Exact Results in Supersymmetric Quantum Field Theory

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Xinyu Zhang

We, the dissertation committe for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Nikita Nekrasov - Dissertation Advisor
Professor, Simons Center for Geometry and Physics

Martin Rocek - Chairperson of Defense
Professor, C. N. Yang Institute for Theoretical Physics

Xu Du
Associate Professor, Department of Physics and Astronomy

Leon Takhtajan
Professor, Department of Mathematics

This dissertation is accepted by the Graduate School

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

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This thesis is devoted to a number of exact computations in various supersymmetric quantum field theories and their applications in studying nonperturbative properties of the theories.

In the first part, we study four-dimensional $\mathcal{N}=2$ supersymmetric theory on the Ω-background. We show that the partition function of $U(N)$ gauge theory with $2N$ fundamental hypermultiplets on the self-dual Ω-background simplifies at special points of the parameter space, and is related to the partition function of two-dimensional Yang-Mills theory on S^2. We also consider the insertion of a Wilson loop operator in two-dimensional Yang-Mills theory, and find the corresponding operator in the four-dimensional theory.

In the second part, we analyze the dynamics of a general two-dimensional $\mathcal{N}=(2,2)$ gauged linear sigma model with semichiral superfields. By computing the elliptic genera, we study the vacuum structure of the model. The result coincides with the model without the semichiral superfields. We also show that the low energy effective twisted superpotential contributed by semichiral superfields vanishes, whether we turn on twisted masses or not.

In the third part, we discuss the supersymmetric localization of the four-dimensional $\mathcal{N}=2$ off-shell gauged supergravity in the background of the

AdS_4 neutral topological black hole, which is the gravity dual of the ABJM theory defined on the boundary $S^1 \times \mathbb{H}^2$. We compute the large-N expansion of the supergravity partition function. The result gives the black hole entropy with the logarithmic correction, which matches the previous result of the entanglement entropy of the ABJM theory up to some stringy effects. Our result is consistent with the previous on-shell one-loop computation of the logarithmic correction to black hole entropy, and it provides a concrete example for the identification of the entanglement entropy of the boundary conformal field theory and the bulk black hole entropy beyond the leading order given by the classical Bekenstein-Hawking formula, which consequently tests the AdS/CFT correspondence at the subleading order.

Dedicated to my parents.

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Publications

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- Partition function of $\mathcal{N} = 2$ supersymmetric gauge theory and two-dimensional Yang-Mills theory,

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- Entanglement Entropy of ABJM Theory and Entropy of Topological Black Hole,

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Part I
Overview

Quantum field theory reconciles the principles of quantum mechanics with those of special relativity using point-like particles as the elementary building blocks. If a Lagrangian description is available, a quantum field theory can be defined in terms of the path integral over an infinite-dimensional space of fields. The theory can be analyzed using the powerful perturbation theory for small coupling constants. Physical quantities are computed order by order as a series expansion in the coupling constant. However, the perturbation theory is practically useless when the coupling is large, and furthermore, there are many important properties of the theory that are beyond the perturbation theory.

Therefore, it is important to be able to perform exact computations in quantum field theory. Typically it is impossible, but the situation improves dramatically if the theory is supersymmetric. Indeed, apart from the potential phenomenological applications, supersymmetric quantum field theories have long been recognized as appealing theoretical laboratories to test various ideas in quantum field theories.

The purpose of this dissertation is to apply the general tools to study in an exact fashion certain quantities in supersymmetric quantum field theories, based on my research work during the last few years. The rest of the thesis is organized as follows:

- **Partition function of $\mathcal{N}=2$ supersymmetric gauge theory and two-dimensional Yang-Mills theory**

  We study four-dimensional $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory with $2N$ fundamental hypermultiplets in the self-dual Ω-background. The partition function simplifies at special points of the parameter space, and is related to the partition function of two-dimensional Yang-Mills theory on S^2. We also consider the insertion of a Wilson loop operator in two-dimensional Yang-Mills theory, and find the corresponding operator in the four-dimensional $\mathcal{N}=2$ gauge theory.

- **Dynamics of two-dimensional $\mathcal{N}=(2,\ 2)$ theories with semichiral superfields**

We analyze the dynamics of a general two-dimensional $\mathcal{N} = (2, 2)$ gauged linear sigma model with semichiral superfields. By computing the elliptic genera, we study the vacuum structure of the model. The result coincides with that of the model without semichiral superfields. We also show that the contribution of the semichiral superfields to the low energy effective twisted superpotential vanishes, both with and without wisted masses.

- **Entanglement Entropy of ABJM Theory and Entropy of Topological Black Hole**

  We discuss the supersymmetric localization of the 4D $\mathcal{N} = 2$ off-shell gauged supergravity in the background of the AdS_4 neutral topological black hole, which is the gravity dual of the ABJM theory defined on the boundary $\mathrm{S}^1 \times \mathbb{H}^2$. We compute the large-$N$ expansion of the supergravity partition function. The result gives the black hole entropy with the logarithmic correction, which matches the previous result of the entanglement entropy of the ABJM theory up to some stringy effects. Our result is consistent with the previous on-shell one-loop computation of the logarithmic correction to black hole entropy. It provides an explicit example of the identification of the entanglement entropy of the boundary conformal field theory with the bulk black hole entropy beyond the leading order given by the classical Bekenstein-Hawking formula, which consequently tests the AdS/CFT correspondence at the subleading order.

Part II
Partition function of $\mathcal{N} = 2$ supersymmetric gauge theory and two-dimensional Yang-Mills theory

1 Introduction

$\mathcal{N} = 2$ supersymmetry in four dimensions imposes powerful constraints on the low energy behavior of supersymmetric theories. All terms with at most two derivatives and four fermions in the Wilsonian effective action are expressed in terms of a single holomorphic quantity, the prepotential $\mathcal{F}$, whose quantum corrections are one-loop exact in the perturbation theory, and generated nonperturbatively only by instantons. The exact form of the prepotential $\mathcal{F}$ was first determined for certain theories by Seiberg and Witten indirectly based on several assumptions on the strong coupling behavior of the theory [1, 2]. It was then extended to more general $\mathcal{N} = 2$ theories (see [3] for a recent review).

It is useful to deform the supersymmetric theories by putting them on nontrivial supergravity backgrounds [4, 5]. The prototypical example is the so-called Ω-background [4], in which the theory is deformed by two parameters ϵ_1, ϵ_2 parametrizing an $SO(4)$ rotation of $\mathbb{R}^4$. The Ω-deformation provides an IR regularization that preserves a part of the deformed supersymmetry. The calculation of the supersymmetric partition function is dramatically simplified and can be performed using equivariant localization techniques. The dependence of the partition function on the parameters ϵ_1, ϵ_2 contains profound physical information. In particular, it gives the prepotential of the low energy effective action of the undeformed theory on $\mathbb{R}^4$, as well as the couplings of the theory to the $\mathcal{N} = 2$ supergravity multiplet.

Soon after the exact computation of the partition function in the Ω-background was done, an interesting relation between supersymmetric gauge theory and topological string theory was discovered [6, 7]. On the gauge theory side, we have the four-dimensional $\mathcal{N} = 2$ supersymmetric $U(N)$

gauge theory with $2N-2$ fundamental hypermultiplets. Its partition function in the self-dual Ω-background simplifies dramatically at a special point of the parameter space and is identified with the disconnected partition function of A-type topological string theory on S^2. The higher Casimir operators in the four-dimensional gauge theory map to gravitational descendants of the Khler form in the topological string theory. It was later further generalized in [8] by adding g adjoint hypermultiplets in the four-dimensional gauge theory and replacing S^2 with a genus g Riemann surface.

Inspired by the previous results, we explore the possible simplification of the partition function of the four-dimensional $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory with $2N$ fundamental hypermultiplets in this paper. We find that the partition function in the self-dual Ω-background at a special point of the parameter space can be related to the partition function of two-dimensional Yang-Mills theory on S^2 [9, 10]. The rank of the gauge group of the two-dimensional theory has nothing to do with the four-dimensional gauge group $U(N)$.

Once the correspondence is established, one may study either side using the information of the other side. In this paper, we consider the Wilson loop operator in the two-dimensional Yang-Mills theory. The exact expectation value of the Wilson loop operator has been known for a long time. We show that inserting a Wilson loop operator in the fundamental representation corresponds to adding a nontrivial operator in the four-dimensional $\mathcal{N} = 2$ gauge theory. The generalization to other representations is more involved and will be discussed in the future.

The structure of this paper is as follows. In Sec. 2, we review the partition function of four-dimensional $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory with $2N$ fundamental hypermultiplets in the Ω-background, and describe the $\mathcal{Y}$-observable that will turn out to be useful in our discussion. We show that the partition function simplifies at special points of the parameter space. In Sec. 3, we show that the simplified partition function can be related to the partition function of two-dimensional Yang-Mills theory on S^2. We then study the effect of inserting a Wilson loop operator in the two-dimensional Yang-Mills theory. Finally, in Sec. 4, we provide some further discussion.

2 Instanton partition function of four-dimensional $\mathcal{N}=2$ gauge theory

In this paper, we are interested in the $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory with $2N$ fundamental hypermultiplets. The Lagrangian and the vacua are parametrized by the coupling constant $q = \exp(2\pi \mathrm{i}\tau)$, the vacuum expectation value $\mathbf{a} = \mathrm{diag}(a_1, \cdots, a_N)$ of the scalar field in the vector multiplet, and the complex masses $\mathbf{m} = \mathrm{diag}(m_1, \cdots, m_{2N})$ of the matter hypermultiplets. We refer to [11] for a detailed analysis and references for the supersymmetric partition function of very general $\mathcal{N}=2$ supersymmetric gauge theories in the Ω-background.

2.1 Partition function in the self-dual Ω-background

Let us first recall the partition function of the four-dimensional $\mathcal{N}=2$ gauge theory in the Ω-background [4]. The Ω-background breaks the translational invariance by deforming the theory in a rotationally covariant way, with parameters ϵ_1, ϵ_2. In the following, we always set $\epsilon_1 = -\hbar$, $\epsilon_2 = \hbar$.

The supersymmetric partition function of $\mathcal{N}=2$ theory consists of three parts: the classical, the one-loop, and the instanton parts,

$$Z(\mathbf{a}, \mathbf{m}, q; \hbar) = Z^{\mathrm{classical}}(\mathbf{a}, q; \hbar)\, Z^{1-\mathrm{loop}}(\mathbf{a}, \mathbf{m}; \hbar)\, Z^{\mathrm{instanton}}(\mathbf{a}, \mathbf{m}, q; \hbar). \quad (1)$$

The classical part is simply

$$Z^{\mathrm{classical}}(\mathbf{a}, q; \hbar) = q^{\frac{1}{2\hbar^2}\sum_{\alpha=1}^{N} a_\alpha^2}. \quad (2)$$

The one-loop part is given as a product of contributions from the vector multiplet and the matter hypermultiplets using the Barnes double gamma function. The one-loop contribution of a vector multiplet is

$$Z^{1-\mathrm{loop}}_{\mathrm{vector}}(\mathbf{a}; \hbar) = \prod_{1 \le i < j \le N} \left[\Gamma_2(a_i - a_j + \hbar | \hbar, -\hbar)\, \Gamma_2(a_i - a_j - \hbar | \hbar, -\hbar)\right]^{-1}, \quad (3)$$

while the one-loop contribution of fundamental hypermultiplets is

$$Z^{1-\mathrm{loop}}_{\mathrm{fund}}(\mathbf{a}, m; \hbar) = \prod_{i=1}^{N} \prod_{f=1}^{2N} \Gamma_2(a_i - m_f | \hbar, -\hbar). \quad (4)$$

The instanton partition function is defined as an equivariant integral over the instanton moduli space. Applying the equivariant localization method, the integral can be reduced to a sum over contributions of the fixed points of the moduli space. There is a one-to-one correspondence between the fixed points and colored partitions $\Lambda = \left(\lambda^{(\alpha)}\right)_{\alpha=1}^{N}$, with each partition $\lambda^{(\alpha)}$ being a weakly decreasing sequence of non-negative integers,

$$\lambda^{(\alpha)} = \left(\lambda_1^{(\alpha)} \geq \lambda_2^{(\alpha)} \geq \cdots \geq \lambda_{\ell\left(\lambda^{(\alpha)}\right)}^{(\alpha)} > \lambda_{\ell\left(\lambda^{(\alpha)}\right)+1}^{(\alpha)} = \cdots = 0\right), \tag{5}$$

whose size is denoted to be $|\lambda^{(\alpha)}| = \sum_i \lambda_i^{(\alpha)}$. Accordingly the instanton partition function becomes a statistical model of random partitions [4],

$$Z^{\text{instanton}}\left(\mathbf{a}, \mathbf{m}, q; \hbar\right) = \sum_{\Lambda} q^{|\Lambda|} \mu_{\Lambda}\left(\mathbf{a}, \mathbf{m}; \hbar\right), \tag{6}$$

where $|\Lambda| = \sum_{\alpha=1}^{N} |\lambda^{(\alpha)}|$. The contribution to the measure of a vector multiplet is given by

$$\mu_{\Lambda \text{vector}}\left(\mathbf{a}; \hbar\right) = \prod_{(\alpha,i)\neq(\beta,j)} \frac{a_\alpha - a_\beta + \hbar\left(\lambda_i^{(\alpha)} - \lambda_j^{(\beta)} + j - i\right)}{a_\alpha - a_\beta + \hbar\left(j - i\right)}, \tag{7}$$

and the contribution to the measure of fundamental hypermultiplets is

$$\begin{aligned} \mu_{\Lambda \text{fund}}\left(\mathbf{a}, \mathbf{m}; \hbar\right) &= \prod_{\alpha=1}^{N} \prod_{f=1}^{2N} \prod_{\square \in \lambda^{(\alpha)}} \left(c_\square - m_f\right) \\ &= \hbar^{2N|\Lambda|} \prod_{\alpha=1}^{N} \prod_{f=1}^{2N} \prod_{i} \frac{\Gamma\left(\frac{a_\alpha - m_f}{\hbar} + 1 + \lambda_i^{(\alpha)} - i\right)}{\Gamma\left(\frac{a_\alpha - m_f}{\hbar} + 1 - i\right)}, \end{aligned} \tag{8}$$

where for each box $\square = (i, j) \in \lambda^{(\alpha)}$, we define its content as

$$c_\square = a_\alpha + \epsilon_1\left(i - 1\right) + \epsilon_2\left(j - 1\right). \tag{9}$$

The contribution to the measure of an antifundamental hypermultiplet with mass m is equal to the contribution to the measure of a fundamental hypermultiplet with mass $-m$ in the self-dual Ω-background.

For the undeformed theory on $\mathbb{R}^4$, we can perturb the theory by adding gauge-invariant chiral operators to the ultraviolet prepotential, while keeping the ultraviolet antiprepotential unchanged,

$$\overline{\mathcal{F}}^{\mathrm{UV}} = \frac{\overline{\tau}}{2}\mathrm{Tr}\overline{\Phi}^2. \tag{10}$$

For example, we can add single-trace operators,

$$\mathcal{F}^{\mathrm{UV}} \to \frac{\tau}{2}\mathrm{Tr}\Phi^2 + \sum_{j=2}^{\infty}\frac{\tau_j}{j}\mathrm{Tr}\Phi^j, \tag{11}$$

which get deformed in the Ω-background. The localization computation still works, and the partition function becomes

$$Z(\mathbf{a},\mathbf{m},q;\tau;\hbar) = Z^{\mathrm{classical}}(\mathbf{a},q;\hbar)\, Z^{1-\mathrm{loop}}(\mathbf{a},\mathbf{m};\hbar)\sum_{\Lambda} q^{|\Lambda|}\mu_{\Lambda}(\mathbf{a},\mathbf{m};\hbar)\exp\left(\frac{1}{\hbar^2}\sum_{j=2}^{\infty}\frac{\tau_j}{j}\mathrm{ch}_j(\mathbf{a},\Lambda)\right) \tag{12}$$

Here $\mathrm{ch}_j(\mathbf{a},\Lambda) = \sum_{\alpha=1}^{N}\mathrm{ch}_j\left(a_\alpha,\lambda^{(\alpha)}\right)$, with

$$\mathrm{ch}_j(a,\lambda) = a^j + \sum_{i=1}^{\infty}\left((a+\hbar(\lambda_i+1-i))^j - (a+\hbar(\lambda_i-i))^j - (a+\hbar(1-i))^j + (a-\hbar i)^j\right). \tag{13}$$

For example,

$$\begin{aligned}\mathrm{ch}_2(a,\lambda) &= a^2 + 2\hbar^2|\lambda|, & (14)\\ \mathrm{ch}_3(a,\lambda) &= a^3 + 6\hbar^2 a|\lambda| + 3\hbar^3\sum_i \lambda_i(\lambda_i+1-2i). & (15)\end{aligned}$$

Multitrace operators can also be added and can be analyzed using the Hubbard-Stratonovich transformation. The full set of gauge-invariant chiral operators can be expressed as

$$\mathcal{F}^{\mathrm{UV}} \to \frac{\tau}{2}\mathrm{Tr}\Phi^2 + \sum_{\vec{k}}^{\infty} t_{\vec{k}}\prod_{j=1}^{\infty}\frac{1}{k_j!}\left(\frac{1}{j}\mathrm{Tr}\Phi^j\right)^{k_j}, \quad \vec{k} = (k_1,k_2,\cdots), \tag{16}$$

and the partition function is deformed to be

$$\begin{aligned}Z(\mathbf{a},\mathbf{m},q;\mathbf{t};\hbar) &= Z^{\mathrm{classical}}(\mathbf{a},q;\hbar)\, Z^{1-\mathrm{loop}}(\mathbf{a},\mathbf{m};\hbar)\times \\ &\times \sum_{\Lambda} q^{|\Lambda|}\mu_{\Lambda}(\mathbf{a},\mathbf{m};\hbar)\exp\left(\frac{1}{\hbar^2}\sum_{\vec{k}}^{\infty} t_{\vec{k}}\prod_{j=1}^{\infty}\frac{1}{k_j!}\left(\frac{1}{j}\mathrm{ch}_j(a,\lambda)\right)^{k_j}\right) \quad (17)\end{aligned}$$

2.2 $\mathcal{Y}$-observable

With the identification of the instanton partition function with a statistical model (6), we can compute the expectation value of observables in the Ω-background as

$$\langle \mathcal{O} \rangle = \frac{\sum_\Lambda q^{|\Lambda|} \mu_\Lambda \mathcal{O}[\Lambda]}{\sum_\Lambda q^{|\Lambda|} \mu_\Lambda}, \tag{18}$$

where $\mathcal{O}[\Lambda]$ is the value of $\mathcal{O}$ at the fixed point labeled by Λ.

An important observable in the analysis of nonperturbative information of four-dimensional $\mathcal{N} = 2$ gauge theory is the $\mathcal{Y}$-observable, which is defined using the gauge-invariant polynomials of the adjoint scalar field ϕ in the vector multiplet, evaluated at the fixed point of the rotational symmetry $SO(4)$,

$$\mathcal{Y}(x) = x^N \exp\left(-\sum_{j=1}^{\infty} \frac{1}{j x^j} \mathrm{Tr}\,(\phi(0))^j \right). \tag{19}$$

Classically, it is given by

$$\mathcal{Y}(x)^{\text{classical}} = \det\left(x - \phi(0)\right) = \prod_{\alpha=1}^{N} \left(x - a_\alpha\right). \tag{20}$$

However, there are quantum corrections due to instantons. Denote the outer and the inner boundaries of the partition λ as $\partial_+\lambda$ and $\partial_-\lambda$, respectively. The value of $\mathcal{Y}(x)$ in the self-dual Ω-background at the fixed point labeled by Λ is [12]

$$\begin{aligned} \mathcal{Y}(x)[\Lambda] &= \prod_{\alpha=1}^{N} \frac{\prod_{\boxplus \in \partial_+\lambda^{(\alpha)}} (x - c_\boxplus)}{\prod_{\boxminus \in \partial_-\lambda^{(\alpha)}} (x - c_\boxminus)} \\ &= \prod_{\alpha=1}^{N} \prod_{i}^{\infty} \frac{x - a_\alpha - \hbar\left(\lambda_i^{(\alpha)} - i + 1\right)}{x - a_\alpha - \hbar\left(\lambda_i^{(\alpha)} - i\right)}. \end{aligned} \tag{21}$$

Notice that the expression (21) is highly redundant, and there can be many cancellations between the numerator and the denominator. For example, the contribution from the box $\left(n+1, \lambda_{n+1}^{(\alpha)} + 1\right) \in \widetilde{\partial_+\lambda^{(\alpha)}}$ cancels the contribution from the box $\left(n, \lambda_n^{(\alpha)}\right) \in \widetilde{\partial_-\lambda^{(\alpha)}}$ for $n > \ell(\lambda^{(\alpha)})$. Hence, $\mathcal{Y}(x)[\Lambda]$ does not change if we truncate the range of the index i to $1 \leq i \leq n$ for an arbitrary integer $n \geq \ell(\lambda^{(\alpha)})$.

2.3 Simplification of partition function

Up to this point we assumed that the expectation values $a_1, \cdots, a_N$ and masses $m_1, \cdots, m_{2N}$ are generic. Then the partition function (6) contains an infinite sum over colored partitions. For a special value of the masses, the partitions Λ that we sum over can be constrained. As a result, the partition function (6) gets simplified.

It is easy to see that if $a_\alpha = m_f$ for some $\alpha \in \{1, 2, \cdots N\}$ and $f \in \{1, 2, \cdots, 2N\}$, then $\lambda^{(\alpha)} = \emptyset$; otherwise (8) is zero. Therefore, if we choose a particular point on the parameter space

$$a_\alpha = m_{2\alpha-1} = m_{2\alpha}, \quad \alpha = 1, \cdots, N, \tag{22}$$

the partitions $\lambda^{(\alpha)} = \emptyset$ for all $\alpha = 1, 2, \cdots, N$, and the instanton partition function is trivially 1. This simplification of the instanton partition function has been known for a long time. Physically, when one of the a_α's is equal to two masses, two of the hypermultiplets become massless, and can be Higgsed so that the $U(N)$ theory with $2N$ flavors is reduced to a $U(N-1)$ theory with $2N-2$ flavors. However, the instanton partition function will not change since it is a Coulomb-branch quantity which is independent of the manipulation on the hypermultiplet side.

Now let us relax the condition (22) a little bit. We still fix

$$a_\alpha = m_{2\alpha-1} = m_{2\alpha}, \quad \alpha = 2, \cdots, N, \tag{23}$$

so that the partitions $\lambda^{(\alpha)} = \emptyset$ for $\alpha = 2, \cdots, N$. We effectively reduce the $U(N)$ gauge theory with $2N$ fundamental hypermultiplets to the $U(1)$ theory with two fundamental hypermultiplets. At the same time, we choose

$$a_1 = m_1 + n\hbar = m_2 + n\hbar, \tag{24}$$

where n is a positive integer. We see from (8) that if $\lambda^{(1)}_{n+1} \geq 1$, then the contribution of the box $\square = (n+1, 1) \in \lambda^{(1)}$ makes $\mu_{\Lambda\text{fund}}$ vanish. Hence, the length of the partition $\lambda^{(1)}$ is at most n. We can set the length of the partition $\lambda^{(1)}$ to be n by adding zeros to the end of the partition if its precise length is less than n. In this case, the measure in the instanton partition function simplifies.

The case $n = 1$ is special, since now $\lambda^{(1)}$ is no longer a two-dimensional partition. The measure of the vector multiplet completely cancels the measure of the fundamental hypermultiplets, and the instanton partition function is

$$Z^{\text{instanton}} = \sum_{\lambda_1^{(1)}=0}^{\infty} q^{\lambda_1^{(1)}} = \frac{1}{1-q}. \tag{25}$$

In the following, we always assume that $n \geq 2$. In this case, the measure of the vector multiplet (7) becomes

$$\begin{aligned}
\mu_{\Lambda\text{vector}} &= \left(\prod_{i\neq j} \frac{\hbar\left(\lambda_i^{(1)} - \lambda_j^{(1)} + j - i\right)}{\hbar\left(j-i\right)}\right)\left(\prod_{\beta=2}^{N}\prod_{i,j}\frac{a_1 - a_\beta + \hbar\left(\lambda_i^{(1)} + j - i\right)}{a_1 - a_\beta + \hbar\left(j-i\right)}\right)^2 \\
&= \left(\prod_{1\leq i<j\leq n}\frac{\lambda_i^{(1)} - \lambda_j^{(1)} + j - i}{j-i}\right)^2 \left(\prod_{i=1}^{n}\frac{\Gamma\left(n+1-i\right)}{\hbar^{\lambda_i^{(1)}}\Gamma\left(n+1+\lambda_i^{(1)}-i\right)}\right)^2 \times \\
&\times \left(\prod_{\beta=2}^{N}\prod_{i=1}^{n}\frac{\Gamma\left(\frac{a_1-a_\beta}{\hbar} - i + 1\right)}{\hbar^{\lambda_i^{(1)}}\Gamma\left(\frac{a_1-a_\beta}{\hbar} - i + \lambda_i^{(1)} + 1\right)}\right)^2,
\end{aligned} \tag{26}$$

while the measure of the fundamental hypermultiplets (8) becomes

$$\begin{aligned}
\mu_{\Lambda\text{fund}} &= \prod_{f=1}^{2N}\prod_{i=1}^{n}\frac{\Gamma\left(\frac{a_1-m_f}{\hbar} + 1 + \lambda_i^{(1)} - i\right)}{\Gamma\left(\frac{a_1-m_f}{\hbar} + 1 - i\right)} \\
&= \hbar^{2N|\lambda^{(1)}|}\left(\prod_{i=1}^{n}\frac{\Gamma\left(n+1+\lambda_i^{(1)}-i\right)}{\Gamma\left(n+1-i\right)}\right)^2 \prod_{\alpha=2}^{N}\left(\prod_{i=1}^{n}\frac{\Gamma\left(\frac{a_1-a_\alpha}{\hbar} + 1 + \lambda_i^{(1)} - i\right)}{\Gamma\left(\frac{a_1-a_\alpha}{\hbar} + 1 - i\right)}\right)^2
\end{aligned} \tag{27}$$

After many cancellations between $\mu_{\Lambda\text{vector}}$ and $\mu_{\Lambda\text{fund}}$, the remaining measure is

$$\mu_\Lambda = \mu_{\Lambda\text{vector}}\mu_{\Lambda\text{fund}} = \left(\prod_{1\leq i<j\leq n}\frac{\lambda_i^{(1)} - \lambda_j^{(1)} + j - i}{j-i}\right)^2. \tag{28}$$

In this case, the $\mathcal{Y}$-observable (21) also simplifies,

$$\begin{aligned}
\mathcal{Y}(x)[\Lambda] &= \frac{\prod_{i=1}^{n+1}\left(x - a_1 - \hbar\left(\lambda_i^{(1)} + 1 - i\right)\right)}{\prod_{i=1}^{n}\left(x - a_1 - \hbar\left(\lambda_i^{(1)} - i\right)\right)} \\
&= \left(x - a_1 + n\hbar\right)\prod_{i=1}^{n}\frac{\left(x - a_1 - \hbar\left(\lambda_i^{(1)} + 1 - i\right)\right)}{\left(x - a_1 - \hbar\left(\lambda_i^{(1)} - i\right)\right)}.
\end{aligned} \tag{29}$$

As we see, at the point (23) (24) of the parameter space, the instanton partition function is independent of the gauge group rank N, and the difference for different N values in the full partition function is an overall constant which is irrelevant to our discussion. Therefore, we shall concentrate on the case $N = 1$ in the following discussion and drop some of the subscripts 1. Notice that the $U(1)$ gauge theory with two fundamental hypermultiplets is nontrivial due to the inexplicit noncommutative deformation.

3 Relation to two-dimensional Yang-Mills theory

In this section, we shall relate the partition function discussed in Sec. 2 to the partition function of two-dimensional Yang-Mills theory on S^2.

3.1 Partition function of two-dimensional Yang-Mills theory

Two-dimensional Yang-Mills theory is an exactly solvable model and has been extensively studied from many different points of view (see [10] for a review). Its partition function on a Riemann surface Σ of genus g is defined as

$$Z_{\Sigma}^{\mathrm{YM2}}\left(\varepsilon, \mathcal{A}(\Sigma), G\right) = \frac{1}{\mathrm{Vol}(G)} \int \mathcal{D}A\mathcal{D}\phi \exp\left(\mathrm{i}\int_{\Sigma} \mathrm{Tr}\phi F_A + \frac{\varepsilon}{2}\int_{\Sigma} d\mu \mathrm{Tr}\phi^2\right), \tag{30}$$

where ε is the coupling constant, $\mathcal{A}(\Sigma)$ is the area of the Riemann surface Σ, and Tr denotes the invariant, negative-definite quadratic form on the Lie algebra $\mathfrak{g}$ of the gauge group G. The partition function (30) can be expressed as a sum over all finite-dimensional irreducible representations R of the gauge group G [9, 13, 14],

$$Z_{\Sigma}^{\mathrm{YM2}}\left(\varrho, G\right) = e^{-\beta(2-2g)-\gamma\varepsilon\mathcal{A}(\Sigma)} \sum_{R} \left(\dim R\right)^{2-2g} \exp\left(-\frac{\varrho}{2} C_2(R)\right), \tag{31}$$

where the prefactor is the regularization-dependent ambiguity, $\dim R$ is the dimension of the representation R, $C_2(R)$ is the quadratic Casimir of the representation R, and $\varrho = \varepsilon\mathcal{A}(\Sigma)$ is the dimensionless coupling constant.

3.2 Matching the parameters

We would like to find the precise relation between the partition function (17) and the partition function of two-dimensional Yang-Mills theory (31), both for the group $SU(n)$ and for the group $U(n)$.

