Technische Universität München
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Bachelor´s Thesis
Quirky composite dark matter

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Chapter 1

Introduction

The topic of this thesis is dark matter, which is a type of matter that is difficult to understand and detect due to its reduced coupling with the standard model interactions. In general there is no need, that it couples at all. The only fundamental force dark matter is known to interact is gravitation. From this we know dark matter exists. One possible explanation for this phenomenon and a candidate for dark matter is the class of weakly interacting massive particles, so called WIMPs. In this thesis a specific type of WIMPs is considered, which is called "quirks", defined in [7]. In the special case of quirks they are bound due to a strong non-Abelian "quirkcolour", which is the hidden nonstandard model force, to form baryonic composite states of two quirks. Although they can carry weak and electric charges, there exist composite states, that do not. These states are the dark matter candidates for the reason, that dark matter must not be charged. The dark matter candidate also has to be stable, which is true for quirks due to being the lightest particles, so they can not decay into others. Also these quirks have a mass m_q, that is bigger than the confinement scale of the quirkcolour Λ_{QC}. At last the feature, that quirks have discrete bound states, leads to the possibility to detect the transitions of quirky dark matter with astrophysical measurements [7, 10].

This thesis has the aim to calculate the energies of the bound states and the energy for transitions between the states of quirky composite dark matter. The following chapters should at first introduce the theory of how to obtain the energy levels of a two-body system with the example of hydrogen. Then the focus gets on the problem of quarkonia, which is a bound state of a quark and an antiquark due to strong interaction, that is much closer to the problem of a bound state through the new strong quirkcolour force than hydrogen. Finally the problem of quirkonia is tackled, where a calculation of the energy splittings in analytical and numerical ways is presented and the transitions for indirect observation of the dark matter are calculated.

Chapter 2

The Hydrogen atom

2.1 The energy levels of the hydrogen atom

At first I will discuss the problem of the Schrödinger equation for the hydrogen atom. First we consider the constituents of the hydrogen atom, which are the proton and the electron. The proton mass m_p is much larger than the electron mass m_e , so in good approximation the reduced mass $\mu = \frac{m_e \cdot m_p}{m_e + m_p} = m_e - \frac{m_e^2}{m_e + m_p} \approx m_e$. In this approximation only the mass and momentum of the electron contribute to the Schrödinger equation. Second the force, which keeps the electron in the proton classical orbit, is simply the electromagnetic force between the charged particles resulting a Coulombic potential with $-\frac{\alpha}{r}$ dependence, where r is the radius between the proton and the electron and α is the fine-structure constant, which is in natural units, that means $c = \hbar = 1$, $\alpha = e^2$ with the elementary charge e [1]. The Schrödinger equation and a way to obtain the radial equation[1]

$$\frac{1}{R(r)} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) R(r) + 2 m_e r^2 \left(\frac{\alpha}{r} + E \right) = l(l+1) \tag{2.1}$$

and the wave function $\psi(r, \vartheta, \varphi) = R(r) Y(\vartheta, \varphi)$, where $R(r)$ is the radial and $Y(\vartheta, \varphi)$ is the spherical part, and properties of the azimuthal quantum number l are in the appendix. So we can start to determine the bound state energy E of the hydrogen atom by the radial equation and substitute with the reduced wave function $u(r) = rR(r)$ and $\frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) R(r) = \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) \frac{u(r)}{r} = \frac{\partial}{\partial r} \left(r \frac{\partial u(r)}{\partial r} - u(r) \right) = r \frac{\partial^2 u(r)}{\partial r^2}$, which leads to[1]

$$\frac{\partial^2 u(r)}{\partial r^2} + \left(2 m_e \left(\frac{\alpha}{r} + E \right) - \frac{l(l+1)}{r^2} \right) u(r) = 0. \tag{2.2}$$

To solve this differential equation it is useful to look at the limits of r and analyze the asymptotic behaviors. For $r \to 0$ the $\frac{1}{r^2}$-part dominates, so the equation simplifies to $\frac{\partial^2 u(r)}{\partial r^2} - \frac{l(l+1)}{r^2} u(r) = 0$, which has the solutions $u(r) =$

$Ar^{l+1} + Br^{-l}$, with the azimuthal quantum number l, that must be an integer to be able to well define the Laguerre polynomials explained in appendix . The limit $r \to 0$ forbids r^{-l} because it cannot be normalized for $l \neq 0$, where $l = 0$ provides only a solution for the limit , so that $u(r) \propto r^{l+1}$. For $r \to \infty$ the simplified equation is $\frac{\partial^2 u(r)}{\partial r^2} + 2m_e E u(r) = 0$. For negative energies the solution is $u(r) = Ae^{kr} + Be^{-kr}$, with $k = \sqrt{-2m_e E}$, where $e^{kr}$is forbidden because it cannot be normalized, so $u(r) \propto e^{-kr}$. With a last substitution, $\rho = kr$, $u(\rho)$ can finally be written as

$$u(\rho) = e^{-\rho}\rho^{l+1}v(\rho) \tag{2.3}$$

with a function $v(\rho)$ and the differential equations for $u(\rho)$ and $v(\rho)$ respectively are [2]

$$\frac{\partial^2 u(\rho)}{\partial \rho^2} + \left(\frac{c}{\rho} - 1 - \frac{l(l+1)}{\rho^2}\right) u(\rho) = 0 \tag{2.4}$$

$$\frac{\rho\partial^2 v(\rho)}{\partial \rho^2} + 2\left((l+1) - \rho\right)\frac{\partial v(\rho)}{\partial \rho} + v(\rho)\left(-2(l+1) + c\right) = 0 \tag{2.5}$$

with the substitution $c = \sqrt{\frac{-2m}{E}}\alpha$. Now I express the function $v(\rho)$ as a power series $v(\rho) = \sum_{i=0}^{\infty} a_i \rho^i$ and insert it in the equation with leads to[2]

$$\sum_{i=0}^{\infty} a_i \left[i(i-1)\rho^{i-1} + 2(l+1)i\rho^{i-1} - 2i\rho^i + \left(-2(l+1) + c\right)\rho^i\right] = 0 \tag{2.6}$$

The condition that this sum is zero, can only be satisfied, if the coefficients of every power are zero, so that we get a recursion relation for a_i[2]

$$a_{i+1}\left(i(i+1) + 2(l+1)(i+1)\right) + a_i\left(c - 2(l+1) - 2i\right) = 0 \tag{2.7}$$

$$a_{i+1} = a_i \frac{2(l+1) + 2i - c}{i(i+1) + 2(l+1)(i+1)} \tag{2.8}$$

From this recursion relation the behavior for the limit $i \to \infty$ can be extracted, which is $\frac{a_{i+1}}{a_i} \to \frac{2}{i}$. But that is the behavior of $e^{2\rho}$, which is in contradiction to the found limit $u(r) \propto e^{-\rho}$. That leads to the conclusion, that the series has to be truncated, which means that $c = 2(l+1+i) = 2n$,where n is an integer, named the principal quantum number, and l is the azimuthal quantum number, which must be smaller than n . Now using this condition in our earlier introduced expression $c = \sqrt{\frac{-2m_e}{E}}\alpha$ and solving it for the energy levels E of the hydrogen atom leads to the expression [2]

$$E_n = -\frac{m_e \alpha^2}{2n^2}. \tag{2.9}$$

2.2 Relativistic corrections of the Schrödinger equation

The Schrödinger equation is a non-relativistic equation. To calculate the relativistic corrections we have to use the Dirac equation. Here we have to decouple the solutions for positive and negative energies of the Dirac equation, which can be achieved with a Foldy-Wouthuysen transformation, where the Dirac Hamiltonian $H\acute{} = e^{iS}(H - \partial_t)e^{-iS}$ and the wave function $\psi = e^{-iS}\psi\acute{}$ are transformed with a matrix S that makes the operators in the Hamiltonian diagonal, so that upper and lower components do not mix[3]. In the case of electromagnetic interaction the Hamiltonian $H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$ gets modified via minimal coupling, where $p \to (p - eA)$ and $i\frac{\partial}{\partial t} \to i\frac{\partial}{\partial t} - e\Phi$ with the potentials A and Φ, so that the Hamiltonian looks like[3]

$$H = \boldsymbol{\alpha} \cdot (\mathbf{p} - \mathbf{eA}) + \beta m + e\Phi. \tag{2.10}$$

