2. \tag{C.3.3c}$$

Boyer–Lindquist functions:

$$g(r) = \frac{\sqrt{\big(\tilde{f}_t\tilde{f}_r\big)^{-1}}\,\tilde{f}_\Omega - F'G'}{\Delta}, \qquad h(r) = \frac{F'}{H(\theta)\Delta}. \tag{C.3.4}$$

When $F' = 0$ the metric takes a simple form

$$\mathrm{d}s^2 = -\tilde{f}_t\big(\mathrm{d}t + G' H\,\mathrm{d}\phi\big)^2 + \tilde{f}_r\,\mathrm{d}r^2 + \tilde{f}_\Omega\,\mathrm{d}\Omega^2. \tag{C.3.5}$$

Appendix D

Group theory

D.1 Group classification

For some elements see [165, app. B].

D.1.1 Symplectic groups

Given a vector space of dimension $2n$ over a field $\mathbb{K}$ endowed with a skew-symmetric product defined by the 2-form Ω, the set of transformations that preserve this product define the symplectic group $\mathrm{Sp}(2n, \mathbb{K}) \subset \mathrm{SL}(2n, \mathbb{K})$

$$S \in \mathrm{Sp}(2n, \mathbb{K}) \Longrightarrow S^t \Omega S = \Omega. \tag{D.1.1}$$

The three symplectic groups of interest to us are: $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{Sp}(n) \equiv \mathrm{USp}(2n)$. The first two are non-compact while the latter is compact: $\mathrm{USp}(2n)$ is the compact form of $\mathrm{Sp}(2n, \mathbb{R})$, both being real Lie groups. On the other hand $\mathrm{Sp}(2n, \mathbb{C})$ is complex. They all have n generators and are of dimension (real or complex) $n(2n+1)$.

The Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$ corresponds to the semi-simple complex algebra C_n, while the others are real forms: $\mathfrak{usp}(n)$ is the compact form and $\mathfrak{sp}(2n, \mathbb{R})$ is the normal (or split) form.

The compact group is isomorphic to

$$\mathrm{U}(n, \mathbb{H}) \equiv \mathrm{USp}(2n) \sim \mathrm{U}(2n) \cap \mathrm{Sp}(2n, \mathbb{C}). \tag{D.1.2}$$

Note also the isomorphism

$$\mathfrak{sp}(1) \sim \mathfrak{su}(2) \sim \mathfrak{so}(3), \qquad \mathfrak{sp}(2) \sim \mathfrak{so}(5) \tag{D.1.3}$$

Group	Matrices	Group type	compact	π_1
$\mathrm{Sp}(2n, \mathbb{R})$	$\mathbb{R}$	real	no	$\mathbb{Z}$
$\mathrm{Sp}(2n, \mathbb{C})$	$\mathbb{C}$	complex	no	1
$\mathrm{Sp}(n) \equiv \mathrm{USp}(2n)$	$\mathbb{H}$	real	yes	1

Table D.1 – Symplectic groups.

D.1.2 Groups on quaternions

Several matrix groups on the quaternions can be defined

$$\mathrm{SO}^*(2n) = \mathrm{O}(n, \mathbb{H}), \tag{D.1.4a}$$
$$\mathrm{SU}^*(2n) = \mathrm{SL}(n, \mathbb{H}), \tag{D.1.4b}$$
$$\mathrm{USp}(2n) = \mathrm{U}(n, \mathbb{H}), \tag{D.1.4c}$$
$$\mathrm{USp}^*(2n_+, 2n_-) = \mathrm{U}(2n_+, 2n_-) \cap \mathrm{Sp}(2n_+, 2n_-, \mathbb{C}). \tag{D.1.4d}$$

D.2 Homogeneous space

A homogeneous space $\mathcal{M}$ of dimension n is a coset manifold

$$\mathcal{M} = \frac{G}{H}, \qquad n = \dim G - \dim H. \tag{D.2.1}$$

It admits $n(n+1)/2$ Killing vectors which is the maximum number in dimension n. In such a space all points are equivalent, i.e. it is always possible to find an isometry transformation that takes a point p to a point p'. Its isometry group is G

$$\mathrm{ISO}(G/H) = G \tag{D.2.2}$$

only if the normalizer of H in G is the trivial group [170, p. 8].

A symmetric space is a homogeneous space for which the algebra of G can be decomposed as [208]

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{k} \tag{D.2.3}$$

with

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}. \tag{D.2.4}$$

Appendix E

Formulas

E.1 Quartic invariant identities

The formulas given in this appendix are a consequence of the Jordan algebra's structure of very special geometry and the fact that the duality groups are of E_7-type [147]. While they can be proved using techniques from [147, sec. 4] (see also [213, sec. 3, 207, sec. 2.2, 148, 211]), they have been determined by matching both sides on Mathematica. Some of them appeared already in [84, 85, 150].

The quartic invariant possesses many identities, some of them being given in [207, sec. 2.2].

Given two vectors A and B, any vectors built from them and from $I'_4(\cdot,\cdot,\cdot)$ can be expanded on the following basis

$$\Big\{A, B, I'_4(A), I'_4(A,A,B), I'_4(A,B,B), I'_4(B), I'_4\big(A,A,I'_4(B)\big), I'_4\big(B,B,I'_4(A)\big)\Big\}, \qquad \text{(E.1.1)}$$

where there are 1, 3 or 5 vectors.

Below is the full list of identities involving respectively $5, 7$ and 9 vectors. They were computed using Mathematica by matching coefficients of both sides by using the explicit expressions of I_4. This has been checked for several cubic models and for the quadratic $n_v = 1$.

We recall two equations involving the section

$$I_4(\operatorname{Re}\mathcal{V}) = I_4(\operatorname{Im}\mathcal{V}) = \frac{1}{16}, \qquad \text{(E.1.2a)}$$

$$\operatorname{Re}\mathcal{V} = 2\,\epsilon_\Omega\, I'_4(\operatorname{Im}\mathcal{V}) = \epsilon_\Omega\,\frac{I'_4(\operatorname{Im}\mathcal{V})}{2\sqrt{I_4(\operatorname{Im}V)}}, \qquad \text{(E.1.2b)}$$

$$I'_4(A,\operatorname{Im}\mathcal{V},\operatorname{Im}\mathcal{V}) = -4\,\langle \operatorname{Im}\mathcal{V}, A\rangle \operatorname{Im}\mathcal{V} - 8\,\langle \operatorname{Re}\mathcal{V}, A\rangle \operatorname{Re}\mathcal{V} - \Omega\mathcal{M}A. \qquad \text{(E.1.2c)}$$

None of these identities changes when $\mathcal{V}$ is multiplied by a phase.

E.1.1 Symplectic product

$$\langle I'_4(A,A,B), I'_4(A)\rangle = -8\, I_4(A)\,\langle A,B\rangle \qquad \text{(E.1.3a)}$$

$$\langle I'_4(A,B,B), I'_4(A)\rangle = -\frac{2}{3}\, I_4(A,A,A,B)\,\langle A,B\rangle \qquad \text{(E.1.3b)}$$

$$\langle I'_4(A,B,B), I'_4(A,A,B)\rangle = 12\,\langle I'_4(A), I'_4(B)\rangle - 4\, I_4(A,A,B,B)\,\langle A,B\rangle \qquad \text{(E.1.3c)}$$

E.1.2 Order 5

$$
\begin{aligned}
I_4'(I_4'(A),A,A) &= -8\,AI_4(A)\\
I_4'(I_4'(A),A,B) &= 2\,I_4'(A)\,\langle A,B\rangle - \frac{1}{3}\,AI_4(A,A,A,B)\\
I_4'(I_4'(A,A,B),A,A) &= -\frac{4}{3}\,AI_4(A,A,A,B) - 8\,I_4'(A)\,\langle A,B\rangle - 16\,BI_4(A)\\
I_4'(I_4'(A,A,B),A,B) &= -\frac{1}{3}\,2\,BI_4(A,A,A,B) - 2\,AI_4(A,A,B,B) + 2\,I_4'(A,A,B)\,\langle A,B\rangle\\
&\quad - 2\,I_4'(I_4'(A),B,B)\\
I_4'(I_4'(A,B,B),A,A) &= -\frac{4}{3}\,BI_4(A,A,A,B) - 4\,I_4'(A,A,B)\,\langle A,B\rangle + 2\,I_4'(I_4'(A),B,B)
\end{aligned}
$$

E.1.3 Order 7

$$
\begin{aligned}
I_4'(I_4'(A),I_4'(A),A) &= 8\,I_4(A)I_4'(A)\\
I_4'(I_4'(A),I_4'(A),B) &= 4\,I_4(A)I_4'(A,A,B) - \frac{2}{3}\,I_4'(A)I_4(A,A,A,B) - 16\,AI_4(A)\,\langle A,B\rangle\\
I_4'(I_4'(A),I_4'(A,A,B),A) &= 2\,I_4'(A)I_4(A,A,A,B) + 16\,AI_4(A)\,\langle A,B\rangle\\
I_4'(I_4'(A),I_4'(A,B,B),A) &= 2\,I_4'(A)I_4(A,A,B,B) + \frac{4}{3}\,AI_4(A,A,A,B)\,\langle A,B\rangle\\
I_4'(I_4'(A),I_4'(A,A,B),B) &= 8\,I_4(A)I_4'(A,B,B) - 2\,I_4'(A)I_4(A,A,B,B) + \frac{1}{3}\,I_4(A,A,A,B)I_4'(A,A,B)\\
&\quad - 16\,BI_4(A)\,\langle A,B\rangle - \frac{8}{3}\,AI_4(A,A,A,B)\,\langle A,B\rangle\\
I_4'(I_4'(A,A,B),I_4'(A,A,B),A) &= -16\,I_4(A)I_4'(A,B,B) + 8\,I_4'(A)I_4(A,A,B,B) + \frac{4}{3}\,I_4(A,A,A,B)I_4'(A,A,B)\\
&\quad + 64\,BI_4(A)\,\langle A,B\rangle + \frac{16}{3}\,AI_4(A,A,A,B)\,\langle A,B\rangle\\
I_4'(I_4'(A),I_4'(B),A) &= \frac{1}{3}\,I_4'(A)I_4(A,B,B,B) + 2\,A\,\langle I_4'(A),I_4'(B)\rangle
\end{aligned}
$$

$$
\begin{aligned}
I_4'(I_4'(A),I_4'(A,B,B),B) &= -\frac{2}{3}\,I_4'(A)I_4(A,B,B,B) + \frac{1}{3}\,I_4(A,A,A,B)I_4'(A,B,B)\\
&\quad - \frac{4}{3}\,BI_4(A,A,A,B)\,\langle A,B\rangle - 8\,A\,\langle I_4'(A),I_4'(B)\rangle + 16\,I_4(A)I_4'(B)\\
I_4'(I_4'(A,A,B),I_4'(A,A,B),B) &= -\frac{16}{3}\,I_4'(A)I_4(A,B,B,B) + \frac{8}{3}\,I_4(A,A,A,B)I_4'(A,B,B)\\
&\quad - 16\,AI_4(A,A,B,B)\,\langle A,B\rangle - \frac{16}{3}\,BI_4(A,A,A,B)\,\langle A,B\rangle\\
&\quad + 32\,A\,\langle I_4'(A),I_4'(B)\rangle + 32\,I_4(A)I_4'(B)\\
I_4'(I_4'(A,A,B),I_4'(A,B,B),A) &= \frac{16}{3}\,I_4'(A)I_4(A,B,B,B) + 2\,I_4(A,A,B,B)I_4'(A,A,B)\\
&\quad - \frac{2}{3}\,I_4(A,A,A,B)I_4'(A,B,B) + \frac{16}{3}\,BI_4(A,A,A,B)\,\langle A,B\rangle\\
&\quad + 8\,AI_4(A,A,B,B)\,\langle A,B\rangle - 8\,A\,\langle I_4'(A),I_4'(B)\rangle - 32\,I_4(A)I_4'(B)
\end{aligned}
$$

E.1.4 Order 9

$$
\begin{aligned}
I_4'(I_4'(A)) &= -16\, I_4(A)^2 A \\
I_4'(I_4'(A), I_4'(A), I_4'(A,A,B)) &= -64\, B I_4(A)^2 - \frac{64}{3}\, I_4(A,A,A,B) A I_4(A) \\
I_4'(I_4'(A), I_4'(A), I_4'(A,B,B)) &= -\frac{16}{3}\, B I_4(A) I_4(A,A,A,B) + \frac{8}{3}\, \langle A,B\rangle\, I_4'(A) I_4(A,A,A,B) \\
&\quad - 16\, I_4(A) I_4(A,A,B,B) A - 16\, \langle A,B\rangle\, I_4(A) I_4'(A,A,B) \\
&\quad + 8\, I_4(A) I_4'(B,B,I_4'(A)) \\
I_4'(I_4'(A), I_4'(A,A,B), I_4'(A,A,B)) &= -\frac{32}{9}\, A I_4(A,A,A,B)^2 - 32\, B I_4(A) I_4(A,A,A,B) \\
&\quad - \frac{16}{3}\, \langle A,B\rangle\, I_4'(A) I_4(A,A,A,B) - 32\, I_4(A) I_4(A,A,B,B) A \\
&\quad + 32\, \langle A,B\rangle\, I_4(A) I_4'(A,A,B) - 16\, I_4(A) I_4'(B,B,I_4'(A)) \\
I_4'(I_4'(A), I_4'(B,B,I_4'(A)), A) &= 32\, A I_4(A)\, \langle A,B\rangle^2 + \frac{4}{3}\, I_4(A,A,A,B) I_4'(A)\, \langle A,B\rangle \\
&\quad - \frac{2}{9}\, I_4(A,A,A,B)^2 A + 8\, A I_4(A) I_4(A,A,B,B) \\
I_4'(I_4'(A), I_4'(A), I_4'(B)) &= -\frac{8}{3}\, I_4(A) A I_4(A,B,B,B) + 4\, \langle I_4'(A), I_4'(B)\rangle\, I_4'(A) \\
&\quad + 4\, I_4(A) I_4'(A,A,I_4'(B)) \\
I_4'(I_4'(A), I_4'(A,A,B), I_4'(A,B,B)) &= -\frac{1}{9}\, 8\, B I_4(A,A,A,B)^2 - \frac{8}{3}\, I_4(A,A,B,B) A I_4(A,A,A,B) \\
&\quad - \frac{4}{3}\, \langle A,B\rangle\, I_4'(A,A,B) I_4(A,A,A,B) \\
&\quad + \frac{4}{3}\, I_4'(B,B,I_4'(A)) I_4(A,A,A,B) - \frac{64}{3}\, I_4(A) I_4(A,B,B,B) A \\
&\quad - 32\, I_4(A) I_4(A,A,B,B) B - 24\, \langle I_4'(A), I_4'(B)\rangle\, I_4'(A) \\
&\quad + 8\, \langle A,B\rangle\, I_4(A,A,B,B) I_4'(A) - 16\, I_4(A) I_4'(A,A,I_4'(B)) \\
&\quad - 16\, \langle A,B\rangle\, I_4(A) I_4'(A,B,B) \\
I_4'(I_4'(A), I_4'(B,B,I_4'(A)), B) &= -32\, I_4(A) B\, \langle A,B\rangle^2 + \frac{2}{9}\, B I_4(A,A,A,B)^2 - 8\, I_4(A) I_4(A,A,B,B) B \\
&\quad + \frac{16}{3}\, A I_4(A) I_4(A,B,B,B) - 12\, \langle I_4'(A), I_4'(B)\rangle\, I_4'(A) \\
&\quad - 8\, I_4(A) I_4'(A,A,I_4'(B)) + \frac{1}{3}\, I_4(A,A,A,B) I_4'(B,B,I_4'(A))
\end{aligned}
$$

$$
\begin{aligned}
I_4'(I_4'(A,A,B),I_4'(B,B,I_4'(A)),A) &= 128\,BI_4(A)\,\langle A,B\rangle^2 + \frac{16}{3}\,AI_4(A,A,A,B)\,\langle A,B\rangle^2 \\
&+ \frac{4}{3}\,I_4(A,A,A,B)I_4'(A,A,B)\,\langle A,B\rangle - \frac{8}{9}\,I_4(A,A,A,B)^2B \\
&+ 32\,BI_4(A)I_4(A,A,B,B) + \frac{16}{3}\,AI_4(A)I_4(A,B,B,B) \\
&+ 48\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A) + 16\,I_4(A)I_4'(A,A,I_4'(B)) \\
&- \frac{2}{3}\,I_4(A,A,A,B)I_4'(B,B,I_4'(A)) \\
I_4'(I_4'(A),I_4'(A,A,I_4'(B)),A) &= \frac{8}{3}\,I_4(A)I_4(A,B,B,B)A - 12\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A) \\
I_4'(I_4'(A),I_4'(B),I_4'(A,A,B)) &= -64\,I_4(A)AI_4(B) - \frac{4}{9}\,I_4(A,A,A,B)I_4(A,B,B,B)A \\
&- \frac{16}{3}\,I_4(A)I_4(A,B,B,B)B + \frac{4}{3}\,\langle A,B\rangle\,I_4(A,B,B,B)I_4'(A) \\
&- 16\,\langle A,B\rangle\,I_4(A)I_4'(B) + 2\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A,A,B) \\
&+ \frac{2}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B)) \\
I_4'(I_4'(A),I_4'(A,B,B),I_4'(A,B,B)) &= -128\,I_4(A)AI_4(B) - \frac{16}{9}\,I_4(A,A,A,B)I_4(A,B,B,B)A \\
&- \frac{8}{3}\,I_4(A,A,A,B)I_4(A,A,B,B)B - \frac{64}{3}\,I_4(A)I_4(A,B,B,B)B \\
&+ \frac{16}{3}\,\langle A,B\rangle\,I_4(A,B,B,B)I_4'(A) - 16\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A,A,B) \\
&- \frac{8}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B)) \\
&- \frac{8}{3}\,\langle A,B\rangle\,I_4(A,A,A,B)I_4'(A,B,B) \\
&+ 4\,I_4(A,A,B,B)I_4'(B,B,I_4'(A)) \\
I_4'(I_4'(A,A,B),I_4'(A,A,B),I_4'(A,B,B)) &= -16\,AI_4(A,A,B,B)^2 - 16\,I_4(A,A,A,B)BI_4(A,A,B,B) \\
&- 8\,I_4'(B,B,I_4'(A))I_4(A,A,B,B) \\
&- \frac{64}{9}\,I_4(A,A,A,B)I_4(A,B,B,B)A + 256\,AI_4(A)I_4(B) \\
&- \frac{64}{3}\,\langle A,B\rangle\,I_4(A,B,B,B)I_4'(A) + 128\,\langle A,B\rangle\,I_4(A)I_4'(B) \\
&+ 16\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A,A,B) + \frac{16}{3}\,\langle A,B\rangle\,I_4(A,A,A,B)I_4'(A,B,B)
\end{aligned}
$$

$$
\begin{aligned}
I_4'(I_4'(A,A,B),I_4'(B,B,I_4'(A)),B) = &-\frac{1}{3}\,16\,I_4(A,A,A,B)B\,\langle A,B\rangle^2 - \frac{16}{3}\,I_4(A,B,B,B)I_4'(A)\,\langle A,B\rangle\\
&+ 64\,I_4(A)I_4'(B)\,\langle A,B\rangle - \frac{16}{3}\,I_4(A)I_4(A,B,B,B)B\\
&+ 128\,AI_4(A)I_4(B) + \frac{8}{9}\,AI_4(A,A,A,B)I_4(A,B,B,B)\\
&- 4\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A,A,B) - \frac{4}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B))\\
&+ 2\,I_4(A,A,B,B)I_4'(B,B,I_4'(A))\\
I_4'(I_4'(A,B,B),I_4'(B,B,I_4'(A)),A) = &\,\frac{32}{3}\,BI_4(A,A,A,B)\,\langle A,B\rangle^2 + 16\,A\,\langle I_4'(A),I_4'(B)\rangle\,\langle A,B\rangle\\
&+ \frac{16}{3}\,I_4(A,B,B,B)I_4'(A)\,\langle A,B\rangle - 64\,I_4(A)I_4'(B)\,\langle A,B\rangle\\
&+ \frac{4}{3}\,I_4(A,A,A,B)I_4'(A,B,B)\,\langle A,B\rangle\\
&- \frac{4}{9}\,I_4(A,A,A,B)I_4(A,B,B,B)A + \frac{64}{3}\,BI_4(A)I_4(A,B,B,B)\\
&+ 16\,\langle I_4'(A),I_4'(B)\rangle\,I_4'(A,A,B) + \frac{4}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B))\\
&- 2\,I_4(A,A,B,B)I_4'(B,B,I_4'(A))
\end{aligned}
$$

$$
\begin{aligned}
I_4'(I_4'(A),I_4'(A,A,I_4'(B)),B) = &\,16\,A\,\langle A,B\rangle\,\langle I_4'(A),I_4'(B)\rangle - 4\,I_4'(A,A,B)\,\langle I_4'(A),I_4'(B)\rangle\\
&- \frac{8}{3}\,I_4(A)I_4(A,B,B,B)B + 64\,AI_4(A)I_4(B)\\
&+ \frac{4}{3}\,\langle A,B\rangle\,I_4(A,B,B,B)I_4'(A) - 32\,\langle A,B\rangle\,I_4(A)I_4'(B)\\
&- \frac{1}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B))\\
I_4'(I_4'(A,A,B),I_4'(A,A,I_4'(B)),A) = &-16\,\langle A,B\rangle\,A\,\langle I_4'(A),I_4'(B)\rangle - 4\,I_4'(A,A,B)\,\langle I_4'(A),I_4'(B)\rangle\\
&+ \frac{32}{3}\,BI_4(A)I_4(A,B,B,B) + \frac{4}{9}\,AI_4(A,A,A,B)I_4(A,B,B,B)\\
&- \frac{16}{3}\,\langle A,B\rangle\,I_4(A,B,B,B)I_4'(A) + 64\,\langle A,B\rangle\,I_4(A)I_4'(B)\\
&+ \frac{2}{3}\,I_4(A,A,A,B)I_4'(A,A,I_4'(B))
\end{aligned}
$$

E.2 Quaternionic gaugings: constraints

For completeness the full set of constraints for the (symplectic) gaugings parameters is listed below [149, sec. 6.1, app. C].

The set of parameters

$$\Theta^\alpha = \{\mathbb{U}, \alpha, \widehat{\alpha}^t, \epsilon_+, \epsilon_0, \epsilon_-\} \tag{E.2.1}$$

reads explicitly

$$
\begin{gathered}
\alpha = \begin{pmatrix}\alpha^\Lambda\\ \alpha_\Lambda\end{pmatrix}, = \begin{pmatrix}\begin{pmatrix}\alpha^{A\Lambda}\\ \alpha^\Lambda_A\end{pmatrix}\\ \begin{pmatrix}\alpha^A_\Lambda\\ \alpha_{A\Lambda}\end{pmatrix}\end{pmatrix}, \qquad \widehat{\alpha} = \begin{pmatrix}\widehat{\alpha}^\Lambda\\ \widehat{\alpha}_\Lambda\end{pmatrix}, = \begin{pmatrix}\begin{pmatrix}\widehat{\alpha}^{A\Lambda}\\ \widehat{\alpha}^\Lambda_A\end{pmatrix}\\ \begin{pmatrix}\widehat{\alpha}^A_\Lambda\\ \widehat{\alpha}_{A\Lambda}\end{pmatrix}\end{pmatrix},\\
\mathbb{U} = \begin{pmatrix}\mathbb{U}^\Lambda\\ \mathbb{U}_\Lambda\end{pmatrix}, \qquad \epsilon_\pm = \begin{pmatrix}\epsilon^\Lambda_\pm\\ \epsilon_{\pm\Lambda}\end{pmatrix}, \qquad \epsilon_0 = \begin{pmatrix}\epsilon^\Lambda_0\\ \epsilon_{0\Lambda}\end{pmatrix},
\end{gathered} \tag{E.2.2}
$$

where $\mathbb{U}^\Lambda$ and $\mathbb{U}_\Lambda$ are matrices whose parameters depend on the model.

The number of parameters is (approx.)

$$\#(\text{params}) = n_v(x + 4n_h + 3), \tag{E.2.3}$$

x being the number of independent isometries of the base (this can be of order n_h^2, n_h or 1).

