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Integrability and Virasoro Symmetry of the Noncritical Baxter/Ising Model

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Abstract

The properties of a noncritical $c = 1$ lattice Virasoro algebra for the exactly solvable Baxter eight-vertex model are studied in the decoupling limit in which the Baxter model is equivalent to a staggered pair of Ising models. The Virasoro operator L_0 is given by the first moment of the XY spin chain density which is proportional to the logarithm of the corner transfer matrix (CTM). The eigenmodes of L_0 are explicitly constructed in terms of lattice fermion operators. They are related to the eigenmodes of the XY Hamiltonian by harmonic analysis on the spectral (rapidity) torus, which arises as the CTM group manifold. At the critical point, the results reduce to a momentum space formulation of space-time conformal symmetry. Two related Virasoro algebras are constructed from the fermionic eigenmode operators of L_0. One is a generalization of the $T = T_c$ conformal algebra away from the critical point. The other represents diffeomorphisms of the real rapidity circle, which is compact due to the presence of the space-time lattice. Both are spectrum generating algebras of the corner transfer matrix. In the scaling limit, the Virasoro operators are expressed in terms of a massive, free Dirac field. For $n \geq -1$, the L_n's are given by integrals of local densities. We discuss the relation between these densities and the integrals of motion and higher spectral flows that characterize the integrability of the system.

I. Introduction

A number of recent developments in two-dimensional field theory have suggested a deep relationship between conformal symmetry and complete quantum integrability for exactly solvable lattice models [1,2]. The application of conformal symmetry arguments to the study of critical statistical mechanical systems has already yielded new insight into their properties.[3] On the other hand, the algebraic structure of non-critical integrable systems is generally expressed in terms of Yang-Baxter relations [4,5], and the relation between this structure and that of conformal field theory is still rather mysterious. One approach to the study of this connection [1] has focused on Baxter's method of corner transfer matrices[6] (hereafter called CTM's), which seem to be generic to integrable quantum systems. In a previous paper[1], we showed that, in the eight-vertex model, the logarithm of the CTM (which is also the first moment of the XYZ spin chain Hamiltonian density) was in fact the central element $\mathcal{L}_0$ of a Virasoro algebra[7]. This provides some insight into the remarkably simple eigenvalue spectrum of the CTM first derived by Baxter[6]. An important consequence of the CTM-Virasoro connection is that, in a certain sense, the infinite dimensional symmetry of the critical eight-vertex model is still present in the non-critical theory, and that the noncritical realization of the Virasoro symmetry is therefore characteristic of the more general phenomenon of integrability. In this paper, we present a detailed discussion of the Virasoro symmetry of the noncritical $(\text{Ising})^2/XY$ spin chain model, which is a special case of the Baxter model with vanishing four-spin coupling. After fermionization, this model becomes the theory of a free massive Dirac fermion on a lattice. It therefore provides a relatively simple framework in which to investigate the significance of the noncritical lattice Virasoro algebra and its connection with the conformal symmetry of the critical theory.

In Section II, we briefly review the relationship between the Baxter eight-vertex model, the XYZ spin chain, and the massive Thirring model[8]. This establishes, for later reference, the connection between the spin variables of the lattice model and the Dirac fermion field operators. Specializing to the $(\text{Ising})^2$/XY-chain/massive-free-fermion case, we begin Section III by examining the well-known diagonalization of the XY spin chain Hamiltonian. This is accomplished by the lattice version of a Bogoliubov rotation. This discussion also serves to introduce the elliptic function (lattice rapidity) parametrization of momentum space on a lattice, which is a central

feature of the entire discussion. Then, we go on to discuss the diagonalization of the corner transfer matrix. Just as the eigenstates of the row-to-row transfer matrix of the Baxter model are the same as those of the XYZ Hamiltonian (the zeroth moment of the Hamiltonian density), the eigenstates of the CTM are the same as those of the first moment of the XYZ Hamiltonian density. Thus, for the (Ising)^2/XY case, we are again faced with the problem of diagonalizing an operator which is quadratic in the fermion operators,

$$\mathcal{L}_0 = \frac{2K}{\pi} \sum_{j=-\infty}^{\infty} j\mathcal{H}(j,j+1) \ , \tag{1.1}$$

$$\mathcal{H}(j,j+1) = -\frac{1}{2}[\sigma_j^x\sigma_{j+1}^x + k\sigma_j^y\sigma_{j+1}^y] \ , \tag{1.2}$$

where k turns out to be the elliptic modulus. The diagonalization of $\mathcal{L}_0$ is in fact closely related to that of the XY Hamiltonian. The eigenmode operators of $\mathcal{L}_0$ are simply the Fourier transforms of the Hamiltonian eigenmode operators over the elliptic rapidity parameter.

In Section IV, we give a more detailed account of the lattice Virasoro algebra constructed in ref. ([1]). Much of this discussion applies to the more general case of the eight-vertex model (massive interacting fermions). However, the explicit construction of CTM eigenstates is more subtle in the interacting case and will not be discussed here. The lattice Virasoro algebra, which we will denote by $\{\{\mathcal{L}_n\}\}$, is constructed in terms of the $\mathcal{L}_0$ eigenmode operators. It is a natural generalization of the exact lattice Poincare algebra [9] in fact, $\mathcal{L}_{-1}$ is a certain linear combination of the conserved charges associated with commuting row-to-row transfer matrices (RTM's). This implies, for example, that the eigenstates of the Virasoro operator $\mathcal{L}_{-1}$ are given by the same Bethe ansatz that diagonalizes the RTM and the XYZ Hamiltonian. This structure, therefore, exhibits a close relation between the Virasoro algebra and the sequence of mutually commuting conserved charges and consequently a relationship between the Virasoro verma module and the Hamiltonian spectrum. This connection persists in the general interacting case, where the Hamiltonian spectrum includes an adjustable number of bound states. As discussed briefly in [10] , these considerations lead, via the ABF models[11], to a relation between the bound state thresholds and the FQS sequence of Virasoro central charges[12,13]. This will be discussed in detail in a subsequent publication.

One crucial ingredient of the lattice Virasoro algebra is the discreteness of the $\mathcal{L}_0$-eigenvalues which is due to the periodicity of momentum space on a lattice. Moreover, lattice momentum space is compact not only in the real but also in the imaginary rapidity direction : the topology of momentum space on the lattice is that of a torus in the rapidity parametrization. (We will refer to this as the spectral torus. See Fig. 1). Because of this double periodicity, it is possible to construct two distinct $\mathcal{L}_0$ eigenmode operators corresponding to the doubly periodic nature of the momentum space topology. The operators constructed by Fourier transformation around the real rapidity direction are the eigenmode operators of the CTM, while those constructed by Fourier transformation around the imaginary direction of the spectral torus become the fermionic oscillator modes of the conformal algebra at $T = T_c$. To construct the eigenmode operators in the imaginary direction, we discuss the extension of the Hamiltonian eigenmode operator into the complex rapidity plane. This entails the introduction of a "harmonic conjugate" mode operator, which is given by the Fourier series conjugate to that of the original Hamiltonian eigenmode operator. The even and odd combinations of the Hamiltonian eigenmode operator and its harmonic conjugate are given by lattice Fourier sums over either the positive or negative x-axis and provide the analytic continuation needed. In this way, we construct the Euclidean version of the original lattice Virasoro algebra. We denote it by $\{\{L_n\}\}$. At the critical point this reduces to either the left- or right-moving conformal algebra, depending on whether we have chosen the fixed real part of the rapidity to correspond to lattice momentum near 0 or near π. Thus, from this point of view, the appearance of two independent (left and right) algebras in the critical theory is related to the familiar species doubling of lattice fermions. (Away from the critical point, the left and right movers are connected by mass terms, and the two algebras do not commute.)

So far, the Virasoro operators we have mentioned have a fairly direct connection with those of the critical conformal theory. Their physical effect may be described as a noncritical, lattice generalization of space-time conformal transformations. There is, however, another construction of a Virasoro algebra from the eigenmode operators of the CTM. This one is motivated by observing the analogy between the spectral (momentum) torus of the noncritical lattice model and a coordinate-space torus in conformal field theory. Here we will also consider this algebra, which is associated with diffeomorphisms of the spectral (rapidity) parameter [14]. This algebra will be denoted by $\{\{\mathcal{L}_n^{diff}\}\}$. The $\mathcal{L}_0$ operators for the two algebras $\{\{\mathcal{L}_n\}\}$ and $\{\{\mathcal{L}_n^{diff}\}\}$

are in fact the same (in each case $\mathcal{L}_0$ is essentially the log of the CTM). This results from the equivalence of rigid rotations in coordinate space and momentum space.

In Section V, we investigate further the relation between the lattice Virasoro algebra and the conformal algebra at the critical point. (We will denote the latter by $\{\{L_n^c\}\}$.) In the low temperature limit, the discrete eigenvalue spectrum of the CTM arises from the periodicity of real rapidity, while the periodicity in the imaginary direction is responsible for the discrete integer eigenvalues in the critical regime. The L_0^c of the critical conformal algebra has discrete integer eigenvalues in the radial quantization scheme due to the periodicity of Euclidean rotations. In the limit $T \to T_c$ we identify the lattice Virasoro algebra $\{\{L_n\}\}$ arising from the periodicity in imaginary rapidity with the critical conformal Virasoro algebra formulated in the analytic momentum plane.

The conventional critical Virasoro algebra is obtained by taking the moments of a conserved traceless stress-energy tensor. Clearly, this structure does not persist away from the critical point since the tracelessness is unique to the theory with zero mass. A main purpose of Section VI is to establish a corresponding - but by no means identical - statement in the noncritical Virasoro algebra. We investigate the near-critical behavior of the model. In the scaling limit, this reduces to the continuum field theory of a massive free fermion. This allows us to study, in a field theoretic context, the role of mass in the noncritical Virasoro symmetry. For this massive continuum limit, we find that the Virasoro operators L_n for $n \geq -1$ can all be expressed as integrals of local space-time dependent conserved densities. Each new Virasoro operator L_n turns out to introduce a new member of the sequence of higher conserved densities associated with integrability. We see, in this way, a direct link between the noncritical Virasoro algebra and the higher integrals of motion representing the integrability of the system [15]. Finally, the relation between the transformations generated by the L_n^{diff}'s and the higher spectral flows is discussed.

In Appendix A, we give some details leading to the elliptic function parametrization used in Section III. In Appendix B, we collect some explicit formulae for the local densities of commuting charges.

Throughout the paper, our essential viewpoint is that the boost operator can be viewed as an evolution operator of the system. There are some indications that this point of view is very natural in many integrable systems. We will refer to the

quantization scheme based on the boost operator as "angular quantization" since surfaces of fixed angular "time" radiate out from the origin in Euclidean space. (In a conformal theory, one may map the $z(= x + iy)$ plane onto a cylinder by a conformal mapping $\tau + i\sigma = \log z$. In this case, the angular quantization scheme is related to the more familiar radial quantization by an interchange of σ and τ.)

II. The Eight-Vertex Model and Massive Dirac Fermions on a Lattice

In the spin or "interaction round a face" (IRF) formulation, the eight-vertex (8V) model consists of a pair of staggered Ising lattices coupled together by a four-spin interaction. In an elementary face shown in Fig. 2, the spins are labeled $\sigma_1, \sigma_2, \sigma_3$, and σ_4 with σ_1 and σ_3 on one Ising sublattice and σ_2 and σ_4 on the other. The Boltzmann weight of a face is given by

$$\exp[K_1\sigma_1\sigma_3 + K_2\sigma_2\sigma_4 + K''\sigma_1\sigma_2\sigma_3\sigma_4] \quad . \tag{2.1}$$

with $\sigma_i = \pm 1$. Here K_1 and K_2 are the horizontal and vertical spin-spin couplings of the two Ising sublattices (which are oriented at 45 degrees to the axes of the 8V model). The four-spin coupling K'' connects the two sublattices and gives rise to a Thirring four-fermion interaction in the corresponding field theory.

The fermionization of the model is most easily carried out in the vertex or "arrow" formulation, where we place up (right) or down (left) arrows on each link of the dual lattice according to whether the spins on either side of a vertical (horizontal) link are the same or opposite, as shown in Fig. 2. A face of the spin lattice becomes a vertex of the arrow lattice, and, in the absence of external field, there are eight different vertex configurations with four possible Boltzman weights a, b, c, and d given by

$$\begin{aligned} a &= \exp(K_1 + K_2 + K'') \ , \\ b &= \exp(-K_1 - K_2 + K'') \ , \\ c &= \exp(-K_1 + K_2 - K'') \ , \end{aligned} \tag{2.2}$$

$$d \;=\; \exp(K_1 - K_2 - K'') \;\; .$$

In the arrow formulation, we may write an elementary vertex as a two-spin operator acting on the upper and right arrows and turning them into the lower and left arrows respectively,

$$V_n = \frac{1}{2}\left[(a+c) + (a-c)\sigma_n^z\sigma_{n+1}^z + (b+d)\sigma_n^x\sigma_{n+1}^x + (b-d)\sigma_n^y\sigma_{n+1}^y\right] \quad . \tag{2.3}$$

To dispel some possible confusion, it may be useful to note that, in the quantum inverse formalism for the eight-vertex model [5,16], a somewhat different notation is used for a vertex. In that formalism, the vertex (called the "L-matrix") is a two-by-two matrix labeled by the horizontal arrows, with each element of the matrix being a one-spin operator acting on the vertical arrow at site n. This formulation is appropriate for the row- o-row transfer matrix, while the description of the vertex as a two-spin operator (2.3) is more natural for the corner transfer matrix. The latter description also provides the connection to the Heisenberg spin chain Hamiltonian, whose zeroth and first moments are a central focus of our discussion. Introducing the elliptic function parametrization of the vertex weights,

$$\begin{aligned} a &= \mathrm{snh}(\lambda - u)/\mathrm{snh}\lambda \;, \quad & b &= \mathrm{snh}u/\mathrm{snh}\lambda \;, \\ c &= 1 \;, & d &= k\,\mathrm{snh}\,u\,\mathrm{snh}(\lambda - u) \;, \end{aligned} \tag{2.4}$$

where k is the elliptic modulus and $\mathrm{snh}u \equiv -i\,\mathrm{sn}(iu)$, we see that for $u = 0, a = c = 1$ and $b = d = 0$. Thus, the vertex (2.3) becomes proportional to the unit operator at $u = 0$ and we may expand around this point to give

$$V_n^{-1}(0)V_n(u) = 1 - u\,\mathcal{H}_{XYZ}(n, n+1) + O(u^2) \;\; , \tag{2.5}$$

where

$$\mathcal{H}_{XYZ}(j, j+1) = J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z \quad . \tag{2.6}$$

Using properties of elliptic functions, it may be shown that the spin chain coefficients are related to the Boltzmann weights by $J_x : J_y : J_z = ab + cd : ab - cd : \frac{1}{2}(a^2 + b^2 - c^2 - d^2) \equiv 1 : \Gamma : \Delta$.

Throughout our discussion, we will be considering operators in the infinite volume limit and manage to avoid any specific reference to boundary conditions. This treatment causes no great difficulties for the (Ising)^2/free fermion case we are considering

here. (As pointed out in [17], this leads to the $h = \frac{1}{8}$ representatation of the $c = 1$ Virasoro algebra, corresponding to fixed boundary conditions.) A discussion of the interacting case requires much more careful attention to boundary conditions. For the case of the row-to-row transfer matrix (RTM), the proper treatment is well understood [18] and leads to the Bethe ansatz equations which determine the spectrum. Much less is known about the diagonalization of the CTM. Although its eigenvalues are completely determined by Baxter's arguments, its eigenstates are not. We suspect that the lattice Virasoro algebra provides a clarification of the structure of these eigenstates and their relation to those of the row-to-row transfer matrix and to the Bethe ansatz equations. For the present free fermion discussion, however, we can avoid these complications and immediately define our operators in an infinite volume. The "sea filling" which gives rise to the Bethe ansatz integral equations in the general case can be accomplished by a simple normal ordering prescription in the free fermion theory. Thus, we take the row-to-row transfer matrix to be an infinite row of vertices,

$$T(u) = \lim_{N\to\infty} \mathrm{Tr}\, V_{-N} V_{-N+1} \dots V_N \quad . \tag{2.7}$$

To define the CTM, we first denote a finite row of vertices by

$$G_j^{(n)} = V_n V_{n-1} V_{n-2} \dots V_j \quad . \tag{2.8}$$

The CTM operator $A_N(u)$ is then defined by stacking rows together,

$$A_N(u) = G_1^{(N)} G_2^{(N)} G_3^{(N)} \dots G_N^{(N)} \quad . \tag{2.9}$$

Physically, this operator represents one quadrant of a lattice and connects a semi-infinite row of arrows with a semi-infinite column of arrows. As Baxter has shown, the CTM is well-defined in the thermodynamic limit $N \to \infty$ if we divide by the largest eigenvalue Λ_N of A_N in the region where u is real and positive (this corresponds to normal ordering the log of the CTM),

$$A(u) = \lim_{N\to\infty} A_N(u)/\Lambda_N \quad . \tag{2.10}$$

(Again we note that, in both (2.7) and (2.9), the naive infinite volume limits taken here will require a more careful treatment for the general eight-vertex model.) We also define an "extended" CTM $\mathcal{A}(u)$, which is the direct product of a lower-right and

an upper-left CTM. (See, Fig. 3.) The extended CTM acts on a full row of arrows and connects it to a full column of arrows.

Expanding the RTM (2.7) and the CTM (2.9) around $u = 0$, it is easy to see that the terms of order u are given, respectively, by the zeroth and first moments of the XYZ spin chain density :

$$T^{-1}(0)T(u) = 1 - u\,H_{XYZ} + \cdots \ , \tag{2.11}$$

$$\mathcal{A}(u) = 1 - u\mathcal{L}_0 + \cdots \ , \tag{2.12}$$

where

$$H_{XYZ} \ = \ \sum_{j=-\infty}^{\infty} \mathcal{H}_{XYZ}(j, j+1) \ , \tag{2.13}$$

(2.14)

$$\mathcal{L}_0 \ = \ \frac{2K}{\pi} \sum_{j=-\infty}^{\infty} j\mathcal{H}_{XYZ}(j, j+1) \ . \tag{2.15}$$

In the expansion of $T(u)$, the higher order terms provide an infinite number of mutually commuting conserved operators. For the CTM, the higher order terms actually exponentiate to give the exact relation

$$\mathcal{A}(u) = \exp(-\frac{\pi u}{2K}\,\mathcal{L}_0) \ . \tag{2.16}$$

where K is the real elliptic quarter-period (complete elliptic integral) associated with modulus k. This exponentiation follows from the remarkable group property of the CTM: $A(u)A(v) = A(u+v)$. (Note that the sum in (2.15) is taken from $-\infty$ to ∞, corresponding to the log of the extended CTM. We have postponed here the problem of normal ordering the extended CTM, which is required to obtain a highest weight representation of the lattice Virasoro algebra. This will be discussed Section IV.)

To conclude this review of the model, we want to recall the precise connection between the XYZ spin chain Hamiltonian and the Hamiltonian of a massive Thirring fermion. We first introduce canonical lattice fermion operators by a Jordan-Wigner transformation,

$$c_j^{x,y} = \frac{1}{\sqrt{2}}\sigma_j^{x,y}(\prod_{l<j}\sigma_l^z) \ . \tag{2.17}$$

The XYZ spin chain Hamiltonian is then written as

$$H_{XYZ} = \sum_{j=-\infty}^{\infty} i(c_j^y c_{j+1}^x - \Gamma c_j^x c_{j+1}^y - 2\Delta c_j^x c_j^y c_{j+1}^y c_{j+1}^x) \ . \tag{2.18}$$

It is convenient to introduce the following complex fermion operators :

$$c_j \equiv \frac{i^j}{\sqrt{2}}\left(c_j^x + ic_j^y\right) \ , \quad c_j^\dagger \equiv \frac{(-i)^j}{\sqrt{2}}\left(c_j^x - ic_j^y\right) \ . \tag{2.19}$$

As is well known, one complex fermion on a (1+1)-dimensional lattice yields a 2-component Dirac fermion (two complex Weyl fermions) in the continuum limit (fermion doubling). Going into momentum space, we identify the upper and lower chiral component of the Dirac fermion in the γ_5 diagonal representation as

$$\psi_1(q) \equiv e^{-\frac{\pi}{4}i}\sqrt{2a}\sum_j e^{-iqja}(-1)^j c_j^\dagger \ , \tag{2.20}$$

$$\psi_2(q) \equiv e^{\frac{\pi}{4}i}\sqrt{2a}\sum_j e^{-iqja} c_j \quad , \quad -\frac{\pi}{2a} < q < \frac{\pi}{2a} \quad , \tag{2.21}$$

respectively. Here a is a lattice spacing and we introduced a phase factor which is designed to give a conventional Dirac mass term. Note that the range of the momentum is half of the entire Brillouin zone. In the γ_0 diagonal basis, the above expression is equivalent to the staggered structure with even and odd sites occupied respectively by the upper and lower component of the Dirac fermion.

