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Superconformal methods in 4 and 6 dimensions

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Voorwoord

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Contents

Chapter 1

Introduction

The standard model and general relativity synthesize our present knowledge of the behavior of elementary particles in terms of four fundamental forces. In the 19th century, only the gravitational and the electro-magnetic force were known. At that time, gravity was described by Newton's law and the classical Maxwell equations enabled the explanation of electro-magnetic phenomena. From the beginning of the 20th century on, it became clear that for a proper description of the short-distance behavior of the elementary constituents of matter, quantum mechanics was needed.

By now, the standard model explains or at least accommodates all experiments done in accelerators so far. It is a renormalizable quantum field theory with gauge group $SU(3)_c \times SU(2)_L \times U(1)$. Another big achievement of the last century is a better description of the gravitational force in terms of general relativity. This geometrical description of space and time replaces Newton's law. It survives all experimental tests done nowadays. Remarkably, the gravitational force is tested only up to distances of centimeters. It is so weak in comparison to the other three forces that the classical, geometrical description is sufficient. No quantum-mechanical formulation is needed to explain the phenomena observed up to now. Even then, there remains the fundamental question for a quantum-mechanical description of the gravitational force. The last couple of years, astrophysicists have found further evidence for the existence of black holes. At the horizon of these classical singularities of spacetime, the gravitational force is much stronger and a quantum-mechanical description is required. The approach to quantize the other forces, using regularization and renormalization, cannot be used for the gravitational force. The gravitational coupling constant is dimensionful and the renormalization program is doomed to fail. This means that another way is needed to formulate a quantum-mechanical theory of gravity. Also the standard model is not fully satisfactory. Its formulation requires almost 20 arbitrary parameters, the difference in masses of the elementary particles begs for an explanation, there is no clear reason for the existence of three generations of leptons, ... The strongest argument to look for a more fundamental theory is that gravity is not included in this standard model. The quest for a unified theory of all the four forces was initiated already many years ago. For several decades, people are trying to scratch the surface of this 'theory of everything'.

1.1 Superstrings and supergravity

1.1.1 Why introduce superstrings?

In these days, there is only one viable (and promising!) candidate to solve some (all?) of the problems with the standard model and general relativity: *string theory* [1, 2]. String theory is a theory where elementary constituents of matter are no point particles any more, but tiny one-dimensional entities. The vibrational modes of these strings can be considered as the different elementary particles[1]. As particles sweep out a worldline in spacetime, these strings sweep out a two-dimensional worldsheet in a higher-dimensional spacetime. A Lorentz-invariant formulation of string theory requires 26 (indeed, twenty-six) spacetime dimensions, but even then, the lowest-energy state is a tachyon, a state with negative mass-squared, signaling that the theory is instable. There is however a way out for these inconveniences.

The introduction of *supersymmetry* allows a stable quantum-mechanical formulation of string theory in ten dimensions. Supersymmetry is a generalization of bosonic symmetries. It relates bosons and fermions, states of different spin. It is clear that at low energies supersymmetry is broken. It is not clear whether at higher energies, supersymmetry is a true, physical symmetry. For the moment, the only indication for its existence comes from the unification of the different gauge couplings at high energy (10^{16} GeV) in the minimal supersymmetric standard model. People hope to find it in the near future at LHC. Supersymmetry helps a lot in discovering properties of superstring theory and supergravity theory. There are some strong arguments in favor of superstring theory. By introducing supersymmetry, there are no tachyonic states any more in the string spectrum. Gravity is automatically included. The string spectrum contains a massless spin-2 particle,

[1]This was also the way in which string theory was introduced into physics. Before the advent of QCD, string theory was used as a model to explain the mesonic resonances.

which can be identified with the graviton[2]. Furthermore, it allows a unified description of gravity and gauge symmetry. This unification is one of the dreams of physicists: one theory that could explain all the fundamental forces. Superstring theory has two aspects that seem awkward at first sight: consistency requires ten spacetime dimensions and there are five different superstring theories. Both of these aspects will turn out to be useful later on. There is the type IIA or type IIB theory. These are theories that contain closed strings and have two spacetime supersymmetries. The heterotic string is a closed string with one supersymmetry that is consistent for the gauge groups $SO(32)$ and $E_8 \times E_8$. Finally, type I string theory describes open strings. At low energies, these superstring theories give rise to supergravity theories. The research in this thesis is situated in this low-energy domain. We study different matter multiplets in different supergravity backgrounds.

1.1.2 Supergravity: the low-energy limit of superstrings

We want to derive that supergravity is the low-energy limit of superstring theory. For that purpose, we start from the description of a superstring theory in a curved spacetime. The worldsheet action of a string is a generalization of the worldline action for a particle. The bosonic theory is an interacting two-dimensional quantum field theory that contains the massless states of string theory (e.g., the spacetime metric $G_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton Φ). These massless states are derived from an analysis in flat spacetime. The fundamental field of this theory is the position of the string $X^\mu(\sigma,\tau)$, which depends on the worldsheet coordinates σ and τ. These fields X^μ span the curved target space. Later, supersymmetric partners are introduced and the target space becomes a superspace. If the kinetic term of the fields in the action is field dependent, this is called a non-linear σ-model. If the characteristic radius of curvature of the target space is R_c, there is an effective dimensionless coupling $\alpha'^{1/2}R_c^{-1}$, where the Regge slope α' has units of spacetime-length-squared. α' is related to the string tension T in the following way: $T = \frac{1}{2\pi\alpha'}$. If R_c is much greater then the characteristic length, this coupling is small and perturbation theory is a useful tool in the two-dimensional field theory. This difference in length scales says also that the internal structure of the string becomes negligible and a description in terms of a low-energy effective field theory is useful. This low energy was already used implicitly by introducing only the massless string states in the worldsheet description. The massive string states have masses of the order of the Planck mass, $M_P \simeq 10^{19}$ GeV and this is much higher than what is (and will be) accessible in accelerators.

[2]When using string theory as a model for the strong interaction, this spin-2 particle was very annoying since there were no massless spin-2 particles there.

Consistency of string theory requires conformal or Weyl invariance of the worldsheet action. This requires that the trace of the energy-momentum tensor vanishes. This can also be derived from an effective spacetime action. In this way, superstring theory gives rise to a supergravity action at low energies. Each of the five string theories has its corresponding supergravity theory. One of the big, modern challenges is to give a microscopic description of the 11-dimensional theory that has 11-dimensional supergravity as its low-energy limit. These supergravity theories are interesting on their own too. They describe gravity and gauge theory at the same time. Even if these supergravity theories are not fundamental, microscopic theories, they reveal already a lot of interesting properties. They are some kind of bridge between the standard model and general relativity on the one side and a real 'theory of everything' on the other side. That is the reason why we are interested in studying their properties.

1.1.3 Compactification

One of the two main problems with superstring theory (and its low-energy limit supergravity) was the presence of ten spacetime dimensions. The idea to solve this problem is to compactify some dimensions. After all, string theory does not dictate that all its ten dimensions should be infinitely extended. This means that our world consists in a string-theoretic framework of four macroscopic, visible dimensions, while the six other dimensions are rolled up in a specific way. These compact dimensions are so small that we cannot see them, at least not at the energies that are at present available. Depending on the structure of the compact manifold, different amounts of supersymmetry survive in the effective theory in lower dimensions. Instead of immediately attacking the full problem of compactifying to four dimensions, people have also been studying compactifications to other dimensions, e.g., to six dimensions. Although it is clear that this has no predictive power about our universe, it contributes to a better understanding of compactifications of supergravity theories. One of the possibilities to compactify four dimensions gives chiral[3] theories in six dimensions. IIB theory on a $K3$ manifold gives rise to chiral $(2,0)$ supergravity in six dimensions. The only possibility for a matter representation in these supersymmetric theories are self-dual tensor multiplets. A self-dual tensor in six dimensions is a two-index tensor that has a real self-dual field strength. We will pay a lot of attention to models with a self-dual tensor in chapter 4 and 5.

Besides, people have very much studied compactification on Calabi–Yau man-

[3] In even dimensions, it is possible to split a spinor (and thus also the supercharges of the supersymmetry algebra) in two different chiralities using a projection. This is explained in chapter 2. $(2,0)$ supersymmetry means that there are two supersymmetries of the same chirality.

ifolds. These six-dimensional manifolds obey certain restrictions such that one quarter of the ten-dimensional supersymmetry survives in four dimensions. The Calabi–Yau compactification of the heterotic superstring gives rise to a chiral theory with one supersymmetry in four dimensions and a certain gauge group. These theories come qualitatively very close to the minimal supersymmetric standard model. The Calabi–Yau compactification of IIA or IIB string theory leads to theories with two supersymmetries in four dimensions. Up to now, nobody has been able to find a realistic superstring vacuum.

1.1.4 Non-perturbative aspects of field theory

The solution of the problem of five, seemingly-different superstring theories requires something more. Up to now, we have only explained some results that are found in a perturbative treatment of string theory, in the same way that the Feynman diagram expansion in particle physics is a perturbative approximation of the theory. Up to some years ago, we knew, in contrast to field theory, only the perturbative approximation of string theory. Only the last couple of years, we have gained some insight into aspects of string theory that go beyond the perturbative level. Still, we have not (yet) a description of the full theory. We first explain something more about non-perturbative aspects of field theory that will later be relevant in string theory. In a next section, we explain more on non-perturbative aspects of string theory. It will also be in the context of these non-perturbative aspects of field theory that we will study vector multiplets in four dimensions.