After three transformations of the Hamiltonian you get finally all correction terms up to order $\frac{1}{m^2}$. The transformed Dirac equation for the positive energies written with a spinor $\psi\acute{} = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}$ reads[3]

$$\begin{aligned} \Bigg(& m + e\Phi + \frac{(\mathbf{p} - \mathbf{eA})^2}{2m} - \frac{e}{2m}\boldsymbol{\sigma} \cdot \mathbf{B} - \frac{\mathbf{p}^4}{8m^3} + \\ & - \frac{e}{4m^2}\boldsymbol{\sigma} \cdot \mathbf{E} \times (\mathbf{p} - \mathrm{e}\mathbf{A}) - \frac{e}{8m^2}\nabla \cdot \mathbf{E}\Bigg)\varphi = i\frac{\partial \varphi}{\partial t} \end{aligned} \tag{2.11}$$

For the hydrogen atom the potential $\mathbf{A} = 0$ and $\mathbf{E} = -\nabla\Phi\left(r\right) = -\frac{1}{r}\frac{d}{dr}V\left(r\right)\mathbf{x}$ with $V(r) = -\frac{\alpha}{r}$, so that we can rewrite $\frac{1}{4m^2}\boldsymbol{\sigma}\cdot\mathbf{E} \times (\mathbf{p} - \mathrm{e}\mathbf{A}) = \frac{1}{2m^2}\frac{\boldsymbol{\sigma}\cdot\mathbf{L}}{r}\frac{d}{dr}V\left(r\right)$[3]. Here we can see the four corrections beyond the potentials and the kinetic energy, where $H_{rm} = -\frac{\mathbf{p}^4}{8m^3}$ is the relativistic mass correction. It is convenient to write it in terms of the unperturbed state $H_0 = \frac{\mathbf{p^2}}{2m} - \frac{\alpha}{r}$ and the potential $\frac{\alpha}{r}$, so that $H_{rm} = -\frac{\mathbf{p}^4}{8m^3} = -\frac{1}{2m}\left(H_0 + \frac{\alpha}{r}\right)^2$[2]. The solution of the first order perturbation theory is

$$E^1_{rm} =< \psi^0 \mid \frac{-1}{2m}\left(H_0 + \frac{\alpha}{r}\right)^2 \mid \psi^0 >= \frac{-1}{2m}\left(E_n^2 + 2\alpha E_n < \frac{1}{r} > + \alpha^2 < \frac{1}{r^2} >\right) \tag{2.12}$$

which leads with $E_n = -\frac{m\alpha^2}{2n^2}$, $< \frac{1}{r} >= \frac{1}{r_b n^2}$, $< \frac{1}{r^2} >= \frac{1}{r_b^2 n^3 \left(l + \frac{1}{2}\right)}$ and the Bohr radius $r_B = \frac{1}{m\alpha}$ to[2]

$$E^1_{rm} = -\frac{m\alpha^4}{2n^4}\left(\frac{n}{l + \frac{1}{2}} - \frac{3}{4}\right). \tag{2.13}$$

The next correction $H_{Darwin} = -\frac{e}{8m^2}div\mathbf{E}$ is the Darwin term. Here it is useful to put in the result of the Poisson equation, that leads to $H_{Darwin} =$

$-\frac{e}{8m^2} div\mathbf{E} = -\frac{e}{8m^2} 4\pi\rho(r)$[4] with the charge density $\rho(r)$. For hydrogen atom the charge density is $\rho(r) = -e\delta^3(r)$, which leads to the perturbation term[2]

$$E^1_{Darwin} =< \psi^0 \mid \frac{\pi e^2}{2m^2}\delta^3(r) \mid \psi^0 >= \frac{\pi e^2}{2m^2} \mid \psi^0_{nlm}(0) \mid^2 = \frac{m\alpha^4}{2n^3}\delta_{l,0} \tag{2.14}$$

with $< \psi^0_{100} \mid \delta(r) \mid \psi^0_{100} >=\mid \psi^0_{100}(0) \mid^2 = \frac{1}{\pi r_B^3}$.

Then there is the spin-orbit coupling resulting from an interaction between the electron spin and the magnetic field of the proton in the rest frame of the electron, with the Hamiltonian[2]

$$H_{Spin-Orbit} = \frac{1}{2m^2}\frac{\mathbf{S}\cdot\mathbf{L}}{r}\frac{d}{dr}V(r)\,. \tag{2.15}$$

For the hydrogen potential $V(r) = -\frac{\alpha}{r}$ the energy correction of this term can be solved by first order perturbation theory by[2]

$$E^1_{Spin-Orbit} =< \psi^0 \mid \frac{\alpha}{2m^2}\frac{\mathbf{S}\cdot\mathbf{L}}{r^3} \mid \psi^0 > . \tag{2.16}$$

Here we have to express $\mathbf{S}\cdot\mathbf{L}$ in terms of $\mathbf{J^2}$, $\mathbf{L^2}$ and $\mathbf{S^2}$, so we can easily put in their known eigenvalues. The total angular momentum is written as the sum of angular momentum and the spin $\mathbf{J^2} = (\mathbf{L}+\mathbf{S})^\mathbf{2} = \mathbf{L^2} + 2\mathbf{L}\cdot\mathbf{S} + \mathbf{S^2}$, so that the eigenvalues of $\mathbf{S}\cdot\mathbf{L} = \frac{1}{2}\left(\mathbf{J^2} - \mathbf{L^2} - \mathbf{S^2}\right) = \frac{1}{2}\left(j(j+1) - l(l+1) - s(s+1)\right)$. For our spin $\frac{1}{2}$-electron the correction looks like[2]

$$E^1_{Spin-Orbit} = \frac{\alpha\hbar^2\left(j(j+1) - l(l+1) - \frac{3}{4}\right)}{4m^2} < \psi^0 \mid \frac{1}{r^3} \mid \psi^0 > . \tag{2.17}$$

The expectation value of $< \psi^0 \mid \frac{1}{r^3} \mid \psi^0 >$is[2]

$$< \psi^0 \mid \frac{1}{r^3} \mid \psi^0 >= \frac{1}{n^3 r_B^3 l\left(l+\frac{1}{2}\right)(l+1)}. \tag{2.18}$$

That leads to the final correction with an α^4-dependence [2]

$$E^1_{Spin-Orbit} = \alpha^4 \frac{m\left(j(j+1) - l(l+1) - \frac{3}{4}\right)}{4n^3 l\left(l+\frac{1}{2}\right)(l+1)}. \tag{2.19}$$

This formula applies only for $l \geq 1$, but diverges for $l = 0$. The solution for $l = 0$ can be obtained, when a spherical potential is treated, which means we consider the proton as composite of quarks and not as point particle. Here the expectation value of $< \psi^0 \mid \frac{1}{r^3} \mid \psi^0 >$ for $l \to 0$ is not divergent anymore, but the expectation value of $< \psi^0 \mid \mathbf{S}\cdot\mathbf{L} \mid \psi^0 >= 0$ [2]. So the expectation value $< \psi^0 \mid H \mid \psi^0 >_{l=0} = 0$ vanishes and we can finally sum up the relativistic corrections $\Delta E_{rel} = E_{rm} + E_{Darwin} + E_{Spin-Orbit}$ to get the total correction [2]

$$\Delta E_{rel} = \frac{m\alpha^4}{2n^4}\left(\frac{3}{4} - \frac{n}{j+\frac{1}{2}}\right) \tag{2.20}$$

2.3 The hyperfine splitting with tensor- and $\mathbf{L}\cdot\mathbf{S}$-terms

The other splitting is the hyperfine splitting, which is caused by couplings between the electron spin and the proton spin, where the magnetic moments are $\mu_{\mathbf{e}} = -\frac{e}{m_e}\mathbf{S_e}$ and $\mu_{\mathbf{p}} = \frac{g_p e}{2m_e}\mathbf{S_p}$ and g_p is the gyromagnetic ratio of the proton, where the gyromagnetic ratio of the electron has already been considered in the expression of the magnetic moment. The resulting field is a composite of the dipole field between the magnetic moments of the proton and the electron and an interaction between the proton magnetic moment and the magnetic field, which is caused by the movement of the electron with angular momentum $\mathbf{L}$, so that the Hamiltonian in the frame of the proton looks like [4]

$$H_{hf} = -\frac{8\pi}{3}\mu_{\mathbf{e}}\cdot\mu_{\mathbf{p}}\delta\left(r\right) + \frac{1}{r^3}\left[\mu_{\mathbf{e}}\cdot\mu_{\mathbf{p}} - 3\frac{\left(\mathbf{r}\cdot\mu_{\mathbf{e}}\right)\left(\mathbf{r}\cdot\mu_{\mathbf{p}}\right)}{r^2} - \frac{e}{m}\mathbf{L}\cdot\mu_{\mathbf{p}}\right]. \quad (2.21)$$