By inverting the above expressions and substituting them into eq. (2.18), we obtain

$$H_{XYZ} = H_0 + H_{mass} + H_{th} \ , \tag{2.22}$$

$$H_0 = \left(\frac{1+\Gamma}{2}\right)\int_{-\pi/2a}^{\pi/2a}\frac{dq}{2\pi}\sin qa\ \left(\tilde{\psi}_1(q)\psi_1(q) - \tilde{\psi}_2(q)\psi_2(q)\right) \ , \tag{2.23}$$

$$H_{mass} = \left(\frac{1-\Gamma}{2}\right)\int_{-\pi/2a}^{\pi/2a}\frac{dq}{2\pi}\cos qa\ \left(\tilde{\psi}_1(q)\psi_2(q) + \tilde{\psi}_2(q)\psi_1(q)\right) \ , \tag{2.24}$$

$$H_{th} = -4a\Delta\int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}}\left(\prod_{i=1}^{4}\frac{dq_i}{2\pi}\right)2\pi\delta(q_1+q_2-q_3-q_4)\cos(q_1+q_2)a$$
$$\tilde{\psi}_1(q_1)\tilde{\psi}_2(q_2)\psi_2(q_3)\psi_1(q_4) \ . \tag{2.25}$$

Here, we denote by $\tilde{\psi}_{1,2}$ the hermitean conjugate of $\psi_{1,2}$ in fixed time quantization. Eqs. (2.22)-(2.25) reduces, in the continuum limit $a \to 0$, to the massive Thirring Hamiltonian. Note that, according to (2.18), the case where the four-spin coupling vanishes (i. e. $ab = cd$) corresponds to $\Gamma = 0$. By an interchange of Pauli matrices $\sigma^y \leftrightarrow \sigma^x$, this is equivalent to the case $\Delta = 0$ which we discuss here.

III. Diagonalization of the Corner Transfer Matrix

In order to proceed with an explicit investigation of the CTM and its eigenvectors we now restrict ourselves to the free fermion case. Let us first consider the diagonalization of the XY spin chain Hamiltonian. The solution to this problem is of course well-known ([19]), but it is useful to review it here, since it turns out to be closely related to the diagonalization of the CTM. The XY spin chain Hamiltonian is

$$H = \sum_{j=-\infty}^{\infty} \mathcal{H}(j, j+1) \ , \tag{3.1}$$

where

$$\mathcal{H}(j, j+1) = -\frac{1}{2}[\sigma_j^x \sigma_{j+1}^x + k\sigma_j^y \sigma_{j+1}^y] \ . \tag{3.2}$$

Upon introducing the Onsager/Baxter elliptic function parametrization, the parameter k will become the elliptic modulus. We rewrite the Hamiltonian in terms of the fermion operators constructed in the preceeding Section,

$$c_j^{x,y} = \frac{1}{\sqrt{2}} \sigma_j^{x,y} (\prod_{l<j} \sigma_l^z) \ . \tag{3.3}$$

The Hamiltonian becomes

$$H = i \sum_j [c_j^y c_{j+1}^x - k c_j^x c_{j+1}^y] \ . \tag{3.4}$$

We will make use of the following elliptic function expressions : let $\alpha = \beta + \frac{iK'}{2} - K$

and define

$$S(\beta) = -i\sqrt{k}\,\mathrm{sn}\alpha = \left[\frac{(1-k)\,\mathrm{sn}\beta + \mathrm{i}\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{(1-\mathrm{k})\,\mathrm{sn}\beta - \mathrm{i}\,\mathrm{cn}\beta\,\mathrm{dn}\beta}\right]^{\frac{1}{2}}$$
$$= \frac{(1-k)\mathrm{sn}\beta + \mathrm{icn}\beta\mathrm{dn}\beta}{1-\mathrm{ksn}^2\beta} \equiv e^{ip} \ , \tag{3.5}$$

$$C(\beta) = i\sqrt{k}\,\mathrm{cn}\alpha = \sqrt{1-k}\left[\frac{i\,\mathrm{sn}\beta\,\mathrm{dn}\beta - \mathrm{cn}\beta}{1-\mathrm{k}\,\mathrm{sn}^2\beta}\right] \ , \tag{3.6}$$

$$D(\beta) = \mathrm{dn}\alpha = \sqrt{1-k}\left[\frac{i\,k\mathrm{sn}\beta\,\mathrm{cn}\beta + \mathrm{dn}\beta}{1-\mathrm{k}\,\mathrm{sn}^2\beta}\right] \ , \tag{3.7}$$

for $-K < \beta < 3K$. In Appendix A, we briefly outline the diagonalization of $\mathcal{L}_0$ and how it leads to this elliptic parametrization. This parametrization is also crucial to understanding the relation between the eigenvectors of the CTM and those of the XY Hamiltonian.

In order to diagonalize the XY Hamiltonian, we introduce the momentum space operators

$$B_{x,y}(\beta) = \sum_j [S(\beta)]^{j-1} c_j^{x,y} \ . \tag{3.8}$$

Using simple properties of elliptic functions, it is easy to show that

$$[H, D(\beta)B_x(\beta)] = \Lambda(\beta)C(\beta)B_y(\beta) \tag{3.9}$$

$$[H, C(\beta)B_y(\beta)] = \Lambda(\beta)D(\beta)B_x(\beta) \ , \tag{3.10}$$

where

$$\Lambda(\beta) \equiv \frac{-iC(\beta)D(\beta)}{S(\beta)} = (1-k)\frac{1+k\mathrm{sn}^2\beta}{1-k\mathrm{sn}^2\beta}$$
$$= \sqrt{(1+k)^2\cos^2 p(\beta) + (1-k)^2\sin^2 p(\beta)} > 0 \ . \tag{3.11}$$

The eigenmode operators of the Hamiltonian are

$$B_\pm(\beta) = \left[D(\beta)B_x(\beta) \pm C(\beta)B_y(\beta)\right]/\sqrt{2} \ , \tag{3.12}$$

which satisfy

$$[H, B_\pm(\beta)] = \pm\Lambda(\beta)B_\pm(\beta) \ . \tag{3.13}$$

For $0 < k < 1$ we see that $\Lambda(\beta) \geq |1-k|$ and there is a finite mass gap (fermion mass). The critical point is reached at $k = 1$ where the mass gap vanishes. The operator $B_{+(-)}(\beta)$ defined over the full elliptic period in the real direction $(-K < \beta < 3K)$ is an annihilation (creation) operator with respect to the physical ground state of H, namely the one obtained by filling up all negative energy states.

We list some useful properties of the functions $S(\beta), C(\beta)$, and $D(\beta)$ which follow from the periodicity properties of the Jacobi elliptic functions [20]:

$$\begin{array}{lll} S(\beta+2K) = -S(\beta)\,, & C(\beta+2K) = -C(\beta)\,, & D(\beta+2K) = D(\beta)\,, \\ S(\beta+iK') = -1/S(\beta)\,, & C(\beta+iK') = -iD(\beta)/S(\beta)\,, & D(\beta+iK') = iC(\beta)/S(\beta)\,, \\ S(\beta+2iK') = S(\beta)\,, & C(\beta+2iK') = -C(\beta)\,, & D(\beta+2iK') = -D(\beta)\,, \end{array} \tag{3.14}$$

$$C(\beta)C(-\beta) = D(\beta)D(-\beta) = \Lambda(\beta)\ . \tag{3.15}$$

Also note that, for β real,

$$S^*(\beta) = -S(-\beta)\,,\ C^*(\beta) = C(-\beta)\,,\ D^*(\beta) = D(-\beta)\ . \tag{3.16}$$

From now on, we let $B_+(\beta) \equiv B(\beta)$. Then $B_-(\beta) = B(-\beta - iK')$ (β real). $B(\beta)$ satisfies the following properties:

$$B^*(\beta) = \pm B(\beta + 2K \mp iK')\ , \tag{3.17}$$

$$B(\beta + 2iK') = -B(\beta)\ , \tag{3.18}$$

as well as canonical anticommutation relations

$$\begin{aligned} \{B(\beta), B(\beta' + 2K - iK')\} &= 2\pi\delta(\beta - \beta') \\ \{B(\beta), B(\beta')\} &= 0\ , \end{aligned} \tag{3.19}$$

for β, β' real. These properties are represented schematically in Fig. 4. The second equation (3.18) expresses the double-valuedness of the fermion operators under Euclidean rotation by 2π. In the subsequent discussion it will be necessary to decompose $B(\beta)$ into two sums over positive and negative directions on the lattice :

$$B(\beta) = B^{>}(\beta) + B^{<}(\beta)\ . \tag{3.20}$$

where $B^>$ and $B^<$ are given by (3.12) with B_x and B_y replaced by

$$B^>_{x,y}(\beta) = \sum_{j>0} [S(\beta)]^{j-1} c_j^{x,y} \quad , \tag{3.21}$$

$$B^<_{x,y}(\beta) = \sum_{j\leq 0} [S(\beta)]^{j-1} c_j^{x,y} \quad . \tag{3.22}$$

Next, we consider the diagonalization of the corner transfer matrix. In reference [21], Baxter diagonalized the corner transfer matrix directly. Here we will consider instead the operator

$$\begin{aligned} \mathcal{L}_0 &= \frac{2K}{\pi} : \sum_{j=-\infty}^{\infty} j\mathcal{H}(j,j+1) : + const. \\ &=: \left(\sum_{j>0} + \sum_{j<0} \right) j\mathcal{H}(j,j+1) : + const. \equiv \mathcal{L}_0^{(+)} + \mathcal{L}_0^{(-)} \quad . \end{aligned} \tag{3.23}$$

which has the same eigenstates as the CTM. In the second line, we separated the sum into positive and negative parts again. The constant will be chosen later. In (3.23) the normal ordering is to be taken with respect to the vacuum (lowest lying eigenstate) of $\mathcal{L}_0$. This vacuum state may be obtained by first diagonalizing $\mathcal{L}_0$ and then taking its negative eigenmodes to be filled. As Baxter has shown, the operator $\mathcal{L}_0^{(+)}$ is related to the (normalized) corner transfer matrix $A^{(+)}(u)$ by $A^{(+)}(u) = \exp[-(\pi u/2K)\mathcal{L}_0^{(+)}]$, and thus has the same eigenstates as the CTM. Again using elementary properties of elliptic functions, it can be shown that

$$[\mathcal{L}_0, D(\beta)B_x(\beta)] = \frac{2iK}{\pi} \frac{\partial}{\partial\beta} (C(\beta)B_y(\beta)) \quad , \tag{3.24}$$

$$[\mathcal{L}_0, C(\beta)B_y(\beta)] = \frac{2iK}{\pi} \frac{\partial}{\partial\beta} (D(\beta)B_x(\beta)) \quad . \tag{3.25}$$

and therefore,

$$[\mathcal{L}_0, B(\beta)] = \frac{2iK}{\pi} \frac{\partial}{\partial\beta} (B(\beta)) \quad . \tag{3.26}$$

Thus, we define the operators

$$\Psi(\ell) \equiv \int_{-K}^{3K} \frac{d\beta}{\sqrt{8\pi K}} \exp\left(-\frac{i\pi\beta\ell}{2K} \right) B(\beta) \tag{3.27}$$

$$\tilde{\Psi}(\ell) \equiv \int_{-K}^{3K} \frac{d\beta}{\sqrt{8\pi K}} \exp\left(\frac{i\pi\beta\ell}{2K}\right) B(\beta + 2K - iK') \ , \tag{3.28}$$

(for ℓ =integer) which obey anticommutation relations

$$\{\Psi(\ell), \tilde{\Psi}(\ell')\} = \delta_{\ell,\ell'} \ , \ \{\Psi(\ell), \Psi(\ell')\} = \{\Psi(\ell), \Psi(\ell')\} = 0 \ . \tag{3.29}$$

and have a complex conjugation property $\Psi^*(\ell) = \tilde{\Psi}(\ell)$. These operators diagonalize $\mathcal{L}_0$:

$$\begin{aligned} [\mathcal{L}_0, \Psi(\ell)] &= -\ell\, \Psi(\ell) \ , \\ \left[\mathcal{L}_0, \tilde{\Psi}(\ell)\right] &= \ell\, \tilde{\Psi}(\ell) \end{aligned} \tag{3.30}$$

Thus, up to a normal ordering constant,

$$\mathcal{L}_0 = \sum_{\ell} \ell\, \tilde{\Psi}(\ell)\Psi(\ell) \ . \tag{3.31}$$

This clearly demonstrates that the eigenvalues of the CTM, in contrast with the eigenvalues of the RTM, stay discrete in the infinite-volume limit. Moreover, the eigenvalues of $(2K/\pi u)\log CTM$ are integers. The periodicity of the Brillouin zone is responsible for this discrete eigenvalue structure.

Eq. (3.27) exhibits a direct relation between the eigenstates of the Hamiltonian created by $B(\beta)$ and those of the CTM created by $\Psi(\ell)$. It is precisely what one would expect from the physical interpretation of the CTM as a lattice Lorentz boost operator which shifts the rapidities of all the particles in a Hamiltonian eigenstate by an equal amount.[9] The states obtained by Fourier transforming the Hamiltonian eigenstates over the rapidity of each particle are thus eigenstates of the CTM. The lattice fermion theory described by the XY spin chain represents a compactification of momentum space which preserves the Lorentz symmetry while discretizing the eigenvalues of the boost operator. Here we have exhibited this structure in the particularly simple context of a free fermion theory. However, the physical picture we have arrived at also applies to more general interacting theories such as the eight-vertex/XYZ model, and should provide a framework for investigating the eigenvectors of the CTM and their relation to the Bethe ansatz equations which describe the eigenstates of the XYZ Hamiltonian.

One of the important results of Baxter's work on corner transfer matrices is that, with the appropriate definition of the spectral parameter, the integer eigenvalue structure we have discussed for the Ising/XY spin chain case is also valid in the full eight-vertex model. Correspondingly, the role of the CTM as the rapidity shift operator and its relation to the RTM and the XYZ Hamiltonian remain intact for the interacting case ([9]). The main complication in constructing eigenstates of the CTM in the full eight-vertex model arises from the filling of the Dirac sea and the attendant Bethe ansatz equations needed to describe eigenstates of the RTM. Although our main focus in this paper is on the Ising case, we will briefly indicate here how the Virasoro structure of the full eight-vertex model may be abstracted from the eigenvalue spectrum of the CTM [1]. However, the explicit construction of the eigenstates of the CTM for the general eight-vertex model will not be discussed here. The eigenvalue structure of the eight-vertex model CTM may be represented as

$$\mathcal{L}_0^{(+)} = \sum_{\ell=1}^{\infty} (\ell/2)(\sigma_\ell^z + 1) \ , \tag{3.32}$$

in the diagonal representation of the CTM [1]. Here, the matrices $\sigma_\ell^{x,y,z}$ are the Pauli matrices, but are physically distinct from the matrices in the arrow representation used in the last section. In other words, the suffix ℓ should not be confused with the label j for the lattice sites. (In the low temperature limit, ℓ does reduce to a site index in what Baxter referred to as the "third-spin representation" [6].) One can define a parity conjugate of $\mathcal{L}_0^{(+)}$ obtained by $\mathcal{L}_0^{(-)} \equiv P\mathcal{L}_0^{(+)}P^{-1}$ with $P\sigma_j^{y,z}P^{-1} = -\sigma_{-j}^{y,z}$, $P\sigma_j^x P^{-1} = \sigma_{-j}^x$. A full boost operator which corresponds to a normalized extended CTM is defined to be $\mathcal{L}_0^{(+)} + \mathcal{L}_0^{(-)}$. Note that this operator is positive semi-definite by construction.

One can apply again the Jordan-Wigner transformation to the Pauli spin operators defined in this diagonal representation of the CTM to construct canonical fermion fields $\Psi(\ell)$ and $\bar{\Psi}(\ell')$ by

$$\Psi(\ell) = \left(\prod_{i<\ell} \sigma_i^z \right) \sigma_\ell^- \ , \tag{3.33}$$

$$\bar{\Psi}(\ell) = \left(\prod_{i=<\ell} \sigma_i^z \right) \sigma_\ell^+ \ . \tag{3.34}$$

with the anticommutators (3.29). In the Ising case, the mode operators $\Psi(\ell), \bar{\Psi}(\ell)$

are given explicitly by (3.27) and (3.28).

IV. Lattice Virasoro Algebra

Let us now turn to the construction of the lattice Virasoro algebra [1]. We require that the central element of the lattice Virasoro algebra is

$$\mathcal{L}_0 \equiv \mathcal{L}_0^{(+)} + \mathcal{L}_0^{(-)} + h = : \sum_{\ell} \ell \, \tilde{\Psi}(\ell)\Psi(\ell) : + h \ . \tag{4.1}$$

Here, h is a constant term and the normal ordering is with respect to the state $\mid h >$ defined by

$$\Psi(\ell) \mid h\rangle = 0 \ , \text{ for } \ell \geq 1 \ , \quad \tilde{\Psi}(\ell) \mid h\rangle = 0 \ , \text{ for } \ell \leq -1 \ . \tag{4.2}$$

To anticipate the form of the Virasoro operators for $n \neq 0$, recall that, classically, the conformal generators may be written as differential operators $\ell_n = -\frac{1}{2} z^{n+1} \frac{d}{dz} - \frac{1}{2} \frac{d}{dz} z^{n+1}$ acting on analytic functions of z. In our case, an analogous operator $\tilde{\ell}_n$ acts on the analytic plane denoted by $\zeta \equiv exp(i\pi\beta/2K)$. Since the algebra relies upon the property $[\frac{d}{dz}, z] = 1$ alone, we may construct the $\tilde{\ell}_n{}'s$ by replacing z by $\frac{d}{d\zeta}$ and $\frac{d}{dz}$ by $-\zeta$:

$$\begin{aligned} \tilde{\ell}_n &= \frac{1}{2} \zeta^{1/2} \left(\frac{d^{n+1}}{d\zeta^{n+1}} + \zeta \frac{d^{n+1}}{d\zeta^{n+1}} \frac{1}{\zeta} \right) \zeta^{1/2} \\ &= \zeta^{1/2} \left(\frac{d^{n+1}}{d\zeta^{n+1}} - \frac{1}{2}(n+1) \frac{d^n}{d\zeta^n} \frac{1}{\zeta} \right) \zeta^{1/2} \quad \text{for } n \geq -1 \ . \end{aligned} \tag{4.3}$$

This formula also holds for $n \leq -2$ if we take negative powers of the derivative to represent indefinite integration,

$$\left(\frac{d}{d\zeta} \right)^{-1} = \int^{\zeta} d\zeta' \ . \tag{4.4}$$

In this way, the lattice Virasoro algebra

$$[\mathcal{L}_n\ ,\ \mathcal{L}_m] \ = \ (n-m)\ \mathcal{L}_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \tag{4.5}$$

is given by

$$\begin{aligned} \mathcal{L}_n \ &= \ :\int_{-K}^{3K} \frac{d\beta}{2\pi} B(\beta+2K-iK')\ \tilde{\ell}_n B(\beta): + h\delta_{n,0}\ , \\ &= \ \sum_{\ell}\left(\ell+\frac{1}{2}n\right)\frac{\Gamma(\ell+n+1/2)}{\Gamma(\ell+1/2)} : \tilde{\Psi}(\ell)\,\Psi(\ell+n): + h\delta_{n,0}\ . \end{aligned} \tag{4.6}$$

The physical Hilbert space based on the angular quantization and built from the state $\mid h >$ forms a highest weight representation of the lattice Virasoro algebra: the eigenvalues of the operator $\mathcal{L}_0$ are bounded from below. Also,the eigenvalues of $\mathcal{L}_0$ are doubly degenerate due to the zero modes of the operators $\Psi(\ell=0)$ and $\tilde{\Psi}(\ell=0)$. The highest weight vector, therefore, forms a two-dimensional representation under parity conjugation. (Ramond sector).

As we will see in the next section, eq. (4.6) is identical in form to the Virasoro operators of the standard critical conformal algebra written in the momentum representation. However, the physical significance of the algebra is more subtle when we are away from the critical temperature. It is only at $T=T_c$ that the eigenmode operator $\Psi(\ell)$ is proportional to the Fourier transform of a local field in z-space (see the next Section). Away from T_c the relation between $\Psi(\ell)$ and the local lattice operators of the theory is more complicated, and is closely related to the structure of the Hamiltonian eigenstates. It is amusing to note that at $T=T_c$ the eigenvalue ℓ is an "angular momentum" variable (i. e. it is Fourier conjugate to the angle variable in coordinate space) while in the low-temperature limit $T\to 0$ it becomes the lattice site number.

The calculation of the central charge c and the highest weight h is done in a standard way [22]. One calculates

$$\langle h \mid [\mathcal{L}_n, \mathcal{L}_{-n}] \mid h\rangle = 2nh + \frac{c}{12}(n^3-n)\ . \tag{4.7}$$

Calculating the left hand side using eq. (4.7), one finds, after a proper treatment of the zero modes, $c=1$ and $h=\frac{1}{8}$.

As is well-known, the expression for ℓ_n is not the only form of the "first quantized" operator satisfying the classical Virasoro algebra. We may construct a one-parameter family of first quantized operators $\ell_n = -(1-\gamma)z^{n+1}\frac{d}{dz} - \gamma\frac{d}{dz}z^{n+1}$. Likewise,

$$\begin{aligned}\tilde{\ell}_n &= \zeta^{1-\delta}\left((1-\gamma)\frac{d^{n+1}}{d\zeta^{n+1}} + \gamma\zeta\frac{d^{n+1}}{d\zeta^{n+1}}\frac{1}{\zeta}\right)\zeta^{\delta} \\ &= \zeta^{1-\delta}\left(\frac{d^{n+1}}{d\zeta^{n+1}} - \gamma(n+1)\frac{d^n}{d\zeta^n}\frac{1}{\zeta}\right)\zeta^{\delta} \quad \text{for } n \geq -1 \ . \end{aligned} \tag{4.8}$$

For $n \leq -2$, we replace the derivative by eq. (4.4). We obtain

$$\mathcal{L}_n = \sum_{\ell}\{(1-\gamma)n + \ell + \delta - \gamma\}\frac{\Gamma(\ell+n+\delta)}{\Gamma(\ell+\delta)}\tilde{\Psi}(\ell)\Psi(\ell+n) \ . \tag{4.9}$$

The highest weight vector is nondegenerate for $\delta \neq \gamma$. (Neveu-Schwarz sector). We find [3]

$$\begin{aligned} c &= -12\gamma^2 + 12\gamma - 2, \\ h &= \frac{1}{2}(\delta-\gamma)^2 - \frac{1}{2}(\delta-\gamma) - \frac{1}{2}\gamma^2 + \frac{1}{2}\gamma \ . \end{aligned} \tag{4.10}$$

The degenerate (Ramond) case corresponds to $\delta = \gamma$ with the same results as above after treating the zero modes differently.

