Hybrid compactifications and brane gravity in six dimensions

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Abstract

We consider a six-dimensional axisymmetric Einstein-Maxwell model of warped braneworlds. The bulk is bounded by two branes, one of which is a conical 3-brane and the other is a 4-brane wrapped around the axis of symmetry. The latter brane is assumed to be our universe. The 3-brane folds the internal two-dimensional space in a narrow cone, making sufficiently small the Kaluza-Klein circle of the 4-brane. An arbitrary energy-momentum tensor can be accommodated on this ring-like 4-brane. We study linear perturbations sourced by matter on the brane, and show that weak gravity is apparently described by a four-dimensional scalar-tensor theory. The extra scalar degree of freedom can be interpreted as the fluctuation of the internal space volume, the effect of which turns out to be suppressed at long distances. Consequently, four-dimensional Einstein gravity is reproduced on the brane.

1 Introduction

Probably one of the most interesting recent developments in particle physics and cosmology has been the idea of braneworlds. Models with extra dimensions are motivated theoretically, as in superstring theory, which is a very promising approach to unification, requiring ten spacetime dimensions. Braneworld scenarios are further motivated by their phenomenologically interesting aspects. Among them are the possible effect of having the fundamental scale as low as the weak scale and some modification of the gravity law on submillimeter scales [1], both of which are accessible by experiments. So far five-dimensional (5D) Randall-Sundrum-type braneworlds [1] have been the most extensively studied examples, whereas more recently there has been growing interest in six- or higher dimensional models. In the present paper we will be focusing on 6D braneworlds with Maxwell fields. Since two extra dimensions are enough to admit flux-stabilized compactifications while keeping the setup as simple as possible, such brane models allow us to explore some of the interesting features which would be less easily addressed in more string theoretical settings. Perhaps the simplest exact solution of this type of warped braneworlds has been constructed in [2]. Codimension two branes are often considered in the above approaches, and they are unfortunately associated with the problem of the localization of matter. Namely, a strict codimension two defect does not allow for arbitrary energy-momentum tensor localized on it. Gravitational aspects of such higher dimensional braneworlds have not been explored thoroughly yet because of this fact. The hybrid Kaluza-Klein / Randall-Sundrum construction of [3] evades this problem by assuming that our universe is a 4-brane in six dimensions, with one of the spatial directions compactified on a circle. Refs. [4, 5, 6, 7] also exploit essentially the same idea to resolve codimension two singularities. The specific model we consider in this paper is most closely similar to that of [3], but not exactly the same. In [3] the bulk with axisymmetry closes regularly at the point where the axial Killing vector vanishes. In contrast, ours does not, permitting a conical singularity there, corresponding to a tensional 3-brane. The conical 3-brane can fold the internal 2D space in a narrow cone, yielding a small Kaluza-Klein circle of the 4-brane wrapped around the symmetry axis. (For this idea we are indebted to [8].) To study in more detail the behavior of weak gravity sourced by matter in the braneworld, we provide a rigorous treatment of metric and matter perturbations in this paper. We use the technique of [9], which was originally developed for studying linear perturbations in the Randall-Sundrum model and was developed by [4, 6] in the context of 6D brane models.

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2 The model

Our 6D bulk is described by the Einstein-Maxwell action. In our setup the bulk cosmological constant may be positive or negative or zero, and so we write $\Lambda_6 = \epsilon\frac{10}{\ell^2}$, $\epsilon = \pm 1,\ 0$. The 6D field equations derived from the above action and they admit the following bulk solution [3]:

$$g_{MN}dx^M dx^N = \xi^2 \eta_{\mu\nu}dx^\mu dx^\nu + \ell^2\left[\frac{d\xi^2}{f(\xi)} + \beta^2 f(\xi)d\theta^2\right], \tag{1}$$