In field theory, we have point-particles with electric charge e. They are coupled to a vector field, a one-form. In four dimensions, there is also another object that couples to this form: the magnetic monopole. This magnetic monopole has magnetic charge g. The Dirac quantization condition,

$$eg = 2\pi n\hbar \,, \tag{1.1.1}$$

imposes that the product of the electric charge and the magnetic charge should be equal to a constant. This means that the electric and the magnetic charge are inversely proportional. These magnetic monopoles are present in $U(1)$ gauge theories. In non-Abelian gauge theories with a $U(1)$ subgroup appear 't Hooft–Polyakov monopoles. The presence of such topologically stable, magnetically charged particles raises the possibility for a symmetry between electricity and magnetism. There is no experimental proof of the existence of magnetic monopoles up to now. So, in the real world, with only the presence of electric charges, this electro-magnetic duality is not present. In vacuum, the Maxwell equations are invariant under the transformation

$$\vec{E} \to \vec{B}\,, \qquad \vec{B} \to -\vec{E}\,, \tag{1.1.2}$$

of the electric field $\vec{E}$ and the magnetic field $\vec{B}$. In the presence of electrically and magnetically charged particles, this would imply that their charges would be interchanged. This transformation can be generalized to rotations over an arbitrary angle

$$\begin{aligned} \vec{E} &\to \cos\theta\,\vec{E} + \sin\theta\,\vec{B}\,, \\ \vec{B} &\to -\sin\theta\,\vec{E} + \cos\theta\,\vec{B}\,. \end{aligned} \tag{1.1.3}$$

The same symmetry works also on the version of the Maxwell equations with the field equation and the Bianchi identity[4]:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0\,, \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0\,, \end{aligned} \tag{1.1.4}$$

where $F_{\mu\nu}$ is the field strength for the vector and $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. This set of equations is again invariant under rotations over an arbitrary angle:

$$\begin{pmatrix} F'^{\mu\nu} \\ \tilde{F}'^{\mu\nu} \end{pmatrix} = \mathcal{S}\begin{pmatrix} F^{\mu\nu} \\ \tilde{F}^{\mu\nu} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F^{\mu\nu} \\ \tilde{F}^{\mu\nu} \end{pmatrix}, \tag{1.1.5}$$

where $\mathcal{S} \in GL(2,\mathbb{R})$. In the presence of m Abelian vector fields, this symmetry generalizes to $GL(2m,\mathbb{R})$. The equations of motion can be derived from an action:

$$\mathcal{L}_1 = \tfrac{1}{4}(\mathrm{Im}\,\mathcal{N}_{IJ})F^I_{\mu\nu}F^{\mu\nu J} - \tfrac{\mathrm{i}}{8}(\mathrm{Re}\,\mathcal{N}_{IJ})\epsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}\,, \tag{1.1.6}$$

where $I,J = 1,...,m$, the matrix $\mathcal{N}_{IJ}$ is symmetric and $F^I_{\mu\nu}$ are the field strengths of the vector fields. In a theory with also scalars, $\mathcal{N}_{IJ}$ may depend on them. Supersymmetry imposes a very specific dependence of the scalars. Because the field strengths transform under electro-magnetic duality transformations, also the action will change. Imposing that the equations of motion are still derivable from the transformed action, implies that the group of symmetry transformations is restricted to symplectic matrices in $Sp(2m,\mathbb{R})$:

$$\mathcal{S}^T\Omega\mathcal{S} = \Omega \qquad \text{with} \qquad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{1.1.7}$$

[4] The Bianchi identity expresses that the field strength is locally derivable form a vector.

This implies that the components of $\mathcal{S}$ satisfy

$$A^T C - C^T A = 0 \quad , \quad B^T D - D^T B = 0 \quad , \quad A^T D - C^T B = \mathbb{1} \,. \tag{1.1.8}$$

The matrix $\mathcal{N}$ must transform into

$$\tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} \tag{1.1.9}$$

by these symplectic transformations. If both electric and magnetic sources are added to the combination of field equations and Bianchi identities, the Schwinger-Zwanziger quantization condition restricts the charges to a lattice. For instance, the charges of two dyons (q_1, g_1) and (q_2, g_2) have to satisfy

$$q_1 g_2 - q_2 g_1 = 2\pi n \hbar \,. \tag{1.1.10}$$

This lattice restricts the symplectic transformations further to its discrete subgroup $Sp(2m, \mathbb{Z})$. These integer symplectic transformations will be essential in the treatment of vector multiplets coupled to $N = 2$ supergravity[5] in four dimensions. Later in this thesis, in chapter 6, we will be interested in the coupling of vector multiplets to supergravity. The symplectic form of the electro-magnetic duality transformations will get a geometrical interpretation there.

A symmetry between the electrically and magnetically charged objects was conjectured in the late 70's by Montonen and Olive. In 1994, Sen gave new evidence for the presence of S-duality in $N = 4$ supersymmetric Yang–Mills theory. This S-duality is a strong-weak coupling duality. It relates a regime of the theory at strong coupling to a regime of the same or another theory at weak coupling where perturbative results can be achieved. The presence of much supersymmetry was essential in his proof, since supersymmetry limits the possible quantum corrections. Indeed it involves relations between coupling constants, which get renormalized. Supersymmetry can protect the relations between them from getting spoiled by renormalization effects.

1.1.5 Non-perturbative aspects of string theory

The ideas of the previous section from quantum field theory can be generalized to higher dimensional theories and also to string theory. In general, an object, extended in p dimensions, couples electrically to a $(p+1)$-form. Its magnetic dual

[5] We will denote the number of supersymmetries in the specific dimension under consideration by N in this thesis.

is extended in $(d-p-4)$ space dimensions. This means that a string in ten dimensions couples to a two-form, the NSNS[6] antisymmetric form $B_{\mu\nu}$. This suggests that there exist also non-perturbative objects that are magnetically charged with respect to this two-form. These have five space dimensions and are called NS5-branes. They are present in each of the five different string theories. The mass of these 5-branes is proportional to $1/g_s^2$, where g_s is the string coupling constant. Also in 11-dimensional supergravity, we have electric and magnetic objects. There is a three-form potential which suggests that there are electrically charged membranes. The membranes have a magnetically dual object with five spatial dimensions: the $M5$-brane. Moreover, different string theories contain also different antisymmetric RR-forms. The objects that are charged with respect to these forms are called Dp-branes[7]. Type IIA/IIB string theory allows Dp-branes for even/odd p. Different points of view are suited to enlarge insight into these objects. A first way to look upon D-branes is to study them as solutions of the supergravity equations of motion. These solutions involve harmonic functions. On the other hand, D-branes are extended objects in open string theory that obey certain Dirichlet boundary conditions. The ends of open superstrings are free to move on the brane. It is possible to determine the long-wavelength behavior of these D-branes in terms of a worldvolume field theory. The generalization of the worldsheet description is a σ-model in $(p+1)$ dimensions that contains the position of the brane in spacetime as part of the fields. The worldvolume theory of a single D-brane in addition contains a vector field and fermions. They reside together in a vector multiplet. A stack of n D-branes on top of each other leads to a field theory with a non-Abelian gauge group $U(n)$.

All these extended objects lead to the conclusion that string theory is a theory of more than strings. String theory teems with extended objects. These branes also allow to clarify lots of intricate relations between the five, at first sight different string theories. These relations are dualities. A first type of duality is T-duality. T-duality in its simplest form relates IIA and IIB string theory. Via a compactification to nine dimensions, it is possible to see a relation between the different antisymmetric forms in IIA and IIB theory. So, D-branes clarify the relations between different string theories. Also mirror symmetry can be seen as T-duality. Mirror symmetry expresses that the compactification of a type II string theories on a certain Calabi–Yau manifold is equivalent to the other type II string theory on a different, but related Calabi–Yau manifold. This type of T-duality is

[6] The bosonic fields in target space are called NSNS-fields (NS for Neveu–Schwarz) if they originate from a spinor bilinear in worldsheet fermions with antiperiodic boundary conditions and RR-fields (R for Ramond) if they come from two periodic worldsheet fermions.

[7] The p in Dp-branes counts the number of spatial dimensions of these non-perturbative objects: a $D0$-brane is a particle, a $D1$-brane is a string, a $D2$-brane a membrane, ... The standard work on D-branes is nowadays [2].

relevant for the compactification of string theory to four dimensions.

Another type of duality is a generalization of S-duality in quantum field theory. By increasing the string coupling constant g_s, the solitonic objects can become the lightest objects in the theory. A perturbative expansion in terms of $1/g_s$ becomes more interesting in such situations. A combination of T- and S-dualities relates all the different string theories. Another remarkable conclusion that was drawn the last couple of years, is that it is also possible to recover an 11-dimensional theory from string theory. Type IIA supergravity at strong coupling gives rise to its unique 11-dimensional cousin. The matrix model [3] is a candidate for the microscopic description of the theory with 11-dimensional supergravity as its low-energy limit. The final version of this microscopic description is not known yet, but it got the name M-theory already. The dualities between all these different string theories suppress the objection that there exist multiple perturbative string theories. People have found that they are all connected at the non-perturbative level.