The energy correction can again be determined by perturbation theory, where for spherical symmetric problems the expectation value of the term between the square brackets is 0, so the energy correction for the ground state is

$$E^1_{hf} = -\frac{4\pi g_p e^2}{3m_e m_p} < \psi^0_{100} \mid \mathbf{S_e}\cdot\mathbf{S_p}\delta\left(r\right) \mid \psi^0_{100} > . \quad (2.22)$$

Here we have again to express $\mathbf{S_e}\cdot\mathbf{S_p}$ by their eigenvalues, so the total spin $\mathbf{S}^2 = \left(\mathbf{S_e}+\mathbf{S_p}\right)^2 = \mathbf{S}^2_{\mathbf{e}} + 2\mathbf{S_e}\cdot\mathbf{S_p} + \mathbf{S}^2_{\mathbf{p}}$ and the eigenvalue of $\mathbf{S_e}\cdot\mathbf{S_p}$ is $\mathbf{S_e}\cdot\mathbf{S_p} = \frac{1}{2}\left(\mathbf{S}^2 - \mathbf{S}^2_{\mathbf{e}} - \mathbf{S}^2_{\mathbf{p}}\right) = \frac{1}{2}\left[s\left(s+1\right) - s_e\left(s_e+1\right) - s_p\left(s_p+1\right)\right]$, where $s_e = s_p = \frac{1}{2}$ and the total spin s can be $s = \begin{cases}1\\0\end{cases}$, which leads to an energy difference between these two possible states of

$$\Delta E_{hf} = -\frac{4\pi g_p e^2}{3m_e m_p} < \psi^0_{100} \mid \delta\left(r\right) \mid \psi^0_{100} > . \quad (2.23)$$

The delta function gives us the solution of the integrand at zero, which leads to a $\frac{m_e^2}{m_p}\alpha^4$-dependent energy correction[4]

$$\Delta E_{hf} = -\frac{4 g_p m_e^2 \alpha^4}{3m_p}. \quad (2.24)$$

For a non spherical term with $l > 0$ the tensor and $\mathbf{L}\cdot\mathbf{S}$ term cause also an energy splitting with can be written in perturbation theory by

$$\Delta E_T = -\frac{e^2}{4\pi m_e^2} < \psi^0 \mid \frac{1}{r^3}\left[S_{12} + \mathbf{L}\cdot\mathbf{S}\right] \mid \psi^0 >, \quad (2.25)$$

where $S_{12} = \frac{3}{r^2}(\mathbf{r} \cdot \mathbf{S}_{\mathrm{e}})(\mathbf{r} \cdot \mathbf{S}_{\mathrm{p}}) - \mathbf{S}_e \cdot \mathbf{S}_p$. Inserting the expectation value $< \psi^0 \mid \frac{1}{r^3} \mid \psi^0 >= \frac{1}{n^3 r_B^3 l \left(l+\frac{1}{2}\right)(l+1)}$ and the equation for the eigenvalues $\mathbf{S} \cdot \mathbf{L} = \frac{1}{2}\left(\mathbf{J^2} - \mathbf{L^2} - \mathbf{S^2}\right) = \frac{1}{2}\left(j\left(j+1\right) - l\left(l+1\right) - s\left(s+1\right)\right)$ it writes finally

$$\Delta E_T = -\frac{m_e \alpha^4}{n^3 l \left(l+\frac{1}{2}\right)\left(l+1\right)} \left(< S_{12} > + \frac{1}{2}\left(j\left(j+1\right) - l\left(l+1\right) - s\left(s+1\right)\right)\right) \tag{2.26}$$

with $< \frac{1}{2} S_{12} >= \begin{cases} -\frac{l+1}{2l-1}, & j = l-1 \\ 1, & j = l \\ -\frac{l}{2l+3}, & j = l+1 \end{cases}$ for $l > 0$ and $s = 1$[9].

Chapter 3

Quarkonia

3.1 Static potential

Quarkonia are pairs of a heavy quark and a heavy antiquark of the same flavour, which are bound by the strong interaction. Here we can see an analogy to the hydrogen atom, where we have the proton and electron, which are bound by the electromagnetic interaction. But to write down the potential we have to think about the differences between the interactions[5]. The first difference is that the quarks and antiquarks have the same and much bigger mass then the electron and proton, so they have a different reduced mass. For the heavier quarks it is $m_c \approx 1.29^{+0.05}_{-0.11}$GeV, $m_b \approx 4.67^{+0.18}_{-0.06}$GeV and $m_t \approx 172.9\pm0.6 \pm 0.9$GeV [15]. The second difference is that the coupling constant α_s is much more energy dependent than α_{em} for hydrogen. Where $\alpha_{em}(0) \approx \frac{1}{137}$ increases only about 5% between $Q = 0$ and $Q = 30$GeV [6], $\alpha_s(Q) = \frac{12\pi}{(33-2n_f)\ln\left(\frac{Q^2}{\Lambda^2_{QCD}}\right)}$ is much bigger and diverging for small Q, where Λ_{QCD} is the QCD scale and n_f is the number of light quarks [5]. But there are also important analogies between quarkonia and hydrogen like they both are non relativistic systems, which means they have the same hierarchy of scales $m \gg m\alpha \gg m\alpha^2$, where the coupling constant α for hydrogen can be substituted with the velocity v of the quark for quarkonia and the mass m is the electron or quark mass[16]. Due to the non relativistic behavior they can be both described by the Schrödinger equation. From this quarkonia can only be treated with perturbation theory, when the energy scale $m\alpha_s^2 \sim mv^2$ is much bigger then Λ_{QCD}, which makes $\alpha_s < 1$, where a typical value for $\alpha_s \approx 0.15 - 0.25$ [5]. Also the coupling constant is now affected with an extra factor $\alpha \to C_F\alpha_s$ the so called colour factor C_F , which changes with the number of colours in the theory. The colour factor is $C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$ for quarks, where N_c is the number of colours, which is 3 [6, 5]. At last there is the phenomenon that quarks are confined, which mean they can not be detected as free particles like the electrons, which results in a linear attracting part in the potential additionally to the Coulombic part, so the quarkonium potential reads [5]

$$V_Q\left(r\right) = -C_F\frac{\alpha_s(r)}{r} + \sigma r, \tag{3.1}$$

where $\sigma \approx 1\frac{GeV}{fm}$ is the string tension which keeps the quarks confined. For the typical radius of the ground states, which is for the J/ψ $r_b \approx 0.2fm$ and also smaller for heavier quarks, the Coulombic term dominates and the potential is given by the pure Coulombic term, so that the energy is calculated the same way like for hydrogen and we get[5]

$$E_{Q,n} = -\frac{\mu\left(C_F\alpha_s(r_b)\right)^2}{2n^2} \tag{3.2}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass which is for quarkonia $\mu = \frac{m_Q}{2}$.

3.2 One loop-correction

The energy derived from the static potential for quarkonia looks similar to hydrogen and has the order $\mu\alpha^2$. For hydrogen the next corrections are of the order $\mu\alpha^4$ due to the fact, that the components of hydrogen are the lightest particles and so are stable. But for quarkonia there are also contributions of $\mu\alpha^3$, which results from the fact, that lighter quarks then the bound ones exist, so they are not stable and loops of lighter quarks are possible. In the same way gluons can couple to each other and also form loops, which in both cases leads to additional terms in the potential [5]. The full $\alpha_s(r)$ and the additional potential ΔV for the one loop correction read[12, 13, 8]

$$\alpha_s\left(r\right) = \alpha_s(r_b)\left(1 + \frac{\alpha_s(r_b)}{4\pi}\left(C_0 + C_1\ln(\frac{r}{r_b})\right)\right) \tag{3.3}$$

$$\Delta V(r) = -C_F\frac{\alpha_s^2(r_b)}{4\pi r}\left(C_0 + C_1\ln(\frac{r}{r_b})\right) \tag{3.4}$$

with the constants

$$\begin{aligned} C_0 &= \frac{31}{9}C_A - \frac{20}{9}T_f n_f + 2\beta_0\gamma_E \\ C_1 &= 2\beta_0 \\ \beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T_f n_f, \end{aligned}$$

where now $r_b = \frac{1}{C_f\alpha_s\mu}$ is the Bohr radius of quarkonium , $T_f = \frac{1}{2}$, $C_A = N_c$ is the Casimir operator of the adjoint representation which is equal to the number of colours , n_f is the number of light fermion flavours and $\gamma_E \approx 0.577216$ is the Euler constant. The energy calculated by first order perturbation theory reads