where $f(\xi) := -\epsilon\xi^2 + \frac{\mu}{\xi^3} - \frac{q^2}{\xi^6}$. Only the $(\xi\theta)$ component of the field strength is nonvanishing, given by $F_{\xi\theta} = 2\sqrt{3}\frac{\beta\ell}{\kappa}\frac{q}{\xi^4}$. Let ξ_0 be the positive zero of $f(\xi)$. We consider the region in which $\xi \geq \xi_0$ and $f(\xi) \geq 0$. More specifically, ξ_0 is the largest positive zero of $f(\xi_0)$ for $\epsilon = -1$. For $\epsilon = 0$, we have $\xi_0 = (q^2/\mu)^{1/3}$. In the $\epsilon = 1$ case, ξ_0 is the second largest positive zero, and we consider the region $\xi_0 \leq \xi < \xi_1$, with ξ_1 being the largest zero. Accordingly, we have a deficit angle $\delta = 2\pi\left[1 - \beta f'(\xi_0)/2\right]$, corresponding to a conical 3-brane placed at $\xi = \xi_0$ with tension $\kappa^2\sigma = 2\pi\left[1 - \frac{\beta f'(\xi_0)}{2}\right]$. As in [3], one may impose $\beta = 2/f'(\xi_0)$, leading to the regular geometry without a 3-brane. In the present paper, however, we do not do so and allow for a conical deficit. We follow the construction of [3] and add a ring-like 4-brane at a point $\xi_* > \xi_0$, which is assumed to be our universe. The brane action is given by $S_{\rm brane} = \int d^5x\sqrt{-\gamma}\,(-\lambda + \mathcal{L}_m)$, where λ is the tension of the 4-brane and $\mathcal{L}_m$ is the matter Lagrangian. We denote by γ_{ab} the induced metric on the brane. Let $\mathcal{M}$ be the spacetime in which ξ ranges from ξ_0 to ξ_*. We impose Z_2 symmetry about ξ_*, and glue $\mathcal{M}$ and a copy of $\mathcal{M}$ together at $\xi = \xi_*$. In so doing we assume that the metric and F_{MN} are continuous across the brane.[3] The first derivative of the metric is subject to the Israel conditions. We now consider a vacuum brane. In this case the Israel conditions determines the brane position as $\xi_* = 2\left(\frac{q^2}{5\mu}\right)^{1/3}$. Since our brane model includes one Kaluza-Klein direction, we must impose that the circumference of the ring, $\mathcal{C} = 2\pi\beta\ell\sqrt{f_*}$, is not too large (say $\mathcal{C} \lesssim 10^{-16}$ cm), whereas if the scale of the "braneworld compactification" is as large as $\ell \sim 10^{-2}$ cm it will be particularly interesting. Clearly, this can be achieved by requiring $\beta\sqrt{f_*} \ll 1$. In other words, if the tension of the conical brane is fine-tuned to be very close to the critical value, $\kappa^2\sigma \simeq 2\pi$, the bulk will look like a narrow sliver with a small Kaluza-Klein circle. The required fine-tuning is $1 - \frac{\kappa^2\sigma}{2\pi} \sim \frac{\mathcal{C}}{\ell}$. We will keep using the conical brane to set the boundary of the system.

We can express the parameters μ and q^2 in terms of ξ_0 and ξ_*: $\mu = -\epsilon\frac{8\xi_0^5}{5\alpha^3-8}$, $q^2 = -\epsilon\frac{5\alpha^3\xi_0^8}{5\alpha^3-8}$, where $\alpha := \frac{\xi_*}{\xi_0}$. Note that the above expression is valid only for $\epsilon \neq 0$. Introducing the new coordinate $z := \xi/\xi_0$, we write $f = \xi_0^2\overline{f}(z)$, where $\overline{f}(z) := -\epsilon\left(z^2 + \frac{8}{5\alpha^3-8}\frac{1}{z^3} - \frac{5\alpha^3}{5\alpha^3-8}\frac{1}{z^6}\right)$. The background solution apparently depends on ξ_0, but it can be eliminated by performing an appropriate coordinate rescaling. Thus, it turns out that the background configuration in the $\epsilon \neq 0$ models is characterized by two parameters, α and the 3-brane tension σ. We see that $(1 <)\ \alpha^3 < 8/5$ for $\epsilon = +1$ and $\alpha^3 > 8/5$ for $\epsilon = -1$. If α is very close to $2/5^{1/3}$, we have a large circumference, $\mathcal{C} \propto |\alpha - 2/5^{1/3}|^{-1/2}$. Large α also tends to give a large Kaluza-Klein radius, $\mathcal{C} \propto \alpha$. Therefore, in what follows we will assume $\alpha \sim O(1)$ but not too close to $2/5^{1/3}$. The special case with $\epsilon = 0$ $(\alpha^3 = 8/5)$ should be considered separately. Since the 6D cosmological constant vanishes, the typical compactification scale is given solely by the Maxwell field: $\kappa^2F^2 = (24/\ell_0^2)z^{-8} \sim 1/\ell_0^2$, where $\ell_0 := \frac{\xi_0^4\ell}{q}$.