1.1.6 The Maldacena conjecture

When putting a large number n of $D3$-branes on top of each other, the classical supergravity solution for the metric becomes a good approximation for the spacetime geometry in string theory. Taking the large n limit in which the string length is taken to zero, gives rise to a near-horizon geometry which is the product of $adS_5 \times S^5$. The isometry group of this space is $SO(4,2) \times SO(6)$. On the other hand, we have seen that n $D3$-branes on top of each other give rise to a $N = 4$ Yang–Mills gauge theory in four dimensions with gauge group $U(n)$. This gauge theory is conformally invariant and has as spacetime symmetry group $SO(4,2)$. Besides, there is the internal R-symmetry group which rotates the different supersymmetry charges into each other: $SU(4)$. As $SU(4)$ is the covering group of $SO(6)$, we find that the isometry group of both theories is the same. These facts were the motivation for Maldacena to conjecture the adS/CFT-correspondence [4]: string theory in an $adS_5 \times S^5$ background[8] is equivalent to a $N = 4$ superconformal Yang–Mills theory in four dimensions in the limit for large n. This field theory contains no gravity. It is defined on the four-dimensional boundary of the adS_5 space. More precisely, the bulk amplitudes in IIB supergravity are functions of the string fields at points on the boundary of adS_5. The conjecture states that these boundary fields are the sources of certain operators in the Yang–Mills theory. The string amplitudes are to be identified with correlation functions in the conformal field theory. By this correspondence, a classical calculation in super-

[8]Less symmetrical spaces than perfect spheres give rise to situations where the correspondence should be valid with less supersymmetry.

gravity theory can lead to the calculation of strong quantum effects in conformal field theory.

The Maldacena conjecture was also formulated for M-theory on $adS_4 \times S^7$ and M-theory on $adS_7 \times S^4$. These cases can be interpreted as the geometry on the horizon of a stack of $M2$-branes or $M5$-branes on top of each other. The worldvolume description of the $M5$-brane contains a self-dual tensor. This implies that the worldvolume description of a stack of $M5$-branes requires an interacting theory for self-dual tensors. It remains one of the challenges of high-energy theoretical physics to find a theory that describes this.

1.2 Overview of the thesis

In this thesis, we study aspects of two different supergravity models. This means that we consider aspects of the low-energy physics from string theory/M-theory. The models under consideration are both studied in a superconformal gravitational background. There are two reasons to do so. A first reason is that we want to study several aspects of models that can be found from compactifications of string theory on manifolds that preserve some amount of supersymmetry. The study of different matter representations in a gravitational background can be done very elegantly using superconformal tensor calculus. With this motivation in the back of the mind, the coupling of matter to a conformal background is only an intermediate step. The ultimate goal, which is aimed to be reached via this step, is the coupling of matter to Poincaré supergravity. Another reason that was motivating us, certainly when studying the coupling of the self-dual two-tensor to chiral six-dimensional gravity, is its possible role in the Maldacena conjecture.

In chapter 2, we stress the role of symmetry in this work. Using the algebraic setup allows to use the presence of certain symmetries (mainly superconformal symmetry). We first repeat the classification of the different supergroups and explain the two supergroups that will be relevant in this thesis, $SU(2,2|2)$ and $OSp(8^*|4)$, in detail. The knowledge of these algebras and their representations are a solid tool in discovering the properties of the two models under consideration in this thesis. In section 2.2, we study the different representations of these and other supergroups. We pay extra attention to chiral bosons in different dimensions, since the six-dimensional model will have a chiral two-form as backbone. Also different types of spinors are classified.

In chapter 3, we explain the two main techniques used in this work. Section 3.1 gives a short overview of superconformal tensor calculus. This technique is used very often in the construction of the coupling of different matter multiplets to Poincaré supergravity. The idea is that one first constructs the coupling of matter to a conformal supergravity background and that this model is gauge equivalent to the model with matter multiplets coupled to Poincaré supersymmetry. The conformal group is bigger than the final symmetry of the model we are after. We introduce compensating, unphysical multiplets which allow us to break the superfluous symmetry by imposing suitable gauge conditions. This technique will be used in both models we study in this thesis. In the model for vector multiplets in four dimensions with two supersymmetries, the breaking of the conformal symmetry will be done explicitly. In the six-dimensional model, the relevant results will be achieved already before the breaking is done and there, the conformal symmetry need not be broken. This section further contains also two examples that illustrate the essential steps in the construction. They clarify the train of thought behind the results in this thesis and are highly recommended for the reader that is not familiar with superconformal tensor calculus. Section 3.2 repeats shortly the Batalin–Vilkovisky formalism. This formalism is suited to treat models with the most general gauge symmetries, both for classical and quantum-mechanical aspects. We will use it in chapter 4 for the gauge symmetries of the self-dual tensor.

In chapter 4 and 5, we study aspects of the first model under consideration in this thesis: the self-dual tensor (multiplet) in six dimensions. This self-dual tensor is a two-form that has a real self-dual field strength in six Lorentzian dimensions. For a long time. the self-duality condition has been the obstacle to formulate a Lorentz-covariant action for this field. In 1996, this was achieved by Pasti, Sorokin and Tonin (PST) in [5]. This action required the introduction of an extra scalar field and two PST gauge symmetries which relied on the existence of the extra scalar. In a first stage of our research, we study the bosonic aspects of these gauge symmetries and two gauge fixings using the Batalin–Vilkovisky formalism. The results can be found in chapter 4 and in [6].

Some supersymmetric aspects of models with a self-dual tensor are studied in chapter 5. We first give an overview of the role of different chiral bosons in different contexts in supersymmetric theories. Section 5.2 sketches the relevant properties of the self-dual tensor multiplet which appear already when there is only rigid supersymmetry. The role of the extra scalar and of the PST gauge symmetries in a supersymmetric context is clarified. In section 5.3, we prepare the coupling of the self-dual tensor multiplet to conformal supergravity by studying the Weyl multiplet for $(2, 0)$ supersymmetry in six dimensions. We give some new arguments for including matter fields in the Weyl multiplet. In section 5.4, we give the Lorentz-covariant action for a self-dual tensor multiplet in a $(2, 0)$ chiral, conformal supergravity background. This requires a combination of the results of chapter 4, section 5.2 and section 5.3. This Lorentz-covariant action was achieved earlier for supersymmetric models and also for self-dual tensor multiplets coupled to Poincaré supergravity. The coupling to a conformal background is new and can also be found in [7]. We comment on how this action leads to similar actions with less supersymmetry or with Poincaré instead of conformal supersymmetry.

In chapter 6, we are studying vector multiplets in four dimensions with two supersymmetries. The coupling of matter multiplets to $N = 2$ supergravity is already a subject of research for more than fifteen years. During this period, it became clear that there is an intricate relation between these theories and the geometrical Calabi–Yau compactifications of string theory. This has lead to attempts to formulate the vector multiplets with $N = 2$ in four dimensions using a geometrical definition which allowed the connection to Calabi–Yau compactifications [8]. This definition does not rely any more on the existence of a prepotential, a homogeneous function of degree two, which in the past was thought to be indispensable for the construction. The geometrical construction heavily relies on the symplectic symmetry, which is a generalization of electro-magnetic duality, as explained earlier in the introduction. A first thing we consider, is the development of the Bianchi identities and the equations of motion of the special case of one vector multiplet, which was not worked out in [8]. The formulation of a geometrical definition reveals that there are two possible definitions for the coupling of one vector multiplet to $N = 2$ supergravity in four dimensions. We work out the model that satisfies only the weak definition and not the strong one. Besides, we construct an explicit model that satisfies the weak definition and not the strong one. These results can be found in chapter 6 and in [9]. In this chapter, we also explain why this model cannot be found from a geometrical Calabi–Yau compactification. So, there exist models for one vector multiplet in $N = 2$ and four dimensions that cannot be found from a geometrical Calabi–Yau compactification. In section 6.5, we investigate which part of the symplectic invariance group can be used for a non-Abelian gauging. We report the (partial) results here for the first time.

Concluding remark

One could ask: "What is the use of all this if all these results seem to contribute nothing to the final aim of string theory? Why doing research in string (supergravity) theory if it is up to now impossible to make contact between the standard model and general relativity on one side and a microscopic description of elementary 'particles' that comprises all the fundamental forces on the other side?"
This digression of high-energy theoretical physicists to problems which are far away from making contact with the standard model, has to my opinion nothing to do with unwillingness. It is rather the conviction of the community that a better understanding of the mathematical structure underlying string theory is needed before we can make reliable predictions. We hope that this thesis will help to make a tiny, tiny step in the right direction.

Chapter 2

Algebraic background

The physical problems that are subject of modern scientific research are phenomena that are very different from our daily experiences. There is a language however, namely that of mathematics, that is extremely suited to describe, understand and interpret the solutions to these problems. Together with a good amount of creativity, a great deal of perseverance and something that maybe can be called 'physical intuition', mathematics, as a pure form of logical deduction, has played an enormous role in taking small steps forward to grasp the world of physics (or the physics of the world) and it will surely remain as important in the future as it is nowadays. This chapter contains some mathematical concepts that are essential to build up this work.

The concept of symmetry is very important for physics in general and also for the models in high-energy theoretical physics under consideration in this thesis. The translation of the concept of symmetry into mathematics requires vocabulary as: algebra, groups, representations, gauge fields, curvatures, ... Since we will concentrate very much on these algebraic properties of certain supersymmetric theories, we start this chapter by a section on the classification of supergroups. We will mainly be interested in superconformal groups in four and six dimensions. The fields in these models are collected into multiplets, representations of these superalgebras.