$$\Delta E_1 =< \psi^0 \mid -C_F \frac{\alpha_s^2(r_b)}{4\pi r}\left(C_0 + C_1 \ln(\frac{r}{r_b})\right) \mid \psi^0 >=$$
$$- C_F \frac{\alpha_s^2(r_b)}{4\pi} C_0 < \psi^0 \mid \frac{1}{r} \mid \psi^0 > -C_F \frac{\alpha_s^2(r_b) C_1}{4\pi} \ln\left(\frac{1}{\nu r_b}\right) < \psi^0 \mid \frac{1}{r} \mid \psi^0 >$$
$$- C_F \frac{\alpha_s^2(r_b) C_1}{4\pi} < \psi^0 \mid \frac{\ln(\nu r)}{r} \mid \psi^0 >= \Delta E_1^{(1)} + \Delta E_1^{(2)} + \Delta E_1^{(3)}, \quad (3.5)$$

where ν is a cut off parameter. The first two expectation values can be solved easily with $< \frac{1}{r} >= \frac{1}{r_b n^2}$, so that

$$\Delta E_1^{(1)} = -C_F \frac{\alpha_s^2(r_b)}{4\pi} C_0 < \psi^0 \mid \frac{1}{r} \mid \psi^0 >= -C_F \frac{\alpha_s^2(r_b)}{4\pi} \frac{C_0}{r_b n^2} = E_{Q,n} \frac{\alpha_s(r_b) C_0}{2\pi} \quad (3.6)$$

$$\Delta E_1^{(2)} = -C_F \frac{\alpha_s^2(r_b) C_1}{4\pi} \ln\left(\frac{1}{\nu r_b}\right) < \psi^0 \mid \frac{1}{r} \mid \psi^0 >= -C_F \frac{\alpha_s^2(r_b) C_1}{4\pi} \frac{\ln\left(\frac{1}{\nu r_b}\right)}{r_b n^2} =$$
$$E_{Q,n} \frac{\alpha_s(r_b) C_1}{2\pi} \ln\left(\frac{1}{\nu r_b}\right). \quad (3.7)$$

The third expectation value with the $\frac{\ln(r)}{r}$ dependence has to be solved explicitly with the hydrogen wave function (5.10), so it reads

$$\Delta E_1^{(3)} = -C_F \frac{\alpha_s^2(r_b) C_1}{4\pi} < \psi^0 \mid \frac{\ln(\nu r)}{r} \mid \psi^0 >= -C_F \frac{\alpha_s^2(r_b) C_1}{4\pi}$$
$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \sqrt{\frac{(2k)^3 (n-l-1)!}{2n\left[(n+l)!\right]}} e^{-kr} (2kr)^l L_{n-l-1}^{2l+1}(2kr) Y_l^{m*}(\vartheta, \varphi) \frac{\ln(\nu r)}{r}$$
$$\sqrt{\frac{(2k)^3 (n-l-1)!}{2n\left[(n+l)!\right]}} e^{-kr} (2kr)^l L_{n-l-1}^{2l+1}(2kr) Y_l^m(\vartheta, \varphi) r^2 \sin(\vartheta) dr d\vartheta d\varphi \quad (3.8)$$

This can be simplified with the orthogonal relation of the spherical harmonics[1]

$$\int_0^\pi \sin(\vartheta) d\vartheta \int_0^{2\pi} d\varphi Y_{l'}^{m'*}(\vartheta, \varphi) Y_l^m(\vartheta, \varphi) = \delta_{l,l'} \delta_{m,m'} \quad (3.9)$$

so it writes

$$\Delta E_1^{(3)} = -C_F \frac{\alpha_s^2(r_b) C_1}{4\pi} \int_0^\infty r \ln(\nu r) \frac{(2k)^3 (n-l-1)!}{2n\left[(n+l)!\right]} e^{-2kr} (2kr)^{2l} \left[L_{n-l-1}^{2l+1}(2kr)\right]^2 dr. \quad (3.10)$$

Here a substitution with $R = 2kr$ and $dR = 2kdr$ and the Euler Γ-gamma function[17] cleans the notation a bit up, so that

$$\Delta E_1^{(3)} = -C_F \frac{\alpha_s^2(r_b) C_1}{4\pi n} \frac{\Gamma(n-l)}{\Gamma(n+l+1)} \int\limits_0^\infty \ln(\frac{\nu R}{2k}) k e^{-R} R^{2l+1} \left[L_{n-l-1}^{2l+1}(R)\right]^2 dR \tag{3.11}$$

where like for hydrogen $k = \sqrt{-2\mu E} = \frac{\mu C_F \alpha_s(r_b)}{n} = \frac{1}{nr_b}$. To solve this integral the identity[8]

$$\int\limits_0^\infty e^{-R} R^n \left[L_m^n(R)\right]^2 dR = \frac{\Gamma(m+n+1)}{\Gamma(m+1)} \tag{3.12}$$

can be used and the $\ln(R)$ can simply be transformed by introducing an auxiliary parameter ε, so that $\ln(R) R^{2l+1} = \frac{d}{d\varepsilon} R^{2l+1+\varepsilon} \mid_{\varepsilon=0}$. After substituting with eq.(3.2) the energy reads

$$\begin{aligned}\Delta E_1^{(3)} &= E_{Q,n} \frac{\alpha_s(r_b) C_1}{2\pi} \frac{\Gamma(n-l)}{\Gamma(n+l+1)} \\ &\left[\left(\ln\left(\frac{\nu n r_b}{2}\right) + \frac{d}{d\varepsilon}\right) \int\limits_0^\infty e^{-R} R^{2l+1+\varepsilon} \left[L_{n-l-1}^{2l+1+\varepsilon}(R)\right]^2 dR\right]\Bigg|_{\varepsilon=0} = \\ &E_{Q,n} \frac{\alpha_s(r_b) C_1}{2\pi} \frac{\Gamma(n-l)}{\Gamma(n+l+1)} \left[\left(\ln\left(\frac{\nu n r_b}{2}\right) + \frac{d}{d\varepsilon}\right) \frac{\Gamma(n+l+1+\varepsilon)}{\Gamma(n-l)}\right]\Bigg|_{\varepsilon=0} = \\ &E_{Q,n} \frac{\alpha_s(r_b) C_1}{2\pi} \left[\ln\left(\frac{\nu n r_b}{2}\right) + \Psi(n+l+1)\right]\end{aligned} \tag{3.13}$$

where $\Psi(x) = \frac{d}{dx} \ln(\Gamma(x)) = \frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)}$ is the polygamma function[18]. Finally the expression for the whole energy difference is given by[9]

$$\Delta E_1 = E_{Q,n} \frac{\alpha_s(r_b)}{2\pi} \left[C_0 + C_1 \left(\ln\left(\frac{n}{2}\right) + \Psi(n+l+1)\right)\right]. \tag{3.14}$$

3.3 Corrections of $\mu\alpha^4$-order

Like in the hydrogen case for quarkonium also the same higher order corrections in the potential apply, which can be obtained through a non relativistic effective field theory of QCD, that provides a definition of the potential. Some of these terms in the potential, which have appeared for hydrogen before and will be needed for later calculations , read

$$\begin{aligned}\Delta^{(2)} V(r) = \frac{C_F \alpha_s(r_b)}{m^2} \Bigg[&-\frac{1}{2}\left\{\frac{1}{r}, \mathbf{p}^2\right\} + \frac{\mathbf{L}^2}{2r^3} + \pi\delta^{(3)}(r) + \\ &\frac{4\pi}{3}\mathbf{S}^2 \delta^{(3)}(r) + \frac{3}{2r^3}\mathbf{L}\cdot\mathbf{S} + \frac{1}{4r^3} S_{12}(\hat{r})\Bigg]\end{aligned} \tag{3.15}$$

with $S_{12}(r) = 3\left(\hat{r}\cdot\sigma_1\right)\left(\hat{r}\cdot\sigma_2\right) - \sigma_1\cdot\sigma_2$ and $\mathbf{S} = \frac{\sigma_1}{2} + \frac{\sigma_2}{2}$, where the matching coefficients are only considered to the $\mu\alpha^4$-order [11]. The energy corrections due to hyperfine, tensor and $\mathbf{L}\cdot\mathbf{S}$ coupling are calculated equally to the hydrogen case, so that

$$\Delta E_{hf,T} = \frac{C_F\alpha_s(r_b)}{m^2} < \psi \mid \frac{4\pi}{3}\mathbf{S}^2\delta^{(3)}(r) + \frac{3}{2r^3}\mathbf{L}\cdot\mathbf{S} + \frac{1}{4r^3}S_{12}(\hat{r}) \mid \psi >=$$
$$\frac{\mu\left(C_F\alpha_s(r_b)\right)^4}{3}\delta_{l0} + \frac{\mu\left(C_F\alpha_s(r_b)\right)^4}{16n^3 l\left(l+\frac{1}{2}\right)\left(l+1\right)}\left[3\left(j\left(j+1\right) - l\left(l+1\right) - 2\right) + < S_{12}(\hat{r}) >\right] \tag{3.16}$$

with $< \frac{1}{2}S_{12} >= \begin{cases} -\frac{l+1}{2l-1}, & j = l-1 \\ 1, & j = l \\ -\frac{l}{2l+3}, & j = l+1 \end{cases}$ for $l > 0$ and $s = 1$[9].