3.2.1 $SU(n)$ theory

For the group $G = SU(n)$, the irreducible representations R are parametrized by the partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0)$. The dimension and the quadratic Casimir of the representation R are

$$\dim R = \prod_{1\leq i<j\leq n} \frac{\lambda_i - \lambda_j + j - i}{j-i}, \tag{32}$$

$$C_2(R) = \sum_{i=1}^{n} \lambda_i \left(\lambda_i - 2i + 1\right) + n|\lambda| - \frac{|\lambda|^2}{n}. \tag{33}$$

We see that both the dimension and the quadratic Casimir are independent of the overall shift of λ's. Therefore, the difference between the summation over $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n \geq 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$ in the partition function is merely an irrelevant overall constant.

To identify the partition function of two-dimensional $SU(n)$ Yang-Mills theory on S^2 with the partition function of the four-dimensional $\mathcal{N} = 2$ $U(1)$ gauge theory with two fundamental hypermultiplets at the degenerate point of the parameter space, we need to set $a = 0$ and turn on operators with couplings $t_{0,1}$, $t_{0,2}$ and $t_{0,0,1}$ in (17). The partition function becomes

$$\begin{aligned}
& Z\left(a=0, m_1=m_2=-n\hbar, q; \tau; \hbar\right) \\
= \; & \Gamma_2\left(n\hbar|\hbar,-\hbar\right)^2 \sum_{\lambda} q^{|\lambda|} \left(\prod_{1\leq i<j\leq n} \frac{\lambda_i - \lambda_j + j - i}{j-i} \right)^2 \times \\
& \times \exp\left\{ \frac{1}{\hbar^2} \left[\frac{t_{0,1}}{2} \mathrm{ch}_2\left(0,\lambda\right) + \frac{t_{0,2}}{8} \left(\mathrm{ch}_2\left(0,\lambda\right)\right)^2 + \frac{t_{0,0,1}}{3} \mathrm{ch}_3\left(0,\lambda\right) \right] \right\}
\end{aligned} \tag{34}$$

$$\begin{aligned}
= \; & \Gamma_2\left(n\hbar|\hbar,-\hbar\right)^2 \sum_{\lambda} q^{|\lambda|} \left(\prod_{1\leq i<j\leq n} \frac{\lambda_i - \lambda_j + j - i}{j-i} \right)^2 \times \\
& \times \exp\left\{ \left[t_{0,1}|\lambda| + \frac{t_{0,2}\hbar^2}{2} |\lambda|^2 + t_{0,0,1}\hbar \sum_{i} \lambda_i \left(\lambda_i + 1 - 2i\right) \right] \right\}.
\end{aligned} \tag{35}$$

Ignoring the unimportant prefactor coming from the one-loop contribution, the partition function is equal to the partition function of two-dimensional Yang-Mills theory on S^2 (31) with gauge group $SU(n)$ when

$$
\begin{aligned}
& \log(q)|\lambda| + t_{0,1}|\lambda| + \frac{t_{0,2}\hbar^2}{2}|\lambda|^2 + t_{0,0,1}\hbar \sum_i \lambda_i \left(\lambda_i + 1 - 2i\right) \\
= & -\frac{\varrho}{2}\left(\sum_{i=1}^{n} \lambda_i \left(\lambda_i - 2i + 1\right) + n|\lambda| - \frac{|\lambda|^2}{n}\right), \qquad (36)
\end{aligned}
$$

which gives

$$
t_{0,1} = -\frac{\varrho n}{2} - \log\left(q\right), \quad t_{0,2} = \frac{\varrho}{n\hbar^2}, \quad t_{0,0,1} = -\frac{\varrho}{2\hbar}. \qquad (37)
$$

3.2.2 $U(n)$ theory

For the group $U(n)$, the irreducible representations $\mathcal{R}$ are parametrized by n integers ($\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n$) without positivity restriction. It is convenient to use the decomposition of the representation $\mathcal{R}$ of $U(n)$ in terms of representation R of $SU(n)$ and the $U(1)$ charge p,

$$
\begin{aligned}
\mu_i &= \lambda_i + r, \quad i = 1, 2, \cdots, n-1 \\
\mu_n &= r, \\
p &= |\lambda| + nr, \quad r \in \mathbb{Z}. \qquad (38)
\end{aligned}
$$

The dimension of representation $\mathcal{R}$ of group $U(n)$ has the same form (32) as the group $SU(n)$, while the quadratic Casimir is given by

$$
C_2\left(\mathcal{R}\right) = C_2\left(R\right) + \frac{p^2}{n} = \sum_{i=1}^{n} \lambda_i \left(\lambda_i - 2i + 1\right) + \left(n + 2r\right)|\lambda| + nr^2. \qquad (39)
$$

To relate the four-dimensional theory to two-dimensional Yang-Mills theory with gauge group $U(n)$, we no longer need to turn on the double-trace operators. Instead, we turn on operators with parameter τ_2 and τ_3 in (12),

$$
\begin{aligned}
& Z\left(a, \mathbf{m}, q; \tau; \hbar\right) \\
= & \Gamma_2\left(n\hbar|\hbar, -\hbar\right)^2 \sum_\lambda \left(\prod_{1\leq i<j\leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}\right)^2 \times \\
& \times \exp\left[\left(\tau_2 + \log(q)\right)\left(\frac{a^2}{2\hbar^2} + |\lambda|\right) + \tau_3\left(\frac{a^3}{3\hbar^2} + 2a|\lambda| + \hbar \sum_i \lambda_i\left(\lambda_i + 1 - 2i\right)\right)\right] \qquad (40)
\end{aligned}
$$

We now set

$$a = m_1 + n\hbar = m_2 + n\hbar = r\hbar, \tag{41}$$

where $r \in \mathbb{Z}$. Ignoring the irrelevant prefactor coming from the one-loop contribution, the partition function becomes

$$\begin{aligned}
& Z\left(r\hbar, (r-n)\hbar, q; \tau; \hbar\right) \\
= & \sum_{\lambda} \left(\prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right)^2 \times \\
& \times \exp\left[(\tau_2 + \log(q)) \left(\frac{r^2}{2} + |\lambda| \right) + \tau_3 \hbar \left(\frac{r^3}{3} + 2r|\lambda| + \sum_i \lambda_i \left(\lambda_i + 1 - 2i \right) \right) \right]
\end{aligned} \tag{42}$$

Now we consider the sum over $r \in \mathbb{Z}$ with a possible weight depending on r,

$$\begin{aligned}
& \sum_{r \in \mathbb{Z}} \exp\left(-f_2 r^2 - f_3 r^3 \right) Z\left(r\hbar, (r-n)\hbar, q; \tau; \hbar\right) \\
= & \sum_{r \in \mathbb{Z}} \sum_{\lambda} \left(\prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right)^2 \times \\
& \times \exp\left[(\tau_2 + \log(q)) \left(\frac{r^2}{2} + |\lambda| \right) + \tau_3 \hbar \left(\frac{r^3}{3} + 2r|\lambda| + \sum_i \lambda_i \left(\lambda_i + 1 - 2i \right) \right) - f_2 r^2 - f_3 r^3 \right]
\end{aligned} \tag{43}$$

which is equal to the partition function of two-dimensional Yang-Mills theory on S^2 (31) with gauge group $U(n)$ when

$$\begin{aligned}
& \tau_3 \hbar \sum_i \lambda_i \left(\lambda_i + 1 - 2i \right) + (\tau_2 + \log(q)) |\lambda| + 2\tau_3 \hbar |\lambda| r + \left(\frac{\tau_2 + \log(q)}{2} - f_2 \right) r^2 + \left(\frac{\tau_3 \hbar}{3} - f_3 \right. \\
= & -\frac{\varrho}{2} \left[\sum_{i=1}^{n} \lambda_i \left(\lambda_i - 2i + 1 \right) + n|\lambda| + 2r|\lambda| + nr^2 \right],
\end{aligned} \quad ($$

which gives that

$$\tau_2 = -\frac{\varrho n}{2} - \log(q), \quad \tau_3 = -\frac{\varrho}{2\hbar}, \quad f_2 = \frac{\varrho n}{4}, \quad f_3 = -\frac{\varrho}{6\hbar}. \tag{45}$$

Therefore, we have the relation

$$\sum_{r \in \mathbb{Z}} \exp\left(-\frac{\varrho n}{4} r^2 + \frac{\varrho}{6\hbar} r^3 \right) Z\left(r\hbar, (r-n)\hbar, q; \tau_2 = -\frac{\varrho n}{2} - \log(q), \tau_3 = -\frac{\varrho}{2\hbar}; \hbar \right) = Z_{S^2}^{\mathrm{YM2}}\left(\varrho, U(n)\right) \tag{46}$$

3.3 Wilson loop operator in two-dimensional Yang-Mills theory

The correspondence was hitherto at the level of the partition functions. We would like to deepen it by studying the Wilson loop operator in the two-dimensional Yang-Mills theory.

Suppose that a loop Γ decomposes S^2 into two disjoint connected components Σ_1 and Σ_2. Associated to the curve Γ we have a representation R_Γ of the gauge group and we define a Wilson loop operator

$$W\left(\Gamma, R_\Gamma\right) = \mathrm{Tr}_{R_\Gamma} P \exp \oint_\Gamma A. \tag{47}$$

The expectation value of the Wilson loop operator $W\left(\Gamma, R_\Gamma\right)$ is given by

$$\begin{aligned} \langle W\left(\Gamma, R_\Gamma\right)\rangle^{\mathrm{YM2}} &= Z_{S^2}^{\mathrm{YM2}}\left(\varepsilon \mathcal{A}\left(\Sigma_1 \cup \Sigma_2\right)\right)^{-1} \sum_{R_1, R_2} \left(\dim R_1\right)\left(\dim R_2\right) \times \\ &\times \exp\left(-\frac{\varepsilon \mathcal{A}(\Sigma_1)}{2} C_2(R_1) - \frac{\varepsilon \mathcal{A}(\Sigma_2)}{2} C_2(R_2)\right) \mathfrak{N}\left(R_1 \otimes R_\Gamma, R_2\right) \end{aligned} \tag{48}$$

where $\mathfrak{N}\left(R_1 \otimes R_\Gamma, R_2\right)$ is the fusion number defined by the decomposition of a tensor product into irreducible representations:

$$R_1 \otimes R_\Gamma = \bigoplus_{R_2} \mathfrak{N}\left(R_1 \otimes R_\Gamma, R_2\right) R_2. \tag{49}$$

In this paper, we are interested in the simple case that R_Γ is the fundamental representation. The fusion number is 1 if the Young diagram associated to R_2 is obtained by adding a box in the Young diagram associated to R_1, and 0 otherwise. We can make an analogy with (18) and write

$$\langle W\left(\Gamma, \square\right)\rangle^{\mathrm{YM2}} = Z_{S^2}^{\mathrm{YM2}}\left(\varepsilon \mathcal{A}\left(\Sigma_1 \cup \Sigma_2\right)\right)^{-1} \sum_{R} \left(\dim R\right)^2 \exp\left(-\frac{\varepsilon \mathcal{A}(\Sigma_1 \cup \Sigma_2)}{2} C_2(R)\right) W\left(\Gamma, \square\right)[R]. \tag{50}$$

Here $W\left(\Gamma, \square\right)[R]$ is the value of $W\left(\Gamma, \square\right)$ evaluated at the representation R,

$$W\left(\Gamma, \square\right)[R] = \sum_{R_+ = R \otimes \square} \frac{\dim R_+}{\dim R} \exp\left(-\frac{\varepsilon \Delta \mathcal{A}}{2}\left(C_2(R_+) - C_2(R)\right)\right), \tag{51}$$

where $\Delta \mathcal{A} = \mathcal{A}(\Sigma_2) - \mathcal{A}(\Sigma_1)$.

First we consider the case when the gauge group is $SU(n)$. Suppose that the Young diagram associated to the representation R is $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$, and becomes the Young diagram associated to the representation R_+ by adding a box in the lth row. From (32) and (33), we obtain that

$$\frac{\dim R_+}{\dim R} = \prod_{i\neq r} \frac{\lambda_i - (\lambda_l + 1) + l - i}{\lambda_i - \lambda_l + l - i}, \tag{52}$$

$$C_2(R_+) - C_2(R) = 2\left(\lambda_l - l + 1\right) + \frac{n^2 - 1 - 2|\lambda|}{n}. \tag{53}$$

It is interesting to notice that

$$\mathrm{Res}_{x=a_1+\hbar\left(\lambda_l^{(1)}+1-l\right)} \left(\frac{x + n\hbar}{\mathcal{Y}(x)[\Lambda]}\right) = \frac{\dim R_+}{\dim R}. \tag{54}$$

The appearance of the $\mathcal{Y}$-observable should not be surprising. Recall that the physical meaning of the $\mathcal{Y}$-observable is to add or remove a pointlike instanton. Hence, the four-dimensional operator corresponding to $W\left(\Gamma, \Box\right)[R]$ is

$$\frac{1}{2\pi \mathrm{i}} \oint dx \frac{x + n\hbar}{\mathcal{Y}(x)[\Lambda]} e^{-\varepsilon \Delta \mathcal{A} x} \exp\left(-\varepsilon \Delta \mathcal{A} \left(\frac{n^2-1}{2n} - \frac{1}{n} q \frac{\partial}{\partial q}\right)\right). \tag{55}$$

For the case of $U(n)$, the equations (52) and (54) still hold. The difference between the Casimirs now is simpler

$$C_2(\mathcal{R}_+) - C_2(\mathcal{R}) = 2\left(\lambda_l - l + 1\right) + n + 2r. \tag{56}$$

Hence, the four-dimensional operator corresponding to $W\left(\Gamma, \Box\right)[R]$ is now

$$\frac{1}{2\pi \mathrm{i}} \oint dx \frac{x + n\hbar}{\mathcal{Y}(x)[\Lambda]} \exp\left(-\varepsilon \Delta \mathcal{A} \left(x + \frac{n}{2}\right)\right). \tag{57}$$

4 Discussions

In this paper, we study a generalization of the correspondence between four-dimensional $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory with $2N-2$ fundamental hypermultiplets and A-type topological string theory on S^2. In our correspondence, the partition function of the four-dimensional $U(N)$ gauge theory with $2N$ fundamental hypermultiplets at a suitable nongeneric point of the parameter space is related to the partition function of two-dimensional

Yang-Mills theory on S^2. We also study the expectation value of a Wilson loop operator in the fundamental representation in the two-dimensional Yang-Mills theory. The corresponding operator in the four-dimensional theory can be found for the fundamental representation. It appears that the correspondence is more complicated than the old correspondence in [6, 7, 8].

The relation between four-dimensional supersymmetric gauge theory and two-dimensional Yang-Mills theory on S^2 was discovered in many other places. For example, the supersymmetric Wilson loops restricted to an S^2 submanifold of four-dimensional space in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [15, 16] can be consistently truncated to a two-dimensional Yang-Mills theory on S^2. However, the number of supersymmetry in four-dimensional gauge theory and the way to identify the Wilson loop operator in their work is quite different from our story. One other similar relation is the identification of the superconformal index of a class of four-dimensional $\mathcal{N} = 2$ theories with a deformation of two-dimensional Yang-Mills theory on punctured Riemann surfaces [17]. However, in their correspondence, the four-dimensional gauge theory is a complicated quiver theory, and there are necessarily a number of punctures in the Riemann surface. Hence, all these old relations are indeed different from ours.

So far, the correspondence discussed in this paper is only a mathematical coincidence of two different partition functions. It will be nice if one can embed our correspondence into a string theory setup and provide a physical interpretation of the results we have got. The procedure (23) and (24) is similar to the approach to introduce surface operators or vortices in the previous discussions of AGT correspondence, and one may effectively describe the surface operator as some two-dimensional gauge theory. One may wonder whether the two-dimensional Yang-Mills theory we discuss is somehow related to the gauge theory in this construction. However, we would like to point out that this is not the case. Notice that if we want to have a surface operator in a $U(N)$ gauge theory, we can consider a two-dimensional gauge theory coupled to the $U(N)$ gauge theory, or we can start with a $U(N) \times U(N')$ theory and tune the Coulomb moduli in the $U(N')$ part of the theory. Furthermore, in this case, the two-dimensional gauge theory lives inside the spacetime of the four-dimensional gauge theory. Instead, we suggest that the proper physical origin of our result should come from the compactification of little string theory. The four-dimensional gauge theory and the two-dimensional Yang-Mills theory live in the perpendicular spaces. This is also the case for the old correspondence between supersymmetric gauge theory and topological string theory [6, 7].

There are many open problems which remain to be answered.

First, we only studied the Wilson loop operator which is inserted in the two-dimensional Yang-Mills theory in the fundamental representation. We can insert Wilson loop operators in arbitrary representations of the gauge group and define a quantity similar to (51),

$$W\left(\Gamma, R_\Gamma\right)[R] = \sum_{R_+} \frac{\dim R_+}{\dim R} \exp\left(-\frac{\varepsilon \Delta \mathcal{A}}{2}\left(C_2(R_+) - C_2(R)\right)\right) \mathfrak{N}\left(R\otimes R_\Gamma, R_+\right). \tag{58}$$

Now $\mathfrak{N}\left(R\otimes R_\Gamma, R_+\right)$ is more complicated. What are the corresponding four-dimensional operators?

Second, we only consider the first nontrivial simplification of the instanton partition function at a nongeneric point of the parameter space in this paper. It is natural to extend our analysis to the cases

$$a_1 = m_1 + n_1\hbar = m_2 + n_1\hbar, \quad a_2 = m_3 + n_2\hbar = m_4 + n_2\hbar, \quad a_3 = m_5 = m_6, \cdots, a_N = m_{2N-1} = m_{2N} \tag{59}$$

Then the length of the partition $\lambda^{(1)}$ is at most n_1, the length of the partition $\lambda^{(2)}$ is at most n_2, while all the other partitions are empty. Similar to the case discussed in the paper, there are many cancellations in the measure. The resulting measure is

$$\begin{aligned} \mu \;=\; & \left(\prod_{1\le i<j\le n_1} \frac{\lambda_i^{(1)} - \lambda_j^{(1)} + j - i}{j-i}\right)^2 \left(\prod_{1\le i<j\le n_2} \frac{\lambda_i^{(2)} - \lambda_j^{(2)} + j - i}{j-i}\right)^2 \times \\ \times\; & \left(\prod_{i=1}^{n_1}\prod_{j=1}^{n_2} \frac{a_1 - a_2 + \hbar\left(\lambda_i^{(1)} - \lambda_j^{(2)} + j - i\right)}{a_1 - a_2 + \hbar\,(j-i)}\right)^2 \times \\ \times\; & \left(\prod_{i=1}^{n_1} \frac{\Gamma\left(\frac{a_1-a_2}{\hbar} + n_2 + 1 + \lambda_i^{(1)} - i\right)}{\Gamma\left(\frac{a_1-a_2}{\hbar} + n_2 + 1 - i\right)}\right)^2 \left(\prod_{i=1}^{n_2} \frac{\Gamma\left(\frac{a_2-a_1}{\hbar} + n_1 + 1 + \lambda_i^{(2)} - i\right)}{\Gamma\left(\frac{a_2-a_1}{\hbar} + n_1 + 1 - i\right)}\right)^2 \end{aligned} \tag{60}$$

What is the physical interpretation of this partition function?

Part III

Dynamics of two-dimensional $\mathcal{N} = (2,\ 2)$ theories with semichiral superfields

5 Introduction

There has been a lot of work on two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric theories. For a review with numerous references, see [18]. Compared to the well-known two-dimensional $\mathcal{N} = (2, 2)$ chiral and twisted chiral superfields, semichiral superfields are less studied in the literature.

Semichiral superfields were first introduced in Ref. [19]. In Ref. [20] it was proved that to have a complete description of off-shell two-dimensional $\mathcal{N} = (2, 2)$ supersymmetry, one needs chiral, twisted chiral and semichiral superfields. However, except for some works, e.g. Ref. [21], the majority of previous works on semichiral superfields focused on mathematical interpretations of sigma models at the classical level, while leaving many problems of quantum dynamics untouched. We shall fill this gap in a series of papers. As a first step, our goal in the present paper is not to study the most general two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric theories with semichiral superfields. We will only consider a special type of models, namely the gauged linear sigma model (GLSM) [22]. More general cases will be discussed in subsequent papers [23]. We also limit ourselves to theories on a flat worldsheet. Theories on a sphere were studied in a separate paper [24].

The first elementary question of a supersymmetric model is whether supersymmetry is spontaneously broken or not. To answer this question, we need to compute the Witten index [25], which gives the number of zero energy bosonic vacuum states minus the number of zero energy fermionic vacuum states. It is important because if supersymmetry is spontaneously broken then there are no zero energy ground states and the Witten index vanishes. It is also useful because it is a quasi-topological quantity, which depends only on F-terms and not on D-terms in the Lagrangian, and is exactly computable. In two dimensions, we can compute a more refined invariant, the elliptic genus, which can give more information about the vacuum structure of the theory. From the purely mathematical point of view, the elliptic genus of a sigma model captures important topological information of the target space.

Next, we want to go beyond the vacuum states. We assume that the vector fields used to gauge the semichiral superfields are ordinary vector fields, which can be equivalently organized using twisted chiral superfields.[1] On the Coulomb branch, the gauge group G is broken down to its Cartan subgroup $\mathrm{U}(1)^r$. In addition, we can turn on generic twisted masses for all the matter fields so that the matter fields become massive. At energies below the mass scales in the theory, we can integrate out both W-bosons and matter fields, and the low energy effective theory is described by a model with only twisted chiral superfields. It is still beyond our scope to compute exactly the full low energy effective action. However, thanks to the special properties of supersymmetry, we can compute the effective twisted superpotential $\widetilde{\mathcal{W}}^{\mathrm{eff}}$ exactly. This quantity plays an essential role in the discussion of the sigma model/Landau-Ginzberg models correspondence [22], and determines the (twisted) chiral ring structure of the theory.

Recently, $\widetilde{\mathcal{W}}^{\mathrm{eff}}$ also appeared in the Bethe/gauge correspondence [26, 27, 28]. In this remarkable correspondence, $\widetilde{\mathcal{W}}^{\mathrm{eff}}$ computed from a two-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theory is identified with the Yang-Yang function $\mathcal{Y}$ of a quantum integrable system. In the previous discussions, the matter multiplets are always built using chiral superfields. It is natural to ask whether semichiral superfields can give new contributions.

The organization of this paper is as follows. In section 6 we review the two-dimensional $\mathcal{N} = (2,2)$ supersymmetric GLSMs with semichiral superfields. In section 7 we compute the elliptic genus. After a general discussion, we work out two important examples, namely the elliptic genus of the Eguchi-Hanson space and the Taub-NUT space. The sigma models built from semichiral superfields give exactly the same results as those without using the semichiral superfields. In section 8 we compute the low energy effective twisted superpotential. We find that the contribution of the semichiral superfield vanishes, even with generic twisted masses. Finally, in section 9 we give a conclusion and discuss possible directions for future work. Since computations with semichiral superfields are unavoidably lengthy, we put some details in the appendices.

[1] Writing the vector multiplet as a twisted chiral superfield, the imaginary part of the highest component is the field strength of the vector field.

6 Two-dimensional $\mathcal{N} = (2,2)$ supersymmetry

In this section, in order to be self-contained we review two-dimensional $\mathcal{N} = (2,2)$ supersymmetric theories. The natural language to describe the theories we will study in a compact form is the two-dimensional $\mathcal{N} = (2,2)$ superspace (see Appendix A). Some detailed formulae written in components are collected in Appendix B.

6.1 Supersymmetric multiplets

Using the two-dimensional $\mathcal{N} = (2,2)$ superspace, we can describe all possible choices of superfields which can appear in a general supersymmetric model. We will be brief in the discussion of the well understood multiplets and focus on semichiral multiplets.

The basic matter multiplet is described by a chiral superfield Φ, which contains a scalar ϕ, a fermion $\psi_{\pm\Phi}$ and an auxiliary field F_Φ. [2] It is defined by the condition

$$\overline{\mathbb{D}}_\pm \Phi = 0\,. \tag{61}$$

Similarly we can define its conjugate to be an anti-chiral superfield $\overline{\Phi}$. Indeed, chiral superfields can be obtained by dimensional reduction from $\mathcal{N} = 1$ chiral superfields in four dimensions.

The basic vector multiplet contains a real gauge field A_μ, a complex scalar $\widehat{\sigma}$, two Weyl fermions $\lambda_\pm$ and an auxiliary real scalar D. It can also be obtained via dimensional reduction from the four-dimensional vector multiplet.

Although the $\mathcal{N} = (2,2)$ supersymmetry algebra in two dimensions can be obtained by dimensional reduction from $\mathcal{N} = 1$ supersymmetry algebra in four dimensions, not all superfields in two dimensions can be obtained simply via dimensional reduction from four dimensions. An important superfield which is unique in two dimensions is the twisted chiral superfield Σ, defined by the conditions

$$\overline{\mathbb{D}}_+ \Sigma = \mathbb{D}_- \Sigma = 0\,. \tag{62}$$

Similarly we can define a twisted anti-chiral superfield $\overline{\Sigma}$. The components of a vector multiplet can be reorganized into a twisted chiral superfield Σ in

[2]Here we add a subscript Φ to distinguish them from components of semichiral superfields.

the following way:

$$
\begin{aligned}
\Sigma| &= \widehat{\sigma}\,, \quad \mathbb{D}_+\Sigma| = i\overline{\lambda}_+\,, \quad \overline{\mathbb{D}}_-\Sigma| = i\lambda_-\,, \quad \overline{\mathbb{D}}_-\mathbb{D}_+\Sigma| = D - iF_{01}\,, \\
\overline{\Sigma}| &= \overline{\widehat{\sigma}}\,, \quad \overline{\mathbb{D}}_+\overline{\Sigma}| = -i\lambda_+\,, \quad \mathbb{D}_-\overline{\Sigma}| = -i\overline{\lambda}_-\,, \quad \mathbb{D}_-\overline{\mathbb{D}}_+\overline{\Sigma}| = -(D + iF_{01})
\end{aligned} \tag{63}
$$

When we construct models with gauge fields, it is more convenient to use the covariant approach, in which the gauge connections are incorporated in the supercovariant derivatives. Accordingly, the anticommutation relations of the supercovariant derivatives are modified as follows

$$
\{\mathcal{D}_\pm, \overline{\mathcal{D}}_\pm\} = -2iD_\pm\,, \quad \{\overline{\mathcal{D}}_+, \mathcal{D}_-\} \equiv \Sigma\,, \quad \{\mathcal{D}_+, \overline{\mathcal{D}}_-\} \equiv \overline{\Sigma}\,, \tag{64}
$$

where $D_\pm \equiv \partial_\pm + A_\pm$ is the gauge covariant derivative, and Σ is the field strength superfield, which is twisted chiral.

However, this is not the end of the story. The remaining building blocks are semichiral superfields. The left-semichiral and the right-semichiral multiplets are defined by

$$
\overline{\mathbb{D}}_+\mathbb{X}_L = 0\,, \quad \overline{\mathbb{D}}_-\mathbb{X}_R = 0\,, \tag{65}
$$

and similarly, we have for their conjugates

$$
\mathbb{D}_+\overline{\mathbb{X}}_L = 0\,, \quad \mathbb{D}_-\overline{\mathbb{X}}_R = 0\,. \tag{66}
$$

In order to have a better understanding of these semichiral superfields, we will expand the superfields and write down their components. It is convenient to treat the left-semichiral and the right-semichiral multiplets simultaneously by imposing a weaker constraint:

$$
\overline{\mathbb{D}}_+\overline{\mathbb{D}}_-\mathbb{X} = 0\,, \quad \mathbb{D}_+\mathbb{D}_-\overline{\mathbb{X}} = 0\,. \tag{67}
$$

Then we define the components

$$
\begin{aligned}
&X = \mathbb{X}|\,, \quad \psi_\pm \equiv \mathbb{D}_\pm\mathbb{X}|\,, \quad \overline{\chi}_\pm = \overline{\mathbb{D}}_\pm\mathbb{X}|\,, \quad F \equiv \mathbb{D}_+\mathbb{D}_-\mathbb{X}|\,, \\
&M_{-+} = \mathbb{D}_+\overline{\mathbb{D}}_-\mathbb{X}|\,, \quad M_{+-} = \mathbb{D}_-\overline{\mathbb{D}}_+\mathbb{X}|\,, \quad M_{\pm\pm} = \mathbb{D}_\pm\overline{\mathbb{D}}_\pm\mathbb{X}|\,, \quad \overline{\eta}_\pm = \mathbb{D}_+\mathbb{D}_-\overline{\mathbb{D}}_\pm\mathbb{X}|\,,
\end{aligned} \tag{68}
$$

and

$$
\begin{aligned}
&\overline{X} = \overline{\mathbb{X}}|\,, \quad \overline{\psi}_\pm = \overline{\mathbb{D}}_\pm\overline{\mathbb{X}}|\,, \quad \chi_\pm = \mathbb{D}_\pm\overline{\mathbb{X}}|\,, \quad \overline{F} = \overline{\mathbb{D}}_+\overline{\mathbb{D}}_-\overline{\mathbb{X}}|\,, \\
&\overline{M}_{-+} = \overline{\mathbb{D}}_+\mathbb{D}_-\overline{\mathbb{X}}|\,, \quad \overline{M}_{+-} = \overline{\mathbb{D}}_-\mathbb{D}_+\overline{\mathbb{X}}|\,, \quad \overline{M}_{\pm\pm} = \overline{\mathbb{D}}_\pm\mathbb{D}_\pm\overline{\mathbb{X}}|\,, \quad \eta = \overline{\mathbb{D}}_+\overline{\mathbb{D}}_-\mathbb{D}_\pm\overline{\mathbb{X}}|\,.
\end{aligned} \tag{69}
$$

We then impose the constraints for each multiplet, and some component fields should vanish,

$$
\begin{aligned}
\text{For } \mathbb{X}_L: &\quad \chi_+ = M_{+-} = M_{++} = \eta_+ = 0\,, \\
\text{For } \mathbb{X}_R: &\quad \chi_- = M_{-+} = M_{--} = \eta_- = 0\,, \\
\text{For } \overline{\mathbb{X}}_L: &\quad \overline{\chi}_+ = \overline{M}_{+-} = \overline{M}_{++} = \overline{\eta}_+ = 0\,, \\
\text{For } \overline{\mathbb{X}}_R: &\quad \overline{\chi}_- = \overline{M}_{-+} = \overline{M}_{--} = \overline{\eta}_- = 0\,.
\end{aligned} \tag{70}
$$

6.2 Gauged linear sigma models

It was shown in Ref. [20] that the most general two-dimensional $\mathcal{N} = (2,2)$ GLSM can be constructed using chiral, twisted chiral and semichiral superfields. The GLSM with chiral and twisted chiral superfields has been exploited at length in the literature. Hence, we will focus on the indispensable but poorly understood building block, the action with semichiral superfields. To obtain a gauged linear linear model with physical kinetic terms, one needs both left-semichiral and right-semichiral superfields simultaneously. Models with only left-semichiral or only right-semichiral superfields turn out to be topological.