From now on, we focus on the $c = 1$ Virasoro algebra. Among the Virasoro operators we have constructed, $\mathcal{L}_0$ and $\mathcal{L}_{-1}$ are known symmetries of the system: $\mathcal{L}_0$ is a rapidity shift (boost) operator and $\mathcal{L}_{-1}$ is a certain linear combination of the sequence of conserved charges. (This can be seen from eq. (4.6) since $\mathcal{L}_{-1}$ is just an integral over the conserved operator $B^*(\beta)B(\beta)$ which describes the particle density in β-space.) In the above construction of the lattice Virasoro algebra, however, the role of the rest of the operators as the realization of an infinite dimensional symmetry of the system is not immediately evident. In later sections, we will try to further clarify this point by i) showing the connection to the conventional Virasoro algebra at the critical point and by ii) revealing a relation with infinite conservation laws in the massive scaling limit. Let us here describe the qualitative behavior at $k \to 0$ ($T \to 0$) and $k \to 1$ ($T \to T_c$). At $k = 0$, the lattice momentum is identified with the lattice

[3] In ref. [1], we used a form corresponding to $\gamma = 0$ by adding half of the number operator to $\mathcal{L}_0$. This produced $c = -2$.

rapidity and the index ℓ is identified as the site index j. The operator $\mathcal{L}_n$ is local in the sense that Ψ and $\tilde{\Psi}$ are separated by n sites. As k deviates from 0, the above structure is modified by some smearing function decaying with the distance. On the other hand, as k goes to 1, the eigenvalues of the CTM become continuous, and one loses the Brillouin zone perodicity. The continuum limit of eq. (4.6) with mode operators defined by eqs. (3.27), (3.28) (or (3.33), (3.34)) is not directly related to the standard Virasoro algebra at the critical point. On the other hand, the periodicity of the imaginary rapidity survives the continuum limit. It is this period which gives rise to the discrete eigenvalue structure at the critical point. We would like to construct CTM eigenmode operators Fourier-transformed over the imaginary rapidity direction. We will refer to these as Euclidean eigenmode operators.

For that purpose, let us first recall that the energy eigenmode operator $B(\beta)$ is given as an operator-valued Fourier series with respect to the lattice momentum (c.f. eqs. (3.21)- (3.22) :

$$B(\beta) \equiv \sum_j f(j) e^{ijp(\beta)} \;=\; B^{>}(\beta) + B^{<}(\beta) \;, \tag{4.11}$$

for β real and real $\pm iK'$. Following a standard procedure of harmonic analysis [23], we introduce a conjugate Fourier series

$$B_{conj}(\beta) = -i \sum_j f(j) \mathrm{sgn}(j) e^{ijp(\beta)} = -i \left(B^{>}(\beta) - B^{<}(\beta) \right) \;. \tag{4.12}$$

Viewed as a complex function of $\omega \equiv e^{ip(\beta)}$, these series have extensions from the unit circle $\mid \omega \mid = 1$ to the unit disc $D = \{\{ \omega \mid \mid \omega \mid < 1\}\}$ through the convolution with the Poisson kernel $\mathcal{P}$ and its harmonic conjugate $\mathcal{Q}$ respectively :

$$\underline{B}(\beta, r) \;\equiv\; \sum_j f(j) r^{|j|} e^{ijp(\beta)} = \mathcal{P} * B \;, \tag{4.13}$$

$$\underline{B}_{conj}(\beta, r) \;\equiv\; -i \sum_j f(j) \mathrm{sgn}(j) r^{|j|} e^{ijp(\beta)} = \mathcal{Q} * B_{conj} \;. \tag{4.14}$$

Here,

$$\mathcal{P} = \mathrm{Re} \frac{1 + re^{ip}}{1 - re^{ip}} \;, \quad \mathcal{Q} = \mathrm{Im} \frac{1 + re^{ip}}{1 - re^{ip}} \;. \tag{4.15}$$

Equivalently, the expressions for $B(\beta) + iB_{conj}(\beta) = 2B^{>}(\beta)$ and $B(\beta) - iB_{conj}(\beta) = 2B^{<}(\beta)$ are convergent at the shaded areas of Fig. 5 and Fig. 6 respectively. They

separately give rise to an (anti-)holomorphic operator in the corresponding regions of the complex ω plane. The extension of $B^{>or<}(\beta)$ to the entire imaginary period of β is done by analytic continuation in the ω plane. All in all, the CTM eigenmode operators are given by

$$\Psi_e(\ell) = e^{\frac{\pi}{4}i}\int_{C>}\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{+\ell+1/2}B^{>}(\beta+K) + e^{\frac{\pi}{4}i}\int_{C<}\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{+\ell+1/2}B^{<}(\beta+K)\ , \tag{4.16}$$

$$\tilde{\Psi}_e(\ell) = e^{\frac{\pi}{4}i}\int_{\tilde{C}>}\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{-\ell-1/2}B^{>}(\beta+3K-iK')+e^{\frac{\pi}{4}i}\int_{\tilde{C}<}\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{-\ell-1/2}B^{<}(\beta+3K-iK')\ . \tag{4.17}$$

Here, $\zeta_e \equiv e^{\frac{\pi\beta}{K'}}$ and the contours $C^>$, $C^<$, $\tilde{C}^>$, $\tilde{C}^<$ are shown in Fig. 7. It is straightforward to check the canonical anticommutation relation (eq. (3.30). The paths in the real direction shown in Fig. 7 do not give rise to any contribution but are introduced to indicate the location of the cuts of $B(\beta)$ in the ζ_e plane. The above eqs. (4.16), (4.17) can be rewritten as

$$\begin{aligned}
\Psi_e(\ell) &= e^{\frac{3}{4}\pi i}\left(\int_{-2K}^{0}+\int_{-iK'}^{-iK'-2K}\right)\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{\ell+1/2}B_{conj}(\beta+K) \\
&+ e^{\frac{\pi}{4}i}\left(\int_{0}^{iK'}+\int_{-iK'-2K}^{-2K}\right)\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{\ell+1/2}B^{>}(\beta+K) \\
&+ e^{\frac{\pi}{4}i}\left(\int_{-iK'}^{0}+\int_{-2K}^{-2K+iK'}\right)\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{\ell+1/2}B^{<}(\beta+K)\ , \qquad (4.18) \\
\tilde{\Psi}_e(\ell) &= e^{\frac{3}{4}\pi i}\left(\int_{0}^{-2K}+\int_{-2K-iK'}^{-iK'}\right)\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{\ell+1/2}B_{conj}(\beta+3K-iK') \\
&+ e^{\frac{\pi}{4}i}\left(\int_{-iK'}^{0}+\int_{-2K}^{-2K+iK'}\right)\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{\ell+1/2}B^{>}(\beta+3K-iK') \\
&+ e^{\frac{\pi}{4}i}\left(\int_{0}^{iK'}+\int_{-2K-iK'}^{-2K}\right)\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{\ell+1/2}B^{<}(\beta+3K-iK')\ . \qquad (4.19)
\end{aligned}$$

(4.20)

This shows that the integrand on the real axis is the conjugate function defined in the positive p region. The Euclidean version of the Virasoro algebra $\{\{L_n\}\}$ is generated by

$$L_n = \sum_{\ell}\left(\ell+\frac{1}{2}n\right)\frac{\Gamma(\ell+n+1/2)}{\Gamma(\ell+1/2)} : \tilde{\Psi}_e(\ell)\,\Psi_e(\ell+n) : +h\delta_{n,0}\ . \tag{4.21}$$

The calculation of the central charge and the highest weight is the same as before, but the state forming the highest weight representation is distinct from the original case.

So far, we have been discussing the Virasoro algebra based on the first quantized operator $\tilde{\ell}_n$. A main reason is its relation to the critical Virasoro alwebra which we will discuss in the next section. However, it is possible to form a Virasoro algebra of a completetely different kind by using the operator $\ell_n = -\frac{1}{2}\zeta^{n+1}\frac{d}{d\zeta} - \frac{1}{2}\frac{d}{d\zeta}\zeta^{n+1}$. This algebra denoted by $\{\{\mathcal{L}_n^{diff}\}\}$ generates diffeomorphisms of the spectral (rapidity) parameter.[14] It is given by

$$\mathcal{L}_n^{diff} = :\int_{-K}^{3K}\frac{d\beta}{2\pi}B\left(\beta+2K-iK'\right)\ell_n B(\beta): \ , \tag{4.22}$$

$$= \sum_{\ell}\left(\ell+\frac{1}{2}n\right):\tilde{\Psi}\left(\ell\right)\Psi\left(\ell+n\right):+h\delta_{n,0} \ . \tag{4.23}$$

The Euclidean version $\{\{L_n^{diff}\}\}$ is constructed by replacing the CTM mode operators in eq. (4.22) by the Euclidean mode operators (eqs. (4.16), (4.17)). From any of the noncritical Virasoro algebras given above, we can construct its parity conjugate by replacing ζ by $-1/\zeta$. This pair of Virasoro algebras, however, do not commute due to the presence of the mass gap.

Note that the hermitean conjugation property $[\Psi(\ell)]^\dagger = \tilde{\Psi}(\ell)$, appropriate to fixed time quantization, leads to the unitarity property $\left[\mathcal{L}_n^{diff}\right]^\dagger = \mathcal{L}_{-n}^{diff}$ for the Virasoro operators defined in (4.22). On the other hand, the Euclidean algebra (4.21) should be understood as a generalization of the radially quantized conformal algebra. In the limit $T \to T_c$, the conjugation property appropriate to radial quantization leads to the unitarity of (4.21), $L_n^\dagger = L_{-n}$, as we will see in the next section.

V. Behavior at the Critical Point and CFT in the Analytic Momentum Plane

In this section, we will consider the form of the Euclidean eigenmode operators $\Psi_\epsilon(\ell)$, $\tilde{\Psi}_\epsilon(\ell)$ and the Virasoro operators L_n at the critical point (*i.e.* $k = 1$ case). This permits us to exhibit the connection of the lattice Virasoro algebra (4.21) with the conventional critical Virasoro algebra generated by massless Dirac fermions. As we showed in the previous section, the eigenmode operators of $\mathcal{L}_0$ (and of the CTM) are simply the Fourier components of $B(\beta)$ around the Brillouin zone period. Since this period is going to infinity at the critical point, the eigenvalues of the CTM, which are discrete for $T \neq T_c$, become continuous in the critical limit. However, the periodicity in the imaginary rapidity direction, corresponding to Euclidean rotations, still survives at the critical point. It is this period which gives rise to the discrete eigenvalue structure of the critical Virasoro algebra. In the standard treatment of the critical theory, this is implemented by defining fermion field operators which are analytic functions of complex Euclidean coordinate $z = x + iy$ and Fourier-analyzing them around the unit circle in the z-plane (radial quantization). We will show that an equivalent momentum space formulation of the critical Virasoro algebra can be obtained by defining analytically continued momentum-space operators and Fourier-transforming them around the imaginary direction of the spectral (rapidity) torus. This momentum space formulation allows the relation between the conformal algebra and the noncritical algebra to be exhibited.

Let us first examine the elliptic function expressions (eqs. (3.5)-(3.7)) again, but this time, at the critical point, $k \to 1$. Since K has gone to infinity, (and $K' \to \pi/2$), one has to carefully locate the lattice rapidity β_c . This can be most clearly seen in eq. (3.11) : the energy- momentum dispersion curve develops a cusp at $p = n\pi \pm \pi/2$, and $\Lambda_c = \mid \cos p \mid$. (See Fig. 8). This leads us to employ another Brillouin zone scheme different from the one adopted in Section III. The proper shift turns out to be $\alpha = \beta_c - 2K + iK'/2$ and $\beta_c + iK'/2$ with $-K < \beta_c < K$. These two distinct shifts cover the original full period $4K$ and we call these two regimes, regime I and regime II respectively. Similarly, regime III and regime IV are the regions $-K < \beta_c < K$ for $\alpha = \beta_c - 2K - iK'/2$ and $\beta_c - iK'/2$ respectively. Eqs. (3.5), (3.6), (3.7) are now

$$S_c^I(\beta_c) \;=\; i\tanh(\beta_c + i\pi/4) = \; e^{ip(\beta_c)} \;, \tag{5.1}$$

$$C_c^I(\beta_c) = -i/\cosh(\beta_c + i\pi/4) = \sqrt{-2\cos p(\beta_c)}\; e^{ip(\beta_c)/2} \ , \tag{5.2}$$

$$D_c^I(\beta_c) = 1/\cosh(\beta_c + i\pi/4) = i\sqrt{-2\cos p(\beta_c)}\; e^{ip(\beta_c)/2} \ , \tag{5.3}$$

in regime I and

$$S_c^{II}(\beta_c) = -i\tanh(\beta_c + i\pi/4) = e^{ip(\beta_c)} \ , \tag{5.4}$$

$$C_c^{II}(\beta_c) = i/\cosh(\beta_c + i\pi/4) = i\sqrt{2\cos p(\beta_c)}\; e^{ip(\beta_c)/2} \ , \tag{5.5}$$

$$D_c^{II}(\beta_c) = 1/\cosh(\beta_c + i\pi/4) = \sqrt{2\cos p(\beta_c)}\; e^{ip(\beta_c)/2} \ , \tag{5.6}$$

in regime II respectively. The expressions at regime III and regime IV are given by replacing β_c by $\beta_c - i\pi/2$ in regime I and in regime II respectively. The energy-eigenmode operator is now

$$B^I(\beta_c) = ie^{iaq_c/2}\sqrt{-\sin(q_c a)/a}\;\psi_1(q_c) \ , \ \text{for } p(\beta_c) = \pi/2 - q_c a \ , \tag{5.7}$$

$$= -e^{iaq_c/2}\sqrt{\sin(q_c a)/a}\;\tilde{\psi}_2(-q_c) \ , \ \text{for } p(\beta_c) = -\pi/2 - q_c a \ , \tag{5.8}$$

$$B^{II}(\beta_c) = -ie^{iaq_c/2}\sqrt{\sin(q_c a)/a}\;\psi_2(q_c) \ , \ \text{for } p(\beta_c) = \pi/2 - q_c a \ , \tag{5.9}$$

$$= e^{iaq_c/2}\sqrt{-\sin(q_c a)/a}\;\tilde{\psi}_1(-q_c) \ , \ \text{for } p(\beta_c) = -\pi/2 - q_c a \ , \tag{5.10}$$

in those regions indicated by the superscripts. We have given two different expressions in one regime. They are the relevant expressions at either $\beta_c \sim +\infty$ or $\beta_c \sim -\infty$ and describe independent fermionic degrees of freedom with either positive ($0 < q_c < \pi/2a$) or negative ($-\pi/2a < q_c < 0$) momenta in the continuum limit. Similarly, in regime III and in regime IV, we have

$$B^{III}(\beta_c) \equiv B^I(\beta_c - iK') \ ,$$

$$= e^{isq_c/2}\sqrt{\sin(q_c a)/a}\;\psi_1(q_c) \ , \text{for } p(\beta_c) = \pi/2 - q_c a \ , \tag{5.11}$$

$$= ie^{isq_c/2}\sqrt{-\sin(q_c a)/a}\;\tilde{\psi}_2(-q_c) \ , \text{for } p(\beta_c) = -\pi/2 - q_c a \ , \tag{5.12}$$

$$B^{IV}(\beta_c) \equiv B^{II}(\beta_c - iK') \ ,$$

$$= e^{isq_c/2}\sqrt{-\sin(q_c a)/a}\;\psi_2(q_c) \ , \text{for } p(\beta_c) = \pi/2 - q_c a \ , \tag{5.13}$$

$$= ie^{isq_c/2}\sqrt{\sin(q_c a)/a}\;\tilde{\psi}_1(-q_c) \ , \text{for } p(\beta_c) = -\pi/2 - q_c a \ . \tag{5.14}$$

These are schematically summarized in Fig. 9. The upper (lower) components of the Dirac fermion are located near the cusp at $p = \frac{\pi}{2}(-\frac{\pi}{2})$ or, equivalently, at $\beta_c \to +\infty(-\infty)$. Positive and negative values of momentum are on opposite sides of each cusp.

Next, we construct the Euclidean eigenmode operators by Fourier transforming the $B(\beta)$ operator extended to the imaginary rapidity direction in Section IV. We locate the full imaginary period at Re $\beta_c \to \pm\infty$ so that they reduce to the proper continuum limit expressions. Thus we have

$$\Psi_c(\ell) = e^{\frac{\pi}{4}i}\int_{C^{I>}} \frac{d\beta_c}{\sqrt{2\pi}}\zeta_c{}^{+\ell+1/2}B^{I>}(\beta_c) + e^{\frac{\pi}{4}i}\int_{C^{I<}} \frac{d\beta_c}{\sqrt{2\pi}}\zeta_c{}^{+\ell+1/2}B^{I<}(\beta_c) \quad (5.15)$$

$$\begin{aligned}\bar{\Psi}_c(\ell) &= e^{\frac{\pi}{4}i}\int_{C^{II>}} \frac{d\beta_c}{\sqrt{2\pi}}\zeta_c{}^{-\ell-1/2}B^{II>}(\beta_c - iK') \\ &\quad + e^{\frac{\pi}{4}i}\int_{C^{II<}} \frac{d\beta_c}{\sqrt{2\pi}}\zeta_c{}^{-\ell-1/2}B^{II<}(\beta_c - iK') \quad . \end{aligned} \quad (5.16)$$

Note that the power $1/2$ is dictated by the fact that the operator $B(\beta)$ is double-valued as an analytic function of $\zeta_c \equiv e^{2\beta_c}$. The contours are shown in Fig. 10. We have extended the original contours by adding two paths in the real direction (which cancel by periodicity) without changing the definition of the mode operators. The difference between $C^{I or II>}$ and $C^{I or II<}$ is dictated by the requirement that the summation over the lattice sites in $B^{I<or II>}(\beta_c)\left(\, B^{I>or II<}(\beta_c)\right)$ is convergent so that the integrand can be analytically continued into the region $0 \le \text{Im}\,\beta_c \le \frac{\pi}{2}$ $\left(-\frac{\pi}{2} \le \text{Im}\,\beta_c \le 0\right)$.

Let us now study the continuum limit of eqs. (5.7)-(5.14). We find

$$B^I(\beta_c) \to i(-q_c)^{1/2}\psi_1(q_c)\ ,\ -(q_c)^{1/2}\bar{\psi}_2(-q_c)\ , \quad (5.17)$$

$$B^{II}(\beta_c) \to -i(q_c)^{1/2}\psi_2(q_c)\ ,\ (-q_c)^{1/2}\bar{\psi}_1(-q_c)\ , \quad (5.18)$$

$$B^{III}(\beta_c) \to (q_c)^{1/2}\psi_1(q_c)\ ,\ i(-q_c)^{1/2}\bar{\psi}_2(-q_c)\ , \quad (5.19)$$

$$B^{IV}(\beta_c) \to (-q_c)^{1/2}\psi_2(q_c)\ ,\ i(q_c)^{1/2}\bar{\psi}_1(-q_c)\ , \quad (5.20)$$

as $a \to 0$, $\beta_c \to \pm\infty$ with $q_c \sim \pm\frac{2}{a}e^{\pm 2\beta_c}$. (The sign may be read off in each individual case from Fig. 9). Rescaling the CTM mode operators and the Virasoro operators by $\Psi_c(\ell) \equiv e^{-\frac{3\pi i}{4}}(-2/a)^{\ell+1/2}\sqrt{a/2}\Psi_c^{res}(\ell)$, $\bar{\Psi}_c(\ell) \equiv e^{+\frac{3\pi i}{4}}(-2/a)^{-\ell-1/2}\sqrt{2/a}\bar{\Psi}_c^{res}(\ell)$,

and $L_n \equiv (-2/a)^n L_n^c$, we obtain, in the continuum limit,

$$\Psi_c^{res}(\ell) = \int \frac{dq_c}{2\pi} \int dx_{C_L^> + C_L^<} q_c{}^{-\ell-1/2} e^{-iq_c x} \psi_1(x) \ , \tag{5.21}$$

$$\tilde{\Psi}_c^{res}(\ell) = \frac{1}{2}(-)^\ell \int \frac{dq_c}{2\pi} \int dx_{C_L^> + C_L^<} q_c{}^{\ell-1/2} e^{-iq_c x} \tilde{\psi}_1(x) \ , \tag{5.22}$$

$$L_n^c = \sum_\ell \left(\ell + \frac{1}{2}n\right) \frac{\Gamma(\ell+n+1/2)}{\Gamma(\ell+1/2)} \tilde{\Psi}_c^{res}(\ell)\, \Psi_c^{res}(\ell+n) \ . \tag{5.23}$$

The contours $C_L^>$ and $C_L^<$ in the q_c plane are shown in Fig. 11. One can open up these contours wrapping the cut so that they lie on the real axis. The fact that we have to choose the cuts in the analytic q_c plane dependent on the phase of x is crucial, as we discuss below.

So far, we have been considering the left mover ψ_1 and the resulting Virasoro algebra. The right moving Virasoro algebra can be constructed in a completely analogous way by starting with the expressions for CTM mode operators with contours seen in Fig. 12. From now on, we restrict our discussion to the left mover (the analytic part).