3 Linear perturbations

Let us now analyze linear perturbations on the brane model. We are interested in a length scale much larger than the circumference of the ring, and hence we focus on perturbations homogeneous in the θ-direction. Linear perturbations are split into scalar, vector, and tensor modes under the Lorentz group in the external spacetime. Here let us consider scalar and tensor perturbations. (Vector modes are of no particular interest.) For the transverse and traceless tensor perturbation, $h_{\mu\nu}$, the Einstein equations

[3] We impose the same boundary condition as in [3] for the Maxwell field. This is different from [4, 5, 6], in which F_{MN} is discontinuous at the 4-brane due to the Stückelberg term included in the brane action.

simply give $\left(\xi^4 f h'_{\mu\nu}\right)' + \xi^2\ell^2\Box h_{\mu\nu} = 0$, where $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu$. For the scalar perturbations, it is convenient to employ the 6D longitudinal gauge. The $(\mu\nu)$, $(\xi\xi)$, and $(\theta\theta)$ components of the Einstein equations are combined to give two basic equations as $\Omega'' + 2\left(\frac{f'}{f} + \frac{5}{\xi}\right)\Omega' + \ldots = 0$ and $\Psi'' + \frac{4}{\xi}\Psi' + \ldots = 0$. The remaining variables are obtained from $\Xi = \Psi + \Omega$ and $\delta A_\theta = \frac{\beta\ell\xi^3}{2\sqrt{3}\kappa q}\left[f\left(\xi\Omega' + 2\Omega\right) + \xi f'(\Omega + 2\Psi)\right]$. We now proceed to discuss boundary conditions. At the point where the geometry pinches off, $\xi = \xi_0$, we impose some regularity conditions on the perturbations. For the tensor mode, we require that both $h_{\mu\nu}$ and $h'_{\mu\nu}$ are regular at $\xi = \xi_0$. The regularity conditions for the scalar modes are $f\Omega|_{\xi_0} = 0$ and $(f\Omega)' + 2f'\Psi|_{\xi_0} = 0$. The perturbed field strength in the Gaussian-normal gauge must be continuous at $\xi = \xi_*$. Since we are assuming the $\mathbb{Z}_2$ symmetry across the ring, it leads to the condition $\delta A_{\theta *} + A'_{\theta *}\zeta = 0$, where the equation is written in terms of the 6D longitudinal gauge perturbations and hence includes the brane bending mode $\zeta = \zeta(x)$. In the 6D longitudinal gauge, the location of the brane is perturbed in general: $\xi_* \to \xi_* + \zeta(x)$. We now investigate the long-distance behavior of weak gravity on the 4-brane. Using the Israel condition we can put the bulk equation of motion and the boundary condition into a single equation with a source term:

$$\mathcal{O}h_{\mu\nu} := \left(\xi^4 f h'_{\mu\nu}\right)' + \xi^2\ell^2\Box h_{\mu\nu} = -\mathcal{S}_{\mu\nu}\delta(\xi - \xi_*), \tag{2}$$

where we define $\mathcal{S}_{\mu\nu} \equiv 2\ell\xi_*^2\sqrt{f_*}\kappa^2\left(T_{\mu\nu} - \frac{1}{3}T_\lambda^{\ \lambda}\gamma_{\mu\nu}\right) + 4\ell^2\xi_*^2\zeta_{,\mu\nu}$. We use the standard Green function method to solve Eq. (2). The Green function is explicitly given by $G_R = -\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \sum \frac{u_i(\xi)u_i(\xi')}{m_i^2 + \mathbf{k}^2 - (\omega + i\epsilon)^2}$, where $u_i(\xi)$ are a complete set of eigenfunctions of