E.2.1 Constraints from abelian algebra

The constraints from the closure of the abelian algebra are

- electric/electric

$$0 = \mathbb{T}(\alpha_\Lambda, \hat{\alpha}_\Sigma) - \mathbb{T}(\alpha_\Sigma, \hat{\alpha}_\Lambda), \tag{E.2.4a}$$
$$0 = -(\mathbb{U}_\Lambda \alpha_\Sigma - \mathbb{U}_\Sigma \alpha_\Lambda) + (\epsilon_{0\Lambda}\alpha_\Sigma - \epsilon_{0\Sigma}\alpha_\Lambda) + (\epsilon_{+\Lambda}\widehat{\alpha}_\Sigma - \epsilon_{+\Sigma}\widehat{\alpha}_\Lambda), \tag{E.2.4b}$$
$$0 = (\mathbb{U}_\Lambda \widehat{\alpha}_\Sigma - \mathbb{U}_\Sigma \widehat{\alpha}_\Lambda) + (\epsilon_{-\Lambda}\alpha_\Sigma - \epsilon_{-\Sigma}\alpha_\Lambda) + (\epsilon_{0\Lambda}\widehat{\alpha}_\Sigma - \epsilon_{0\Sigma}\widehat{\alpha}_\Lambda), \tag{E.2.4c}$$
$$0 = \alpha_\Lambda^t \mathbb{C} \alpha_\Sigma + 2(\epsilon_{+\Sigma}\epsilon_{0\Lambda} - \epsilon_{+\Lambda}\epsilon_{0\Sigma}), \tag{E.2.4d}$$
$$0 = (\widehat{\alpha}_\Lambda^t \mathbb{C}\alpha_\Sigma - \alpha_\Lambda^t \mathbb{C}\widehat{\alpha}_\Sigma) + 2(\epsilon_{+\Sigma}\epsilon_{-\Lambda} - \epsilon_{+\Lambda}\epsilon_{-\Sigma}), \tag{E.2.4e}$$
$$0 = \widehat{\alpha}_\Lambda^t \mathbb{C}\widehat{\alpha}_\Sigma + 2(\epsilon_{0\Lambda}\epsilon_{-\Sigma} - \epsilon_{0\Sigma}\epsilon_{-\Lambda}). \tag{E.2.4f}$$

- electric/magnetic

$$0 = \mathbb{T}(\alpha_\Lambda, \hat{\alpha}^\Sigma) - \mathbb{T}(\alpha^\Sigma, \hat{\alpha}_\Lambda), \tag{E.2.4g}$$
$$0 = -(\mathbb{U}_\Lambda \alpha^\Sigma - \mathbb{U}^\Sigma \alpha_\Lambda) + (\epsilon_{0\Lambda}\alpha^\Sigma - \epsilon_0^\Sigma\alpha_\Lambda) + (\epsilon_{+\Lambda}\widehat{\alpha}^\Sigma - \epsilon_+^\Sigma\widehat{\alpha}_\Lambda), \tag{E.2.4h}$$
$$0 = (\mathbb{U}_\Lambda \widehat{\alpha}^\Sigma - \mathbb{U}^\Sigma \widehat{\alpha}_\Lambda) + (\epsilon_{-\Lambda}\alpha^\Sigma - \epsilon_-^\Sigma\alpha_\Lambda) + (\epsilon_{0\Lambda}\widehat{\alpha}^\Sigma - \epsilon_0^\Sigma\widehat{\alpha}_\Lambda), \tag{E.2.4i}$$
$$0 = \alpha_\Lambda^t \mathbb{C} \alpha^\Sigma + 2(\epsilon_+^\Sigma\epsilon_{0\Lambda} - \epsilon_{+\Lambda}\epsilon_0^\Sigma), \tag{E.2.4j}$$
$$0 = (\widehat{\alpha}_\Lambda^t \mathbb{C}\alpha^\Sigma - \alpha_\Lambda^t \mathbb{C}\widehat{\alpha}^\Sigma) + 2(\epsilon_+^\Sigma\epsilon_{-\Lambda} - \epsilon_{+\Lambda}\epsilon_-^\Sigma), \tag{E.2.4k}$$
$$0 = \widehat{\alpha}_\Lambda^t \mathbb{C}\widehat{\alpha}^\Sigma + 2(\epsilon_{0\Lambda}\epsilon_-^\Sigma - \epsilon_0^\Sigma\epsilon_{-\Lambda}). \tag{E.2.4l}$$

- magnetic/magnetic

$$0 = \mathbb{T}(\alpha^\Lambda, \hat{\alpha}^\Sigma) - \mathbb{T}(\alpha^\Sigma, \hat{\alpha}^\Lambda), \tag{E.2.4m}$$
$$0 = -(\mathbb{U}^\Lambda \alpha^\Sigma - \mathbb{U}^\Sigma \alpha^\Lambda) + (\epsilon_0^\Lambda\alpha^\Sigma - \epsilon_0^\Sigma\alpha^\Lambda) + (\epsilon_+^\Lambda\widehat{\alpha}^\Sigma - \epsilon_+^\Sigma\widehat{\alpha}^\Lambda), \tag{E.2.4n}$$
$$0 = (\mathbb{U}^\Lambda \widehat{\alpha}^\Sigma - \mathbb{U}^\Sigma \widehat{\alpha}^\Lambda) + (\epsilon_-^\Lambda\alpha^\Sigma - \epsilon_-^\Sigma\alpha^\Lambda) + (\epsilon_0^\Lambda\widehat{\alpha}^\Sigma - \epsilon_0^\Sigma\widehat{\alpha}^\Lambda), \tag{E.2.4o}$$
$$0 = \alpha^{t\,\Lambda} \mathbb{C} \alpha^\Sigma + 2(\epsilon_+^\Sigma\epsilon_0^\Lambda - \epsilon_+^\Lambda\epsilon_0^\Sigma), \tag{E.2.4p}$$
$$0 = (\widehat{\alpha}^{t\,\Lambda} \mathbb{C}\alpha^\Sigma - \alpha^{t\,\Lambda} \mathbb{C}\widehat{\alpha}^\Sigma) + 2(\epsilon_+^\Sigma\epsilon_-^\Lambda - \epsilon_+^\Lambda\epsilon_-^\Sigma), \tag{E.2.4q}$$
$$0 = \widehat{\alpha}^{t\,\Lambda} \mathbb{C}\widehat{\alpha}^\Sigma + 2(\epsilon_0^\Lambda\epsilon_-^\Sigma - \epsilon_0^\Sigma\epsilon_-^\Lambda). \tag{E.2.4r}$$

We recall the expression of the matrix

$$\mathbb{T}_{\alpha,\hat{\alpha}} = (\alpha^t\partial_\xi)(\hat{\alpha}^t\partial_\xi)\,\mathbb{S}. \tag{E.2.5}$$

The number of (electric) constraints is (approx.)

$$\#(\text{constraints}) = \frac{n_v(n_v-1)}{2}\,(x + 2n_h + 3)\,, \tag{E.2.6}$$

where the front factor comes from the antisymmetric equations on (Λ, Σ), and x is the number of independent entries in the matrix $\mathbb{S}$ (this can be of order n_h^2, n_h or 1).

E.2.2 Locality constraints

The constraints from locality are

$$0 = \langle \alpha, \alpha^t \rangle = \alpha^\Lambda \alpha^t_\Lambda - \alpha_\Lambda \alpha^{t\Lambda}, \tag{E.2.7a}$$
$$0 = \langle \alpha, \widehat{\alpha}^t \rangle = \alpha^\Lambda \widehat{\alpha}^t_\Lambda - \alpha_\Lambda \widehat{\alpha}^{t\Lambda}, \tag{E.2.7b}$$
$$0 = \langle \widehat{\alpha}, \widehat{\alpha}^t \rangle = \widehat{\alpha}^\Lambda \widehat{\alpha}^t_\Lambda - \widehat{\alpha}_\Lambda \widehat{\alpha}^{t\Lambda}, \tag{E.2.7c}$$
$$0 = \langle \alpha, \epsilon_+ \rangle = \alpha^\Lambda \epsilon_{+\Lambda} - \alpha_\Lambda \epsilon_+^\Lambda, \tag{E.2.7d}$$
$$0 = \langle \alpha, \epsilon_0 \rangle = \alpha^\Lambda \epsilon_{0\Lambda} - \alpha_\Lambda \epsilon_0^\Lambda, \tag{E.2.7e}$$
$$0 = \langle \alpha, \epsilon_- \rangle = \alpha^\Lambda \epsilon_{-\Lambda} - \alpha_\Lambda \epsilon_-^\Lambda, \tag{E.2.7f}$$
$$0 = \langle \widehat{\alpha}, \epsilon_+ \rangle = \widehat{\alpha}^\Lambda \epsilon_{+\Lambda} - \widehat{\alpha}_\Lambda \epsilon_+^\Lambda, \tag{E.2.7g}$$
$$0 = \langle \widehat{\alpha}, \epsilon_0 \rangle = \widehat{\alpha}^\Lambda \epsilon_{0\Lambda} - \widehat{\alpha}_\Lambda \epsilon_0^\Lambda, \tag{E.2.7h}$$

$$0 = \langle \widehat{\alpha}, \epsilon_- \rangle = \widehat{\alpha}^\Lambda \epsilon_{-\Lambda} - \widehat{\alpha}_\Lambda \epsilon_-^\Lambda, \tag{E.2.7i}$$
$$0 = \langle \epsilon_+, \epsilon_- \rangle = \epsilon_+^\Lambda \epsilon_{-\Lambda} - \epsilon_{+\Lambda} \epsilon_-^\Lambda, \tag{E.2.7j}$$
$$0 = \langle \epsilon_+, \epsilon_0 \rangle = \epsilon_+^\Lambda \epsilon_{0\Lambda} - \epsilon_{+\Lambda} \epsilon_0^\Lambda, \tag{E.2.7k}$$
$$0 = \langle \epsilon_0, \epsilon_- \rangle = \epsilon_0^\Lambda \epsilon_{-\Lambda} - \epsilon_{0\Lambda} \epsilon_-^\Lambda, \tag{E.2.7l}$$
$$0 = \langle \mathbb{U}, \epsilon_+ \rangle = \alpha^\Lambda \epsilon_{+\Lambda} - \alpha_\Lambda \epsilon_+^\Lambda, \tag{E.2.7m}$$
$$0 = \langle \mathbb{U}, \epsilon_0 \rangle = \alpha^\Lambda \epsilon_{0\Lambda} - \alpha_\Lambda \epsilon_0^\Lambda, \tag{E.2.7n}$$
$$0 = \langle \mathbb{U}, \epsilon_- \rangle = \alpha^\Lambda \epsilon_{-\Lambda} - \alpha_\Lambda \epsilon_-^\Lambda, \tag{E.2.7o}$$
$$0 = \langle \mathbb{U}, \alpha \rangle = \alpha^\Lambda \epsilon_{0\Lambda} - \alpha_\Lambda \epsilon_0^\Lambda, \tag{E.2.7p}$$
$$0 = \langle \mathbb{U}, \widehat{\alpha} \rangle = \alpha^\Lambda \epsilon_{-\Lambda} - \alpha_\Lambda \epsilon_-^\Lambda \tag{E.2.7q}$$

where

$$\langle \alpha, \alpha^t \rangle = \begin{pmatrix} \langle \alpha^A, \alpha^B \rangle & \langle \alpha^A, \alpha_B \rangle \\ \langle \alpha_A, \alpha^B \rangle & \langle \alpha_A, \alpha_B \rangle \end{pmatrix}, \qquad \langle \alpha, \epsilon_+ \rangle = \begin{pmatrix} \langle \alpha^A, \epsilon_+ \rangle \\ \langle \alpha_A, \epsilon_+ \rangle \end{pmatrix} \tag{E.2.8}$$

and similarly for the others. The notation $\langle \mathbb{U}, X \rangle$ is a shortcut for the product of X with all parameters of $\mathbb{U}$ (by linearity). For example with a cubic prepotential one of the constraint is

$$\langle \beta, X \rangle = 0, \qquad \beta = \begin{pmatrix} \beta^\Lambda \\ \beta_\Lambda \end{pmatrix}. \tag{E.2.9}$$

The numbers of locality constraints is (approx.)

$$\#(\text{locality constraints}) = 3(n_h + 1)^2 + x\, n_h (2n_h + 3). \tag{E.2.10}$$

Appendix F

Computations

In this section we are collecting long and cumbersome computations.

F.1 Quaternionic isometries: Killing algebra

F.1.1 Computations: duality and extra commutators

The non-vanishing commutators of the algebra are

$$[k_0, k_+] = 2k_+, \qquad [k_0, k_\alpha] = k_\alpha, \qquad \left[k_\alpha, k^t_\alpha\right] = \mathbb{C}\, k_+, \qquad [k_\mathbb{U}, k_\alpha] = \mathbb{U}\, k_\alpha. \tag{F.1.1}$$

The evaluation of the last commutator proceeds as

$$\begin{aligned}\left[k_\mathbb{U}, k^A\right] &= \frac{1}{2}(\mathbb{U}\xi)^B(\partial_B \xi^A)\partial_a - \left(\left[\partial^A(\mathbb{U}\xi)^B\right]\partial_B + \left[\partial^A(\mathbb{U}\xi)_B\right]\partial^B\right)\\ &= \frac{1}{2}\left(v^A{}_B\xi^B + t^{AB}\tilde{\xi}_B\right)\partial_a - t^{BA}\partial_B - u_B{}^A\partial^B\\ &= v^A{}_B\left(\partial^B + \frac{1}{2}\,\xi^B\partial_a\right) - t^{AB}\left(\partial_B - \frac{1}{2}\,\tilde{\xi}_B\partial_a\right).\end{aligned}$$

In components we have

$$\begin{gathered}\left[k_A, h^B\right] = -\delta_A{}^B k_+, \qquad \left[k_0, k^A\right] = k^A, \qquad [k_0, k_A] = k_A,\\ \left[k_\mathbb{U}, k^A\right] = (\mathbb{U}\mathbb{C}h)^A, \qquad [k_\mathbb{U}, k_A] = (\mathbb{U}\mathbb{C}h)_A.\end{gathered} \tag{F.1.2}$$

F.1.2 Computations: hidden and mixed commutators

We now compute the commutators between hidden and duality symmetries

$$\begin{gathered}[k_0, k_-] = -2\,k_-, \qquad [k_0, k_{\hat\alpha}] = -k_{\hat\alpha}, \qquad [k_-, k_\alpha] = -k_{\hat\alpha},\\ [k_+, k_-] = -k_0, \qquad [k_+, k_{\hat\alpha}] = k_\alpha, \qquad [k_\mathbb{U}, k_{\hat\alpha}] = \mathbb{U}\, k_{\hat\alpha},\\ \left[k_{\hat\alpha}, k^t_{\hat\alpha}\right] = \mathbb{C}\, k_-, \qquad \left[\widehat{\alpha}^t k_{\hat\alpha}, \alpha^t k_\alpha\right] = \frac{1}{2}\,\widehat{\alpha}\mathbb{C}\alpha\, k_0 + k_{\mathbb{T}_{\alpha,\hat\alpha}}\end{gathered} \tag{F.1.3}$$

where

$$\mathbb{T}_{\alpha,\hat\alpha} = (\alpha^t\mathbb{C}\partial_\xi)(\hat\alpha^t\mathbb{C}\partial_\xi)\,\mathbb{S} = -\frac{1}{2}\,\mathbb{C}(\hat\alpha\alpha^t + \alpha\hat\alpha^t) + \frac{1}{4}\,H''_{\alpha,\hat\alpha}\mathbb{C}, \tag{F.1.4a}$$

$$H''_{\alpha,\hat\alpha} = \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h''_{\alpha,\hat\alpha})^t = (\alpha^t\mathbb{C}\partial_\xi)(\hat\alpha^t\mathbb{C}\partial_\xi)H, \tag{F.1.4b}$$

$$h''_{\alpha,\hat\alpha} = (\alpha^t\mathbb{C}\partial_\xi)(\hat\alpha^t\mathbb{C}\partial_\xi)h. \tag{F.1.4c}$$

We have

$$\big[k^A, k_-\big] = a\partial^A - (\mathbb{C}\partial_\xi \partial^A W)^t \partial_\xi - (\partial^A \mathbb{S}\, Z)^t \partial_Z + \text{c.c.} - \frac{1}{2}\xi^A\, \partial_\phi + a\xi^A \partial_a \\ + \frac{1}{2}\,\xi^A \xi^t \partial_\xi - \frac{1}{2}(a\xi^A - \partial^A W)\partial_a. \tag{F.1.5}$$

Another commutator:

$$\begin{aligned}
[k_0, k_-] &= 4\,\mathrm{e}^{-4\phi}\,\partial_a - 2a(-\partial_\phi + 2a\partial_a + \xi^t\partial_\xi) + (\xi^t\partial_\xi - \partial_\phi)W\,\partial_a \\
&\quad - \big(a\xi - \mathbb{C}(\xi^t\partial_\xi - \partial_\phi)\partial_\xi W\big)^t\partial_\xi + \big((\xi^t\partial_\xi)\mathbb{S}Z\big)^t\partial_Z + \text{c.c.} \\
&\quad + 2(a^2 - \mathrm{e}^{-4\phi} - W)\partial_a + (a\xi - \mathbb{C}\partial_\xi W)^t\partial_\xi \\
&= 4(\mathrm{e}^{-4\phi} - a^2)\partial_a + 2a\partial_\phi - 2a\,\xi^t\partial_\xi + 4W\,\partial_a - \big(a\xi - 3\mathbb{C}\partial_\xi W\big)^t\partial_\xi \\
&\quad + 2(\mathbb{S}Z)^t\partial_Z + \text{c.c.} + 2(a^2 - \mathrm{e}^{-4\phi} - W)\partial_a + (a\xi - \mathbb{C}\partial_\xi W)^t\partial_\xi \\
&= -2\Big[-a\partial_\phi + (a^2 - \mathrm{e}^{-4\phi} - W)\partial_a + (a\xi - \mathbb{C}\partial_\xi W)^t\partial_\xi - (\mathbb{S}Z)^t\partial_Z + \text{c.c.}\Big] \\
&= -2\,k_-,
\end{aligned}$$

where we used the "homogeneity" of W (9.1.27).

Introducing a set of parameters $\alpha, \widehat{\alpha}$, then we have

$$\begin{aligned}
\big[\alpha^t\,k_\alpha, \widehat{\alpha}^t\,k_{\hat{\alpha}}\big] &= \frac{1}{2}\,(-\partial_\phi + a\partial_a)\,(\alpha^t\mathbb{C}\partial_\xi)\hat{\alpha}^t\xi - \frac{1}{2}\,(\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\mathbb{C}\partial_\xi W)\,\partial_a \\
&\quad + \frac{1}{2}\,(\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\xi\xi^t\partial_\xi) - (\alpha^t\mathbb{C}\partial_\xi)\,\big[\hat{\alpha}^t\mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t\partial_\xi\big] \\
&\quad - (\alpha^t\mathbb{C}\partial_\xi)\big[\hat{\alpha}^t(\mathbb{C}\partial_\xi\mathbb{S}\,Z)^t\partial_Z\big] + \cancel{\frac{1}{4}\,(\alpha^t\xi)\,(\hat{\alpha}^t\xi)}\,\partial_a + \frac{1}{2}\,(\alpha^t\xi)\,(\hat{\alpha}^t\mathbb{C}\partial_\xi) \\
&\quad - \frac{a}{2}\,(\hat{\alpha}^t\mathbb{C}\partial_\xi)\,\alpha^t\xi\,\partial_a - \frac{1}{2}\left[\hat{\alpha}^t\left(\cancel{\frac{1}{2}\xi\xi^t} - \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t\right)\partial_\xi\right](\alpha^t\xi)\,\partial_a.
\end{aligned}$$

The two terms cancel because

$$\hat{\alpha}^t\xi\xi^t\alpha = (\hat{\alpha}^t\xi)(\xi^t\alpha) = (\alpha^t\xi)(\xi^t\hat{\alpha}). \tag{F.1.6}$$

We have

$$(\alpha^t\mathbb{C}\partial_\xi)\hat{\alpha}^t\xi = \alpha^t\mathbb{C}\hat{\alpha}^t \tag{F.1.7}$$

as can be seen by writing the indices explicitly

$$\alpha_i\mathbb{C}_{ij}\partial_j\ \hat{\alpha}_k\xi_k = \alpha_i\mathbb{C}_{ij}\delta_{jk}\hat{\alpha}_k = \alpha_i\mathbb{C}_{ij}\hat{\alpha}_j. \tag{F.1.8}$$

Moreover we can rewrite

$$\hat{\alpha}^t\xi\xi^t\partial_\xi = (\hat{\alpha}^t\xi)(\xi^t\partial_\xi). \tag{F.1.9}$$

and then

$$(\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\xi\xi^t\partial_\xi) = (\alpha^t\mathbb{C}\hat{\alpha})(\xi^t\partial_\xi) + (\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\xi). \tag{F.1.10}$$

The expression simplifies to

$$\begin{aligned}
[\alpha\,k_\alpha, \widehat{\alpha}\,k_{\hat{\alpha}}] &= -\,\frac{1}{2}\,\alpha^t\mathbb{C}\hat{\alpha}\,(\partial_\phi - 2a\partial_a - \xi^t\partial_\xi) - \frac{1}{2}\,\cancel{(\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\mathbb{C}\partial_\xi W)}\,\partial_a \\
&\quad + \frac{1}{2}\,(\alpha^t\mathbb{C}\partial_\xi)(\hat{\alpha}^t\xi) + \frac{1}{2}\,(\alpha^t\xi)(\hat{\alpha}^t\mathbb{C}\partial_\xi) \\
&\quad - (\alpha^t\mathbb{C}\partial_\xi)\,\big[\hat{\alpha}^t\mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t\partial_\xi\big] - (\alpha^t\mathbb{C}\partial_\xi)\big[\hat{\alpha}^t(\mathbb{C}\partial_\xi\mathbb{S}\,Z)^t\partial_Z\big] \\
&\quad + \frac{1}{2}\cancel{\big((\hat{\alpha}^t\mathbb{C}\partial_\xi)(\mathbb{C}\partial_\xi W)^t\alpha\big)}\,\partial_a.
\end{aligned}$$

The cancellation occurs since

$$(\hat{\alpha}^t \mathbb{C}\partial_\xi)(\mathbb{C}\partial_\xi W)^t \alpha = (\hat{\alpha}^t \mathbb{C}\partial_\xi)(\alpha^t \mathbb{C}\partial_\xi W)^t = (\hat{\alpha}^t \mathbb{C}\partial_\xi)(\alpha^t \mathbb{C}\partial_\xi W) \tag{F.1.11}$$

the last parenthesis being just a number.

The penultimate in the first expression gives a factor 2 in $2a\partial_a$ since

$$-\frac{a}{2}\,(\hat{\alpha}^t \mathbb{C}\alpha)\;\partial_a = \frac{a}{2}\,(\alpha^t \mathbb{C}\hat{\alpha})\;\partial_a \tag{F.1.12}$$

by antisymmetry of $\mathbb{C}$.

Then we can write

$$\begin{aligned}(\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \xi) + (\alpha^t \xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi) &= (\xi^t \hat{\alpha})(\alpha^t \mathbb{C}\partial_\xi) + (\xi^t \alpha)(\hat{\alpha}^t \mathbb{C}\partial_\xi)\\ &= \xi^t(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t)\mathbb{C}\,\partial_\xi\\ &= -\big[\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t)\xi\big]^t\,\partial_\xi.\end{aligned}$$

We need to simplify the terms with W and $\mathbb{S}$. Starting with W: this function contains quartic and quadratic terms in ξ, so $(\alpha^t \mathbb{C}\partial_\xi)\,(\hat{\alpha}^t \mathbb{C}\partial_\xi)(\mathbb{C}\partial_\xi W)^t$ is linear in ξ, which implies that it is homogeneous of first order. This linear term is given by the third derivative of h, such that

$$(\alpha^t \mathbb{C}\partial_\xi)\,\big[\hat{\alpha}^t \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W) = \frac{1}{4}\,\mathbb{C}\partial_\xi h''_{\alpha,\hat{\alpha}} \tag{F.1.13}$$

and we have defined

$$h''_{\alpha,\hat{\alpha}} = (\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)h. \tag{F.1.14}$$

As we said its derivative is homogeneous, thus

$$\mathbb{C}\partial_\xi h''_{\alpha,\hat{\alpha}} = \xi^t \partial_\xi(\mathbb{C}\partial_\xi h''_{\alpha,\hat{\alpha}})^t = -\xi^t \mathbb{C} H''_{\alpha,\hat{\alpha}}. \tag{F.1.15}$$

The new symbol we have defined is

$$H''_{\alpha,\hat{\alpha}} = \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi h''_{\alpha,\hat{\alpha}})^t = (\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)H. \tag{F.1.16}$$

Note that the matrix $H''_{\alpha,\hat{\alpha}}$ is constant and symmetric.

Using all this we can simplify the W term as

$$(\alpha^t \mathbb{C}\partial_\xi)\,\big[\hat{\alpha}^t \mathbb{C}\partial_\xi(\mathbb{C}\partial_\xi W)^t \partial_\xi\big] = \frac{1}{4}\,(H''_{\alpha,\hat{\alpha}}\mathbb{C}\xi)^t \partial_\xi. \tag{F.1.17}$$

After all this the computation for $\mathbb{S}$ is straightforward:

$$\begin{aligned}(\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)\mathbb{S} &= \frac{1}{2}(\alpha^t \mathbb{C}\partial_\xi)(\hat{\alpha}^t \mathbb{C}\partial_\xi)\left(\xi\xi^t + \frac{1}{2}\,H\right)\mathbb{C}\\ &= -\frac{1}{2}\,\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t) + \frac{1}{4}\,H''_{\alpha,\hat{\alpha}}\mathbb{C}.\end{aligned}$$

The new expression is

$$\begin{aligned}[\alpha\, k_\alpha, \widehat{\alpha}\, k_{\hat{\alpha}}] = &-\frac{1}{2}\,k_0 + \frac{1}{2}\big[\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t)\xi\big]^t\,\partial_\xi\\ &-\frac{1}{4}\,(H''_{\alpha,\hat{\alpha}}\mathbb{C}\xi)^t\partial_\xi + \frac{1}{2}\big[\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t)Z\big]^t\,\partial_Z - \frac{1}{4}\,(H''_{\alpha,\hat{\alpha}}\mathbb{C}Z)^t\partial_Z.\end{aligned}$$

We recognize the vector $k_{-\mathbb{U}_{\alpha,\hat{\alpha}}}$ with parameters

$$\mathbb{U}_{\alpha,\hat{\alpha}} = -\frac{1}{2}\,\mathbb{C}(\hat{\alpha}\alpha^t + \alpha\hat{\alpha}^t) + \frac{1}{4}\,H''_{\alpha,\hat{\alpha}}\mathbb{C}. \tag{F.1.18}$$

F.2 Gauged supergravity

F.2.1 Computations : constraints from algebra closure

We compute first the various pieces:

$$
\begin{aligned}
[k_{\mathbb{U}_\Lambda}, k_\Sigma] &= \Big[k_{\mathbb{U}_\Lambda}, \alpha^t_\Sigma \mathbb{C} k_\alpha + \widehat{\alpha}^t_\Sigma \mathbb{C} \widehat{k}_\alpha\Big] = \alpha^t_\Sigma \mathbb{C}\mathbb{U}_\Lambda\, k_\alpha + \widehat{\alpha}^t_\Sigma \mathbb{C}\mathbb{U}_\Lambda\, \widehat{k}_\alpha, \\
\big[\alpha^t_\Lambda \mathbb{C} k_\alpha, k_\Sigma\big] &= \Big[\alpha^t_\Lambda \mathbb{C} k_\alpha, k_{\mathbb{U}_\Sigma} + \alpha^t_\Sigma \mathbb{C} k_\alpha + \widehat{\alpha}^t_\Sigma \mathbb{C} \widehat{k}_\alpha + \epsilon_{0\Sigma} k_0 + \epsilon_{-\Sigma} k_-\Big] \\
&= -\alpha^t_\Lambda \mathbb{C}\mathbb{U}_\Sigma\, k_\alpha + \alpha^t_\Lambda \mathbb{C}\alpha_\Sigma\, k_+ - \frac{1}{2}\,\alpha^t_\Lambda \mathbb{C}\widehat{\alpha}_\Sigma\, k_0 - k_{\mathbb{U}(\alpha_\Lambda, \hat{\alpha}_\Sigma)} \\
&\qquad - \epsilon_{0\Sigma}\alpha^t_\Lambda \mathbb{C}\, k_\alpha + \epsilon_{-\Sigma}\alpha^t_\Lambda \mathbb{C}\, \widehat{k}_\alpha, \\
\Big[\widehat{\alpha}^t_\Lambda \mathbb{C} \widehat{k}_\alpha, k_\Sigma\Big] &= \Big[\widehat{\alpha}^t_\Lambda \mathbb{C} \widehat{k}_\alpha, k_{\mathbb{U}_\Sigma} + \alpha^t_\Sigma \mathbb{C} k_\alpha + \widehat{\alpha}^t_\Sigma \mathbb{C} \widehat{k}_\alpha + \epsilon_{+\Sigma} k_+ + \epsilon_{0\Sigma} k_0\Big] \\
&= -\widehat{\alpha}^t_\Lambda \mathbb{C}\mathbb{U}_\Sigma\, \widehat{k}_\alpha + \frac{1}{2}\,\widehat{\alpha}^t_\Lambda \mathbb{C}\alpha_\Sigma\, k_0 + k_{\mathbb{T}(\alpha_\Lambda, \hat{\alpha}_\Sigma)} + \widehat{\alpha}^t_\Lambda \mathbb{C}\widehat{\alpha}_\Sigma\, k_- \\
&\qquad - \epsilon_{+\Sigma}\widehat{\alpha}^t_\Lambda \mathbb{C}\, k_\alpha + \epsilon_{0\Sigma}\widehat{\alpha}^t_\Lambda \mathbb{C}\, \widehat{k}_\alpha, \\
[\epsilon_{+\Lambda} k_+, k_\Sigma] &= \Big[\epsilon_{+\Sigma} k_+, \widehat{\alpha}^t_\Sigma \mathbb{C} \widehat{k}_\alpha + \epsilon_{0\Sigma} k_0 + \epsilon_{-\Sigma} k_-\Big] \\
&= \epsilon_{+\Lambda}\widehat{\alpha}^t_\Sigma \mathbb{C}\, k_\alpha - 2\epsilon_{+\Lambda}\epsilon_{0\Sigma}\, k_+ - \epsilon_{+\Lambda}\epsilon_{-\Sigma}\, k_0, \\
[\epsilon_{0\Lambda} k_0, k_\Sigma] &= \epsilon_{0\Lambda}\alpha^t_\Sigma \mathbb{C}\, k_\alpha - \epsilon_{0\Lambda}\widehat{\alpha}^t_\Sigma \mathbb{C}\, \widehat{k}_\alpha + 2\epsilon_{+\Sigma}\epsilon_{0\Lambda}\, k_+ - 2\epsilon_{0\Lambda}\epsilon_{-\Sigma}\, k_-, \\
[\epsilon_{-\Lambda} k_-, k_\Sigma] &= \big[\epsilon_{-\Sigma} k_-, \mathbb{C}\alpha^t_\Sigma k_\alpha + \epsilon_{0\Sigma} k_0 + \epsilon_{+\Sigma} k_+\big] \\
&= -\epsilon_{-\Lambda}\alpha^t_\Sigma \mathbb{C}\, \widehat{k}_\alpha + 2\epsilon_{0\Sigma}\epsilon_{-\Lambda}\, k_- + \epsilon_{+\Sigma}\epsilon_{-\Lambda}\, k_0.
\end{aligned}
$$