After the continuum limit is taken, the system is described by the Lagrangian:

$$\mathcal{L} = \frac{i}{2}\tilde{\psi}_1 \left(-i\,\frac{\overleftrightarrow{\partial}}{\partial t_E} + \frac{\overleftrightarrow{\partial}}{\partial x}\right) \psi_1 + \frac{i}{2}\tilde{\psi}_2 \left(-i\,\frac{\overleftrightarrow{\partial}}{\partial t_E} - \frac{\overleftrightarrow{\partial}}{\partial x}\right) \psi_2 \ . \tag{5.24}$$

In the remainder of this section, we show how the above expressions reduce to the conventional framework of the conformal field theory of massless Dirac fermions. Let us very briefly sketch the standard radial quantization based on the complex coordinates $z = x + it_E = \exp(i\sigma + \tau)$, $\bar{z} = x - it_E$. Here, τ is taken to be a time variable. The mode expansions of the analytic fields read $\psi_1(z) = \sum_n b_n z^{-n-1/2}$, $\tilde{\psi}_1(z) = \sum_n \tilde{b}_n z^{-n-1/2}$, with $\{b_n, \tilde{b}_m\} = \delta_{n+m,0}$. Following the discussion in Section IV, we consider the case $n \in \mathcal{Z}$, namely the double-valued (Ramond) field. We suppress the subscript 1. The stress-energy tensor leading to the Virasoro algebra with the central charge $c = 1$ is

$$T(z) = -\frac{1}{2}\tilde{\psi}(z)\,\frac{\overleftrightarrow{d}}{dz}\,\psi(z) = \sum_n L_n^{CFT} z^{-n-2} \ , \tag{5.25}$$

with

$$L_n^{CFT} = \sum_\ell (\ell + n/2)\, \tilde{b}_{-\ell} b_{\ell+n} \ . \tag{5.26}$$

The natural hermiticity condition is with respect to the states constructed on the unit circle :

$$\begin{aligned} \left(\psi(z)(dz)^{1/2}\right)^\dagger_{z=1/z} &= \pm i\tilde{\psi}(z)(dz)^{1/2} \\ i.e. \quad \tilde{b}_\ell &= \pm b^\dagger_{-\ell} \ . \end{aligned} \tag{5.27}$$

Motivated by eqs. (5.21)-(5.22), we introduce the following double-contour integrals

$$\Psi_c^{CFT}(\ell) \equiv \frac{i}{(2\pi)^2} \int_C dpdz(-ip)^{-\ell-1/2} e^{-ipz}\psi(z) \ , \tag{5.28}$$

$$\tilde{\Psi}_c^{CFT}(\ell) \equiv \pi(-1)^\ell \frac{i}{(2\pi)^2} \int_C dpdz(-ip)^{-\ell-1/2} e^{-ipz}\tilde{\psi}(z) \ . \tag{5.29}$$

Here, we denote by C a two dimensional surface of the four dimensional space defined by the direct product of the complex p plane and the complex z plane. The integrand contains square root branch cuts both in the p-plane and the z-plane. The cut forms a two-dimensional surface in the four-dimensional space whose orientation is determined by the requirement that the factor e^{-ipz} is always exponentially decaying asymptotically. For any fixed value of the one integration variable, the contour C in the other variable is defined to wrap around this surface in the counterclockwise direction. (See, Fig. 13.) In the special case where the z plane is restricted to the real line, the above expression reduces to eqs. (5.21),(5.22) up to an irrelevant phase. (The factor π in eq. (5.28) is due to the difference in the normalization of the anticommutators in fixed time and radial quantization schemes.)

To identify $\Psi_c^{CFT}(\ell)$, let us perform the p integration first. The p integration is nothing but the Hankel representation of the inverse Γ function. The subsequent z integration does not have a cut in the integrand. We find

$$\Psi_c^{CFT}(\ell) = \frac{b_\ell}{\Gamma(\ell + 1/2)} \ . \tag{5.30}$$

Thus, $\Psi_c^{CFT}(\ell)$ is equal, up to a normalization factor, to the mode operator in the radial quantization scheme. On the other hand, one can carry out the z integration

first and subsequently, the p integration. Eq. (5.27) then reads

$$a(p) \equiv \frac{i}{2\pi}\int_{C_z} dz e^{-ipz}\psi(z) \ , \tag{5.31}$$

$$\Psi_c^{CFT}(\ell) = \oint \frac{dp}{2\pi}(-ip)^{-\ell-1/2}a(p) \ . \tag{5.32}$$

Again, the contour C_z of the z integration wraps around the cut of the double valued field $\psi(z)$ which is chosen such that e^{-ipz} is exponentially decaying. Eq. (5.30) is similar to a one-side Laplace transform but the origin of the integrand is avoided so that it is applicable to fields introduced by Laurent expansion. Substituting these expressions into eq. (5.30), we see

$$\begin{aligned} a(p) &= \sum_n \frac{(ip)^{n-1/2}}{\Gamma(n+1/2)} b_n \\ &= \sum_n (ip)^{n-1/2}\Psi_c^{CFT}(n) \ . \end{aligned} \tag{5.33}$$

The operator $a(p)(ip)^{1/2}$ is Laurent expandable. The field $a(p)$ can be viewed as a weight $1/2$ conformal field in the analytic momentum plane and is double valued. The Ψ_c^{CFT} is the mode operator in the analytic momentum plane. Inverting eq. (5.32), we obtain

$$\psi(z) = \oint \frac{dp}{2\pi} a(p) \sum_n \Gamma(n+1/2)(ipz)^{-n-1/2} \ . \tag{5.34}$$

Here, the contour is around the origin. Writing $p = e^{\alpha}$ we see that the integration is over a period in the direction of imaginary rapidity. One can similarly proceed to the conjugate field $\tilde{\psi}$. We just give the corresponding formulae :

$$\tilde{\Psi}_c^{CFT}(\ell) = \pi(-1)^{\ell}\frac{i}{(2\pi)^2}\int_C dp dz (-ip)^{-\ell-1/2}e^{-ipz}\tilde{\psi}(z) \tag{5.35}$$

$$= \pi(-1)^{\ell}\frac{\tilde{b}_{\ell}}{\Gamma(\ell+1/2)} \tag{5.36}$$

$$= \pi(-1)^{\ell}\int \frac{dp}{2\pi}(-ip)^{-\ell-1/2}\tilde{a}(p) \ , \tag{5.37}$$

$$\tilde{a}(p) \equiv \frac{i}{2\pi}\int dz e^{-ipz}\tilde{\psi}(z) \tag{5.38}$$

$$= \frac{1}{\pi}\sum_n (-1)^n (ip)^{n-1/2}\tilde{\Psi}_c^{CFT}(n) \ , \tag{5.39}$$

$$\tilde{\psi}(z) \;=\; \oint \frac{dp}{2\pi}\tilde{a}(p)\sum_n \Gamma(n+1/2)(ipz)^{-n-1/2} \;. \tag{5.40}$$

The additional normalization factor in eq. (5.34) is necessary so that the canonical commutator

$$\left\{\tilde{\Psi}_c^{CFT}(\ell'),\Psi_c^{CFT}(\ell)\right\} = \delta_{\ell+\ell',0} \;. \tag{5.41}$$

follows from eqs. (5.28)-(5.29) and the reflection formula for the Γ function.

We are now ready to express the Virasoro operators L_n^{CFT}'s in the momentum representation. In terms of the momentum mode operators, they read

$$L_n^{CFT} = \sum_\ell (\ell+\frac{1}{2}n)\frac{\Gamma(\ell+n+1/2)}{\Gamma(\ell+1/2)}\tilde{\Psi}_c^{CFT}(-\ell)\Psi_c^{CFT}(\ell+n) \;, \tag{5.42}$$

This corresponds precisely with the expression we obtained for the XY spin chain at the critical point, i.e. eq. (5.23).

VI. Scaling Limit and Massive Free Fermion Algebra

In this section, we study the massive scaling limit of the XY spin chain. As we stated in the Introduction, our motivation for this study is that restriction to the scaling regime allows us to more easily study the connection between the noncritical Virasoro algebra and the infinitely many mutually commuting charges in more familiar field-theoretic terms. First let us describe the procedure for taking the continuum limit of the XY spin chain and recovering a massive Dirac fermion theory. We must take the elliptic modulus $k \to 1$ and simultaneously scale the lattice spacing a to zero in such a way that the mass of the fermion remains finite. Note that, as $k \to 1$,

$$\begin{aligned} \mathrm{sn}\,\beta &\to \tanh\beta \;, \\ \mathrm{cn}\,\beta, \mathrm{dn}\,\beta &\to \mathrm{sech}\,\beta \;. \end{aligned} \tag{6.1}$$

Using this and eqs. (3.5)-(3.7), we find the continuum limits for $S(\beta), C(\beta)$, and $D(\beta)$:

$$S(\beta) \to ie^{-iq} \;, \quad q \equiv \frac{k'^2}{4}\sinh 2\beta \;, \tag{6.2}$$

$$C(\beta) \to \frac{k'}{2}[e^{-\frac{\pi}{4}i+\beta} + e^{+\frac{\pi}{4}i-\beta}] \ , \tag{6.3}$$

$$D(\beta) \to \frac{k'}{2}[e^{+\frac{\pi}{4}i+\beta} + e^{-\frac{\pi}{4}i-\beta}] \ . \tag{6.4}$$

Here, $k' = \sqrt{1-k^2}$ is the complimentary modulus and the the range of β originally from $-K$ to $+K$ is now extended to the real line while the imaginary period $2K'$ becomes π. Note that $\frac{\pi\beta}{K'} \to 2\beta \equiv \alpha$ is the continuum rapidity of a one particle state.

The above scaling limit is described by the relativistic Lagrangian

$$\mathcal{L} = \frac{i}{2}\tilde{\psi}_1 \left(\frac{\overleftrightarrow{\partial}}{\partial t} + \frac{\overleftrightarrow{\partial}}{\partial x}\right) \psi_1 + \frac{i}{2}\tilde{\psi}_2 \left(\frac{\overleftrightarrow{\partial}}{\partial t} - \frac{\overleftrightarrow{\partial}}{\partial x}\right) \psi_2 - m\left(\tilde{\psi}_1\psi_2 + \tilde{\psi}_2\psi_1\right) \ . \tag{6.5}$$

In fact, it is straightforward to check from eqs. (2.22)-(2.25) that in the continuum limit, $a \to 0, k \to 1$, with fixed k'^2/a, the quantity $(H_0 + H_{mass})/a$ goes over to the Dirac Hamiltonian, with the fermion mass given by $k'^2/4a$. The diagonalization of the Hamiltonian naturally leads us to define a set of operators

$$\chi_{\frac{1}{2}} = \sqrt{\frac{m}{2}} \int_{-\infty}^{\infty} dx e^{-imx\sinh\alpha} \left(\pm e^{\pm\alpha/2}\psi_1(x) + e^{\mp\alpha/2}\psi_2(x)\right) \ , \tag{6.6}$$

$$\tilde{\chi}_{\frac{1}{2}} = \sqrt{\frac{m}{2}} \int_{-\infty}^{\infty} dx e^{+imx\sinh\alpha} \left(\pm e^{\pm\alpha/2}\tilde{\psi}_1(x) + e^{\mp\alpha/2}\tilde{\psi}_2(x)\right) \ . \tag{6.7}$$

Here, the signs are associated with the upper and the lower entries of the subscripts for χ and $\tilde{\chi}$. For later purpose, we also introduce

$$\chi_{conj\frac{1}{2}} = -i\sqrt{\frac{m}{2}} \int_{-\infty}^{\infty} dx \epsilon(x) e^{-imx\sinh\alpha} \left(\pm e^{\pm\alpha/2}\psi_1(x) + e^{\mp\alpha/2}\psi_2(x)\right) \ , \tag{6.8}$$

$$\tilde{\chi}_{conj\frac{1}{2}} = +i\sqrt{\frac{m}{2}} \int_{-\infty}^{\infty} dx \epsilon(x) e^{+imx\sinh\alpha} \left(\pm e^{\pm\alpha/2}\tilde{\psi}_1(x) + e^{\mp\alpha/2}\tilde{\psi}_2(x)\right) \ . \tag{6.9}$$

From eqs. (6.2)-(6.4), we find that the eigenmode operators $B(\beta)$ reduce to the above ones in the scaling limit :

$$B(\beta) \to -i\chi_2(\alpha) \ , \tag{6.10}$$

$$B(-\beta + 2K) \to \tilde{\chi}_1(\alpha) \ , \tag{6.11}$$

$$B(-\beta \mp iK') \to \mp\chi_1(\alpha) \ , \tag{6.12}$$

$$B(\beta + 2K \mp iK') \to \pm i\tilde{\chi}_2(\alpha) \ . \tag{6.13}$$

These are schematically represented in Fig. 14. The diagonalized Hamiltonian reads

$$H = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \cosh\alpha \, (\tilde{\chi}_1(\alpha), \tilde{\chi}_2(\alpha)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi_1(\alpha) \\ \chi_2(\alpha) \end{pmatrix} \ . \tag{6.14}$$

The operators $\tilde{\chi}_{1,(2)}$, $\chi_{1,(2)}$ are respectively creation and annihilation operators of positive (negative) energy states with respect to the empty Dirac sea. They are a simple (albeit somewhat trivial) example of the Bethe ansatz creation and annihilation operators. Furthermore, these positive and negative energy states are succinctly described by extending the domain of α to the line $\alpha = \text{real} + i\pi$. Define

$$\chi(\alpha) \equiv \chi_1(\alpha) \tag{6.15}$$

over this extended contour denoted as C. (See Fig. 15.) Then, $\chi(\alpha) = -i\chi_2(i\pi - \alpha)$ for α the real $+ i\pi$ line. Similarly, define

$$\tilde{\chi}(\alpha) \equiv \tilde{\chi}_1(\alpha) \tag{6.16}$$

over the real line, and

$$\tilde{\chi}(\alpha) \equiv \tilde{\chi}_1(\alpha - 2\pi i) = i\tilde{\chi}_2(i\pi - \alpha) \tag{6.17}$$

over $\alpha = \text{real} \ + i\pi$ line. The canonical anticommutators are now given as

$$\{\chi(\alpha), \tilde{\chi}(\alpha')\} = 2\pi\delta(\alpha - \alpha') \ , \tag{6.18}$$

over the C contour, with the δ-function taken to vanish when $\alpha - \alpha'$ has a nonzero imaginary part. The Hamiltonian is given by

$$H = \int_C \frac{d\alpha}{2\pi} \cosh\alpha \ \epsilon \ \tilde{\chi}(\alpha)\chi(\alpha) \ . \tag{6.19}$$

Here ϵ is 1 for α real and -1 for α real $+ i\pi$.

In the Bethe ansatz solution to a relativistic fermion model, the construction of the physical Hilbert space associated with the Hamiltonian (6.19) is accomplished by first diagonalizing the Hamiltonian in an empty Dirac sea, and then filling up the negative energy modes. In a similar way, we construct a highest weight representation of the Virasoro algebra by diagonalizing the lattice boost operator L_0 and taking its

negative eigenmodes to be filled. Of course, in this approach we must be careful to compute the central charge anomaly in the filled vacuum.[4] Some expressions we will use below in the fixed time quantization are, therefore, strictly speaking formal ones mainly to reveal the locality of the algebra.

Before considering the scaling limit of the lattice Virasoro algebra, it is useful to consider the form of the infinite sequence of conserved densities. This sequence results from the fact that not only the total momentum, but the entire momentum distribution, is conserved in time. Thus, we define a set of operators

$$Q_n \equiv \int_C \frac{d\alpha}{2\pi} e^{n\alpha} \tilde{\chi}(\alpha)\chi(\alpha) \quad n \in \mathcal{Z} \ . \tag{6.20}$$

One may verify the action of the Boost operators on Q_n :

$$[L_0, Q_n] = nQ_n \ . \tag{6.21}$$

This means that, for an arbitrary eigenstate $\mid h \rangle$ labeled by L_0 eigenvalue h,

$$Q_n \mid h \rangle = \mid n+h \rangle \quad \text{or} \quad Q_n \mid h \rangle = 0 \ . \tag{6.22}$$

One can perform the α or $\zeta \equiv e^{\alpha}$ integration, substituting eqs. (6.6), (6.7) into eq. (6.20). The following formula due to the symmetry under $\zeta \to (-1/\zeta)$ is fundamental to the locality of conserved densities :

$$\int \frac{d\zeta}{2\pi\zeta} \zeta^n e^{-i\frac{m}{2}x(\zeta - 1/\zeta)} = P_n\left(\frac{2i}{m}\frac{d}{dx}\right)\frac{2}{m}\delta(x) \quad \text{for } n \in \mathcal{Z} \ . \tag{6.23}$$

Here the integration is over the real axis and we introduced a set of polynomials $P_n(x)$ defined by

$$\begin{aligned} P_n\left(x \equiv \zeta - \frac{1}{\zeta}\right) &\equiv \frac{\zeta^n - (-1/\zeta)^n}{\zeta + (1/\zeta)} \\ &= xP_{n-1}(x) + P_{n-2}(x) \ . \end{aligned} \tag{6.24}$$

[4]Note that the eigenmode operator $B(\beta)$ along real β is automatically a creation operator with respect to the filled vacuum. On the other hand, in the Bethe ansatz or quantum inverse formalism, one starts out from the pseudovacuum and builds the physical vacuum as a state with filled negative energy modes.

$$+ \quad e^{+\pi i/4}\int_{-3iK'/2}^{iK'/2}\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{-\ell-\frac{1}{2}}B^<(-\beta-K+iK') \ , \qquad (6.29)$$

$$\tilde{\Psi}_e(\ell) \ = \ e^{+\pi i/4}\int_{-3iK'/2}^{iK'/2}\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{\ell+\frac{1}{2}}B^>(-\beta+K)$$

$$+ \quad e^{+\pi i/4}\int_{-iK'/2}^{3iK'/2}\frac{d\beta}{\sqrt{4\pi K'}}\zeta_e{}^{\ell+\frac{1}{2}}B^<(-\beta+K) \ . \qquad (6.30)$$

Here, the contours in the above expressions are shown in Fig.16. Since the integrands are all periodic, one can add to the original contours two paths of length $2K$ extending to the real direction without changing anything. (See Fig. 16.) Substituting eqs. (6.29), (6.30) into eq. (4.21) and performing the ℓ summation, we obtain

$$L_n \ = \ \oint\frac{d\zeta_e}{2\pi\zeta_e}B(-\beta+K)\ \tilde{\ell}_n\ B(-\beta-K+iK') \ . \qquad (6.31)$$

$$\tilde{\ell}_n \ = \ \zeta_e\left\{\left(\frac{d}{d\zeta_e}\right)^{n+1}-\frac{1}{2}(n+1)\left(\frac{d}{d\zeta_e}\right)^n\zeta_e{}^{-1}\right\} \quad \text{for } n \ \geq -1 \ . \qquad (6.32)$$

and the derivative is replaced by the integration for $n \leq -2$. One can check that, as K becomes large, the integrands in eqs. (6.29)- (6.31) have exponentially decaying contributions at the shaded areas in Fig. 16. We can then effectively turn the full period in the imaginary direction into a half period so that the two paths running in the real directions are separated by $K' \to \frac{\pi}{2}$. Rescaling the Virasoro operators L_n by $L_n \equiv e^{\pi nK/K'}L_n^{sc}$, we obtain the scaling limit $\zeta_e \to \zeta \equiv e^{\alpha}$:

$$L_n^{sc} = \int_C\frac{d\alpha'}{2\pi}\int_C\frac{d\alpha}{2\pi}\tilde{\chi}_{conj}(\alpha')\tilde{\ell}_n(\alpha',\alpha)\chi_{conj}(\alpha) \ . \qquad (6.33)$$

Here $\tilde{\ell}_n(\alpha',\alpha)$ is , along the C contour this time, given by

$$\zeta'\left\{\left(\frac{d}{d\zeta'}\right)^{n+1}-\frac{1}{2}(n+1)\left(\frac{d}{d\zeta'}\right)^n\zeta'^{-1}\right\}\delta(\alpha'-\alpha) \ , \ \text{for} \ \ n \geq -1 \ . \qquad (6.34)$$

For $n \leq -2$, the integrations are explicitly performed to give

$$\sum_{\ell=1}^{|m+1|}\left(\prod_{\substack{k=1\\k\neq\ell}}^{|m+1|}\frac{1}{k-\ell}\right)\zeta'^{|m+1|+\frac{1}{2}+\ell}\zeta^{\ell+\frac{1}{2}}+\frac{1}{2}\mid m+1\mid\sum_{\ell=0}^{|m+1|}\left(\prod_{\substack{k=0\\k\neq\ell}}^{|m+1|}\frac{1}{k-\ell}\right)\zeta'^{|m+1|+\frac{1}{2}+\ell}\zeta^{(\ell+\frac{1}{2})} \ , \qquad (6.35)$$

for $\alpha' \geq \alpha$ and 0 for $\alpha' \leq \alpha$. We extend this expression to the other regions by demanding the ordering $Re\zeta' \geq Re\zeta$ Eq. (6.35) is also succinctly represented as

$$\tilde{\ell}_n(\alpha',\ \alpha) = \int_{-\infty}^{+\infty} ds(is + 1/2 - (n+1)/2)\frac{\Gamma(is+1/2)}{\Gamma(is-n+1/2)}\zeta'^{\,is-n}\zeta^{-is}\ , \qquad (6.36)$$

for $n \in \mathcal{Z}$ and α', α real. This formula indicates that the α ,α' contour is now deformed to the real axis.