The reason for studying these superconformal algebras and their representations is twofold:

- A first reason is valid for each of the two models in chapter 5 and in chapter 6. The aim is to study the coupling of matter to gravity with Poincaré supersymmetry. This is achieved by studying first theories with superconformal invariance. Since the final goal is in this case *not* to study theories with superconformal invariance, we will break the conformal to Poincaré supersymmetry. Besides breaking the dilatations, this involves also the breaking of two other symmetries. This procedure is called superconformal tensor calculus. In section 3.1, we will explain the essential ingredients to achieve this.

- A second, more recent, motivation is the role of conformal supergravity in the adS/CFT-correspondence. Here, the conformal symmetry is an essential ingredient of the theory and it is not broken. This will be a focus in the study of the six-dimensional model.

Section 2.1 contains the introductory notions, relevant for the algebraic approach. We will first give the different spacetime algebras that underlie gravitational theories. Then we discuss the classification of superalgebras, starting with a short review about the most general groups that have the right spacetime symmetry group as a subgroup of the bosonic subgroup. Since the advent of supersymmetry the last decades, this implies the development of the concept of super Lie group. We end with a classification of conformal supergroups.

In section 2.2, we study different representations of these algebras. We develop the different irreducible spinor representations for different dimensions and signatures, after introducing Clifford algebras. Then we pay some attention to chiral bosons, antisymmetric p-forms with real (anti)self-dual field strengths. The spinor representations allow to write down the two superalgebras $OSp(8^*|4)$ and $SU(2,2|2)$, relevant for models with $(2,0)$ chiral supersymmetry in six dimensions and models with $N = 2$ in four dimensions. Other reviews on superalgebras and spinors can be found in [10, 11].

2.1 Supergroups

The analysis of a physical system is often much easier when using a symmetry present in the problem. The mathematical translation of this symmetry principle introduces the concept of a group. Often, this group is a continuous Lie group with a Lie algebra underlying it. Lie groups are used for instance in classical and quantum mechanics, in solid state physics and of course in the study of particle physics by using quantum field theory, but this list is far from exhaustive. Good introductions into group theory can be found in [12].

2.1.1 Different bosonic algebras and their relation

Poincaré and (anti) de Sitter algebras

$ISO(1,d-1)$, the Poincaré algebra in d dimensions, has translation generators P_μ and Lorentz rotation generators $M_{\mu\nu}$:

$$\begin{aligned}[M_{\mu\nu},M_{\rho\sigma}] &= \eta_{\mu[\rho}M_{\sigma]\nu}-\eta_{\nu[\rho}M_{\sigma]\mu}\,,\\ [P_\mu,M_{\nu\rho}] &= \eta_{\mu[\nu}P_{\rho]}\,,\\ [P_\mu,P_\nu] &= 0\,.\end{aligned} \tag{2.1.1}$$

This is the symmetry algebra in general relativity with a vanishing cosmological constant. The symmetries of spaces with a non-trivial cosmological constant can be described by adjusting the last line in the Poincaré algebra. Spaces with negative cosmological constant are called anti de Sitter (adS) spaces [13]. Their algebra is

$$\begin{aligned}[M_{\mu\nu},M_{\rho\sigma}] &= \eta_{\mu[\rho}M_{\sigma]\nu}-\eta_{\nu[\rho}M_{\sigma]\mu}\,,\\ [P_\mu,M_{\nu\rho}] &= \eta_{\mu[\nu}P_{\rho]}\,,\\ [P_\mu,P_\nu] &= \frac{1}{2R^2}M_{\mu\nu}\,,\end{aligned} \tag{2.1.2}$$

where R is the radius of curvature of the adS-space. The supersymmetric version of this algebra is very important in the adS/CFT-conjecture. By defining $R\,P_\mu = M_{d\mu} = -M_{\mu d}$, it is possible to collect all the generators of the anti de Sitter space into $M_{\hat\mu\hat\nu} = -M_{\hat\nu\hat\mu}$, with $\hat\mu = 0,\dots,d$. Using the metric $\eta_{\hat\mu\hat\nu} = (-+\dots+-)$, these $M_{\hat\mu\hat\nu}$ span a $SO(d-1,2)$ algebra:

$$[M_{\hat\mu\hat\nu},M_{\hat\rho\hat\sigma}] = \eta_{\hat\mu[\hat\rho}M_{\hat\sigma]\hat\nu}-\eta_{\hat\nu[\hat\rho}M_{\hat\sigma]\hat\mu}\,. \tag{2.1.3}$$

Its relation with the conformal algebra will be clarified later on.

The last couple of years, astrophysical experiments have collected increasing evidence that our universe has a slightly positive cosmological constant [14]. This would mean that our universe is a de Sitter space. Then, the commutator of two translations is

$$[P_\mu,P_\nu] = -\frac{1}{2R^2}M_{\mu\nu}\,. \tag{2.1.4}$$

Furthermore, it is possible to derive the Poincaré algebra from the (anti) de Sitter algebra by taking $R\to\infty$, the infinite-radius limit of the (anti) de Sitter space. This limiting procedure is called an 'Inönü–Wigner' contraction.

The conformal algebra

Conformal transformations are defined to keep angles fixed. This implies that a conformal transformation can transform the metric:

$$ds^2(x) \to \Omega(x) ds^2(x) \,. \tag{2.1.5}$$

Under infinitesimal coordinate transformations $x^\mu \to x^\mu + \xi^\mu$, the metric transforms as

$$ds^2(x) \to ds^2(x) + 2\partial_{(\mu}\xi_{\nu)} dx^\mu dx^\nu \,. \tag{2.1.6}$$

Compatibility with (2.1.5) requires that

$$\partial_{(\mu}\xi_{\nu)} - \tfrac{1}{d}\eta_{\mu\nu}\partial\cdot\xi = 0 \tag{2.1.7}$$

in d dimensions. It also follows from (2.1.7) that

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu)\partial\cdot\xi = 0 \,. \tag{2.1.8}$$

For $d > 2$, (2.1.7) and (2.1.8) require that ξ^μ is at most quadratic in x^ν. For ξ^μ zeroth order in x^ν, we have

$$\xi^\mu = a^\mu \,, \tag{2.1.9}$$

i.e., ordinary translations independent of x. There are two solutions for which ξ^μ is linear in x^ν:

$$\begin{aligned} \xi^\mu &= \lambda^\mu{}_\nu x^\nu \,, \\ \xi^\mu &= \Lambda_D x^\mu \,, \end{aligned} \tag{2.1.10}$$

where $\lambda^{\mu\nu}$ is antisymmetric in $\mu\nu$. These solutions correspond to Lorentz rotations and dilatations. Finally, when ξ^μ is quadratic in x^ν we have

$$\xi^\mu = \Lambda_K^\mu x^2 - 2x^\mu \Lambda_K \cdot x \,, \tag{2.1.11}$$

the so-called special conformal transformations. This conformal algebra has $d + d(d-1)/2 + 1 + d = (d+1)(d+2)/2$ generators. The most general conformal transformation can thus be written as

$$\delta_C = a^\mu P_\mu + \lambda^{\mu\nu} M_{\mu\nu} + \Lambda_D D + \Lambda_K^\mu K_\mu \,, \tag{2.1.12}$$

in terms of the parameters times the generators. With these transformations, one can obtain the algebra with as non-zero commutators

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu[\rho} M_{\sigma]\nu} - \eta_{\nu[\rho} M_{\sigma]\mu} \, , \\
[P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu} P_{\rho]} \, , \\
[K_\mu, M_{\nu\rho}] &= K_{[\rho} \eta_{\nu]\mu} \, , \\
[P_\mu, K_\nu] &= -2(\eta_{\mu\nu} D + 2M_{\mu\nu}) \, , \\
[D, P_\mu] &= P_\mu \, , \\
[D, K_\mu] &= -K_\mu \, . \qquad (2.1.13)
\end{aligned}$$

The case of two dimensions is special in (2.1.7) and (2.1.8). For $d = 2$, (2.1.7) becomes the Cauchy–Riemann equation and it is natural to write everything in terms of complex coordinates. Two-dimensional conformal transformations thus coincide with the analytic coordinate transformations, which have an algebra with infinitely many generators. These conformal algebras in two dimensions are a cornerstone of string theory and studied for instance in [15].

The $SO(d, 2)$ algebra (2.1.13) can be rewritten as

$$M^{\hat\mu\hat\nu} = \begin{pmatrix} M^{\mu\nu} & \frac{1}{4}(P^\mu - K^\mu) & \frac{1}{4}(P^\mu + K^\mu) \\ -\frac{1}{4}(P^\nu - K^\nu) & 0 & -\frac{1}{2}D \\ -\frac{1}{4}(P^\nu + K^\nu) & \frac{1}{2}D & 0 \end{pmatrix} \, , \qquad (2.1.14)$$

where the metric is $\eta_{\mu\nu} = (- + \ldots + -)$. This is the same algebra as the anti de Sitter algebra (2.1.2) in $d + 1$ dimensions, so

$$Conf_d = adS_{d+1} \, . \qquad (2.1.15)$$

The fact that these two algebras, and also their supersymmetric extension, are the same ones is the cornerstone of the adS/CFT-conjecture.