$$\left(\xi^4 f u_i'\right)' = -\xi^2\ell^2 m_i^2 u_i. \tag{3}$$

The eigenfunctions must be normalized. We are mainly interested in the long-range gravity on the brane and hence the zero-mode solution of (3) is the most important. Setting $m_0^2 = 0$ and integrating once, we obtain $u_0' = \xi^{-4}f^{-1}U$, where U is an integration constant. However, from the regularity condition at $\xi = \xi_0$ we must impose $U = 0$. Therefore, the zero-mode solution is given by $u_0 = L^{-1} =$ constant. The normalization is determined as $L = \ell\sqrt{\frac{2}{3}(\xi_*^3 - \xi_0^3)}$. The zero-mode truncation of the Green function leads to $h_{\mu\nu} \approx -\frac{1}{L^2}\Box^{-1}\mathcal{S}_{\mu\nu}$. Now we would like to compute the Ricci tensor $R^{(4)}_{\mu\nu}$ of the 4D metric $\overline{g}_{\mu\nu} = \xi_*^2[(1 + 2\overline{\Psi}_*)\eta_{\mu\nu} + h_{\mu\nu}]$. Here $\overline{\Psi}_*$ is the metric perturbation in the Gaussian-normal gauge, which is related to the longitudinal gauge quantities. We write $R^{(4)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} - \frac{2\xi_*^2\ell^2}{L^2}\zeta_{,\mu\nu} - \frac{\ell^2}{L^2}\gamma_{\mu\nu}\Box\zeta - (2\partial_\mu\partial_\nu + \eta_{\mu\nu}\Box)\Upsilon$, where we defined $\Upsilon := \overline{\Psi}_* - \frac{\ell^2}{L^2}\xi_*^2\zeta$. Finally, we can evaluate

$$R^{(4)}_{\mu\nu} \approx \kappa_4^2\left(\overline{T}_{\mu\nu} - \frac{1}{2}\overline{T}_\lambda^{\ \lambda}\gamma_{\mu\nu}\right) - (2\partial_\mu\partial_\nu + \eta_{\mu\nu}\Box)\Upsilon, \tag{4}$$

where $\overline{T}_{ab} := \mathcal{C}T_{ab}$ is the energy-momentum tensor integrated along the θ-direction, and we defined the 4D Newton constant as $\kappa_4^2 := \frac{\xi_*^2\kappa^2}{2\pi L^2\beta}$. Thus, we see that the first three terms help to recover a 4D gravitational theory. However, brane gravity looks different from Einstein gravity at this stage because of the additional scalar degree of freedom encoded in Υ. It should be stressed here that the brane bending mode is crucial for reproducing the 4D tensor structure. The role of the brane bending here is the same as that of the Randall-Sundrum braneworld [9], and it has been shown that the same mechanism works in a slightly different setup of 6D braneworlds [4, 6].

Let us evaluate the effect of Υ. For $\ell^2\Box = 0$ we have the exact solutions, where four integration constants $c_1(x), \cdots$ etc. are to be determined by the boundary conditions. In general cases with $T_{ab} \neq 0$ we have nonzero integration constants. From the regularity conditions, one can express c_3 and c_4 in terms of c_1 and c_2. Then, we can write ζ in terms of c_1 and c_2. For $\epsilon \neq 0$ we find $\Upsilon = \frac{(\xi_*^3 - \xi_0^3)(\xi_*^3 + 8\xi_0^3)}{72\xi_*^3\xi_0^6}\hat{c}(x)$, where $\hat{c} := 8\xi_0^3 c_1 - c_2$. Using the Israel conditions, we finally arrive at

$$\Upsilon = \mathcal{F}(\alpha)\ell^2\ \kappa_4^2\left(\frac{1}{3}\overline{T}_\lambda^{\ \lambda} - \overline{T}_\theta^{\ \theta}\right), \tag{5}$$

