Quantization of Anomalous Gauge Theories: The Chiral Schwinger Model

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Abstract

We implement the proposal of Faddeev and quantize the anomalous, chiral Schwinger model. We carry out a Schrodinger representation quantization,on a circle, in the Hamiltonian formulation. We make a special emphasis to uncover the structure of the fermionic Hilbert bundle over the space of gauge fields. We find that although a unitary and consistent quantum field theory is obtained, Lorentz invariance is lost.

I. Introduction

Anomalous gauge theories /1/ pose a perplexing puzzle to quantum field theorists. First it was anticipated and demonstrated /2/ that standard methods of quantization would not yield a consistent quantum field theory. The principle of cancellation of anomalies was a foundation of any grand unified model. Recently however, our understanding of anomalies has greatly improved. In the Hamiltonian interpretation of anomalies /3/, it was shown that anomalies constitute an obstruction to projecting the Hilbert bundle of fermions over the space of all gauge field configurations, to a bundle over the true configuration space, the space of all gauge field configurations factored by the group of gauge transformations. The reason for the obstruction is that the fermions transform under a non-trivial projective representation of the gauge group. Consequently, non-degenerate energy eigenstates, the fermionic Fock vacuum for example, form non-trivial complex line bundles over the gauge group. Accordingly, no gauge invariant definition of these states can be made. However the question still remains, whether or not one can modifiy the quantization procedure and further quantize the gauge degrees of freedom in a gauge invariant manner.

In the Hamiltonian formulation of gauge theories /4/, gauge invariance is educed by imposing Gauss' law as a constraint to characterize physical states,

1 $$G_a(x)\,|\Psi_{phys}\rangle = 0$$

In an anomalous gauge theory, the Gauss operator satisfies the algebra /5/

$$\left[G_a(x)\,,\ G_b(y)\right] = i f_{abc}\,\delta(x-y)\,G_c(x) + S_{ab}(x,y;A) \tag{2}$$

making (1) inconsistent, as $S_{ab}(x,y;A)$ does not generally annihilate any states. In light of the algebra (2), Faddeev made the proposal /6/ on how to effect the further quantization of the gauge fields, such that (1) is in some sense obeyed. He proposed to decompose $G_a(x)$ as

$$G_a(x) = G_a^+(x) + G_a^-(x) \tag{3}$$

with

$$G_a^+(x) = \left(G_a^-(x)\right)^\dagger \tag{4}$$

and impose only

$$G_a^-(x)\,|\,\Psi_{phys}\rangle = 0 \quad . \tag{5}$$

Then Gauss' law (1) is regained as a statement valid in matrix elements between physical states

$$\langle \Phi_{phys}\,|\,G_a(x)\,|\,\Psi_{phys}\rangle = 0 \quad . \tag{6}$$

Of course the decomposition must satisfy

$$\left[G_a^\pm(x)\,,\ G_b^\pm(y)\right] = i f_{abc}\,\delta(x-y)\,G_c^\pm(x) \tag{7}$$

for (5) to be sensible, and

$$\left[G_a^+(x)\,,\ G_b^-(y)\right] + \left[G_a^-(x)\,,\ G_b^+(y)\right] = S_{ab}(x,y;A) \quad . \tag{8}$$

We present below the application of this program to the quantization of the anomalous, chiral Schwinger model. We work on a circle of length 2π and in momentum space, more from a constructive viewpoint, along lines developed by Manton /7/ for the usual Schwinger model.

II. Momentum Space, Schrodinger Representation Quantization

By the chiral Schwinger model we mean the theory ostensibly described by the Lagrangian

$$L = \int d^2x \left(\psi_L^\dagger(x)\, i\sigma^\mu(\partial_\mu + iA_\mu)\psi_L(x) - \frac{1}{4}\left(F_{\mu\nu}\right)^2\right) \tag{9}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $\sigma^0 = 1, \sigma^1 = -1$, $\psi_L(x)$ is a single component, left handed fermion, and we work with $A_0 = 0$. Thus

10 $$L = \int d^2x \left(\psi_L^\dagger(x)\, i\sigma^\mu \partial_\mu \psi_L(x) + \psi_L^\dagger(x)\psi_L(x) A_1(x) + \frac{1}{2}\left(\dot{A}_1(x)\right)^2 \right).$$

The equations of motion obtained from this Lagrangian include the Dirac equation and Ampere's law, but Gauss' law is absent. However, due to the invariance of the Lagrangian under arbitrary, time independent gauge transformations, the Gauss operator

11 $$G(x) = \partial_x \dot{A}_1(x) - \psi_L^\dagger(x)\psi_L(x)$$

is conserved.