Chapter 4

Quirkonia

4.1 Bound state energies of quirkonia

The topic of this thesis is the calculation of the energy levels of quirkonia, which are bound states of weakly interacting massive particles named quirks. They are a dark matter candidate, which was proposed in [7]. The simplest model for a bound state of quirks consists of two quirk flavours u and d transforming under a $SU(2)_Q$ group, which is the so called quirkcolour [7]. They also form a weak isospin doublet, where for$u$ the third component of the isospin I has $I_3 = \frac{1}{2}$ and d has $I_3 = -\frac{1}{2}$ as isospin. They carry the hyper charge $U(1)_Y$, which is $\frac{1}{2}$ for u and $-\frac{1}{2}$ for d, and a quirky baryon number $U(1)_{QBN}$, that is $\frac{1}{2}$ for both u and d [7]. So a suitable choice for a quirk bound state, which can also be a dark matter candidate, is the bound state of a u and d quirk. This is, due to the quirkcolour and baryon number, a baryonic state, which is stable and electric neutral, but has a small electric dipole element, which is small enough to not have an effect to rule it out as dark matter[7]. Another reason for this choice is, that the considered ground state should be scalar, where a mesonic ground state, e. g. $u\bar{u}$, would be pseudoscalar[7]. The mass $m_{u,d}$ of the quirks, which is generated through the Higgs mechanism, is in the range of $100 - 500$GeV [7].

The calculation of the energy of the bound states will be done perturbative like for quarkonia. This can be done for quirkonia due to the hierarchy of the scales $m_q \gg m_q v \gg m_q v^2$ with the quirk mass m_q and the coupling $\alpha_Q \sim v$ similar to the quirk velocity v in a non relativistic bound state. So for quirkonia $\alpha_Q \ll 1$ is small and calculations can be done perturbative [7]. So the only difference to quarkonia is, that the coupling constant for quirkcolour changes the colour factor due to the different number of colours. For obtaining the energy levels of the quirk bound state, similar to quarkonia, we have to start with the quirkcolour static potential[7]

$$V(r) = -\frac{\bar{\alpha}(r)}{r}, \tag{4.1}$$

where $\bar{\alpha}(r) = C_F \alpha_Q(r) = \frac{3}{4}\alpha_Q(r)$ with a colour factor C_F for two colours and a perturbative but large coupling constant $\alpha_Q(r) = \alpha_Q(r_b)$ at the Bohr radius r_b. The potential can be inserted in the Schrödinger equation and can be solved. So we obtain the same wave function like for hydrogen and the binding energies for the static potential, which read[7]

$$E_0 = -\frac{\bar{\alpha}^2(r_b)\mu}{2n^2}, \tag{4.2}$$

where $\mu = \frac{m_u m_d}{m_u + m_d}$ is the reduced mass of the bound state and the Bohr radius $r_b = \frac{1}{\mu\bar{\alpha}(r_b)}$. Next the energy levels for the one-loop potential can be calculated. Therefore we write down the potential of the one-loop correction potential ΔV and the coupling constant $\alpha_Q(r)$, like shown in the quarkonia part, with the new values of the constants C_0 and C_1 for quirkonia. The expressions read than[7, 12, 13, 8]

$$\alpha_Q(r) = \alpha_Q(r_b)\left\{1 + \frac{\alpha_Q(r_b)}{4\pi}\left[\left(\frac{62}{9} + \frac{44}{3}\gamma_E\right) + \frac{44}{3}\ln(\frac{r}{r_b})\right]\right\} \tag{4.3}$$

$$\Delta V(r) = -C_F\frac{\alpha_s^2(r_b)}{4\pi r}\left(\left(\frac{62}{9} + \frac{44}{3}\gamma_E\right) + \frac{44}{3}\ln(\frac{r}{r_b})\right), \tag{4.4}$$

where the values $C_1 = \frac{44}{3}$ and $C_0 = \frac{62}{9} + \frac{44}{3}\gamma_E$ are obtained, when the number of colours N_c is 2, resulting from the $SU(2)_Q$ quirkcolour, and the number of light quirks n_f is 0, which is reasonable for a model, where there are no light quirks. This potential can be evaluated now and is written with a factor k_{nl} compared to the unperturbed pure Coulombic state like [7]

$$E_{nl} = -k_{nl}\frac{\bar{\alpha}^2(r_b)\mu}{2n^2}. \tag{4.5}$$

First the energy levels are obtained by an analytical approach, where the energy for the potential ΔV has been calculated by first order perturbation theory, like for quarkonia, which leads to the energy difference[8, 9]

$$\Delta E_1 = E_0\frac{\alpha_Q(r_b)}{2\pi}\left[\left(\frac{62}{9} + \frac{44}{3}\gamma_E\right) + \frac{44}{3}\left(\ln\left(\frac{n}{2}\right) + \Psi(n+l+1)\right)\right] \tag{4.6}$$

and the total energy

$$E = E_0\left\{1 + \frac{\alpha_Q\left(r_b\right)}{2\pi}\left[\left(\frac{62}{9} + \frac{44}{3}\gamma_E\right) + \frac{44}{3}\left(\ln\left(\frac{n}{2}\right) + \Psi(n+l+1)\right)\right]\right\} = E_0 k_{nl}. \tag{4.7}$$

The only value in the energy we still do not know is the value of the coupling constant $\bar{\alpha}$, that is unknown, but there is the constraint to be small enough to be perturbative. With this condition the results of the analytical calculation are shown in table(4.1), which gives the values of k_{nl} for some values of $\bar{\alpha}(r_b) = (0.1, 0.2, 0.3, 0.4)$.

$\bar{\alpha}$	k_{10}	k_{20}	k_{21}	k_{30}	k_{31}	k_{32}
0.1	1.24	1.61	1.72	1.84	1.92	1.98
0.2	1.48	2.23	2.43	2.69	2.84	2.97
0.3	1.73	2.84	3.15	3.53	3.76	3.95
0.4	1.97	3.45	3.87	4.37	4.68	4.93

Table 4.1: analytical energy coefficient k_{nl} for the full potential

To check the results, the Schrödinger equation with the perturbative potential can also be solved exactly by a numerical calculation. For the same values of $\bar{\alpha}(r_b) = (0.1, 0.2, 0.3, 0.4)$ and $n = (1, 2, 3)$ following numerical values have been obtained and a diagram of the energy splitting is given in figure(4.1).

$\bar{\alpha}$	k_{10}	k_{20}	k_{21}	k_{30}	k_{31}	k_{32}
0.1	1.25	1.63	1.76	1.91	2.00	2.08
0.2	1.50	2.30	2.57	2.90	3.14	3.32
0.3	1.75	3.00	3.42	3.98	4.39	4.69
0.4	2.01	3.72	4.32	5.12	5.71	6.16

Table 4.2: numerical energy coefficient k_{nl} for the full potential

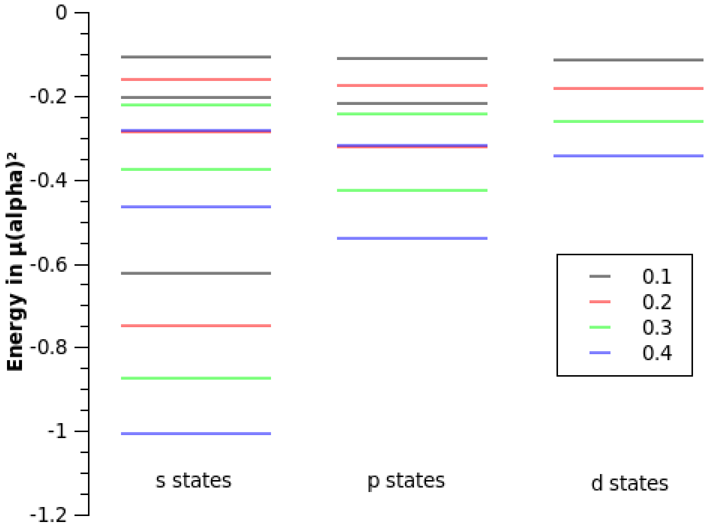

Figure 4.1: Here the energy splitting of the quirkonium states due to the one loop potential is shown in terms of $\mu\alpha^2$. The splittings are shown in different colours for the values of alpha and the lowest three s states, the lowest two p states and the lowest d state.