where $\mathcal{F}(\alpha) := -\frac{\epsilon}{1440}\alpha^2(5\alpha^3 - 8)(\alpha^3 + 8)$. Eqs. (4) and (5) imply that the effect of Υ is suppressed on scales much greater than $\sqrt{\mathcal{F}}\ell$. For $\alpha \sim O(1)$, the coefficient $\sqrt{\mathcal{F}}$ is not large, so that the critical scale may be given by ℓ. The critical scale becomes large for $\alpha \gg 1$, but this is not the case we are considering. In the $\epsilon = 0$ case, a straightforward computation can be similarly done and the effect of Υ is negligible on scales much greater than ℓ_0. To illustrate the geometrical interpretation of the scalar mode Υ, we compute the perturbations of the internal space volume and the circumference of the brane. It then turns out that $\delta\mathcal{V} \propto \delta\mathcal{C} \propto \hat{c}$. Namely, Υ ($\propto \hat{c}$) can be interpreted as the perturbations of the internal space volume and the circumference of the ring. It is reasonable that standard 4D gravity is recovered when the matter fields on the brane do not perturb the internal space much.

So far we have seen that the zero-mode sector of perturbations can reproduce standard 4D gravity on the brane. Basically, the effect of discrete Kaluza-Klein modes are Yukawa-suppressed, and hence we can safely neglect these massive modes at long distances. In this subsection, we compute the mass spectrum of the Kaluza-Klein modes for completeness. To do so we rewrite Eq. (3) in terms of z and $\overline{f}(z)$, so that we would like to solve $\frac{d}{dz}\left[z^4\overline{f}(z)\frac{du_i}{dz}\right] + \nu_i^2 z^2 u_i = 0$, $\nu_i^2 := \frac{m_i^2\ell^2}{\xi_0^2}$, supplemented with the boundary conditions. For $\epsilon = 0$ we replace ℓ^2 in ν_i^2 by ℓ_0^2. In the case of $\epsilon = 0$ we have analytic solutions for the Kaluza-Klein mode functions. The Kaluza-Klein mass spectrum can be calculated from the boundary condition and we can find $\nu_1 \simeq 7.42$, $\nu_2 \simeq 13.6$, $\nu_3 \simeq 19.7, \cdots$. The Kaluza-Klein masses measured by an observer on the ring are $\nu_i\ell_0^{-1}(\xi_0/\xi_*) \simeq 0.855 \times \nu_i\ell_0^{-1}$. In the case of $\epsilon \neq 0$ we compute the mass spectra fully numerically. As before, the Kaluza-Klein masses measured by an observer on the ring are $\nu_i\ell^{-1}\alpha^{-1}$. We are considering the case with $\alpha \sim O(1)$, and so we have $m_i/\xi_* \gtrsim \ell^{-1}$.

4 Summary

We have considered a warped braneworld in six dimensions. The background is given by the model of [3] with a slight modification, in which our universe is assumed to be a 4-brane wrapped around the axisymmetric internal space. Since the codimension of the brane is one, this construction allows for localized matter on the brane. We have performed a linearized perturbation analysis in order to study the long-distance behavior of weak gravity sourced by arbitrary matter on the brane. We have found that there are two scalar modes, ζ and Υ, relevant to brane gravity. The first one, ζ, describes the shift of the brane position and plays an important role in recovering the tensor structure of 4D gravity. The mode Υ encodes the fluctuation of the volume of the internal space (or that of the circumference of the 4-brane) and signals a scalar-tensor theory of gravity. However, the effect of Υ was shown to be suppressed on scales greater than ℓ (or ℓ_0). Discrete Kaluza-Klein modes are Yukawa-suppressed at long distances. Thus, we have successfully obtained standard 4D gravity on the brane. The hybrid braneworld does not eliminate the hierarchy problem with relatively "large" extra dimensions, because one of the extra dimensions will be quite small compared to the other. Indeed, we find $M_{\rm Pl}^2 = (M_6^4)\ell\mathcal{C}\,\frac{2(\xi_*^3-\xi_0^3)}{3\xi_*^2\sqrt{f_*}} \sim (M_6^4)\ell\mathcal{C}$. (For $\epsilon = 0$, we find $M_{\rm Pl}^2 = 2(M_6^4)\ell_0\mathcal{C}/\sqrt{15}$.) The circumference of the ring must be $\mathcal{C} \lesssim 10^{-16}$ cm. Thus, for $\ell \lesssim 10^{-2}$ cm we get the fundamental scale $M_6 \gtrsim 10^7$ GeV.

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