Adding everything we get

$$
\begin{aligned}
[k_\Lambda, k_\Sigma] = k_{\mathbb{T}(\alpha_\Lambda, \hat{\alpha}_\Sigma)} &+ \big(\alpha^t_\Sigma \mathbb{C}\mathbb{U}_\Lambda + \epsilon_{+\Lambda}\widehat{\alpha}^t_\Sigma \mathbb{C} + \epsilon_{0\Lambda}\alpha^t_\Sigma \mathbb{C}\big)\, k_\alpha \\
&+ \big(\widehat{\alpha}^t_\Sigma \mathbb{C}\mathbb{U}_\Lambda + \epsilon_{-\Sigma}\alpha^t_\Lambda \mathbb{C} + \epsilon_{0\Sigma}\widehat{\alpha}^t_\Lambda \mathbb{C}\big)\, \widehat{k}_\alpha \\
&+ \big(\alpha^t_\Lambda \mathbb{C}\alpha_\Sigma + 2\epsilon_{+\Sigma}\epsilon_{0\Lambda}\big)\, k_+ + \left(\frac{1}{2}\,\widehat{\alpha}^t_\Lambda \mathbb{C}\alpha_\Sigma + \epsilon_{+\Sigma}\epsilon_{-\Lambda}\right) k_0 \\
&+ \big(\widehat{\alpha}^t_\Lambda \mathbb{C}\widehat{\alpha}_\Sigma + 2\epsilon_{0\Lambda}\epsilon_{-\Sigma}\big)\, k_- - (\Lambda \leftrightarrow \Sigma).
\end{aligned}
\tag{F.2.1}
$$

We will take the transpose and use that

$$
\mathbb{U}^t\mathbb{C} + \mathbb{C}\mathbb{U} = 0. \tag{F.2.2}
$$

F.3 Static BPS solutions

F.3.1 Ansatz

We take the following ansatz for the metric and the gauge fields

$$
\mathrm{d}s^2 = \mathrm{e}^{2U}\mathrm{d}t^2 - \mathrm{e}^{-2U}\mathrm{d}r^2 - \mathrm{e}^{2(V-U)}\,\mathrm{d}\Sigma_g^2, \tag{F.3.1a}
$$

$$
A^\Lambda = \tilde{q}^\Lambda\,\mathrm{d}t - \kappa p^\Lambda F'(\theta)\mathrm{d}\phi. \tag{F.3.1b}
$$

The functions $U, V, \tilde{q}$ and p depend only on r. The space Σ_g is a Riemann surface.[1]

[1] The convention are slightly different from the one in the appendix A.7. One needs to make the replacement $(H, H') \to (-\kappa H', H)$.

Ansatz: Vierbein and spin connections

Recall the metric

$$ds^2 = \mathrm{e}^{2U}\mathrm{d}t^2 - \mathrm{e}^{-2U}\mathrm{d}r^2 - \mathrm{e}^{2(V-U)}\big(\mathrm{d}\theta^2 + F^2\mathrm{d}\phi^2\big). \tag{F.3.2}$$

We introduce the following vierbein

$$e^0 = \mathrm{e}^U\mathrm{d}t, \qquad e^1 = \mathrm{e}^{-U}\mathrm{d}r, \qquad e^2 = \mathrm{e}^{V-U}\mathrm{d}\theta, \qquad e^3 = F\,\mathrm{e}^{V-U}\mathrm{d}\phi. \tag{F.3.3}$$

We compute the differential

$$\begin{aligned}
\mathrm{d}e^0 &= U'\mathrm{d}r\wedge e^0,\\
\mathrm{d}e^1 &= 0,\\
\mathrm{d}e^2 &= (V'-U')\,\mathrm{e}^{V-U}\mathrm{d}r\wedge\mathrm{d}\theta,\\
\mathrm{d}e^3 &= F\,(V'-U')\,\mathrm{e}^{V-U}\mathrm{d}r\wedge\mathrm{d}\phi + F'\,\mathrm{e}^{V-U}\mathrm{d}\theta\wedge\mathrm{d}\phi.
\end{aligned}$$

Using (A.7.4) and the vierbein expressions (F.3.3), we can replace all the differentials by the vierbein

$$\mathrm{d}e^0 = U'\,\mathrm{e}^U e^1\wedge e^0, \tag{F.3.4a}$$

$$\mathrm{d}e^1 = 0, \tag{F.3.4b}$$

$$\mathrm{d}e^2 = (V'-U')e^U e^1\wedge e^2, \tag{F.3.4c}$$

$$\mathrm{d}e^3 = (V'-U')e^U e^1\wedge e^3 + \frac{F'}{F}\,\mathrm{e}^{U-V}e^2\wedge e^3. \tag{F.3.4d}$$

Using Cartan formula

$$\mathrm{d}e^a + \omega^a{}_b\wedge e^b = 0 \tag{F.3.5}$$

we obtain the following spin connections

$$\begin{aligned}
\omega^0{}_1 = U'\,\mathrm{e}^U e^0, \qquad \omega^2{}_1 = (V'-U')\,\mathrm{e}^U e^2,\\
\omega^3{}_1 = (V'-U')\,\mathrm{e}^U e^3, \qquad \omega^3{}_2 = \frac{F'}{F}\,\mathrm{e}^{U-V}e^3.
\end{aligned} \tag{F.3.6}$$

The explicit components

$$\omega^a{}_b = \omega_\mu{}^a{}_b\,\mathrm{d}x^\mu \tag{F.3.7}$$

are

$$\omega_{001} = U'\,\mathrm{e}^U, \qquad \omega_{212} = \omega_{313} = (V'-U')\,\mathrm{e}^U, \qquad \omega_{323} = \frac{F'}{F}\,\mathrm{e}^{U-V}. \tag{F.3.8}$$

Field strength

Recall the gauge fields

$$A^\Lambda = \tilde{q}^\Lambda\,\mathrm{d}t - \kappa p^\Lambda F'\,\mathrm{d}\phi. \tag{F.3.9}$$

In terms of the vierbein (F.3.3) we have

$$A^\Lambda = \tilde{q}^\Lambda\,\mathrm{e}^U e^0 - \kappa\,\frac{F'}{F}\,\mathrm{e}^{U-V}p^\Lambda\,e^3. \tag{F.3.10}$$

Now we compute the field strength as

$$F^\Lambda = \mathrm{d}A^\Lambda = \tilde{q}'^\Lambda\,\mathrm{d}r\wedge\mathrm{d}t + (p^\Lambda - 2b\tilde{q}^\Lambda)F\,\mathrm{d}\theta\wedge\mathrm{d}\phi - \kappa p'^\Lambda F'\,\mathrm{d}r\wedge\mathrm{d}\phi \tag{F.3.11a}$$

$$= -\tilde{q}'^\Lambda\,e^0\wedge e^1 - \kappa\,\frac{F'}{F}\,p'^\Lambda\,\mathrm{e}^{2U-V}\,e^1\wedge e^3 + p^\Lambda\,\mathrm{e}^{2(U-V)}e^2\wedge e^3. \tag{F.3.11b}$$

The Hodge dual field strength is

$$\star F^\Lambda = \tilde{q}'^\Lambda\, e^2 \wedge e^3 + \kappa\, \frac{F'}{F}\, p'^\Lambda\, \mathrm{e}^{2U-V}\, e^0 \wedge e^2 + p^\Lambda\, \mathrm{e}^{2(U-V)} e^0 \wedge e^1 \tag{F.3.12a}$$

$$= \tilde{q}'^\Lambda\, \mathrm{e}^{2(V-U)} F\, \mathrm{d}\theta \wedge \mathrm{d}\phi + \kappa\, \frac{F'}{F}\, p'^\Lambda\, \mathrm{e}^{2U}\, \mathrm{d}t \wedge \mathrm{d}\theta - p^\Lambda\, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \mathrm{d}t. \tag{F.3.12b}$$

Finally the anti-self dual form is

$$\mathcal{F}^{-\Lambda} = \frac{1}{2}\big(F^\Lambda - i \star F^\Lambda\big) = \tilde{F}^\Lambda (e^0 \wedge e^1 + i\, e^2 \wedge e^3) + \frac{F'}{F}\, \tilde{G}^\Lambda (e^1 \wedge e^3 + i\, e^0 \wedge e^2) \tag{F.3.13}$$

where

$$\tilde{F}^\Lambda = -\frac{1}{2}\, \tilde{q}^\Lambda - \frac{i}{2}\, p^\Lambda\, \mathrm{e}^{2(U-V)}, \qquad \tilde{G}^\Lambda = -\kappa\, \mathrm{e}^{2U-V} p'^\Lambda. \tag{F.3.14}$$

The symplectic dual G_Λ of F^Λ is defined by

$$G_\Lambda = \frac{\delta \mathcal{L}}{\delta F^\Lambda} = \mathcal{R}_{\Lambda\Sigma}\, F^\Sigma - \mathcal{I}_{\Lambda\Sigma} \star F^\Sigma. \tag{F.3.15}$$

It reads explictly (with a matrix/vector notation)

$$\begin{aligned} G = \mathcal{R}\, &(\tilde{q}'\, \mathrm{d}r \wedge \mathrm{d}t + p\, F\, \mathrm{d}\theta \wedge \mathrm{d}\phi - \kappa p' F'\, \mathrm{d}r \wedge \mathrm{d}\phi) \\ &- \mathcal{I} \left(\tilde{q}'\, \mathrm{e}^{2(V-U)} F\, \mathrm{d}\theta \wedge \mathrm{d}\phi + \kappa\, \frac{F'}{F}\, p'\, \mathrm{e}^{2U}\, \mathrm{d}t \wedge \mathrm{d}\theta - p\, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \mathrm{d}t \right), \end{aligned} \tag{F.3.16}$$

or after simplification

$$\begin{aligned} G = \big(\mathcal{R}\tilde{q}' + \mathcal{I} p\, \mathrm{e}^{2(U-V)}\big) \mathrm{d}t + \big(\mathcal{R}p - \mathcal{I}\tilde{q}'\, \mathrm{e}^{2(V-U)}\big) F\, \mathrm{d}\theta \wedge \mathrm{d}\phi \\ - \kappa\, F' \big(\mathcal{R}\, \mathrm{d}r \wedge \mathrm{d}\phi + \mathcal{I}\, \mathrm{e}^{2U}\, \mathrm{d}t \wedge \mathrm{d}\theta\big)\, p'. \end{aligned} \tag{F.3.17}$$

The "conserved" electric and magnetic charges are defined by [62]

$$p^\Lambda = \frac{1}{4\pi} \int_{S^2} F^\Lambda, \qquad q_\Lambda = \frac{1}{4\pi} \int_{S^2} G_\Lambda. \tag{F.3.18}$$

The pair

$$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \tag{F.3.19}$$

forms the correct symplectic vector of charges.[2]

We obtain the explicit expressions

$$q_\Lambda = \mathcal{R}_{\Lambda\Sigma}\, p^\Sigma - \mathrm{e}^{2(V-U)} \mathcal{I}_{\Lambda\Sigma} \tilde{q}'^\Sigma. \tag{F.3.20}$$

We can solve for $\tilde{q}'^\Lambda$ in terms of p^Λ and q_Λ

$$\tilde{q}'^\Lambda = \mathrm{e}^{2(U-V)} (\mathcal{I}^{-1})^{\Lambda\Sigma} (\mathcal{R}_{\Sigma\Delta}\, p^\Delta - q_\Sigma). \tag{F.3.21}$$

If $p'^\Lambda = 0$ we can obtain the field strength and its Hodge dual in terms of the symplectic charges (we use a matrix/vector notation)

$$\begin{aligned} F &= \mathrm{e}^{2(U-V)} (\mathcal{I}^{-1}\mathcal{R}\, p - \mathcal{I}^{-1} q)\, \mathrm{d}r \wedge \mathrm{d}t + p\, F\, \mathrm{d}\theta \wedge \mathrm{d}\phi, \\ \star F &= -p\, \mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \mathrm{d}t + \mathcal{I}^{-1} (\mathcal{R}p - q)\, F \mathrm{d}\theta \wedge \mathrm{d}\phi. \end{aligned}$$

[2] Note that [62] forgets to add κ in the formula: the presence of κ here can be traced to the fact that it is absent in (F.3.1b), and ultimately the reason is that the gauge field should be defined with the integral of F, and not its derivative; see [76] for comparison.

From here we compute the symplectic dual of F^Λ

$$G = \mathcal{R}\left(\mathrm{e}^{2(U-V)}\mathcal{I}^{-1}(\mathcal{R}p - q)\,\mathrm{d}r \wedge \mathrm{d}t + p\,F\,\mathrm{d}\theta \wedge \mathrm{d}\phi\right) - \mathcal{I}\left(-p\,\mathrm{e}^{2(U-V)}\mathrm{d}r \wedge \mathrm{d}t + \mathcal{I}^{-1}(\mathcal{R}p - q)\,F\mathrm{d}\theta \wedge \mathrm{d}\phi\right) \tag{F.3.23}$$

and after replacing the charges

$$G = \mathrm{e}^{2(U-V)}\big((\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})p - \mathcal{R}\mathcal{I}^{-1}q\big)\mathrm{d}t + qF\,\mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{F.3.24}$$

We can gather both vectors into a symplectic vector using the expression of $\mathcal{M}$ [165, p. 515]

$$\mathcal{F} = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} = \mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{Q}\,\mathrm{d}r \wedge \mathrm{d}t + \mathcal{Q}\,F\,\mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{F.3.25}$$

Note that it does not seem possible to write such an expression if $p' \neq 0$.

Dirac quantization condition implies that [62, sec. 2]

$$p^\Lambda P^3_\Lambda \in \mathbb{Z}, \qquad p^\Lambda k^u_\Lambda \in \mathbb{Z}. \tag{F.3.26}$$

Supersymmetry restricts the integers to be

$$p^\Lambda P^3_\Lambda = \kappa, \qquad p^\Lambda k^u_\Lambda = 0. \tag{F.3.27}$$

It seems that for $P^1, P^2 \neq 0$ one has [149, app. D]

$$(p^\Lambda P^x_\Lambda)^2 = \kappa^2. \tag{F.3.28}$$

F.3.2 Symplectic extension

Almost all the BPS equations we obtained in the previous sections are already symplectic invariant since they are given in terms of symplectic invariant quantities.

We replace the charges by $\mathcal{Q}$. To replace $\tilde{q}'^\Lambda$ we note that

$$\mathrm{e}^{-2(U-V)}\tilde{q}'^\Lambda = (\mathcal{I}^{-1})^{\Lambda\Sigma}(\mathcal{R}_{\Sigma\Delta}\,p^\Delta - q_\Sigma) \tag{F.3.29}$$

corresponds to the first component of $-\mathcal{M}\mathcal{Q}$.

The symplectic invariant equations are

$$\langle \mathcal{Q}, \mathcal{G}\rangle = -\kappa, \tag{F.3.30a}$$

$$\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) = \mathrm{e}^{2(U-V)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) \tag{F.3.30b}$$

$$\psi' = -A_r + 2\,\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}), \tag{F.3.30c}$$

$$2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) = -8\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V}) - \mathcal{Q} - \mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{G}, \tag{F.3.30d}$$

$$(\mathrm{e}^V)' = -2\,\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}). \tag{F.3.30e}$$

We also have the equation

$$2\,\partial_r\big(\mathrm{e}^U\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V})\big) = \mathrm{e}^{2(U-V)}\mathcal{M}\mathcal{Q} + \mathcal{G}. \tag{F.3.31}$$

The second term cannot be seen from the original equation since g^Λ was set to zero, but we could get it by computing explicitly the derivative of M_Λ.

The equation (F.3.30d) can be modified using (F.3.30e) to include one factor e^V inside the derivative. The LHS is

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) &= 2\,\mathrm{e}^V\partial_r\big(\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) - 2\,\mathrm{e}^{V-U}\partial_r(\mathrm{e}^V)\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &= 2\,\mathrm{e}^V\partial_r\big(\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) + 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \end{aligned}$$

and it combines with the RHS to

$$
\begin{aligned}
2\,\mathrm{e}^{V}\partial_r\big(\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) = &- 8\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V}) \\
&- 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \\
&- \mathcal{Q} - \mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{G}.
\end{aligned} \tag{F.3.32}
$$

Finally we recall the equations for U' and z'^i

$$
(\mathrm{e}^{U})' = -g_\Lambda\tilde{p}^\Lambda\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}), \tag{F.3.33a}
$$

$$
(z^i)' = \mathrm{e}^{-U}\,\mathrm{e}^{i\psi}g^{i\bar{\jmath}}\big(\mathrm{e}^{2(U-V)}\,\mathrm{D}_{\bar{\jmath}}\mathcal{Z} + i\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{D}_{\bar{\jmath}}\mathcal{L}\big). \tag{F.3.33b}
$$

F.3.3 Fayet–Iliopoulos gauging

We write

$$
\mathcal{G} = \mathcal{P}^3 = \begin{pmatrix} g^\Lambda \\ g_\Lambda \end{pmatrix} \tag{F.3.34}
$$

to really distinguish between non-constant and constant prepotentials.

Equations from special geometry

We can use several identities involving the quartic invariant in order to express all equations in terms of $\mathrm{Im}\,\mathcal{V}$ and V uniquely.

We define

$$
\widetilde{\mathcal{V}} = \mathrm{e}^{V-U}\,\mathrm{e}^{-i\psi}\,\mathcal{V}. \tag{F.3.35}
$$

The first step is to use the identity (6.3.16) in (F.3.32)

$$
2\,\mathrm{e}^{V}\partial_r\,\mathrm{Im}\,\widetilde{\mathcal{V}} = -\mathcal{Q} + I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{G}), \tag{F.3.36}
$$

Then using (5.3.16) and (6.3.17) as

$$
I_4(\mathrm{Im}\,\widetilde{\mathcal{V}}) = \frac{1}{16}\,\mathrm{e}^{4(V-U)}, \qquad \mathrm{Re}\,\widetilde{\mathcal{V}} = -2\,\mathrm{e}^{2(U-V)}\,I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}). \tag{F.3.37}
$$

we can replace $\mathrm{Re}(\widetilde{\mathcal{V}})$ and e^U

$$
\mathrm{e}^{2U-V}\,\mathrm{Re}\,\widetilde{\mathcal{V}} = -2\,\mathrm{e}^{4U-3V}\,I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}) = -\frac{1}{8}\,\mathrm{e}^{V}\,\frac{I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}})}{I_4(\mathrm{Im}\,\widetilde{\mathcal{V}})}. \tag{F.3.38}
$$

In terms of this new variable the equations (F.3.30d) and (F.3.30e) become

$$
2\,\mathrm{e}^{V}\partial_r\big(\mathrm{Im}\,\widetilde{\mathcal{V}})\big) = -\mathcal{Q} + I_4'(\mathrm{Im}\,\widetilde{\mathcal{V}}, \mathrm{Im}\,\widetilde{\mathcal{V}}, \mathcal{G}), \tag{F.3.39a}
$$

$$
(\mathrm{e}^{V})' = -2\left\langle \mathcal{G}, \mathrm{Im}\,\widetilde{\mathcal{V}} \right\rangle. \tag{F.3.39b}
$$

F.4 NUT black hole

F.4.1 Ansatz

We consider $N=2$ gauged supergravity with n_v vector multiplets. Fayet–Iliopoulos gaugings are denoted by g_Λ.

We take the following ansatz for the metric and the gauge fields[3]

$$
\mathrm{d}s^2 = \mathrm{e}^{2U}\big(\mathrm{d}t + 2\kappa n F'(\theta)\,\mathrm{d}\phi\big)^2 - \mathrm{e}^{-2U}\mathrm{d}r^2 - \mathrm{e}^{2(V-U)}\big(\mathrm{d}\theta^2 + F(\theta)^2\mathrm{d}\phi^2\big), \tag{F.4.1a}
$$

$$
A^\Lambda = \tilde{q}^\Lambda\big(\mathrm{d}t + 2\kappa n F'(\theta)\,\mathrm{d}\phi\big) - \kappa\tilde{p}^\Lambda F'(\theta)\mathrm{d}\phi. \tag{F.4.1b}
$$

[3]Nick is defining $N = \kappa n$.

$U, V, \tilde{q}$ and p are only function of r, while

$$F(\theta) = \begin{cases} \sin\theta & \kappa = 1 \\ \theta & \kappa = 0 \\ \sinh\theta & \kappa = -1 \end{cases}, \qquad \kappa = \mathrm{sign}(1-g) \tag{F.4.2}$$

where g is the genus of the surface. We note that the second derivative of F satisfies

$$F'' = -\kappa F. \tag{F.4.3}$$

F.4.2 Vierbein and spin connections

Recall the metric

$$\mathrm{d}s^2 = \mathrm{e}^{2U}\big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big)^2 - \mathrm{e}^{-2U}\mathrm{d}r^2 - \mathrm{e}^{2(V-U)}\big(\mathrm{d}\theta^2 + F^2\mathrm{d}\phi^2\big). \tag{F.4.4}$$

We introduce the following vierbein

$$e^0 = \mathrm{e}^U\big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big), \qquad e^1 = \mathrm{e}^{-U}\mathrm{d}r, \qquad e^2 = \mathrm{e}^{V-U}\mathrm{d}\theta, \qquad e^3 = F\,\mathrm{e}^{V-U}\mathrm{d}\phi. \tag{F.4.5}$$

We compute the differential

$$\begin{aligned}
\mathrm{d}e^0 &= U'\mathrm{d}r\wedge e^0 + 2\kappa n\, F''\,\mathrm{e}^U\mathrm{d}\theta\wedge\mathrm{d}\phi,\\
\mathrm{d}e^1 &= 0,\\
\mathrm{d}e^2 &= (V'-U')\,\mathrm{e}^{V-U}\mathrm{d}r\wedge\mathrm{d}\theta,\\
\mathrm{d}e^3 &= F\,(V'-U')\,\mathrm{e}^{V-U}\mathrm{d}r\wedge\mathrm{d}\phi + F'\,\mathrm{e}^{V-U}\mathrm{d}\theta\wedge\mathrm{d}\phi.
\end{aligned}$$

Using (F.4.3) and the vierbein expressions (F.4.5), we can replace all the differential by the vierbein

$$\mathrm{d}e^0 = U'\,\mathrm{e}^U e^1\wedge e^0 - 2n\,\mathrm{e}^{3U-2V}e^2\wedge e^3, \tag{F.4.6a}$$

$$\mathrm{d}e^1 = 0, \tag{F.4.6b}$$

$$\mathrm{d}e^2 = (V'-U')e^U e^1\wedge e^2, \tag{F.4.6c}$$

$$\mathrm{d}e^3 = (V'-U')e^U e^1\wedge e^3 + \frac{F'}{F}\,\mathrm{e}^{U-V}e^2\wedge e^3. \tag{F.4.6d}$$

Using the Cartan formula

$$\mathrm{d}e^a + \omega^a{}_b\wedge e^b = 0 \tag{F.4.7}$$

we obtain the following spin connections

$$\begin{gathered}
\omega^0{}_1 = U'\,\mathrm{e}^U e^0, \qquad \omega^0{}_2 = -n\,\mathrm{e}^{3U-2V}e^3, \qquad \omega^0{}_3 = n\,\mathrm{e}^{3U-2V}e^2,\\
\omega^2{}_1 = (V'-U')\,\mathrm{e}^U e^2, \qquad \omega^3{}_1 = (V'-U')\,\mathrm{e}^U e^3,\\
\omega^3{}_2 = \frac{F'}{F}\,\mathrm{e}^{U-V}e^3 + n\,\mathrm{e}^{3U-2V}e^0.
\end{gathered} \tag{F.4.8}$$

The last term in $\omega^3{}_2$ comes from the fact that

$$0 = \mathrm{d}e^3 + \omega^3{}_2 e^2 + \omega^3{}_0 e^0 = \mathrm{d}e^3 + \omega^3{}_2 e^2 + n\,\mathrm{e}^{3U-2V}e^2\wedge e^0. \tag{F.4.9}$$

since $\omega^3{}_0 = \omega^0{}_3$.