Here we can see that the numerical and analytical values for the bound states with a lower $\bar{\alpha}$ are very close together, while the two results differ for bigger values. This results from the way the analytical values are calculated, where only first order perturbations theory was used. So the energy values for higher values of $\bar{\alpha}$, which are not so perturbative anymore, have to be calculated with higher orders for better coincidence with the exact values.

In general the binding energies increase for a larger coupling constant $\bar{\alpha}$, where the excited states are more affected by the change of the value of the coupling constant. So the energy of the ground state increases only about 60%, where the energy of the excited state E_{32} increases nearly about its double value, when the coupling constant $\bar{\alpha}$ is increased from 0.1 to 0.4. This is a result of the bigger contribution of the perturbation to the whole energy for excited states. The ground state has a smaller energy correction term, so the term grows slower with the coupling constant. For the bigger corrections of excited states this terms grow faster and the whole binding energy increases also faster.

Second the binding energies of states with a higher azimuthal quantum number l are always slightly higher for a fixed principal quantum number n. In the analytical approach we can see, that this l dependence comes from the logarithmic potential in the perturbation, where the Coulombic potential only depends on n.

4.2 Bound state energies of quirkonia with different conditions

Additionally to the calculation with the full potential for quirk bound states in the chapter before now calculations are presented with a different potential considered in [7]. The coupling constant reads there[7]

$$\alpha_Q(r) = \alpha_Q(r_b)\left(1 + \frac{\alpha_Q(r_b)}{3\pi}(11 - n_f)\ln(\frac{r}{r_b})\right), \tag{4.8}$$

where they have ignored the $C_0 = 0$ term and just considered the term $C_1 = \frac{4}{3}(11 - n_f)$[12, 13]. Also they did not fix the number of light flavours n_f at 0, but considered it is 2[7]. The potential for this coupling constant can also be treated with first order perturbation theory, so analytical values for the binding energies are obtained by the formula[9]

$$E = E_{nl}\left\{1 + \frac{\alpha_Q\left(r_b\right)}{3\pi}\left(22 - 2n_f\right)\left(\ln\left(\frac{n}{2}\right) + \Psi(n + l + 1)\right)\right\} = E_{nl}k_{nl}. \tag{4.9}$$

Again an exact numerical solution of the Schrödinger equation has been done to compare with the results of the first order perturbation theory. The energy will always be given by the value of k_{nl} for the strength of the binding energy. The energy values have been obtained for $n = (1, 2, 3)$, $l = (0, 1, 2)$, $\bar{\alpha}(r_b) = (0.1, 0.2, 0.3, 0.4)$ and $n_f = 2$, which are presented in the tables(4.3),(4.4).

$\bar{\alpha}$	k_{10}	k_{20}	k_{21}	k_{30}	k_{31}	k_{32}
0.1	0.93	1.23	1.32	1.42	1.49	1.54
0.2	0.86	1.47	1.64	1.85	1.97	2.08
0.3	0.79	1.70	1.96	2.27	2.46	2.61
0.4	0.72	1.94	2.28	2.69	2.95	3.15

Table 4.3: analytical energy coefficient k_{nl} for $n_f = 2$

$\bar{\alpha}$	k_{10}	k_{20}	k_{21}	k_{30}	k_{31}	k_{32}
0.1	0.96	1.24	1.33	1.43	1.49	1.56
0.2	0.95	1.48	1.65	1.87	2.02	2.15
0.3	0.97	1.72	1.97	2.31	2.56	2.75
0.4	1.01	1.97	2.30	2.77	3.12	3.38

Table 4.4: numerical energy coefficient k_{nl} for $n_f = 2$

Additional to the calculation done in [7] here we consider also a only running potential without light flavours for the reason, that the bound state quirks are already the lightest quirks. So the values of the factor k_{nl} have been calculated again with the values for $\bar{\alpha}(r_b) = (0.1, 0.2, 0.3, 0.4)$ and the number of light fermion flavours $n_f = 0$ and presented in tables(4.5),(4.6).

$\bar{\alpha}$	k_{10}	k_{20}	k_{21}	k_{30}	k_{31}	k_{32}
0.1	0.92	1.29	1.39	1.52	1.60	1.66
0.2	0.83	1.57	1.78	2.03	2.19	2.31
0.3	0.75	1.86	2.17	2.55	2.79	2.97
0.4	0.66	2.15	2.56	3.07	3.38	3.63

Table 4.5: analytical energy coefficient k_{nl} for $n_f = 0$

$\bar{\alpha}$	k_{10}	k_{20}	k_{21}	k_{30}	k_{31}	k_{32}
0.1	0.95	1.29	1.39	1.53	1.62	1.69
0.2	0.96	1.58	1.79	2.06	2.26	2.41
0.3	1.00	1.88	2.19	2.62	2.93	3.17
0.4	1.05	2.19	2.60	3.19	3.62	3.95

Table 4.6: numerical energy coefficient k_{nl} for $n_f = 0$

Here can be seen that the ground state of quirkonium is less bound than its analog. This is a result of the repelling logarithm term in the perturbation, which shortens the depth of the potential, so that such deep energies can not be reached. There is also a difference in the results between the numerical and the analytical calculations, so that the binding in the numerical approach gets slightly stronger with an increasing coupling constant, but in the analytical way it keeps falling constantly. This is caused by the analytical formula of the first order perturbation theory, where we get for fixed numbers n and l a constant term times the coupling constant for the perturbation. In the ground state we simply do not have a behavior, that can be approximated linear. Here we have to use higher order perturbation theory to obtain more exact values. For the most perturbative coupling constant $\bar{\alpha} = 0.1$ we get analytical results, that fit to the numerical solutions very well, so we can trust at least these results. For the excited states the results of both calculations are always very close together, so there we have a nearly linear behavior and the values make sense.

The general behavior of the binding energy for these types of quirkonia with only the running term in the potential is similar to the one considered in the previous chapter. So the binding energies also increase for larger coupling constants $\bar{\alpha}$ faster for excited states and the binding energies for states with fixed n increase with larger l. The only difference is the strength of the binding in the different models. Where the strength increases with a larger constant term in the coupling constant, that makes a larger difference to the pure Coulombic binding energies. The constant in the model considered in the previous chapter has the additional factor C_0, which makes it much larger so we get the largest values for the binding energy. In the models of this chapter C_0 is considered to be 0 and the only difference in the potential comes from the number of light quirks considered. When we consider no light quirks the potential is larger, so the binding energy is larger. We see this behavior also in the results.

4.3 Transition lines of quirks

After the energy levels have been calculated, which are shown in figure (4.2), the energy of transitions between two states can also be easily calculated. In the following calculations only the quirkonium type with the full potential from chapter 4.1 is considered. At first the energy of the Lyman transitions, which are transitions between an excited p-state and the ground state, are given by the energy difference[7]

$$E_L = E_{n1} - E_{10} = \left(\frac{k_{10}}{2} - \frac{k_{n1}}{2n^2} \right) \bar{\alpha}^2 \mu, \tag{4.10}$$

where some values for $\left(\frac{k_{10}}{2} - \frac{k_{n1}}{2n^2} \right)$ are given in Table (4.7).

$\bar{\alpha}$	k_{21}	k_{31}
0.1	0.41	0.51
0.2	0.43	0.58
0.3	0.45	0.63
0.4	0.47	0.69

Table 4.7: values for $\left(\frac{k_{10}}{2} - \frac{k_{n1}}{2n^2}\right)$

The values here don't depend much on the coupling constant, so that the transition energy is merely $\bar{\alpha}^2\mu$-dependent.