2.1.2 Classifying superalgebras

The Coleman–Mandula theorem

In the 60's and the 70's, people were studying which was the most general combination of four-dimensional spacetime and internal symmetries allowing non-trivial scattering amplitudes in the S-matrix approach. Coleman and Mandula [16] achieved the most general no-go theorem at that moment. They showed that if the symmetry group is a Lie group with an internal group G mixing in a non-trivial way with the Poincaré group, then the S-matrix for all processes would be $\mathbb{1}$. Among other assumptions, the theorem is only valid when there are *no massless* particles. An implication of the direct product of the Poincaré group and the internal group for a non-trivial scattering matrix is that G cannot relate eigenstates with different mass or spin.

This theorem can be extended to the massless case (modulo infrared problems): a non-trivial S-matrix then requires the direct product of the four-dimensional conformal group with an internal Lie group: $SO(2,4)\times G$. These conformal transformations leave the lightcone invariant. The presence of masslike parameters would spoil the invariance under these transformations. The internal symmetry group again has to commute with the conformal group.

This is a purely bosonic result, even if Coleman and Mandula talk about a supermultiplet in their article. Anno 1967, a supermultiplet was a set of states transformed by the bosonic internal group G into one another (and thus with the same mass and spin).

The Haag–Łopuszanski–Sohnius theorem

The Coleman–Mandula theorem did not allow to relate eigenstates with the same mass, but with different spin. This is only allowed if one includes symmetry generators that change the spin of the states. Graded Lie algebras are suited to do this. They contain two types of generators: bosonic and fermionic ones. The Lie product of Lie groups is generalized to an anticommutator for the product of two fermionic generators:

$$[\delta(\epsilon_1),\delta(\epsilon_2)] = \epsilon_2^B\epsilon_1^A(Q_AQ_B + Q_BQ_A) = \epsilon_2^B\epsilon_1^A\,\{Q_A, Q_B\}\ , \tag{2.1.16}$$

where $\delta(\varepsilon_1)$ is a fermionic transformation with parameter ε_1^A and fermionic generator Q_A. If the Lie product of two fermionic generators gives rise to a bosonic one, they are called $\mathbb{Z}_2$ graded algebras[1]. These fermionic generators allow to bypass the no-go theorem of Coleman and Mandula. This graded algebra also has to satisfy Jacobi-identities

$$\left[\epsilon_1^A T_A, \left[\epsilon_2^B T_B, \epsilon_3^C T_C\right]\right] + \text{ cyclic in 1, 2, 3 } = 0\,. \tag{2.1.17}$$

For the generators, this translates into

$$\left[[T_A, T_B\right\}, T_C\} = [T_A, [T_B, T_C\}\} - (-)^{AB}\,[T_B, [T_A, T_C\}\}\ , \tag{2.1.18}$$

[1] Also $\mathbb{Z}_3$ graded algebras are studied, for instance in [17].

where the notation $[,\}$ means an anticommutator when both generators are odd (fermionic) and a commutator if at least one of them is even (bosonic).

Models with this type of relations were first proposed by Gol'fand and Likhtman [18]. Volkov and Akulov were the first to write down a non-linear realization of supersymmetry [19]. The first renormalizable field theoretical model was presented by Wess and Zumino [20][2]. It contained two scalars with spin 0 and a fermion with spin 1/2, sitting in the same supermultiplet. This model contained supersymmetry generators which transformed states with spin j into states with spin $j+1/2$. Such models cannot be described by Lie groups and are therefore not ruled out by the Coleman–Mandula theorem. Haag, Łopuszański and Sohnius were able to generalize the result of Coleman–Mandula. They proved that also the supersymmetric generalization must be split into two cases:

- In the presence of massive states, the most general symmetry group of the four-dimensional S-matrix has the same $ISO(1,3)\otimes G$ as a bosonic subgroup as in the purely bosonic treatment. The fermionic operators transform as spinors under the homogeneous Lorentz group and commute with translations. The commutator of two supersymmetries gives always rise to a translation. The fermionic generators in the algebra allow the presence of central charges Z^{LM}:
$$\{Q^L_\alpha, Q^M_\beta\} = \varepsilon_{\alpha\beta} Z^{LM}\,. \qquad (2.1.19)$$

- With only massless particles, two types of supersymmetries are present. The bosonic subgroup of the graded Lie algebra is the direct product of the conformal group $SO(2,4)$ and an internal group G. This internal group is the direct product of the R-symmetry group (which rotates the supersymmetry generators into each other) and another internal symmetry group G'.

Classification of superalgebras

In the first half of the 70's, the analysis of supergroups was only relevant for four-dimensional physical models. The advent of superstrings and supergravities in 10 and 11 dimensions and their compactifications to all different dimensions with different amounts of residual supersymmetry asked for a classification of Lie supergroups in different dimensions. The simple Lie superalgebras have been classified in [21]. The analysis of different properties of the graded Lie algebra contains a lot of subtleties, for instance a semi-simple superalgebra is not always the direct sum of simple ones. The main properties of superalgebras and lots of useful tables

[2]The 'supergauge symmetry' in their title was later called 'supersymmetry'.

can be found in [22]. The classical Lie superalgebras are a subset of the simple Lie subalgebras, whose fermionic generators are in an irreducible or a completely reducible representation of the bosonic algebra. Simple Lie superalgebras that are not classical are called Cartan-type superalgebras [22]. The list of classical superalgebras and their real forms [23] can be found in table 2.1.

Name	Range	Bosonic algebra	Defining repres.	Number of generators
$SU(m\|n)$	$m \geq 2$ $m \neq n$ $m = n$	$SU(m) \oplus SU(n)$ $\oplus U(1)$ no $U(1)$	$(m, \bar{n}) \oplus$ $(\bar{m}, n)$	$m^2 + n^2 - 1$, $2mn$ $2(m^2 - 1), 2m^2$
$S\ell(m\|n)$ $SU(m-p, p\|n-q, q)$ $SU^*(2m\|2n)$		$S\ell(m) \oplus S\ell(n)$ $\oplus SO(1,1)$ $SU(m-p,p) \oplus SU(n-q,q)$ $\oplus U(1)$ $SU^*(2m) \oplus SU^*(2n)$ $\oplus SO(1,1)$		} if $m \neq n$
$OSp(m\|n)$	$m \geq 1$ $n = 2, 4, ..$	$SO(m) \oplus Sp(n)$	(m, n)	$\frac{1}{2}(m^2 - m +$ $n^2 + n), mn$
$OSp(m-p, p\|n)$ $OSp(m^*\|n-q, q)$		$SO(m-p,p) \oplus Sp(n)$ $SO^*(m) \oplus USp(n-q,q)$		n even m, n, q even
$D(2,1,\alpha)$	$0 < \alpha \leq 1$	$SO(4) \oplus S\ell(2)$	$(2,2,2)$	9, 8
$D^p(2,1,\alpha)$		$SO(4-p,p) \oplus S\ell(2)$		$p = 0, 1, 2$
$F(4)$		$\overline{SO(7)} \oplus S\ell(2)$	$(8, 2)$	21, 16
$F^p(4)$		$SO(7-p,p) \oplus S\ell(2)$		$p = 0, 1, 2, 3$
$G(3)$		$G_2 \oplus S\ell(2)$	$(7, 2)$	14, 14
$G_p(3)$		$G_{2,p} \oplus S\ell(2)$		$p = -14, 2$
$P(m-1)$	$m \geq 3$	$S\ell(m)$	$(m \otimes m)$	$m^2 - 1, m^2$
$Q(m-1)$	$m \geq 3$	$SU(m)$	Adjoint	$m^2 - 1, m^2 - 1$
$Q(m-1)$ $Q((m-1)^*)$ $UQ(p, m-1-p)$		$S\ell(m)$ $SU^*(m)$ $SU(p, m-p)$		

Table 2.1: Lie superalgebras of classical type. Under 'name' is given the algebra over ℂ. The 'defining representation' gives the bosonic representation in which the fermionic generators transform. The number of generators gives first the bosonic and then the fermionic ones. Before starting with a new algebra over ℂ, the different real forms are given, together with their respective bosonic subalgebras.

The first series of SU-groups is a supersymmetric generalization of the bosonic SU-groups (the A-series in the classification using Dynkin diagrams). The OSp-

series contains supersymmetric generalizations of the B-, the C- and the D-series: for $m=1$, one finds the 'supersymmetric C-series', for other odd values of m, the B-series is recovered and for m even, the D-series is discovered. The exceptional Lie superalgebras $F(4)$ and $G(3)$ contain the special groups F_4 and G_2 in their bosonic subgroup. The other exceptional group $D(2,1,\alpha)$ has three bosonic factors $sl(2)$ in its bosonic subgroup. They appear with relative weights 1, α and $-1-\alpha$ in the anticommutator of the fermionic generators. Its real forms contain $SO(4) = SU(2)\times SU(2)$, $SO(3,1) = Sl(2,\mathbb{C})$ and $SO(2,2) = Sl(2)\times Sl(2)$. In the first and last case, α should be real and for $p=1$, $\alpha = 1+ia$ with a real is required. In the limit that $\alpha = 1$, one finds $D^p(2,1,1) = OSp(4-p,p)$. $P(n)$ and $Q(n)$ are called strange superalgebras.