In this paper, we gauge the semichiral superfields using the constrained semichiral vector multiplets [29], which will be reviewed in the following. Let us first discuss using the semichiral vector multiplet to gauge the semichiral superfields. Here we only consider the abelian case, and the nonabelian case is discussed in Ref. [30]. An abelian semichiral vector multiplet can be described by three real vector superfields (V_L, V_R, V') [31]. If we define

$$
\mathbb{V} \equiv \frac{1}{2}(-V' + i(V_L - V_R))\,, \quad \widetilde{\mathbb{V}} \equiv \frac{1}{2}(-V' + i(V_L + V_R))\,, \tag{71}
$$

the action for a pair of semichiral superfields is

$$
S = \int d^2x\, d^4\theta\, K\,, \tag{72}
$$

where

$$
K = \overline{\mathbb{X}}_L e^{V_L} \mathbb{X}_L + \overline{\mathbb{X}}_R e^{V_R} \mathbb{X}_R + \alpha \overline{\mathbb{X}}_L e^{i\overline{\widetilde{\mathbb{V}}}} \mathbb{X}_R + \alpha \overline{\mathbb{X}}_R e^{-i\widetilde{\mathbb{V}}} \mathbb{X}_L\,, \tag{73}
$$

with $|\alpha| > 1$. This action is invariant under the gauge transformations:

$$
\delta\mathbb{X}_L = e^{i\Lambda_L}\mathbb{X}_L\,, \quad \delta\mathbb{X}_R = e^{i\Lambda_R}\mathbb{X}_R\,, \tag{74}
$$

$$\delta V_L = i(\overline{\Lambda}_L - \Lambda_L)\,, \quad \delta V_R = i(\overline{\Lambda}_R - \Lambda_R)\,, \quad \delta V' = \Lambda_R + \overline{\Lambda}_R - \Lambda_L - \overline{\Lambda}_L\,, \tag{75}$$

or equivalently,

$$\delta \mathbb{V} = \Lambda_L - \Lambda_R\,, \quad \delta \widetilde{\mathbb{V}} = \Lambda_L - \overline{\Lambda}_R\,. \tag{76}$$

To review the constrained semichiral vector multiplet, we first see that one can define two independent gauge invariant field strengths for the semichiral vector multiplet

$$\mathbb{F} \equiv \overline{\mathbb{D}}_+ \overline{\mathbb{D}}_- \mathbb{V}\,, \quad \widetilde{\mathbb{F}} \equiv \overline{\mathbb{D}}_+ \mathbb{D}_- \widetilde{\mathbb{V}}\,. \tag{77}$$

Here $\mathbb{F}$ is a chiral superfield, and $\widetilde{\mathbb{F}}$ is a twisted chiral superfield. The constrained semichiral vector multiplet can be viewed as a semichiral vector multiplet [29] with an additional term:

$$\int d^2\theta\, \widehat{\Phi}\mathbb{F} + c.c.\,, \tag{78}$$

where $\widehat{\Phi}$ is a chiral Lagrange multiplier, and it imposes the constraint

$$\mathbb{F} = 0\,. \tag{79}$$

Since this additional term is a F-term, which is SUSY exact, it does not affect the result of localization, as long as it does not introduce some additional constraints for instance on the R-charges. Therefore, in many cases we can use the constrained semichiral vector multiplet to replace the vector multiplet without changing the result of localization.

We can perform a partial gauge fixing $V' = V_L - V_R = 0$; this leaves just a chiral gauge invariance as a residual gauge invariance. The theory given in Eq. (73) then becomes

$$K = \overline{\mathbb{X}}_L e^V \mathbb{X}_L + \overline{\mathbb{X}}_R e^V \mathbb{X}_R + \alpha \overline{\mathbb{X}}_L e^V \mathbb{X}_R + \alpha \overline{\mathbb{X}}_R e^V \mathbb{X}_L\,, \tag{80}$$

The vector superfield V can be viewed as a constrained semichiral vector multiplet after partially gauge fixing the full semichiral gauge freedom, and $\mathbb{X}_L$ and $\mathbb{X}_R$ have the same gauge charge. We can expand the action into component fields. The result is quite lengthy, and is written down in Appendix B for interested readers. To summarize, it is more natural to use the semichiral vector multiplet to gauge the semichiral multiplets, but after a partial gauge fixing the constrained semichiral vector multiplet is equivalent to an ordinary vector multiplet. In this sense, one can also use the ordinary vector multiplet to gauge the semichiral multiplets. Hence, the ordinary vector multiplet is a special case of the constrained semichiral vector multiplet.

Now we generalize the above discussion to a theory with gauge group $\mathrm{U}(1)^N$ and N_F flavors. One can turn on twisted mass deformations for the model with flavor symmetry. To add the most general twisted masses, one first gauges the flavor symmetry using the semichiral vector superfield and then sets the scalar component of the semichiral vector superfield to a nonzero constant value, with other components of the semichiral vector superfield vanishing. Supersymmetry is not broken by twisted masses. If we write explicit color and flavor indices, we have

$$\begin{aligned} K = & \overline{\mathbb{X}}^L_{a,i} \left(e^V\right)^{ab} \left(e^{V_L}\right)^{ij} \mathbb{X}^L_{b,j} + \overline{\mathbb{X}}^R_{a,i} \left(e^V\right)^{ab} \left(e^{V_R}\right)^{ij} \mathbb{X}^R_{b,j} \\ & + \alpha \left[\overline{\mathbb{X}}^L_{a,i} \left(e^V\right)^{ab} \left(e^{i\overline{\widetilde{\mathbb{V}}}}\right)^{ij} \mathbb{X}^R_{b,j} + \overline{\mathbb{X}}^R_{a,i} \left(e^V\right)^{ab} \left(e^{-i\widetilde{\mathbb{V}}}\right)^{ij} \mathbb{X}^L_{b,j}\right] , \end{aligned} \tag{81}$$

where $a, b = 1, \cdots, N$, $i, j = 1, \cdots, N_F$.

7 Elliptic genus

The elliptic genus can be computed both using the Hamiltonian formalism [32] and the path integral formalism [33, 34, 35]. In this section, we will compute the elliptic genus of the GLSM with semichiral superfields using both methods. Our discussion will be restricted to the Abelian GLSM.

7.1 Hamiltonian formalism

The elliptic genus is defined in the Hamiltonian formalism as a refined Witten index,

$$Z = \mathrm{Tr}_{\mathrm{RR}}(-1)^F q^{H_L} \overline{q}^{H_R} y^J \prod_a x_a^{K_a} , \tag{82}$$

where the trace is taken in the RR sector, in which fermions have periodic boundary conditions, and F is the fermion number. In Euclidean signature, $H_L = \frac{1}{2}(H + iP)$ and $H_R = \frac{1}{2}(H - iP)$ are the left- and the right-moving Hamiltonians. J and K_a are the R-symmetry and the a-th flavor symmetry generators, respectively. It is standard to also define

$$q \equiv e^{2\pi i\tau} , \quad x_a \equiv e^{2\pi i u_a} , \quad y \equiv e^{2\pi i z} . \tag{83}$$

If $u_a = z = 0$ the elliptic genus reduces to the Witten index, and computes the Euler characteristic of the target space if there is a well-defined geometric description.

The contributions from different multiplets can be computed independently, and we will only consider the unexplored contribution from the semichiral multiplet. As we have seen in Appendix B, the physical component fields of the semichiral superfield $\mathbb{X}$ are two complex scalars X_L and X_R, and spinors $\psi'_\pm$, χ^L_- and χ^R_+. All fields have the same flavor symmetry charge Q. The R-charges of $(X_L, X_R, \psi'_+, \psi'_- \chi^L_-, \chi^R_+)$ are $(\frac{R}{2}, \frac{R}{2}, \frac{R}{2}-1, \frac{R}{2}, \frac{R}{2}, \frac{R}{2}+1)$.

Let us consider the fermionic zero modes first. We denote the zero modes of ψ'_+ and $\overline{\psi}'_+$ as $\psi'_{+,0}$ and $\overline{\psi}'_{+,0}$, respectively. They satisfy

$$\{\psi'_{+,0}\,,\,\overline{\psi}'_{+,0}\} = 1\,, \tag{84}$$

which can be represented in the space spanned by $|\downarrow\rangle$ and $|\uparrow\rangle$ with

$$\psi'_{+,0}\,|\downarrow\rangle = |\uparrow\rangle\,, \quad \overline{\psi}'_{+,0}\,|\uparrow\rangle = |\downarrow\rangle\,. \tag{85}$$

One of $|\downarrow\rangle$ and $|\uparrow\rangle$ can be chosen to be bosonic, while the other is fermionic. Under the $U(1)_R$ the zero modes transform as

$$\psi'_{+,0} \to e^{-i\pi z(\frac{R}{2}-1)}\psi'_{-,0}\,, \quad \overline{\psi}'_{+,0} \to e^{i\pi z(\frac{R}{2}-1)}\overline{\psi}'_{-,0}\,, \tag{86}$$

while under $U(1)_f$ they transform as

$$\psi'_{+,0} \to e^{-i\pi uQ}\psi'_{-,0}\,, \quad \overline{\psi}'_{+,0} \to e^{i\pi uQ}\overline{\psi}'_{-,0}\,, \tag{87}$$

These two states contribute a factor

$$e^{-i\pi z(\frac{R}{2}-1)}\,e^{-i\pi uQ} - e^{i\pi z(\frac{R}{2}-1)}\,e^{i\pi uQ} \tag{88}$$

to the elliptic genus. Similarly, the contributions of the other zero modes are

$$\begin{aligned}
(\psi'_{-,0}, \overline{\psi}'_{-,0}): &\quad e^{i\pi z\frac{R}{2}}\,e^{i\pi uQ} - e^{-i\pi z\frac{R}{2}}\,e^{-i\pi uQ}\,,\\
(\chi^L_{-,0}, \overline{\chi}^L_{-,0}): &\quad e^{i\pi z\frac{R}{2}}\,e^{i\pi uQ} - e^{-i\pi z\frac{R}{2}}\,e^{-i\pi uQ}\,,\\
(\chi^R_{+,0}, \overline{\chi}^R_{+,0}): &\quad e^{-i\pi z(\frac{R}{2}+1)}\,e^{-i\pi uQ} - e^{i\pi z(\frac{R}{2}+1)}\,e^{i\pi uQ}\,.
\end{aligned} \tag{89}$$

The contributions from the bosonic zero modes are relatively simple. They are

$$\frac{1}{\left[\left(1 - e^{i\pi z\frac{R}{2}}\,e^{i\pi uQ}\right)\cdot\left(1 - e^{-i\pi z\frac{R}{2}}\,e^{-i\pi uQ}\right)\right]^2}\,. \tag{90}$$

Bringing all the factors together, we obtain the zero mode part of the elliptic genus:

$$\frac{\left[1-e^{i\pi(R-2)z}\,e^{2i\pi uQ}\right]\cdot\left[1-e^{i\pi(R+2)z}\,e^{2i\pi uQ}\right]}{\left(1-e^{i\pi Rz}\,e^{2i\pi uQ}\right)^2}=\frac{\left(1-y^{\frac{R}{2}-1}x^Q\right)\cdot\left(1-y^{\frac{R}{2}+1}x^Q\right)}{(1-y^{\frac{R}{2}}x^Q)^2}\,. \tag{91}$$

We then consider the nonzero modes. The contribution from the fermionic sector $(\psi'_\pm,\chi^L_-,\chi^R_+)$ is

$$\begin{aligned}\prod_{n=1}^{\infty}&\left(1-q^ne^{2i\pi z(\frac{R}{2}-1)}\,e^{2i\pi uQ}\right)\cdot\left(1-q^ne^{-2i\pi z(\frac{R}{2}-1)}\,e^{-2i\pi uQ}\right)\\&\cdot\left(1-\overline{q}^ne^{2i\pi z\frac{R}{2}}\,e^{2i\pi uQ}\right)\cdot\left(1-\overline{q}^ne^{-2i\pi z\frac{R}{2}}\,e^{-2i\pi uQ}\right)\\&\cdot\left(1-q^ne^{2i\pi z(\frac{R}{2}+1)}\,e^{2i\pi uQ}\right)\cdot\left(1-q^ne^{-2i\pi z(\frac{R}{2}+1)}\,e^{-2i\pi uQ}\right)\\&\cdot\left(1-\overline{q}^ne^{2i\pi z\frac{R}{2}}\,e^{2i\pi uQ}\right)\cdot\left(1-\overline{q}^ne^{-2i\pi z\frac{R}{2}}\,e^{-2i\pi uQ}\right)\,,\end{aligned} \tag{92}$$

while the contribution from the bosonic sector (X_L,X_R) is

$$\begin{aligned}\prod_{n=1}^{\infty}&\frac{1}{\left(1-q^ne^{2i\pi z\frac{R}{2}}\,e^{2i\pi uQ}\right)\cdot\left(1-q^ne^{-2i\pi z\frac{R}{2}}\,e^{-2i\pi uQ}\right)}\\&\cdot\frac{1}{\left(1-\overline{q}^ne^{2i\pi z\frac{R}{2}}\,e^{2i\pi uQ}\right)\cdot\left(1-\overline{q}^ne^{-2i\pi z\frac{R}{2}}\,e^{-2i\pi uQ}\right)}\\&\cdot\frac{1}{\left(1-q^ne^{2i\pi z\frac{R}{2}}\,e^{2i\pi uQ}\right)\cdot\left(1-q^ne^{-2i\pi z\frac{R}{2}}\,e^{-2i\pi uQ}\right)}\\&\cdot\frac{1}{\left(1-\overline{q}^ne^{2i\pi z\frac{R}{2}}\,e^{2i\pi uQ}\right)\cdot\left(1-\overline{q}^ne^{-2i\pi z\frac{R}{2}}\,e^{-2i\pi uQ}\right)}\,.\end{aligned} \tag{93}$$

Hence, the nonzero modes contribute to the elliptic genus a factor

$$\prod_{n=1}^{\infty}\frac{\left(1-q^n e^{2i\pi z(\frac{R}{2}-1)}\,e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z(\frac{R}{2}-1)}\,e^{-2i\pi uQ}\right)}{\left(1-q^n e^{2i\pi z\frac{R}{2}}\,e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z\frac{R}{2}}\,e^{-2i\pi uQ}\right)}$$
$$\cdot\frac{\left(1-q^n e^{2i\pi z(\frac{R}{2}+1)}\,e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z(\frac{R}{2}+1)}\,e^{-2i\pi uQ}\right)}{\left(1-q^n e^{2i\pi z\frac{R}{2}}\,e^{2i\pi uQ}\right)\cdot\left(1-q^n e^{-2i\pi z\frac{R}{2}}\,e^{-2i\pi uQ}\right)}$$
$$=\prod_{n=1}^{\infty}\frac{\left(1-q^n y^{\frac{R}{2}-1}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}-1}x^Q)^{-1}\right)}{\left(1-q^n y^{\frac{R}{2}}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}}x^Q)^{-1}\right)}\cdot\frac{\left(1-q^n y^{\frac{R}{2}+1}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}+1}x^Q)^{-1}\right)}{\left(1-q^n y^{\frac{R}{2}}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}}x^Q)^{-1}\right)}\,. \tag{94}$$

Taking both the zero modes (91) and the nonzero modes (94) into account, we obtain

$$\frac{\left(1-y^{\frac{R}{2}-1}x^Q\right)\cdot\left(1-y^{\frac{R}{2}+1}x^Q\right)}{(1-y^{\frac{R}{2}}x^Q)^2}\cdot\prod_{n=1}^{\infty}\frac{\left(1-q^n y^{\frac{R}{2}-1}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}-1}x^Q)^{-1}\right)}{\left(1-q^n y^{\frac{R}{2}}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}}x^Q)^{-1}\right)}$$
$$\cdot\prod_{n=1}^{\infty}\frac{\left(1-q^n y^{\frac{R}{2}+1}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}+1}x^Q)^{-1}\right)}{\left(1-q^n y^{\frac{R}{2}}x^Q\right)\cdot\left(1-q^n(y^{\frac{R}{2}}x^Q)^{-1}\right)}\,. \tag{95}$$

Using the formula

$$\vartheta_1(\tau,z)=-iy^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1-yq^{n+1})(1-y^{-1}q^n)\,, \tag{96}$$

where

$$q\equiv e^{2\pi i\tau}\,,\quad y\equiv e^{2\pi iz}\,, \tag{97}$$

we can rewrite (95) as

$$Z^{1-loop}(\tau,\,u,\,z)=\frac{\vartheta_1(\tau,\,z\left(\frac{R}{2}+1\right)+uQ)}{\vartheta_1(\tau,\,z\frac{R}{2}+uQ)}\cdot\frac{\vartheta_1(\tau,\,z\left(\frac{R}{2}-1\right)+uQ)}{\vartheta_1(q,\,z\frac{R}{2}+uQ)}\,. \tag{98}$$

Comparing to the contribution of a chiral superfield [33, 34], we see that the 1-loop determinant of the elliptic genus for one pair of semichiral superfields is equal to the product of the 1-loop determinants for two chiral superfields with the opposite R-charge and the opposite flavor charge, which is consistent with the result of the semichiral gauged linear sigma model localized on the two-sphere [24].

7.2 Path integral formalism

The elliptic genus can be equivalently described in the path integral formalism as a twisted partition function on the torus, we may apply the technique of localization to compute it.

Recall that the Witten index is expressed in the path integral formalism as the partition function of the theory on a torus, with periodic boundary conditions for both bosons and fermions. To deform the Witten index into the elliptic genus, we should specify twisted boundary conditions for all fields. Equivalently, we can keep the periodic boundary conditions and introduce background gauge fields A^R and $A^{f,a}$ for the R-symmetry and the a-th flavor-symmetry, respectively. They are related to the parameters in the definition of elliptic genus via

$$z \equiv \oint A_1^R \, dx_1 - \tau \oint A_2^R \, dx_2 \,, \quad u_a \equiv \oint A_1^{f,a} dx_1 - \tau \oint A_2^{f,a} dx_2 \,. \tag{99}$$

Following the general principle of localization, if we regard the background gauge fields as parameters in the theory, we only need the free part of the Lagrangian in order to compute the elliptic genus. The free part of the Lagrangian in the Euclidean signature is

$$\begin{aligned}\mathcal{L}^{\text{free}} = {} & D_\mu \overline{X}^I D^\mu X_I + i\overline{X}^I D X_I + \overline{F}^I F_I \\ & - \overline{M}^{++,I} M_{++,I} - \overline{M}^{--,I} M_{--,I} - \overline{M}^{+-,I} M_{-+,I} - \overline{M}^{-+,I} M_{+-,I} \\ & - \overline{M}^{++,I}(-2iD_+ X_I) - \overline{M}^{--,I} 2iD_- X_I + \overline{X}^I(-2iD_+ M_I^{++}) + \overline{X}^I(2iD_- M_I^{--}) \\ & - i\overline{\psi}^I \gamma^\mu D_\mu \psi_I - \overline{\eta}^I \psi_I - \overline{\psi}^I \eta_I + i\overline{\chi}^I \gamma^\mu D_\mu \chi_I \,,\end{aligned} \tag{100}$$

where the covariant derivative is defined as

$$D_\mu \equiv \partial_\mu - \widehat{Q} u_\mu - \widehat{R} z_\mu \,, \tag{101}$$

and the operators $\widehat{Q}$ and $\widehat{R}$ acting on different fields give their corresponding

$U(1)_f$ and $U(1)_R$ charges as follows:

	X	ψ_+	ψ_-	F	χ_+	χ_-	M_{++}	M_{--}	M_{+-}	M_{-+}	η_+	η_-
$\widehat{Q}$	Q	Q	Q	Q	Q	Q	Q	Q	Q	Q	Q	Q
$\widehat{R}$	$\frac{R}{2}$	$\frac{R}{2}-1$	$\frac{R}{2}$	$\frac{R}{2}-1$	$\frac{R}{2}+1$	$\frac{R}{2}$	$\frac{R}{2}$	$\frac{R}{2}$	$\frac{R}{2}+1$	$\frac{R}{2}-1$	$\frac{R}{2}$	$\frac{R}{2}-1$

(102)

The BPS equations are obtained by setting the SUSY transformations of fermions to zero. The solutions to the BPS equtions provide the background that can perserve certain amount of supersymmetry. In this case, the BPS equations have only trivial solutions, i.e., all the fields in the semichiral multiplets are vanishing.

We adopt the metric on the torus

$$ds^2 = g_{ij}\, dx^i\, dx^j\,, \tag{103}$$

where

$$g_{ij} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}\,, \tag{104}$$

and $\tau = \tau_1 + i\tau_2$ is the complex structure, and we expand all the fields in the modes

$$e^{2\pi i(nx_1 - mx_2)}\,,$$

where $n,\, m \in \mathbb{Z}$. Then we can integrate out the auxiliary fields, and calculate the 1-loop determinant of the free part of the Lagrangian on the torus. The result is

$$Z^{1-loop} = \prod_{m,\, n\in\mathbb{Z}} \frac{\big(m + n\tau - Qu - (\frac{R}{2}+1)z\big)\cdot\big(m + n\tau - Qu - (\frac{R}{2}-1)z\big)}{\big(m + n\tau - (Qu + \frac{R}{2}z)\big)\cdot\big(m + n\tau - (Qu + \frac{R}{2}z)\big)}\,. \tag{105}$$

After regularization, this expression can be written in terms of theta functions:

$$Z^{1-loop}(\tau,\, u,\, z) = \frac{\vartheta_1(\tau,\, z\left(\frac{R}{2}+1\right) + uQ)}{\vartheta_1(\tau,\, z\frac{R}{2} + uQ)}\cdot\frac{\vartheta_1(\tau,\, z\left(\frac{R}{2}-1\right) + uQ)}{\vartheta_1(q,\, z\frac{R}{2} + uQ)}\,. \tag{106}$$

Using the localization technique, Refs. [34, 35] have shown that for a large class of 2-dimensional $\mathcal{N} = (0,2)$ GLSM's the elliptic genus is given by

$$Z = \frac{1}{|W|}\sum_{u_*\in\mathfrak{M}^*_{\text{sing}}} \text{JK-Res}_{u_*}(Q(u_*),\eta)\, Z_{1-loop}(u)\,, \tag{107}$$

where u is the holonomy of the gauge field on the spacetime torus T^2:

$$u \equiv \oint A_t \, dt - \tau \oint A_s \, ds \quad (t, s : \text{temporal and spatial direction}) \tag{108}$$

which is different from the fugacities u_a for the flavor symmetries defined in Eq. (99). As shown in Refs. [34, 35], the final result is the Jeffrey-Kirwan residue (see also Appendix D).

7.3 Eguchi-Hanson space

Eguchi-Hanson space is the simplest example of the ALE spaces, and can be constructed via hyperkähler quotient in terms of semichiral superfields [36]:

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2e^2} \int d^4\theta (\overline{\widetilde{\mathbb{F}}}\widetilde{\mathbb{F}} - \overline{\mathbb{F}}\mathbb{F}) + \left(i \int d^2\theta \, \Phi \, \mathbb{F} + c.c. \right) + \left(i \int d^2\widetilde{\theta} \, t \, \widetilde{\mathbb{F}} + c.c. \right) \\ &- \int d^4\theta \left[\overline{\mathbb{X}}_i^L \, e^{Q_i V_L} \, \mathbb{X}_i^L + \overline{\mathbb{X}}_i^R \, e^{Q_i V_R} \, \mathbb{X}_i^R + \alpha (\overline{\mathbb{X}}_i^L \, e^{i Q_i \overline{\widetilde{\mathbb{V}}}} \, \mathbb{X}_i^R + \overline{\mathbb{X}}_i^R \, e^{-i Q_i \widetilde{\mathbb{V}}} \, \mathbb{X}_i^L) \right] , \end{aligned} \tag{109}$$

where $i = 1, 2$, and for simplicity we set $t = 0$.

The model (109) has $\mathcal{N} = (4,4)$ supersymmetry, and the R-symmetry is $SO(4) \times SU(2) \cong SU(2)_1 \times SU(2)_2 \times SU(2)_3$ [37]. Hence, we can assign the R-charges (Q_1, Q_2, Q_R), where Q_R corresponds to the $U(1)_R$ charge that we discussed in the previous section. Similar to Ref. [37], we choose the supercharges $\mathcal{Q}_-$ and $\mathcal{Q}_+$ to be in the representation $(2,2,1)$ and $(2,1,2)$ respectively under the R-symmetry group. Moreover, the flavor symmetry Q_f now becomes $SU(2)_f$. In this case, the fields appearing in the model (109), which are relevant for the elliptic genus, have the following charge assignments:

	X_1^L	X_1^R	$\psi_{1+}^{(2)}$	$\psi_{1-}^{(2)}$	χ_{1+}^R	χ_{1-}^L	X_2^L	X_2^R	$\psi_{2+}^{(2)}$	$\psi_{2-}^{(2)}$	χ_{2+}^R	χ_{2-}^L
$Q_1 - Q_2$	-1	-1	0	-1	0	-1	-1	-1	0	-1	0	-1
Q_R	0	0	-1	0	1	0	0	0	-1	0	1	0
Q_f	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1

(110)

The components of the chiral and the twisted chiral field strength, $\mathbb{F}$ and $\widetilde{\mathbb{F}}$, have the following charge assignments:

$$\begin{array}{c||c|c|c||c|c|c|c} & \widetilde{\phi} & \widetilde{\psi}_+ & \widetilde{\psi}_- & \sigma & \overline{\lambda}_+ & \lambda_- & A_\mu \\ \hline Q_1 - Q_2 & 1 & 2 & 1 & -1 & 0 & -1 & 0 \\ \hline Q_R & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline Q_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{111}$$

The fugacities corresponding to Q_f, $Q_1 - Q_2$ and Q_R are denoted by ξ_1, ξ_2 and z respectively.

As we discussed before, the constrained semichiral vector multiplet and the unconstrained semichiral vector multiplet differ by a F-term, which does not show up in the result of localization, hence we can make use of the 1-loop determinant from the previous section. Then for the GLSM given by Eq. (109), the 1-loop determinant is

$$Z^{EH}_{1-loop} = Z_{\widetilde{\mathbb{F}},\mathbb{F}} \cdot Z_1^{L,R} \cdot Z_2^{L,R}\,, \tag{112}$$

where

$$\begin{aligned} Z_{\widetilde{\mathbb{F}},\mathbb{F}} &= \frac{i\eta(q)^3}{\vartheta_1(\tau,\xi_2 - z)} \cdot \frac{\vartheta_1(\tau,2\xi_2)}{\vartheta_1(\tau,\xi_2+z)}\,,\\ Z_1^{L,R} &= \frac{\vartheta_1(\tau,u+\xi_1-z)}{\vartheta_1(\tau,u+\xi_1-\xi_2)} \cdot \frac{\vartheta_1(\tau,u+\xi_1+z)}{\vartheta_1(\tau,u+\xi_1+\xi_2)}\,,\\ Z_2^{L,R} &= \frac{\vartheta_1(\tau,u-\xi_1-z)}{\vartheta_1(\tau,u-\xi_1-\xi_2)} \cdot \frac{\vartheta_1(\tau,u-\xi_1+z)}{\vartheta_1(\tau,u-\xi_1+\xi_2)}\,. \end{aligned} \tag{113}$$

Then the elliptic genus is given by

$$Z^{EH}(\tau;z,\xi) = \frac{1}{|W|} \sum_{u_* \in \mathfrak{M}^*_{\text{sing}}} \text{JK-Res}_{u_*}(Q(u_*),\eta)\, Z_{1-loop}(u)\,, \tag{114}$$

where "JK-Res" denotes the Jeffrey-Kirwan residue, which was briefly reviewed in Appendix D. In practice, the Jeffrey-Kirwan residue can be calculated as follows:

$$Z = -\sum_{u_j \in \mathfrak{M}^+_{\text{sing}}} \oint_{u=u_j} du\, Z_{1-loop}\,, \tag{115}$$

where we choose $\eta > 0$ for the vector η discussed in Appendix D. The poles are at

$$Q_i u + \frac{R_i}{2} z + P_i(\xi) = 0 \quad (\text{mod } \mathbb{Z} + \tau\mathbb{Z})\,, \tag{116}$$

where ξ denotes the holonomy of the flavor symmetry on the torus, and P_i are the flavor charges under the maximal torus of the flavor symmetry group G_F. The poles with $Q_i > 0$ and $Q_i < 0$ are grouped in to $\mathfrak{M}^+_{\text{sing}}$ and $\mathfrak{M}^-_{\text{sing}}$ respectively. In the Eguchi-Hanson case, for instance for the phase where the intersection of $H_X = \{u + \xi_1 - \xi_2 = 0\}$ and $H_Y = \{u - \xi_1 - \xi_2 = 0\}$ contributes,

$$\mathfrak{M}^+_{\text{sing}} = \{-\xi_1 + \xi_2,\, \xi_1 + \xi_2\}\,. \tag{117}$$

Hence, the elliptic genus equals

$$\begin{aligned} Z^{EH}(\tau; z, \xi) &= \frac{\vartheta_1(\tau, -2\xi_1 + \xi_2 - z) \cdot \vartheta_1(\tau, 2\xi_1 - \xi_2 - z)}{\vartheta_1(\tau, -2\xi_1) \cdot \vartheta(\tau, 2\xi_1 - 2\xi_2)} \\ &+ \frac{\vartheta_1(\tau, 2\xi_1 + \xi_2 - z) \cdot \vartheta_1(\tau, -2\xi_1 - \xi_2 - z)}{\vartheta_1(\tau, 2\xi_1) \cdot \vartheta_1(\tau, -2\xi_1 - 2\xi_2)}\,, \end{aligned} \tag{118}$$

which is the same as the result obtained in Ref. [37].