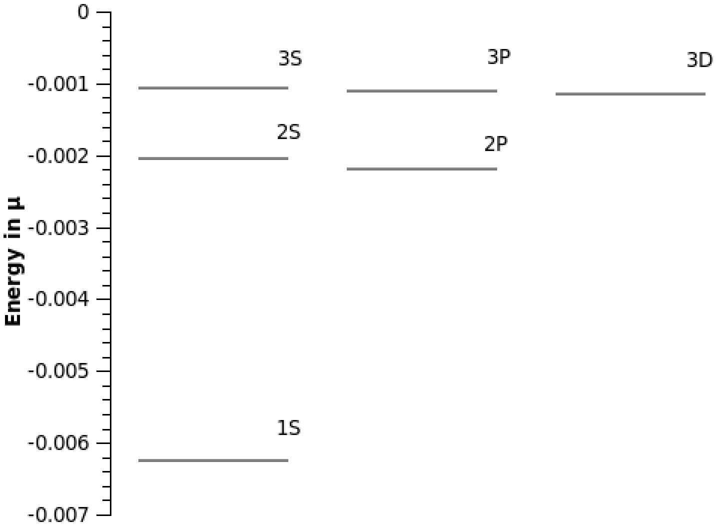

Figure 4.2: Here the energy splitting of quirkonium with the coupling constant $\bar{\alpha} = 0.1$ is shown in units of the reduced mass μ for some well-chosen states. The notation of the states is nL , where n is the principal quantum number and L is the azimuthal quantum number.

The width of the Lyman transitions is given by[7]

$$\Gamma_L = \frac{4}{9} q^2 \alpha_{em} E_L^3 \mid < R_{n1} \mid r \mid R_{10} > \mid^2, \tag{4.11}$$

where the charge $q = \frac{1}{2}$ for quirks and numerical calculated values for the dipole matrix element for quirkonia are given in Table (4.8).

$\bar{\alpha}$	$< R_{21} \mid r \mid R_{10} > /r_b$	$< R_{31} \mid r \mid R_{10} > /r_b$
pure Coulombic	1.29	0.52
0.1	1.45	0.59
0.2	1.51	0.52
0.3	1.56	0.48
0.4	1.59	0.43

Table 4.8: In this table the radial part of the dipole matrix element between the ground state and the lowest p states is shown in terms of r_b for different values of $\bar{\alpha}$ in numerical calculation and a value for the pure Coulombic potential.

The obtained values for the dipole matrix elements of the alpha transitions are larger then in pure Coulombic approximation and increasing for bigger alpha. Where for beta transitions the value for weak coupling is larger then the Coulombic, it is decreasing and smaller for stronger coupling. This comes from the effect, that the wave functions form sharper peaks under the perturbative potential, so that the overlap of different wave functions gets bigger for wave functions with same number of nodes, but decreases for different number of nodes. In the case of very strong coupling the peaks of different wave functions can on the other hand be so sharp, that the overlap decreases, when the peaks are not very near together. This case does not apply here for a perturbative potential.

The other transition, that should be calculated, is the hyperfine transition between the ground states with total spin 1 and 0. Its energy can be obtained in the same way used for quarkonia, see eq. (3.16), with the only difference that the wave function of quirkonium differs to the pure Coulombic wave function due to it's C_0 and C_1 potential terms. This leads to a correction factor c_{nl}, so that wave function and hyperfine energy are defined by[7]

$$| \psi_{nl} (0) |^2 = c_{nl} \frac{(\bar{\alpha} (r_b) \mu)^3}{\pi} \tag{4.12}$$

$$E_{hf} = c_{10} \frac{2}{3} \bar{\alpha}^4 \mu, \tag{4.13}$$

where the values of c_{10}, which were obtained numerically, for different $\bar{\alpha}$ are in table(4.9).

$\bar{\alpha}$	$c_{1,0}$
0.1	0.71
0.2	0.46
0.3	0.33
0.4	0.25

Table 4.9: In this table the constant $c_{1,0}$, which indicates the difference between the square of the static and perturbed wave function at the origin, is given for specific $\bar{\alpha}$.

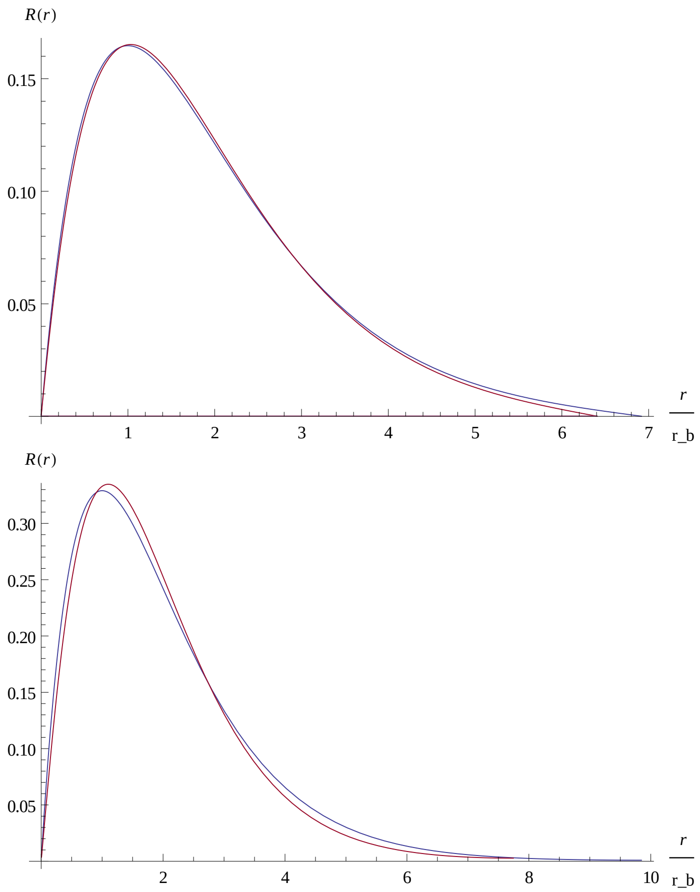

Figure 4.3: This graphs show the difference of the reduced normalized wave functions of the Coulombic (blue) and one loop potential (red), where the coupling constant in the upper picture is $\bar{\alpha} = 0.1$ and in the lower $\bar{\alpha} = 0.4$.

Comparing the results for c_{10} with figure (4.3) we see, that the reduced wave function concentrates around r_b for bigger $\bar{\alpha}$. This also applies to the wave function and gives it a different normalization factor, which decreases with higher peaks at larger values. This lower normalization factor causes lower values at the origin of the wave function. The obtained numbers also fit in their behavior with the results of [7], but they are always little smaller, which comes from the additional C_0 term considered here.

The decay rate is given by the formula[7, 14]

$$\Gamma_{hf} = \frac{4}{3} q^2 \alpha_{em} \frac{E_{hf}^3}{\mu^2} = \frac{8}{81} \alpha_{em} c_{10}^3 \bar{\alpha}^{12} \mu \tag{4.14}$$

with the quirk charge $q = \frac{1}{2}$.

4.4 Conclusion

In this thesis the energy levels of the quirkonium have been calculated, where a potential induced by a new type of strong interaction has been consider to one loop. Also the quirks, which couple to quirkonium, are considered to be the lightest quirks of the theory. To get the energy the Schrödinger equation for the pure Coulombic static potential has been solved, which is possible for the non relativistic system of quirkonia. After obtaining the unperturbed energy and wave function, the energy of the one loop potential has been solved analytically by first order perturbation theory, where the unknown coupling constant $\bar{\alpha}$ of the new interaction must be small enough to be perturbative. Additionally the Schrödinger equation for the full potential has been solved numerically to compare the obtained results and to get the accuracy of the first order perturbation theory for different $\bar{\alpha}$.

In the second part the original type of quirkonium, which has been considered in [7], has been treated the same way. This type of quirkonia differs to the type considered before only in its coupling constant. The coupling constant does only run with $\ln(\frac{r}{r_b})$, but it has no constant part. The number of light quirk flavour there has been considered to be 2, which changes also the coupling constant. So the energy values have been calculated for the original potential with 0 and 2 light quirk flavours numerically and analytically by first order perturbation theory.

In the third part some transition lines of quirkonium have been calculated only for the potential, that has been considered here. At first the transition energies and widths of Lyman transitions have been calculated, which are transitions between the ground state and excited p-states. Second the energies and decay rates of the hyperfine transition between the ground state with spin 1 and 0 have been calculated.

At last a comparison with the result of [7] will be given. The obtained binding energy values of the quirkonium type with the full potential are always larger then the values of the type without the constant. But the here obtained results for the binding energies of their type of quirkonium are in comparison with the results of [7] always significantly smaller, where here the accuracy of the calculations is higher and also analytical calculations have been performed. The Lyman transition, which was there solved only in Coulombic approximation for Lyman alpha, has been calculated here also with numerically solved dipole matrix elements. The second calculated transition was the hyperfine transition, where energy values were calculated, which are little smaller then in [7], which comes from the additional considered C_0 term, so the results are consistent.