We first discuss the case $n \geq -1$. It is easy to see, from eqs. (6.33), (6.34) that L_n^{sc}'s are spatial integrals over local operators. The lowest few cases are

$$L_{-1}^{sc} = \int dx j_0^{(+1)}(x)\ , \qquad (6.37)$$

$$L_0^{sc} = \int dx \left(\frac{-imx}{2}\right)(j_0^{(+1)}(x) + j_0^{(-1)}(x)) \equiv \int dx(-ix)\mathcal{H}\ . \qquad (6.38)$$

The operators L_0^{sc} and L_{-1}^{sc} are the boost and the light-cone Hamiltonian respectively and are elements of the Poincare algebra. The L_1^{sc} is the first unfamiliar operator arising from the Virasoro algebra:

$$\begin{aligned} L_{+1}^{sc} &= \int dx \left(\frac{-imx}{2}\right)^2 \bullet \left(j_0^{(+1)}(x) + 2j_0^{(-1)}(x) + j_0^{(-3)}(x)\right) - \left(\frac{-imx}{2}\right) \bullet \left(j_0^{(0)}(x) + j_0^{(-2)}(x)\right) \\ &+ \left(\tilde{\psi}_1(x), \tilde{\psi}_2(x)\right) \begin{pmatrix} 2P_{+1}\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right)\ , & 2P_{+2}\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right) \\ 0\ , & -2P_{+3}\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right) \end{pmatrix} \left(\frac{-imx}{2}\right) \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}\ , \\ &+ \left(\tilde{\psi}_1(x), \tilde{\psi}_2(x)\right) \begin{pmatrix} 0\ , & P_{+1}\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right) \\ 0\ , & -P_{+2}\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right) \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}\ . \end{aligned} \qquad (6.39)$$

Here, the $\bullet$ indicates that the power of x is inserted right after the derivative operators such as $P_n\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right)$. In general, the term with the highest power of x in L_n^{sc}, $(n \geq -1)$ is a linear combination of conserved densities $j_0^{(n)}(x)$ alone seen in eq. (6.26) :

$$\left(\frac{-imx}{2}\right)^{n+1} \sum_{k=0}^{n+1} c_{k,n+1}\ j_0^{(1-2k)}(x)\ . \qquad (6.40)$$

The terms with lower powers of x do not have such a direct relation to the higher conserved densities. Eq. (6.40) provides a direct link between the infinite conservation laws and the Virasoro operators.

Let us denote the local expression for the Virasoro charges by

$$L_n^{lc} = \int dx \mathcal{J}_0^{(n)}(x, t=0) \ , \quad n \geq -1 \ . \tag{6.41}$$

For massive free fermions, we have demonstrated the locality of the Virasoro operators as well as their relationship with the higher conservation laws. Each Virasoro operator requires us to introduce a new member of the sequence of conserved densities. This is in contrast with the standard conformal Virasoro algebra where the Virasoro operators are generated by a single conserved traceless light-cone stress-energy density. The demonstration made here provides a reason why the lattice Virasoro algebra is a dynamically realized symmetry despite the non-traceless stress-energy tensor of the system.

One might ask in what sense the densities for the Virasoro operators are conserved. Since the Virasoro algebra is a spectrum generating algebra, one should not expect $\mathcal{J}_0(x, t=0)$ to produce a conserved quantity in the usual sense, as for example the conserved operators Q_n. This is because the Virasoro charges contain an explicit space-time dependence. Nonetheless, they obey a local conservation law,

$$\frac{d}{dt}\mathcal{J}_0^{(n)}(x,t) + \frac{d}{dx}\mathcal{J}_1^{(n)}(x,t) = 0 \ , \tag{6.42}$$

Note that, for $t \neq 0$, the densities contain explicit powers of t as well as x. The simplest well-known case is

$$\mathcal{J}_0^{(n=0)}(x,t) = (-ix)\mathcal{H} + (it)\mathcal{P} \ . \tag{6.43}$$

Let us write generically

$$\mathcal{J}_\mu^{(n)}(x,t) = \sum_{\ell=0}^{\ell_0}\sum_{\ell'=0}^{n+1}\left(\frac{-imt}{2}\right)^{\ell}\left(\frac{-imx}{2}\right)^{\ell'}\mathcal{T}_\mu^{(n)\ \ell,\ell'}(x) \ , \mu = 0,1 \tag{6.44}$$

and demonstrate, by a recursive procedure, that the series in fact terminates due to the fact that $\mathcal{T}_0^{(n)\ \ell=0,\ell'=n+1}$ is a conserved density. (Lorentz invariance alone tells us that we must have $\ell_0 = n+1$.) From eqs. (6.42) and (6.44), we obtain a recursive equation

$$(\ell+1)\mathcal{T}_0^{(n)\ \ell+1,\ell'} + \frac{2i}{m}\frac{d}{dt}\mathcal{T}_0^{(n)\ \ell,\ell'} + \frac{2i}{m}\frac{d}{dx}\mathcal{T}_1^{(n)\ \ell,\ell'} + (\ell'+1)\mathcal{T}_1^{(n)\ \ell,\ell'+1} = 0 \ . \tag{6.45}$$

This recursion formula is represented by the arrow in Fig. 17. The last term is absent for the $\ell' = n+1$ line, and the first term is absent for the $\ell = \ell_0$ line. The quantities $\mathcal{T}_0^{(n)\ \ell=0,\ell'}$ $(\ell' = 0$ to $n+1)$ are inputs. One begins with solving $\ell' = n+1$ line and leads to the conclusion that $\mathcal{T}_0^{(n)\ \ell,\ell'}$ $(\ell = 1 \text{to} \ell_0)$ is vanishing up to total derivatives. The lines $\ell' = m \quad (0 \leq m \leq n)$ are then solved one by one from the right and $\mathcal{T}_0^{(n)\ \ell=n+1,\ell'}(\ell' = 1 \sim n)$ are determined up to total derivatives. Line $\ell = \ell_0$ is simply used to determine $\mathcal{T}_1^{(n)\ \ell=n+1,\ell'}$ $(\ell' = 1 \sim n+1)$. Finally, we obtain a nontrivial constraint

$$\frac{2i}{m}\frac{d}{dt}\mathcal{T}_0^{(n)\ \ell=\ell_0,\ell'=n+1}(x) + \frac{2i}{m}\frac{d}{dx}\mathcal{T}_1^{(n)\ \ell=\ell_0,\ell'=n+1}(x) = 0 \ . \tag{6.46}$$

This can be easily implemented since $\mathcal{T}_0^{(n)\ \ell=\ell_0,\ell'=n+1}(x)$ is a total derivative of an arbitrary function. If $\mathcal{T}_0^{(n)\ \ell=0,\ell'=n+1}(x)$ were not a conserved density, however, it would not be possible to do so, and the entire recursive procedure would not terminate. In this way, we see that the conservation law (eq. (6.42)) is guaranteed to hold. (Actually, eq. (6.44) is more general than is necessary. By Lorentz covariance, nonvanishing elements up to total derivatives are in $\ell + \ell' \leq n+1$. The operators $\mathcal{T}_0^{(n)\ \ell=n+1-\ell',\ell'}(x)$ are going to be conserved densities.)

The above discussion is sufficient to demonstrate the local densities of the Virasoro operators L_n^{sc} $(n \geq -1)$ as conserved quantities. Let us now discuss the case $n \leq -2$ briefly. A detailed study using eq. (6.23) shows that the operators L_n^{sc} with $n \leq -2$ have nonlocal expressions in the scaling limit. The basic reason is that the inversion of the momentum rapidity relation $q = m(\zeta - \frac{1}{\zeta})/2$ has a branch so that the ordering with respect to ζ does not translate into the ordering with respect the momenta which will guarantee the locality. The nonlocality is order m and becomes irrelevant in the massless limit. (See, Fig. 18.)

In the previous section, we argued that there is another Virasoro algebra parity-conjugate to the original one and that they correspond to the left and right Virasoro algebras at the critical point. The parity-conjugate Virasoro operators $L_n^{(P)sc}$ are obtained by replacing ζ by $-1/\zeta$ in eqs. (6.33)-(6.36) and eqs. (6.37), (6.38) are replaced by

$$L_{-1}^{(P)sc} = \int dx j_0^{(-1)}(x) \ , \tag{6.47}$$

$$L_0^{(P)sc} = \int dx \left(\frac{imx}{2}\right)(j_0^{(+1)}(x) + j_0^{(-1)}(x)) = \int dx(+ix)\mathcal{H} \ . \tag{6.48}$$

So far we have discussed the continuum scaling limit of the lattice Virasoro algebra. This continuum algebra can naturally be viewed as an integrable extension of the conventional Virasoro algebra away from the critical point. We have shown that there is such an integrable extension and it originates from the remarkable integer eigenvalue structure of the CTM.

As we discussed in Section IV, there is another Virasoro algebra defined on a lattice which represents the diffeomorphisms of the spectral parameter. Let us study the scaling limit of this algebra. One starts out from the expression L_n^{diff} with mode operators given by eqs. (4.16),(4.17). Going through a similar procedure leading to eqs. (6.33), (6.34), we obtain the scaling limit of the Virasoro operators $L_n^{diff,sc}$:

$$L_n^{diff,sc} = \int_C \frac{d\alpha}{2\pi} \tilde{\chi}_{conj}(\alpha) \ell_n \chi_{conj}(\alpha) \ . \tag{6.49}$$

$$\ell_n = -\zeta^{n+1}\frac{d}{d\zeta} - \frac{1}{2}(n+1)\zeta^n \ . \tag{6.50}$$

This expression is local for all $n \in \mathcal{Z}$:

$$L_n^{diff,sc} = \int dx \left(\frac{+imx}{2}\right) \bullet \left(j_0^{(n+1)}(x) + j_0^{(n+1)}(x)\right) - \Delta_n(x) \tag{6.51}$$

$$\Delta_n(x) \equiv \frac{(n+1)}{2} j_0^{(n)}(x) + \frac{1}{2}\left(\tilde{\psi}_1, \tilde{\psi}_2\right) \begin{pmatrix} P_{n+1}\left(\frac{-i}{m}\overleftrightarrow{\frac{d}{dx}}\right) , & -P_n\left(\frac{-i}{m}\overleftrightarrow{\frac{d}{dx}}\right) \\ P_n\left(\frac{-i}{m}\overleftrightarrow{\frac{d}{dx}}\right) , & -P_{n-1}\left(\frac{-i}{m}\overleftrightarrow{\frac{d}{dx}}\right) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{6.52}$$

This time, only terms with first moment and zeroth moments are generated. The operator taking the first moment is again a linear combination of the conserved currents, guaranteeing that these operators are symmetries of the system. The operators $L_n^{diff,sc}$ can be viewed as generators of the higher spectral flows in the angular quantization scheme. We conclude this section with a discussion of this point. (From now on, we omit the superscript *sc*.)

In conformal field theory, the physical significance of the Virasoro generators L_n^{CFT} is that they describe analytic distortions of complex Euclidean space-time, $z = x + iy$. This is seen directly from the form of the local conformal charge $z^{n+1}T(z)$, noting that the light-cone stress tensor $T(z)$ generates local motions of the coordinate z, e.g.

$$[T(z), \psi(z')] = \delta(z - z')\frac{\partial}{\partial z'}\psi(z') \tag{6.53}$$

Similar considerations applied to the non-critical Virasoro operators provide some insight into their physical significance. In particular, the connection discussed in this section between the Virasoro charges and higher conserved charges provides a geometrical relation between the Virasoro generators and higher spectral flows. To exhibit this relation, we again consider the massive scaling limit of the $XY/(\text{Ising})^2$ model. Let us begin by discussing the geometrical transformation generated by L_0, which is the boost operator for massive fermions at fixed time $t = 0$,

$$L_0 = i \int dx \; x\mathcal{H}(x) \quad , \tag{6.54}$$

where $\mathcal{H}$ is the Hamiltonian density,

$$\mathcal{H} = \frac{-i}{2}\left(\tilde{\psi}_1 \overleftrightarrow{\frac{\partial}{\partial x}} \psi_1 - \tilde{\psi}_2 \overleftrightarrow{\frac{\partial}{\partial x}} \psi_2\right) + m(\tilde{\psi}_1\psi_2 + \tilde{\psi}_2\psi_1) \quad . \tag{6.55}$$

This, of course, yields the familiar Lorentz transformation properties of the Dirac field,

$$\begin{aligned} [L_0, \psi_1(x)] &= -ix\left(-i\frac{\partial}{\partial x}\psi_1 + m\psi_2\right) - \frac{1}{2}\psi_1 \\ &= x\frac{\partial}{\partial t}\psi_1 - \frac{1}{2}\psi_1 \qquad (6.56) \\ [L_0, \psi_2(x)] &= -ix\left(i\frac{\partial}{\partial x}\psi_2 + m\psi_1\right) + \frac{1}{2}\psi_2 \\ &= x\frac{\partial}{\partial t}\psi_2 + \frac{1}{2}\psi_2 \qquad (6.57) \end{aligned}$$

where the last expressions follow from the equations of motion. The first term on the right hand side of each equation (proportional to x) arises from the $t = 0$ coordinate shift $t \to t + \epsilon x$. The second term represents the induced transformation on the internal (spin) degree of freedom of the Dirac field. We may think of the space-time transformation generated by L_0 as an "angular time evolution," with each point on the x-axis at $t = 0$ evolving in the time direction, the evolution being proportional to the distance from the origin.

Now let us apply a similar reasoning to interpret the space-time significance of the higher Virasoro operators. We will consider the algebra $\{\{L_n^{diff}\}\}$ which appears to

have the simplest interpretation in terms of higher spectral flows. Using the expression (6.49), we can write the transformation induced on the fields by L_n^{diff} at $t = 0$ as

$$[L_n^{diff}, \psi_i] = ix\ [q_n, \psi_i] + \mathcal{O}(1) \quad , \tag{6.58}$$

where

$$q_n \equiv \frac{m}{2}(Q_{n+1} + Q_{n-1}) \quad , \tag{6.59}$$

and the $\mathcal{O}(1)$ term in (6.58) represents terms which do not contain an explicit factor of x. The operators Q_n are the higher conserved densities defined before in this section. We may gain some insight into the significance of (6.58 by recalling the role of higher conserved operators in classical soliton theory. [24] There it is often advantageous to regard each of these operators as a Hamiltonian, and to consider not only the time evolution generated by the original equation of motion, but rather an infinite set of possible time evolutions arising from the same initial data. For example, this "many time variables" approach was exploited by the Date, et al. [25] to reveal the profound role of vertex operators and Kac-Moody algebras in the theory of classical KdV solitons. For our case, let us define the infinite set of time variables t_n whose evolution is generated by the n^{th} conserved charge q_n,

$$i[q_n, \psi_i] = \frac{\partial}{\partial t_n}\psi_i \quad . \tag{6.60}$$

Then the transformation of the fields under L_n^{diff} may be written

$$[L_n^{diff}, \psi_i] = x\frac{\partial}{\partial t_n}\psi_i + \mathcal{O}(1) \quad . \tag{6.61}$$

Thus, just as L_0 could be interpreted as the angular time evolution operator associated with the original time variable t_0, the Virasoro operators L_n for $n > 0$ are angular time evolution operators associated with the heirarchy of time variables t_n. Instead of a single space-time plane as in conformal field theory, we are led to consider an infinite sequence of space time planes, all of which intersect on the x-axis. The relation between the higher Virasoro operators L_n^{diff} and the higher conserved charges q_n is completely analogous to that between L_0 and the Dirac Hamiltonian. The $\mathcal{O}(1)$ terms in (6.58) arise in a manner similar to the spin terms in (6.56)-(6.57), but we do not have any direct physical interpretation of them.

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APPENDIX A

In this Appendix we derive the canonical transformation that diagonalizes the operator

$$L_0 = i\sum_{j=1}^{\infty} j\left\{c_j^y c_{j+1}^x - k c_j^x c_{j+1}^y\right\} \qquad (A.1)$$

where c_j^x and c_j^y are real lattice fermions satisfying $\left\{c_j^a, c_\ell^b\right\} = \delta_{ab}\delta_{j\ell}$. We want to construct a linear transformation

$$\Psi(\ell) = \sum_{j=1}^{\infty}\left\{A_{\ell j}c_j^x + B_{\ell j}c_j^y\right\} \qquad (A.2)$$

where we require $\Psi(\ell)$ to be an eigenmode operator of L_0

$$[\Psi(\ell), L_0] = \lambda_\ell \Psi(\ell) \qquad (A.3)$$

where λ_ℓ is the eigenvalue associated with mode ℓ. Inserting (A.3) into (A.2), we obtain two sets of equations

$$jA_{\ell,j+1} + k(j-1)A_{\ell,j-1} = i\lambda_\ell B_{\ell j} \qquad (A.4a)$$

$$(j-1)B_{\ell,j-1} + kjB_{\ell,j+1} = -i\lambda_\ell A_{\ell j} \qquad (A.4b)$$

Introducing the Fourier transformed coefficients,

$$A_\ell(p) = \sum_{j=1}^{\infty} e^{-ipj}A_{\ell j} \qquad (A.5a)$$

$$B_\ell(p) = \sum_{j=1}^{\infty} e^{-ipj}B_{\ell j} \qquad (A.5b)$$

we find that $A_\ell(p)$ and $B_\ell(p)$ must obey the following equations

$$\frac{\partial}{\partial p}\left(e^{ip}A_\ell(p)\right) + ke^{-ip}\frac{\partial}{\partial p}A_\ell(p) = \lambda_\ell B_\ell(p) \qquad (A6.a)$$

$$e^{-ip}\frac{\partial}{\partial p}B_\ell(p) + k\frac{\partial}{\partial p}\left(e^{ip}B_\ell(p)\right) = -\lambda_\ell A_\ell(p) \qquad (A6.b)$$

Now we want to find a change of variables which will simplify these equations. This is done by introducing a parametrization of momentum space $p \rightarrow p(\alpha)$ where α will be the spectral or lattice rapidity parameter. We write

$$e^{ip} = g(\alpha) \tag{A.7}$$

The specific form of the function $g(\alpha)$ will be chosen in the course of the following derivation.

Equations (A.6) become

$$\frac{(g^2+k)}{g'}\frac{\partial A_\ell}{\partial \alpha} + gA_\ell = -i\lambda_\ell B_\ell \tag{A.9a}$$

$$\frac{(1+kg^2)}{g'}\frac{\partial B_\ell}{\partial \alpha} + kg\ B_\ell = i\lambda_\ell A_\ell \tag{A.9b}$$

These equations can be further simplified by taking out integration factors,

$$A_\ell(\alpha) = f_1(\alpha)\, P_\ell(\alpha) \tag{A.10a}$$

$$B_\ell(\alpha) = f_2(\alpha)\, Q_\ell(\alpha) \tag{A.10b}$$

and choosing $f_1(\alpha)$ and $f_2(\alpha)$ to satisfy the first order differential equations

$$f_1'/f_1 = -\frac{gg'}{g^2+k} \tag{A.11a}$$

$$f_2'/f_2 = -\frac{gg'}{g^2+k^{-1}} \tag{A.11b}$$

Thus, we choose

$$f_1 = C_1\left[g^2+k\right]^{-1/2} \tag{A.12a}$$

$$f_2 = C_2\left[g^2+k^{-1}\right]^{-1/2} \tag{A.12b}$$

where C_1 and C_2 are constants. Choosing $C_1 = 1$ and $C_2 = \sqrt{k}$, Eqs. (A.9) simplify to

$$h(\alpha)P_\ell'(\alpha) = -i\lambda_\ell(\alpha)Q_\ell(\alpha) \tag{A.13a}$$

$$h(\alpha)Q_\ell'(\alpha) = i\lambda_\ell(k)P_\ell(k) \tag{A.13b}$$

where

$$h(\alpha) = \sqrt{k}\left[\left(g^2+k\right)\left(g^2+k^{-1}\right)\right]^{1/2}/g' \qquad (A.14)$$

Up until now, we have not chosen the function $g(\alpha)$ which defines the rapidity parametrization (A.7). The Onsager-Baxter elliptic parametrization is obtained by requiring that $L(\alpha)$ in (A.13) and (A.14) be a constant. Note that if we choose (c.f. Ref. [21]

$$e^{ip} \equiv g(\alpha) = -i\sqrt{k}\mathrm{sn}\alpha \qquad (A.15)$$

Then

$$\left(g^2+k\right)^{1/2} = \sqrt{k}\mathrm{cn}\alpha \qquad (A.16)$$

$$\left(g^2+k^{-1}\right)^{1/2} = \frac{1}{\sqrt{k}}\mathrm{dn}\alpha \qquad (A.17)$$

and

$$g'(\alpha) = -i\sqrt{k}\mathrm{cn}\alpha\mathrm{dn}\alpha \qquad (A.18)$$

Thus

$$h(\alpha) = i \qquad (A.19)$$

and

$$P_\ell'(\alpha) = -\lambda_\ell Q_\ell(\alpha) \qquad (A.1)$$

$$Q_\ell'(\alpha) = \lambda_\ell P_\ell(\alpha) \qquad (A.2)$$

Thus we obtain the eigenvalues

$$\lambda_\ell = \pi\ell/2K \qquad (A.3)$$

where ℓ is constrained by periodicity to be an integer. The solution to (A.1) leads to the result (3.27) of the text, where the integration contour α = real $+iK'/2$ is dictated by the requirement that the momentum $p(\alpha)$ be real.