From our construction of the ALE space using semichiral GLSM, it is also clear that the elliptic genus for the ALE space coincides with the one for the six-dimensional conifold space. The reason is following. As we discussed before, to obtain an ALE space through a semichiral GLSM we need the semichiral vector multiplet, which has three real components, while to construct a conifold (or resolved conifold when the FI parameter $t \neq 0$) one should use the constrained semichiral vector multiplet, which has only one real component. However, these two vector multiplets differ only by a superpotential term, which does not affect the result of the localization. Hence, the result that we obtained using localization give us the elliptic genus both for the ALE space and for the conifold.[3]

[3]We would like to thank P. Marcos Crichigno for discussing this.

7.4 Taub-NUT space

Taub-NUT space is an example of the ALF space, and can be constructed by semichiral GLSM as follows [36]:

$$\mathcal{L} = \int d^4\theta \Big[-\frac{1}{2e^2}(\overline{\widetilde{\mathbb{F}}}\widetilde{\mathbb{F}} - \overline{\mathbb{F}}\mathbb{F}) + \overline{\mathbb{X}}_1^L\, e^{V_L}\, \mathbb{X}_1^L + \overline{\mathbb{X}}_1^R\, e^{V_R}\, \mathbb{X}_1^R + \alpha(\overline{\mathbb{X}}_1^L\, e^{i\overline{\widetilde{\mathbb{V}}}}\, \mathbb{X}_1^R + \overline{\mathbb{X}}_1^R\, e^{-i\widetilde{\mathbb{V}}}\, \mathbb{X}_1^L)$$
$$+ \frac{1}{2}\left(\mathbb{X}_2^L + \overline{\mathbb{X}}_2^L + V_L\right)^2 + \frac{1}{2}\left(\mathbb{X}_2^R + \overline{\mathbb{X}}_2^R + V_R\right)^2$$
$$+ \frac{\alpha}{2}\left(\mathbb{X}_2^L + \overline{\mathbb{X}}_2^R - i\widetilde{\mathbb{V}}\right)^2 + \frac{\alpha}{2}\left(\mathbb{X}_2^R + \mathbb{X}_2^L + i\overline{\widetilde{\mathbb{V}}}\right)^2 \Big]$$
$$+ \left(\int d^2\theta\, \Phi\, \mathbb{F} + c.c.\right) - \left(\int d^2\widetilde{\theta}\, t\, \widetilde{\mathbb{F}} + c.c.\right), \tag{119}$$

where for simplicity we set $t = 0$.

Using the results from the previous section, and assigning the same R-symmetry and the flavor symmetry charges as in the Eguchi-Hanson case (110) (111), we can write down immediately the 1-loop contribution from the semichiral vector multiplet, $\widetilde{\mathbb{F}}$ and $\mathbb{F}$, as well as the one from the semichiral multiplet, $\mathbb{X}_1^L$ and $\mathbb{X}_1^R$, of the model (119):

$$Z_{\widetilde{\mathbb{F}},\mathbb{F}} = \prod_{m,n\in\mathbb{Z}} \frac{n + \tau m - 2\xi_2}{(n + \tau m - \xi_2 + z)\cdot(n + \tau m - \xi_2 - z)} \cdot \prod_{(m,n)\neq(0,0)} (n + m\tau)\,, \tag{120}$$

$$Z_1^{L,R} = \frac{\vartheta_1(\tau, u + \xi_1 - z)}{\vartheta_1(\tau, u + \xi_1 - \xi_2)} \cdot \frac{\vartheta_1(\tau, u + \xi_1 + z)}{\vartheta_1(\tau, u + \xi_1 + \xi_2)}\,. \tag{121}$$

However, to obtain the full 1-loop determinant, we still have to work out the part of the model from semichiral Stückelberg fields, and localize it to obtain its contribution to the 1-loop determinant. Let us start with the Lagrangian for the Stückelberg field in the superspace:

$$\mathcal{L}_{St} = \int d^4\theta \Big[\frac{1}{2}\left(\mathbb{X}_2^L + \overline{\mathbb{X}}_2^L + V_L\right)^2 + \frac{1}{2}\left(\mathbb{X}_2^R + \overline{\mathbb{X}}_2^R + V_R\right)^2$$
$$+ \frac{\alpha}{2}\left(\mathbb{X}_2^L + \overline{\mathbb{X}}_2^R - i\widetilde{\mathbb{V}}\right)^2 + \frac{\alpha}{2}\left(\mathbb{X}_2^R + \mathbb{X}_2^L + i\overline{\widetilde{\mathbb{V}}}\right)^2 \Big]. \tag{122}$$

Expanding the Lagrangian into components and integrate out auxiliary fields (see Appendix C), we obtain

$$\mathcal{L}_{St} = \frac{\alpha-1}{\alpha}\left(\overline{r}_1\Box r_1 + \overline{\gamma}_1\Box\gamma_1\right) + \frac{\alpha+1}{\alpha}\left(\overline{r}_2\Box r_2 + \overline{\gamma}_2\Box\gamma_2\right)$$
$$+\frac{i}{2}(\frac{1}{\alpha^2}-\alpha^2)\overline{\psi}_+^2 D_-\psi_+^2 - \frac{i}{2}(\frac{1}{\alpha^2}-\alpha^2)\overline{\psi}_-^2 D_+\psi_-^2 + \overline{\chi}_-^L 2iD_+\chi_-^L - \overline{\chi}_+^R 2iD_-\chi_+^R\,. \quad (123)$$

As discussed in Appendix C, among the real components $r_{1,2}$ and $\gamma_{1,2}$ only r_2 transforms under the gauge transformations. We can assign the following charges to the components of the Stückelberg field:

	r_1	r_2	γ_1	γ_2	ψ_+	ψ_-	χ_+	χ_-
Q_1-Q_2	-2	0	-2	0	-1	-2	-1	0
Q_R	0	0	0	0	-1	0	1	0
Q_f	0	0	0	0	0	0	0	0

(124)

Taking both the momentum and the winding modes into account, we obtain the contribution from the Stückelberg field to the 1-loop determinant

$$Z_{St} = \prod_{m,n\in\mathbb{Z}} \frac{(n+\tau m+\xi_2+z)\cdot(n+\tau m+\xi_2-z)}{n+\tau m+2\xi_2}\cdot\prod_{(m,n)\neq(0,0)}\frac{1}{n+m\tau}\cdot\sum_{v,w\in\mathbb{Z}} e^{-\frac{g^2\pi}{\tau_2}|u+v+\tau w|^2}\,. \quad (125)$$

Together with Eq. (120) and Eq. (121), we obtain the full 1-loop determinant of the elliptic genus for the Taub-NUT space

$$Z^{TN}_{1-loop} = \frac{\vartheta_1(\tau,u+\xi_1-z)}{\vartheta_1(\tau,u+\xi_1-\xi_2)}\cdot\frac{\vartheta_1(\tau,u+\xi_1+z)}{\vartheta_1(\tau,u+\xi_1+\xi_2)}\cdot\sum_{v,w\in\mathbb{Z}} e^{-\frac{g^2\pi}{\tau_2}|u+v+\tau w|^2}\,. \quad (126)$$

The elliptic genus for the Taub-NUT space is given by

$$Z^{TN} = g^2\int_{E(\tau)}\frac{du\,d\overline{u}}{\tau_2}\,\frac{\vartheta_1(\tau,u+\xi_1-z)}{\vartheta_1(\tau,u+\xi_1-\xi_2)}\cdot\frac{\vartheta_1(\tau,u+\xi_1+z)}{\vartheta_1(\tau,u+\xi_1+\xi_2)}\cdot\sum_{v,w\in\mathbb{Z}} e^{-\frac{g^2\pi}{\tau_2}|u+v+\tau w|^2}\,, \quad (127)$$

where $E(\tau)=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. This result is the same as the one in Ref. [37] obtained from the chiral GLSM. We would like to emphasize that the result Eq. (127) cannot be included in the result presented in Eq. (114). The reason is that one needs the semichiral Stückelberg field to describe the Taub-NUT space, which has the holomorphic anomaly in the elliptic genus.

Similar to the ALE space, the elliptic genus for the ALF space should coincide with the one for some six-dimensional space. In semichiral GLSM language, one is obtained using the unconstrained semichiral vector multiplet, while the other is constructed using the constrained semichiral vector multiplet. However, as far as we know, this kind of six-dimensional space is not well studied in the literature as the conifold. We would like to investigate it in more detail in the future.

8 Low energy effective twisted superpotential

In this section, we attempt to study the low energy physics of a general GLSM.

$$\frac{1}{2\pi}\int \Delta\mathcal{L}_E^{(1)} d^2x = \log\det\Delta_{\mathrm{bos}} - \log\det\Delta_{\mathrm{ferm}}\,, \tag{128}$$

where

$$\Delta_{\mathrm{bos}} \equiv \begin{pmatrix} \Box + D + \alpha^2|\widehat{\sigma}|^2 & \frac{1}{\alpha}\Box + \alpha D + \alpha|\widehat{\sigma}|^2 \\ \frac{1}{\alpha}\Box + \alpha D + \alpha|\widehat{\sigma}|^2 & \Box + D + \alpha^2|\widehat{\sigma}|^2 \end{pmatrix} \tag{129}$$

is the 2×2-matrix appearing in $\mathcal{L}_{\mathrm{bos}}$, while Δ_{ferm} is the corresponding matrix from the fermionic sector, which is irrelevant for the calculation of the 1-loop coupled to the field D. Up to an irrelevant constant due to field rescaling, we obtain

$$\begin{aligned}\log\det\Delta_{\mathrm{bos}} &= \log\left(\alpha D + D_\mu D^\mu + \alpha^2\sigma^2\right) + \log\left(-\alpha D + D_\mu D^\mu + \alpha^2\sigma^2\right)\\ &= \frac{\alpha D}{D_\mu D^\mu + \alpha^2\sigma^2} + \frac{-\alpha D}{D_\mu D^\mu + \alpha^2\sigma^2} + (\text{higher-order terms in } D)\,.\end{aligned} \tag{130}$$

Since the terms linear in D have opposite signs, they cancel out exactly and do not show up in $\Delta\mathcal{L}_E^{(1)}$. Therefore, the effective twisted superpotential $\widetilde{W}$ is zero, and do not have terms like $\Sigma\log(\Sigma)$.

As discussed before, we can turn on two types of twisted masses m_i and $\widetilde{m}_i$ using the semichiral vector multiplet. However, m_i can be viewed as a chiral superfield and cannot enter the twisted superpotential. On the other hand, the effect of $\widetilde{m}_i$ is merely a shift of $\widehat{\sigma}$

$$\widehat{\sigma} \to \widehat{\sigma} + \widetilde{m}_i\,.$$

Although the shifts $\widetilde{m}_i$ are generically different for different flavors, the nontrivial contributions to the effective twisted superpotential cancel out within each flavor, as one can see from Eq. (130). Hence, similar to the case without twisted masses, after turning on the twisted masses the effective twisted superpotential still remains zero.

We may ask whether there is a better way to understand why the effective twisted superpotential vanishes. Indeed, a more conceptual reasoning goes as follows. We notice that the theory without twisted mass deformations is invariant under a larger supersymmetry, namely it has $\mathcal{N}=(4,4)$ rather than just $\mathcal{N}=(2,2)$ supersymmetry [38]. Since a nontrivial twisted superpotential is not compatible with $\mathcal{N}=(4,4)$ supersymmetry, we cannot generate an effective twisted superpotential term at the low energy. This fact is not completely new. If we construct a GLSM using chiral superfields in such a way that the target space is a hyperkähler manifold, the contributions from chiral superfields will cancel in pairs and the final result is zero. However, in a GLSM with chiral superfields, after we turning on twisted masses, the $\mathcal{N}=(4,4)$ supersymmetry algebra is broken down to its $\mathcal{N}=(2,2)$ subalgebra, and we obtain a nonzero effective twisted superpotential. Hence the real question is why the twisted mass deformations for semichiral superfields do not break $\mathcal{N}=(4,4)$ supersymmetry. To answer this question, we take a slightly different point of view of the twisted masses. Instead of setting component fields to the required constant values by hand, we introduce Lagrange multipliers $\widehat{\Sigma}$ and $\widehat{\widehat{\Sigma}}$, [4]

$$\int d^2\theta\, \widehat{\Sigma}(\mathbb{F}-m)+\int d^2\widetilde{\theta}\, \widehat{\widehat{\Sigma}}(\widetilde{\mathbb{F}}-\widetilde{m})\,, \tag{131}$$

where $\widehat{\Sigma}$ is a chiral superfield, while $\widehat{\widehat{\Sigma}}$ is a twisted chiral superfield. Since now we focus on the flavor symmetry group, let us suppress the color indices at the moment, then the Kähler potential part of the Lagrangian is

$$\int d^4\theta\, \left[\overline{\mathbb{X}}_L e^{V_L}\mathbb{X}_L+\overline{\mathbb{X}}_R e^{V_R}\mathbb{X}_R+\alpha(\overline{\mathbb{X}}_L e^{i\widetilde{\overline{\mathbb{V}}}}\mathbb{X}_R+\overline{\mathbb{X}}_R e^{-i\widetilde{\mathbb{V}}}\mathbb{X}_L)\right]\,, \tag{132}$$

or written in covariant approach as

$$\int d^4\theta\, \left[\overline{\mathbb{X}}_L\mathbb{X}_L+\overline{\mathbb{X}}_R\mathbb{X}_R+\alpha(\overline{\mathbb{X}}_L\mathbb{X}_R+\overline{\mathbb{X}}_R\mathbb{X}_L)\right]\,, \tag{133}$$

[4]We would like to thank Martin Roček for discussing this.

where now

$$\mathbb{F} = \{\overline{\mathbb{D}}_+, \overline{\mathbb{D}}_-\}\,, \quad \widetilde{\mathbb{F}} = \{\overline{\mathbb{D}}_+, \mathbb{D}_-\}\,. \tag{134}$$

Next, we need to work out the expression of the Kähler potential part of the Lagrangian in components, and integrating out the Lagrange multipliers $\widehat{\Sigma}$ and $\widehat{\widehat{\Sigma}}$ will set the lowest components $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ to some constant values m and $\widetilde{m}$ respectively, which are the twisted masses in our model. To be precise, the superfields carry flavor indices as follows:

$$\int d^4\theta\,\left[\overline{\mathbb{X}}^i_L\mathbb{X}_{Li} + \overline{\mathbb{X}}^i_R\mathbb{X}_{Ri} + \alpha(\overline{\mathbb{X}}^i_L\mathbb{X}_{Ri} + \overline{\mathbb{X}}^i_R\mathbb{X}_{Li})\right] + \int d^2\theta\,\widehat{\Sigma}^{ij}(\mathbb{F}_{ij} - m_{ij}) + \int d^2\widetilde{\theta}\,\widehat{\widehat{\Sigma}}^{ij}(\widetilde{\mathbb{F}}_{ij} - \widetilde{m}_{ij})\,. \tag{135}$$

However, the terms

$$\int d^2\theta\,\widehat{\Sigma}^{ij} m_{ij} \quad \text{and} \quad \int d^2\widetilde{\theta}\,\widehat{\widehat{\Sigma}}^{ij}\widetilde{m}_{ij}$$

are similar to the FI term, which is gauge invariant only for Abelian groups. Hence, only diagonal parts of the matrices m_{ii} and $\widetilde{m}_{ii}$ preserve the gauge symmetry, and we only need to consider

$$\int d^2\theta\,\widehat{\Sigma}^i(\mathbb{F}_i - m_i) + \int d^2\widetilde{\theta}\,\widehat{\widehat{\Sigma}}^i(\widetilde{\mathbb{F}}_i - \widetilde{m}_i)\,. \tag{136}$$

$m_i \neq 0$ will break the R-symmetry. Nevertheless, since m_i is part of the superpotential, it does not enter the effective twisted superpotential in the end. On the other hand, $\widetilde{m}_i$ plays the same role as Coulomb branch moduli $\widehat{\sigma}$, which is essentially the VEV of the scalar in the vector multiplet. Since using the semichiral vector multiplets to gauge semichiral multiplets preserves $\mathcal{N} = (4,4)$ supersymmetry, it is impossible to generate a nonzero twisted superpotential term in this way.

9 Conclusion and future directions

In this paper, we have studied the dynamics of GLSMs with semichiral superfields on the flat space.

We have computed the elliptic genus using both the Hamiltonian formalism and the path integral formalism. We have also worked out two important examples, namely the Eguchi-Hanson space and the Taub-NUT space. The results agree with the previous computations using GLSMs without using semichiral superfields [37].

It is natural to ask whether our computation can be generalized to other models. There are many interesting cases which do not have known realization in terms of GLSMs. For example, we may construct Wess-Zumino-Witten models with manifest $\mathcal{N} = (2, 2)$ supersymmetry. The elliptic genus can be again computed with a minor modification. It will be described in detail in the subsequent paper [23].

Futhermore, we have also computed the low energy effective twisted superpotential $\widetilde{\mathcal{W}}^{\text{eff}}$ of the GLSMs on the Coulomb branch. Unfortunately, the contribution from semichiral superfields to $\widetilde{\mathcal{W}}^{\text{eff}}$ vanishes. Therefore, the low energy behavior of the GLSM with semichiral superfields is determined only by the generalized Kähler potential, which is not protected by supersymmetry and is difficult to compute exactly. It will be interesting if one can figure out some other methods to describe some exact properties of the low energy effective theory.

Part IV
Entanglement Entropy of ABJM Theory and Entropy of Topological Black Hole

10 Introduction

The black hole entropy is one of the central problems in theoretical physics. The celebrated AdS/CFT correspondence [39] provides us with a new insight into the problem of the black hole entropy. Based on this principle, the conformal field theory defined on the boundary of an AdS space should capture some features of the gravity theory in the bulk. Hence, it is really tempting to identify the black hole entropy in the bulk and the entanglement entropy of the conformal field theory on the boundary [40]. When the boundary conformal field theory and the bulk gravity both have certain amount of supersymmetries, the technique of supersymmetric localization allows to to compute the entropy on both sides and test the identification precisely. In this paper, we would like to study a concrete example towards this direction, i.e. the ABJM theory via the supergravity localization, to test this proposal.

As a generalization of entanglement entropy, supersymmetric Rényi entropy S_q was first defined on a q-branched three-sphere [41]. It can be computed exactly using the technique of supersymmetric localization, and the result can be expressed in terms of the partition function of the the 3D superconformal field theory defined on a squashed sphere S^3_b with the squashing parameter b [41]. Using the technique of supersymmetric localization on curved manifolds [42], one can further express this partition function into a matrix integral [43, 44, 45, 46, 47]. In some cases, one can even evaluate the matrix integral to obtain a relatively simple result. For instance, neglecting the nonperturbative effects at large N, the matrix integral for the ABJM theory on some compact manifolds can be evaluated using the Fermi gas approach [48, 49].

Since it can be computed exactly, the supersymmetric Rényi entropy provides a new quantitiy to test various dualities. For instance, one can use it to test the AdS/CFT correspondence more precisely. Before doing it, one has to first find the holographic way of computing the supersymmetric Rényi entropy. As explained in Ref. [41], the technical problem is the conical singularity caused by the branched sphere. To resolve the conical singularity, it was proposed in Refs. [50, 51] to perform a conformal transformation, which maps the branched three-sphere into $S^1 \times \mathbb{H}^2$, i.e.,

$$\begin{aligned} ds^2 &= d\theta^2 + q^2 \sin^2\theta \, d\tau^2 + \cos^2\theta \, d\phi^2 \\ &= \sin^2\theta \left[d\widetilde{\tau}^2 + du^2 + \sinh^2 u \, d\phi^2 \right] , \end{aligned} \tag{137}$$

where $\tau \in [0,\, 2\pi)$, $\widetilde{\tau} = q\tau \in [0,\, 2q\pi)$, and

$$\sinh u = -\cot\theta \,. \tag{138}$$

After the conformal transformation, one finds that $S^1 \times \mathbb{H}^2$ can be viewed as the boundary of the AdS_4 topological black hole (TBH). Hence, in principle the supersymmetric Rényi entropy can be computed in the topological black hole holographically. The free energy and the Killing spinor equations can also be evaluated in the bulk gravity theory, which support the holographic interpretation [50, 51]. In particular, the parameter q coming from the branched sphere can be viewed as a deformation parameter of the original theory, and it is related to the mass and the charge of the topological black hole. This relation was called the TBH/qSCFT correspondence [50, 52], and it provides another precise test of the AdS/CFT correspondence. Later, these works were generalized to other dimensions, and similar results were found [52, 53, 54, 55, 56, 57, 58, 59].

The partition functions and consequently the supersymmetric Rényi entropies of some superconformal field theories can be computed exactly using the technique of supersymmetric localization. In fact, this technique can also be applied to some supergravity theories. Different backgrounds ($AdS_2 \times S^2$, AdS_4) have been studied [60, 61, 62]. In particular, the localization of the 4D $\mathcal{N} = 2$ off-shell supergravity on AdS_4 corresponds to the ABJM theory on the boundary S^3 [62], and the partition functions of both theories can be expressed in terms of Airy function. From the partition function, we can compute the entanglement entropy of the ABJM theory across a circle S^1 on the boundary, which matches the previous results [41].

It is then natural to consider the supergravity localization on 4D topological black holes, whose boundaries are $S^1 \times \mathbb{H}^2$. From the supergravity localization we should be able to compute holographically the supersymmetric Rényi entropies of the corresponding superconformal field theories on the boundary. Comparing the results from the bulk and the known results from the boundary provides an exact test of the AdS/CFT correspondence, and at the same time one can also check the proposal of identifying these entropies as the black hole entropies in this framework concretely.

As a starting point, in this paper we study the localization of the 4D $\mathcal{N} = 2$ off-shell gauged supergravity on the neutral topological black hole, which corresponds to the entanglement entropy of the superconformal ABJM theory across a circle S^1 on the boundary. The logic of our computation is as follows. The gravity dual of the ABJM theory is the 11-dimensional M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ [63]. We neglect all the stringy effects and consistently truncate the 11-dimensional supergravity to a 4-dimensional $\mathcal{N} = 2$ gauged supergravity theory, which has an off-shell formalism using superconformal gauged supergravity. We fix the values of fields in the Weyl multiplet, and apply the localization method to evaluate the supersymmetric partition function by integrating over the vector multiplets and the hypermultiplets. Our localization calculation is similar to the standard field theory localization, except that the background spacetime is noncompact. We find that the entropy of the AdS_4 neutral topological black hole and the entanglement entropy of the ABJM theory on the boundary coincide in the large-N expansion up to some stringy effects. More precisely,

$$S_{EE}^{\mathrm{ABJM}} = S_{BH} = -\frac{\sqrt{2}\pi}{3} k^{1/2} N^{3/2} - \frac{1}{4} \log(N) + \mathcal{O}(N^0)\,. \qquad (139)$$

Meanwhile, using the supergravity localization we obtain the logarithmic correction to the classical result of the black hole entropy given by the Bekenstein-Hawking formula [64, 65, 66], and this correction is consistent with the on-shell 1-loop computation from the Euclidean 11-dimensional supergravity on $\mathrm{AdS}_4 \times X_7$ [67].

This paper is organized as follows. In Section 11 we review some facts about the supersymmetric Rényi entropy and the ABJM theory. The gravity dual of the supersymmetric Rényi entropy will be reviewed in Section 12. In Section 13 we discuss the localization of the 4D $\mathcal{N} = 2$ off-shell supergravity on the AdS_4 neutral topological black hole with the boundary $\mathrm{S}^1 \times \mathbb{H}^2$. The bulk black hole entropy and the boundary entanglement entropy of the ABJM theory can be read off from the results of the supergravity localization, which is presented in Section 14. Some further discussions will be made in Section 15. We also present some details of the calculations in a few appendices. In Appendix E we review the 4D $\mathcal{N}=2$ off-shell supergravity, while in Appendix F the Killing spinors and the convenction of the Gamma matrices are discussed. For the supergravity localization, the explicit form of the localization action will be presented in Appendix G, and we will evaluate the action along the localization locus in Appendix H.

11 Supersymmetric Rényi Entropy of ABJM Theory

11.1 Supersymmetric Rényi Entropy

We start with the well-known definitions of entanglement entropy and Rényi entropy. Suppose the space on which the theory is defined can be divided into a piece A and its complement $\overline{A}=B$, and correspondingly the Hilbert space factorizes into a tensor product $\mathcal{H}_A\otimes\mathcal{H}_B$. If the density matrix over the whole Hilbert space is ρ, then the reduced density matrix is defined as

$$\rho_A \equiv \mathrm{tr}_B\rho\,. \tag{140}$$

The entanglement entropy is the von Neumann entropy of ρ_A,

$$S_E \equiv -\mathrm{tr}\rho_A \log\rho_A\,, \tag{141}$$

while the Rényi entropies are defined to be

$$S_n \equiv \frac{1}{1-n}\log\mathrm{tr}(\rho_A)^n\,. \tag{142}$$

Assuming an analytic continuation of S_n can be obtained, the entanglement entropy can alternately be expressed as a limit of the Rényi entropy:

$$\lim_{n\to 1} S_n = S_E\,. \tag{143}$$

The Rényi entropy can be calculated using the so-called "replica trick":

$$S_n = \frac{1}{1-n}\log\left(\frac{Z_n}{(Z_1)^n}\right)\,, \tag{144}$$

where Z_n is the Euclidean partition function on a n-covering space branched along A.

The concept of the supersymmetric Rényi entropy is a generalization of Rényi entropy. It was first introduced in Ref. [41] for the 3-dimensional supersymmetric field theories as follows:

$$S_q^{\mathrm{SUSY}} \equiv \frac{1}{1-q}\left[\log\left(\frac{Z_{\text{singular space}}(q)}{(Z_{\mathrm{S}^3})^q}\right)\right], \tag{145}$$

where Z_{S^3} is the partition function of a supersymmetric theory on a three-sphere S^3, while $Z_{\text{singular space}}(q)$ is the partition function on the q-covering of a three-sphere, S^3_q, which is also called the q-branched sphere given by the metric

$$ds^2 = L^2(d\theta^2 + q^2\sin^2\theta\, d\tau^2 + \cos^2\theta\, d\phi^2) \tag{146}$$

with $\theta \in [0, \pi/2]$, $\tau \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$. In the limit $q \to 1$, the q-branched sphere returns to the round sphere, and the supersymmetric Rényi entropy becomes the entanglement entropy. Initially, the supersymmetric Rényi entropy was defined for 3D $\mathcal{N} = 2$ superconformal field theories [41], and later it was generalized to other dimensions [52, 53, 54, 55, 56, 57, 58, 59].

Using the supersymmetric localization, it was derived explicitly in Ref. [41] that in the definition of the supersymmetric Rényi entropy of 3D superconformal field theories $Z_{\text{singular space}}(q)$ can be written as:

$$Z_{\text{singular space}}(q) = \frac{1}{|W|}\int \prod_{i=1}^{\mathrm{rank}\, G} d\sigma_i\, e^{\pi i k \mathrm{Tr}(\sigma^2)} \prod_{\alpha} \frac{1}{\Gamma_h(\alpha(\sigma))} \prod_I \prod_{\rho\in\mathcal{R}_I} \Gamma_I\left(\rho(\sigma) + i\omega\Delta_I\right) \tag{147}$$

with

$$\omega = \frac{\omega_1+\omega_2}{2}, \quad \omega_1 = \sqrt{q}, \quad \omega_2 = \frac{1}{\sqrt{q}}, \tag{148}$$

and

$$\Gamma_h(z) \equiv \Gamma_h(z;\, i\omega_1, i\omega_2) \tag{149}$$

is a hyperbolic gamma function. σ_i parametrize the localization locus of the Coulomb branch. k stands for the Chern-Simons level, and I is the index for the chiral multiplets. α and ρ denote the root of the adjoint representation and the weight of the representation $\mathcal{R}_I$ of the gauge group G respectively. Δ_I is the R-charge of the scalar in the chiral multiplet. It turns out that the partition function $Z_{\text{singular space}}(q)$ equals the partition function of the same theory on a squashed three-sphere S^3_b with $b = \sqrt{q}$.