Chapter 5

Appendix

5.1 The wave function of the hydrogen atom

The Schrödinger equation looks like[1]

$$H\psi = \left(-\frac{\hbar^2}{2m_e}\nabla^2 - \frac{\alpha}{r}\right)\psi = E\psi \tag{5.1}$$

and with $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\left(\frac{\partial^2}{\partial\varphi^2}\right)$ in spherical coordinates it changes to

$$\begin{split}-\frac{\hbar^2}{2m_e}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\left(\frac{\partial^2}{\partial\varphi^2}\right)\right]\psi \\ = \left(E + \frac{\alpha}{r}\right)\psi.\end{split} \tag{5.2}$$

The next step is to make a separation of variables, which means to split $\psi(r,\vartheta,\varphi)$ in a radial and a spherical function $\psi(r,\vartheta,\varphi) = R(r)\,Y(\vartheta,\varphi)$ and to split the whole equation into a radial and a spherical part[1]

$$\begin{split}\frac{R(r)}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)Y(\vartheta,\varphi) + \frac{R(r)}{r^2\sin^2\vartheta}\left(\frac{\partial^2}{\partial\varphi^2}\right)Y(\vartheta,\varphi) \\ = -\frac{Y(\vartheta,\varphi)}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R(r) - \frac{2m_e}{\hbar^2}\left(\frac{\alpha}{r} + E\right)R(r)\,Y(\vartheta,\varphi).\end{split} \tag{5.3}$$

Here I transferred the potential to the right side, multiplied with $-\frac{2m_e}{\hbar^2}$ and inserted $\psi(r,\vartheta,\varphi) = R(r)\,Y(\vartheta,\varphi)$. When the equation gets multiplied with $\frac{r^2}{R(r)Y(\vartheta,\varphi)}$ it can finally be split in two parts, where the first part is only dependent on r the second on (ϑ,φ)[1]

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R(r)+\frac{2m_er^2}{\hbar^2}\left(\frac{\alpha}{r}+E\right)=$$
$$\frac{-1}{Y(\vartheta,\varphi)\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)Y(\vartheta,\varphi)-\frac{1}{Y(\vartheta,\varphi)\sin^2\vartheta}\left(\frac{\partial^2}{\partial\varphi^2}\right)Y(\vartheta,\varphi)\,. \tag{5.4}$$

If I change r with keeping (ϑ,φ) constant, the right side is still constant, so the left side must also be constant for all r and vice versa, so I can split it in two parts with a constant $l(l+1)$, where l is here the azimuthal quantum number and it appears, that $l(l+1)$ is eigenvalue of the angular momentum square $\mathbf{L}^2$, like seen later in the appendix, so that[1]

$$\frac{1}{Y(\vartheta,\varphi)\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)Y(\vartheta,\varphi)+\frac{1}{Y(\vartheta,\varphi)\sin^2\vartheta}\left(\frac{\partial^2}{\partial\varphi^2}\right)Y(\vartheta,\varphi)$$
$$=-l(l+1) \tag{5.5}$$

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R(r)+\frac{2m_er^2}{\hbar^2}\left(\frac{\alpha}{r}+E\right)=l(l+1). \tag{5.6}$$

To determine the wave function $\psi(r,\vartheta,\varphi)=R(r)Y(\vartheta,\varphi)$ I solve at first $R(r)$. So we need the solution of $v(\rho)=\sum_{i=0}^{\infty}a_i\rho^i$, which is $v(\rho)=L_{n-l-1}^{2l+1}(2\rho)$ the so called Laguerre polynomial, which is defined by the formula $L_{n-m}^{m}(x)=\frac{1}{n!}\left(-1\frac{d}{dx}\right)^m e^x\left(\frac{d}{dx}\right)^n(e^{-x}x^n)$[17]. Inserting this into the earlier substitutions leads to $R_{nl}(r)=\frac{u(r)}{r}=e^{-kr}k^{l+2}r^lL_{n-l-1}^{2l+1}(2kr)$ with $k=\frac{1}{\hbar}\sqrt{-2mE}$.

To determine $Y(\vartheta,\varphi)$ we have to go back to equation (5.5) and make a separation of variables again with $Y(\vartheta,\varphi)=\Theta(\vartheta)\Phi(\varphi)$, which leads to[1]

$$\frac{\sin\vartheta}{\Theta(\vartheta)}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)\Theta(\vartheta)+\sin^2\vartheta l(l+1)=\frac{-1}{\Phi(\varphi)}\left(\frac{\partial^2}{\partial\varphi^2}\right)\Phi(\varphi). \tag{5.7}$$

This is again an equation, which is dependent on differentials of different variables on each side, so I can split it with an constant m^2, where m is the magnetic quantum number, which is an eigenvalue to the z-component of the angular momentum, like seen in the appendix[1].

$$\frac{\sin\vartheta}{\Theta(\vartheta)}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)\Theta(\vartheta)+\sin^2\vartheta l(l+1)=m^2, \tag{5.8}$$

$$\frac{1}{\Phi(\varphi)}\left(\frac{\partial^2}{\partial\varphi^2}\right)\Phi(\varphi)=-m^2. \tag{5.9}$$

The solution of equation (5.9) is simply $\Phi(\varphi)=Ae^{im\varphi}+Be^{-im\varphi}$, but with a positive or negative $\mid m \mid$ and a later done normalization it can be also be simplified to $\Phi(\varphi)=e^{im\varphi}$.

The solution of equation (5.8) is $\Theta(\vartheta) = AP_l^m(\cos\vartheta)$, where $P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|}(\frac{d}{dx})^l(x^2-1)^l$ is the so named Legendre polynomial, here we can see that the polynomial is only nonzero, if $\mid m \mid\leq l$, so the magnetic quantum number must not be bigger than the azimuthal to get a physical solution.

Finally the spherical harmonics $Y_l^m(\vartheta,\varphi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}e^{im\varphi}P_l^m(\cos\vartheta)$ are the solution of the whole angular equation and the normalized wave function of the hydrogen atom by the condition $\int \mid \psi\left(r,\vartheta,\varphi\right)\mid^2 r^2\sin\vartheta dr d\vartheta d\varphi = 1$ is[1]

$$\psi_{nlm} = \sqrt{\frac{(2k)^3\left(n-l-1\right)!}{2n\left[(n+l)!\right]}}e^{-kr}\left(2kr\right)^l L_{n-l-1}^{2l+1}\left(2kr\right)Y_l^m(\vartheta,\varphi). \tag{5.10}$$

5.2 Eigenfunctions and eigenvalues of the angular momentum

The angular momentum $\mathbf{L} = \frac{\hbar}{i}\left(\mathbf{r}\times\nabla\right)$ in quantum mechanics is an operator with eigenvalues and eigenfunctions. To find the eigenfunctions we write the operator in spherical coordinates with $\nabla = \mathbf{r}\frac{\partial}{\partial r} + \boldsymbol{\vartheta}\frac{1}{r}\frac{\partial}{\partial\vartheta} + \boldsymbol{\varphi}\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}$, so that $\mathbf{L} = \frac{\hbar}{i}\left(\boldsymbol{\varphi}\frac{\partial}{\partial\vartheta} - \boldsymbol{\vartheta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)$[1]. Here we can split the angular momentum again in parts of Cartesian coordinates with $\boldsymbol{\varphi} = -\left(\sin\varphi\right)\mathbf{x} + \left(\cos\varphi\right)\mathbf{y}$ and $\boldsymbol{\vartheta} = \left(\cos\varphi\cos\vartheta\right)\mathbf{x} + \left(\cos\vartheta\sin\varphi\right)\mathbf{y} - \left(\sin\vartheta\right)\mathbf{z}$, so that $\mathbf{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$. In equation (5.9) we have exactly this operator with the eigenfunction $\Phi\left(\varphi\right)$, which is the φ-dependent part of the spherical harmonics, with the eigenvalue $\hbar$m. So we can say the spherical harmonics are the eigenfunctions of $\mathbf{L}_z$ with eigenvalues $\hbar m$[1].

In the same way we can determine the value of the square $\mathbf{L}^2 = -\hbar^2\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right)$, which has already appeared in equation (5.5). There we had as solution the spherical harmonics as eigenfunctions and $\hbar^2 l\left(l+1\right)$ as eigenvalues. So the spherical harmonics are also eigenfunctions to $\mathbf{L}^2$ with the eigenvalues $\hbar^2 l\left(l+1\right)$[1].

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