2.1.3 Superconformal algebras

As we have seen in section 2.1.1, the conformal algebra in d dimensions is equivalent to the adS algebra in $d+1$ dimensions:

$$Conf_d = adS_{d+1}\,. \tag{2.1.20}$$

It is also possible to do an Inönü–Wigner contraction of an (a)dS algebra in d dimensions to a Poincaré algebra in d dimensions. Furthermore, we start the derivation mostly from different superconformal algebras in this thesis. Therefore, it is sufficient to study the classification of the conformal superalgebras. We always first construct representations of the superconformal algebra and only afterwards break sometimes to Poincaré supersymmetry. Also for the adS/CFT-conjecture, it is sufficient to classify the superconformal (or equivalently the anti de Sitter) algebras.

We first discuss the general structure of the superconformal algebra. Solving (2.1.7) and (2.1.8), we found that the bosonic algebra already contained translations, Lorentz rotations, dilatations, and special conformal transformations. Extending this to the supersymmetric context immediately implies that a second supersymmetry has to be introduced. Indeed, the commutator of the special conformal transformations and supersymmetry gives rise to a fermionic generator in a $\mathbb{Z}_2$-graded algebra. This supersymmetry is called 'special supersymmetry' and denoted by S. Schematically, this can be written as: $[K_a, Q] = \Gamma_a S$. Furthermore, the anticommutator $\{Q,S\}$ gives rise to new bosonic generators, mostly called R-symmetry generators. The whole superconformal algebra can be represented schematically in a supermatrix as

$$\begin{pmatrix} SO(d,2) & Q+S \\ Q-S & R \end{pmatrix}\,. \tag{2.1.21}$$

d	superalgebra	R	number of fermionic generators
1	$OSp(N\|2)$	$O(N)$	$2N$
	$SU(N\|1,1)$	$SU(N)\times U(1)$ for $N\neq 2$	$4N$
	$SU(2\|1,1)$	$SU(2)$	8
	$OSp(4^*\|2N)$	$SU(2)\times USp(2N)$	$8N$
	$G(3)$	G_2	14
	$F^0(4)$	$SO(7)$	16
	$D^0(2,1,\alpha)$	$SU(2)\times SU(2)$	8
3	$OSp(N\|4)$	$SO(N)$	$4N$
4	$SU(2,2\|N)$	$SU(N)\times U(1)$ for $N\neq 4$	$8N$
	$SU(2,2\|4)$	$SU(4)$	32
5	$F^2(4)$	$SU(2)$	16
6	$OSp(8^*\|N)$	$USp(N)$ (N even)	$8N$

Table 2.2: Super adS_{d+1} or $conf_d$ algebras.

The diagonal elements are the bosonic generators (the conformal algebra and the R-symmetry generators), the fermionic generators (for supersymmetry and special supersymmetry) are off-diagonal. Nahm classified the conformal algebras[3] in [24]. He imposed $SO(2,d)$ to be a factor of the factorizable bosonic subgroup and moreover, the fermionic generators should transform as spinors of the conformal group. Using the algebra isomorphisms[4],

$$\begin{aligned} Sp(2)=SO(2,1) &= SU(1,1)=Sl(2)\,,\\ SO^*(4) &= SU(1,1)\oplus SU(2)\,,\\ SO(2,2) &= Sl(2)\oplus Sl(2)\,,\\ SO(3,2) &= Sp(4)\,,\\ SO(4,2) &= SU(2,2)\,,\\ SO(6,2) &= SO^*(8)\,, \end{aligned} \tag{2.1.22}$$

one ends up with the conformal algebras with compact R-symmetry group in diverse dimensions of table 2.2. The maximal dimension for a superconformal algebra is six.

The first line of the real forms of the $OSp(m|n)$ groups of table 2.1 contains a group with bosonic subgroup $SO(6,2)$ and a *non-compact* R-symmetry group

[3] His classification also includes the de Sitter algebras.

[4] These equality signs are not correct for the corresponding groups. There one should use the appropriate covering groups. $SO^*(2n)$ is a complex matrix that is equivalent to the quaternionic matrix $O(n,Q)$.

$Sp(N)$. We would like to have a supergroup with a compact R-symmetry. Due to the isomorphism between the bosonic groups $SO^*(8)$ and $SO(6,2)$ we find that $OSp(8^*|4)$ is a superconformal algebra with a compact R-symmetry group in six dimensions.

The conformal algebra in two dimensions is $SO(2,2) \sim SO(2,1) \oplus SO(2,1)$. The superconformal algebras in two dimensions with a finite number of generators can be constructed as the sum of two algebras in one dimension of table 2.2. The superconformal algebras in two dimensions with infinitely many generators are omnipresent in the worldsheet description of string theory. They are classified in [25].

The Haag–Łopuszanski–Sohnius theorem allowed for a central extension Z_{LM} which was a scalar under Lorentz transformations. This theorem was formulated in the context of S-matrix theory in four-dimensional quantum field theory. In the meantime, we know that string theory is not a theory of strings alone. Extended objects (D-branes and NS-branes) are also present. Their description requires corresponding 'central charges' with different numbers of indices. These 'central charges' are not Lorentz-invariant. This relaxation of the Haag–Łopuszanski–Sohnius theorem was introduced already in [26] and is reviewed in [27].

Two of these superconformal algebras will be used to construct matter couplings to super-Poincaré theories. Then, one first couples matter to a conformal gravity background and later breaks local superconformal symmetry to local Poincaré symmetry. In this thesis, the analysis is restricted to two types of models. The first one is the construction of an action for the $(2,0)$ self-dual tensor multiplet in six dimensions. Then, the supergroup $OSp(8^*|4)$ is relevant. The other supergroup used, is $SU(2,2|2)$, which corresponds to $N=2$ supersymmetry in four dimensions. Using superconformal tensor calculus, we will construct the most general coupling of vector multiplets to $N=2$ supergravity in four dimensions.

2.2 Spinors and chiral bosons

In supersymmetric theories, the supercharges transform bosons (like scalars, vectors, metrics, chiral bosons, etc.) into fermions and vice versa. This transformation of a boson (B) into a fermion (F) can generically be denoted by:

$$\delta B = \bar{\varepsilon} F \,. \tag{2.2.1}$$

A dimensional analysis learns that the dimension of the supersymmetry parameter ε is $-\frac{1}{2}$. This implies that

$$\delta F = (\partial B)\varepsilon\,. \tag{2.2.2}$$

These two rules lead very schematically to the result that the commutator of two supersymmetries gives rise to a translation. These schematic rules also argue that a supermultiplet, a representation of an algebra that contains fermionic generators, will contain both bosonic and fermionic fields. In [28] is argued the following: *there are an equal number of bosonic and fermionic degrees of freedom in any realization of the supersymmetry algebra when translations are an invertible operation.*
This equality in number of fermionic and bosonic states holds both for on-shell as for off-shell representations. All this implies that we need a good understanding of spinors in diverse dimensions. We will also discuss the different possibilities of spaces where chiral bosons can exist.

A free electron is described by a Dirac fermion satisfying the Dirac equation [29]. This equation contains the Dirac matrices which span a Clifford algebra. So, the knowledge of Clifford algebras is indispensable for the description of spinors. For certain dimensions with specific signatures, it is possible to impose chirality (Weyl) or reality (Majorana or symplectic Majorana) conditions on these spinors such that they become irreducible representations. We list these possibilities to retrieve later the analogy with chiral bosons: exactly the same dimensions and signatures allow chiral bosons and (symplectic) Majorana–Weyl spinors. A similar analysis on Clifford algebras and spinors can also be found in [30].

2.2.1 Clifford algebras

We do our analysis in a space with flat metric $\eta_{ab} = \mathrm{diag}(- \ldots - + \ldots +)$ writing first t timelike and then s spacelike directions in $d = t + s$ spacetime dimensions. The Clifford algebra is defined by

$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2\eta_{ab}\,. \tag{2.2.3}$$

The purely spacelike (signature $(0, d)$) realization of this algebra in terms of the Hermitian Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{2.2.4}$$

can be built in the following way:

$$\Gamma_1 \;=\; \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots$$

$$\begin{aligned}
\Gamma_2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\Gamma_3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \ldots \\
\Gamma_4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \ldots \\
\Gamma_5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \ldots \\
\ldots &= \ldots .
\end{aligned} \tag{2.2.5}$$

For even spacetime dimensions, this Clifford algebra has dimension[5] $2^{d/2}$. For odd spacetime dimensions, the σ_1 in the matrix Γ_d is not needed. This implies that the dimension of the Clifford algebra for odd d is $2^{(d-1)/2}$. For non-Euclidean signatures, the first t matrices are multiplied with a factor i. The timelike Γ-matrices are anti-Hermitian, the spacelike ones Hermitian:

$$\Gamma_t^\dagger = -\Gamma_t \,, \qquad \Gamma_s^\dagger = \Gamma_s \,. \tag{2.2.6}$$

This can be written in one equation:

$$\Gamma_a^\dagger = (-)^t A \Gamma_a A^{-1} \,, \tag{2.2.7}$$

where $A = \Gamma_1 \ldots \Gamma_t$. The $\Gamma_a^\dagger$ also span the Clifford algebra (2.2.3) (or equivalently, one could have multiplied the timelike Γ's of (2.2.5) with $-$i).