The explicit components

$$\omega^a{}_b = \omega_\mu{}^a{}_b\mathrm{d}x^\mu \tag{F.4.10}$$

are

$$\begin{gathered}
\omega_{001} = U'\,\mathrm{e}^U, \qquad \omega_{203} = -\omega_{302} = n\,\mathrm{e}^{3U-2V},\\
\omega_{212} = \omega_{313} = (V'-U')\,\mathrm{e}^U, \qquad \omega_{323} = \frac{F'}{F}\,\mathrm{e}^{U-V},\\
\omega_{023} = n\,\mathrm{e}^{3U-2V}.
\end{gathered} \tag{F.4.11}$$

F.4.3 Gauge fields

Recall the gauge fields

$$A^\Lambda = \tilde{q}^\Lambda \big(\mathrm{d}t + 2\kappa n F' \,\mathrm{d}\phi\big) - \kappa \tilde{p}^\Lambda F' \,\mathrm{d}\phi \tag{F.4.12a}$$

$$= \tilde{q}^\Lambda \,\mathrm{d}t - \kappa p^\Lambda F' \,\mathrm{d}\phi. \tag{F.4.12b}$$

where we have defined

$$p^\Lambda = \tilde{p}^\Lambda - 2n\tilde{q}^\Lambda. \tag{F.4.13}$$

For $n = 0$ we obviously recover the formula from [62], and for this reason formulas written in terms of $^\Lambda$ in terms of $\tilde{p}^\Lambda$ should be equivalent to this case.

In terms of the vierbein (F.4.5) we have

$$A^\Lambda = \tilde{q}^\Lambda \,\mathrm{e}^U e^0 - \kappa \,\frac{F'}{F}\,\mathrm{e}^{U-V} \tilde{p}^\Lambda \, e^3. \tag{F.4.14}$$

Field strengths

Electric field strength Now we compute the field strength

$$F^\Lambda = \mathrm{d}A^\Lambda \tag{F.4.15}$$

and we get

$$F^\Lambda = \tilde{q}'^\Lambda \,\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \,\mathrm{d}\phi\big) + p^\Lambda F \,\mathrm{d}\theta \wedge \mathrm{d}\phi - \kappa \tilde{p}'^\Lambda F' \,\mathrm{d}r \wedge \mathrm{d}\phi \tag{F.4.16a}$$

$$= -\tilde{q}'^\Lambda \,\mathrm{d}t \wedge \mathrm{d}r + p^\Lambda \, F \,\mathrm{d}\theta \wedge \mathrm{d}\phi - \kappa p'^\Lambda F' \,\mathrm{d}r \wedge \mathrm{d}\phi, \tag{F.4.16b}$$

or in terms of the tetrads

$$F^\Lambda = -\tilde{q}'^\Lambda \, e^0 \wedge e^1 + p^\Lambda \,\mathrm{e}^{2(U-V)} e^2 \wedge e^3 - \kappa \,\tilde{p}'^\Lambda \frac{F'}{F} \,\mathrm{e}^{2U-V} \, e^1 \wedge e^3. \tag{F.4.16c}$$

In particular it is trivial to see that the Bianchi identity is satisfied

$$\mathrm{d}F = p'^\Lambda \, F \,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi + p'^\Lambda F \,\mathrm{d}\theta \wedge \mathrm{d}r \wedge \mathrm{d}\phi = 0. \tag{F.4.17}$$

Hodge field strength Using the facts that

$$\star(e^\mu \wedge e^\nu) = \frac{1}{2}\,\varepsilon^{\mu\nu}{}_{\rho\sigma}\, e^\rho \wedge e^\sigma, \tag{F.4.18}$$

and

$$\varepsilon^{01}{}_{23} = \varepsilon^{13}{}_{02} = -1, \qquad \varepsilon^{23}{}_{01} = 1, \tag{F.4.19}$$

the Hodge dual field strength is found to be

$$\star F^\Lambda = p^\Lambda \,\mathrm{e}^{2(U-V)} e^0 \wedge e^1 + \tilde{q}'^\Lambda \, e^2 \wedge e^3 + \kappa \,\tilde{p}'^\Lambda \frac{F'}{F} \,\mathrm{e}^{2U-V} \, e^0 \wedge e^2 \tag{F.4.20a}$$

or by replacing the tetrads

$$\begin{aligned}\star F^\Lambda = &- p^\Lambda \,\mathrm{e}^{2(U-V)} \mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F' \,\mathrm{d}\phi\big) + \tilde{q}'^\Lambda \,\mathrm{e}^{2(V-U)} F \,\mathrm{d}\theta \wedge \mathrm{d}\phi \\ &- \kappa \,\tilde{p}'^\Lambda \frac{F'}{F} \,\mathrm{e}^{2U} \,\big(\mathrm{d}t + 2\kappa n F' \,\mathrm{d}\phi\big) \wedge \mathrm{d}\theta.\end{aligned} \tag{F.4.20b}$$

We can also expand in order to get all components

$$\begin{aligned}\star F^\Lambda = p^\Lambda \,\mathrm{e}^{2(U-V)} \mathrm{d}t \wedge \mathrm{d}r &+ \left(\tilde{q}'^\Lambda \,\mathrm{e}^{2(V-U)} + 2n \,\tilde{p}'^\Lambda \frac{F'^2}{F^2} \,\mathrm{e}^{2U} \right) F \,\mathrm{d}\theta \wedge \mathrm{d}\phi \\ &- 2\kappa n \, p^\Lambda \,\mathrm{e}^{2(U-V)} F' \,\mathrm{d}r \wedge \mathrm{d}\phi - \kappa \,\tilde{p}'^\Lambda \frac{F'}{F} \,\mathrm{e}^{2U} \,\mathrm{d}t \wedge \mathrm{d}\theta.\end{aligned} \tag{F.4.20c}$$

(Anti-)self dual field strength The anti-self dual form is

$$\mathcal{F}^{-\Lambda} = \frac{1}{2}\big(F^\Lambda - i\star F^\Lambda\big) = \tilde{F}^\Lambda(e^0\wedge e^1 + i\,e^2\wedge e^3) + \frac{F'}{F}\,\tilde{G}^\Lambda(e^1\wedge e^3 + i\,e^0\wedge e^2) \qquad \text{(F.4.21)}$$

where

$$\tilde{F}^\Lambda = -\frac{1}{2}\,\tilde{q}^\Lambda - \frac{i}{2}\,p^\Lambda\,\mathrm{e}^{2(U-V)}, \qquad \tilde{G}^\Lambda = -\kappa\,\mathrm{e}^{2U-V}\tilde{p}'^\Lambda. \qquad \text{(F.4.22)}$$

Magnetic field strength The symplectic dual G_Λ of F^Λ is defined by

$$G_\Lambda = \star\left(\frac{\delta\mathcal{L}}{\delta F^\Lambda}\right) = \mathcal{R}_{\Lambda\Sigma}\,F^\Sigma - \mathcal{I}_{\Lambda\Sigma}\,\star F^\Sigma. \qquad \text{(F.4.23)}$$

It reads explictly (with a matrix/vector notation)

$$\begin{aligned} G = \mathcal{R}\,&\big(\tilde{q}'\,\mathrm{d}r\wedge\big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + p\,F\,\mathrm{d}\theta\wedge\mathrm{d}\phi - \kappa\tilde{p}'F'\,\mathrm{d}r\wedge\mathrm{d}\phi\big) \\ &- \mathcal{I}\bigg(\tilde{q}'\,\mathrm{e}^{2(V-U)}F\,\mathrm{d}\theta\wedge\mathrm{d}\phi + \kappa\,\frac{F'}{F}\,\tilde{p}'\,\mathrm{e}^{2U}\,\big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big)\wedge\mathrm{d}\theta \\ &\qquad + p\,\mathrm{e}^{2(U-V)}\mathrm{d}r\wedge\big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big)\bigg), \end{aligned} \qquad \text{(F.4.24)}$$

or after simplification (in the last term we moved $\tilde{p}'$ in front of the expression since all matrices are symmetric)

$$\begin{aligned} G = &\big(\mathcal{R}\tilde{q}' + \mathcal{I}p\,\mathrm{e}^{2(U-V)}\big)\,\mathrm{d}r\wedge\big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + \big(\mathcal{R}p - \mathcal{I}\tilde{q}'\,\mathrm{e}^{2(V-U)}\big)F\,\mathrm{d}\theta\wedge\mathrm{d}\phi \\ &- \kappa\,\tilde{p}'\,F'\,\big(\mathcal{R}\,\mathrm{d}r\wedge\mathrm{d}\phi + \mathcal{I}\,\mathrm{e}^{2U}\,\big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big)\wedge\mathrm{d}\theta\big)\,. \end{aligned} \qquad \text{(F.4.25)}$$

Electromagnetic charges

The electric and magnetic charges are defined by [62]

$$p^\Lambda = \frac{1}{4\pi}\int_{S^2} F^\Lambda, \qquad q_\Lambda = \frac{1}{4\pi}\int_{S^2} G_\Lambda. \qquad \text{(F.4.26)}$$

The pair

$$\mathcal{Q} = (p^\Lambda, q_\Lambda) \qquad \text{(F.4.27)}$$

forms the correct symplectic vector of charges.[4]

We obtain the explicit expressions

$$p^\Lambda = \tilde{p}^\Lambda - 2n\,\tilde{q}^\Lambda, \qquad \text{(F.4.28a)}$$

$$q_\Lambda = \mathcal{R}_{\Lambda\Sigma}\,p^\Sigma - \,\mathrm{e}^{2(V-U)}\mathcal{I}_{\Lambda\Sigma}\tilde{q}'^\Sigma + 2n\,\mathcal{I}_{\Lambda\Sigma}\tilde{p}'^\Sigma\,\mathrm{e}^{2U}\int\frac{F'^2}{F}\,\mathrm{d}\theta, \qquad \text{(F.4.28b)}$$

which justify a posteriori that we identified p^Λ above.

The last integral can be done as

$$\int_0^{\theta_{\max}}\frac{F'^2}{F}\,\mathrm{d}\theta = \int_0^{F_{\max}}\frac{F'}{F}\,\mathrm{d}F = \ln F(\theta_{\max}) - \ln F(0). \qquad \text{(F.4.29)}$$

Since $F(0) = 0$ the last piece is divergent so we should require that

$$n = 0 \qquad \text{or} \qquad \tilde{p}'^\Lambda = 0. \qquad \text{(F.4.30)}$$

[4] Note that [62] forgets to add κ in the formula: the presence of κ here can be traced to the fact that it is absent in (F.4.1b), and ultimately the reason is that the gauge field should be defined with the integral of F, and not its derivative; see [76] for comparison.

Since we want that our black holes carry a NUT charge we require

$$\boxed{\tilde{p}'^{\Lambda} = 0.} \tag{F.4.31}$$

Another evidence for imposing this equation is that the field strength (F.4.16) and its dual (F.4.20) do not respect the isometries of the spacetime if $\tilde{p}^{\Lambda} \neq 0$. Moreover if this equation does not hold it is not possible to construct the symplectic vector of field strengths. Finally we will see that supersymmetry imposes naturally this constraint. For the rest of the section we will consider that this term is absent.

Imposing (F.4.31) we obtain the electromagnetic charges

$$p^{\Lambda} = \tilde{p}^{\Lambda} - 2n\,\tilde{q}^{\Lambda}, \tag{F.4.32a}$$

$$q_{\Lambda} = \mathcal{R}_{\Lambda\Sigma}\,p^{\Sigma} - \,\mathrm{e}^{2(V-U)}\mathcal{I}_{\Lambda\Sigma}\tilde{q}'^{\Sigma}. \tag{F.4.32b}$$

We can solve for $\tilde{q}'^{\Lambda}$ in terms of p^{Λ} and q_{Λ}

$$\boxed{\tilde{q}'^{\Lambda} = \,\mathrm{e}^{2(U-V)}(\mathcal{I}^{-1})^{\Lambda\Sigma}(\mathcal{R}_{\Sigma\Delta}\,p^{\Delta} - q_{\Sigma}).} \tag{F.4.33}$$

We note that the above relation corresponds to

$$\tilde{q}'^{\Lambda} = -\,\mathrm{e}^{2(U-V)}(\mathcal{M}\mathcal{Q})^{\Lambda}, \tag{F.4.34}$$

and we may use this relation for obtaning symplectic covariant formulas.

Symplectic field strengths

Imposing the condition (F.4.31), the expression (F.4.16) for the field strength becomes

$$F^{\Lambda} = \tilde{q}'^{\Lambda}\,\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + p^{\Lambda} F\,\mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{F.4.35}$$

The Bianchi identity reads

$$\mathrm{d}F^{\Lambda} = (p' + 2n\,\tilde{q}'^{\Lambda})F\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi = \tilde{p}'^{\Lambda} F\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi = 0 \tag{F.4.36}$$

which is solved by (F.4.31) and this is consistent.

The Hodge dual (F.4.20) reads

$$\star F^{\Lambda} = -p^{\Lambda}\,\mathrm{e}^{2(U-V)}\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + \tilde{q}'^{\Lambda}\,\mathrm{e}^{2(V-U)} F\,\mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{F.4.37}$$

Finally the magnetic field strength (F.4.25) is

$$G = \big(\mathcal{R}\tilde{q}' + \mathcal{I}p\,\mathrm{e}^{2(U-V)}\big)\,\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + \big(\mathcal{R}p - \mathcal{I}\tilde{q}'\,\mathrm{e}^{2(V-U)}\big)F\,\mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{F.4.38}$$

Then we can use the expression (F.4.33) for removing $\tilde{q}'$ in F^{Λ} and G_{Λ} (we use a matrix/vector notation)

$$F = \,\mathrm{e}^{2(U-V)}(\mathcal{I}^{-1}\mathcal{R}\,p - \mathcal{I}^{-1}q)\,\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + p\,F\,\mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{F.4.39a}$$

$$G = \,\mathrm{e}^{2(U-V)}\big((\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})p - \mathcal{R}\mathcal{I}^{-1}q\big)\,\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + q\,F\,\mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{F.4.39b}$$

where G is obtained from the simplification of

$$\begin{aligned} G = \mathcal{R}\,\Big(&\mathrm{e}^{2(U-V)}\mathcal{I}^{-1}(\mathcal{R}p - q)\,\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + p\,F\,\mathrm{d}\theta \wedge \mathrm{d}\phi\Big) \\ &- \mathcal{I}\,\Big(-p\,\mathrm{e}^{2(U-V)}\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + \mathcal{I}^{-1}(\mathcal{R}p - q)\,F\mathrm{d}\theta \wedge \mathrm{d}\phi\Big)\,. \end{aligned} \tag{F.4.40}$$

Note that we also have

$$\star F = -p\,\mathrm{e}^{2(U-V)}\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + \mathcal{I}^{-1}(\mathcal{R}p - q)\,F\mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{F.4.41}$$

Looking at (F.4.39) we can gather F and G into a symplectic vector using (F.4.34)

$$\mathcal{F} = \begin{pmatrix} F^{\Lambda} \\ G_{\Lambda} \end{pmatrix} = \,\mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{Q}\,\mathrm{d}r \wedge \big(\mathrm{d}t + 2\kappa n F'\,\mathrm{d}\phi\big) + \mathcal{Q}\,F\,\mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{F.4.42}$$

As explained above we cannot obtain this symplectic vector if $\tilde{p}' \neq 0$.

Maxwell equation

Maxwell equation reads

$$\mathrm{d}G_\Lambda = 0. \tag{F.4.43}$$

From the expression (F.4.39) we obtain

$$\mathrm{d}G = \Big[2n\,\mathrm{e}^{2(U-V)}\big((\mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R})p - \mathcal{R}\mathcal{I}^{-1}q\big) + q'\Big]\,F\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi, \tag{F.4.44}$$

or in components

$$q' = -2n\,\mathrm{e}^{2(U-V)}\big((\mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R})p - \mathcal{R}\mathcal{I}^{-1}q\big). \tag{F.4.45}$$

This computation is much more complicated if one keeps $\tilde{p}' \neq 0$ (the hope would be to get $\tilde{p}' = 0$ as a second equation).

The constraint (F.4.31) and the Bianchi identity

$$\mathrm{d}F^\Lambda = 0 \tag{F.4.46}$$

both read

$$\tilde{p}' = p' + 2n\,\tilde{q}' = 0. \tag{F.4.47}$$

Using the expression (F.4.33) one obtains

$$p' = -2n\ \mathrm{e}^{2(U-V)}\mathcal{I}^{-1}(\mathcal{R}\,p - q). \tag{F.4.48}$$

The equations for p' and q' can be gathered into a symplectic equation as

$$\mathcal{Q}' = -2n\,\mathrm{e}^{2(U-V)}\,\mathcal{M}\mathcal{Q} \tag{F.4.49}$$

using the expression for $\mathcal{M}$. This result can also be straightforwardly derived from the symplectic field strength (F.4.42).

Central charge

The central charge is defined by

$$\mathcal{Z} = \langle \mathcal{Q}, \mathcal{V}\rangle = p^\Lambda M_\Lambda - q_\Lambda L^\Lambda. \tag{F.4.50}$$

where $\mathcal{Q} = (p^\Lambda, q_\Lambda)$. Using (F.4.32), the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and $M_\Lambda = \mathcal{N}_{\Lambda\Sigma}L^\Sigma$ we can find another expression

$$\mathcal{Z} = p^\Lambda(\mathcal{R}_{\Lambda\Sigma} + i\mathcal{I}_{\Lambda\Sigma})L^\Sigma - \big(\mathcal{R}_{\Lambda\Sigma}\,p^\Lambda - \mathrm{e}^{2(V-U)}\mathcal{I}_{\Lambda\Sigma}\tilde{q}'^\Lambda\big)L^\Sigma,$$

and after simplifcation we get

$$\mathcal{Z} = \mathcal{I}_{\Lambda\Sigma}\big(\mathrm{e}^{2(V-U)}\tilde{q}'^\Lambda + ip^\Lambda\big)L^\Sigma. \tag{F.4.51}$$

Now we can deduce its relation with $\tilde{F}^\Lambda$ from (F.4.22)

$$\mathcal{Z} = -2\,\mathrm{e}^{2(V-U)}\mathcal{I}_{\Lambda\Sigma}\tilde{F}^\Lambda L^\Sigma. \tag{F.4.52}$$

Let's now compute the derivative of the central charge

$$\mathcal{Z}_i \equiv \mathrm{D}_i\mathcal{Z} = \langle \mathcal{Q}, U_i\rangle\,. \tag{F.4.53}$$

We have

$$\mathcal{Z}_i = p^\Lambda(\mathcal{R}_{\Lambda\Sigma} - i\mathcal{I}_{\Lambda\Sigma})f_i^\Sigma - \big(\mathcal{R}_{\Lambda\Sigma}\,p^\Lambda - \mathrm{e}^{2(V-U)}\mathcal{I}_{\Lambda\Sigma}\tilde{q}'^\Lambda\big)f_i^\Sigma,$$

since now $h_{i\Lambda} = \bar{\mathcal{N}}_{\Lambda\Sigma}f_i^\Sigma$, simplification gives

$$\mathcal{Z}_i = \mathcal{I}_{\Lambda\Sigma}\big(\mathrm{e}^{2(V-U)}\tilde{q}'^\Lambda - ip^\Lambda\big)f_i^\Sigma. \tag{F.4.54}$$

On the other hand we will have

$$\mathcal{Z}_{\bar{\imath}} = \mathcal{I}_{\Lambda\Sigma}\big(\,\mathrm{e}^{2(V-U)}\tilde{q}'^{\Lambda} + ip^{\Lambda}\big)\bar{f}^{\Sigma}_{\bar{\imath}}. \tag{F.4.55}$$

Finally we introduce a last quantity

$$\mathcal{L} = \langle \mathcal{G}, \mathcal{V}\rangle = g^{\Lambda}M_{\Lambda} - g_{\Lambda}L^{\Lambda}. \tag{F.4.56}$$

where $\mathcal{G} = (g^{\Lambda}, g_{\Lambda})$ (recall that $g^{\Lambda} = 0$ for the moment).

Inverting (F.4.50) we get

$$\mathcal{I}_{\Lambda\Sigma}\tilde{F}^{\Lambda}L^{\Sigma} = -\frac{1}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}. \tag{F.4.57}$$

We also define

$$\mathcal{I}_{\Lambda\Sigma}\tilde{G}^{\Lambda}L^{\Sigma} = -\frac{1}{2}\,\mathcal{Y}. \tag{F.4.58}$$

F.5 BPS equations for NUT black hole

We obtain the equation from [62] by taking $P^3_{\Lambda} = g_{\Lambda}$. We take the scalars and spinors to depend only on r. The ansatz for the spinors is

$$\varepsilon_A(r) = \mathrm{e}^{\frac{1}{2}(H+i\alpha)}\varepsilon_{0A} \tag{F.5.1}$$

with H and α both functions of r, and ε_{0A} is a constant spinor.

F.5.1 Gravitino equation

The gravitino variation is

$$\delta\psi_{\mu A} = \mathcal{D}_{\mu}\varepsilon_A + iS_{AB}\,\gamma_{\mu}\varepsilon^{B} + T^{-}_{\mu\nu}\gamma^{\nu}\varepsilon_{AB}\varepsilon^{B} = 0 \tag{F.5.2}$$

where

$$\mathcal{D}_{\mu}\varepsilon_A = D_{\mu}\varepsilon_A + \frac{i}{2}\,g_{\Lambda}A^{\Lambda}_{\mu}\sigma^{3}{}_{A}{}^{B}\varepsilon_B, \tag{F.5.3a}$$

$$D_{\mu} = \partial_{\mu} + \frac{1}{4}\,\omega_{\mu ab}\gamma^{ab} + \frac{i}{2}\,A_{\mu}, \tag{F.5.3b}$$

$$A_{\mu} = \frac{1}{2i}(K_i\partial_{\mu}z^i - K_{\bar{\imath}}\partial_{\mu}z^{\bar{\imath}}), \tag{F.5.3c}$$

$$S_{AB} = -\frac{i}{2}\,\mathcal{L}\,\sigma^{3}{}_{A}{}^{C}\varepsilon_{BC}, \tag{F.5.3d}$$

$$T^{-}_{\mu\nu} = 2i\,\mathcal{I}_{\Lambda\Sigma}L^{\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}. \tag{F.5.3e}$$

More precisely we will look at the components of $\gamma^a\delta\psi_{aA}$ (no sum over a).

We can obtain another expression for T^- from (F.4.21)

$$\begin{aligned} T^{-} &= 2i\,\mathcal{I}_{\Lambda\Sigma}L^{\Sigma}\mathcal{F}^{-\Lambda} \\ &= 2i\,\mathcal{I}_{\Lambda\Sigma}\tilde{F}^{\Lambda}L^{\Sigma}(e^0\wedge e^1 + i\,e^2\wedge e^3) + 2i\,\frac{F'}{F}\,\mathcal{I}_{\Lambda\Sigma}\tilde{G}^{\Lambda}L^{\Sigma}(e^1\wedge e^3 + i\,e^0\wedge e^2) \\ &= -i\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,(e^0\wedge e^1 + i\,e^2\wedge e^3) - i\,\frac{F'}{F}\,\mathcal{Y}\,(e^1\wedge e^3 + i\,e^0\wedge e^2) \end{aligned}$$

using the expressions (F.4.52) and (F.4.58). By contracting this expression with γ^b and multiplying by γ^a (thus with no sum over a) with

$$\gamma^a\gamma^b = \frac{1}{2}\,\gamma^{ab}, \tag{F.5.4}$$

we can see that only one term will remain for each value of a, and the factor will be ± 1 or $\pm i$.