11.2 Results for ABJM Theory

As an example of the 3D superconformal field theory, the ABJM theory has been intensively studied. As first discussed by Aharony, Bergman, Jafferis and Maldacena in Ref. [63], the ABJM theory is a 3D $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theory with the gauge group $U(N)_k \times U(N)_{-k}$, where k stands for the Chern-Simons level. The theory describes the low-energy dynamics of N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$, and it has 4 bi-fundamental chiral multiplets, two of them in the $(N, \overline{N})$ representation and the other two in the $(\overline{N}, N)$ representation. The matter content of the ABJM theory can be illustrated using the following quiver diagram:

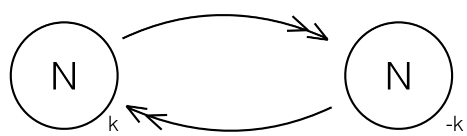

Figure 1: The quiver diagram for the ABJM theory

With the development of the supersymmetric localization on curved manifolds [42], the partition functions of some 3D supersymmetric gauge theories including the ABJM theory were studied in Ref. [68], and they can be expressed as matrix integrals. In particular, the partition function of the ABJM theory is reduced to the following matrix model:

$$Z_{\text{ABJM}} = \int \prod_i d\sigma_i \, d\widetilde{\sigma}_i \, e^{-ik\pi(\sigma_i^2 - \widetilde{\sigma}_i^2)} \frac{\prod_{i<j} \left(4 \sinh(\pi \sigma_{ij}) \sinh(\pi \widetilde{\sigma}_{ij})\right)^2}{\prod_{i,j} \left(2 \cosh(\pi(\sigma_i - \widetilde{\sigma}_j))\right)^2} \,, \tag{150}$$

where $\sigma_{ij} \equiv \sigma_i - \sigma_j$ and $\widetilde{\sigma}_{ij} \equiv \widetilde{\sigma}_i - \widetilde{\sigma}_j$ are the roots of $U(N)_k$ and $U(N)_{-k}$ respectively, and the weights in the representations $(N, \overline{N})$ and $(\overline{N}, N)$ are

$$\begin{aligned} \rho_{i,j}^{(N,\overline{N})} &= \sigma_i - \widetilde{\sigma}_j \,, \\ \rho_{i,j}^{(\overline{N},N)} &= -\sigma_i + \widetilde{\sigma}_j \,. \end{aligned} \tag{151}$$

To evaluate these integrals, one still has to solve the matrix model, which sometimes can be nontrivial. To proceed the computation, on the one hand the matrix integral were evaluated directly under some approximations for the 3D superconformal field theories [69, 70], while on the other hand using the Fermi gas approach one can obtain the final results of the perturbative contributions, which was done in Ref. [48]. The result for the partition function of the ABJM theory from the Fermi gas approach was obtained by Mariño and Putrov, which can be written in terms of the Airy function [48]:

$$Z^{\text{ABJM}} \propto \text{Ai}\left[\left(\frac{\pi^2 k}{2}\right)^{1/3}\left(N - \frac{k}{24} - \frac{1}{3k}\right)\right] . \tag{152}$$

As discussed above, the supersymmetric Rényi entropies of some 3D superconformal field theories can be expressed in terms of partition functions of these theories on squashed three sphere S_b^3, and these partition functions can still be written as matrix models using the technique of localization. For the ABJM theory, the partition function on a squashed three-sphere can be written as

$$Z_{b^2}^{\text{ABJM}} = \frac{1}{(N!)^2}\int \prod_i d\sigma_i\, d\widetilde{\sigma}_i\, e^{-ik\pi(\sigma_i^2 - \widetilde{\sigma}_i^2)}\, Z_{b^2}^{\text{vec}}\, Z_{b^2}^{\text{bi-fund}} , \tag{153}$$

where

$$\begin{aligned} Z_{b^2}^{\text{vec}} &= \prod_{i<j} 4\sinh(\pi b \sigma_{ij})\sinh(\pi b^{-1}\sigma_{ij})\, 4\sinh(\pi b \widetilde{\sigma}_{ij})\sinh(\pi b^{-1}\widetilde{\sigma}_{ij}) , \\ Z_{b^2}^{\text{bi-fund}} &= \prod_{i,j} \frac{s_b\left(\sigma_i - \widetilde{\sigma}_j + iQ/4\right)^2}{s_b\left(\sigma_i - \widetilde{\sigma}_j - iQ/4\right)^2} , \end{aligned} \tag{154}$$

where $Q = b + 1/b$, and $s_b(x)$ is the double sine function. In the limit $b \to 1$,

$$\begin{aligned} Z_{b^2=1}^{\text{vec}} &= \prod_{i<j} \left(4\sinh(\pi\sigma_{ij})\sinh(\pi\widetilde{\sigma}_{ij})\right)^2 , \\ Z_{b^2=1}^{\text{bi-fund}} &= \prod_{i,j} \frac{1}{\left(2\cosh(\pi(\sigma_i - \widetilde{\sigma}_j))\right)^2} , \end{aligned} \tag{155}$$

which reproduce the partition function of the ABJM theory on the round three-sphere found in Ref. [68].

Recently, Hatsuda studied the partition function of the ABJM theory on a squashed three-spheres S_b^3 [49], and found that for some cases the matrix model can be greatly simplified and evaluated analytically at large N using the Fermi gas approach. For instance, when $k = 1$ and $b^2 = 3$, the leading contribution to the partition function is

$$Z^{\text{ABJM}}_{b^2=3} = C_3^{-1/3}\, e^{A_3}\, \text{Ai}\left[C_3^{-1/3}(N - B_3)\right] + \cdots \,, \tag{156}$$

where

$$A_3 = -\frac{\zeta(3)}{3\pi^2} + \frac{\log 3}{6}\,, \quad B_3 = \frac{1}{8}\,, \quad C_3 = \frac{9}{8\pi^2}\,, \tag{157}$$

With these results, one can study the supersymmetric Rényi entropy at large N beyond the leading order.

12 Gravity Dual of Supersymmetric Rényi Entropy

The gravity dual of the supersymmetric Rényi entropies of 3D superconformal field theories (including the ABJM theory) has been constructed in Refs. [50, 51]. Later, it was generalized to other dimensions [52, 53, 54, 55, 56, 57, 58, 59]. In this section, we briefly review the gravity dual theory found in Refs. [50, 51].

As discussed in Ref. [41], due to the conical singularity one has to turn on a R-symmetry gauge field in order to preserve supersymmetry. In the spirit of the AdS/CFT correspondence, instead of finding an AdS space with the branched sphere as the boundary, one can first perform the conformal transformation introduced in Section 10 to the branched sphere, which maps the branched three-sphere into $\mathrm{S}^1 \times \mathbb{H}^2$, i.e.

$$\begin{aligned} ds^2 &= d\theta^2 + q^2 \sin^2\theta\, d\tau^2 + \cos^2\theta\, d\phi^2 \\ &= \sin^2\theta \left[d\widetilde{\tau}^2 + du^2 + \sinh^2 u\, d\phi^2\right]\,, \end{aligned} \tag{158}$$

where $\tau \in [0,\, 2\pi)$, $\widetilde{\tau} = q\tau \in [0,\, 2q\pi)$, and

$$\sinh u = -\cot\theta\,. \tag{159}$$

Next, one can find an AdS_4 topological black hole with the metric [71]:

$$ds^2 = -f(r)\, dt^2 + \frac{1}{f(r)} dr^2 + r^2\, d\Sigma(\mathbb{H}^2)\,, \tag{160}$$

whose boundary is $\mathbb{R}^1 \times \mathbb{H}^2$, where

$$f(r) = \frac{r^2}{L^2} - 1 - \frac{2m}{r} + \frac{Q^2}{r^2}\,, \tag{161}$$

and

$$d\Sigma(\mathbb{H}^2) = du^2 + \sinh^2 u\, d\phi^2\,. \tag{162}$$

This metric can be viewed as solutions to the 4D $\mathcal{N} = 2$ gauged supergravity given by the effective action [72], whose bosonic part is

$$I = -\frac{1}{2\,\ell_P^2}\int d^4x\,\sqrt{-g}\left(2\Lambda + R - \frac{1}{g^2}F_{\mu\nu}F^{\mu\nu}\right)\,. \tag{163}$$

The gauge field is given by

$$A = \left(\frac{Q}{r} - \frac{Q}{r_h}\right) dt\,, \tag{164}$$

where r_h is the horizon radius of the black hole determined by $f(r_h) = 0$.

As explained in Refs. [73, 50], to preserve the supersymmetry, the condition

$$m^2 + Q^2 = 0 \tag{165}$$

holds for both the charged case ($Q \neq 0$) and the neutral case ($Q = 0$).

For $Q^2 = -m^2 \neq 0$,

$$f(r) = \frac{r^2}{L^2} - \left(1 + \frac{m}{r}\right)^2\,, \tag{166}$$

the metric (160) corresponds to a charged topological black hole. As shown in Refs. [50, 51], the Bekenstein-Hawking entropy of the charged topological black hole equals the Rényi entropy of the superconformal field theory on the boundary. In particular, the result from the gravity dual recovers the relation between the Rényi entropy S_q and the entanglement entropy S_1 for the 3D superconformal field theories:

$$S_q = \frac{3q+1}{4q}S_1\,. \tag{167}$$

For $Q = m = 0$,

$$f(r) = \frac{r^2}{L^2} - 1\,, \tag{168}$$

and $r_h = L$. The gravity solution is dual to a 3D superconformal field theory on $S^1 \times \mathbb{H}^2$ with $q = 1$. Correspondingly, the black hole entropy in this case equals the entanglement entropy of the boundary superconformal field theory. The evaluation of the gravity free energy at classical level supports this identification.

As we know, the Bekenstein-Hawking entropy corresponds to the classical result of the gravity. Using supersymmetric localization, we can obtain more precise result and go beyond the classical result. Hence, in this way we can test the gravity dual and the AdS/CFT correspondence more precisely.

In this paper, we would like to consider the neutral topological black hole, which gives the entanglement entropy of the superconformal field theory on the boundary. For this case, the branching parameter $q = 1$, which corresponds to the round three sphere. However, one can nevertheless perform the conformal transformation (159), and the hyperbolic AdS_4 space becomes an AdS_4 neutral topological black hole, and the entanglement entropy of the superconformal field theory on the boundary is supposed to equal the bulk black hole entropy, which can be tested more precisely using the results of the localization of supergravity.

13 4D $\mathcal{N} = 2$ Off-Shell Gauged Supergravity and Its Localization

In this section, we discuss the localization of the 4D $\mathcal{N} = 2$ off-shell supergravity on AdS_4 topological black hole with the boundary $S^1 \times \mathbb{H}^2$. The steps are similar to the ones in Ref. [62], however, there are some subtle differences which consequently lead to different final results.

13.1 4D $\mathcal{N} = 2$ Off-Shell Gauged Supergravity

The 4D $\mathcal{N} = 2$ off-shell supergravity theory can be obtained as a consistent truncation of M-theory on a Sasaki-Einstein manifold X_7. The theory was originally constructed in Ref. [74] and also reviwed in Ref. [62]. We also briefly summarize the theory in Appendix E.

The $\mathcal{N} = 2$ superconformal algebra has the generators:

$$P_a,\ M_{ab},\ D,\ K_a,\ Q_i,\ S^i,\ U_{ij}, \tag{169}$$

which correspond to the generators of translations, Lorentz rotations, dilatations, special conformal transformations, usual supersymmetry transformations, special conformal supersymmetry transformations and the 4D $SU(2)$ R-symmetry respectively. The gauge fields corresponding to these generators are

$$e_\mu^a,\, \omega_\mu^{ab},\, b_\mu,\, f_\mu^a,\, \psi_\mu^i,\, \phi_\mu^i,\, \mathcal{V}_\mu^{ij} \tag{170}$$

respectively.

To construct a 4D off-shell $\mathcal{N} = 2$ supergravity, one needs the Weyl multiplet $\mathbf{W}$:

$$\mathbf{W} = \left(e_\mu^a,\, \psi_\mu^i,\, b_\mu,\, A_\mu,\, \mathcal{V}_{\mu j}^i,\, T_{ab}^{ij},\, \chi^i,\, D\right), \tag{171}$$

the vector multiplet $\mathbf{X}^I$:

$$\mathbf{X}^I = \left(X^I,\, \Omega_i^I,\, W_\mu^I,\, Y_{ij}^I\right), \tag{172}$$

and the hypermultiplet $(A_i{}^\alpha,\, \zeta^\alpha)$. More details about these multiplets and their supersymmetric transformations can be found in Appendix E.

Given a prepotential $F(X)$, the two-derivative off-shell action for the bosonic fields is given by

$$\begin{aligned} S = \int d^4x\sqrt{g}\Bigg[& N_{IJ}\overline{X}^I X^J\left(\frac{R}{6} + D\right) + N_{IJ}\partial\overline{X}^I\partial X^J - \frac{1}{8}N_{IJ}Y^{ijI}Y_{ij}^J \\ & + \Bigg(- \nabla A^i{}_\beta \nabla A_i{}^\alpha - \left(\frac{R}{6} - \frac{D}{2}\right) A^i{}_\beta A_i{}^\alpha + F^i{}_\beta F_i{}^\alpha + 4g^2 A^i{}_\beta \overline{X}^\alpha{}_\gamma X^\gamma{}_\delta A_i{}^\delta \\ & + gA^i{}_\beta (Y^{jk})_\gamma{}^\alpha A_k{}^\gamma \epsilon_{ij}\Bigg) d_\alpha{}^\beta\Bigg], \end{aligned} \tag{173}$$

where

$$N_{IJ} \equiv \frac{1}{2i}\left(F_{IJ} - \overline{F}_{IJ}\right), \quad F_{IJ} \equiv \partial_I\partial_J F(X), \tag{174}$$

and $F_i{}^\alpha$ is related to the field $A_i{}^{\alpha(z)}$ discussed in Appendix E in the following way:

$$F_i{}^\alpha = aA_i{}^{\alpha(z)}. \tag{175}$$

The term $\sim RA^2$ provides a negative cosmological constant for the AdS_4 space.

13.2 Localization of Supergravity

As discussed before, to find the gravity dual of the supersymmetric Rényi entropy, one can perform a conformal transformation (159) on the boundary, which maps the branched three-sphere into $\mathrm{S}^1 \times \mathbb{H}^2$. Correspondingly, the metric in the bulk now should be the AdS_4 topological black hole given by the metric (160) to match the boundary.

In this section, we discuss the localization of the 4D $\mathcal{N} = 2$ off-shell gauged supergravity on the background of the AdS_4 neutral topological black hole (160). In other words, we focus on the case with the branching parameter $q = 1$, which corresponds to the entanglement entropy of the ABJM theory across a circle S^1 on the boundary. The discussions are similar to the supergravity localization on the hyperbolic AdS_4 in Ref. [62].

13.2.1 BPS Equations

Let us first consider the BPS equations from different supergravity multiplets. For the Weyl multiplet, by setting $\delta\psi_\mu{}^i = 0$ one obtains the Killing spinor equation:

$$2\nabla_\mu \epsilon^i + i A_\mu \epsilon^i - \frac{1}{8} T_{ab}{}^{ij} \gamma^{ab} \gamma_\mu \epsilon_j = \gamma_\mu \eta^i \,, \tag{176}$$

where we set the background $b_\mu = \mathcal{V}_\mu{}^{ij} = 0$, and

$$\nabla_\mu \epsilon^i \equiv \partial_\mu \epsilon^i + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab}\, \epsilon^i \,. \tag{177}$$

For $q = 1$, one can further set $A_\mu = T_{ab}{}^{ij} = 0$ in Eq. (176), and the Killing spinor equation becomes

$$\nabla_\mu \epsilon^i = \frac{1}{2} \gamma_\mu \eta^i \,. \tag{178}$$

The upper and the lower indices for ϵ denote the positive and the negative chirality respectively, and the opposite for η. We can use the Dirac notation to combine different components into Dirac spinors

$$\xi = (\xi_+^i,\, \xi_-^i)\,, \quad \eta = (\eta_+^i,\, \eta_-^i)\,, \tag{179}$$

where

$$\xi_+^i \equiv \epsilon^i\,, \quad \epsilon_i \equiv i \epsilon_{ij} \xi_-^j\,, \quad \eta_i \equiv -\epsilon_{ij} \eta_+^j\,, \quad \eta^i \equiv i \eta_-^i\,. \tag{180}$$

Using these notations, we can rewrite the Killing spinor equation (178) for $q = 1$ as follows:

$$\nabla_\mu \xi^i = \frac{i}{2}\gamma_\mu \eta^i \,. \tag{181}$$

We would like to recover the Killing spinor equation for the topological black hole discussed in Ref. [50] for $q = 1$. To do so, let us first consider the general Killing spinor equation for the AdS_4 topological black hole [50]:

$$\nabla_\mu \epsilon - igA_\mu \epsilon + \frac{i}{4}F_{ab}\gamma^{ab}\gamma_\mu \epsilon = -\frac{1}{2}g\gamma_\mu \epsilon \,, \tag{182}$$

where

$$A = \left(\frac{Q}{r} - \frac{Q}{r_h}\right) dt \tag{183}$$

with r_h denoting the position of the horizon, and consequently only the components F_{rt} and F_{tr} are nonvanishing. For the branching paramter $q = 1$ considered in this paper, the black hole is neutral, i.e. $Q = 0$, hence both A_μ and $F_{\mu\nu}$ vanish for this case. As discussed in Appendix F, using the charge conjugation matrix B one can define the charge conjugate spinor satisfying another Killing spinor equation:

$$\nabla_\mu \epsilon^c + igA_\mu \epsilon^c + \frac{i}{4}F_{ab}\gamma^{ab}\gamma_\mu \epsilon^c = \frac{1}{2}g\gamma_\mu \epsilon^c \,. \tag{184}$$

The coupling constant g is related to the AdS radius [50]:

$$L = \frac{1}{g} \,. \tag{185}$$

For $q = 1$, the two Killing spinor equations are

$$\nabla_\mu \epsilon = -\frac{1}{2L}\gamma_\mu \epsilon \,, \quad \nabla_\mu \epsilon^c = \frac{1}{2L}\gamma_\mu \epsilon^c \,, \tag{186}$$

which can be written into a more compact form using the Dirac notation:

$$\nabla_\mu \widetilde{\xi}^i = \frac{1}{2L}\gamma_\mu (\sigma_3)^i{}_j \widetilde{\xi}^j \,, \tag{187}$$

with

$$\widetilde{\xi}^1 \equiv \epsilon^c \,, \quad \widetilde{\xi}^2 \equiv \epsilon \,. \tag{188}$$

Defining

$$\xi^i \equiv \frac{1+i\gamma_5}{2}\,\widetilde{\xi}^i\,, \tag{189}$$

one can further obtain an equivalent expression for the Killing spinor equation at $q=1$:

$$\nabla_\mu \xi^i = \frac{i}{2L}\gamma_5\gamma_\mu(\sigma_3)^i{}_j\xi^j\,. \tag{190}$$

Comparing this equation with Eq. (181), we should identify

$$\eta^i = -\frac{1}{L}\gamma_5(\sigma_3)^i{}_j\xi^j\,. \tag{191}$$

Eq. (190) will be the Killing spinor equation used throughout the rest of this paper.

Next, for the vector multiplet, the BPS equations are obtained from $\delta\Omega_i = 0$. Setting $F_{\mu\nu} = T_{\mu\nu} = 0$ and distinguishing different chiralities, we obtain

$$\begin{aligned}
\delta\Omega^i_+ &= -i\partial\!\!\!/ X\xi^i_- - \frac{1}{2}Y^i{}_j\xi^j_+ + X\eta^i_+ = 0\,,\\
\delta\Omega^i_- &= -i\partial\!\!\!/ \overline{X}\xi^i_+ - \frac{1}{2}Y^i{}_j\xi^j_- + \overline{X}\eta^i_- = 0\,,
\end{aligned} \tag{192}$$

which can be combined into

$$-i\partial\!\!\!/(H-i\gamma_5 J)\xi^i - \frac{1}{2}Y^i{}_j\xi^j - \frac{1}{L}(H+i\gamma_5 J)\gamma_5(\sigma_3)^i{}_j\xi^j = 0\,, \tag{193}$$

where we have parametrized $X = H+iJ$ and used the expression for η^i given above. For constant H and J, the BPS equations above have the solution:

$$H=0\,,\quad Y^1{}_1 = -Y^2{}_2 = -\frac{2i}{L}J\,,\quad Y^1{}_2 = -Y^2{}_1 = 0\,. \tag{194}$$

The BPS equation for the hypermultiplet can be obtained by setting the modified supersymmetric transformation $\delta\zeta = 0$ (see Appendix E), which leads to

$$\begin{aligned}
\delta\zeta_{\alpha+} &= i\nabla\!\!\!\!/ A^i{}_\alpha\epsilon_{ij}\xi^j_- + 2g\overline{X}_\alpha{}^\beta A^i{}_\beta\epsilon_{ij}\xi^j_+ - A^i{}_\alpha\epsilon_{ij}\eta^j_+ + F^i{}_\alpha\epsilon_{ij}\xi^j_+ = 0\,,\\
\delta\zeta^\alpha_- &= \nabla\!\!\!\!/ A_i{}^\alpha\xi^i_+ - 2giX^\alpha{}_\beta A_i{}^\beta\xi^i_- + iA_i{}^\alpha\eta^i_- - iF_i{}^\alpha\xi^i_- = 0\,,
\end{aligned} \tag{195}$$

where $F_i{}^\alpha \equiv aA_i{}^{\alpha(z)}$ satisfying $F_i{}^\alpha = (F^i{}_\alpha)^* = \epsilon_{ij}\epsilon^{\alpha\beta}F^j{}_\beta$. One can combine these two equations using the Dirac notation in the following way:

$$\nabla\!\!\!\!/ A_i{}^\alpha\xi^i - 2gi(H^I - i\gamma_5 J^I)(t_I)^\alpha{}_\beta A_i{}^\beta\xi^i + iA_i{}^\alpha\eta^i - iF_i{}^\alpha\xi^i = 0\,. \tag{196}$$

We consider the model with the charges

$$t_I A_i{}^{\alpha} = P_I (i\sigma_3)^{\alpha}{}_{\beta} A_i{}^{\beta}\,, \tag{197}$$

where P_I are moment maps on the hyperkähler manifold with the scalars in the hypermultiplet as sections. In the gauge $A^i{}_{\alpha} \propto \delta^i{}_{\alpha}$, using the relation (191) one can express the BPS equation for the hypermultiplet as

$$\left[2g(H\cdot P) - 2gi\gamma_5(J\cdot P) - \frac{i}{L}\gamma_5\right] A_i{}^{\alpha}(\sigma_3)^i{}_j \xi^j - iF_i{}^{\alpha}\xi^i = 0\,, \tag{198}$$

which leads to the solution

$$F_j{}^{\alpha} = -2igA_i{}^{\alpha}(\sigma_3)^i{}_j(H\cdot P)\,, \quad 2g(J\cdot P) = -\frac{1}{L}\,. \tag{199}$$

13.2.2 Attractor Solution

As we discussed before, given a prepotential $F(X)$, the two-derivative off-shell action for the bosonic fields is given by Eq. (173). Now let us take a closer look at the theory and analyze its attractor solution. Later in the localization procedure, the localization locus will fluctuate around the attractor solution discussed in this subsection.

First, the field D plays the role of a Lagrange multiplier, which imposes the condition:

$$N_{IJ}\overline{X}^I X^J + \frac{1}{2}A^i{}_{\beta}A_i{}^{\alpha}d_{\alpha}{}^{\beta} = 0\,. \tag{200}$$

By requiring that the terms containing the Ricci scalar reproduce the Einstein-Hilbert action, we obtain

$$\frac{1}{6}N_{IJ}\overline{X}^I X^J - \frac{1}{6}A^i{}_{\beta}A_i{}^{\alpha}d_{\alpha}{}^{\beta} = \frac{1}{16\pi G}\,, \tag{201}$$

where G is the Newton's constant. These two equations lead to

$$N_{IJ}\overline{X}^I X^J = \frac{1}{8\pi G}\,, \quad A^i{}_{\beta}A_i{}^{\alpha}d_{\alpha}{}^{\beta} = -\frac{1}{4\pi G}\,. \tag{202}$$

In the gauge $A_i{}^{\alpha} \propto \delta_i^{\alpha}$, the second equation above implies

$$A_i{}^{\alpha} = \frac{1}{\sqrt{8\pi G}}\delta_i^{\alpha}\,, \tag{203}$$

where we have used ${d_\alpha}^\beta = -{\delta_\alpha}^\beta$ discussed in Appendix E.

By analyzing the field equations of various fields in the action (173), we arrive at the same solution that we found before from the BPS equations (199):

$$2g(J \cdot P) = -\frac{1}{L}\,, \tag{204}$$

more precisely,

$$8gJ^0 P_0 = -\frac{1}{L}\,, \quad 8gJ^1 P_1 = -\frac{3}{L}\,. \tag{205}$$

For the prepotential $F(X) = \sqrt{X^0(X^1)^3}$, the first one of Eq. (202) becomes

$$\frac{1}{4i}|X^0|^2 \left(\sqrt{\frac{X^1}{X^0}} - \overline{\sqrt{\frac{X^1}{X^0}}} \right)^3 = \frac{1}{8\pi G}\,, \tag{206}$$

which consequently leads to

$$(J^0)^{1/2}(J^1)^{3/2} = \frac{i}{16\pi G}\,, \tag{207}$$

where we choose $\sqrt{-1} = -i$.

13.2.3 Localization Action

As in the standard localization procedure, we can add a SUSY-exact term to the action without changing the partition function of the theory. The SUSY-exact term is called the localization action. In our case, we choose the following localization action for the vector multiplet:

$$\Delta S = \delta((\delta\Omega)^\dagger \Omega)\,, \tag{208}$$

where Ω denotes the gaugino field in the vector multiplet. The bosonic part of the localization action is

$$(\Delta S)_{\text{bos}} = (\delta\Omega)^\dagger \delta\Omega\,. \tag{209}$$

We can solve $(\Delta S)_{\text{bos}} = 0$ to find the localization locus. Some details are presented in Appendix G.

When expanding the localization action, we choose the Killing spinor found in Ref. [50] for the topological black hole with $q = 1$:

$$\epsilon = e^{-\frac{i}{2qL}\tau_E}\, e^{i\frac{u}{2}\gamma_4\gamma_1\gamma_2}\, e^{\frac{\phi}{2}\gamma_{23}}\, \widetilde{\epsilon}(r) \tag{210}$$

with

$$\widetilde{\epsilon}(r) = \left(\sqrt{\frac{r}{L} + \sqrt{f(r)}} - i\gamma_4\sqrt{\frac{r}{L} - \sqrt{f(r)}}\right)\left(\frac{1-\gamma_1}{2}\right)\epsilon'_0\,, \tag{211}$$

where ϵ'_0 is an arbitrary constant spinor, and $f(r)$ is the factor appearing in the metric of the topological black hole (160):

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}dr^2 + r^2\,d\Sigma(\mathbb{H}^2)\,.$$

In principle, there are 8 independent Killing spinors ϵ. Moreover, ϵ and $\epsilon^c \equiv B\epsilon^*$ satisfy the Killing spinor equations (186):

$$\nabla_\mu \epsilon = -\frac{1}{2L}\gamma_\mu \epsilon\,, \quad \nabla_\mu \epsilon^c = \frac{1}{2L}\gamma_\mu \epsilon^c\,.$$

As we discussed in Subsection 13.2.1, it is more convenient to work with the Killing spinors

$$\xi^i \equiv \frac{1+i\gamma_5}{2}\,\widetilde{\xi}^i\,, \tag{212}$$

with

$$\widetilde{\xi}^1 \equiv \epsilon^c\,, \quad \widetilde{\xi}^2 \equiv \epsilon\,. \tag{213}$$

and they satisfy the equivalent Killing spinor equation (190):

$$\nabla_\mu \xi^i = \frac{i}{2L}\gamma_5\gamma_\mu(\sigma_3)^i{}_j\xi^j\,.$$

The Killing spinors ξ^i generate the Killing vector

$$v = \xi^\dagger\gamma^\mu\xi\partial_\mu = \mathcal{L}_{U(1)}\,, \tag{214}$$

which is a linear combination of the compact $U(1)$'s along the compact directions τ and ϕ in the metric (324).

Using the Killing spinor discussed above with the special choice of the constant spinor $\epsilon'_0 = (1, 0, i, 0)^T$, we can compute various Killing spinor bilinears, and expand the localization action (209) explicitly. The localization action (209) can be expressed as a sum of some squares (338). By requiring these squares vanish, we obtain the following solutions:

$$H = \frac{C}{\cosh(\eta)}\,, \quad Y^1{}_1 = \frac{2C}{\cosh^2(\eta)}\,, \qquad \text{for } u = 0\,; \tag{215}$$

$$J = \text{const}\,, \quad F_{ab}V^b = 0\,, \tag{216}$$

where C is an arbitrary constant, and the constant value of J is fixed by the attractor solutions (205) (207). Together with the BPS solutions found in Subsection 13.2.1, these form the localization locus:

$$\begin{aligned} X^I &= H^I + iJ^I = \frac{C^I}{\cosh(\eta)} + iJ^I = \frac{J^I h^I}{\cosh(\eta)} + iJ^I\,, \\ (Y^I)^1{}_1 &= -(Y^I)^2{}_2 = \frac{2C^I}{\cosh^2(\eta)} - \frac{2i}{L}J^I = \frac{2J^I h^I}{\cosh^2(\eta)} - \frac{2i}{L}J^I\,, \end{aligned} \tag{217}$$

where we have written the gauge index I explicitly and used the parametrization $C^I = J^I h^I$, and again the values of J^I are fixed to be the attractor solutions given by Eq. (205) and Eq. (207).