Define for $n = 0, 1, \ldots, d$ the matrices

$$\Gamma^{(n)} = \Gamma_{a_1 \ldots a_n} = \Gamma_{[a_1} \Gamma_{a_2} \ldots \Gamma_{a_n]} \,. \tag{2.2.8}$$

In even spacetime dimensions[6], the set $\{\Gamma^{(n)}\}$ is a complete set of $2^{d/2} \times 2^{d/2}$-matrices where $\Gamma^{(0)} = \mathbb{1}$ is the only one with non-trivial trace. The last matrix $\Gamma^{(d)}$ is related to

$$\Gamma_* = (-\mathrm{i})^{d/2+t} \Gamma_1 \ldots \Gamma_d \,, \tag{2.2.9}$$

such that $\Gamma_* \Gamma_* = \mathbb{1}$. If the Γ_a of (2.2.5) are used, Γ_* is independent of the spacetime signature because of the factor $(-\mathrm{i})^t$ in (2.2.9). Γ_* anticommutes with

[5]The word 'dimension' here refers to the number of rows and columns of the irreducible representation of the algebra in matrix form. The irreducibility of the representation also guarantees that a notion of the transpose of Γ_a can be defined. The dimension of the Clifford algebra, in the sense of the number of generators, is 2^d, where d is used for the spacetime dimension.

[6]We put more emphasis on even spacetime dimensions, since the calculations that use these explicit realizations of the algebras and their representations are in four and six spacetime dimensions.

d	1	2	3	4	5	6	7	8
η	-	$\pm$	+	$\pm$	-	$\pm$	+	$\pm$
ε	-	$\pm$	+	+	+	$\mp$	-	-

Table 2.3: The symmetry properties related to $\mathcal{C}$: values of ε and η for different dimensions.

all Γ_a. Therefore, it can be used as Γ_{d+1} for the next odd spacetime dimension. For even spacetime dimensions, we have the following relation between Γ-matrices:

$$\Gamma_{a_1\ldots a_n} = \tfrac{1}{(d-n)!}\varepsilon_{a_1\ldots a_d}\mathrm{i}^{d/2+t}\Gamma_*\Gamma^{a_d\ldots a_{n+1}} \,. \qquad (2.2.10)$$

Furthermore, for the irreducible representations it is always possible to define a charge conjugation matrix $\mathcal{C}$ which satisfies

$$\mathcal{C}^T = -\varepsilon\,\mathcal{C}\,, \qquad \Gamma_a^T = -\eta\,\mathcal{C}\,\Gamma_a\,\mathcal{C}^{-1}\,, \qquad \mathcal{C}\,\mathcal{C}^\dagger = \mathbb{1}\,, \qquad (2.2.11)$$

for $\varepsilon = \pm 1$ and $\eta = \pm 1$. $\mathcal{C}$ is always unitary. For even dimensions, two possibilities exist for $\mathcal{C}$. For the representation (2.2.5), these are:

$$\begin{aligned} \mathcal{C}_+ &= \sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\ldots \qquad \eta = 1\,, \\ \mathcal{C}_- &= \sigma_1\otimes\sigma_2\otimes\sigma_1\otimes\ldots \qquad \eta = -1\,. \end{aligned} \qquad (2.2.12)$$

Remark that in this representation $\mathcal{C}_- = \mathcal{C}_+\Gamma_*$ and $\mathcal{C}_\pm = \mathcal{C}_\pm^\dagger$. For odd dimensions, only one of the two possibilities of one dimension lower remains. All possibilities for ε and η are summarized in table 2.3. η and ε are independent of the spacetime signature and the same for d and $d+8$ dimensions.

2.2.2 Irreducible spinors in different dimensions

The next ingredient in our theories is the construction of the irreducible spinor representations for different dimensions and signatures. A spinor is a representation of the universal covering group of the relevant SO-group. One talks about Euclidean, Minkowski, or conformal spinors in d dimensions if the symmetry group of spacetime rotations is $SO(d)$, $SO(1,d-1)$, or $SO(2,d)$. Using the Γ-matrices of the last section, the infinitesimal transformation of a spinor under spacetime rotations is

$$\delta\psi = -\tfrac{1}{4}\Gamma_{ab}\lambda^{ab}\psi\,, \qquad (2.2.13)$$

where $\Gamma_{ab} = \Gamma^{(2)}$ of (2.2.8). The spinor in (2.2.13) is a priori a Dirac spinor. It has[7] $2 \cdot 2^{[d/2]}$ real off-shell components. Imposing the Dirac equation gives $2^{[d/2]}$ real on-shell components. The Dirac spinor is often a reducible representation. Two types of projections are possible to find the irreducible fermionic representations.

For all even dimensions, it is always possible to use Γ_*. Since

$$\left(\frac{1 \pm \Gamma_*}{2}\right)^2 = \frac{1 \pm \Gamma_*}{2}\,, \tag{2.2.14}$$

it is possible to project onto left- or righthanded spinors:

$$\lambda_L = \left(\frac{1 + \Gamma_*}{2}\right)\lambda_L\,; \qquad \lambda_R = \left(\frac{1 - \Gamma_*}{2}\right)\lambda_R\,. \tag{2.2.15}$$

In a representation of the Clifford algebra where Γ_* is of the form

$$\Gamma_* = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{2.2.16}$$

this gives rise to

$$\lambda_L = \begin{pmatrix} \lambda' \\ 0 \end{pmatrix} \qquad \lambda_R = \begin{pmatrix} 0 \\ \lambda'' \end{pmatrix}. \tag{2.2.17}$$

This projection is compatible with Lorentz transformations since $[\frac{1 \pm \Gamma_*}{2}, \Gamma_{ab}] = 0$. Spinors obeying one of these conditions are called Weyl spinors.

Another projection imposes a reality condition. From (2.2.7) and (2.2.11), one obtains

$$\Gamma_a^* = -\eta(-)^t B \Gamma_a B^{-1}\,, \tag{2.2.18}$$

where

$$B^T = \mathcal{C}\,A^{-1}\,, \tag{2.2.19}$$

and

$$B^*\,B = -\varepsilon\,\eta^t\,(-)^{\frac{t(t+1)}{2}}\,. \tag{2.2.20}$$

[7] $[d/2] = d/2$ for d even and equals $(d-1)/2$ for d odd.

Imposing a reality condition on spinors,

$$\lambda^* = \tilde{B}\lambda\,, \tag{2.2.21}$$

must survive Lorentz transformations:

$$\left(-\tfrac{1}{4}\Gamma_{ab}\lambda\right)^* = -\tfrac{1}{4}\tilde{B}\Gamma_{ab}\lambda\,. \tag{2.2.22}$$

Using (2.2.18), implies that

$$B\,\Gamma_{ab}\,B^{-1}\,\tilde{B} = \tilde{B}\,\Gamma_{ab}\,, \tag{2.2.23}$$

which is solved by

$$\tilde{B} = B\alpha\,, \tag{2.2.24}$$

where α can in general be a matrix in the space of all the spinors, but is a scalar in spinor space. This leads to the condition that

$$[\alpha, \Gamma_{ab}] = 0\,. \tag{2.2.25}$$

Consistency requires that $\lambda^{**} = \lambda$. This means that $\tilde{B}^*\tilde{B} = \mathbb{1}$. Choosing $\alpha = e^{i\phi}\mathbb{1}$ leads to

$$\tilde{B}^*\tilde{B} = \left(-\varepsilon\,\eta^t\,(-)^{\frac{t(t+1)}{2}}\right)\mathbb{1} = \mathbb{1}\,, \tag{2.2.26}$$

and this is solved if the right hand side of (2.2.20)

$$-\varepsilon\,\eta^t\,(-)^{\frac{t(t+1)}{2}} = 1\,. \tag{2.2.27}$$

Checking the different possibilities for ε and η in table 2.3, gives that this condition is satisfied for

$$\begin{aligned} s-t &= 0,1,7 \mod 8\,,\\ s-t &= 2 \mod 8 \ \text{ with } \eta=+1\,,\\ s-t &= 6 \mod 8 \ \text{ with } \eta=-1\,. \end{aligned} \tag{2.2.28}$$

The spinors that obey a reality condition like $\lambda^* = B\lambda$ are called Majorana spinors. There is another possibility to denote this reality condition: the Majorana conjugate should equal the Dirac conjugate or

$$\bar{\lambda} \equiv \lambda^T\mathcal{C}\,, \qquad \bar{\lambda}^C \equiv \lambda^\dagger A\alpha^{-1}\,. \tag{2.2.29}$$

For the case of Minkowski signature, the Dirac conjugate is the familiar $\lambda^\dagger\Gamma_0$.

Even if (2.2.20) is -1, it is possible to impose a 'reality condition'. For that purpose, one needs a non-trivial R-symmetry group. Choose α in $\tilde{B}$ equal to Ω_{ij}, an antisymmetric matrix such that $\Omega\Omega^* = -\mathbb{1}$. Ω_{ij} acts on the different spinors in a representation of the R-symmetry group, so it still commutes with the Γ-matrices. So starting from complex spinors in a representation of the R-symmetry group, equation (2.2.21) becomes:

$$\lambda_i^* \equiv \left(\lambda^i\right)^* = B\Omega_{ij}\lambda^j\,, \qquad (2.2.30)$$

The advantage of spinors with this type of reality condition is that they transform nicely in a representation of the R-symmetry group of the superalgebra. These symplectic Majorana spinors will be relevant for simple and extended chiral supersymmetry in six dimensions (with R-symmetry group $SU(2)$ and $USp(4)$) in chapter 5. We will use four symplectic Majorana–Weyl spinors in the **4**-representation of R-symmetry group $USp(4)$ instead of two chiral spinors without a reality constraint.