The components of the variation read

$$\begin{aligned}\gamma^0\delta\psi_{0A} &= \frac{1}{2}\big(U'\,\mathrm{e}^U\gamma^1 + n\,\mathrm{e}^{3U-2V}\gamma^{023}\big)\varepsilon_A + \frac{i}{2}\,g_\Lambda\tilde{q}^\Lambda\ \mathrm{e}^{-U}\gamma^0\sigma^3{}_A{}^B\varepsilon_B + iS_{AB}\varepsilon^B \qquad \text{(F.5.5a)}\\ &\quad - \frac{i}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{01}\varepsilon_{AB}\varepsilon^B - \frac{F'}{F}\,\mathcal{Y}\,\gamma^{02}\varepsilon_{AB}\varepsilon^B,\\ \gamma^1\delta\psi_{1A} &= e^U\left(\partial_r + \frac{i}{2}\,A_r\right)\gamma^1\varepsilon_A + iS_{AB}\varepsilon^B - \frac{i}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{01}\varepsilon_{AB}\varepsilon^B \qquad \text{(F.5.5b)}\\ &\quad + i\,\frac{F'}{F}\,\mathcal{Y}\,\gamma^{13}\varepsilon_{AB}\varepsilon^B,\\ \gamma^2\delta\psi_{2A} &= \frac{1}{2}\big((V'-U')e^U\gamma^1 - n\,\mathrm{e}^{3U-2V}\gamma^{023}\big)\varepsilon_A + iS_{AB}\varepsilon^B \qquad \text{(F.5.5c)}\\ &\quad + \frac{1}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{23}\varepsilon_{AB}\varepsilon^B - \frac{F'}{F}\,\mathcal{Y}\,\gamma^{02}\varepsilon_{AB}\varepsilon^B,\\ \gamma^3\delta\psi_{3A} &= \frac{1}{2}\left((V'-U')e^U\gamma^1 - n\,\mathrm{e}^{3U-2V}\gamma^{023} + \frac{F'}{F}\ \mathrm{e}^{U-V}\gamma^2\right)\varepsilon_A + iS_{AB}\varepsilon^B \qquad \text{(F.5.5d)}\\ &\quad - \frac{i}{2}\,\frac{F'}{F}\,\kappa\,g_\Lambda\tilde{p}^\Lambda\ \mathrm{e}^{U-V}\gamma^3\sigma^3{}_A{}^B\varepsilon_B + \frac{1}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{23}\varepsilon_{AB}\varepsilon^B + i\,\frac{F'}{F}\,\mathcal{Y}\,\gamma^{13}\varepsilon_{AB}\varepsilon^B.\end{aligned}$$

We use the fact that $\gamma^a\gamma^b = \gamma^{ab}/2$ in all the last terms. Also we introduce curved index r for derivatives by using the inverse tetrad for the 1-component. We can rewrite γ^{023} and γ^{13}, and we simplify the equations

$$\begin{aligned}\gamma^0\delta\psi_{0A} &= \frac{\mathrm{e}^U}{2}\big(U' - in\,\mathrm{e}^{2(U-V)}\big)\gamma^1\varepsilon_A + \frac{i}{2}\,g_\Lambda\tilde{q}^\Lambda\ \mathrm{e}^{-U}\gamma^0\sigma^3{}_A{}^B\varepsilon_B + iS_{AB}\varepsilon^B \qquad \text{(F.5.6a)}\\ &\quad - \frac{i}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{01}\varepsilon_{AB}\varepsilon^B - \frac{F'}{F}\,\mathcal{Y}\,\gamma^{02}\varepsilon_{AB}\varepsilon^B,\\ \gamma^1\delta\psi_{1A} &= e^U\left(\partial_r + \frac{i}{2}\,A_r\right)\gamma^1\varepsilon_A + iS_{AB}\varepsilon^B - \frac{i}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{01}\varepsilon_{AB}\varepsilon^B \qquad \text{(F.5.6b)}\\ &\quad - \frac{F'}{F}\,\mathcal{Y}\,\gamma^{02}\varepsilon_{AB}\varepsilon^B,\\ \gamma^2\delta\psi_{2A} &= \frac{\mathrm{e}^U}{2}\big((V'-U') + in\,\mathrm{e}^{2(U-V)}\big)\gamma^1\varepsilon_A + iS_{AB}\varepsilon^B + \frac{i}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^{01}\varepsilon_{AB}\varepsilon^B \qquad \text{(F.5.6c)}\\ &\quad - \frac{F'}{F}\,\mathcal{Y}\,\gamma^{02}\varepsilon_{AB}\varepsilon^B,\\ \gamma^3\delta\psi_{3A} &= \gamma^2\delta\psi_{2A} + \frac{1}{2}\,\frac{F'}{F}\ \mathrm{e}^{U-V}\big(\gamma^2\varepsilon_A - i\kappa\,g_\Lambda\tilde{p}^\Lambda\,\gamma^3\sigma^3{}_A{}^B\varepsilon_B\big). \qquad \text{(F.5.6d)}\end{aligned}$$

First we see that each equation contains a θ-dependent term which should vanish since we have only r-dependent functions, thus

$$\mathcal{Y} = \mathcal{I}_{\Lambda\Sigma}\tilde{G}^\Lambda L^\Sigma = 0 \Longrightarrow \mathcal{I}_{\Lambda\Sigma}\tilde{p}'^\Lambda L^\Sigma = 0. \qquad \text{(F.5.7)}$$

We note that (F.5.6d) and (F.5.6c) differ only by a θ-dependent term, which gives a first projector equation

$$\gamma^2\varepsilon_A - i\kappa\,g_\Lambda\tilde{p}^\Lambda\,\gamma^3\sigma^3{}_A{}^B\varepsilon_B = 0. \qquad \text{(F.5.8)}$$

Taking the difference of (F.5.6a) and (F.5.6b) gives

$$\mathrm{e}^U\left(\partial_r + \frac{i}{2}\,A_r\right)\varepsilon_A = \frac{\mathrm{e}^U}{2}\big(U' - in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A + \frac{i}{2}\,g_\Lambda\tilde{q}^\Lambda\ \mathrm{e}^{-U}\gamma^{01}\sigma^3{}_A{}^B\varepsilon_B. \qquad \text{(F.5.9)}$$

Finally we need to take (F.5.6a) minus (F.5.6c)

$$\big(2U' - V' - 2in\,\mathrm{e}^{2(U-V)}\big)\gamma^1\varepsilon_A + i\,g_\Lambda\tilde{q}^\Lambda\ \mathrm{e}^{-2U}\,\gamma^0\sigma^3{}_A{}^B\varepsilon_B - 2i\,\mathrm{e}^{U-2V}\mathcal{Z}\gamma^{01}\varepsilon_{AB}\varepsilon^B = 0. \qquad \text{(F.5.10)}$$

We multiply (F.5.6c) by gamma matrices and we replace S_{AB} to get

$$\frac{i}{2}\,\mathcal{L}\,\gamma^{01}\sigma^{3}{}_{A}{}^{C}\varepsilon_{BC}\varepsilon^{B} = \frac{1}{2}\,\mathrm{e}^{2(U-V)}\mathcal{Z}\,\varepsilon_{AB}\varepsilon^{B} + \frac{i}{2}\,\mathrm{e}^{U}\big(V' - U' + in\,\mathrm{e}^{2(U-V)}\big)\gamma^{0}\varepsilon_{A}. \tag{F.5.11}$$

Let's summarize the equations we need to solve[5]

$$0 = \mathcal{I}_{\Lambda\Sigma}\tilde{p}'^{\Lambda}L^{\Sigma}, \tag{F.5.12a}$$

$$\left(\partial_r + \frac{i}{2}\,A_r\right)\varepsilon_A = \frac{1}{2}\big(U' - in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A + \frac{i}{2}\,g_\Lambda\tilde{q}^{\Lambda}\;\mathrm{e}^{-2U}\gamma^{01}\sigma^{3}{}_{A}{}^{B}\varepsilon_B, \tag{F.5.12b}$$

$$\big(2U' - V' - 2in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A = -2i\,\mathrm{e}^{U-2V}\mathcal{Z}\gamma^{0}\varepsilon_{AB}\varepsilon^{B} - i\,g_\Lambda\tilde{q}^{\Lambda}\;\mathrm{e}^{-2U}\,\gamma^{01}\sigma^{3}{}_{A}{}^{B}\varepsilon_B, \tag{F.5.12c}$$

$$\varepsilon_A = -\kappa\,g_\Lambda\tilde{p}^{\Lambda}\,\gamma^{01}\sigma^{3}{}_{A}{}^{B}\varepsilon_B, \tag{F.5.12d}$$

$$i\,\mathcal{L}\,\gamma^{01}\sigma^{3}{}_{A}{}^{C}\varepsilon_{BC}\varepsilon^{B} = \mathrm{e}^{2(U-V)}\mathcal{Z}\,\varepsilon_{AB}\varepsilon^{B} - i\,\mathrm{e}^{U}\big(V' - U' + in\,\mathrm{e}^{2(U-V)}\big)\gamma^{0}\varepsilon_{A}. \tag{F.5.12e}$$

These equations are equivalent to the ones in [62] if we replace

$$U' \longrightarrow U' - in\,\mathrm{e}^{2(U-V)}. \tag{F.5.13}$$

There are four equations with projectors, and we need to reduce two of them to bosonic equations in order to get 1/4-BPS solutions.

We can plug (F.5.12d) into itself and find the following consistency condition[6]

$$(\kappa g_\Lambda\tilde{p}^{\Lambda})^2 = 1 \Longrightarrow \boxed{g_\Lambda\tilde{p}^{\Lambda} = \pm\kappa.} \tag{F.5.14}$$

For simplicity we will keep the expression

$$\boxed{\varepsilon_A = -\kappa\,g_\Lambda\tilde{p}^{\Lambda}\gamma^{01}\sigma^{3}{}_{A}{}^{B}\varepsilon_B} \tag{F.5.15}$$

for the projector and simplify the sign only at the end. If g_Λ is fixed, then we can pick a sign and obtain the other just by inverting the other charges. An equivalent formulation gives

$$\kappa\,g_\Lambda\tilde{p}^{\Lambda}\,\varepsilon_A = -\gamma^{01}\sigma^{3}{}_{A}{}^{B}\varepsilon_B \tag{F.5.16}$$

by multiplying (F.5.15) on both side by $\kappa g_\Lambda p^\Lambda$ and using (F.5.14).

We can use it to simplify (F.5.12c)

$$\big(2U' - V' - 2in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A = -2i\,\mathrm{e}^{U-2V}\mathcal{Z}\,\gamma^{0}\varepsilon_{AB}\varepsilon^{B} + i\,c\,\varepsilon_A \tag{F.5.17}$$

where we have introduced the shortcut notation

$$c = \kappa\,g_\Lambda\tilde{p}^{\Lambda}\,g_\Sigma\tilde{q}^{\Sigma}\,\mathrm{e}^{-2U} = \pm g_\Lambda\tilde{q}^{\Lambda}\,\mathrm{e}^{-2U}. \tag{F.5.18}$$

We rewrite the equation as

$$\big(2U' - V' - i\tilde{c}\big)\varepsilon_A = -2i\,\mathrm{e}^{U-2V}\mathcal{Z}\,\gamma^{0}\varepsilon_{AB}\varepsilon^{B} \tag{F.5.19}$$

where

$$\tilde{c} = c + 2n\,\mathrm{e}^{2(U-V)} = \kappa\,g_\Lambda\tilde{p}^{\Lambda}\,g_\Sigma\tilde{q}^{\Sigma}\,\mathrm{e}^{-2U} + 2n\,\mathrm{e}^{2(U-V)}. \tag{F.5.20}$$

Hence we can interpret the effect of n as shifting c instead of U'.

We can now look for consistency of this last equation by plugging it into itself. First take the complex conjugate

$$(2U' - V' + i\tilde{c})\varepsilon^{A} = 2i\,\mathrm{e}^{U-2V}\bar{\mathcal{Z}}\,\gamma^{0}\varepsilon^{AB}\varepsilon_B. \tag{F.5.21}$$

[5] We obtain five equations from four because we got one additional constraint by requiring that the θ-dependent term in each equation vanishes.

[6] We could have not included κ into this equation but this choice allows to remove all κ from the equations, and it appears that it is necessary for finding a solution.

Now use this result into the first equation

$$|2U' - V' + i\tilde{c}|^2 = 4\,|\mathcal{Z}|^2\, e^{2U-4V}, \tag{F.5.22}$$

or written differently

$$|\mathcal{Z}|^2 = \frac{e^{4V-2U}}{4}\big((2U' - V')^2 + \tilde{c}^2\big). \tag{F.5.23}$$

We define the phase[7] $\psi(r)$ by the equation

$$2\, e^{U-2V}\, e^{-i\psi}\mathcal{Z} = 2U' - V' - i\tilde{c}, \tag{F.5.24}$$

or by replacing $\tilde{c}$

$$\boxed{2\, e^{U-2V}\, e^{-i\psi}\mathcal{Z} = 2U' - V' - i\big(\kappa\, g_\Lambda \tilde{p}^\Lambda\, g_\Sigma \tilde{q}^\Sigma\, e^{-2U} + 2n\, e^{2(U-V)}\big).} \tag{F.5.25}$$

The real and imaginary parts of this equation are respectively

$$2\, e^{U-2V}\, \mathrm{Re}(\, e^{-i\psi}\mathcal{Z}) = 2U' - V', \tag{F.5.26a}$$

$$2\, e^{U-2V}\, \mathrm{Im}(\, e^{-i\psi}\mathcal{Z}) = -\kappa\, g_\Lambda \tilde{p}^\Lambda\, g_\Sigma \tilde{q}^\Sigma\, e^{-2U} - 2n\, e^{2(U-V)}. \tag{F.5.26b}$$

The second equation will help us to replace $\tilde{q}^\Lambda$ everywhere.

The projector then becomes

$$\boxed{\varepsilon_A = i\, e^{i\psi}\, \gamma^0 \varepsilon_{AB} \varepsilon^B.} \tag{F.5.27}$$

The version with indices up is

$$\varepsilon^A = i\, e^{-i\psi}\, \gamma^0 \varepsilon^{AB} \varepsilon_B. \tag{F.5.28}$$

The phase ψ which appears here is the same as the one of the spinor in (F.5.1), as can be seen by comparing the phases of (F.5.27), thus

$$\alpha = \psi. \tag{F.5.29}$$

Inserting the projector (F.5.15) into (F.5.12b) turns it into a bosonic equation

$$\partial_r \varepsilon_A = \frac{1}{2}\Big(U' - i\big(A_r + c + n\, e^{2(U-V)}\big)\Big)\varepsilon_A \tag{F.5.30a}$$

$$= \frac{1}{2}\Big(U' - i\big(A_r + \tilde{c} - n\, e^{2(U-V)}\big)\Big)\varepsilon_A. \tag{F.5.30b}$$

Plugging the ansatz (F.5.1) for the spinor, we get a differential equation for the phase

$$\psi' = -\big(A_r + c + n\, e^{2(U-V)}\big) \tag{F.5.31}$$

from the imaginary part, while the real part tells us that $H' = U'$, and setting to zero the integration constant we have

$$H = U. \tag{F.5.32}$$

Replacing c we have

$$\psi' = -\big(A_r + \kappa\, g_\Lambda \tilde{p}^\Lambda\, g_\Sigma \tilde{q}^\Sigma\, e^{-2U} + n\, e^{2(U-V)}\big). \tag{F.5.33}$$

and it simplifies with (F.5.26b)

$$\boxed{\psi' = -A_r + 2\, e^{U-2V}\, \mathrm{Im}(\, e^{-i\psi}\mathcal{Z}) + n\, e^{2(U-V)}.} \tag{F.5.34}$$

[7] We know that both sides of the equation differ by this phase because of the above value for $|\mathcal{Z}|$.

The last step is to simplify (F.5.12e)

$$\begin{aligned}
i\,\mathcal{L}\,\gamma^{01}\sigma^3{}_A{}^C\varepsilon_{BC}\varepsilon^B &= \mathrm{e}^{2(U-V)}\mathcal{Z}\,\varepsilon_{AB}\varepsilon^B - i\,\mathrm{e}^U\big(V'-U'+in\,\mathrm{e}^{2(U-V)}\big)\gamma^0\varepsilon_A,\\
-i\,\mathcal{L}\,\gamma^{01}\sigma^3{}_A{}^C\gamma^0\varepsilon_{CB}\varepsilon^B &= \mathrm{e}^{2(U-V)}\mathcal{Z}\,\gamma^0\varepsilon_{AB}\varepsilon^B - i\,\mathrm{e}^U\big(V'-U'+in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A,\\
-\,\mathrm{e}^{-i\psi}\mathcal{L}\,\gamma^{01}\sigma^3{}_A{}^C\varepsilon_C &= -i\,\mathrm{e}^{2(U-V)}\,\mathrm{e}^{-i\psi}\mathcal{Z}\,\varepsilon_A - i\,\mathrm{e}^U\big(V'-U'+in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A,\\
\kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{e}^{-i\psi}\mathcal{L}\,\varepsilon_A &= -i\,\mathrm{e}^{2(U-V)}\,\mathrm{e}^{-i\psi}\mathcal{Z}\,\varepsilon_A - i\,\mathrm{e}^U\big(V'-U'+in\,\mathrm{e}^{2(U-V)}\big)\varepsilon_A.
\end{aligned}$$

In the first step we multiplied by γ^0 and reversed ε_{BC}, then we used the projector (F.5.27), and finally we used the other projector (F.5.16). After simplifcation we obtain a bosonic equation

$$i\kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{e}^{-i\psi}\mathcal{L} = \mathrm{e}^{2(U-V)}\,\mathrm{e}^{-i\psi}\mathcal{Z} + \mathrm{e}^U\Big(V'-U'+in\,\mathrm{e}^{2(U-V)}\Big). \tag{F.5.35}$$

The real part and imaginary parts read

$$\kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}) = -\,\mathrm{e}^{2(U-V)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}) - \mathrm{e}^U(V'-U'), \tag{F.5.36a}$$

$$\boxed{\kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) = \mathrm{e}^{2(U-V)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{3U-2V}.} \tag{F.5.36b}$$

From the equation (F.5.26a)

$$\mathrm{e}^U V' = 2\big((\mathrm{e}^U)' - \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z})\big), \tag{F.5.37}$$

we can simplify the first equation

$$\kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}) = -\,\mathrm{e}^{2(U-V)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}) - \big(2(\mathrm{e}^U)' - 2\,\mathrm{e}^{2(U-V)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}) - (\mathrm{e}^U)'\big) \tag{F.5.38}$$

and get a differential equation for U'

$$\boxed{(\mathrm{e}^U)' = -\kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}).} \tag{F.5.39}$$

Plugging this equation back we obtain a differential equation for V'

$$\boxed{(\mathrm{e}^V)' = -2\kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}).} \tag{F.5.40}$$

We can solve these two equations instead of (F.5.26a) and (F.5.36a).

Adding (F.5.35) to (F.5.25) gives

$$\mathrm{e}^{2(U-V)}\,\mathrm{e}^{-i\psi}\mathcal{Z} + i\kappa\,g_\Lambda\tilde{p}^\Lambda\,\mathrm{e}^{-i\psi}\mathcal{L} = \mathrm{e}^U\Big(U' - i\big(\kappa\,g_\Lambda\tilde{p}^\Lambda\,g_\Sigma\tilde{q}^\Sigma\,\mathrm{e}^{-2U} + n\,\mathrm{e}^{2(U-V)}\big)\Big). \tag{F.5.41}$$

This equation is just a rewriting of previous equations.

F.5.2 Gaugino variation

The gaugino variation is given by

$$\delta\lambda^{iA} = i\partial_\mu z^i\,\gamma^\mu\varepsilon^A - g^{i\bar{\jmath}}\bar{f}^{\Sigma}_{\bar{\jmath}}\,\mathcal{I}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\,\gamma^{\mu\nu}\varepsilon^{AB}\varepsilon_B + ig_\Lambda g^{i\bar{\jmath}}\bar{f}^{\Lambda}_{\bar{\jmath}}\,\sigma^3{}_C{}^B\varepsilon^{CA}\varepsilon_B = 0. \tag{F.5.42}$$

The variation becomes[8]

$$\begin{aligned}
\delta\lambda^{iA} = i\,\mathrm{e}^U\partial_r z^i\,\gamma^1\varepsilon^A &+ \frac{1}{2}\,\mathrm{e}^{2(U-V)}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}(\gamma^{01}+i\gamma^{23})\varepsilon^{AB}\varepsilon_B + ig^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\,\sigma^3{}_C{}^B\varepsilon^{CA}\varepsilon_B\\
&+ 2\,\frac{F'}{F}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Y}(\gamma^{13}+i\,\gamma^{02})\varepsilon^{AB}\varepsilon_B.
\end{aligned} \tag{F.5.43}$$

[8]The contraction is antisymmetric and should give a factor 2; but we wrote $\tilde{F}e^0e^1$, and we did not write the component e^1e^0, thus we do not take it into account (or we could by multiplying by a factor 1/2).

The last term is the only θ dependence and it should cancel

$$g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Y} = g^{i\bar{\jmath}}\,\mathcal{I}_{\Lambda\Sigma}\tilde{G}^{\Lambda}\,\bar{f}_{\bar{\jmath}}^{\Sigma} = 0 \Longrightarrow \mathcal{I}_{\Lambda\Sigma}\tilde{G}^{\Lambda}\,\bar{f}_{\bar{\jmath}}^{\Sigma} = 0. \tag{F.5.44}$$

Adding this to the previous equation (F.5.12a), we see that G^{Λ} is orthogonal to the $n_v + 1$ base vectors $(L^{\Lambda}, f_{\bar{\jmath}}^{\Lambda})$ which implies that it vanishes. We deduce that

$$\boxed{\tilde{p}'^{\Lambda} = 0 \Longrightarrow \tilde{p} = \mathrm{cst.}} \tag{F.5.45}$$

We can simplify the rest of (F.5.43)

$$\begin{aligned}
i\,\mathrm{e}^{U}\partial_r z^i\;\gamma^1\varepsilon^A &= -\frac{1}{2}\,\mathrm{e}^{2(U-V)}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}(\gamma^{01}+i\gamma^{23})\varepsilon^{AB}\varepsilon_B - ig^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\;\sigma^3{}_C{}^B\varepsilon^{CA}\varepsilon_B\\
i\,\mathrm{e}^{U}\partial_r z^i\;\gamma^1\varepsilon^A &= -\,\mathrm{e}^{2(U-V)}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}\;\gamma^{01}\varepsilon^{AB}\varepsilon_B - ig^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\;\sigma^3{}_C{}^B\varepsilon^{CA}\varepsilon_B\\
i\,\mathrm{e}^{U}\partial_r z^i\;\varepsilon^A &= \;\mathrm{e}^{2(U-V)}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}\;\gamma^{0}\varepsilon^{AB}\varepsilon_B + ig^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\;\gamma^0\varepsilon^{CA}\gamma^{01}\sigma^3{}_C{}^B\varepsilon_B\\
i\,\mathrm{e}^{U}\partial_r z^i\;\varepsilon^A &= i\,\mathrm{e}^{2(U-V)}\,\mathrm{e}^{i\psi}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}\,\varepsilon^A - i\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\;\gamma^0\varepsilon^{CA}\varepsilon_C\\
i\,\mathrm{e}^{U}\partial_r z^i\;\varepsilon^A &= i\,\mathrm{e}^{2(U-V)}\,\mathrm{e}^{i\psi}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{Z}\,\varepsilon^A - \kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,\mathrm{e}^{i\psi}\,g^{i\bar{\jmath}}\mathrm{D}_{\bar{\jmath}}\mathcal{L}\,\varepsilon^A.
\end{aligned}$$

First we replaced γ^{23} by γ^{01}, then we multiplied by γ^1 and we introduced $(\gamma^0)^2 = 1$, after what we used projectors (F.5.27) and (F.5.16) respectively for the first and second terms of the RHS, and finally we used again (F.5.27) for the last term after changing $\varepsilon^{CA} = -\varepsilon^{AC}$.

Cleaning up this equation gives finally

$$\boxed{\mathrm{e}^{-i\psi}\,\mathrm{e}^{U}\partial_r z^i = g^{i\bar{\jmath}}\Big(\mathrm{e}^{2(U-V)}\,\mathrm{D}_{\bar{\jmath}}\mathcal{Z} + i\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,\mathrm{D}_{\bar{\jmath}}\mathcal{L}\Big).} \tag{F.5.46}$$

We want to rewrite it in terms of the sections. It is easier to proceed if we replace

$$\mathrm{D}_i\mathcal{Z} = \mathcal{I}_{\Lambda\Sigma}\big(\mathrm{e}^{2(V-U)}\tilde{q}'^{\Lambda} + ip^{\Lambda}\big)f_i^{\Sigma}, \qquad \mathrm{D}_i\mathcal{L} = -g_{\Sigma}f_i^{\Sigma}, \tag{F.5.47}$$

using (F.4.54), to get

$$\mathrm{e}^{-i\psi}\,\mathrm{e}^{U}\partial_r z^i = g^{i\bar{\jmath}}\bar{f}_{\bar{\jmath}}^{\Sigma}\Big(\mathrm{e}^{2(U-V)}\,\mathcal{I}_{\Lambda\Sigma}\big(\mathrm{e}^{2(V-U)}\tilde{q}'^{\Lambda} + ip^{\Lambda}\big) - i\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,g_{\Sigma}\Big) \tag{F.5.48}$$

We contract both sides with f_i^{Δ}. Using the relation

$$-\,g^{i\bar{\jmath}}\bar{f}_{\bar{\jmath}}^{\Sigma}f_i^{\Delta} = \frac{1}{2}\,(\mathcal{I}^{-1})^{\Sigma\Delta} + L^{\Sigma}\bar{L}^{\Delta} \tag{F.5.49}$$

we find

$$\begin{aligned}
\mathrm{e}^{-i\psi}\,\mathrm{e}^{U}f_i^{\Delta}\partial_r z^i &= -\left(\frac{1}{2}\,(\mathcal{I}^{-1})^{\Sigma\Delta} + L^{\Sigma}\bar{L}^{\Delta}\right)\Big(\mathrm{e}^{2(U-V)}\,\mathcal{I}_{\Lambda\Sigma}\big(\mathrm{e}^{2(V-U)}\tilde{q}'^{\Lambda} + ip^{\Lambda}\big) - i\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,g_{\Sigma}\Big)\\
&= -\frac{1}{2}\big(\tilde{q}'^{\Delta} + i\,\mathrm{e}^{2(U-V)}p^{\Delta}\big) + \frac{i}{2}\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,(\mathcal{I}^{-1})^{\Sigma\Delta}g_{\Sigma} + i\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,\bar{L}^{\Delta}\,L^{\Sigma}g_{\Sigma}\\
&\quad -\,\mathrm{e}^{2(U-V)}\,\mathcal{I}_{\Lambda\Sigma}L^{\Sigma}\bar{L}^{\Delta}\big(\mathrm{e}^{2(V-U)}\tilde{q}'^{\Lambda} + ip^{\Lambda}\big)\\
&= -\frac{1}{2}\big(\tilde{q}'^{\Delta} + i\,\mathrm{e}^{2(U-V)}p^{\Delta}\big) + \frac{i}{2}\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,(\mathcal{I}^{-1})^{\Sigma\Delta}g_{\Sigma} - i\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,\bar{L}^{\Delta}\,\mathcal{L}\\
&\quad -\,\mathrm{e}^{2(U-V)}\,\bar{L}^{\Delta}\mathcal{Z}
\end{aligned}$$

where we used the expression of $\mathcal{Z}$ and $\mathcal{L}$

$$\begin{aligned}
&= -\frac{1}{2}\big(\tilde{q}'^{\Delta} + i\,\mathrm{e}^{2(U-V)}p^{\Delta}\big) + \frac{i}{2}\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,(\mathcal{I}^{-1})^{\Sigma\Delta}g_{\Sigma}\\
&\quad -\,\bar{L}^{\Delta}\big(\mathrm{e}^{2(U-V)}\,\mathcal{Z} + i\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,\mathcal{L}\big)\\
&= -\frac{1}{2}\big(\tilde{q}'^{\Delta} + i\,\mathrm{e}^{2(U-V)}p^{\Delta}\big) + \frac{i}{2}\,\kappa\,g_{\Lambda}\tilde{p}^{\Lambda}\,(\mathcal{I}^{-1})^{\Sigma\Delta}g_{\Sigma}\\
&\quad -\,\bar{L}^{\Delta}\,\mathrm{e}^{i\psi}\,\mathrm{e}^{U}\Big(U' - i\big(c + n\,\mathrm{e}^{2(U-V)}\big)\Big)
\end{aligned}$$

by using (F.5.41). We now consider the LHS

$$f_i^\Lambda \partial_r z^i = \partial_r z^i \left(\partial_i L^\Lambda + \frac{1}{2}(\partial_i K)L^\Lambda\right) = \partial_r L^\Lambda + \frac{1}{2}(z'^i\partial_i K - z'^{\bar\imath}\partial_{\bar\imath}K)L^\Lambda$$
$$= \partial_r L^\Lambda + iA_r L^\Lambda = \partial_r L^\Lambda - i\big(\psi' + c + n\, e^{2(U-V)}\big)L^\Lambda$$

from (F.5.34) and from

$$\partial_r L^\Lambda = z'^i\partial_i L^\Lambda + z'^{\bar\imath}\partial_{\bar\imath}L^\Lambda = z'^i\partial_i L^\Lambda + z'^{\bar\imath}\partial_{\bar\imath}(\, e^{\frac{K}{2}}X^\Lambda) = z'^i\partial_i L^\Lambda + \frac{1}{2}\, z'^{\bar\imath}L^\Lambda\partial_{\bar\imath}K$$

(explained with words, L^Λ depends on $\bar z$ by the Kähler potential).