For the hypermultiplet, as discussed in Appendix G, we require $\delta\zeta = 0$ for all 8 Killing spinors, which leads to the solutions

$$F_i{}^\alpha = -\frac{2ig}{\sqrt{8\pi G}}(\sigma_3)^\alpha{}_j(H\cdot P)\,, \quad 2g(J\cdot P) = -\frac{1}{L} \tag{218}$$

with $F_i{}^\alpha$ and P_I given by

$$F_i{}^\alpha = aA_i{}^{\alpha(z)}\,, \quad t_I A_i{}^\alpha = P_I(i\sigma_3)^\alpha{}_\beta A_i{}^\beta\,. \tag{219}$$

These solutions coincide with the solutions (199) to the BPS equations under the attractor solution (203).

13.2.4 Action on Localization Locus

Now we would like to evaluate the action (173) at the localization locus (217) obtained in the previous subsection. We distinguish the action for the vector multiplet and the action for the hypermultiplet:

$$S_{\text{vec}} = \int d^4x\sqrt{g}\left[N_{IJ}\overline{X}^I X^J\frac{R}{6} + N_{IJ}\partial\overline{X}^I\partial X^J - \frac{1}{8}N_{IJ}Y^{ijI}Y^J_{ij}\right], \tag{220}$$

$$\begin{aligned} S_{\text{hyp}} = \int d^4x\sqrt{g}\Bigg[\Bigg(&-\frac{R}{6}A^i{}_\beta A_i{}^\alpha + F^i{}_\beta F_i{}^\alpha + 4g^2 A^i{}_\beta\overline{X}^\alpha{}_\gamma X^\gamma{}_\delta A_i{}^\delta \\ &+ gA^i{}_\beta(Y^{jk})_\gamma{}^\alpha A_k{}^\gamma\epsilon_{ij}\Bigg)d_\alpha{}^\beta\Bigg]\,. \end{aligned} \tag{221}$$

However, there is a subtle difference between the AdS_4 case considered in Ref. [62] and the case considered in this paper. In Ref. [62], the AdS_4 metric is given by

$$ds^2 = L^2(d\eta^2 + \sinh^2(\eta)\, d\Omega_3^2)\,, \tag{222}$$

which leads to $d^4x\,\sqrt{g} = L^4 d\Omega_3 dr(r^2-1)$ with $r \equiv \cosh(\eta)$. In our case, the Euclidean AdS_4 neutral topological black hole is given by the metric (see Appendix F):

$$ds^2 = f(r)\, d\tau^2 + \frac{1}{f(r)}\, dr^2 + r^2\, d\Sigma(\mathbb{H}^2)\,, \tag{223}$$

where

$$f(r) = \frac{r^2}{L^2} - 1\,, \tag{224}$$

therefore, the measure becomes $d^4x\sqrt{g} = d\tau\, d\Sigma_2\, dr\, r^2$. Besides the volume form, we see that the measure for the integral dr differs for the two cases.

Although in our case the integrand evaluated at the localization locus is the same as the AdS_4 case discussed in Ref. [62], the final result is not the same due to the difference in the measure. Some details of the computation are presented in Appendix H. The final results for S_{vec} and S_{hyp} are

$$\begin{aligned} S_{\text{vec}} = \frac{\Omega_3^{\text{reg}} L^2}{32\pi G}\Bigg[& -4r_0^3 + \frac{r_0}{2}\left((h^0)^2 - 3h^1(4i+h^1) - 2h^0(2i+3h^1)\right) \\ & - 2(h^1-i)^{3/2}(h^0-i)^{1/2} + 6(h^1+i)^{3/2}(h^0+i)^{1/2} \\ & - 6i(h^1+i)^{1/2}(h^0+i)^{1/2} - 2i(h^1+i)^{3/2}(h^0+i)^{-1/2} + \mathcal{O}(1/r_0)\Bigg]\,, \end{aligned} \tag{225}$$

$$S_{\text{hyp}} = \frac{i\Omega_3^{\text{reg}}\, L^2}{16\pi G}(r_0-1)\left(h^0 + 3h^1\right)\,, \tag{226}$$

where r_0 and Ω_3^{reg} are the cutoff and the regularized volume of the boundary $S^1 \times \mathbb{H}^2$ respectively, and we have used the attractor solutions given by Eq. (205):

$$8gJ^0P_0 = -\frac{1}{L}\,, \quad 8gJ^1P_1 = -\frac{3}{L}\,,$$

and Eq. (207):

$$(J^0)^{1/2}(J^1)^{3/2} = \frac{i}{16\pi G}\,.$$

Altogether, the action evaluated at the localization locus is

$$
\begin{aligned}
S &= S_{\rm vec} + S_{\rm hyp} \\
&= -\frac{\Omega_3^{\rm reg} L^2}{32\pi G}\Bigg[4r_0^3 - \frac{r_0}{2}\left((h^0)^2 - 3(h^1)^2 - 6h^0h^1\right) \\
&\qquad + 2(h^1-i)^{3/2}(h^0-i)^{1/2} - 6(h^1+i)^{3/2}(h^0+i)^{1/2} + 2i(h^0+3h^1) \\
&\qquad + 6i(h^1+i)^{1/2}(h^0+i)^{1/2} + 2i(h^1+i)^{3/2}(h^0+i)^{-1/2} + \mathcal{O}(1/r_0)\Bigg] .
\end{aligned} \tag{227}
$$

13.2.5 Holographic Renormalization

To remove the divergence depending on the cutoff r_0 in the action (227), which is

$$
-\frac{\Omega_3^{\rm reg} L^2}{8\pi G}\left[r_0^3 - \frac{r_0}{8}\left((h^0)^2 - 3(h^1)^2 - 6h^0h^1\right)\right], \tag{228}
$$

we apply the standard holographic renormalization by adding some boundary counter-terms:

$$
S_{\rm ct} = S_{\rm GH} + \frac{1}{2}S_{B2} . \tag{229}
$$

$S_{\rm GH}$ is the Gibbons-Hawking term given by

$$
\begin{aligned}
S_{\rm GH} &= \int d^3x \sqrt{g_3}\, N_{IJ}\overline{X}^I X^J \frac{\kappa}{3} \\
&= \Omega_3^{\rm reg} L^2 \left(r_0^3 - \frac{2}{3}r_0\right)\left(N_{IJ}\overline{X}^I X^J\right)\Big|_{r_0} \\
&= \frac{\Omega_3^{\rm reg} L^2}{8\pi G}\left(r_0^3 - \frac{2}{3}r_0\right)\left[1 - \frac{(h^0)^2 - 3(h^1)^2 - 6h^0h^1}{8r_0^2} + \mathcal{O}(r_0^{-4})\right],
\end{aligned} \tag{230}
$$

where κ is the extrinsic curvature, which for the metric (324) has the value:

$$
\kappa = \frac{f'(r)}{2\sqrt{f(r)}} + \frac{2\sqrt{f(r)}}{r} . \tag{231}
$$

S_{B2} is the boundary term proportional to the boundary scalar curvature: [5]

$$\begin{aligned} S_{B2} &= -\int d^3x\,\sqrt{g_3} N_{IJ}\overline{X}^I X^J \frac{2L\mathcal{R}}{3} \\ &= \Omega_3^{\text{reg}} L^2 \left(\frac{4}{3} r_0 + \mathcal{O}(r_0^{-1})\right) \left(N_{IJ}\overline{X}^I X^J\right)\Big|_{r_0} \\ &= \frac{\Omega_3^{\text{reg}} L^2}{8\pi G} \left(\frac{4}{3} r_0 + \mathcal{O}(r_0^{-1})\right) \left[1 - \frac{(h^0)^2 - 3(h^1)^2 - 6h^0 h^1}{8r_0^2} + \mathcal{O}(r_0^{-4})\right] , \end{aligned} \tag{232}$$

where the boundary scalar curvature is $\mathcal{R} = -2/r^2$ for the boundary $\mathrm{S}^1 \times \mathbb{H}^2$. Therefore,

$$\begin{aligned} S_{\text{ct}} &= S_{\text{GH}} + \frac{1}{2} S_{B2} \\ &= \frac{\Omega_3^{\text{reg}} L^2}{8\pi G} \left[r_0^3 - \frac{r_0}{8}\left((h^0)^2 - 3(h^1)^2 - 6h^0 h^1\right) + \mathcal{O}(r_0^{-1})\right] , \end{aligned} \tag{233}$$

which cancels exactly the divergnce (228) depending on the cutoff r_0 in the action (227).

However, like in the AdS$_4$ case, the coupling between the boundary curvatures and the hypermultiplet introduces new divergence depending on r_0:

$$S' = \int d^3x\,\sqrt{g_3}(-A^2)\left(\frac{\kappa}{3} - \frac{L\mathcal{R}}{3}\right) = -\frac{\Omega_3^{\text{reg}} L^2}{4\pi G} r_0^3 + \mathcal{O}(r_0^{-1}) , \tag{234}$$

where we have used $A^2 = 1/(4\pi G)$. To cancel this divergence, we have to take into account a boundary term for the flux:

$$S_{\text{flux}} = -i\frac{N}{3\pi^2} \int_{\mathrm{S}^1 \times \mathbb{H}^2} C_3 , \tag{235}$$

where C_3 has the following form:

$$C_3 = \frac{a}{3}(r^3 - 1) d\Omega_3 \tag{236}$$

[5] Compared to Ref. [62], in this paper there is an extra factor 4 in S_{B2}, because for the AdS$_4$ case considered in Ref. [62] the boundary scalar curvature is $24/(L^2 \sinh^2(\eta))$, which is $6/(L^2 \sinh^2(\eta))$ according to our convention.

with a constant a, such that C_3 vanishes at the horizon and the field strength $F_4 = dC_3$ satisfies

$$F_4 = a\,\omega_{\mathrm{AdS}_4}\,, \tag{237}$$

where ω_{AdS_4} is the volume form of the AdS_4 neutral topological black hole in our case. As discussed in Ref. [63], for ABJM theory, $X_7 = \mathrm{S}^7/\mathbb{Z}_k$, which is a Hopf fibration over $M_6 = \mathbb{CP}^3$. Integrating $*F_4$ over X_7 gives

$$\int_{X_7} *F_4 = 6iL^6\,\mathrm{Vol}(X_7)\,, \tag{238}$$

which can also be related to the flux N through [63, 67]

$$N = \frac{6L^6\,\mathrm{Vol}(X_7)}{(2\pi\ell_P)^6}\,. \tag{239}$$

The condition (238) consequently fixes the constant $a = 3iL^3/8$.

From the dimensional reduction of the 11-dimensional M-theory to 4 dimensions, we obtain

$$\frac{L^7\,\mathrm{Vol}(X_7)}{64\pi G_{11}} = \frac{1}{16\pi G_4}\,, \tag{240}$$

where $16\pi G_{11} = (2\pi)^8$ in the unit $\ell_P = 1$. Moreover, using the relation $\mathrm{Vol}(X_7) \sim \mathrm{Vol}(M_6)/k$ as well as the relation between N and $\mathrm{Vol}(X_7)$ discussed above, one can express $\Omega_3^{\mathrm{reg}} L^2/(4\pi G_4)$ in terms of k and N. After choosing an appropriate normalization factor, we have

$$-\frac{\Omega_3^{\mathrm{reg}} L^2}{8\pi G_4} = \frac{\sqrt{2}\pi}{3} k^{1/2} N^{3/2}\,. \tag{241}$$

Altogether, the divergence appearing in S' (234) is canceled by the flux term S_{flux} (235), and a finite contribution from S_{flux} remains:

$$S_{\mathrm{flux}} \supset -\frac{\Omega_3^{\mathrm{reg}} L^2}{4\pi G_4} = \frac{2\sqrt{2}\pi}{3} k^{1/2} N^{3/2}\,. \tag{242}$$

Finally, after the holographic renormalization the remaining finite part of the action is

$$S_{\text{finte}} = \frac{\sqrt{2}\pi}{6} k^{1/2} N^{3/2} \Bigg[(h^1 - i)^{3/2} (h^0 - i)^{1/2} + i(h^0 + 3h^1) + 4$$
$$- 3(h^1 + i)^{3/2} (h^0 + i)^{1/2} + 3i(h^1 + i)^{1/2} (h^0 + i)^{1/2} + i(h^1 + i)^{3/2} (h^0 + i)^{-1/2} \Bigg]$$
$$= \frac{\sqrt{2}\pi}{6} k^{1/2} N^{3/2} \Bigg[- (1 + ih^1)^{3/2} (1 + ih^0)^{1/2} + i(h^0 + 3h^1) + 4$$
$$+ 3(1 - ih^1)^{3/2} (1 - ih^0)^{1/2} - 3(1 - ih^1)^{1/2} (1 - ih^0)^{1/2} - (1 - ih^1)^{3/2} (1 - ih^0)^{-1/2} \Bigg] . \tag{243}$$

As a check, we can also turn off all the fluctuations h^0 and h^1 in Eq. (243), which gives us

$$S_{\text{finite}}(h^0 = 0, h^1 = 0) = -\frac{\Omega_3^{\text{reg}} L^2}{8\pi G_4} = \frac{\pi L^2}{2G_4} = \frac{\sqrt{2}\pi}{3} k^{1/2} N^{3/2} , \tag{244}$$

where we have used the regularized volume for $\mathrm{S}^1 \times \mathbb{H}^2$ [75, 76], $\Omega_3^{\text{reg}} = -4\pi^2$. This expression of the finite action is exactly equal to the on-shell action of the AdS_4 neutral topological black hole [50, 51].

13.2.6 Evaluation of the Integral

Finally, let us evaluate the path integral with the finte part of the action S_{finte} after holographic renormalization given by Eq. (243). Unlike the AdS_4 case discussed in Ref. [62], the result of the path integral in this case is not an Airy function. Instead, we can apply the steepest descent method (see e.g. [77]) to obtain the asymptotic expression, which suffices for our purpose of computing the entanglement entropy in the large-N expansion.

We first find that S_{finte} in Eq. (243) has only one critical point $(h^0,\, h^1) = (0,\, 0)$, hence there are no Stokes phenomena in this case. S_{finite} evaluated at this critical point is equal to the on-shell value shown above in Eq. (244). Next, we can expand S_{finite} around this critical pont and perform the integration over $h^{0,1}$ to obtain the asymptotic expression for the partition function Z, which is given by

$$Z = \int Dh^0\, Dh^1\, e^{-S_{\text{finite}}} . \tag{245}$$

In principle, there can also be some nontrivial Jacobian for $h^{0,1}$ in the path integral.

Around the critical point $(h^0,\, h^1) = (0,\, 0)$, for $h^0 \sim \epsilon$, $h^1 \sim \epsilon$ the expansion to the order $\mathcal{O}(\epsilon^2)$ is given by

$$S_{\text{finite}} = \frac{\sqrt{2}\pi}{6} k^{1/2} N^{3/2} \left[2 + \frac{1}{4}(h^0)^2 - \frac{3}{2} h^0 h^1 - \frac{3}{4}(h^1)^2 + o(\epsilon^2) \right] , \qquad (246)$$

where the constant 2 in the brackets gives the on-shell contribution. Let us compare this expansion with the one for the AdS_4 case. For the AdS_4 case discussed in Ref. [62], the finite part of the action after holographic renormalization is given by

$$S_{\text{ren}} = -\frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2} \left[(1 - i h^1)^{3/2} \sqrt{1 - i h^0} + \frac{i}{2}(3 h^1 + h^0) - 2 \right] . \qquad (247)$$

It has also only one critical point $(h^0,\, h^1) = (0,\, 0)$. We can similarly expand S_{ren} around its critical point and obtain the expansion

$$S_{\text{ren}} = -\frac{\sqrt{2}\pi}{3} k^{1/2} N^{3/2} \left[-1 + \frac{1}{8}(h^0)^2 - \frac{3}{4} h^0 h^1 - \frac{3}{8}(h^1)^2 + o(\epsilon^2) \right] . \qquad (248)$$

We see that the expansion (248) for the AdS_4 case has exactly the same on-shell contribution as the expansion (246) for the topological black hole case, while at the order $\mathcal{O}(\epsilon^2)$ they only differ by a sign, which can be compensated by rotating the contours in the integrals. The true discrepancy takes place at higher orders $\sim o(\epsilon^2)$, which is not just a sign difference.

Although the partition function for the AdS_4 neutral topological black hole is not exactly given by an Airy function as in the AdS_4 case, based on the comparison above, we expect that the asymptotic expressions of the partition function for both cases should coincide at leading orders up to a phase factor. However, assuming a flat measure for $h^{0,1}$ in the path integral (245), a direct computation of the partition function Z by integrating $h^{0,1}$ in S_{finite} (246) does not give the result that we expect. Instead, if we adopt the assumptions made in Ref. [62] and perform the same procedure by dropping some constants from the Jacobian and the Gaussian integral, due to the same expansion of the renormalized action around the critical point, for the AdS_4 neutral topological black hole the partition function of the supergravity has the expected asymptotic expansion for $z \to \infty$:

$$Z \sim \frac{\exp\left(-\frac{2}{3} z^{3/2}\right)}{2\sqrt{\pi}(-z)^{1/4}} \left(1 + \mathcal{O}(z^{-3/2})\right) , \qquad (249)$$

where $z \equiv N(\pi^2 k/2)^{1/3}$.

Therefore, we find the partition function (249) of the $4D$ $\mathcal{N}=2$ off-shell gauged supergravity on the background of the AdS_4 neutral topological black hole. Although this result is not exactly the same as the partition function for the ABJM theory on round S^3 [48] or the partition function of the 4D $\mathcal{N}=2$ off-shell gauged supergravity on AdS_4 [62], which are equal to $\mathrm{Ai}(z)$ perturbatively and have the asymptotic expansion for $-\pi < \arg(z) < \pi$:

$$\mathrm{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n\, \Gamma\left(n+\frac{5}{6}\right)\, \Gamma\left(n+\frac{1}{6}\right)\, \left(\frac{3}{4}\right)^n}{2\pi\, n!\, z^{3n/2}} \right], \tag{250}$$

all these theories share the same asymptotic expansion at the leading order up to a phase given by Eq. (249), which consequently leads to the free energy:

$$F = -\log Z = \frac{\sqrt{2}\pi}{3} k^{1/2} N^{3/2} + \frac{1}{4}\log(N) + \mathcal{O}(N^0)\,, \tag{251}$$

where we single out the logarithmic term, whose coefficient is universal and has some physical meaning that we will briefly discuss in the next section.

14 Black Hole Entropy and Entanglement Entropy

After computing the partition function and the free energy of the 4D $\mathcal{N}=2$ off-shell gauged supergravity on the background of the AdS_4 neutral topological black hole, we can relate them to the black hole entropy according to Refs. [50, 51]:

$$S_{BH} = -F = \log Z = -\frac{\sqrt{2}\pi}{3} k^{1/2} N^{3/2} - \frac{1}{4}\log(N) + \mathcal{O}(N^0)\,. \tag{252}$$

The leading term in the expression above corresponds to the contribution from the Bekenstein-Hawking entropy formula of the black hole [64, 65, 66], i.e. $A/(4G)$, while the second term is the logarithmic correction. Therefore, the supergravity localization indeed provides a way of computing the logarithmic corrections of the black hole entropy.

Based on the AdS/CFT correspondence or more precisely the gravity dual of supersymmetric Rényi entropy discussed in Refs. [50, 51], we can interpret the entropy of the AdS_4 neutral topological black hole obtained above from the supergravity localization as the entanglement entropy of the ABJM theory across a circle S^1 on the boundary, i.e.,

$$S_{EE}^{\text{ABJM}} = S_{BH} = -\frac{\sqrt{2}\pi}{3} k^{1/2} N^{3/2} - \frac{1}{4}\log(N) + \mathcal{O}(N^0)\,. \tag{253}$$

This interpretation is also consistent with the Ryu-Takayanagi formula [40, 78] of the entanglement entropy for the boundary conformal field theory.

The entanglement entropy of the ABJM theory can also be obtained directly from the field theory side by evaluating the S^3-partition function of the ABJM theory [68, 48] and expanding it at large N:

$$S_{EE}^{\text{ABJM}} = -\frac{\sqrt{2}\pi}{3} k^{1/2} N^{3/2} + \frac{\sqrt{2}\pi(8+k^2)}{48} k^{-1/2} N^{1/2} - \frac{1}{4}\log(N) + \mathcal{O}(N^0)\,. \tag{254}$$

In particular, we can consider a special case $k = 1$:

$$S_{EE,\,k=1}^{\text{ABJM}} = -\frac{\sqrt{2}\pi}{3} N^{3/2} + \frac{3\sqrt{2}\pi}{16} N^{1/2} - \frac{1}{4}\log(N) + \mathcal{O}(N^0)\,. \tag{255}$$

This result also coincides with the $q \to 1$ limit of the supersymmetric Rényi entropy for $k = 1$, which was discussed in Refs. [41, 49].

Comparing the results Eq. (253) and Eq. (254), we see that they differ in two places. First, the result from supergravity localization (253) does not have the contribution of the order $\sim N^{1/2}$. This is due to the fact that this term corresponds to the stringy corrections, which cannot be taken into account within supergravity, as discussed in Ref. [62]. More precisely, the resulf of supergravity localization on AdS_4 differs by a shift in N compared to the one from field theory localization, i.e., instead of N/k the field theory localization has

$$\frac{N}{k}\left(1 - \frac{1}{24}\frac{k}{N} - \frac{1}{3}\frac{1}{Nk}\right) \sim \frac{N}{k}\left(1 + C_1 \frac{\ell^4}{L^4} + C_2\, g_4^2\right)\,, \tag{256}$$

where

$$\frac{N}{k} \sim \frac{L^4}{\ell^4}\,, \quad \frac{1}{Nk} \sim g_4^2\,, \tag{257}$$

with g_4, L and $g_4\ell$ denoting the 4D string coupling, the AdS_4 radius and the 4D Planck length respectively. Hence, these shifts correspond to some stringy effects, and to reproduce them requires some stringy computation in the bulk. The similar thing happens in our calculations, i.e., the supergravity localization on the AdS_4 neutral topological black hole cannot reproduce some stringy corrections, which includes the term $\sim N^{1/2}$.

Second difference between Eq. (253) and Eq. (254) is that the expression (254) comes from the expansion of the Airy function, while Eq. (253) does not. Although they have the same leading-order expression and the logarithmic correction, the discrepancy emerges at higher orders, since the result from supergravity localization (253) does not exactly reproduce an Airy function. This fact is closely related to the recent works of field theory localization on noncompact manifolds [79, 80, 81], whose results depend on the boundary conditions. In particular, Ref. [81] has discussed the localization of the ABJM theory on $S^1 \times \mathbb{H}^2$, which is precisely the field theory dual of the gravity considered in this paper. Instead of the untwisted partition function, in Ref. [81] the authors focused on the topologically twisted index of the ABJM theory on $S^1 \times \mathbb{H}^2$ similar to Ref. [82] and evaluated it at the leading order in N. We expect that the large-N result of the topologically twisted index in a special limit could match exactly our result obtained from the gravity side, which requires an analysis of the matrix integral beyond the leading order of N.

Concerning the black hole entropy, there have already been some works devoted to this topic using the supergravity localization technique (e.g. [60, 61, 83, 84, 85, 86]). Our work provides a concrete example and relates the bulk black hole entropy to the entanglement entropy of the boundary conformal field theory. In particular, the near-horizon geometry of the AdS_4 neutral topological black hole (160) that we discussed is $AdS_2 \times \mathbb{H}^2$. According to Ref. [84], the quantum black hole entropy has the general form:

$$S_{\mathrm{BH}}^{\mathrm{qu}} = \frac{A}{4} + a_0 \log A + \cdots \,, \tag{258}$$

where a_0 is the coefficient that can be computed by the supergravity localization for each multiplet on the black hole background or by direct evaluation of the 1-loop determinant around the classical attractor background [87, 88, 89, 90]. We can compare the general expression (258) with our result (252). The identification of the leading terms implies that

$$A \sim N^{3/2} \,. \tag{259}$$

Together with the identification of the logarithmic terms, we deduce that in our case

$$a_0 = -\frac{1}{6}\,, \tag{260}$$

which is twice the contribution of one vector multiplet $a_0^{\text{vec}} = -1/12$ [84]. This is consistent with the fact in our case only $(n_v+1) = 2$ vector multiplets have fluctuations, while the metric and the hypermultiplet are fixed to be the attractor solutions.

Moreover, the logarithmic correction to the black hole entropy was also obtained on-shell from the 1-loop computation in the Euclidean 11-dimensional supergravity on $\text{AdS}_4 \times X_7$, and the result coincides with the logarithmic term in the large-N expansion of the partition function of the ABJM theory on S^3 [67], which is also consistent with our result from the localization of the 4D $\mathcal{N}=2$ off-shell supergravity on the AdS_4 neutral topological black hole.

15 Conclusion and Discussion

In this paper we have calculated the partition function of the 4D $\mathcal{N}=2$ off-shell gauged supergravity in the background of the AdS_4 neutral topological black hole via supersymmetric localization. The free energy of the theory is related to the black hole entropy, and using the localization we obtain the logarithmic correction to the leading order result given by the Bekenstein-Hawking formula. Moreover, we compare the black hole entropy with the entanglement entropy of the ABJM theory across a circle S^1 on the boundary and find an exact match up to some stringy effects, which provides a precise test of the AdS/CFT correspondence beyond the leading order.

There are many more interesting extensions of this work for the future research. For instance, one can compute supersymmetric Rényi entropy of the boundary ABJM theory via supergravity localization on backgrounds of charged topological black holes, which generalizes the classical results of Refs. [50, 51] on the gravity side and can also be compared with the exact results on the field theory side discussed in Ref. [49]. Another possible extension is to compute the supersymmetric Wilson loop via supergravity localization. The result can be compared with the exact result on the field theory side [68], and also generalizes the classical result of Ref. [51].

Related to the recent works on the supersymmetric localization of field theories on noncompact manifolds [80, 79] and in particular the topologically twisted index of the ABJM theory on $\mathrm{S}^1 \times \mathbb{H}^2$ discussed in Ref. [81], we expect that the localization result from the field theory side can reproduce our result obtained from the supergravity side beyond the leading order in N. In general, the study of the field theory localization on noncompact manifolds is very important, not only because it provides another precise test of the AdS/CFT correspondence, also because it is directly related to some exact computations of entanglement entropy, supersymmetric Rényi entropy as well as the bulk black hole entropy.

Recently, in Ref. [82] Benini, Hristov and Zaffaroni have found a new relation between the topologically twisted index of the ABJM theory on $\mathrm{S}^2 \times \mathrm{S}^1$ and the entropy of the 4D STU black hole, which in principle allows one to count the microstates of the black hole in the dual field theory. As we mentioned earlier, this work has also been generalized to the ABJM theory on $S^1 \times \mathbb{H}^2$ and correspondingly the AdS_4 hyperbolic black hole [81]. Using the technique of localization of supergravity to the near-horizon geometry of the 4D STU black hole, we should be able to test this new correspondence beyond the leading order and compare the results also to the supersymmetric Rényi entropy.

Finally, supergravity localization itself still needs more study. As we have seen from the text, the metric is fixed as a background, i.e., we have not taken into account the fluctuations of the Weyl multiplet. Although some indirect result can be deduced [84], as far as we know, there is still no direct computation of the localization of the Weyl multiplet in the literature. In some sense, we are studying supergravity as a special kind of quantum field theory on a curved manifold, which shares the same spirit of the work by Festuccia and Seiberg [91]. More detailed study is definitedly required to truly understand the behavior of supergravity. We would like to investigate this open problem in the future research.

A Two-dimensional $\mathcal{N} = (2, 2)$ superspace

The bosonic coordinates of the superspace are $x^\mu, \mu = 0, 1$. We take the flat Minkowski metric to be $\eta_{\mu\nu} = \mathrm{diag}(-1, 1)$. The fermionic coordinates of the superspace are θ^+, θ^-, $\overline{\theta}^+$ and $\overline{\theta}^-$, with the complex conjugation relation $(\theta^\pm)^* = \overline{\theta}^\pm$. The indices $\pm$ stand for the chirality under a Lorentz transformation. To raise or lower the spinor index, we use

$$\psi_\alpha = \epsilon_{\alpha\beta}\,\psi^\beta\,, \quad \psi^\alpha = \epsilon^{\alpha\beta}\,\psi_\beta\,, \tag{261}$$

where

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \alpha,\beta = -,+\,. \tag{262}$$

Hence, we have $\psi_+ = \psi^-$, $\psi_- = -\psi^+$.