APPENDIX B

In this appendix, we collect some explicit formulas for local densities

$$j_0^{(n)}(x) = \left(\tilde{\psi}_1, \tilde{\psi}_2\right) \begin{pmatrix} P_{n+1}\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right) , & P_n\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right) \\ P_n\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right) , & P_{n-1}\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \quad \text{(B.1)}$$

$$j_1^{(n)}(x) = \tilde{\psi}_1 P_{n+1}\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right)\psi_1 - \tilde{\psi}_2 P_{n-1}\left(\frac{-i}{m}\frac{\overleftrightarrow{d}}{dx}\right)\psi_2 . \quad \text{(B.2)}$$

for the sequence of conserved charges

$$Q_n \equiv \int_C \frac{d\alpha}{2\pi} e^{n\alpha}\tilde{\chi}(\alpha)\chi(\alpha) \quad n \in \mathcal{Z} , \quad \text{(B.3)}$$

$$\equiv \int dx j_0^{(n)}(x) . \quad \text{(B.4)}$$

inroduced in the text. Here, a set of polynomials $P_n(x)$ is defined in eq. (6.24). The first several terms are

$$P_0(x) = 0,\ P_1(x) = 1,\ P_2(x) = x,\ P_3(x) = 1 + x^2 \text{ and } P_4(x) = 2x + x^3 . \quad \text{(B.5)}$$

Also, it satisfies

$$P_{-n}(x) = (-)^{n+1} P_n(x) . \quad \text{(B.6)}$$

As one expects, the lowest few currents correspond to familiar conserved quantities;

$$\begin{aligned} j_0^{(n=0)} &= \tilde{\psi}_1\psi_1 + \tilde{\psi}_2\psi_2 \ , j_1^{(n=0)} = \tilde{\psi}_1\psi_1 - \tilde{\psi}_2\psi_2 \\ j_0^{(n=\pm 1)} &= \frac{1}{m}(\mathcal{H} \pm \mathcal{P}) \ , j_1^{(n=\pm 1)} = \frac{1}{m}(\mathcal{P} \pm \mathcal{S}) , \end{aligned} \quad \text{(B.7)}$$

where $\mathcal{H} = T_{00}, \mathcal{P} = T_{01} = T_{10}$ and $\mathcal{S} = T_{11}$ are the Hamiltonian density, the momentum density and the momentum flux respectively and they are the components of the stress energy tensor $T_{\mu\nu}$:

$$\mathcal{H} = -\frac{i}{2}\tilde{\psi}_1 \frac{\overleftrightarrow{d}}{dx}\psi_1 + \frac{i}{2}\tilde{\psi}_2 \frac{\overleftrightarrow{d}}{dx}\psi_2 + m\left(\tilde{\psi}_1\psi_2 + \tilde{\psi}_2\psi_1\right) , \quad \text{(B.8)}$$

$$\mathcal{P} = -\frac{i}{2}\tilde{\psi}_1 \overleftrightarrow{\frac{d}{dx}} \psi_1 - \frac{i}{2}\tilde{\psi}_2 \overleftrightarrow{\frac{d}{dx}} \psi_2 \tag{B.9}$$

$$\text{and } \mathcal{S} = -\frac{i}{2}\tilde{\psi}_1 \overleftrightarrow{\frac{d}{dx}} \psi_1 + \frac{i}{2}\tilde{\psi}_2 \overleftrightarrow{\frac{d}{dx}} \psi_2 \ . \tag{B.10}$$

The next few currents are explicitly given by

$$j_0^{(n=+2)}(x) = \left(\tilde{\psi}_1, \tilde{\psi}_2\right) \begin{pmatrix} 1 - \frac{1}{m^2}\left(\overleftrightarrow{\frac{d}{dx}}\right)^2 , & \frac{-i}{m}\overleftrightarrow{\frac{d}{dx}} \\ \frac{-i}{m}\overleftrightarrow{\frac{d}{dx}} , & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \tag{B.11}$$

$$j_1^{(n=+2)}(x) = \tilde{\psi}_1(1 - \frac{1}{m^2}\left(\overleftrightarrow{\frac{d}{dx}}\right)^2)\psi_1 - \tilde{\psi}_2\psi_2 \ . \tag{B.12}$$

$$j_0^{(n=-2)}(x) = \left(\tilde{\psi}_1, \tilde{\psi}_2\right) \begin{pmatrix} 1 , & \left(\frac{\pm i}{m}\overleftrightarrow{\frac{d}{dx}}\right) \\ \left(\frac{\pm i}{m}\overleftrightarrow{\frac{d}{dx}}\right) = , & 1 - \frac{1}{m^2}\left(\overleftrightarrow{\frac{d}{dx}}\right)^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \tag{B.13}$$

$$j_1^{(n=-2)}(x) = \tilde{\psi}_1\psi_1 - \tilde{\psi}_2(1 - \frac{1}{m^2}\left(\overleftrightarrow{\frac{d}{dx}}\right)^2)\psi_2 \ . \tag{B.14}$$

References

[1] H. Itoyama and H. B. Thacker, Phys. Rev. Lett. 14, 1395 (1987).

[2] E. Date, M. Jimbo, T. Miwa, and M. Okado, Phys. Rev. B35, 2105 (1987).

[3] J. Cardy, Nucl. Phys. B270, 186 (1986);H.W. Blote, J. Cardy and M.P. Nightingale, Phys. Rev. Lett. 52, 1575 (1984) ; I. Affleck, Phys. Rev. Lett.52, 746 (1984).

[4] C. N. Yang, Phys. Rev. 168, 1920 (1968).

[5] R. J. Baxter, Ann. Phys. (N. Y.) 70, 193 (1972).

[6] R. J. Baxter, J. Stat. Phys. 15, 485 (1976); 17, 1 (1977); 19, 461 (1978).

[7] M. A. Virasoro, Phys. Rev. D1, 2933 (1970); I. M. Gelfand and D. B. Fuks, Funktz. Analiz. 2, 92 (1968).

[8] A. Luther, Phys. Rev. B14, 2153 (1975).

[9] H. B. Thacker, Physica 18D, 348 (1986).

[10] H. Itoyama and H. B. Thacker, Fermilab-PUB-88/41-T.

[11] G.E. Andrews, R.J. Baxter, and P.J. Forrester, J. Stat. Phys. 35, 193 (1984).

[12] D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. 52, 1575 (1984).

[13] D. Friedan, Z. Qiu, and S. Shenker in Vertex Operators in Mathematics and Physics, edited by J. Lepowsky, S. Mandelstam, and I. M. Singer, Springer-Verlag (New York, 1985).

[14] J. L. Gervais, Phys. Lett. 160B, 277,279 (1985).

[15] A. B. Zamolodchikov, JETP Lett.46, 161 (1987) for a different approach to the higher constants of motion

[16] L. D. Faddeev and L. A. Takhtadzhan, Uspekhi Mat. Nauk 34:5, 13 (1979) [Russian Math. Surveys 34:5, 11 (1979)].

[17] H. Saleur and M. Bauer, Saclay Spht/88-080

[18] J. D. Johnson, S. Krinsky, and B. M. McCoy, Phys. Rev. A8, 2526 (1973).

[19] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. 16, 407 (1961).

[20] See for example Handbook of Mathematical Functions, M. Abramowitz and I. Stegun, eds., Dover (1965).

[21] R. J. Baxter, J. Stat. Phys. 17, 1 (1977).

[22] D. Mattis and E. Lieb, Phys. Rev. 6, 304 (1965).

[23] See, for example Y. Katznelson, An Introduction to Harmonic Analysis, J. Wiley & Sons, (1968).

[24] For a review, see A. C. Newell, Solitons in Mathematics and Physics, Society for Industrial and Applied Mathematics (1985).

[25] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, Proc. RIMS Symposium on Nonlinear Integrable Systems - Classical and Quantum Theory, M. Jimbo and T. Miwa, eds., World Scientific Press (1983).

Figure Captions

Fig. 1: A picture of the spectral torus

Fig. 2: An elementary face of the eight-vertex model in the spin formulation

Fig. 3: A picture of the extended CTM $\mathcal{A}$

Fig. 4: A schematic representation of the Hamiltonian eigenmode operator $B(\beta)$ in the β plane : the operators located at two points connected by the dotted line are canonically conjugate to each other.

Fig. 5: $B^{>}(\beta)$ is convergent in the shaded areas of this figure.

Fig. 6: $B^{<}(\beta)$ is convergent in the shaded areas of this figure.

Fig. 7: The contours used in eqs. (4.16), (4.17)

Fig. 8: This figure illustrates how the energy-momentum dispersion curve develops a cusp in the critical limit.

Fig. 9: A schematic representation identifying the Hamiltonian eigenmode operator with the complex fermion operators defined in eqs. (2.20), (2.21) in the critical limit.

Fig. 10: The contours used in eqs. (5.15), (5.16)

Fig. 11: The contours $C_L^{>}$, $C_L^{<}$ in the complex q_e plane

Fig. 12: The contours used for the right moving Virasoro algebra

Fig. 13: This figure illustrates how the two dimensional surface C is defined. As $\arg z$ increases, the cut in the complex p plane also rotates.

Fig. 14: This figure illustrates the identification of $B(\beta)$ with eigenmode operators in the massive continuum limit in various regions of β.

Fig. 15: The contour C is defined to consist of the real line $+$ the real line $+ i\pi$ in the α plane.

Fig. 16: The contours in eqs. (6.29), (6.30)

Fig. 17: An illustration of the recursive procedure given by eq. (6.45)

Fig. 18: This figure illustrates that the ordering with respect to the momentum q leads to a discontinuity when two points in the curve are on different branches.

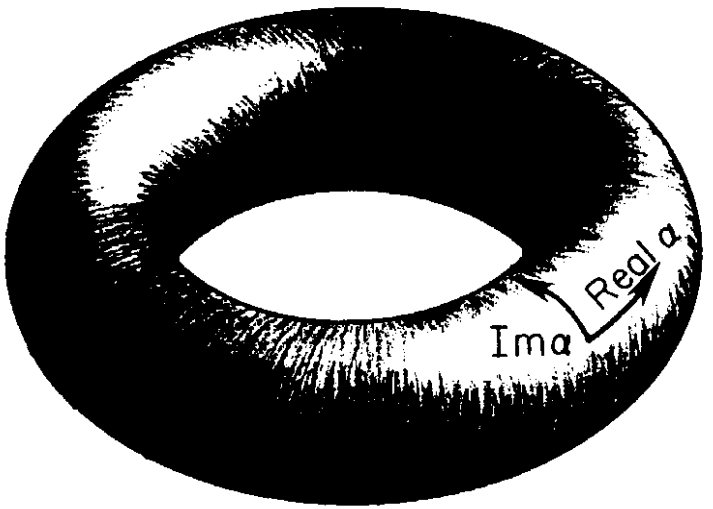


Fig. 1

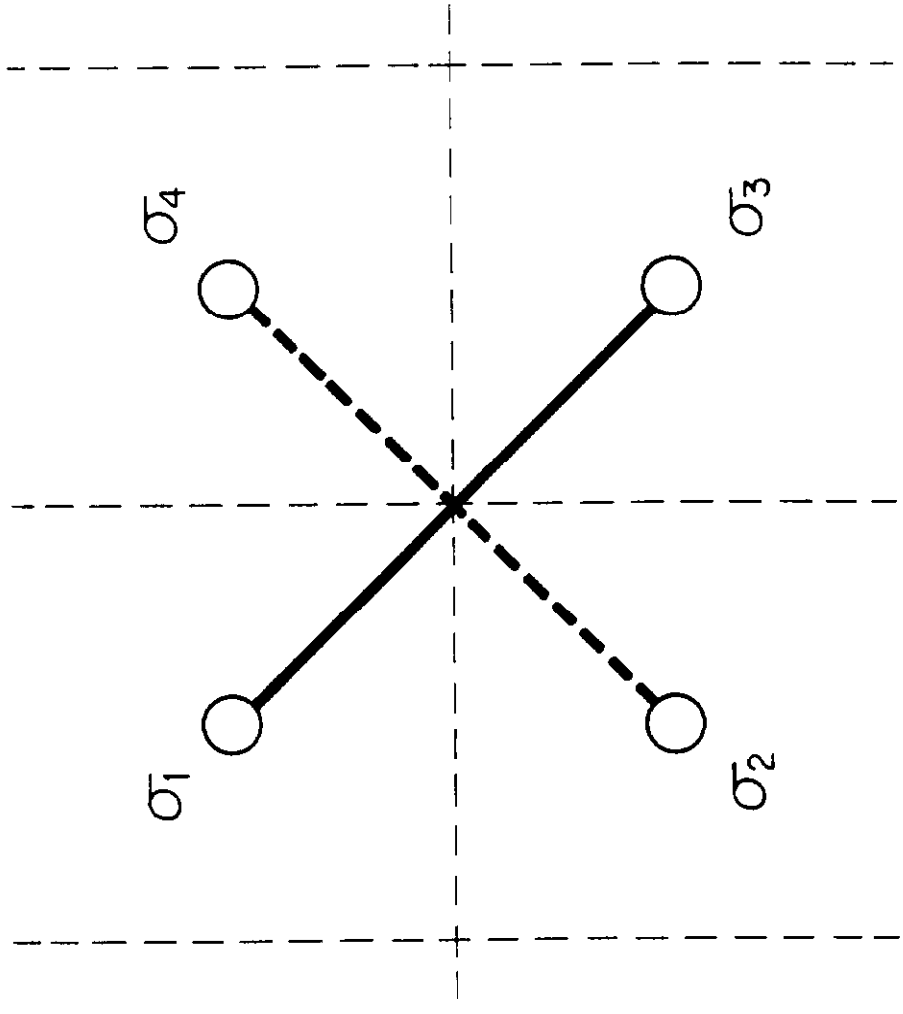


Fig. 2

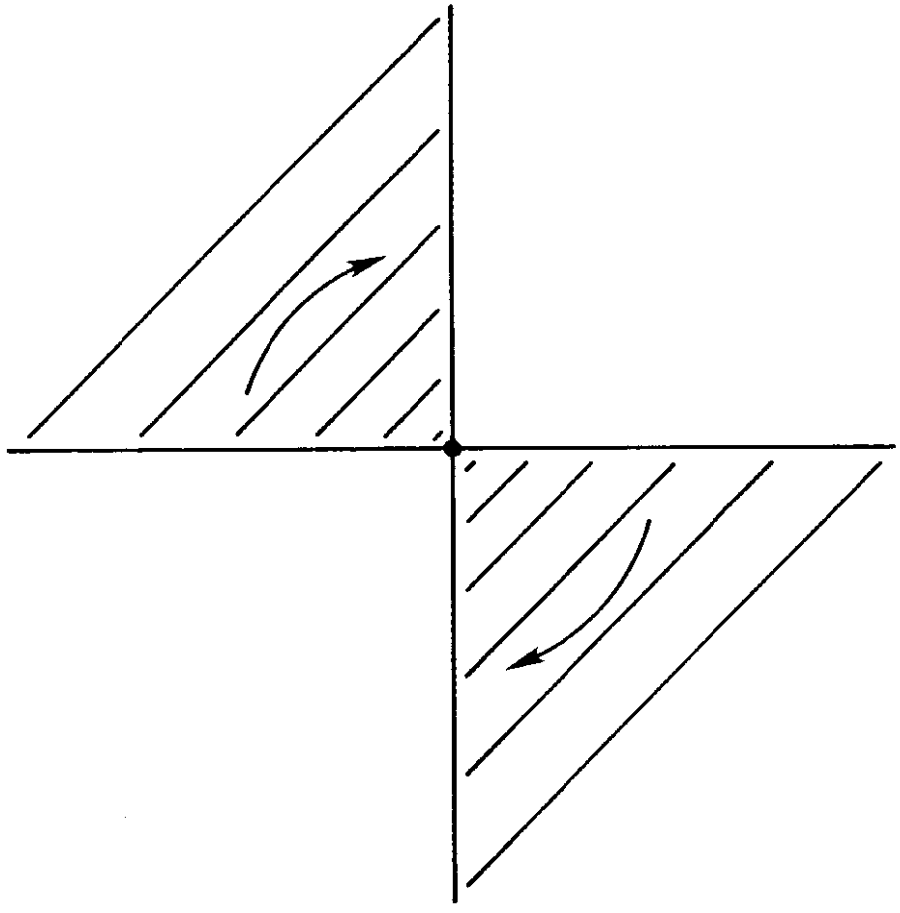


Fig. 3

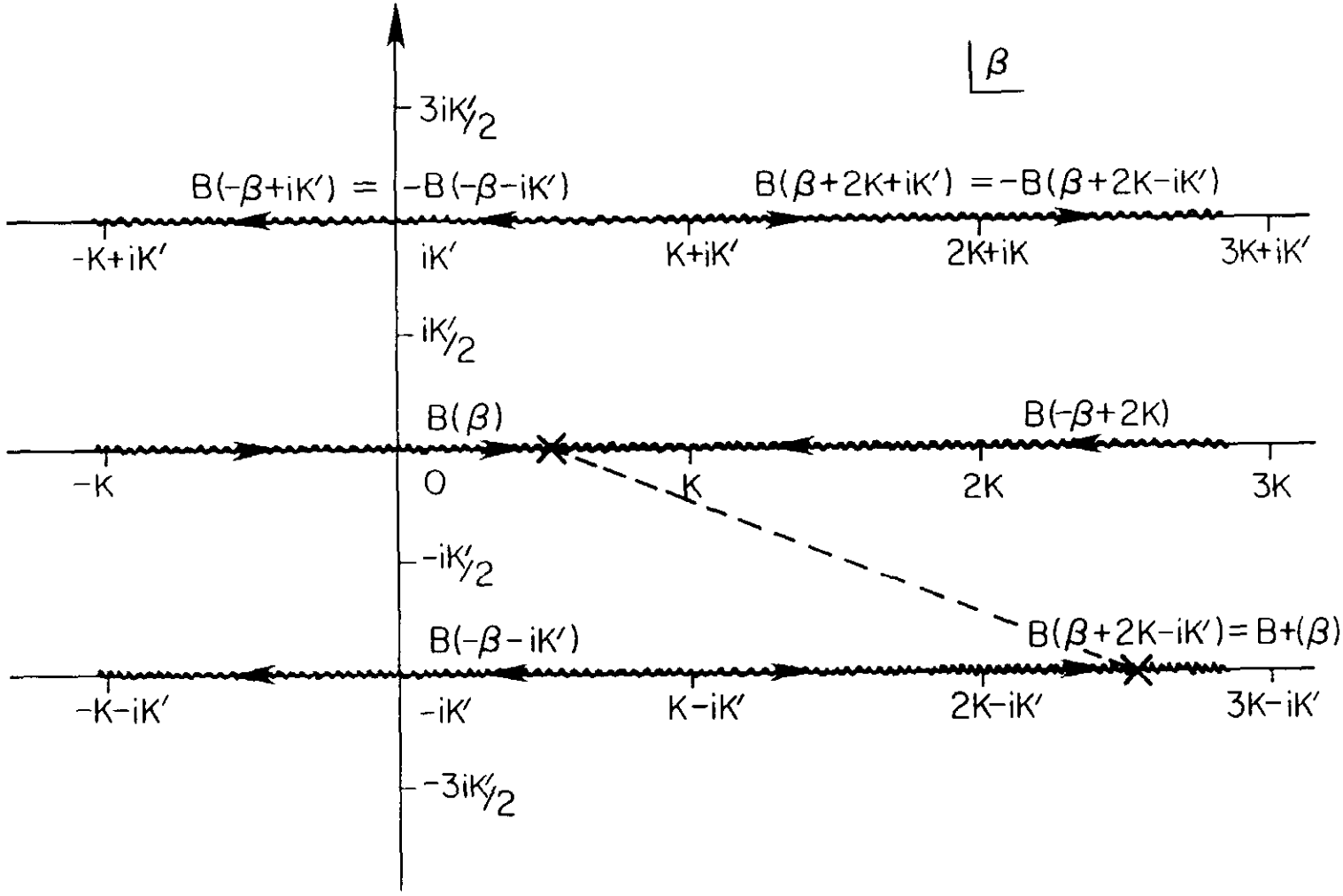
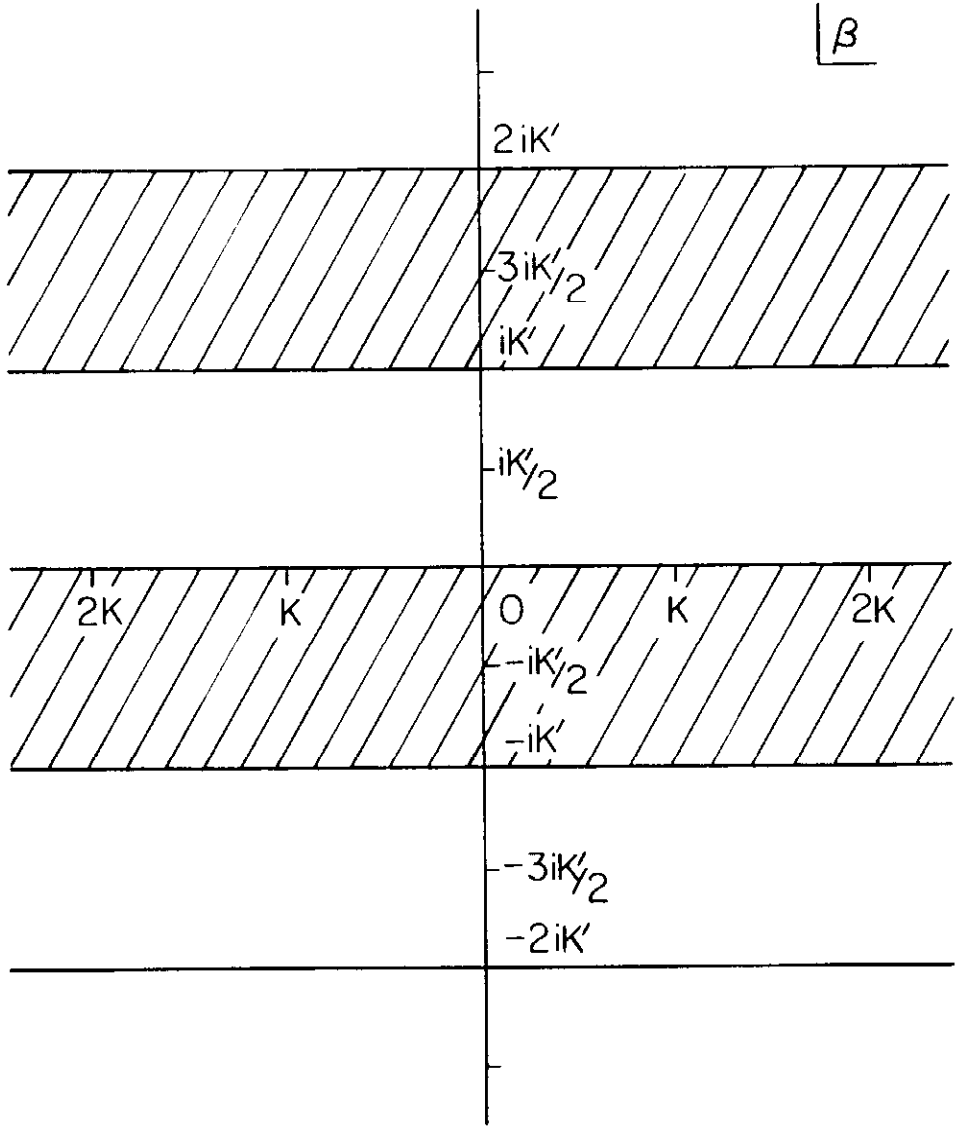
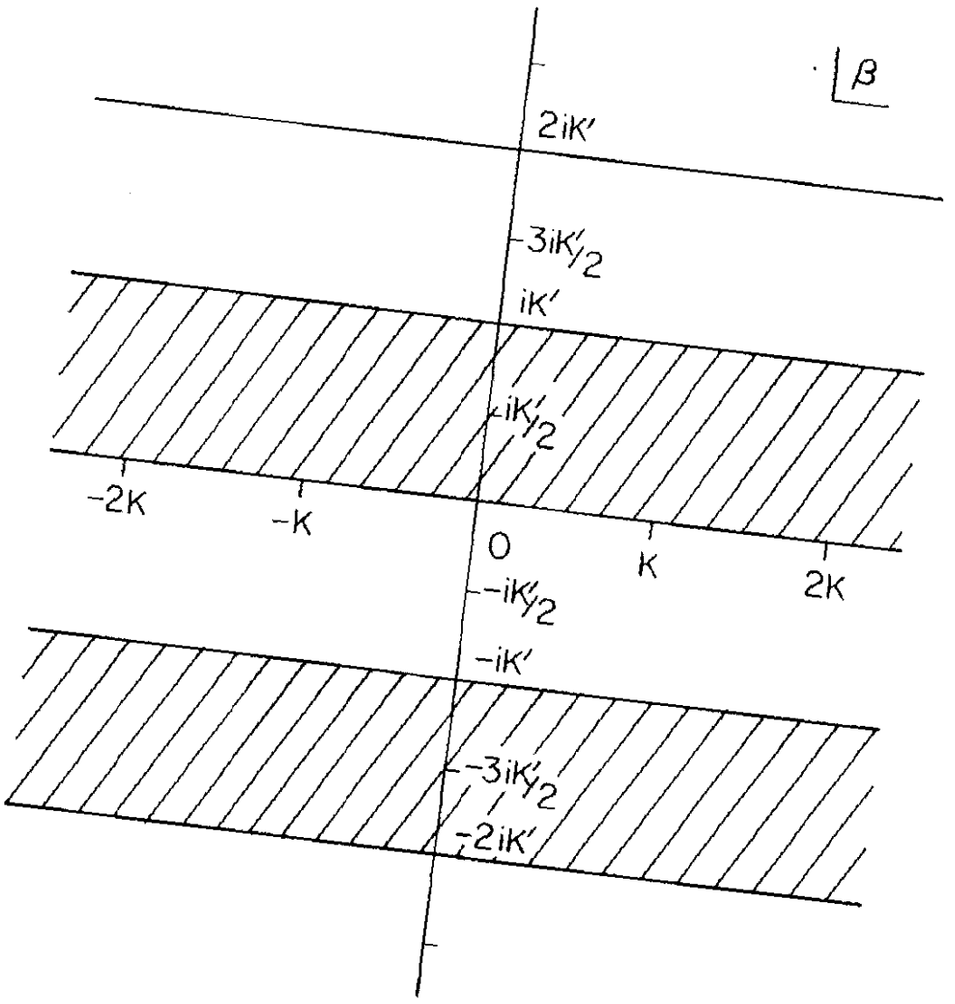