Gluing the two sides we find

$$e^{-i\psi}\, e^U\Big(\partial_r L^\Delta - i\big(\psi' + c + n\, e^{2(U-V)}\big)L^\Delta\Big) + \bar L^\Delta\, e^{i\psi}\, e^U\Big(U' - i\big(c + n\, e^{2(U-V)}\big)\Big)$$
$$= -\frac{1}{2}\big(\tilde q'^\Delta + i\, e^{2(U-V)}p^\Delta\big) + \frac{i}{2}\,\kappa\, g_\Lambda\tilde p^\Lambda\,(\mathcal{I}^{-1})^{\Sigma\Delta}g_\Sigma. \quad \text{(F.5.50)}$$

We focus on the LHS

$$\begin{aligned}
&e^{-i\psi}\, e^U\Big(\partial_r L^\Delta - i\big(\psi' + c + n\, e^{2(U-V)}\big)L^\Delta\Big) + \bar L^\Delta\, e^{i\psi}\, e^U\Big(U' - i\big(c + n\, e^{2(U-V)}\big)\Big)\\
&= e^{-i\psi}\, e^U\big(\partial_r L^\Delta - i\psi' L^\Delta\big) + U'\, e^U\, e^{i\psi}\bar L^\Delta - i\, e^U\big(c + n\, e^{2(U-V)}\big)\big(\, e^{-i\psi}L^\Delta + e^{i\psi}\bar L^\Delta\big)\\
&= e^U\partial_r\big(\, e^{-i\psi}L^\Delta\big) + U'\, e^U\, e^{i\psi}\bar L^\Delta + 2i\, e^U\big(2\, e^{U-2V}\,\mathrm{Im}(\, e^{-i\psi}\mathcal{Z}) + n\, e^{2(U-V)}\big)\,\mathrm{Re}(\, e^{-i\psi}L^\Delta)\\
&= e^U\partial_r\big(\, e^{-i\psi}L^\Delta\big) + U'\, e^U\big(\,\mathrm{Re}(\, e^{-i\psi}L^\Delta) - i\,\mathrm{Im}(\, e^{-i\psi}L^\Delta)\big)\\
&\quad + 2i\, e^U\big(2\, e^{U-2V}\,\mathrm{Im}(\, e^{-i\psi}\mathcal{Z}) + n\, e^{2(U-V)}\big)\,\mathrm{Re}(\, e^{-i\psi}L^\Delta)
\end{aligned}$$

using (F.5.26b) and that $\mathrm{Im}(x^*) = -\mathrm{Im}\, x$ to replace c. We multiply each side by 2 and using the fact that $(\, e^{\pm U})' = \pm U'\, e^U$ we decompose this equation into real and imaginary parts[9]

$$2\,\partial_r\big(\, e^U\,\mathrm{Re}(\, e^{-i\psi}L^\Lambda)\big) = -\tilde q'^\Lambda, \quad \text{(F.5.51a)}$$
$$2\, e^{2V}\partial_r\big(\, e^{-U}\,\mathrm{Im}(\, e^{-i\psi}L^\Lambda)\big) = -p^\Lambda + \kappa\, e^{2(V-U)}g_\Delta\tilde p^\Delta\,(\mathcal{I}^{-1})^{\Sigma\Lambda}g_\Sigma - 4\big(2\,\mathrm{Im}(\, e^{-i\psi}\mathcal{Z}) + n\, e^U\big)\,\mathrm{Re}(\, e^{-i\psi}L^\Delta). \quad \text{(F.5.51b)}$$

The first equation is directly integrated to give

$$\boxed{\tilde q^\Lambda = -2\, e^U\,\mathrm{Re}(\, e^{-i\psi}L^\Lambda).} \quad \text{(F.5.52)}$$

Finally we can use (F.5.36b) to get

$$\boxed{\begin{aligned}2\, e^{2V}\partial_r\big(\, e^{-U}\,\mathrm{Im}(\, e^{-i\psi}L^\Lambda)\big) = &-4\big(2\kappa g_\Delta\tilde p^\Delta\, e^{2(V-U)}\,\mathrm{Re}(\, e^{-i\psi}\mathcal{L}) - n\, e^U\big)\,\mathrm{Re}(\, e^{-i\psi}L^\Delta)\\ &- p^\Lambda + \kappa\, e^{2(V-U)}g_\Delta\tilde p^\Delta\,(\mathcal{I}^{-1})^{\Sigma\Lambda}g_\Sigma.\end{aligned}}$$
(F.5.53)

F.5.3 Summary

We found two projectors

$$\varepsilon^A = i\, e^{-i\psi}\,\gamma^0\varepsilon^{AB}\varepsilon_B, \quad \text{(F.5.54a)}$$
$$\varepsilon_A = -\kappa g_\Lambda\tilde p^\Lambda\,\gamma^{01}\sigma^3{}_A{}^B\varepsilon_B. \quad \text{(F.5.54b)}$$

[9] For the imaginary part we need to multiply by $e^{2(V-U)}$.

We have algebraic

$$g_\Lambda \tilde{p}^\Lambda = \varepsilon_D \kappa, \tag{F.5.55a}$$

$$\kappa\, g_\Lambda \tilde{p}^\Lambda\, \mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L}) = \mathrm{e}^{2(U-V)}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{3U-2V} \tag{F.5.55b}$$

and differential equations

$$\psi' = -A_r + 2\,\mathrm{e}^{U-2V}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{2(U-V)}, \tag{F.5.55c}$$

$$(\,\mathrm{e}^{U})' = -\kappa\, g_\Lambda \tilde{p}^\Lambda\, \mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{Z}), \tag{F.5.55d}$$

$$(\,\mathrm{e}^{V})' = -2\kappa\, g_\Lambda \tilde{p}^\Lambda\, \mathrm{e}^{V-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}\mathcal{L}), \tag{F.5.55e}$$

$$(z^i)' = \mathrm{e}^{-U}\,\mathrm{e}^{i\psi} g^{i\bar{\jmath}}\big(\,\mathrm{e}^{2(U-V)}\,\mathrm{D}_{\bar{\jmath}}\mathcal{Z} + i\,\kappa\, g_\Lambda \tilde{p}^\Lambda\,\mathrm{D}_{\bar{\jmath}}\mathcal{L}\big). \tag{F.5.55f}$$

We have

$$\varepsilon_D = \pm 1 \tag{F.5.56}$$

and both signs correspond to different branches of BPS solutions. In general one can study the solution with $\varepsilon_D = -1$ [46, 52, 84] and the other branch can be found by flipping the sign of the charges – and apparently e^U – once $\mathcal{G}$ is fixed (see [81, app. B, 62, p. 6]). In particular this choice agrees with [58, p. 8]. Note that setting κ to the RHS is necessary (if one wants a solution) even if we do not see this from the equations.

The equations (F.5.55d) and (F.5.55f) can be gathered into

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\,\mathrm{e}^{-U}\,\mathrm{Im}(\,\mathrm{e}^{-i\psi}L^\Lambda)\big) = &- 8\kappa\, g_\Delta \tilde{p}^\Delta\, \mathrm{e}^{2(V-U)}\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}L^\Lambda) \\ &+ 4n\,\mathrm{e}^U\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}L^\Lambda) - p^\Lambda + \kappa\, g_\Delta \tilde{p}^\Delta\, \mathrm{e}^{2(V-U)}(\mathcal{I}^{-1})^{\Sigma\Lambda} g_\Sigma. \end{aligned} \tag{F.5.57}$$

One needs also to impose Maxwell equations (F.4.49)

$$\mathcal{Q}' = -2n\,\mathrm{e}^{2(U-V)}\,\mathcal{M}\mathcal{Q}. \tag{F.5.58}$$

It includes the equation

$$\tilde{p}'^\Lambda = 0 \tag{F.5.59}$$

and the charges $\tilde{q}^\Lambda$ are given by the equation (F.5.52)

$$\tilde{q}^\Lambda = -2\,\mathrm{e}^U\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}L^\Lambda). \tag{F.5.60}$$

Note that (F.5.55a) reduces to Dirac quantization condition from [62] when $n = 0$. Using the definition (F.4.32)

$$\tilde{p}^\Lambda = p^\Lambda + 2n\,\tilde{q}^\Lambda \tag{F.5.61}$$

and the equation (F.5.60)

$$\tilde{q}^\Lambda = -2\,\mathrm{e}^U\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}L^\Lambda), \tag{F.5.62}$$

we obtain[10] a new expression for (F.5.55a) which depends only on the electromagnetic charges

$$g_\Lambda p^\Lambda - 4n\,\mathrm{e}^U g_\Lambda\,\mathrm{Re}(\,\mathrm{e}^{-i\psi}L^\Lambda) = \kappa. \tag{F.5.63}$$

We can use (F.5.55b) in order to get an expression for $\mathrm{e}^{i\psi}$. This last expression will not help to solve the equation since it is complicated, but it means that we can always integrate the differential equation for the phase (F.5.55c), and we can obtain the expression if we know all other quantities. From (F.5.55b) we have[11]

$$\big(\,\mathrm{e}^{-i\psi}\mathcal{L} + \mathrm{e}^{i\psi}\bar{\mathcal{L}}\big) = -i\,\mathrm{e}^{2(U-V)}\big(\,\mathrm{e}^{-i\psi}\mathcal{Z} - \mathrm{e}^{i\psi}\bar{\mathcal{Z}}\big) + 2n\,\mathrm{e}^{3U-2V}. \tag{F.5.64}$$

[10] Since the formula contained $\tilde{q}$ and not $\tilde{q}'$ we could not use (F.4.34) to replace it.
[11] To lighten notations we take $g_\Lambda \tilde{p}^\Lambda = \kappa$

We multiply by $\mathrm{e}^{i\psi}$ in order to get a second order equation

$$\mathrm{e}^{2i\psi}\big(\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\bar{\mathcal{Z}}\big) - 2n\,\mathrm{e}^{3U-2V}\,\mathrm{e}^{i\psi} + \big(\mathcal{L} + i\,\mathrm{e}^{2(U-V)}\mathcal{Z}\big) = 0 \tag{F.5.65}$$

whose solutions are

$$\mathrm{e}^{i\psi} = -\frac{n\,\mathrm{e}^{3U-2V}}{\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\bar{\mathcal{Z}}} \pm 2\sqrt{\left(\frac{n\,\mathrm{e}^{3U-2V}}{\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\bar{\mathcal{Z}}}\right)^2 - \frac{\mathcal{L} + i\,\mathrm{e}^{2(U-V)}\mathcal{Z}}{\bar{\mathcal{L}} - i\,\mathrm{e}^{2(U-V)}\bar{\mathcal{Z}}}}. \tag{F.5.66}$$

For $n = 0$ it reduces to [52, eq. (2.39)]

$$\mathrm{e}^{2i\psi} = \frac{\mathrm{e}^{2(U-V)}\mathcal{Z} - i\mathcal{L}}{\mathrm{e}^{2(U-V)}\bar{\mathcal{Z}} + i\bar{\mathcal{L}}}. \tag{F.5.67}$$

F.5.4 Symplectic extension

Almost all the BPS equations we obtained in the previous sections are already symplectic invariant since they are given in terms of symplectic invariant quantities. The symplectic covariant expression of Dirac quantization condition can be read from (F.5.63).

The symplectic invariant equations are

$$\langle \mathcal{Q}, \mathcal{G}\rangle + 4n\,\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) = \varepsilon_D\,\kappa, \tag{F.5.68a}$$

$$\varepsilon_D\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) = \mathrm{e}^{2(U-V)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) + n\,\mathrm{e}^{3U-2V} \tag{F.5.68b}$$

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) &= \Big(4n\,\mathrm{e}^{U} - 8\varepsilon_D\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L})\Big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &\quad - \mathcal{Q} - \varepsilon_D\,\mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{G}, \end{aligned} \tag{F.5.68c}$$

$$(\mathrm{e}^{V})' = -2\varepsilon_D\,\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}), \tag{F.5.68d}$$

$$\mathcal{Q}' = -2n\,\mathrm{e}^{2(U-V)}\,\mathcal{M}\mathcal{Q}. \tag{F.5.68e}$$

We also have the derivative of equation (F.5.60)

$$2\,\partial_r\big(\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V})\big) = -\mathcal{G} - \mathrm{e}^{2(U-V)}\mathcal{M}\mathcal{Q}. \tag{F.5.69}$$

The first term cannot be seen from (F.5.60) since g^Λ was set to zero, but we could get it by computing explicitly the derivative of M_Λ.

Finally we recall the equations for ψ', U' and z'^i

$$\psi' = -A_r - 2\,\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) - n\,\mathrm{e}^{2(U-V)}, \tag{F.5.70a}$$

$$(\mathrm{e}^{U})' = -\varepsilon_D\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{2(U-V)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{Z}), \tag{F.5.70b}$$

$$(z^i)' = \mathrm{e}^{-U}\,\mathrm{e}^{i\psi}g^{i\bar{\jmath}}\big(\mathrm{e}^{2(U-V)}\,\mathrm{D}_{\bar{\jmath}}\bar{\mathcal{Z}} + i\,\mathrm{D}_{\bar{\jmath}}\bar{\mathcal{L}}\big). \tag{F.5.70c}$$

Other equations

The equation (F.5.68c) can be modified using (F.5.68d) to include one factor e^{V} inside the derivative. The LHS is

$$\begin{aligned} 2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) &= 2\,\mathrm{e}^{V}\partial_r\big(\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) - 2\,\mathrm{e}^{V-U}\partial_r(\mathrm{e}^{V})\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &= 2\,\mathrm{e}^{V}\partial_r\big(\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) + 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \end{aligned}$$

and it combines with the RHS to

$$\begin{aligned} 2\,\mathrm{e}^{V}\partial_r\big(\mathrm{e}^{V-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) &= 4\,\big(n\,\mathrm{e}^{U} - 2\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L})\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &\quad - 4\,\mathrm{e}^{2(V-U)}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{L})\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V}) \\ &\quad - \mathcal{Q} - \mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{G}. \end{aligned} \tag{F.5.71}$$

One can also use Maxwell equation (F.5.68e) to rewrite (F.5.69) as

$$2\,\partial_r\big(\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V})\big) = \frac{1}{2n}\mathcal{Q}' - \mathcal{G}. \tag{F.5.72}$$

It is then straightforward to integrate this equation

$$4n\,\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V}) = \mathcal{Q} - 2n\,\mathcal{G}\,r - \widehat{\mathcal{Q}} \tag{F.5.73}$$

where $\widehat{\mathcal{Q}}$ is the integration constant. In turn one can use this to get the expression for $\mathcal{Q}$ if one knows the other quantities. Moreover plugging this result into Dirac quantization equation (F.5.68a) gives

$$\langle\mathcal{Q},\mathcal{G}\rangle + 4n\,\mathrm{e}^{U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) = \left\langle\widehat{\mathcal{Q}},\mathcal{G}\right\rangle = \varepsilon_D\,\kappa \tag{F.5.74}$$

which shows that the LHS of Dirac equation is constant.

Finally one can use this expression for $\mathcal{Q}$ in order to rewrite the equation (F.5.68c) for the imaginary part of $\mathcal{V}$

$$\begin{aligned}2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) = &\Big(8n\,\mathrm{e}^{U} - 8\varepsilon_D\,\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L})\Big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V})\\ &- 2n\,\mathcal{G}\,r - \widehat{\mathcal{Q}} - \varepsilon_D\,\mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{G}.\end{aligned} \tag{F.5.75}$$

The main advantage is that $\mathcal{Q}$ has been replaced by the constant $\widehat{\mathcal{Q}}$, while the extra term $\mathcal{G}\,r$ is not a big problem.

Another formulation

We can use the second equation to replace n everywhere: we then get a set of equations which is the same as for $n = 0$, and any solution of this set should satisfy the additional constraint (F.5.68b). The new equations are

$$\psi' = -A_r + \mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z}) + \mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}), \tag{F.5.76a}$$

$$\begin{aligned}2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\big) = &-4\big(\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z})\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V})\\ &- \mathcal{Q} - \mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{G},\end{aligned} \tag{F.5.76b}$$

$$\mathcal{Q}' = 2\big(\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) - \mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z})\big)\,\mathcal{M}\mathcal{Q}. \tag{F.5.76c}$$

If we multiply (F.5.76b) by $\mathcal{M}$ (which is real) we get

$$\begin{aligned}2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{M}\mathcal{V})\big) = &-2\big(\mathrm{e}^{2(V-U)}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z})\big)\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{M}\mathcal{V})\\ &- \mathcal{M}\mathcal{Q} + \mathrm{e}^{2(V-U)}\mathcal{M}\mathcal{M}\mathcal{G}\\ &+ 2\,\mathrm{e}^{-U}\,\mathrm{Im}\,\big(\mathrm{e}^{-i\psi}\partial_r(\mathrm{e}^{2V}\mathcal{M})\mathcal{V}\big)\\ -2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Im}(i\,\mathrm{e}^{-i\psi}\mathcal{V})\big) = &-2\big(\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z})\big)\,\mathrm{Re}(i\,\mathrm{e}^{-i\psi}\mathcal{V})\\ &- \mathcal{M}\mathcal{Q} - \mathrm{e}^{2(V-U)}\mathcal{G} + 2\,\mathrm{e}^{-U}\,\mathrm{Im}\,\big(\mathrm{e}^{-i\psi}\partial_r(\mathrm{e}^{2V}\mathcal{M})\mathcal{V}\big)\end{aligned}$$

since $\mathcal{M}^2 = -1$. We obtain

$$\begin{aligned}2\,\mathrm{e}^{2V}\partial_r\big(\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{V})\big) = &-2\big(\mathrm{e}^{-U}\,\mathrm{Re}(\mathrm{e}^{-i\psi}\mathcal{L}) + \mathrm{e}^{U-2V}\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{Z})\big)\,\mathrm{Im}(\mathrm{e}^{-i\psi}\mathcal{V})\\ &+ \mathcal{M}\mathcal{Q} + \mathrm{e}^{2(V-U)}\mathcal{G} + 2\,\mathrm{e}^{-U}\,\mathrm{Im}\,\big(\mathrm{e}^{-i\psi}\partial_r(\mathrm{e}^{2V}\mathcal{M})\mathcal{V}\big).\end{aligned} \tag{F.5.77}$$

Equations from special geometry

We can use several identities involving the quartic invariant in order to express all equations in terms of $\operatorname{Im}\mathcal{V}$ and V uniquely.

We define

$$\widetilde{\mathcal{V}} = \mathrm{e}^{V-U}\,\mathrm{e}^{-i\psi}\,\mathcal{V}. \tag{F.5.78}$$

The first step is to use the identity (E.1.2c) in (F.5.71)

$$2\,\mathrm{e}^{V}\partial_r \operatorname{Im}\widetilde{\mathcal{V}} = -\mathcal{Q} + I_4'(\operatorname{Im}\widetilde{\mathcal{V}}, \operatorname{Im}\widetilde{\mathcal{V}}, \mathcal{G}) + 4n\ \mathrm{e}^{2U-V}\operatorname{Re}\widetilde{\mathcal{V}}, \tag{F.5.79}$$

Then using (E.1.2a) and (E.1.2b) as

$$I_4(\operatorname{Im}\widetilde{\mathcal{V}}) = \frac{1}{16}\ \mathrm{e}^{4(V-U)}, \qquad \operatorname{Re}\widetilde{\mathcal{V}} = -2\,\mathrm{e}^{2(U-V)}\,I_4'(\operatorname{Im}\widetilde{\mathcal{V}}). \tag{F.5.80}$$

we can replace $\operatorname{Re}(\widetilde{\mathcal{V}})$ and e^{U}

$$\mathrm{e}^{2U-V}\operatorname{Re}\widetilde{\mathcal{V}} = -2\,\mathrm{e}^{4U-3V}\,I_4'(\operatorname{Im}\widetilde{\mathcal{V}}) = -\frac{1}{8}\,\mathrm{e}^{V}\,\frac{I_4'(\operatorname{Im}\widetilde{\mathcal{V}})}{I_4(\operatorname{Im}\widetilde{\mathcal{V}})}. \tag{F.5.81}$$

In terms of this new variable the equations (F.5.68c) and (F.5.68d) become

$$2\,\mathrm{e}^{V}\partial_r\big(\operatorname{Im}\widetilde{\mathcal{V}})\big) = -\mathcal{Q} + I_4'(\operatorname{Im}\widetilde{\mathcal{V}}, \operatorname{Im}\widetilde{\mathcal{V}}, \mathcal{G}) - \frac{n}{2}\ \mathrm{e}^{V}\,\frac{I_4'(\operatorname{Im}\widetilde{\mathcal{V}})}{I_4(\operatorname{Im}\widetilde{\mathcal{V}})}, \tag{F.5.82a}$$

$$(\,\mathrm{e}^{V})' = -2\left\langle \mathcal{G}, \operatorname{Im}\widetilde{\mathcal{V}} \right\rangle. \tag{F.5.82b}$$

Bibliography

[1] L. Randall and R. Sundrum. "A Large Mass Hierarchy from a Small Extra Dimension" (May 1999).
DOI: 10.1103/PhysRevLett.83.3370.
arXiv: hep-ph/9905221.

[2] L. Randall and R. Sundrum. "An Alternative to Compactification" (June 1999).
DOI: 10.1103/PhysRevLett.83.4690.
arXiv: hep-th/9906064.

[3] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper, and R. Sundrum. "A Small Cosmological Constant from a Large Extra Dimension" (Jan. 2000).
DOI: 10.1016/S0370-2693(00)00359-2.
arXiv: hep-th/0001197.

[4] S. Coleman and J. Mandula. "All Possible Symmetries of the S Matrix". *Phys. Rev.* 159.5 (July 1967), pp. 1251–1256.
DOI: 10.1103/PhysRev.159.1251.

[5] R. Haag, J. T. Łopuszański, and M. Sohnius. "All Possible Generators of Supersymmetries of the S-matrix". *Nuclear Physics B* 88.2 (Mar. 1975), pp. 257–274.
DOI: 10.1016/0550-3213(75)90279-5.

[6] N. Seiberg and E. Witten. "Monopoles, Duality and Chiral Symmetry Breaking in N=2 Supersymmetric QCD" (Aug. 1994).
arXiv: hep-th/9408099.

[7] N. Seiberg and E. Witten. "Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory" (July 1994).
DOI: 10.1016/0550-3213(94)90124-4.
arXiv: hep-th/9407087.

[8] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov, and J. Trnka. "Scattering Amplitudes and the Positive Grassmannian" (Dec. 2012).
arXiv: 1212.5605.

[9] N. Arkani-Hamed and J. Trnka. "The Amplituhedron" (Dec. 2013).
arXiv: 1312.2007.

[10] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, L. Freyhult, N. Gromov, R. A. Janik, V. Kazakov, T. Klose, G. P. Korchemsky, C. Kristjansen, M. Magro, T. McLoughlin, J. A. Minahan, R. I. Nepomechie, A. Rej, R. Roiban, S. Schafer-Nameki, C. Sieg, M. Staudacher, A. Torrielli, A. A. Tseytlin, P. Vieira, D. Volin, and K. Zoubos. "Review of AdS/CFT Integrability: An Overview" (Dec. 2010).
DOI: 10.1007/s11005-011-0529-2.
arXiv: 1012.3982.

[11] S. Ferrara and S. Sabharwal. "Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces". *Nuclear Physics B* 332.2 (Mar. 1990), pp. 317–332.
DOI: 10.1016/0550-3213(90)90097-W.

[12] B. de Wit and A. Van Proeyen. “Symmetries of dual-quaternionic manifolds”. *Physics Letters B* 252.2 (Dec. 1990), pp. 221–229.
DOI: 10.1016/0370-2693(90)90864-3.

[13] B. de Wit and A. Van Proeyen. “Special geometry, cubic polynomials and homogeneous quaternionic spaces”. *Communications in Mathematical Physics* 149.2 (Oct. 1992), pp. 307–333.
DOI: 10.1007/BF02097627.
arXiv: hep-th/9112027.

[14] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban. “The Ultraviolet Behavior of N=8 Supergravity at Four Loops”. *Physical Review Letters* 103.8 (Aug. 2009).
DOI: 10.1103/PhysRevLett.103.081301.
arXiv: 0905.2326.

[15] Z. Bern, J. J. Carrasco, L. Dixon, H. Johansson, and R. Roiban. “Amplitudes and Ultraviolet Behavior of N=8 Supergravity”. *Fortschritte der Physik* 59.7-8 (July 2011), pp. 561–578.
DOI: 10.1002/prop.201100037.
arXiv: 1103.1848.

[16] Z. Bern, S. Davies, T. Dennen, and Y.-t. Huang. “Absence of Three-Loop Four-Point Divergences in N=4 Supergravity”. *Physical Review Letters* 108.20 (May 2012).
DOI: 10.1103/PhysRevLett.108.201301.
arXiv: 1202.3423.

[17] Z. Bern, S. Davies, T. Dennen, A. V. Smirnov, and V. A. Smirnov. “The Ultraviolet Properties of N=4 Supergravity at Four Loops”. *Physical Review Letters* 111.23 (Dec. 2013).
DOI: 10.1103/PhysRevLett.111.231302.
arXiv: 1309.2498.

[18] J. F. Plebański. “A class of solutions of Einstein-Maxwell equations”. *Annals of Physics* 90.1 (Mar. 1975), pp. 196–255.
DOI: 10.1016/0003-4916(75)90145-1.

[19] J. F. Plebański and M. Demiański. “Rotating, charged, and uniformly accelerating mass in general relativity”. *Annals of Physics* 98.1 (May 1976), pp. 98–127.
DOI: 10.1016/0003-4916(76)90240-2.

[20] A. Strominger and C. Vafa. “Microscopic Origin of the Bekenstein-Hawking Entropy”. *Physics Letters B* 379.1-4 (June 1996), pp. 99–104.
DOI: 10.1016/0370-2693(96)00345-0.
arXiv: hep-th/9601029.

[21] L. Andrianopoli, R. D’Auria, S. Ferrara, and M. Trigiante. “Extremal Black Holes in Supergravity” (Nov. 2006).
arXiv: hep-th/0611345.

[22] S. Ferrara, R. Kallosh, and A. Strominger. “N=2 Extremal Black Holes” (Aug. 1995).
arXiv: hep-th/9508072.

[23] S. Ferrara and R. Kallosh. “Universality of Sypersymmetric Attractors”. *Physical Review D* 54.2 (July 1996), pp. 1525–1534.
DOI: 10.1103/PhysRevD.54.1525.
arXiv: hep-th/9603090.

[24] S. Ferrara and R. Kallosh. “Supersymmetry and Attractors”. *Physical Review D* 54.2 (July 1996), pp. 1514–1524.
DOI: 10.1103/PhysRevD.54.1514.
arXiv: hep-th/9602136.

[25] A. Strominger. “Macroscopic Entropy of $N = 2$ Extremal Black Holes”. *Physics Letters B* 383.1 (Aug. 1996), pp. 39–43.
DOI: 10.1016/0370-2693(96)00711-3.
arXiv: hep-th/9602111.

[26] S. Ferrara, G. W. Gibbons, and R. Kallosh. “Black Holes and Critical Points in Moduli Space”. *Nuclear Physics B* 500.1-3 (Sept. 1997), pp. 75–93.
DOI: 10.1016/S0550-3213(97)00324-6.
arXiv: hep-th/9702103.

[27] S. A. Hartnoll. “Lectures on holographic methods for condensed matter physics” (Mar. 2009).
DOI: 10.1088/0264-9381/26/22/224002.
arXiv: 0903.3246.