The supercharges and the supercovariant derivative operators are

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\partial_\pm\,, \quad \overline{Q}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\partial_\pm\,, \tag{263}$$

$$\mathbb{D}_\pm = \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\partial_\pm\,, \quad \overline{\mathbb{D}}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\partial_\pm\,, \tag{264}$$

where

$$\partial_\pm \equiv \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right). \tag{265}$$

They satisfy the anti-commutation relations

$$\{Q_\pm, \overline{Q}_\pm\} = -2i\partial_\pm\,, \quad \{\mathbb{D}_\pm, \overline{\mathbb{D}}_\pm\} = 2i\partial_\pm\,, \tag{266}$$

with all the other anti-commutators vanishing. In particular,

$$\{Q_\pm, \mathbb{D}_\pm\} = 0\,. \tag{267}$$

B Gauged linear sigma model with semichiral superfields in components

If we expand the theory (80) in components, we obtain the Lagrangian

$$
\begin{aligned}
\mathcal{L}_{SC} = {} & \overline{X}^L 2iD_- 2iD_+ X^L - \overline{X}^L(\sigma_1^2+\sigma_2^2)X^L + \overline{X}^L D X^L + \overline{F}^L F^L \\
& - \overline{M}^L_{-+} M^L_{-+} - \overline{M}^L_{--} 2iD_+ X^L - \overline{X}^L 2iD_+ M^L_{--} + \overline{M}^L_{-+}\overline{\widehat{\sigma}} X^L + \overline{X}^L \widehat{\sigma} M^L_{-+} \\
& + \overline{\psi}^L_- 2iD_+ \psi^L_- + \overline{\psi}^L_+ 2iD_- \psi^L_+ - \overline{\psi}^L_- \widehat{\sigma} \psi^L_+ - \overline{\psi}^L_+ \overline{\widehat{\sigma}} \psi^L_- \\
& + \overline{X}^L i\lambda_+ \psi^L_- - \overline{X}^L i\lambda_- \psi^L_+ + \overline{\psi}^L_+ i\overline{\lambda}_- X^L - \overline{\psi}^L_- i\overline{\lambda}_+ X^L - \overline{\eta}^L_- \psi^L_+ - \overline{\psi}^L_+ \eta^L_- \\
& - \overline{\chi}^L_- 2iD_+ \chi^L_- + \overline{X}^L i\overline{\lambda}_+ \chi^L_- - \overline{\chi}^L_- i\lambda_+ X^L \\
& + \overline{X}^R 2iD_- 2iD_+ X^R - \overline{X}^R(\sigma_1^2+\sigma_2^2)X^R + \overline{X}^R D X^R + \overline{F}^R F^R \\
& - \overline{M}^R_{+-} M^R_{+-} - \overline{M}_{++} 2iD_- X^R - \overline{X}^R 2iD_- M^R_{++} + \overline{M}^R_{+-} \widehat{\sigma} X^R + \overline{X}^R \overline{\widehat{\sigma}} M^R_{+-} \\
& + \overline{\psi}^R_- 2iD_+ \psi^R_- + \overline{\psi}^R_+ 2iD_- \psi^R_+ - \overline{\psi}^R_- \widehat{\sigma} \psi^R_+ - \overline{\psi}^R_+ \overline{\widehat{\sigma}} \psi^R_- \\
& + \overline{X}^R i\lambda_+ \psi^R_- - \overline{X}^R i\lambda_- \psi^R_+ + \overline{\psi}^R_+ i\overline{\lambda}_- X^R - \overline{\psi}^R_- i\overline{\lambda}_+ X^R + \overline{\eta}^R_+ \psi^R_- + \overline{\psi}^R_- \eta^R_+ \\
& - \overline{\chi}^R_+ 2iD_- \chi^R_+ - \overline{X}^R i\overline{\lambda}_- \chi^R_+ + \overline{\chi}^R_+ i\lambda_- X^R \\
& + \alpha\overline{X}^L 2iD_- 2iD_+ X^R - \alpha\overline{X}^L(\sigma_1^2+\sigma_2^2)X^R + \alpha\overline{X}^L D X^R + \alpha\overline{F}^L F^R \\
& + \alpha\overline{M}^L_{--} M^R_{++} - \alpha\overline{M}^L_{--} 2iD_+ X^R - \alpha\overline{X}^L 2iD_- M^R_{++} + \alpha\overline{M}^L_{-+} \overline{\widehat{\sigma}} X^R + \alpha\overline{X}^L \overline{\widehat{\sigma}} M^R_{+-} \\
& + \alpha\overline{\psi}^L_- 2iD_+ \psi^R_- + \alpha\overline{\psi}^L_+ 2iD_- \psi^R_+ - \alpha\overline{\psi}^L_- \widehat{\sigma} \psi^R_+ - \alpha\overline{\psi}^L_+ \overline{\widehat{\sigma}} \psi^R_- \\
& + \alpha\overline{X}^L i\lambda_+ \psi^R_- - \alpha\overline{X}^L i\lambda_- \psi^R_+ + \alpha\overline{\psi}^L_+ i\overline{\lambda}_- X^R - \alpha\overline{\psi}^L_- i\overline{\lambda}_+ X^R - \alpha\overline{\eta}^L_- \psi^R_+ + \alpha\overline{\psi}^L_- \eta^R_+ \\
& + \alpha\overline{\chi}^L_- \overline{\widehat{\sigma}} \chi^R_+ - \alpha\overline{X}^L i\overline{\lambda}_- \chi^R_+ - \alpha\overline{\chi}^L_- i\lambda_+ X^R \\
& + \alpha\overline{X}^R 2iD_- 2iD_+ X^L - \alpha\overline{X}^R(\sigma_1^2+\sigma_2^2)X^L + \alpha\overline{X}^R D X^L + \alpha\overline{F}^R F^L \\
& + \alpha\overline{M}^R_{++} M^L_{--} - \alpha\overline{M}^R_{++} 2iD_- X^L - \alpha\overline{X}^R 2iD_+ M^L_{--} + \alpha\overline{M}^R_{+-} \widehat{\sigma} X^L + \alpha\overline{X}^R \widehat{\sigma} M^L_{-+} \\
& + \alpha\overline{\psi}^R_- 2iD_+ \psi^L_- + \alpha\overline{\psi}^R_+ 2iD_- \psi^L_+ - \alpha\overline{\psi}^R_- \widehat{\sigma} \psi^L_+ - \alpha\overline{\psi}^R_+ \overline{\widehat{\sigma}} \psi^L_- \\
& + \alpha\overline{X}^R i\lambda_+ \psi^L_- - \alpha\overline{X}^R i\lambda_- \psi^L_+ + \alpha\overline{\psi}^R_+ i\overline{\lambda}_- X^L - \alpha\overline{\psi}^R_- i\overline{\lambda}_+ X^L + \alpha\overline{\eta}^R_+ \psi^L_- - \alpha\overline{\psi}^R_+ \eta^L_- \\
& + \alpha\overline{\chi}^R_+ \widehat{\sigma} \chi^L_- + \alpha\overline{X}^R i\overline{\lambda}_+ \chi^L_- + \alpha\overline{\chi}^R_+ i\lambda_- X^L \,.
\end{aligned}
\tag{268}
$$

The supersymmetry transformation laws for the abelian vector multiplet are

$$
\begin{aligned}
\delta A_\mu &= \frac{i}{2}\epsilon\sigma_\mu\overline{\lambda} + \frac{i}{2}\overline{\epsilon}\sigma_\mu\lambda\,, \\
\delta\widehat{\sigma} &= -i\epsilon_-\overline{\lambda}_+ - i\overline{\epsilon}_+\lambda_-\,, \\
\delta\overline{\widehat{\sigma}} &= -i\epsilon_+\overline{\lambda}_- - i\overline{\epsilon}_-\lambda_+\,, \\
\delta\lambda_+ &= 2\epsilon_-\partial_+\overline{\widehat{\sigma}} + i\epsilon_+D - \epsilon_+F_{01}\,, \\
\delta\lambda_- &= 2\epsilon_+\partial_-\widehat{\sigma} + i\epsilon_-D + \epsilon_-F_{01}\,, \\
\delta\overline{\lambda}_+ &= 2\overline{\epsilon}_-\partial_+\widehat{\sigma} - i\overline{\epsilon}_+D - \overline{\epsilon}_+F_{01}\,, \\
\delta\overline{\lambda}_- &= 2\overline{\epsilon}_+\partial_-\overline{\widehat{\sigma}} - i\overline{\epsilon}_-D + \overline{\epsilon}_-F_{01}\,, \\
\delta D &= \epsilon_+\partial_-\overline{\lambda}_+ + \epsilon_-\partial_+\overline{\lambda}_- - \overline{\epsilon}_+\partial_-\lambda_+ - \overline{\epsilon}_-\partial_+\overline{\lambda}_-\,,
\end{aligned}
\tag{269}
$$

where $F_{01} = \partial_0 A_1 - \partial_1 A_0$, and

$$
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{270}
$$

The supersymmetry transformations for the components of semichiral multiplets $\mathbb{X}$ are

$$
\begin{aligned}
\delta X &= \epsilon\psi + \overline{\epsilon}\chi\,, \\
\delta\psi_+ &= -\epsilon_+F - \overline{\epsilon}_+\overline{\widehat{\sigma}}X + \overline{\epsilon}_+M_{-+} + \overline{\epsilon}_-2iD_+X - \overline{\epsilon}_-M_{++}\,, \\
\delta\psi_- &= -\epsilon_-F - \overline{\epsilon}_+2iD_-X + \overline{\epsilon}_+M_{--} + \overline{\epsilon}_-\widehat{\sigma}X - \overline{\epsilon}_-M_{+-}\,, \\
\delta F &= \overline{\epsilon}_+2iD_-\psi_+ + \overline{\epsilon}_-2iD_+\psi_- - \overline{\epsilon}_+\eta_- + \overline{\epsilon}_-\eta_+ - \overline{\epsilon}_+\overline{\widehat{\sigma}}\psi_- - \overline{\epsilon}_-\widehat{\sigma}\psi_+ + i\overline{\epsilon}_+\overline{\lambda}_-X - i\overline{\epsilon}_-\overline{\lambda}_+X\,, \\
\delta\chi_+ &= -\epsilon_-M_{++} + \epsilon_+M_{+-}\,, \\
\delta\chi_- &= -\epsilon_-M_{-+} + \epsilon_+M_{--}\,, \\
\delta M_{+-} &= -\epsilon_-\eta_+ + \overline{\epsilon}_-\widehat{\sigma}\chi_+ - \overline{\epsilon}_+2iD_-\chi_+\,, \\
\delta M_{-+} &= -\epsilon_+\eta_- + \overline{\epsilon}_-2iD_+\chi_- - \overline{\epsilon}_+\overline{\widehat{\sigma}}\chi_-\,, \\
\delta M_{++} &= -\epsilon_+\eta_+ + \overline{\epsilon}_-2iD_+\chi_+ - \overline{\epsilon}_+\overline{\widehat{\sigma}}\chi_+\,, \\
\delta M_{--} &= -\epsilon_-\eta_- + \overline{\epsilon}_-\widehat{\sigma}\chi_- - \overline{\epsilon}_+2iD_-\chi_-\,, \\
\delta\eta_+ &= \overline{\epsilon}_-2iD_+M_{+-} - \overline{\epsilon}_-i\overline{\lambda}_+\chi_+ - \overline{\epsilon}_-\widehat{\sigma}M_{++} - \overline{\epsilon}_+\overline{\widehat{\sigma}}M_{+-} + \overline{\epsilon}_+i\overline{\lambda}_-\chi_+ + \overline{\epsilon}_+2iD_-M_{++}\,, \\
\delta\eta_- &= \overline{\epsilon}_-2iD_+M_{--} - \overline{\epsilon}_-i\overline{\lambda}_+\chi_- - \overline{\epsilon}_-\widehat{\sigma}M_{-+} - \overline{\epsilon}_+\overline{\widehat{\sigma}}M_{--} + \overline{\epsilon}_+i\overline{\lambda}_-\chi_- + \overline{\epsilon}_+2iD_-M_{-+}\,,
\end{aligned}
\tag{271}
$$

and similarly for $\overline{\mathbb{X}}$

$$\begin{aligned}
\delta\overline{X} &= \epsilon\overline{\chi} + \overline{\epsilon}\overline{\psi}\,,\\
\delta\overline{\psi}_+ &= \epsilon_- 2iD_+\overline{X} + \epsilon_-\overline{M}_{++} - \epsilon_+\widehat{\sigma}\overline{X} - \epsilon_+\overline{M}_{-+} + \overline{\epsilon}_+\overline{F}\,,\\
\delta\overline{\psi}_- &= \epsilon_-\overline{\widehat{\sigma}}\overline{X} + \epsilon_-\overline{M}_{+-} - \epsilon_+ 2iD_-\overline{X} - \epsilon_+\overline{M}_{--} + \overline{\epsilon}_-\overline{F}\,,\\
\delta\overline{F} &= \epsilon_+ 2iD_-\overline{\psi}_+ + \epsilon_- 2iD_+\overline{\psi}_- + \epsilon_+\overline{\eta}_- - \epsilon_-\overline{\eta}_+ - \epsilon_+\widehat{\sigma}\overline{\psi}_- - \epsilon_-\overline{\widehat{\sigma}}\overline{\psi}_+ - \epsilon_+ i\lambda_-\overline{X} + \epsilon_- i\lambda_+\overline{X}\,,\\
\delta\overline{\chi}_+ &= \overline{\epsilon}_-\overline{M}_{++} - \overline{\epsilon}_+\overline{M}_{+-}\,,\\
\delta\overline{\chi}_- &= \overline{\epsilon}_-\overline{M}_{-+} - \overline{\epsilon}_+\overline{M}_{--}\,,\\
\delta\overline{M}_{+-} &= \epsilon_-\overline{\widehat{\sigma}}\overline{\chi}_+ - \epsilon_+ 2iD_-\overline{\chi}_+ + \overline{\epsilon}_-\overline{\eta}_+\,,\\
\delta\overline{M}_{-+} &= \epsilon_- 2iD_+\overline{\chi}_- - \epsilon_+\widehat{\sigma}\overline{\chi}_- + \overline{\epsilon}_+\overline{\eta}_-\,,\\
\delta\overline{M}_{++} &= \epsilon_- 2iD_+\overline{\chi}_+ - \epsilon_+\widehat{\sigma}\overline{\chi}_+ + \overline{\epsilon}_+\overline{\eta}_+\,,\\
\delta\overline{M}_{--} &= \epsilon_-\overline{\widehat{\sigma}}\overline{\chi}_- - \epsilon_+ 2iD_-\overline{\chi}_- + \overline{\epsilon}_-\overline{\eta}_-\,,\\
\delta\overline{\eta}_+ &= \epsilon_- 2iD_+\overline{M}_{+-} - \epsilon_- i\overline{\lambda}_+\overline{\chi}_+ - \epsilon_-\overline{\widehat{\sigma}}\overline{M}_{++} - \epsilon_+\widehat{\sigma}\overline{M}_{+-} + \epsilon_+ i\overline{\lambda}_-\overline{\chi}_+ + \epsilon_+ 2iD_-\overline{M}_{++}\,,\\
\delta\overline{\eta}_- &= \epsilon_- 2iD_+\overline{M}_{--} - \epsilon_- i\overline{\lambda}_+\overline{\chi}_- - \epsilon_-\overline{\widehat{\sigma}}\overline{M}_{-+} - \epsilon_+\widehat{\sigma}\overline{M}_{--} + \epsilon_+ i\overline{\lambda}_-\overline{\chi}_- + \epsilon_+ 2iD_-\overline{M}_{-+}\,.
\end{aligned} \tag{272}$$

The transformation laws are written in the general form, and one should set some fields to be zero after imposing the constraints.

Varying the fields M^L_{--}, M^L_{-+}, M^R_{++}, M^R_{+-}, $\overline{M}^L_{--}$, $\overline{M}^L_{-+}$, $\overline{M}^R_{++}$ and $\overline{M}^R_{+-}$, we obtain

$$0 = \alpha\overline{M}^R_{++} + 2iD_+\overline{X}^L + \alpha 2iD_+\overline{X}^R\,, \tag{273}$$

$$0 = -\overline{M}^L_{-+} + \overline{X}^L\widehat{\sigma} + \alpha\overline{X}^R\widehat{\sigma}\,, \tag{274}$$

$$0 = \alpha\overline{M}^L_{--} + 2iD_-\overline{X}^R + \alpha 2iD_-\overline{X}^L\,, \tag{275}$$

$$0 = -\overline{M}^R_{+-} + \overline{X}^R\overline{\widehat{\sigma}} + \alpha\overline{X}^L\overline{\widehat{\sigma}}\,, \tag{276}$$

$$0 = \alpha M^R_{++} - 2iD_+X^L - \alpha 2iD_+X^R\,, \tag{277}$$

$$0 = -M^L_{-+} + \overline{\widehat{\sigma}}X^L + \alpha\overline{\widehat{\sigma}}X^R\,, \tag{278}$$

$$0 = \alpha M^L_{--} - 2iD_-X^R - \alpha 2iD_-X^L\,, \tag{279}$$

$$0 = -M^R_{+-} + \widehat{\sigma}X^R + \alpha\widehat{\sigma}X^L\,. \tag{280}$$

Similarly, varying the fields η_-^L, η_+^R, $\overline{\eta}_-^L$ and $\overline{\eta}_+^R$, we obtain

$$0 = -\overline{\psi}_+^L - \alpha\overline{\psi}_+^R \equiv -\sqrt{\alpha^2+1}\,\overline{\psi}_+^1\,, \tag{281}$$

$$0 = \overline{\psi}_-^R + \alpha\overline{\psi}_-^L \equiv \sqrt{\alpha^2+1}\,\overline{\psi}_-^1\,, \tag{282}$$

$$0 = -\psi_+^L - \alpha\psi_+^R \equiv -\sqrt{\alpha^2+1}\,\psi_+^1\,, \tag{283}$$

$$0 = \psi_-^R + \alpha\psi_-^L \equiv \sqrt{\alpha^2+1}\,\psi_-^1\,. \tag{284}$$

Orthogonal to these fields, we can define

$$\overline{\psi}_+^2 \equiv \frac{1}{\sqrt{\alpha^2+1}}(\alpha\overline{\psi}_+^L - \overline{\psi}_+^R)\,, \tag{285}$$

$$\overline{\psi}_-^2 \equiv \frac{1}{\sqrt{\alpha^2+1}}(\overline{\psi}_-^L - \alpha\overline{\psi}_-^R)\,, \tag{286}$$

$$\psi_+^2 \equiv \frac{1}{\sqrt{\alpha^2+1}}(\alpha\psi_+^L - \psi_+^R)\,, \tag{287}$$

$$\psi_-^2 \equiv \frac{1}{\sqrt{\alpha^2+1}}(\psi_-^L - \alpha\psi_-^R)\,. \tag{288}$$

We can regard them as the physical fermionic fields. Let us call them $\psi'_\pm$ and $\overline{\psi}'_\pm$.

Integrating out these auxiliary fields will give us the on-shell Lagrangian consisting of three parts, the kinetic terms for the bosons and fermions, and their interaction,

$$\mathcal{L}_{\text{bos}} = \begin{pmatrix} \overline{X}^L & \overline{X}^R \end{pmatrix} \cdot \begin{pmatrix} \Box + D + \alpha^2|\widehat{\sigma}|^2 & \frac{1}{\alpha}\Box + \alpha D + \alpha|\widehat{\sigma}|^2 \\ \frac{1}{\alpha}\Box + \alpha D + \alpha|\widehat{\sigma}|^2 & \Box + D + \alpha^2|\widehat{\sigma}|^2 \end{pmatrix} \cdot \begin{pmatrix} X^L \\ X^R \end{pmatrix}$$
$$+ \overline{F}^L F^L + \overline{F}^R F^R + \alpha\overline{F}^L F^R + \alpha\overline{F}^R F^L\,, \tag{289}$$

$$\mathcal{L}_{\text{ferm}} = -\frac{\alpha^2-1}{\alpha^2+1}\overline{\psi}'_- 2iD_+\psi'_- - \frac{\alpha^2-1}{\alpha^2+1}\overline{\psi}'_+ 2iD_-\psi'_+ - \chi_-^L 2iD_+\overline{\chi}_-^L - \chi_+^R 2iD_-\overline{\chi}_+^R\,, \tag{290}$$

$$
\begin{aligned}
\mathcal{L}_{\rm int} = & -\overline{\psi}^L_-\widehat{\sigma}\psi^L_+ - \overline{\psi}^L_+\overline{\widehat{\sigma}}\psi^L_- + \overline{X}^L i(\lambda\psi^L) - i(\overline{\psi}^L\overline{\lambda})X^L + \overline{X}^L i\overline{\lambda}_+\chi^L_- - \overline{\chi}^L_- i\lambda_+ X^L \\
& - \overline{\psi}^R_-\widehat{\sigma}\psi^R_+ - \overline{\psi}^R_+\overline{\widehat{\sigma}}\psi^R_- + \overline{X}^R i(\lambda\psi^R) - i(\overline{\psi}^R\overline{\lambda})X^R - \overline{X}^R i\overline{\lambda}_-\chi^R_+ + \overline{\chi}^R_+ i\lambda_- X^R \\
& - \alpha\overline{\psi}^L_-\widehat{\sigma}\psi^R_+ - \alpha\overline{\psi}^L_+\overline{\widehat{\sigma}}\psi^R_- + \alpha\overline{X}^L i(\lambda\psi^R) - \alpha i(\overline{\psi}^L\overline{\lambda})X^R \\
& + \alpha\overline{\chi}^L_-\overline{\widehat{\sigma}}\chi^R_+ - \alpha\overline{X}^L i\overline{\lambda}_-\chi^R_+ - \alpha\overline{\chi}^L_- i\lambda_+ X^R \\
& - \alpha\overline{\psi}^R_-\widehat{\sigma}\psi^L_+ - \alpha\overline{\psi}^R_+\overline{\widehat{\sigma}}\psi^L_- + \alpha\overline{X}^R i(\lambda\psi^L) - \alpha i(\overline{\psi}^R\overline{\lambda})X^L \\
& + \alpha\overline{\chi}^R_+\widehat{\sigma}\chi^L_- + \alpha\overline{X}^R i\overline{\lambda}_+\chi^L_- + \alpha\overline{\chi}^R_+ i\lambda_- X^L\,.
\end{aligned} \tag{291}
$$

C Semichiral Stückelberg field

Expanding the Lagrangian of the semichiral Stückelberg field (122) in components, we obtain

$$
\begin{aligned}
\mathcal{L}_{St} = & -4(D_+D_-X_L)(X_L+\overline{X}_L) - 4(D_-X_L)(D_+X_L) - \overline{M}^L_{-+}M^L_{-+} + F_L\overline{F}_L \\
& + 2i(D_+M^L_{--})(X_L+\overline{X}_L) + M^L_{--}2i(D_+X_L) - \overline{M}^L_{--}2i(D_+X_L) \\
& + \overline{\psi}^L_+ 2i(D_-\psi^L_+) - \overline{\psi}^L_- 2i(D_+\psi^L_-) + \chi^L_- 2i(D_+\overline{\chi}^L_-) - \overline{\eta}^L_-\overline{\psi}^L_+ - \eta^L_-\psi^L_+ \\
& + 2iD^0 r^0_L + i\lambda^0_-\overline{\psi}^{L0}_+ - i\overline{\lambda}^0_-\psi^{L0}_+ \\
& - 4(D_-D_+X_R)(X_R+\overline{X}_R) - 4(D_-X_R)(D_+X_R) - \overline{M}^R_{+-}M^R_{+-} + F_R\overline{F}_R \\
& - 2i(D_-M^R_{++})(X_R+\overline{X}_R) - 2i(D_-X_R)M^R_{++} + 2i(D_-X_R)\overline{M}^R_{++} \\
& - \overline{\psi}^R_- 2i(D_+\psi^R_-) + \overline{\psi}^R_+ 2i(D_-\psi^R_+) - \chi^R_+ 2i(D_-\overline{\chi}^R_+) + \overline{\eta}^R_+\overline{\psi}^R_- + \eta^R_+\psi^R_- \\
& + 2iD^0 r^0_R + i\lambda^0_-\overline{\psi}^{R0}_+ - i\overline{\lambda}^0_-\psi^{R0}_+ \\
& - 4\alpha(D_+D_-X_L)(X_L+\overline{X}_R) + \alpha(2iD_-X_L + M^L_{--})(2iD_+X_L + \overline{M}^R_{++}) \\
& + 2i\alpha(D_+M^L_{--})(X_L+\overline{X}_R) + \alpha F_L F_R \\
& + \alpha\overline{\psi}^R_+ 2i(D_-\psi^L_+) - \alpha\overline{\psi}^R_- 2i(D_+\psi^L_-) - \alpha\overline{\eta}^L_-\overline{\psi}^R_+ - \alpha\psi^L_-\eta^R_+ \\
& + i\alpha D^0(X_L+\overline{X}_R)^0 + i\alpha\lambda^0_-\overline{\psi}^{R0}_+ \\
& - 4\alpha(D_-D_+X_R)(X_R+\overline{X}_L) + \alpha(2iD_-X_R - \overline{M}^L_{--})(2iD_+X_R - M^R_{++}) \\
& - 2i\alpha(D_-M^R_{++})(X_R+\overline{X}_L) + \alpha F_R\overline{F}_L \\
& - \alpha\overline{\psi}^L_- 2i(D_+\psi^R) + \alpha\overline{\psi}^L_+ 2i(D_-\psi^R_+) - \alpha\overline{\psi}^L_-\overline{\eta}^R_+ - \alpha\eta^L_-\psi^R_+ \\
& + i\alpha D^0(X_R+\overline{X}_L)^0 + i\alpha\lambda^0_-\overline{\psi}^{L0}_+\,.
\end{aligned} \tag{292}
$$

where $r_{L,R}$ stand for the real part of $\mathbb{X}_2^{L,R}$, and the upper index 0 denotes the zero mode. Varying the fields M^L_{--}, $\overline{M}^L_{--}$, M^R_{++} and $\overline{M}^R_{++}$, we obtain

$$
\begin{aligned}
0 &= -2iD_+\overline{X}_L - 2i\alpha D_+\overline{X}_R + \alpha\overline{M}^R_{++}\,,\\
0 &= -2iD_+X_L - 2i\alpha D_+X_R + \alpha M^R_{++}\,,\\
0 &= 2iD_-\overline{X}_R + 2i\alpha D_-\overline{X}_L + \alpha\overline{M}^L_{--}\,,\\
0 &= 2iD_-X_R + 2i\alpha D_-X_L + \alpha M^L_{--}\,. \qquad (293)
\end{aligned}
$$

Similarly, varying the fields η^L_-, $\overline{\eta}^L_-$, η^R_+ and $\overline{\eta}^R_+$ will give us

$$
\begin{aligned}
0 &= -\psi^L_+ - \alpha\psi^R_+ \equiv -\sqrt{1+\alpha^2}\psi^1_+\,,\\
0 &= -\overline{\psi}^L_+ - \alpha\overline{\psi}^R_+ \equiv -\sqrt{1+\alpha^2}\overline{\psi}^1_+\,,\\
0 &= -\psi^R_- - \alpha\psi^L_- \equiv -\sqrt{1+\alpha^2}\psi^1_-\,,\\
0 &= \overline{\psi}^R_- + \alpha\overline{\psi}^L_- \equiv -\sqrt{1+\alpha^2}\overline{\psi}^1_-\,. \qquad (294)
\end{aligned}
$$

We can define

$$
\begin{aligned}
\psi^2_+ &\equiv \frac{1}{\sqrt{1+\alpha^2}}\psi^L_+ - \alpha\psi^R_+\,,\\
\overline{\psi}^2_+ &\equiv \frac{1}{\sqrt{1+\alpha^2}}\overline{\psi}^L_+ - \alpha\overline{\psi}^R_+\,,\\
\psi^2_- &\equiv \frac{1}{\sqrt{1+\alpha^2}}\psi^R_- - \alpha\psi^L_-\,,\\
\overline{\psi}^2_- &\equiv \frac{1}{\sqrt{1+\alpha^2}}\overline{\psi}^R_- - \alpha\overline{\psi}^L_-\,. \qquad (295)
\end{aligned}
$$

Integrating out the auxiliary fields, we obtain

$$\begin{aligned}
\mathcal{L}_{St} &= \begin{pmatrix} \overline{X}_L & \overline{X}_R \end{pmatrix} \begin{pmatrix} \Box & \frac{1}{\alpha}\Box \\ \frac{1}{\alpha}\Box & \Box \end{pmatrix} \begin{pmatrix} X_L \\ X_R \end{pmatrix} \\
&\quad + \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\overline{\psi}_+^2 D_- \psi_+^2 - \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\overline{\psi}_-^2 D_+ \psi_-^2 + \overline{\chi}_-^L 2iD_+ \chi_-^L - \overline{\chi}_+^R 2iD_- \chi_+^R \\
&= \frac{\alpha - 1}{\alpha}\, \overline{X}_1 \Box X_1 + \frac{\alpha + 1}{\alpha}\, \overline{X}_2 \Box X_2 \\
&\quad + \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\overline{\psi}_+^2 D_- \psi_+^2 - \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\overline{\psi}_-^2 D_+ \psi_-^2 + \overline{\chi}_-^L 2iD_+ \chi_-^L - \overline{\chi}_+^R 2iD_- \chi_+^R \\
&= \frac{\alpha - 1}{\alpha}\, (\overline{r}_1 \Box r_1 + \overline{\gamma}_1 \Box \gamma_1) + \frac{\alpha + 1}{\alpha}\, (\overline{r}_2 \Box r_2 + \overline{\gamma}_2 \Box \gamma_2) \\
&\quad + \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\overline{\psi}_+^2 D_- \psi_+^2 - \frac{i}{2}(\frac{1}{\alpha^2} - \alpha^2)\overline{\psi}_-^2 D_+ \psi_-^2 + \overline{\chi}_-^L 2iD_+ \chi_-^L - \overline{\chi}_+^R 2iD_- \chi_+^R \,,
\end{aligned} \tag{296}$$

where

$$X_1 \equiv \frac{-X_L + X_R}{\sqrt{2}} \,, \quad X_2 \equiv \frac{X_L + X_R}{\sqrt{2}} \,, \tag{297}$$

while $r_{1,2}$ and $\gamma_{1,2}$ denote the real parts and the imaginary parts of $X_{1,2}$ respectively. Among these real components only one of them, r_2, transforms under the gauge transformations.

D Jeffrey-Kirwan Residue

In the computation of section 7, we need the Jeffrey-Kirwan residue. Here we give a brief discussion following [34, 35, 37] and the references therein.