β
3iK′/2
B(−β+iK′) = −B(−β−iK′)
B(β+2K+iK′) = −B(β+2K−iK′)
−K+iK′
iK′
K+iK′
2K+iK
3K+iK′
iK′/2
B(β)
B(−β+2K)
−K
0
K
2K
3K
−iK′/2
B(−β−iK′)
B(β+2K−iK′) = B+(β)
−K−iK′
−iK′
K−iK′
2K−iK′
3K−iK′
−3iK′/2

Fig. 4



for $B^{>}(\beta)$

Fig. 5



for $B^{<}(\beta)$

Fig. 6

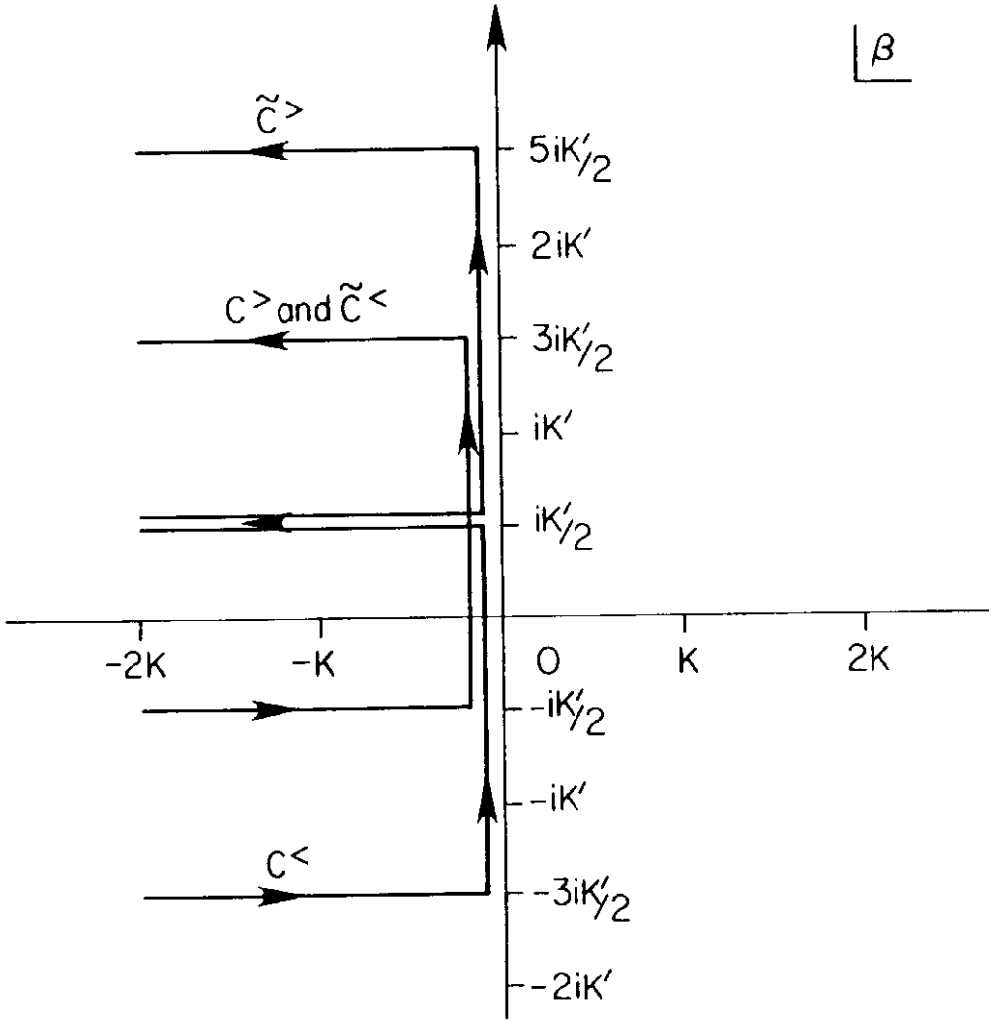

β
$\tilde{C}^{>}$
$C^{>}$ and $\tilde{C}^{<}$
$C^{<}$
5iK′/2
2iK′
3iK′/2
iK′
iK′/2
-iK′/2
-iK′
-3iK′/2
-2iK′
-2K
-K
0
K
2K

Fig. 7

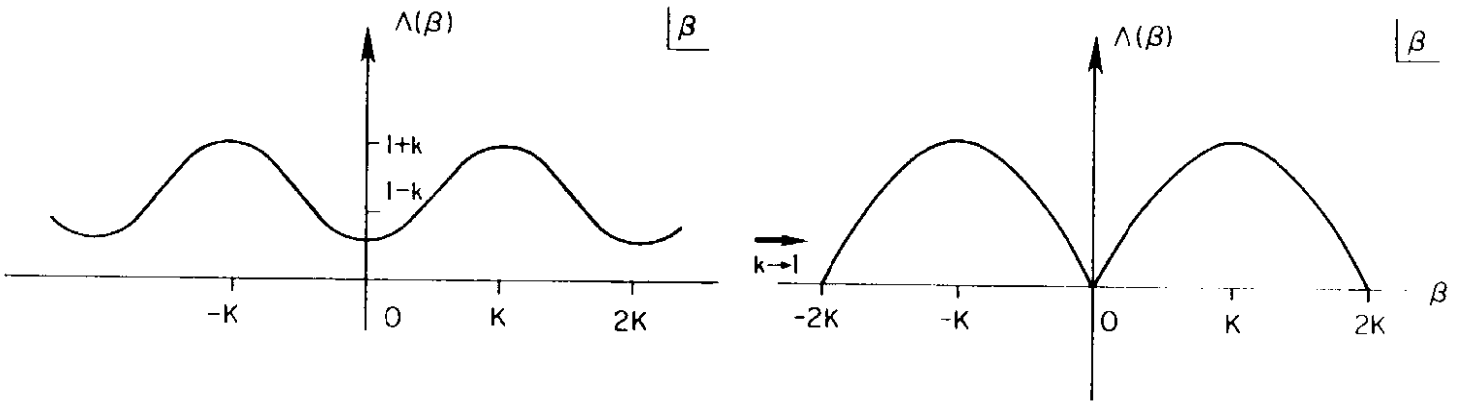


Fig. 8

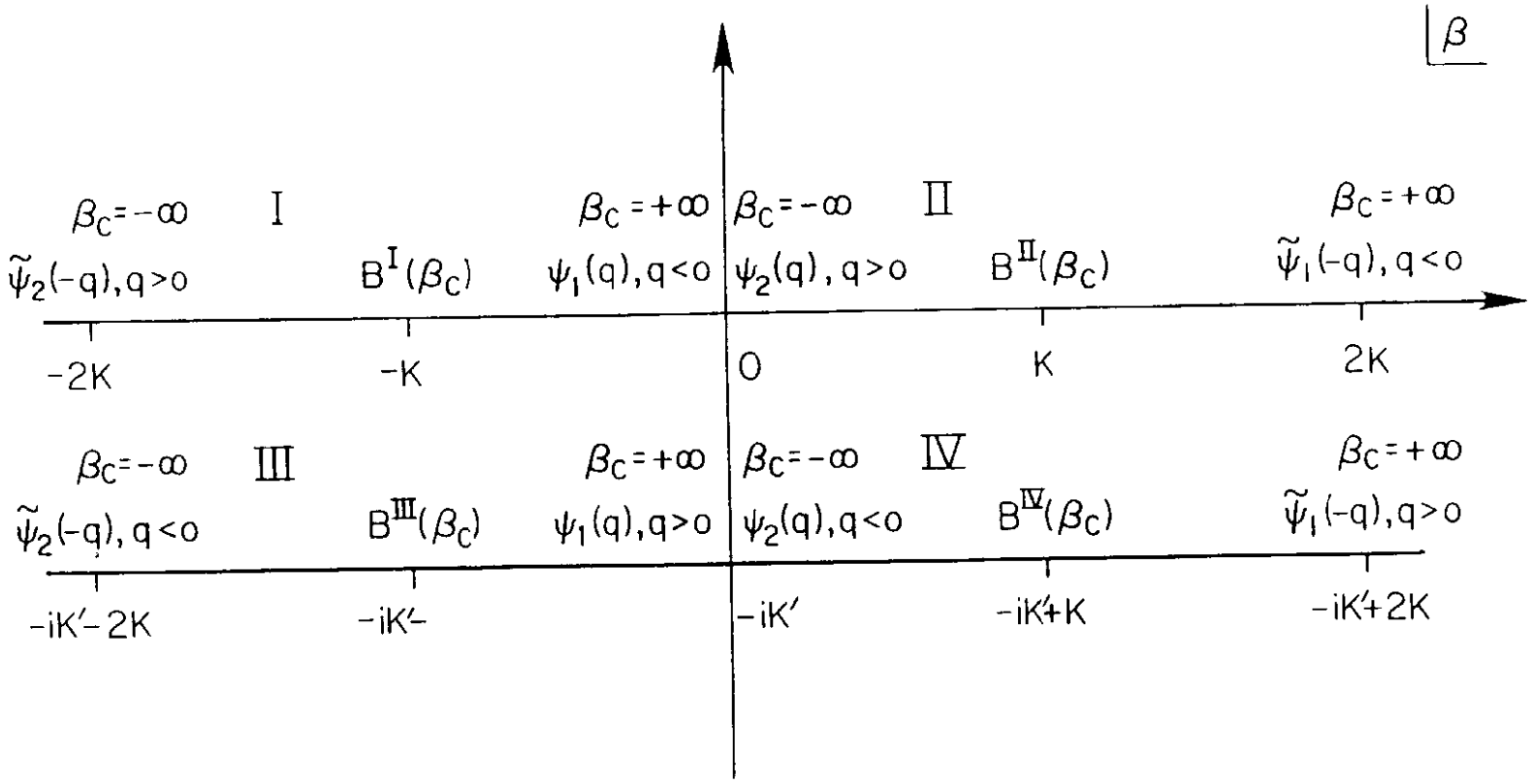

β
I
II
III
IV
$\beta_C=-\infty$
$\beta_C=+\infty$
$\tilde{\psi}_2(-q), q>0$
$B^{I}(\beta_C)$
$\psi_1(q), q<0$
$\psi_2(q), q>0$
$B^{II}(\beta_C)$
$\tilde{\psi}_1(-q), q<0$
$\tilde{\psi}_2(-q), q<0$
$B^{III}(\beta_C)$
$\psi_1(q), q>0$
$\psi_2(q), q<0$
$B^{IV}(\beta_C)$
$\tilde{\psi}_1(-q), q>0$
-2K
-K
0
K
2K
$-iK'-2K$
$-iK'-K$
$-iK'$
$-iK'+K$
$-iK'+2K$

Fig. 9

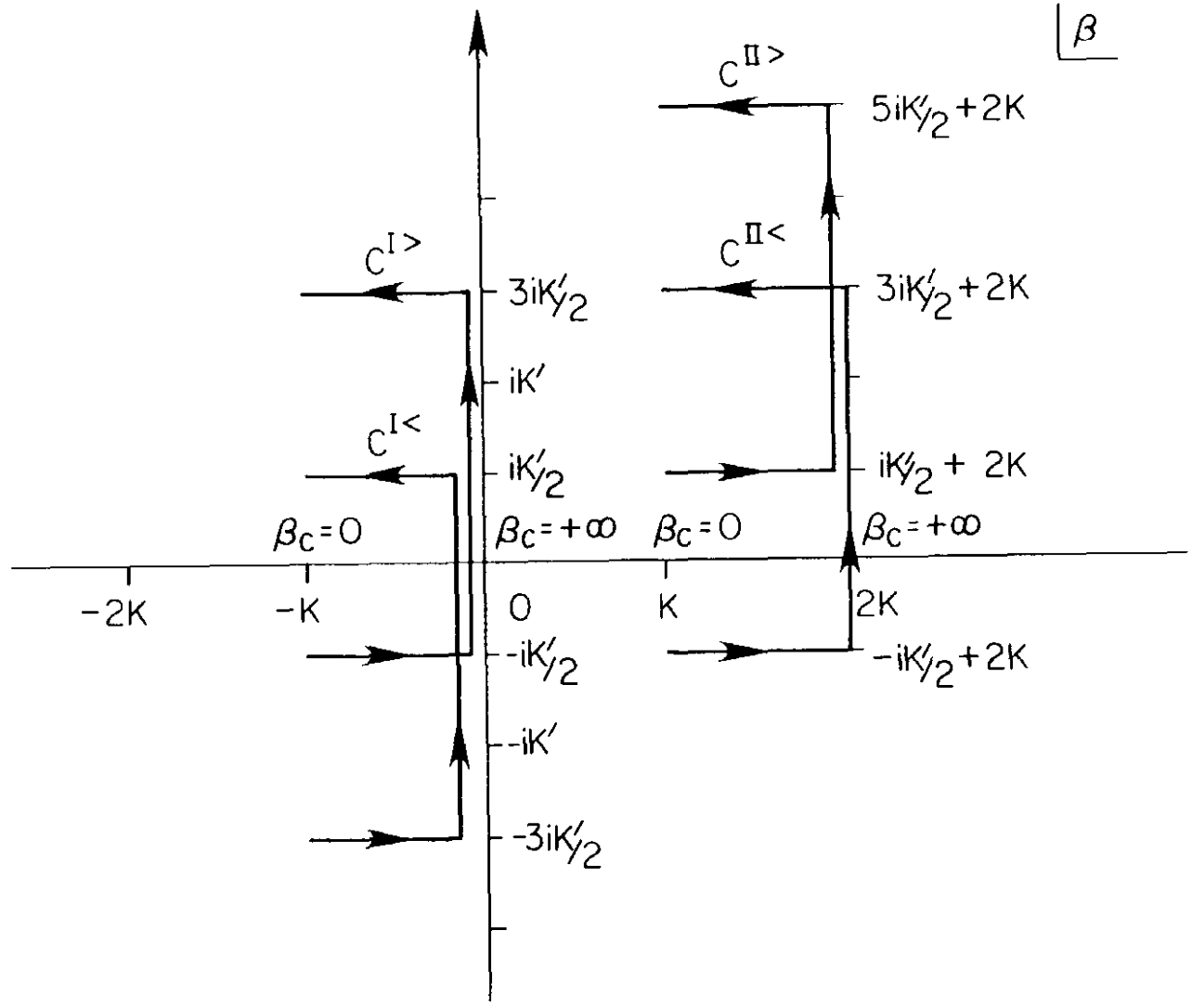

β
C^II>
5iK'/2+2K
C^I>
3iK'/2
C^II<
3iK'/2+2K
iK'
C^I<
iK'/2
iK'/2 + 2K
βc=0
βc=+∞
βc=0
βc=+∞
-2K
-K
0
K
2K
-iK'/2
-iK'/2+2K
-iK'
-3iK'/2