[28] J. McGreevy. “Holographic duality with a view toward many-body physics” (Sept. 2009).
arXiv: 0909.0518.

[29] S. Sachdev. “Condensed matter and AdS/CFT” (Feb. 2010).
arXiv: 1002.2947.

[30] S. W. Hawking and D. N. Page. “Thermodynamics of black holes in anti-de Sitter space”. *Communications in Mathematical Physics* 87.4 (Dec. 1983), pp. 577–588.
DOI: 10.1007/BF01208266.

[31] L. J. Romans. “Supersymmetric, cold and lukewarm black holes in cosmological Einstein-Maxwell theory”. *Nuclear Physics B* 383.1-2 (Sept. 1992), pp. 395–415.
DOI: 10.1016/0550-3213(92)90684-4.
arXiv: hep-th/9203018.

[32] P. Breitenlohner and D. Z. Freedman. “Positive energy in anti-de Sitter backgrounds and gauged extended supergravity”. *Physics Letters B* 115.3 (Sept. 1982), pp. 197–201.
DOI: 10.1016/0370-2693(82)90643-8.

[33] P. Breitenlohner and D. Z. Freedman. “Stability in gauged extended supergravity”. *Annals of Physics* 144.2 (Dec. 1982), pp. 249–281.
DOI: 10.1016/0003-4916(82)90116-6.

[34] D. Astefanesei, R. B. Mann, and E. Radu. “Nut Charged Space-times and Closed Timelike Curves on the Boundary”. *Journal of High Energy Physics* 2005.01 (Jan. 2005), pp. 049–049.
DOI: 10.1088/1126-6708/2005/01/049.
arXiv: hep-th/0407110.

[35] Y. Bardoux, M. M. Caldarelli, and C. Charmousis. “Integrability in conformally coupled gravity: Taub-NUT spacetimes and rotating black holes” (Nov. 2013).
arXiv: 1311.1192.

[36] K. Behrndt, D. Lüst, and W. A. Sabra. “Stationary solutions of $N = 2$ supergravity”. *Nuclear Physics B* 510.1-2 (Jan. 1998), pp. 264–288.
DOI: 10.1016/S0550-3213(97)00633-0.
arXiv: hep-th/9705169.

[37] M. Hübscher, P. Meessen, and T. Ortín. “Supersymmetric solutions of N=2 d=4 sugra: the whole ungauged shebang”. *Nuclear Physics B* 759.1-2 (Dec. 2006), pp. 228–248.
DOI: 10.1016/j.nuclphysb.2006.10.004.
arXiv: hep-th/0606281.

[38] P. Meessen and T. Ortín. “The supersymmetric configurations of N=2, d=4 supergravity coupled to vector supermultiplets”. *Nuclear Physics B* 749.1-3 (Aug. 2006), pp. 291–324.
DOI: 10.1016/j.nuclphysb.2006.05.025.
arXiv: hep-th/0603099.

[39] T. Ortín. *Gravity and strings*. English. Cambridge University Press, 2004.

[40] B. L. Cerchiai, S. Ferrara, A. Marrani, and B. Zumino. “Duality, Entropy and ADM Mass in Supergravity” (Feb. 2009).
arXiv: 0902.3973.

[41] A. Ceresole and G. Dall’Agata. “Flow Equations for Non-BPS Extremal Black Holes”. *Journal of High Energy Physics* 2007.03 (Mar. 2007), pp. 110–110.
DOI: 10.1088/1126-6708/2007/03/110.
arXiv: hep-th/0702088.

[42] A. Ceresole, G. Dall’Agata, S. Ferrara, and A. Yeranyan. “Universality of the superpotential for d = 4 extremal black holes”. *Nuclear Physics B* 832.1-2 (June 2010), pp. 358–381.
DOI: 10.1016/j.nuclphysb.2010.02.015.
arXiv: 0910.2697.

[43] D. D. K. Chow and G. Compère. “Seed for general rotating non-extremal black holes of N=8 supergravity”. *Classical and Quantum Gravity* 31.2 (Jan. 2014), p. 022001.
DOI: 10.1088/0264-9381/31/2/022001.
arXiv: 1310.1925.

[44] D. D. K. Chow and G. Compère. “Black holes in N=8 supergravity from SO(4,4) hidden symmetries”. *Physical Review D* 90.2 (July 2014), p. 025029.
DOI: 10.1103/PhysRevD.90.025029.
arXiv: 1404.2602.

[45] S. de Alwis, J. Louis, L. McAllister, H. Triendl, and A. Westphal. “On Moduli Spaces in **AdS_4** Supergravity” (Dec. 2013).
arXiv: 1312.5659.

[46] A. Gnecchi and N. Halmagyi. “Supersymmetric Black Holes in AdS4 from Very Special Geometry” (Dec. 2013).
arXiv: 1312.2766.

[47] K. Hristov, H. Looyestijn, and S. Vandoren. “Maximally supersymmetric solutions of D=4 N=2 gauged supergravity”. *Journal of High Energy Physics* 2009.11 (Nov. 2009), pp. 115–115.
DOI: 10.1088/1126-6708/2009/11/115.
arXiv: 0909.1743.

[48] J. Louis, P. Smyth, and H. Triendl. “Supersymmetric Vacua in N=2 Supergravity” (Apr. 2012).
arXiv: 1204.3893.

[49] B. de Wit and M. van Zalk. “Electric and magnetic charges in N=2 conformal supergravity theories” (July 2011).
arXiv: 1107.3305.

[50] D. Cassani, S. Ferrara, A. Marrani, J. F. Morales, and H. Samtleben. “A special road to AdS vacua” (Nov. 2009).
arXiv: 0911.2708.

[51] J. Louis, P. Smyth, and H. Triendl. “Spontaneous N=2 to N=1 Supersymmetry Breaking in Supergravity and Type II String Theory” (Nov. 2009).
arXiv: 0911.5077.

[52] G. Dall'Agata and A. Gnecchi. "Flow equations and attractors for black holes in N = 2 U(1) gauged supergravity" (Dec. 2010).
arXiv: 1012.3756.

[53] N. Halmagyi. "BPS Black Hole Horizons in N=2 Gauged Supergravity" (Aug. 2013).
arXiv: 1308.1439.

[54] M. M. Caldarelli and D. Klemm. "All Supersymmetric Solutions of N=2, D=4 Gauged Supergravity". *Journal of High Energy Physics* 2003.09 (Sept. 2003), pp. 019–019.
DOI: 10.1088/1126-6708/2003/09/019.
arXiv: hep-th/0307022.

[55] D. Klemm and E. Zorzan. "All null supersymmetric backgrounds of N=2, D=4 gauged supergravity coupled to abelian vector multiplets". *Classical and Quantum Gravity* 26.14 (July 2009), p. 145018.
DOI: 10.1088/0264-9381/26/14/145018.
arXiv: 0902.4186.

[56] D. Klemm and E. Zorzan. "The timelike half-supersymmetric backgrounds of N=2, D=4 supergravity with Fayet-Iliopoulos gauging". *Physical Review D* 82.4 (Aug. 2010).
DOI: 10.1103/PhysRevD.82.045012.
arXiv: 1003.2974.

[57] P. Meessen and T. Ortín. "Supersymmetric solutions to gauged N=2 d=4 sugra: the full timelike shebang" (Apr. 2012).
arXiv: 1204.0493.

[58] S. L. Cacciatori and D. Klemm. "Supersymmetric AdS_4 black holes and attractors" (Nov. 2009).
arXiv: 0911.4926.

[59] A. H. Chamseddine and W. A. Sabra. "Magnetic and Dyonic Black Holes in D=4 Gauged Supergravity". *Physics Letters B* 485.1-3 (July 2000), pp. 301–307.
DOI: 10.1016/S0370-2693(00)00652-3.
arXiv: hep-th/0003213.

[60] K. Hristov and S. Vandoren. "Static supersymmetric black holes in AdS_4 with spherical symmetry" (Dec. 2010).
arXiv: 1012.4314.

[61] "BPS black holes in N=2 D=4 gauged supergravities" ().

[62] "BPS Black Holes in AdS4 from M-theory" ().

[63] W. A. Sabra. "Anti-De Sitter BPS Black Holes in N=2 Gauged Supergravity". *Physics Letters B* 458.1 (July 1999), pp. 36–42.
DOI: 10.1016/S0370-2693(99)00564-X.
arXiv: hep-th/9903143.

[64] K. Hristov, C. Toldo, and S. Vandoren. "On BPS bounds in D=4 N=2 gauged supergravity" (Oct. 2011).
arXiv: 1110.2688.

[65] K. Hristov. "On BPS bounds in D=4 N=2 gauged supergravity II: general matter couplings and black hole masses" (Dec. 2011).
arXiv: 1112.4289.

[66] S. Bellucci, S. Ferrara, A. Marrani, and A. Yeranyan. "d=4 Black Hole Attractors in N=2 Supergravity with Fayet-Iliopoulos Terms". *Physical Review D* 77.8 (Apr. 2008).
DOI: 10.1103/PhysRevD.77.085027.
arXiv: 0802.0141.

[67] S. Chimento, D. Klemm, and N. Petri. "Supersymmetric black holes and attractors in gauged supergravity with hypermultiplets" (Mar. 2015).
arXiv: 1503.09055.

[68] M. Hübscher, P. Meessen, T. Ortín, and S. Vaula. "Supersymmetric N=2 Einstein-Yang-Mills monopoles and covariant attractors". *Physical Review D* 78.6 (Sept. 2008).
DOI: 10.1103/PhysRevD.78.065031.
arXiv: 0712.1530.

[69] S. Kachru, R. Kallosh, and M. Shmakova. "Generalized Attractor Points in Gauged Supergravity". *Physical Review D* 84.4 (Aug. 2011).
DOI: 10.1103/PhysRevD.84.046003.
arXiv: 1104.2884.

[70] T. Kimura. "Non-supersymmetric Extremal RN-AdS Black Holes in N=2 Gauged Supergravity". *Journal of High Energy Physics* 2010.9 (Sept. 2010).
DOI: 10.1007/JHEP09(2010)061.
arXiv: 1005.4607.

[71] J. F. Morales and H. Samtleben. "Entropy function and attractors for AdS black holes". *Journal of High Energy Physics* 2006.10 (Oct. 2006), pp. 074–074.
DOI: 10.1088/1126-6708/2006/10/074.
arXiv: hep-th/0608044.

[72] D. Klemm, V. Moretti, and L. Vanzo. "Rotating Topological Black Holes" (Oct. 1997).
arXiv: gr-qc/9710123.

[73] L. Vanzo. "Black holes with unusual topology" (May 1997).
arXiv: gr-qc/9705004.

[74] R. B. Mann. "Topological Black Holes – Outside Looking In" (Sept. 1997).
arXiv: gr-qc/9709039.

[75] M. M. Caldarelli and D. Klemm. "Supersymmetry of Anti-de Sitter Black Holes" (Aug. 1998).
DOI: 10.1016/S0550-3213(98)00846-3.
arXiv: hep-th/9808097.

[76] N. Alonso-Alberca, P. Meessen, and T. Ortín. "Supersymmetry of topological Kerr-Newmann-Taub-NUT-aDS spacetimes". *Classical and Quantum Gravity* 17.14 (July 2000), pp. 2783–2797.
DOI: 10.1088/0264-9381/17/14/312.
arXiv: hep-th/0003071.

[77] D. Klemm and M. Nozawa. "Supersymmetry of the C-metric and the general Plebanski-Demianski solution". *JHEP* 1305 (Mar. 2013), p. 123.
DOI: 10.1007/JHEP05(2013)123.
arXiv: 1303.3119.

[78] A. Kostelecky and M. Perry. "Solitonic Black Holes in Gauged N=2 Supergravity" (Dec. 1995).
DOI: 10.1016/0370-2693(95)01607-4.
arXiv: hep-th/9512222.

[79] Z.-W. Chong, M. Cvetic, H. Lu, and C. N. Pope. "Charged Rotating Black Holes in Four-Dimensional Gauged and Ungauged Supergravities". *Nuclear Physics B* 717.1-2 (June 2005), pp. 246–271.
DOI: 10.1016/j.nuclphysb.2005.03.034.
arXiv: hep-th/0411045.

[80] D. D. K. Chow and G. Compère. “Dyonic AdS black holes in maximal gauged supergravity”. *Physical Review D* 89.6 (Mar. 2014), p. 065003.
DOI: 10.1103/PhysRevD.89.065003.
arXiv: 1311.1204.

[81] A. Gnecchi and C. Toldo. “On the non-BPS first order flow in N=2 U(1)-gauged Supergravity” (Nov. 2012).
arXiv: 1211.1966.

[82] A. Gnecchi, K. Hristov, D. Klemm, C. Toldo, and O. Vaughan. “Rotating black holes in 4d gauged supergravity” (Nov. 2013).
arXiv: 1311.1795.

[83] C. Toldo and S. Vandoren. “Static nonextremal AdS4 black hole solutions” (July 2012).
arXiv: 1207.3014.

[84] N. Halmagyi. “Static BPS Black Holes in AdS4 with General Dyonic Charges” (Aug. 2014).
arXiv: 1408.2831.

[85] S. Katmadas. “Static BPS black holes in U(1) gauged supergravity” (May 2014).
arXiv: 1405.4901.

[86] M. Colleoni and D. Klemm. “Nut-charged black holes in matter-coupled N=2, D=4 gauged supergravity”. *Physical Review D* 85.12 (June 2012).
DOI: 10.1103/PhysRevD.85.126003.
arXiv: 1203.6179.

[87] M. M. Caldarelli, R. G. Leigh, A. C. Petkou, P. M. Petropoulos, V. Pozzoli, and K. Siampos. “Vorticity in holographic fluids” (June 2012).
arXiv: 1206.4351.

[88] R. G. Leigh, A. C. Petkou, and P. M. Petropoulos. “Holographic Fluids with Vorticity and Analogue Gravity” (May 2012).
arXiv: 1205.6140.

[89] R. G. Leigh, A. C. Petkou, and P. M. Petropoulos. “Holographic Three-Dimensional Fluids with Nontrivial Vorticity”. *Physical Review D* 85.8 (Apr. 2012).
DOI: 10.1103/PhysRevD.85.086010.
arXiv: 1108.1393.

[90] P. M. Petropoulos. “Gravitational duality, topologically massive gravity and holographic fluids” (June 2014).
arXiv: 1406.2328.

[91] D. Martelli, A. Passias, and J. Sparks. “The gravity dual of supersymmetric gauge theories on a squashed three-sphere” (Oct. 2011).
arXiv: 1110.6400.

[92] D. Martelli and J. Sparks. “The gravity dual of supersymmetric gauge theories on a biaxially squashed three-sphere” (Nov. 2011).
arXiv: 1111.6930.

[93] D. Martelli and A. Passias. “The gravity dual of supersymmetric gauge theories on a two-parameter deformed three-sphere” (June 2013).
arXiv: 1306.3893.

[94] D. Martelli, A. Passias, and J. Sparks. “The supersymmetric NUTs and bolts of holography” (Dec. 2012).
arXiv: 1212.4618.

[95] R. G. Leigh, A. C. Petkou, P. M. Petropoulos, and P. K. Tripathy. “The Geroch group in Einstein spaces” (Mar. 2014).
arXiv: 1403.6511.

[96] E. T. Newman and A. I. Janis. “Note on the Kerr Spinning-Particle Metric”. *Journal of Mathematical Physics* 6.6 (June 1965), pp. 915–917.
DOI: 10.1063/1.1704350.

[97] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence. “Metric of a Rotating, Charged Mass”. *Journal of Mathematical Physics* 6.6 (June 1965), pp. 918–919.
DOI: 10.1063/1.1704351.

[98] G. Giampieri. “Introducing Angular Momentum into a Black Hole using Complex Variables”. *Gravity Research Foundation* (1990).
URL: http://gravityresearchfoundation.org/pdf/awarded/1990/giampieri.pdf.

[99] H. Erbin. “Janis-Newman algorithm: simplifications and gauge field transformation”. *General Relativity and Gravitation* 47.3 (Mar. 2015).
DOI: 10.1007/s10714-015-1860-1.
arXiv: 1410.2602.

[100] M. Demiański. “New Kerr-like space-time”. *Physics Letters A* 42.2 (Nov. 1972), pp. 157–159.
DOI: 10.1016/0375-9601(72)90752-9.

[101] S. P. Drake and P. Szekeres. “Uniqueness of the Newman-Janis algorithm in generating the Kerr-Newman metric”. *General Relativity and Gravitation* 32.3 (2000), pp. 445–457.
DOI: 10.1023/A:1001920232180.
arXiv: gr-qc/9807001.

[102] M. Azreg-Aïnou. “From static to rotating to conformal static solutions: Rotating imperfect fluid wormholes with(out) electric or magnetic field”. *The European Physical Journal C* 74.5 (May 2014).
DOI: 10.1140/epjc/s10052-014-2865-8.
arXiv: 1401.4292.

[103] C. J. Talbot. “Newman-Penrose approach to twisting degenerate metrics”. *Commun. Math. Phys.* 13.1 (Mar. 1969), pp. 45–61.
DOI: 10.1007/BF01645269.

[104] M. Gürses and F. Gürsey. “Lorentz covariant treatment of the Kerr–Schild geometry”. *Journal of Mathematical Physics* 16.12 (Dec. 1975), pp. 2385–2390.
DOI: 10.1063/1.522480.

[105] M. M. Schiffer, R. J. Adler, J. Mark, and C. Sheffield. “Kerr geometry as complexified Schwarzschild geometry”. *Journal of Mathematical Physics* 14.1 (Jan. 1973), pp. 52–56.
DOI: 10.1063/1.1666171.

[106] E. T. Newman. “Complex coordinate transformations and the Schwarzschild-Kerr metrics”. *Journal of Mathematical Physics* 14.6 (June 1973), pp. 774–776.
DOI: doi:10.1063/1.1666393.

[107] E. T. Newman and J. Winicour. “A curiosity concerning angular momentum”. *Journal of Mathematical Physics* 15.7 (July 1974), pp. 1113–1115.
DOI: doi:10.1063/1.1666761.

[108] E. T. Newman. “Heaven and its properties”. en. *Gen Relat Gravit* 7.1 (Jan. 1976), pp. 107–111.
DOI: 10.1007/BF00762018.

[109] R. Ferraro. “Untangling the Newman-Janis algorithm”. *General Relativity and Gravitation* 46.4 (Apr. 2014).
DOI: 10.1007/s10714-014-1705-3.
arXiv: 1311.3946.

[110] T. Adamo and E. T. Newman. "The Kerr-Newman metric: A Review". *Scholarpedia* 9 (Oct. 2014), p. 31791.
DOI: 10.4249/scholarpedia.31791.
arXiv: 1410.6626.

[111] F. J. Ernst. "New Formulation of the Axially Symmetric Gravitational Field Problem". *Phys. Rev.* 167.5 (Mar. 1968), pp. 1175–1178.
DOI: 10.1103/PhysRev.167.1175.

[112] F. J. Ernst. "New Formulation of the Axially Symmetric Gravitational Field Problem. II". *Phys. Rev.* 168.5 (Apr. 1968), pp. 1415–1417.
DOI: 10.1103/PhysRev.168.1415.

[113] H. Quevedo. "Complex transformations of the curvature tensor". en. *Gen Relat Gravit* 24.7 (July 1992), pp. 693–703.
DOI: 10.1007/BF00760076.

[114] H. Quevedo. "Determination of the metric from the curvature". en. *Gen Relat Gravit* 24.8 (Aug. 1992), pp. 799–819.
DOI: 10.1007/BF00759087.

[115] S. Yazadjiev. "Newman-Janis method and rotating dilaton-axion black hole". *General Relativity and Gravitation* 32.12 (2000), pp. 2345–2352.
DOI: 10.1023/A:1002080003862.
arXiv: gr-qc/9907092.

[116] S. P. Drake and R. Turolla. "The Application of the Newman-Janis Algorithm in Obtaining Interior Solutions of the Kerr Metric". *Classical and Quantum Gravity* 14.7 (July 1997), pp. 1883–1897.
DOI: 10.1088/0264-9381/14/7/021.
arXiv: gr-qc/9703084.

[117] E. N. Glass and J. P. Krisch. "Kottler-Lambda-Kerr Spacetime" (May 2004).
arXiv: gr-qc/0405143.

[118] L. Herrera and J. Jiménez. "The complexification of a nonrotating sphere: An extension of the Newman–Janis algorithm". *Journal of Mathematical Physics* 23.12 (Dec. 1982), pp. 2339–2345.
DOI: 10.1063/1.525325.

[119] N. Ibohal. "Rotating metrics admitting non-perfect fluids in General Relativity". *General Relativity and Gravitation* 37.1 (Jan. 2005), pp. 19–51.
DOI: 10.1007/s10714-005-0002-6.
arXiv: gr-qc/0403098.

[120] S. Viaggiu. "Interior Kerr solutions with the Newman-Janis algorithm starting with static physically reasonable space-times". *International Journal of Modern Physics D* 15.09 (Sept. 2006), pp. 1441–1453.
DOI: 10.1142/S0218271806009169.
arXiv: gr-qc/0603036.

[121] M. Azreg-Aïnou. "Generating rotating regular black hole solutions without complexification". *Physical Review D* 90.6 (Sept. 2014).
DOI: 10.1103/PhysRevD.90.064041.
arXiv: 1405.2569.

[122] D.-Y. Xu. "Exact solutions of Einstein and Einstein-Maxwell equations in higher-dimensional spacetime". en. *Class. Quantum Grav.* 5.6 (June 1988), p. 871.
DOI: 10.1088/0264-9381/5/6/008.

[123] H. Kim. "Notes on Spinning *AdS*_3 Black Hole Solution" (June 1997).
arXiv: hep-th/9706008.

[124] H. Kim. "Spinning BTZ Black Hole versus Kerr Black Hole: A Closer Look". *Physical Review D* 59.6 (Feb. 1999), p. 064002.
DOI: 10.1103/PhysRevD.59.064002.
arXiv: gr-qc/9809047.

[125] S. Capozziello, M. De Laurentis, and A. Stabile. "Axially symmetric solutions in $f(R)$-gravity". *Class.Quant.Grav.* 27 (2010), p. 165008.
DOI: 10.1088/0264-9381/27/16/165008.
arXiv: 0912.5286.

[126] F. Caravelli and L. Modesto. "Spinning Loop Black Holes". *Classical and Quantum Gravity* 27.24 (Dec. 2010), p. 245022.
DOI: 10.1088/0264-9381/27/24/245022.
arXiv: 1006.0232.

[127] N. Dadhich and S. G. Ghosh. "Rotating black hole in Einstein and pure Lovelock gravity" (July 2013).
arXiv: 1307.6166.

[128] S. G. Ghosh, S. D. Maharaj, and U. Papnoi. "Radiating Kerr-Newman black hole in $f(R)$ gravity". *Eur. Phys. J. C* 73.6 (June 2013), pp. 1–11.
DOI: 10.1140/epjc/s10052-013-2473-z.
arXiv: 1208.3028.

[129] S. G. Ghosh and U. Papnoi. "Spinning Higher Dimensional Einstein-Yang-Mills black holes" (Sept. 2013).
arXiv: 1309.4231.

[130] R. Mallett. "Metric of a rotating radiating charged mass in a de Sitter space". *Physics Letters A* 126.4 (Jan. 1988), pp. 226–228.
DOI: 10.1016/0375-9601(88)90750-5.

[131] R. Whisker. "Braneworld Black Holes". PhD thesis. University of Durham, Oct. 2008.
arXiv: 0810.1534.

[132] M. Azreg-Aïnou. "Comment on "Spinning loop black holes" [arXiv:1006.0232]". *Classical and Quantum Gravity* 28.14 (July 2011), p. 148001.
DOI: 10.1088/0264-9381/28/14/148001.
arXiv: 1106.0970.

[133] D. Xu. "Radiating metric, retarded time coordinates of Kerr-Newman-de Sitter black holes and related energy-momentum tensor". *Sci. China Ser. A-Math.* 41.6 (June 1998), pp. 663–672.
DOI: 10.1007/BF02876237.

[134] M. Demiański and E. T. Newman. "Combined Kerr-NUT Solution of the Einstein Field Equations". *Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys.* 14 (1966), pp. 653–657.
URL: http://www.osti.gov/energycitations/product.biblio.jsp?osti_id=4444917.

[135] L. K. Patel. "Radiating Demianski-type space-times". *Indian J. Pure Appl. Math* 9 (1978), p. 1019.
URL: http://www.new1.dli.ernet.in/data1/upload/insa/INSA_2/20005a68_1019.pdf.

[136] K. D. Krori, T. Chaudhury, and R. Bhattacharjee. "Charged Demianski metric". *Journal of Mathematical Physics* 22.10 (Oct. 1981), pp. 2235–2236.
DOI: 10.1063/1.524792.

[137] L. K. Patel, R. P. Akabari, and U. K. Dave. "Radiating Demianski-type metrics and the Einstein-Maxwell fields". *The ANZIAM Journal* 30.01 (July 1988), pp. 120–126.
DOI: 10.1017/S0334270000006081.

[138] Y. F. Pirogov. "Towards the rotating scalar-vacuum black holes" (June 2013).
arXiv: 1306.4866.

[139] D. Hansen and N. Yunes. "Applicability of the Newman-Janis Algorithm to Black Hole Solutions of Modified Gravity Theories". *Physical Review D* 88.10 (Nov. 2013), p. 104020.
DOI: 10.1103/PhysRevD.88.104020.
arXiv: 1308.6631.

[140] D. Cirilo-Lombardo. "The Newman-Janis Algorithm, Rotating Solutions and Einstein-Born-Infeld Black Holes" (Dec. 2006).
arXiv: gr-qc/0612063.

[141] J. H. Horne and G. T. Horowitz. "Rotating Dilaton Black Holes". *Physical Review D* 46.4 (Aug. 1992), pp. 1340–1346.
DOI: 10.1103/PhysRevD.46.1340.
arXiv: hep-th/9203083.

[142] R. D'Inverno. *Introducing Einstein's Relativity*. Anglais. Clarendon Press, Aug. 1992.

[143] J. F. Reed. "Some imaginary tetrad-transformations of Einstein spaces". PhD thesis. Rice University, 1974.
URL: http://scholarship.rice.edu/bitstream/handle/1911/15078/7421322.PDF?sequence=1.

[144] B. de Wit, F. Vanderseypen, and A. Van Proeyen. "Symmetry structure of special geometries" (Oct. 1992).
arXiv: hep-th/9210068.

[145] B. de Wit and A. Van Proeyen. "Hidden symmetries, special geometry and quaternionic manifolds" (Oct. 1993).
arXiv: hep-th/9310067.

[146] B. de Wit and A. Van Proeyen. "Isometries of special manifolds" (May 1995).
arXiv: hep-th/9505097.