To make the transcription to momentum space, for the fermions, we need an infinite set of creation and annihilation operators

12 $$\{\alpha_p, \alpha_q^\dagger\} = \delta_{p,q}\,, \{\alpha_p, \alpha_q\} = \{\alpha_p^\dagger, \alpha_q^\dagger\} = 0\,,\ p, q \in \mathbb{Z}\,.$$

The fermionic field operator (distribution)

13 $$\psi_L(x) = \frac{1}{\sqrt{2\pi}} \sum_{q \in \mathbb{Z}} e^{iqx} \alpha_q$$

satisfies the canonical anti-commutation relations. The Fock vacuum is defined as the state annihilated by

13 $$\begin{aligned} \alpha_q^\dagger\,|0\rangle &= 0 \qquad & q \geq 0 \\ \alpha_q\,|0\rangle &= 0 \qquad & q < 0 \end{aligned}\,.$$

The operators corresponding to the fourier transform of the charge density are

14 $$\rho(p) = \sum_{q \in \mathbb{Z}} \alpha_{p+q}^\dagger \alpha_q \qquad , \quad p \neq 0\,.$$

These are well defined operators when acting on states which only have a finite number of excitations relative to the Fock vacuum, which form a dense domain. For p = 0 we have the total charge operator

15 $$Q = \sum_{q<0} \alpha_q^\dagger \alpha_q - \sum_{q \geq 0} \alpha_q^\dagger \alpha_q$$

which we must define with a normal ordering. The free Hamiltonian is also defined with a normal ordering,

16 $$H_0 = \sum_{q \geq 0} q\, \alpha_q \alpha_q^\dagger + \sum_{q<0} |q|\, \alpha_q^\dagger \alpha_q \quad .$$

Q commutes with $\rho(p)$ and H_0, while

17 $$[\rho(p), \rho(q)] = p\, \delta_{p,-q}$$

and

18 $$[H_0, \rho(p)] = -p\, \rho(p) \quad .$$

These commutation relations can be rigourously derived without recourse to any regularization or any ad hoc prescription /7/.

With

19 $$A_1(x) = \frac{2}{\sqrt{2\pi}} \sum_{p>0} \left(\gamma_1(p) \cos(px) + \gamma_2(p) \sin(px)\right) + \frac{\gamma_0}{\sqrt{2\pi}}$$

the full fermionic Hamiltonian becomes

20 $$H_F = H_0 - \sum_{p>0} \left(\sqrt{2p}\, \gamma_i(p)\, Q_i(p) - \gamma_i^2(p)\right) - \gamma_0 Q + \frac{1}{2} \gamma_0^2$$

where

21 $$Q_1(p) = \left(\rho(p) + \rho(-p)\right)/\sqrt{2p}$$
$$Q_2(p) = -i\left(\rho(p) - \rho(-p)\right)/\sqrt{2p}$$

and the c-number constants are determined by insisting that H commute with the Gauss operator. The fourier components of the Gauss operator are

22 $$G_i = i \frac{d}{d\gamma_i} - \sqrt{\frac{2}{p}}\, \epsilon_{ij} Q_j$$

which satisfy the (anomalous) commutator (1),

23 $$[G_i, G_j] = i \frac{2}{p} \epsilon_{ij} \quad .$$

The appearance of $\frac{d}{d\gamma_i}$ in G_i stems from the functional Schrodinger representation quantization of the gauge degrees of freedom, whence

24 $$E(x) = \frac{-i}{\sqrt{2\pi}} \sum_{p>0} \left(\cos(px) \frac{d}{d\gamma_1} + \sin(px) \frac{d}{d\gamma_2} \right) - \frac{i}{\sqrt{2\pi}} \frac{d}{d\gamma_0} \quad .$$

This lends the interpretation of the Gauss operator as the generator of a parallel transport of the fermionic Hilbert space over the space of gauge transformations. Finally including the kinetic part of the gauge fields, we find