Suppose n hyperplanes intersect at $u_* = 0 \in \mathbb{C}^r$, which are given by

$$H_i = \{u \in \mathbb{C}^r | Q_i(u) = 0\} \,, \tag{298}$$

where $i = 1, \cdots, n$ and $Q_i \in (\mathbb{R}^r)^*$. In the GLSM, Q_i correspond to the charges, and they define the hyperplanes as well as their orientations. Then for a vector $\eta \in (\mathbb{R}^r)^*$, the Jeffrey-Kirwan residue is defined as

$$\text{JK-Res}_{u=0}(Q_*, \eta)\, \frac{dQ_{j_1}(u)}{Q_{j_1}(u)} \wedge \cdots \wedge \frac{dQ_{j_r}(u)}{Q_{j_r}(u)} = \begin{cases} \text{sign}\det(Q_{j_1} \cdots Q_{j_r}) \,, & \text{if } \eta \in \text{Cone}(Q_{j_1} \cdots Q_{j_r}) \\ 0 \,, & \text{otherwise} \,, \end{cases} \tag{299}$$

where $Q_* = Q(u_*)$, and $\text{Cone}(Q_{j_1} \cdots Q_{j_r})$ denotes the cone spanned by the vectors $Q_{j_1}, \cdots, Q_{j_r}$. For instance, for the case $r = 1$,

$$\text{JK-Res}_{u=0}(\{q\}, \eta)\, \frac{du}{u} = \begin{cases} \text{sign}(q)\,, & \text{if } \eta q > 0\,, \\ 0\,, & \text{if } \eta q < 0\,. \end{cases} \tag{300}$$

To obtain the elliptic genus, we still have to evaluate the contour integral over u. Since in the paper we often encounter the function $\vartheta_1(\tau, u)$, its residue is very useful in practice:

$$\frac{1}{2\pi i} \oint_{u=a+b\tau} du\, \frac{1}{\vartheta_1(\tau, u)} = \frac{(-1)^{a+b}\, e^{i\pi b^2 \tau}}{2\pi\, \eta(q)^3}\,, \tag{301}$$

where $q = e^{2\pi i \tau}$. This relation can be derived by combining the properties

$$\vartheta_1'(\tau, 0) = 2\pi\, \eta(q)^3\,, \tag{302}$$

and

$$\vartheta_1(\tau, u + a + b\tau) = (-1)^{a+b}\, e^{-2\pi i b u - i\pi b^2 \tau} \vartheta_1(\tau, u) \tag{303}$$

for $a, b \in \mathbb{Z}$ and the fact that $\vartheta_1(\tau, u)$ has only simple zeros at $u = \mathbb{Z} + \tau\mathbb{Z}$ but no poles.

E Review of 4D $\mathcal{N} = 2$ Off-Shell Gauged Supergravity

We review the 4D $\mathcal{N} = 2$ off-shell gauged supergravity theory in this appendix. It can be obtained as a consistent truncation of M-theory on a Sasaki-Einstein manifold X_7. The theory was originally constructed in Ref. [74] and also reviwed in Ref. [62]. We follow these references closely.

The nearly massless fields consist of the supergravity multiplet, a single vector multiplet and a universal hypermultiplet (the dualized tensor multiplet). To obtain an off-shell super-Poincaré gravity theory, one can start with a superconformal gravity theory and then use gauge fixing to reduce it to the super-Poincaré gravity theory.

The $\mathcal{N} = 2$ superconformal algebra has the generators:

$$P_a,\, M_{ab},\, D,\, K_a,\, Q_i,\, S^i,\, U_{ij}, \tag{304}$$

where $\{P_a,\ M_{ab},\ D,\ K_a\}$ are the generators of translations, Lorentz rotations, dilatations, special conformal transformations respectively, whlie Q_i and S^i are the usual supersymmetry and the special conformal supersymmetry generator respectively, and U_{ij} is the generator of the 4D $SU(2)$ R-symmetry. The gauge fields corresponding to these generators are

$$e_\mu^a,\ \omega_\mu^{ab},\ b_\mu,\ f_\mu^a,\ \psi_\mu^i,\ \phi_\mu^i,\ \mathcal{V}_\mu^{ij} \tag{305}$$

respectively.

Let us review different $\mathcal{N}=2$ supersymmetry multiplets in the following:

- Weyl multiplet:

  The Weyl multiplet, denoted by $\mathbf{W}$, contains the following field components:

$$\mathbf{W} = \left(e_\mu^a,\ \psi_\mu^i,\ b_\mu,\ A_\mu,\ \mathcal{V}_{\mu j}^i,\ T_{ab}^{ij},\ \chi^i,\ D\right), \tag{306}$$

  where e_μ^a is the vielbein, ψ_μ^i is the (left-handed) gravitino doublet, b_μ and A_μ are the gauge fields of dilatations and chiral $U(1)$ R-symmetry transformations repectively, while $\mathcal{V}_{\mu j}^i$ is the gauge field of the $SU(2)$ R-symmetry. The auxiliary fields include the antisymmetric anti-selfdual field T_{ab}^{ij}, the $SU(2)$ doublet Majorana spinor χ_i and the real scalar D. Altogether, there are 24 bosonic degrees of freedom and 24 fermionic degrees of freedom in the Weyl multiplet. They satisfy the following supersymmetric transformations:

$$\begin{aligned}
\delta e_\mu{}^a &= \bar{\epsilon}^i\gamma^a\psi_{\mu i} + \bar{\epsilon}_i\gamma^a\psi_\mu{}^i\,,\\
\delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i - \frac{1}{8}T_{ab}{}^{ij}\gamma^{ab}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\,,\\
\delta b_\mu &= \frac{1}{2}\bar{\epsilon}^i\phi_{\mu i} - \frac{3}{4}\bar{\epsilon}^i\gamma_\mu\chi_i - \frac{1}{2}\bar{\eta}^i\psi_{\mu i} + \text{h.c.} + \Lambda_K^a e_{\mu a}\,,\\
\delta A_\mu &= \frac{i}{2}\bar{\epsilon}^i\phi_{\mu i} + \frac{3i}{4}\bar{\epsilon}^i\gamma_\mu\chi_i + \frac{i}{2}\bar{\eta}^i\psi_{\mu i} + \text{h.c.}\,,\\
\delta\mathcal{V}_\mu{}^i{}_j &= 2\bar{\epsilon}_j\phi_\mu{}^i - 3\bar{\epsilon}_j\gamma_\mu\chi^i + 2\bar{\eta}_j\psi_\mu{}^i - (\text{h.c.})\,,\\
\delta T_{ab}{}^{ij} &= 8\bar{\epsilon}^{[i}R(Q)_{ab}{}^{j]}\,,\\
\delta\chi_i &= -\frac{1}{12}\gamma^{ab}\not{D}T_{ab}{}^{ij}\epsilon_j + \frac{1}{6}R(\mathcal{V})_{\mu\nu}{}^i{}_j\gamma^{\mu\nu}\epsilon^j - \frac{i}{3}R_{\mu\nu}(A)\gamma^{\mu\nu}\epsilon^i + D\epsilon^i + \frac{1}{12}\gamma_{ab}T^{abij}\eta_j\,,\\
\delta D &= \bar{\epsilon}^i\not{D}\chi_i + \bar{\epsilon}_i\not{D}\chi^i\,,
\end{aligned} \tag{307}$$

where ϵ, η and Λ_K^a denote the parameters of Q, S and K_a respectively, and

$$\mathcal{D}_\mu \epsilon^i \equiv \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{cd}\gamma_{cd} + \frac{1}{2}b_\mu + \frac{i}{2}A_\mu\right)\epsilon^i + \frac{1}{2}\mathcal{V}_\mu{}^i{}_j\epsilon^j\,. \tag{308}$$

- Vector multiplet:

  The vector multiplet, denoted by $\mathbf{X}^I$ with the index I labelling the gauge group generators, contains the following field components:

$$\mathbf{X}^I = \left(X^I,\, \Omega_i^I,\, W_\mu^I,\, Y_{ij}^I\right), \tag{309}$$

  where X^I is a complex scalar, Ω_i^I is the gaugino that are the $SU(2)$ doublet of chiral fermions, and W_μ^I is the vector field. The auxiliary field Y_{ij}^I is an $SU(2)$ triplet with

$$Y_{ij} = Y_{ji}\,, \quad Y_{ij} = \epsilon_{ik}\,\epsilon_{jl}\,Y^{kl}\,. \tag{310}$$

  Altogether, there are 8 bosonic degrees of freedom and 8 fermionic degrees of freedom in the vector multiplet for each index I. They have the following supersymmetric transformations:

$$\begin{aligned}
\delta X &= \bar{\epsilon}^i\Omega_i\,,\\
\delta\Omega_i &= 2\not{D}X\epsilon_i + \frac{1}{2}\epsilon_{ij}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + Y_{ij}\epsilon^j + 2X\eta_i\,,\\
\delta W_\mu &= \epsilon^{ij}\bar{\epsilon}_i(\gamma_\mu\Omega_j + 2\psi_{\mu j}X) + \epsilon_{ij}\bar{\epsilon}^i(\gamma_\mu\Omega^j + 2\psi_\mu{}^j\overline{X})\,,\\
\delta Y_{ij} &= 2\bar{\epsilon}_{(i}\not{D}\Omega_{j)} + 2\epsilon_{ij}\epsilon_{jl}\bar{\epsilon}^{(k}\not{D}\Omega^{l)}\,,
\end{aligned} \tag{311}$$

  where

$$\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu} - \frac{1}{4}\left(\overline{X}\epsilon_{ij}T_{\mu\nu}{}^{ij} + \text{h.c.}\right) + (\text{fermionic terms})\,, \tag{312}$$

  and $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$.

- Hypermultiplet:

  The hypermultiplet of the 4D $\mathcal{N}=2$ supersymmetry is a little special, because it is well-known that for this multiplet the off-shell closure of the supersymmetry algebra cannot be achieved with finite number of fields. One can start from one hypermultiplet, and then add infinite sequence of fields to obtain the off-shell closure of the supersymmetry algebra.

A single hypermultiplet contains scalars $A_i{}^\alpha$ and spinors ζ^α, where the scalars are doublets of the $SU(2)$ R-symmetry, and all the fields transform in the fundamental representation of $Sp(2r)$, i.e. the index α runs from 1 to $2r$. Moreover, the scalars satisfy the reality condition

$$A^i_\alpha = (A^\alpha_i)^* = \epsilon^{ij}\,\rho_{\alpha\beta}\,A^\beta_j \tag{313}$$

with

$$\rho_{\alpha\beta}\,\rho^{\beta\gamma} = -\delta^\gamma_\alpha\,. \tag{314}$$

The supersymmetric transformations are given by

$$\begin{aligned}\delta A_i{}^\alpha &= 2\overline{\epsilon}_i\zeta^\alpha + 2\rho^{\alpha\beta}\epsilon_{ij}\overline{\epsilon}^j\zeta_\beta\,,\\ \delta\zeta^\alpha &= \not{D}A_i{}^\alpha\epsilon^i + 2gX^\alpha{}_\beta A_i{}^\beta\epsilon^{ij}\epsilon_j + A_i{}^\alpha\eta^i\,,\end{aligned} \tag{315}$$

where g is the coupling constant,

$$X^\alpha{}_\beta \equiv X^I(t_I)^\alpha{}_\beta\,,\quad \overline{X}^\alpha{}_\beta \equiv \overline{X}^I(t_I)^\alpha{}_\beta \tag{316}$$

with $t_\alpha{}^\beta\rho_{\beta\gamma} = \rho_{\alpha\beta}t^\beta{}_\gamma$, and

$$D_\mu A_i{}^\alpha \equiv \partial_\mu A_i{}^\alpha + \frac{1}{2}\mathcal{V}_\mu^j{}_i A_j{}^\alpha - b_\mu A_i{}^\alpha - gW^\alpha_{\mu\beta}A_i{}^\beta - \overline{\psi}_{\mu i}\zeta^\alpha - \rho^{\alpha\beta}\epsilon_{ij}\overline{\psi}^j_\mu\zeta_\beta\,. \tag{317}$$

As discussed in Refs. [74, 62], to realized the off-shell supersymmetry for the hypermultiplet, one needs to introduce an infinite tower of hypermultiplets $(A_i{}^\alpha,\,\zeta^\alpha)$, $(A_i{}^\alpha,\,\zeta^\alpha)^{(z)}$, $(A_i{}^\alpha,\,\zeta^\alpha)^{(zz)}$, $\cdots$. The closure of the superconformal algebra will then impose an infinite set of constraints, and in the end only $(A_i{}^\alpha,\,\zeta^\alpha, A_i{}^{\alpha(z)})$ are independent. Consequently, the supersymmetric transformation (315) for ζ^α will be modified to

$$\delta\zeta^\alpha = \not{D}A_i{}^\alpha\epsilon^i + 2gX^\alpha{}_\beta A_i{}^\beta\epsilon^{ij}\epsilon_j + A_i{}^\alpha\eta^i + aA_i{}^{a(z)}\epsilon^{ij}\epsilon_j\,, \tag{318}$$

where a is the scalar field in the vector multiplet associated with the central charge translation, which can be set to $a = 1$. In the main text, we also denote $A_i{}^{\alpha(z)}$ by $F_i{}^\alpha$ using

$$F_i{}^\alpha = aA_i{}^{\alpha(z)}\,. \tag{319}$$

After constructing a linear multiplet coupled to $(A_i{}^\alpha, \zeta^\alpha)$, $(A_i{}^\alpha, \zeta^\alpha)^{(z)}$ and imposing the constraints, one can find the supersymmetric Lagrangian for the hypermultiplet [74, 62]:

$$\mathcal{L}_{\text{hyp}} = \Big[- D_\mu A^i{}_\beta D^\mu A_i{}^\alpha - \frac{1}{6} R A^i{}_\beta A_i{}^\alpha + \frac{1}{2} D A^i{}_\beta A_i{}^\alpha + (|a|^2 + W^z_\mu W^{\mu z}) A^i{}_\beta{}^{(z)} A_i{}^{\alpha(z)} \\ + 4g^2 A^i{}_\beta \overline{X}^\alpha{}_\gamma X^\gamma{}_\delta A_i{}^\delta + g A^i{}_\beta (Y^{ij})_\gamma{}^\alpha A_k{}^\gamma \epsilon_{ij} \Big] d_\alpha{}^\beta + (\text{fermionic terms}) \, , \tag{320}$$

where $d_\alpha{}^\beta$ satisfies

$$\overline{d_\alpha{}^\beta} = d_\beta{}^\alpha \, , \tag{321}$$

$$d_\alpha{}^\beta = \epsilon_{\gamma\alpha} \epsilon^{\delta\beta} d_\delta{}^\gamma \, , \tag{322}$$

$$t^\alpha{}_\alpha d_\gamma{}^\beta + d_\alpha{}^\gamma t_\gamma{}^\beta = 0 \, . \tag{323}$$

As discussed in Refs. [74, 62], one can set $d_\alpha{}^\beta = -\delta_\alpha{}^\beta$.

F Killing Spinors and Gamma Matrices

To localize the 4D $\mathcal{N} = 2$ off-shell gauged supergravity on the neutral topological black hole, we need to find the Killing spinors in this space. They are explicitly constructed in Ref. [50], and we review the results in this and next appendix.

The metric of the Euclidean AdS$_4$ topological black hole is

$$ds^2 = f(r)\, d\tau^2 + \frac{1}{f(r)}\, dr^2 + r^2\, d\Sigma(\mathbb{H}^2) = f(r)\, d\tau^2 + \frac{1}{f(r)}\, dr^2 + r^2\, (du^2 + \sinh^2 u\, d\phi^2) \, , \tag{324}$$

where

$$f(r) = \frac{r^2}{L^2} + \kappa - \frac{2m}{r} + \frac{Q^2}{r^2} \, , \tag{325}$$

and 2κ is the constant curvature of the 2-dimensional Riemann surface, which implies that $\kappa = -1$ for $\mathbb{H}^2$.

Near the boundary ($r \to \infty$), we keep the terms $\sim \mathcal{O}(1)$ in $f(r)$, and the metric approaches

$$ds^2 = d\eta^2 + \sinh^2\eta\, d\tau^2 + \cosh^2\eta (du^2 + \sinh^2 u\, d\phi^2) \, . \tag{326}$$

where

$$r = \cosh\eta\,, \tag{327}$$

and for simplicity we set $L = 1$. The ranges of the variables are

$$\tau \in [0,\, 2\pi q)\,, \quad \eta \in [0,\, \infty)\,, \quad \phi \in [0,\, 2\pi)\,. \tag{328}$$

Hence, at $r \to \infty$ or $\eta \to \infty$ the boundary of the Euclidean AdS_4 topological black hole is $\text{S}^1 \times \mathbb{H}^2$ as expected.

In this paper, we adopt the convention of the γ-matrices used in Ref. [50]. For the Lorentz signature:

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\,, \quad \gamma_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,, \tag{329}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\,, \quad \gamma_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\,. \tag{330}$$

For the Euclidean signature, one can choose $\gamma_4 = -i\gamma_0$ with the same γ_i's. The charge conjugation matrix B satisfies

$$B^{-1}\gamma_\mu B = -\gamma_\mu^*\,, \quad BB^* = -\mathbb{I}\,. \tag{331}$$

More explicitly, in this paper we choose $B = \gamma_0$.

The Killing spinor equation for the AdS_4 topological black hole is [50]

$$\nabla_\mu \epsilon - igA_\mu \epsilon + \frac{i}{4} F_{ab}\gamma^{ab}\gamma_\mu \epsilon = -\frac{1}{2} g\gamma_\mu \epsilon\,. \tag{332}$$

Using the charge conjugation matrix B, one can construct the charge conjugate spinor $\epsilon^c \equiv B\epsilon^*$, which satisfies a different Killing spinor equation:

$$\nabla_\mu \epsilon^c + igA_\mu \epsilon^c + \frac{i}{4} F_{ab}\gamma^{ab}\gamma_\mu \epsilon^c = \frac{1}{2} g\gamma_\mu \epsilon^c\,. \tag{333}$$

G Localization Action

As discussed in Subsection 13.2.3, we choose the Killing spinor found in Ref. [50]:

$$\epsilon = e^{-\frac{i}{2qL}\tau_E}\, e^{i\frac{u}{2}\gamma_4\gamma_1\gamma_2}\, e^{\frac{\phi}{2}\gamma_{23}}\, \widetilde{\epsilon}(r) \tag{334}$$

with

$$\widetilde{\epsilon}(r) = \left(\sqrt{\frac{r}{L} + \sqrt{f(r)}} - i\gamma_4\sqrt{\frac{r}{L} - \sqrt{f(r)}}\right)\left(\frac{1-\gamma_1}{2}\right)\epsilon_0'\,, \tag{335}$$

where ϵ_0' is an arbitrary constant spinor, and in this paper we choose $\epsilon_0' = (1,\, 0,\, i,\, 0)^T$. We define

$$\xi^i \equiv \frac{1+i\gamma_5}{2}\,\widetilde{\xi}^i \tag{336}$$

with

$$\widetilde{\xi}^1 \equiv \epsilon^c\,, \quad \widetilde{\xi}^2 \equiv \epsilon\,. \tag{337}$$

and use $\xi^i = (\xi^1,\, \xi^2)^T$ as the Killing spinor in the expansion of the localization action.

With the Killing spinor chosen above, we can work out all the Killing spinor bilinears explicitly, and use them to expand the localization action. After some steps, we found that the localization action in this case can be written into a sum of some squares as follows:

$$
\begin{aligned}
(\delta\Omega)^\dagger\delta\Omega = & \frac{1}{2\cosh(\eta+u)}\left(F_{ab}\cosh(\eta+u) - \frac{1}{2}\epsilon_{abcd}F^{cd} - 2\,\epsilon_{abcd}\,\partial^c J\, V^d - 2\,\Theta_{ab}\,J\right)^2 \\
& + \frac{1}{\cosh(\eta+u)}\left(-\frac{1}{2}\epsilon_{abcd}F^{bc}V^d + 2\partial_a\left(J\cosh(\eta+u)\right)\right)^2 + \frac{4}{\cosh(\eta+u)}(V^a\partial_a J)^2 \\
& + \frac{1}{\cosh(\eta+u)}\left(F_{ab}V^b - 2\,\partial_a J\right)^2 + 8\cosh(\eta+u)\,(\partial_i H)^2 \\
& + 8\cosh(\eta+u)\left(\partial_\eta H + \frac{\sinh(\eta+u)}{\cosh(\eta+u)}H\right)^2 + 2\cosh(\eta+u)\left(Y^1{}_1 - \frac{2H}{\cosh(\eta+u)}\right)^2 \\
& + \frac{1}{2\cosh(\eta-u)}\left(F_{ab}\cosh(\eta-u) - \frac{1}{2}\epsilon_{abcd}F^{cd} - 2\,\epsilon_{abcd}\,\partial^c J\, V^d - 2\,\Theta_{ab}\,J\right)^2 \\
& + \frac{1}{\cosh(\eta-u)}\left(-\frac{1}{2}\epsilon_{abcd}F^{bc}V^d + 2\partial_a\left(J\cosh(\eta-u)\right)\right)^2 + \frac{4}{\cosh(\eta-u)}(V^a\partial_a J)^2 \\
& + \frac{1}{\cosh(\eta-u)}\left(F_{ab}V^b - 2\,\partial_a J\right)^2 + 8\cosh(\eta-u)\,(\partial_i H)^2 \\
& + 8\cosh(\eta-u)\left(\partial_\eta H + \frac{\sinh(\eta-u)}{\cosh(\eta-u)}H\right)^2 + 2\cosh(\eta-u)\left(Y^1{}_1 - \frac{2H}{\cosh(\eta-u)}\right)^2 ,
\end{aligned}
\tag{338}
$$

where η is defined by $r = \cosh(\eta)$, and V_a and Θ_{ab} are defined as

$$V_a = \frac{1}{4}(\xi^i)^\dagger\gamma_a\xi^i\,, \tag{339}$$

$$\Theta_{ab} = -\frac{i}{4}(\xi^\dagger)^\dagger\gamma_{ab}\gamma_5\eta^i\,. \tag{340}$$

By requiring all the squares in the sum of the localization action to vanish, we obtain the localization locus of the theory. We see that there is nonvanishing localization locus only at $u = 0$:

$$H = \frac{C}{\cosh(\eta)}\,,\quad Y^1{}_1 = \frac{2C}{\cosh^2(\eta)}\,, \tag{341}$$

where C is an arbitrary constant, and in Section 13 we also use the parametrization $C = Jh$. We make the gauge choice $A_t = 0$, and in this gauge the field J and F_{ab} satisfy

$$J = \text{const}\,,\quad F_{ab}V^b = 0\,, \tag{342}$$

where the constant value of J is fixed by the attractor solutions (205) (207).

For the hypermultiplet, as discussed in Appendix E, in principle we need an infinite tower $(A_i{}^\alpha, \zeta^\alpha)$, $(A_i{}^\alpha, \zeta^\alpha)^{(z)}$, $(A_i{}^\alpha, \zeta^\alpha)^{(zz)}$, $\cdots$ with constraints to realize the off-shell supersymmetry. To look for the BPS solutions, we should require

$$\delta\zeta^\alpha = 0\,, \quad \delta\zeta^{\alpha(z)} = 0\,, \quad \delta\zeta^{\alpha(zz)} = 0\,, \quad \cdots \tag{343}$$

with respect to the constraints, which is rather involved. Instead we follow the approach applied in Ref. [62] by requiring $\delta\zeta = 0$ for all 8 Killing spinors, which consequently leads to the solutions

$$F_i{}^\alpha = -\frac{2ig}{\sqrt{8\pi G}}(\sigma_3)^\alpha{}_j(H\cdot P)\,, \quad 2g(J\cdot P) = -\frac{1}{L} \tag{344}$$

with $F_i{}^\alpha$ and P_I given by

$$F_i{}^\alpha = aA_i{}^{\alpha(z)}\,, \quad t_I A_i{}^\alpha = P_I(i\sigma_3)^\alpha{}_\beta A_i{}^\beta\,. \tag{345}$$

These solutions coincide with the solutions (199) to the BPS equations under the attractor solution (203).

H Evaluation of the Action

In this appendix, we evaluate the action (173) along the localization locus found in Appendix G. As we explained in Subsection 13.2.4, up to the volume of the boundary manifolds, the integrals over the radial direction have the same integrand for the AdS$_4$ case discussed in Ref. [62] and the AdS$_4$ neutral topological black hole considered in this paper. However, the discrepancy comes from the measure $d^4x\,\sqrt{g}$, which differs for the hyperbolic AdS$_4$ and the AdS$_4$ neutral topological black hole.

Let us briefly list the results in the following. For the choice of the prepotential $F(X) = \sqrt{X^0(X^1)^3}$, one can compute the tensor N_{IJ} defined by

$$N_{IJ} \equiv \frac{1}{2i}\left(F_{IJ} - \overline{F}_{IJ}\right)\,, \quad F_{IJ} \equiv \partial_I\partial_J F(X)\,. \tag{346}$$

The explicit expressions are

$$N_{00} = \frac{i}{8}\left(\frac{J^1}{J^0}\right)^{\frac{3}{2}}\left(t^3 + \bar{t}^3\right)\,, \quad N_{11} = -\frac{3i}{8}\left(\frac{J^1}{J^0}\right)^{-\frac{1}{2}}\left(\frac{1}{t} + \frac{1}{\bar{t}}\right)\,, \tag{347}$$

$$N_{01} = -N_{10} = -\frac{3i}{8}\left(\frac{J^1}{J^0}\right)^{\frac{1}{2}}\left(t+\bar{t}\right) . \tag{348}$$

Plugging these expressions into S_{vec}, we obtain the following integral:

$$\begin{aligned} S_{\text{vec}} = \Omega_3^{\text{reg}} L^2 \frac{J^0 (J^1)^3}{2i} \int dr\, r^2 \Bigg[& - \left(1 + \frac{(h^0)^2}{r^2}\right)(t+\bar{t})^3 + \frac{3}{4}(h^1)^2 \left(\frac{1}{t} + \frac{1}{\bar{t}}\right) \frac{r^2-1}{r^4} \\ & + \frac{3}{2} h^1 h^0 (t+\bar{t}) \frac{r^2-1}{r^4} - \frac{1}{4}(h^0)^2 \left(t^3 + \bar{t}^3\right) \frac{r^2-1}{r^4} - \frac{3}{4}\left(\frac{1}{t}+\frac{1}{\bar{t}}\right)\left(1 + i\frac{h^1}{r^2}\right)^2 \\ & - \frac{3}{2}(t+\bar{t})\left(1 + i\frac{h^1}{r^2}\right)\left(1 + i\frac{h^0}{r^2}\right) + \frac{1}{4}\left(t^3+\bar{t}^3\right)\left(1 + i\frac{h^0}{r^2}\right)^2 \Bigg], \end{aligned} \tag{349}$$

where Ω_3^{reg} is the regularized volume of the boundary $\mathrm{S}^1 \times \mathbb{H}^2$. As we have seen in Appendix G, the nontrivial localization locus is only supported by $u = 0$, while the unregularized volume of the noncompact manifold $\mathbb{H}^2$ is divergent. Combining these two factors, we assume that the regularized volume Ω_3^{reg} of of the boundary manifold $\mathrm{S}^1 \times \mathbb{H}^2$ is finite.

The integral appearing in S_{vec} can be evaluated explicitly without the integration limits, and the result is

$$\begin{aligned} \mathcal{I} = \Omega_3^{\text{reg}} L^2 \frac{J^0 (J^1)^3}{2i} \Bigg[& \sqrt{\frac{1+ih^1/r}{1+ih^0/r}} \left(ir(h^1(-3+r) + 2ir^2) - h^0(2h^1(-2+r) + ir(1+r))\right) \\ & + \sqrt{\frac{1-ih^1/r}{1-ih^0/r}} \left(ir(-h^1(3+r) + 2ir^2) - h^0(2h^1(2+r) - ir(-1+r))\right) \Bigg]. \end{aligned} \tag{350}$$

Taking the integration limits into account, we will consider $r \in [1, r_0]$ in Subsection 13.2.4, where r_0 is a cutoff, i.e.,

$$\begin{aligned} S_{\text{vec}} &= \mathcal{I}(r = r_0) - \mathcal{I}(r = 1) \\ &= \Omega_3^{\text{reg}} L^2 \frac{J^0 (J^1)^3}{2i} \Bigg[- 4r_0^3 + \frac{r_0}{2}\left((h^0)^2 - 3h^1(4i + h^1) - 2h^0(2i + 3h^1)\right) \\ & \qquad - 2(h^1 - i)^{3/2}(h^0 - i)^{1/2} + 6(h^1+i)^{3/2}(h^0+i)^{1/2} \\ & \qquad - 6i(h^1+i)^{1/2}(h^0+i)^{1/2} - 2i(h^1+i)^{3/2}(h^0+i)^{-1/2} \Bigg]. \end{aligned} \tag{351}$$

Similar to the vector multiplet, for the hypermultiplet action S_{hyp}, up to the volume of the boundary manifold, the integral in the radial direction has the same integrand as the AdS$_4$ case, but with a different measure from $\sqrt{g}$. In the end, for the AdS$_4$ neutral topological black hole considered in this paper, S_{hyp} can be expressed as

$$\begin{aligned} S_{\text{hyp}} &= -i\Omega_3^{\text{reg}} L^4 \int_1^{r_0} dr\, r^2 \frac{1}{r^2} \frac{g}{2\pi G L} \left(h^0 J^0 P_0 + h^1 J^1 P_1\right) \\ &= -i \frac{\Omega_3^{\text{reg}}\, g L^3}{2\pi G} (r_0 - 1) \left(h^0 J^0 P_0 + h^1 J^1 P_1\right) \\ &= \frac{i\Omega_3^{\text{reg}}\, L^2}{16\pi G} (r_0 - 1) \left(h^0 + 3h^1\right) \,, \end{aligned} \tag{352}$$

where again r_0 is a cutoff, and we have used the attractor solutions Eq. (205):

$$8gJ^0 P_0 = -\frac{1}{L} \,, \quad 8gJ^1 P_1 = -\frac{3}{L} \,.$$

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