[147] R. B. Brown. "Groups of type E7". und. *Journal für die reine und angewandte Mathematik* 236 (1969), pp. 79–102.
URL: https://eudml.org/doc/150932.

[148] S. Ferrara, R. Kallosh, and A. Marrani. "Degeneration of Groups of Type E7 and Minimal Coupling in Supergravity" (Feb. 2012).
arXiv: 1202.1290.

[149] H. Erbin and N. Halmagyi. "Abelian Hypermultiplet Gaugings and BPS Vacua in N = 2 Supergravity". *Journal of High Energy Physics* 2015.5 (May 2015).
DOI: 10.1007/JHEP05(2015)122.
arXiv: 1409.6310.

[150] H. Erbin and N. Halmagyi. "Quarter-BPS Black Holes in AdS_4-NUT from $N = 2$ Gauged Supergravity". *Accepted in JHEP* (Mar. 2015).
arXiv: 1503.04686.

[151] S. Ferrara, A. Marrani, and A. Yeranyan. "Freudenthal Duality and Generalized Special Geometry" (Feb. 2011).
arXiv: 1102.4857.

[152] P. Fré, A. S. Sorin, and M. Trigiante. "The c-map, Tits Satake subalgebras and the search for $\mathcal{N} = 2$ inflaton potentials" (July 2014).
arXiv: 1407.6956.

[153] A. Ceresole, G. Dall'Agata, S. Ferrara, M. Trigiante, and A. Van Proeyen. "A Search for an N=2 Inflaton Potential" (Apr. 2014).
arXiv: 1404.1745.

[154] H. Lu and J. F. Vazquez-Poritz. "C-metrics in Gauged STU Supergravity and Beyond" (Aug. 2014).
arXiv: 1408.6531.

[155] H. Lu and J. F. Vazquez-Poritz. “Dynamic C-metrics in (Gauged) Supergravities”. *Physical Review D* 91.6 (Mar. 2015).
DOI: 10.1103/PhysRevD.91.064004.
arXiv: 1408.3124.

[156] A. J. Keane. “An extension of the Newman-Janis algorithm”. *Classical and Quantum Gravity* 31.15 (Aug. 2014), p. 155003.
DOI: 10.1088/0264-9381/31/15/155003.
arXiv: 1407.4478.

[157] A. Sen. “Rotating Charged Black Hole Solution in Heterotic String Theory”. *Physical Review Letters* 69.7 (Aug. 1992), pp. 1006–1009.
DOI: 10.1103/PhysRevLett.69.1006.
arXiv: hep-th/9204046.

[158] H. Erbin and L. Heurtier. “Supergravity, complex parameters and the Janis-Newman algorithm”. *Class. Quant. Grav.* 32.16 (July 2015), p. 165005.
DOI: 10.1088/0264-9381/32/16/165005.
arXiv: 1501.02188.

[159] H. Erbin. “Deciphering and generalizing Demiański-Janis-Newman algorithm”. *Submitted to Class. Quantum Grav.* (Nov. 2014).
arXiv: 1411.2909.

[160] H. Erbin and L. Heurtier. “Five-dimensional Janis-Newman algorithm”. *Class. Quant. Grav.* 32.16 (July 2015), p. 165004.
DOI: 10.1088/0264-9381/32/16/165004.
arXiv: 1411.2030.

[161] M. J. Perry. “Black holes are coloured”. *Physics Letters B* 71.1 (Nov. 1977), pp. 234–236.
DOI: 10.1016/0370-2693(77)90786-9.

[162] E. Bergshoeff, R. Kallosh, and T. Ortín. “Stationary Axion/Dilaton Solutions and Supersymmetry”. *Nuclear Physics B* 478.1-2 (Oct. 1996), pp. 156–180.
DOI: 10.1016/0550-3213(96)00408-7.
arXiv: hep-th/9605059.

[163] R. Myers and M. Perry. “Black holes in higher dimensional space-times”. *Annals of Physics* 172.2 (Dec. 1986), pp. 304–347.
DOI: 10.1016/0003-4916(86)90186-7.

[164] J. C. Breckenridge, R. C. Myers, A. W. Peet, and C. Vafa. “D-branes and Spinning Black Holes”. *Physics Letters B* 391.1-2 (Jan. 1997), pp. 93–98.
DOI: 10.1016/S0370-2693(96)01460-8.
arXiv: hep-th/9602065.

[165] D. Z. Freedman and A. Van Proeyen. *Supergravity*. English. Cambridge University Press, May 2012.

[166] M. Graña. “Flux compactifications in string theory: a comprehensive review” (Sept. 2005).
arXiv: hep-th/0509003.

[167] A. Strominger. “Special geometry”. *Communications in Mathematical Physics* 133.1 (1990), pp. 163–180.
DOI: 10.1007/BF02096559.

[168] O. Aharony, M. Berkooz, J. Louis, and A. Micu. “Non-Abelian structures in compactifications of M-theory on seven-manifolds with SU(3) structure”. *Journal of High Energy Physics* 2008.09 (Sept. 2008), pp. 108–108.
DOI: 10.1088/1126-6708/2008/09/108.
arXiv: 0806.1051.

[169] D. Cassani, P. Koerber, and O. Varela. "All homogeneous N=2 M-theory truncations with supersymmetric AdS4 vacua" (Aug. 2012).
arXiv: 1208.1262.

[170] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré, and T. Magri. "N=2 Supergravity and N=2 Super Yang-Mills Theory on General Scalar Manifolds: Symplectic Covariance, Gaugings and the Momentum Map". *Journal of Geometry and Physics* 23.2 (Sept. 1997), pp. 111–189.
DOI: 10.1016/S0393-0440(97)00002-8.
arXiv: hep-th/9605032.

[171] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, and P. Fré. "General Matter Coupled N=2 Supergravity". *Nuclear Physics B* 476.3 (Sept. 1996), pp. 397–417.
DOI: 10.1016/0550-3213(96)00344-6.
arXiv: hep-th/9603004.

[172] A. Van Proeyen. *N = 2 supergravity in d = 4, 5, 6 and its matter couplings*. Nov. 2012.
URL: http://itf.fys.kuleuven.be/~toine/LectParis.pdf.

[173] A. Gnecchi. "Ungauged and gauged Supergravity Black Holes: results on U-duality". PhD thesis. Università Di Padova, 2012.

[174] K. Hristov. "Lessons from the Vacuum Structure of 4d N=2 Supergravity". PhD thesis. July 2012.
arXiv: 1207.3830.

[175] C. Toldo. "Anti-de Sitter black holes in gauged supergravity. Supergravity flow, thermodynamics and phase transitions". en. PhD thesis. Utrecht University, June 2014.
URL: http://dspace.library.uu.nl/handle/1874/294324.

[176] L. Andrianopoli, R. D'Auria, L. Sommovigo, and M. Trigiante. "D=4, N=2 Gauged Supergravity coupled to Vector-Tensor Multiplets". *Nuclear Physics B* 851.1 (Oct. 2011), pp. 1–29.
DOI: 10.1016/j.nuclphysb.2011.05.007.
arXiv: 1103.4813.

[177] D. Butter and J. Novak. "Component reduction in N=2 supergravity: the vector, tensor, and vector-tensor multiplets" (Jan. 2012).
arXiv: 1201.5431.

[178] R. D'Auria, L. Sommovigo, and S. Vaula. "N=2 Supergravity Lagrangian Coupled to Tensor Multiplets with Electric and Magnetic Fluxes" (Sept. 2004).
arXiv: hep-th/0409097.

[179] B. de Wit, H. Samtleben, and M. Trigiante. "Magnetic charges in local field theory" (July 2005).
arXiv: hep-th/0507289.

[180] L. Sommovigo and S. Vaulá. "D=4, N=2 Supergravity with Abelian electric and magnetic charges". *Physics Letters B* 602.1-2 (Nov. 2004), pp. 130–136.
DOI: 10.1016/j.physletb.2004.09.058.
arXiv: hep-th/0407205.

[181] L. Sommovigo. "Poincaré dual of D=4 N=2 Supergravity with Tensor Multiplets". *Nuclear Physics B* 716.1-2 (June 2005), pp. 248–260.
DOI: 10.1016/j.nuclphysb.2005.03.029.
arXiv: hep-th/0501048.

[182] U. Theis and S. Vandoren. "N=2 Supersymmetric Scalar-Tensor Couplings". *Journal of High Energy Physics* 2003.04 (Apr. 2003), pp. 042–042.
DOI: 10.1088/1126-6708/2003/04/042.
arXiv: hep-th/0303048.

[183] P. Claus, B. de Wit, M. Faux, B. Kleijn, R. Siebelink, and P. Termonia. "N=2 Supergravity Lagrangians with Vector-Tensor Multiplets". *Nuclear Physics B* 512.1-2 (Feb. 1998), pp. 148–178.
DOI: 10.1016/S0550-3213(97)00781-5.
arXiv: hep-th/9710212.

[184] M. Gunaydin, S. McReynolds, and M. Zagermann. "The R-map and the Coupling of N=2 Tensor Multiplets in 5 and 4 Dimensions". *Journal of High Energy Physics* 2006.01 (Jan. 2006), pp. 168–168.
DOI: 10.1088/1126-6708/2006/01/168.
arXiv: hep-th/0511025.

[185] E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit, and L. Girardello. "Vector multiplets coupled to N=2 supergravity: Super-Higgs effect, flat potentials and geometric structure". *Nuclear Physics B* 250.1–4 (1985), pp. 385–426.
DOI: 10.1016/0550-3213(85)90488-2.

[186] B. Craps, F. Roose, W. Troost, and A. Van Proeyen. "What is Special Kähler Geometry ?" (Mar. 1997).
DOI: 10.1016/S0550-3213(97)00408-2.
arXiv: hep-th/9703082.

[187] M. K. Gaillard and B. Zumino. "Duality rotations for interacting fields". *Nuclear Physics B* 193.1 (Dec. 1981), pp. 221–244.
DOI: 10.1016/0550-3213(81)90527-7.

[188] A. Ceresole, R. D'Auria, and S. Ferrara. "The Symplectic Structure of N=2 Supergravity and its Central Extension". *Nuclear Physics B - Proceedings Supplements* 46.1-3 (Mar. 1996), pp. 67–74.
DOI: 10.1016/0920-5632(96)00008-4.
arXiv: hep-th/9509160.

[189] A. Ceresole, R. D'Auria, S. Ferrara, and A. Van Proeyen. "Duality Transformations in Supersymmetric Yang-Mills Theories coupled to Supergravity". *Nuclear Physics B* 444.1-2 (June 1995), pp. 92–124.
DOI: 10.1016/0550-3213(95)00175-R.
arXiv: hep-th/9502072.

[190] J. Bagger and E. Witten. "Matter couplings in N = 2 supergravity". *Nuclear Physics B* 222.1 (July 1983), pp. 1–10.
DOI: 10.1016/0550-3213(83)90605-3.

[191] B. de Wit and A. Van Proeyen. "Broken sigma-model isometries in very special geometry". *Physics Letters B* 293.1-2 (Oct. 1992), pp. 94–99.
DOI: 10.1016/0370-2693(92)91485-R.
arXiv: hep-th/9207091.

[192] R. D'Auria, S. Ferrara, and P. Fré. "Special and quaternionic isometries: General couplings in N = 2 supergravity and the scalar potential". *Nuclear Physics B* 359.2–3 (Aug. 1991), pp. 705–740.
DOI: 10.1016/0550-3213(91)90077-B.

[193] A. Van Proeyen. "Supergravity with Fayet-Iliopoulos terms and R-symmetry" (Oct. 2004).
arXiv: hep-th/0410053.

[194] H. Samtleben. “Lectures on Gauged Supergravity and Flux Compactifications” (Aug. 2008).
arXiv: 0808.4076.

[195] H. Lu, Y. Pang, and C. N. Pope. “An ω Deformation of Gauged STU Supergravity”. *Journal of High Energy Physics* 2014.4 (Apr. 2014).
DOI: 10.1007/JHEP04(2014)175.
arXiv: 1402.1994.

[196] J. Michelson. “Compactifications of Type IIB Strings to Four Dimensions with Non-trivial Classical Potential” (Oct. 1996).
arXiv: hep-th/9610151.

[197] M. Nakahara. *Geometry, Topology and Physics*. 2nd edition. Institute of Physics Publishing, June 2003.

[198] K. Becker, M. Becker, and J. H. Schwarz. *String Theory and M-Theory: A Modern Introduction*. 1st ed. Cambridge University Press, Dec. 2006.

[199] S. Vandoren. *Lectures on Riemannian Geometry, Part II: Complex Manifolds*. 2009.
URL: http://www.staff.science.uu.nl/~vando101/MRIlectures.pdf.

[200] K. Galicki. “A generalization of the momentum mapping construction for quaternionic Kähler manifolds”. en. *Commun.Math. Phys.* 108.1 (Mar. 1987), pp. 117–138.
DOI: 10.1007/BF01210705.

[201] D. S. Freed. “Special Kähler Manifolds” (Dec. 1997).
DOI: 10.1007/s002200050604.
arXiv: hep-th/9712042.

[202] S. Bellucci, A. Marrani, and R. Roychowdhury. “Topics in Cubic Special Geometry” (Nov. 2010).
arXiv: 1011.0705.

[203] L. Andrianopoli, R. D’Auria, and S. Ferrara. “Flat Symplectic Bundles of N-Extended Supergravities, Central Charges and Black-Hole Entropy” (July 1997).
arXiv: hep-th/9707203.

[204] P. Claus, K. Van Hoof, and A. Van Proeyen. “A symplectic covariant formulation of special Kahler geometry in superconformal calculus”. *Classical and Quantum Gravity* 16.8 (Aug. 1999), pp. 2625–2649.
DOI: 10.1088/0264-9381/16/8/305.
arXiv: hep-th/9904066.

[205] A. Ceresole, S. Ferrara, A. Gnecchi, and A. Marrani. “d-Geometries Revisited” (Oct. 2012).
arXiv: 1210.5983.

[206] S. Bellucci, A. Marrani, and R. Roychowdhury. “On Quantum Special Kähler Geometry”. *International Journal of Modern Physics A* 25.09 (Apr. 2010), pp. 1891–1935.
DOI: 10.1142/S0217751X10049116.
arXiv: 0910.4249.

[207] G. Bossard and S. Katmadas. “Duality covariant multi-centre black hole systems” (Apr. 2013).
arXiv: 1304.6582.

[208] E. Cremmer and A. V. Proeyen. “Classification of Kahler manifolds in N=2 vector multiplet-supergravity couplings”. en. *Class. Quantum Grav.* 2.4 (July 1985), p. 445.
DOI: 10.1088/0264-9381/2/4/010.

[209] B. de Wit and A. Van Proeyen. “Potentials and symmetries of general gauged N = 2 supergravity-Yang-Mills models”. *Nuclear Physics B* 245 (1984), pp. 89–117.
DOI: 10.1016/0550-3213(84)90425-5.

[210] A. Ceresole, G. Dall'Agata, S. Ferrara, and A. Yeranyan. "First order flows for N=2 extremal black holes and duality invariants". *Nuclear Physics B* 824.1-2 (Jan. 2010), pp. 239–253.
DOI: 10.1016/j.nuclphysb.2009.09.003.
arXiv: 0908.1110.

[211] A. Ceresole, S. Ferrara, A. Marrani, and A. Yeranyan. "Small Black Hole Constituents and Horizontal Symmetry" (Apr. 2011).
arXiv: 1104.4652.

[212] S. Ferrara, A. Marrani, and A. Yeranyan. "On Invariant Structures of Black Hole Charges" (Oct. 2011).
arXiv: 1110.4004.

[213] L. Andrianopoli, R. D'Auria, S. Ferrara, A. Marrani, and M. Trigiante. "Two-Centered Magical Charge Orbits" (Jan. 2011).
arXiv: 1101.3496.

[214] M. Günaydin, G. Sierra, and P. K. Townsend. "The geometry of N = 2 Maxwell-Einstein supergravity and Jordan algebras". *Nuclear Physics B* 242.1 (Aug. 1984), pp. 244–268.
DOI: 10.1016/0550-3213(84)90142-1.

[215] S. Cecotti. "Homogeneous Kähler manifolds and T-algebras in $N = 2$ supergravity and superstrings". *Comm. Math. Phys.* 124.1 (1989), pp. 23–55.
URL: http://projecteuclid.org/euclid.cmp/1104179074.

[216] J. F. Luciani. "Coupling of O(2) supergravity with several vector multiplets". *Nuclear Physics B* 132.3–4 (Jan. 1978), pp. 325–332.
DOI: 10.1016/0550-3213(78)90123-2.

[217] K. Galicki. "Quaternionic Kähler and hyper-Kähler non-linear sigma-models". *Nuclear Physics B* 271.3–4 (1986), pp. 402–416.
DOI: 10.1016/S0550-3213(86)80017-7.

[218] R. D'Auria and S. Ferrara. "On Fermion Masses, Gradient Flows and Potential in Supersymmetric Theories" (Mar. 2001).
DOI: 10.1088/1126-6708/2001/05/034.
arXiv: hep-th/0103153.

[219] V. Y. Kraines. "Topology of quaternionic manifolds". EN. *Bull. Amer. Math. Soc.* 71.Number 3, Part 1 (May 1965), pp. 526–527.
URL: http://projecteuclid.org/euclid.bams/1183526914.

[220] A. Gray. "A note on manifolds whose holonomy group is a subgroup of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$." *Michigan Math. J.* 16.2 (July 1969), pp. 125–128.
DOI: 10.1307/mmj/1029000212.

[221] S. Ishihara. "Quaternion Kählerian manifolds". EN. *J. Differential Geom.* 9.4 (1974), pp. 483–500.
URL: http://projecteuclid.org/euclid.jdg/1214432544.

[222] S. Salamon. "Quaternionic Kähler manifolds". en. *Invent Math* 67.1 (Feb. 1982), pp. 143–171.
DOI: 10.1007/BF01393378.

[223] A. Swann. "Hyperkähler and Quaternionic Kähler Geometry". PhD thesis. Oxford University, 1990.
URL: http://home.math.au.dk/swann/thesisafs.pdf.

[224] "HyperKähler and quaternionic Kähler geometry" ().

[225] D. V. Alekseevsky, S. Marchiafava, and M. Pontecorvo. "Compatible Almost Complex Structures on Quaternion Kähler Manifolds". en. *Annals of Global Analysis and Geometry* 16.5 (Oct. 1998), pp. 419–444.
DOI: 10.1023/A:1006574700453.

[226] B. de Wit, M. Rocek, and S. Vandoren. "Gauging Isometries on Hyperkahler Cones and Quaternion-Kahler Manifolds". *Physics Letters B* 511.2-4 (July 2001), pp. 302–310.
DOI: 10.1016/S0370-2693(01)00636-0.
arXiv: hep-th/0104215.

[227] D. V. Alekseevsky. "Classification of Quaternionic Spaces with a Transitive Solvable Group of Motions". en. *Math. USSR Izv.* 9.2 (Apr. 1975), p. 297.
DOI: 10.1070/IM1975v009n02ABEH001479.

[228] V. Cortés. "Alekseevskian spaces". *Differential Geometry and its Applications* 6.2 (July 1996), pp. 129–168.
DOI: 10.1016/0926-2245(96)89146-7.

[229] J. Wolf. "Complex Homogeneous Contact Manifolds and Quaternionic Symmetric Spaces". *Journal of Mathematics and Mechanics* 14.6 (1965).

[230] G. Compère, S. de Buyl, E. Jamsin, and A. Virmani. "G2 Dualities in D=5 Supergravity and Black Strings" (Mar. 2009).
arXiv: 0903.1645.

[231] C. W. Misner. "The Flatter Regions of Newman, Unti, and Tamburino's Generalized Schwarzschild Space". *Journal of Mathematical Physics* 4.7 (July 1963), pp. 924–937.
DOI: 10.1063/1.1704019.

[232] M. Shmakova. "Calabi-Yau Black Holes" (Dec. 1996).
arXiv: hep-th/9612076.

[233] A. Donos, J. P. Gauntlett, N. Kim, and O. Varela. "Wrapped M5-branes, consistent truncations and AdS/CMT" (Sept. 2010).
arXiv: 1009.3805.

[234] M. Visser. "The Kerr spacetime: A brief introduction". *The Kerr Spacetime. Rotating Black Holes in General Relativity*. Ed. by D. L. Wiltshire, M. Visser, and S. M. Scott. Cambridge University Press, Feb. 2009.
arXiv: 0706.0622.

[235] A. Krasiński. *Inhomogeneous Cosmological Models*. English. Cambridge University Press, Nov. 2006.

[236] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers. "Large N Phases, Gravitational Instantons and the Nuts and Bolts of AdS Holography". *Physical Review D* 59.6 (Feb. 1999).
DOI: 10.1103/PhysRevD.59.064010.
arXiv: hep-th/9808177.

[237] C. V. Johnson. "Thermodynamic Volumes for AdS-Taub-NUT and AdS-Taub-Bolt". *Class.Quant.Grav.* 31 (Nov. 2014), p. 235003.
DOI: 10.1088/0264-9381/31/23/235003.
arXiv: 1405.5941.

[238] S. M. Carroll. *Spacetime and geometry: an introduction to general relativity*. English. Addison Wesley, 2004.

[239] J. D. Bekenstein. "Exact solutions of Einstein-conformal scalar equations". *Annals of Physics* 82.2 (Feb. 1974), pp. 535–547.
DOI: 10.1016/0003-4916(74)90124-9.

[240] N. M. Bocharova, K. A. Bronnikov, and V. N. Melnikov. “An exact solution of the system of Einstein equations and mass-free scalar field”. *Vestn. Mosk. Univ. Fiz. Astro.* 6 (1970), p. 706.

[241] R. Emparan and H. S. Reall. “Black Holes in Higher Dimensions”. *Living Rev.Rel.* 11 (Jan. 2008), p. 6.
arXiv: 0801.3471.

[242] A. N. Aliev. “Rotating black holes in higher dimensional Einstein-Maxwell gravity”. *Phys. Rev. D* 74.2 (July 2006), p. 024011.
DOI: 10.1103/PhysRevD.74.024011.

[243] M. Bañados, C. Teitelboim, and J. Zanelli. “The Black Hole in Three Dimensional Space Time”. *Physical Review Letters* 69.13 (Sept. 1992), pp. 1849–1851.
DOI: 10.1103/PhysRevLett.69.1849.
arXiv: hep-th/9204099.

[244] Z.-W. Chong, M. Cvetic, H. Lu, and C. N. Pope. “General Non-Extremal Rotating Black Holes in Minimal Five-Dimensional Gauged Supergravity”. *Physical Review Letters* 95.16 (Oct. 2005), p. 161301.
DOI: 10.1103/PhysRevLett.95.161301.
arXiv: hep-th/0506029.

[245] R. Emparan and H. S. Reall. “A rotating black ring in five dimensions”. *Physical Review Letters* 88.10 (Feb. 2002), p. 101101.
DOI: 10.1103/PhysRevLett.88.101101.
arXiv: hep-th/0110260.

[246] J. P. Gauntlett, R. C. Myers, and P. K. Townsend. “Black Holes of D=5 Supergravity”. *Classical and Quantum Gravity* 16.1 (Jan. 1999), pp. 1–21.
DOI: 10.1088/0264-9381/16/1/001.
arXiv: hep-th/9810204.

[247] H. S. Reall. “Higher dimensional black holes and supersymmetry”. *Physical Review D* 68.2 (July 2003), p. 024024.
DOI: 10.1103/PhysRevD.68.024024.
arXiv: hep-th/0211290.

[248] A. N. Aliev. “Superradiance and black hole bomb in five-dimensional minimal ungauged supergravity” (Aug. 2014).
arXiv: 1408.4269.

[249] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall. “All supersymmetric solutions of minimal supergravity in five dimensions”. *Classical and Quantum Gravity* 20.21 (Nov. 2003), pp. 4587–4634.
DOI: 10.1088/0264-9381/20/21/005.
arXiv: hep-th/0209114.

[250] G. W. Gibbons, D. Kastor, L. A. J. London, P. K. Townsend, and J. Traschen. “Supersymmetric Self-Gravitating Solitons”. *Nuclear Physics B* 416.3 (Apr. 1994), pp. 850–880.
DOI: 10.1016/0550-3213(94)90558-4.
arXiv: hep-th/9310118.

[251] A. Puhm. “Black Holes in String Theory: Guides to Quantum Gravity”. PhD thesis. Université Pierre et Marie Curie - Paris VI, 2013.
URL: http://ipht.cea.fr/Docspht//search/article.php?IDA=10667.

[252] F. R. Tangherlini. “Schwarzschild field in dimensions and the dimensionality of space problem”. en. *Nuovo Cim* 27.3 (Feb. 1963), pp. 636–651.
DOI: 10.1007/BF02784569.

[253] G. W. Gibbons and C. A. R. Herdeiro. "Supersymmetric Rotating Black Holes and Causality Violation". *Classical and Quantum Gravity* 16.11 (Nov. 1999), pp. 3619–3652.
DOI: 10.1088/0264-9381/16/11/311.
arXiv: hep-th/9906098.

[254] A. N. Aliev and D. K. Ciftci. "Note on Rotating Charged Black Holes in Einstein-Maxwell-Chern-Simons Theory". *Physical Review D* 79.4 (Feb. 2009), p. 044004.
DOI: 10.1103/PhysRevD.79.044004.
arXiv: 0811.3948.

[255] B. Ett and D. Kastor. "An Extended Kerr-Schild Ansatz". *Classical and Quantum Gravity* 27.18 (Sept. 2010), p. 185024.
DOI: 10.1088/0264-9381/27/18/185024.
arXiv: 1002.4378.

[256] T. Málek. "Extended Kerr-Schild spacetimes: General properties and some explicit examples". *Classical and Quantum Gravity* 31.18 (Sept. 2014), p. 185013.
DOI: 10.1088/0264-9381/31/18/185013.
arXiv: 1401.1060.

[257] C. Martinez, C. Teitelboim, and J. Zanelli. "Charged Rotating Black Hole in Three Spacetime Dimensions". *Physical Review D* 61.10 (Apr. 2000), p. 104013.
DOI: 10.1103/PhysRevD.61.104013.
arXiv: hep-th/9912259.

[258] P.-H. Lambert. "Conformal symmetries of gravity from asymptotic methods: further developments" (Sept. 2014).
arXiv: 1409.4693.

[259] P. M. R. Binétruy. *Supersymmetry*. Oxford University Press, Jan. 2007.

[260] B. F. Schutz. *Geometrical Methods of Mathematical Physics*. Anglais. Cambridge University Press, Jan. 1980.

[261] G. W. Gibbons, H. Lu, D. N. Page, and C. N. Pope. "The General Kerr-de Sitter Metrics in All Dimensions". *Journal of Geometry and Physics* 53.1 (Jan. 2005), pp. 49–73.
DOI: 10.1016/j.geomphys.2004.05.001.
arXiv: hep-th/0404008.