Fig. 10

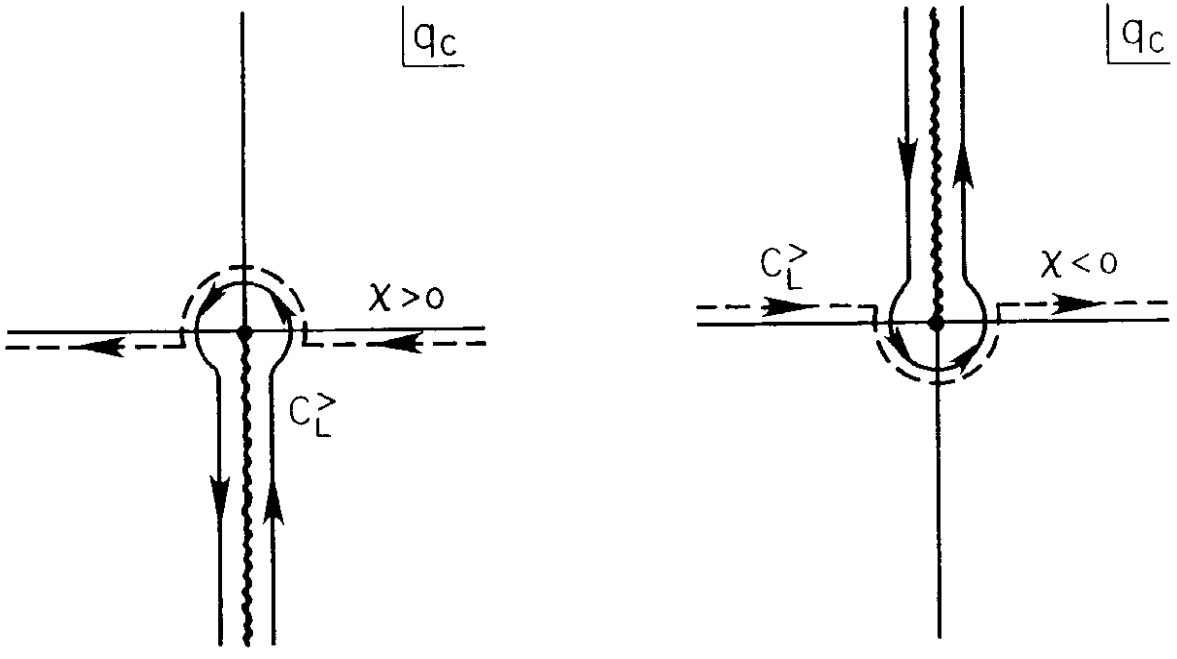


Fig. 11

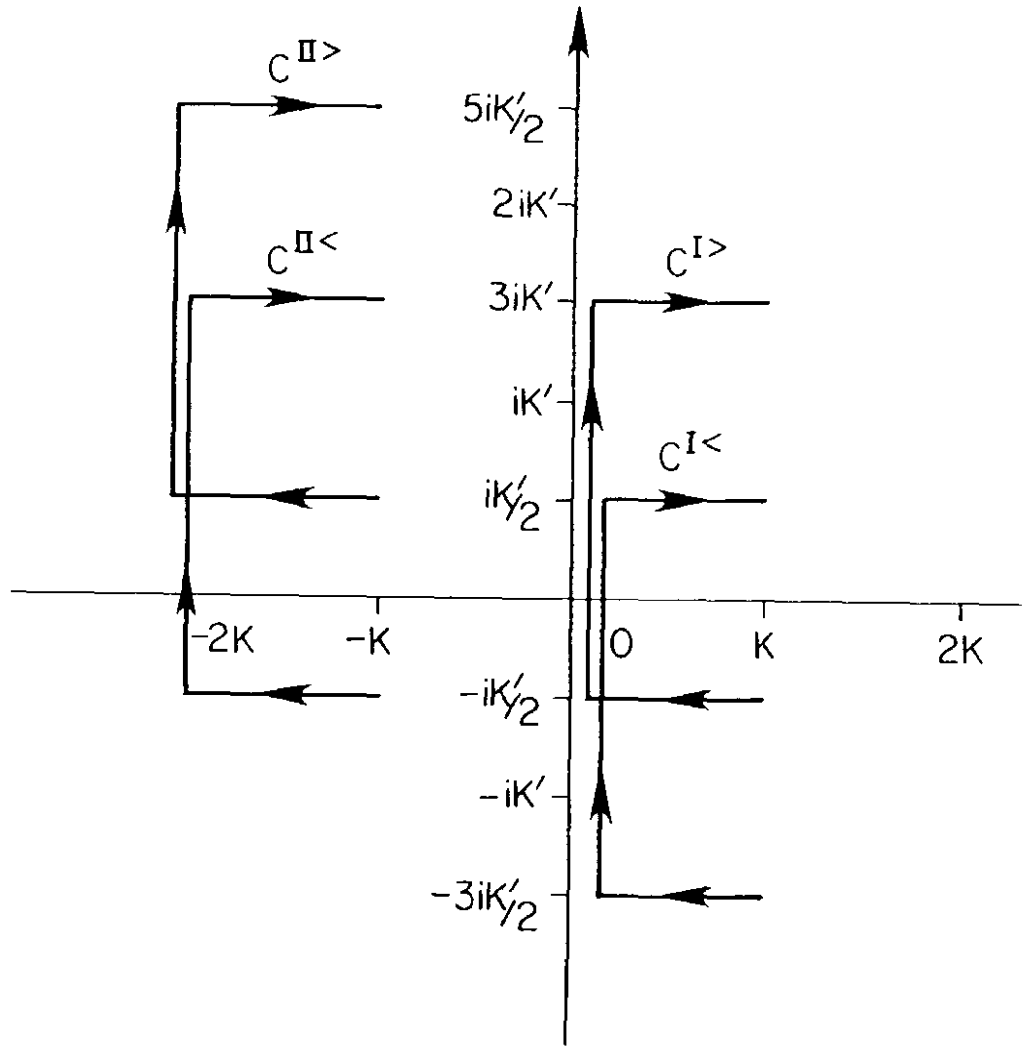


Fig. 12

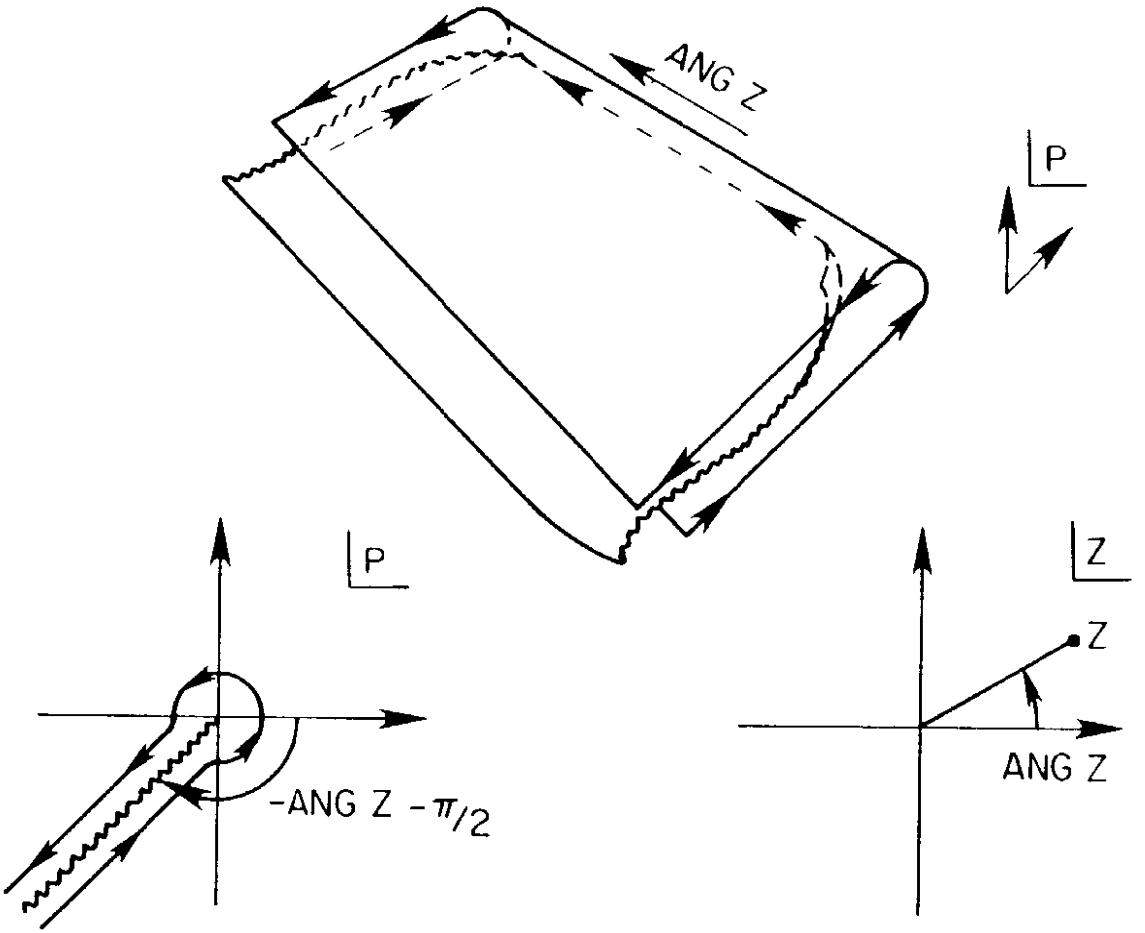


Fig. 13

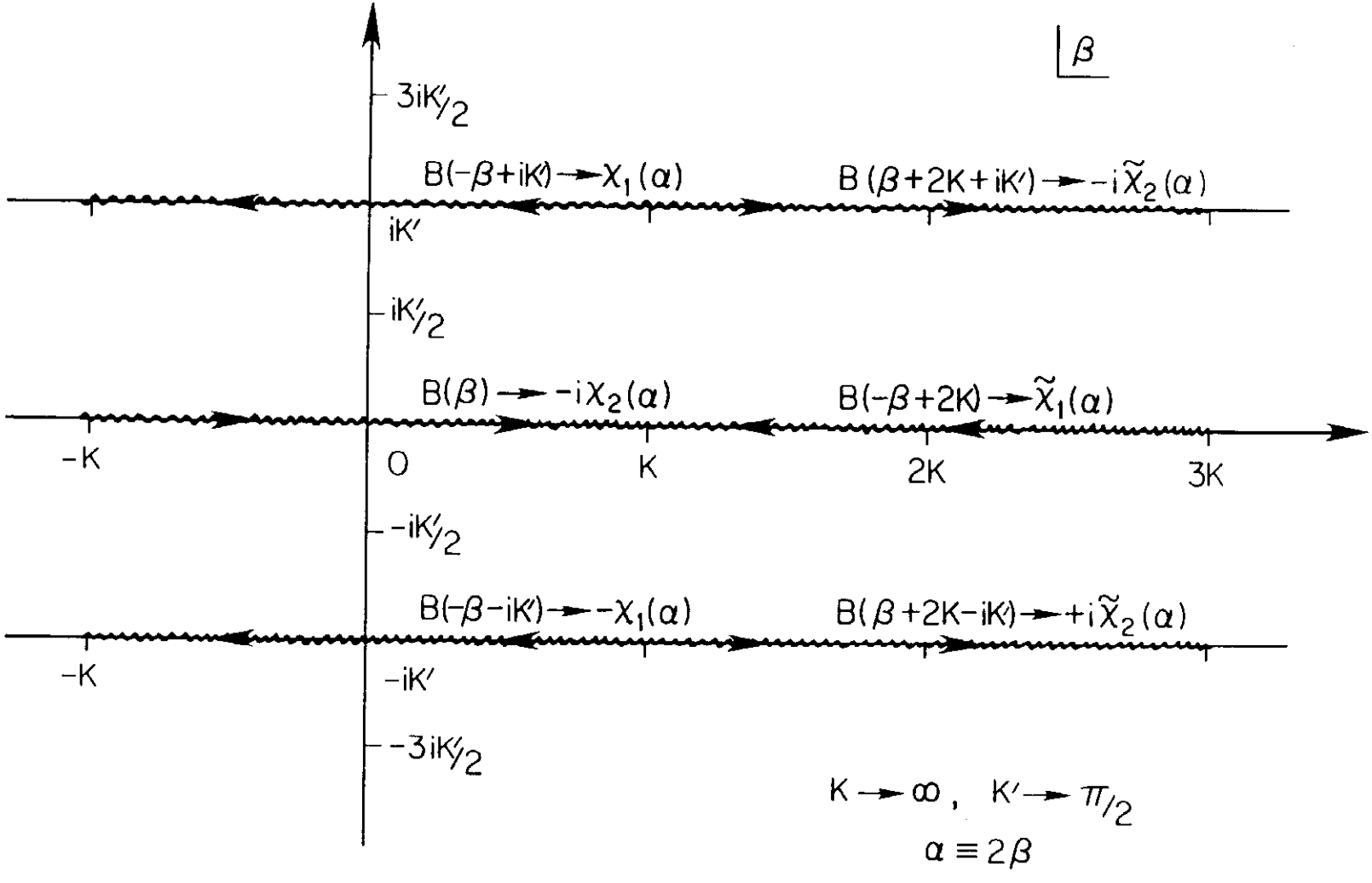

β
3iK′/2
B(−β+iK′)→χ1(α)
B(β+2K+iK′)→−iχ̃2(α)
iK′
iK′/2
B(β)→−iχ2(α)
B(−β+2K)→χ̃1(α)
−K
0
K
2K
3K
−iK′/2
B(−β−iK′)→−χ1(α)
B(β+2K−iK′)→+iχ̃2(α)
−K
−iK′
−3iK′/2
K→∞, K′→π/2
α≡2β

Fig. 14

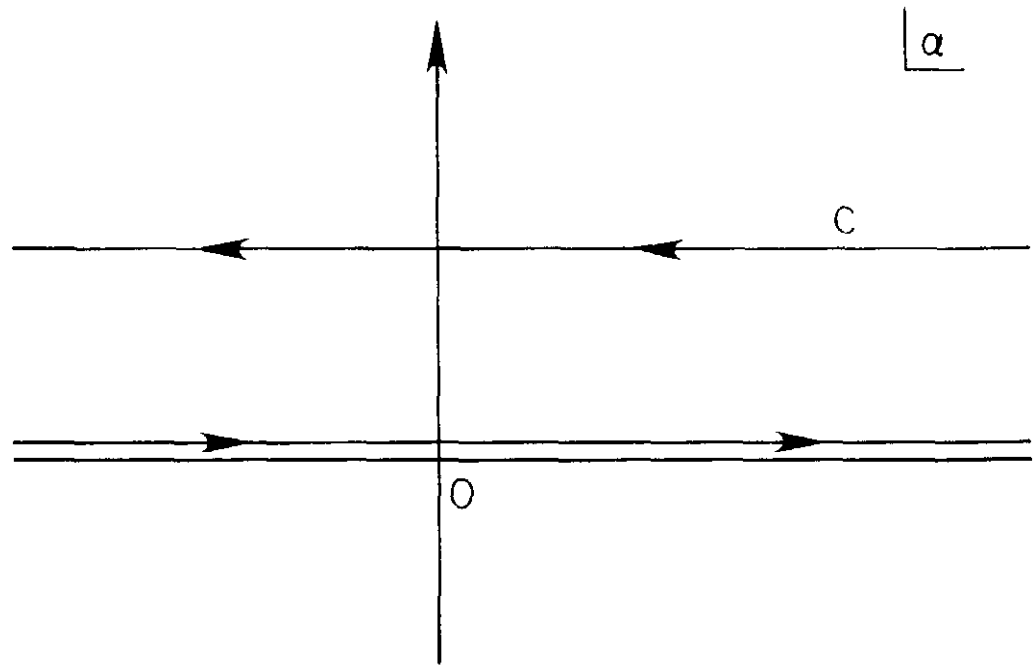


Fig. 15

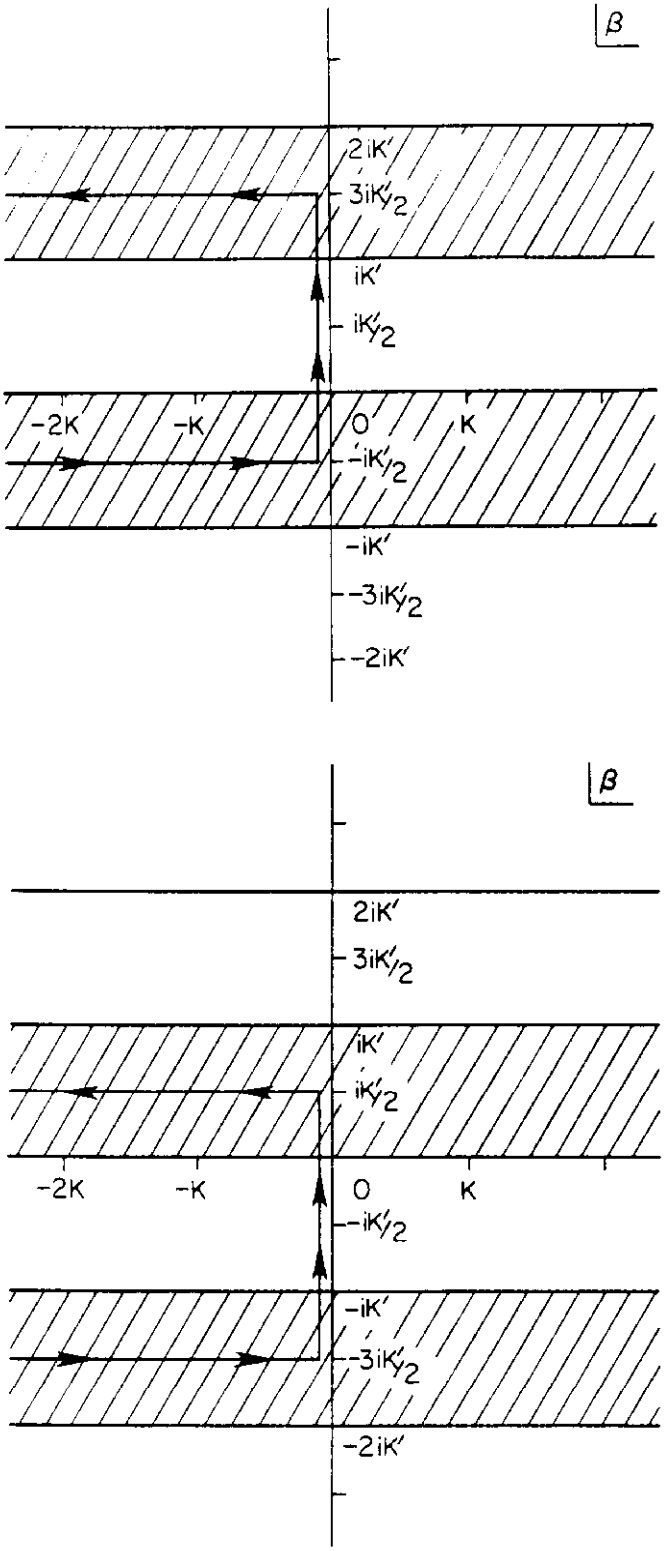

β
2iK′
3iK′/2
iK′
iK′/2
-2K
-K
0
K
-iK′/2
-iK′
-3iK′/2
-2iK′
β
2iK′
3iK′/2
iK′
iK′/2
-2K
-K
0
K
-iK′/2
-iK′
-3iK′/2
-2iK′

Fig. 16

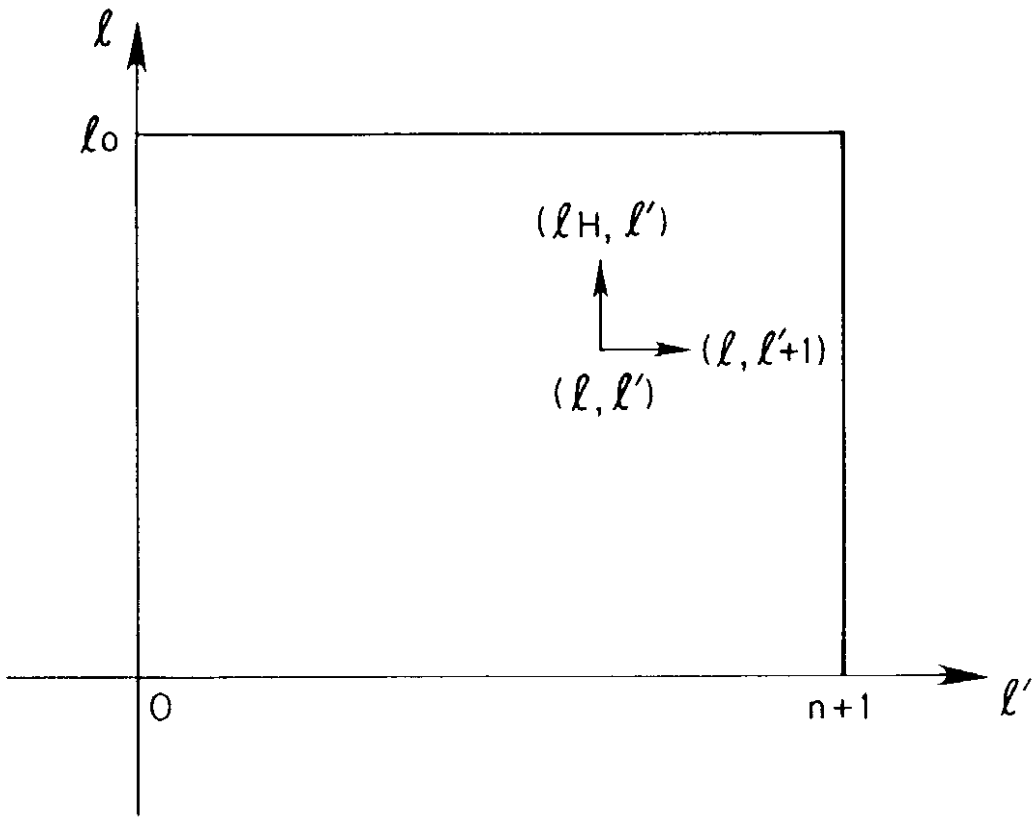


Fig. 17

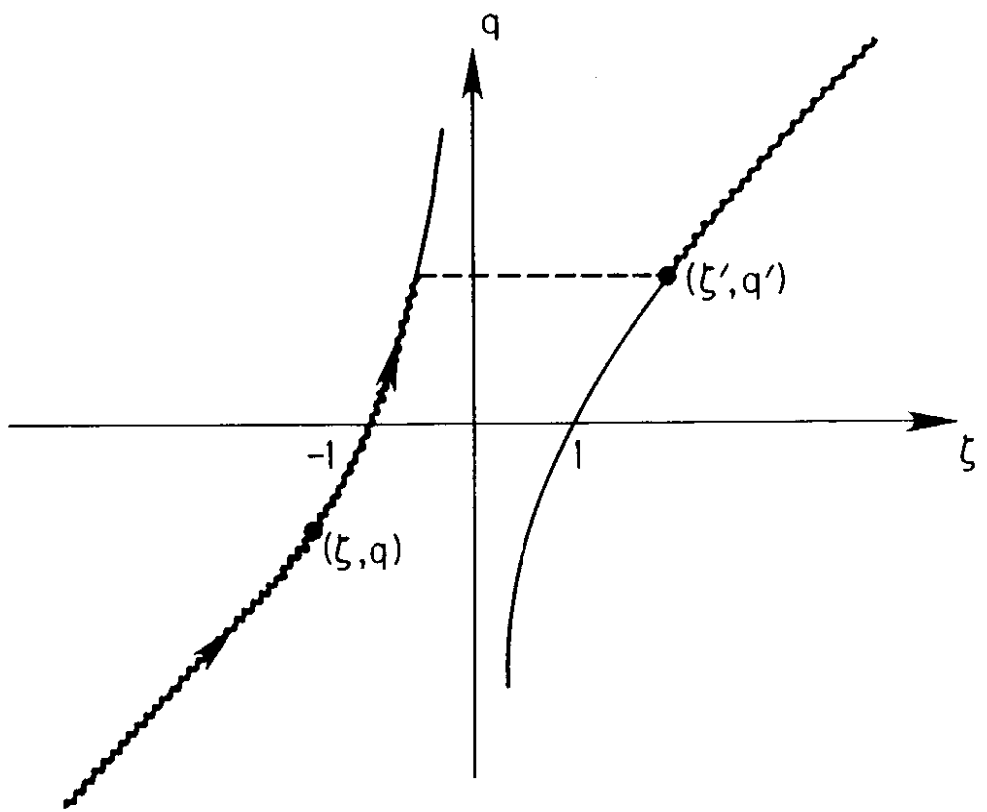


Fig. 18