25 $$H = H_0 - \sum_i \left(\sqrt{2p}\, \gamma_i Q_i - \gamma_i^2 + \frac{1}{2} \frac{d^2}{d\gamma_i^2} \right) - \gamma_0 Q + \frac{1}{2} \gamma_0^2 - \frac{1}{2} \frac{d^2}{d\gamma_0^2} \quad .$$

III. Fermionic Hilbert Bundle

The full fermionic Hamiltonian can be diagonalized by a unitary transformation,

26 $$H_F = U H_0 U^{+}$$
$$U = e^{-i\sqrt{\frac{2}{p}}\, \gamma_i \epsilon_{ij} Q_j} \quad .$$

The full eigenstates $U|\varepsilon\rangle$, where $|\varepsilon\rangle$ is an eigenstate of H_0, can be thought of as having been obtained from the free eigenstates by parallely transporting these in the space of γ_i with a (fermionic) operator valued connection

27 $$\mathcal{A}_i = U^{+} \frac{d}{d\gamma_i} U = -i\, \epsilon_{ij} \left(\sqrt{\frac{2}{p}}\, Q_j + \frac{1}{p} \gamma_j \right) \quad .$$

Clearly this is a pure gauge, and the corresponding field strength vanishes. Therfore the fermionic Hilbert spaces form a trivial bundle over the space of gauge fields γ_i. However if we consider the reduction of this bundle to each energy eigensubspace, the resulting bundle is non-trivial. The relevant (U(1)) connection is that defined by Berry /8/

28 $$i\, A_i = \langle \varepsilon | \mathcal{A}_i | \varepsilon \rangle = \frac{-i}{p}\, \epsilon_{ij} \gamma_j$$

and

29 $$B = \frac{1}{2} \epsilon_{ij} F_{ij} = \frac{1}{2} \epsilon_{ij} \left(\frac{d}{d\gamma_i} A_j - \frac{d}{d\gamma_j} A_i \right) = \frac{2}{p} \quad .$$

In more concrete terms, such a connection means that parallely transporting an eigenstate about a closed loop in the space yeilds back the same state multiplied by a phase. The phase is exactly the flux of the magnetic field (29) passing through the loop. This serves as a rigorous example of the bundle structure proposed by Alvarez-Gaume and Nelson /3/ and established by Niemi and Semenoff /8/ and by Sonoda /9/.

IV. Faddeev's Proposal and Full Quantization

To implement Faddeev's proposal is straight-forward. With

30 $$G_1 = \sqrt{\frac{2}{p}}\left(\frac{a + a^\dagger}{\sqrt{2}}\right) \quad , \quad G_2 = -i\sqrt{\frac{2}{p}}\left(\frac{a - a^\dagger}{\sqrt{2}}\right) ,$$

31 $$\left[a \, , \, a^\dagger\right] = 1$$

for (23). This is the appropriate decomposition. Then

32 $$a \mid \Psi_{phys} \rangle = 0$$

is equivalent to imposing Gauss' law as an equation between matrix elements. The full Hamiltonian is diagonalized by linear canonical transformatiions, after bosonization of the fermionic kinetic parts. The details will be published elsewhere. The result is

33 $$H(p) = \frac{1}{2}\omega_+\left(\hat{Q}^2 + \hat{P}^2\right) + \frac{1}{2}\omega_-\left(\hat{q}_2^{\,2} + \hat{p}_2^{\,2}\right)$$

with frequencies

34 $$\omega_\pm = \pm\frac{p}{2} + \frac{2}{p}\sqrt{\left(\left(\frac{p}{2}\right)^2 + 1\right)^2 - 1} \; .$$

V. Conclusion

It is evident that the spectrum (34) is not Lorentz invariant. It is clear that this is not a consequence of having worked on a circle. Everything can be carried out equally well on the line, only one must work mostly with operator valued distributions, which just adds surmountable technical difficulties. However a perfectly consistent and unitary quantum field theory has emerged. It is important to investigate exactly why Lorentz invariance is lost, and if there is some way to overcome this problem. We suggest that our techniques may be extended to a manifestly Lorentz invariant scheme for quantization, perhaps along the lines of the Gupta-Bleuler quantization of Q E D, or incorporating the BRST formalism.

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