A last possibility is to combine the chirality and the reality condition. Combining (2.2.9) and (2.2.18), one finds that

$$(\Gamma_*)^* = (-)^{d/2+t}\,B\,\Gamma_*\,B^{-1}\,. \qquad (2.2.31)$$

This is compatible with the reality condition if $d/2+t$ is even or if $s-t=0 \mod 4$. All the possibilities are collected in table 2.4. Spinors obeying a combination of the chirality and reality condition are called (symplectic) Majorana–Weyl spinors.

2.2.3 Chiral bosons

Chiral bosons are antisymmetric p-forms in $2p+2$ dimensions with real (anti)self-dual field strengths. Using the definitions for the Levi-Civita tensor ε of appendix A.1, the dual of an n-tensor in $2n$ dimensions is

$$\tilde{F}_{a_1\cdots a_n} = \mathrm{i}^{d/2+t}\tfrac{1}{n!}\varepsilon_{a_1\cdots a_d}F^{a_{n+1}\cdots a_d}\,. \qquad (2.2.32)$$

With this definition, the dual of the dual is the original tensor. The (anti)self-dual component of an n-form is defined as

$$F^{\pm}_{a_1\cdots a_n} = \tfrac{1}{2}\left(F_{a_1\cdots a_n} \pm \tilde{F}_{a_1\cdots a_n}\right)\,. \qquad (2.2.33)$$

d=t+s \ t	0		1		2		3	
1	M	1	M	1				
2	M^-	2	MW	1	M^+	2		
3		4	M	2	M	2		4
4	SMW	4	M^+	4	MW	2	M^-	4
5		8		8	M	4	M	4
6	M^+	8	SMW	8	M^-	8	MW	4
7	M	8		16		16	M	8
8	MW	8	M^-	16	SMW	16	M^+	16
9	M	16	M	16		32		32
10	M^-	32	MW	16	M^+	32	SMW	32
11		64	M	32	M	32		64
12	SMW	64	M^+	64	MW	32	M^-	64

Table 2.4: Spinors in various dimensions, and for various number of time directions (modulo 4). For even dimensions, Weyl (W) spinors are always possible. M stands for Majorana spinors. For even dimensions, $M^{\pm}$ indicates which sign of η (table 2.3) should be used. MW indicates the possibility of Majorana–Weyl spinors. SMW indicates the possibility of symplectic Majorana–Weyl spinors. They are always possible when the Majorana condition is not possible. The numbers indicate the real dimension of the minimal off-shell spinor.

This means that real (anti)self-dual field strengths are possible in spacetimes where $d/2 + t$ is even. So, chiral bosons appear in two, six, ten, ... dimensions for Minkowski spacetimes and multiples of four for Euclidean spacetimes. All possibilities are collected in table 2.5.

Comparing the different dimensions and signatures where chiral bosons are possible (table 2.5) and where (symplectic) Majorana–Weyl spinors are allowed (table 2.4), leads to exactly the same spacetimes. So, the smallest irreducible representations for both fermions and bosons appear in the same spacetimes. When discussing supersymmetry, they will be together in supersymmetric multiplets.

Later in this thesis, we will study two different models. The first model is the $(2, 0)$ self-dual tensor multiplet in six Minkowski dimensions. This self-dual tensor multiplet contains a chiral two-form, four symplectic Majorana–Weyl spinors and five scalars. This means that we have to look in the row with $d = 6$ and the column $t = 1$ for this case. The other model will be vector multiplets with two supersymmetries in four Minkowski dimensions. The fermions will be Majorana fermions as can be seen in table 2.4. Table 2.5 shows that chiral bosons are not possible for Minkowski signature in four dimensions.

d=t+s \ t	0	1	2	3
2		X		
4	X		X	
6		X		X
8	X		X	
10		X		X
12	X		X	

Table 2.5: Possible chiral bosons in various dimensions, and for various number of time directions (modulo 4) are indicated by X .

2.2.4 Superconformal algebras in 4 and 6 dimensions

The superalgebra $SU(2,2|2)$

The superalgebra $SU(2,2|2)$ is one of the possibilities of four-dimensional conformal algebras of table 2.2. We have $N=2$, two supersymmetries in four dimensions. The R-symmetry group is $SU(2)\times U(1)$. The algebra has ten generators in the Poincaré algebra: four translations P_a and six rotations M_{ab}. The bosonic conformal algebra further contains the dilatation generator D and four special conformal generators K_a. They form the algebra[8]

$$\begin{aligned}
[M_{ab},M_{cd}] &= -\eta_{a[c}M_{d]b}+\eta_{b[c}M_{d]a}\,,\\
[P_a,M_{bc}] &= -\eta_{a[b}P_{c]}\,,\\
[K_a,M_{bc}] &= -\eta_{a[b}K_{c]}\,,\\
[P_a,K_b] &= (\eta_{ab}D-2M_{ab})\,,\\
[D,P_a] &= P_a\,,\\
[D,K_a] &= -K_a\,.
\end{aligned} \tag{2.2.34}$$

Unlike the algebras for different spacetimes in section 2.1.1, the generators here carry tangent-space indices. The supersymmetry charges Q_i and special supersymmetry charges S_i carry an $SU(2)$-index $i=1,2$ counting the number of supersymmetries. The place of the index denotes the chirality:

$$\gamma_5 Q_{\alpha i}=Q_{\alpha i}\,,\qquad \gamma_5 Q^i_\alpha=-Q^i_\alpha\,,\qquad \gamma_5 S_{\alpha i}=-S_{\alpha i}\,,\qquad \gamma_5 S^i_\alpha=S^i_\alpha\,. \tag{2.2.35}$$

[8]The factors in this algebra differ slightly from the ones used in (2.1.13) and in (2.2.38). With these conventions, the symplectic structure is achieved more clear and we remain in contact with the literature for $N=2$, d=4. The following conversion gives rise to the bosonic subalgebra in (2.1.13): change the sign of P_a and of M_{ab} and transform K_a into $-\frac{1}{2}K_a$.

The R-symmetry generators are denoted by $U_i{}^j$ and $U(1)$. The $SU(2)$-generators $U_i{}^j$ are anti-Hermitian. They satisfy

$$U_i{}^j \equiv (U^i{}_j)^* = -U^j{}_i, \qquad (2.2.36)$$

where * means complex conjugation. The algebra has the following non-trivial (anti)commutators:

$$\begin{aligned}
\big[M_{ab}, Q^i_\alpha\big] &= \tfrac{1}{4}(\gamma_{ab}Q^i)_\alpha\,, \qquad \big[M_{ab}, S^i_\alpha\big] = \tfrac{1}{4}(\gamma_{ab}S^i)_\alpha\,,\\
\big[K_a, Q^i_\alpha\big] &= (\gamma_a S^i)_\alpha\,, \qquad \big[P_a, S^i_\alpha\big] = \tfrac{1}{2}(\gamma_a Q^i)_\alpha\,,\\
\big[D, Q^i_\alpha\big] &= \tfrac{1}{2}Q^i_\alpha\,, \qquad \big[D, S^i_\alpha\big] = -\tfrac{1}{2}S^i_\alpha\,,\\
\big[U(1), Q^i_\alpha\big] &= -\tfrac{\mathrm{i}}{2}Q^i_\alpha\,, \qquad \big[U(1), S^i_\alpha\big] = -\tfrac{\mathrm{i}}{2}S^i_\alpha\,,\\
\{Q^i_\alpha, \bar{Q}^\beta_j\} &= -\delta_j{}^i(\gamma^a)_\alpha{}^\beta P_a\,,\\
\{S^i_\alpha, \bar{S}^\beta_j\} &= -\tfrac{1}{2}\delta_j{}^i(\gamma^a)_\alpha{}^\beta K_a\,,\\
\{Q^i_\alpha, \bar{S}^\beta_j\} &= -\tfrac{1}{2}\delta_j{}^i\delta_\alpha{}^\beta D + \tfrac{1}{2}\delta_j{}^i(\gamma^{ab})_\alpha{}^\beta M_{ab} - \tfrac{\mathrm{i}}{2}(\gamma_5)_\alpha{}^\beta U(1) - \delta_\alpha{}^\beta U_j{}^i\,,\\
\big[U_i{}^j, Q^k_\alpha\big] &= -\delta_i{}^k Q^j_\alpha + \tfrac{1}{2}\delta_i{}^j Q^k_\alpha\,, \qquad \big[U_i{}^j, S^k_\alpha\big] = -\delta_i{}^k S^j_\alpha + \tfrac{1}{2}\delta_i{}^j S^k_\alpha\,,\\
\big[U_i{}^j, U_k{}^l\big] &= -\delta_i{}^l U_k{}^j + \delta_k{}^j U_i{}^l\,.
\end{aligned} \qquad (2.2.37)$$

In the righthand side of the (anti)commutators, (anti)chiral projections are left out. It should be easy to add them, since the chirality of the supercharges is given in (2.2.35). This corresponds to the algebra used in [31, 32], up to factors: the (special) supersymmetry generators there should be multiplied with $\sqrt{2}$. This algebra will be used in chapter 6 to study the most general coupling of vector multiplets to supergravity.

The superalgebra $OSp(8^*|4)$

The conformal group in 6 dimensions consists of the 21 generators of the Poincaré algebra (6 translations P_a and 15 rotations M_{ab}), the dilatation-generator D and 6 special conformal generators K_a with the following algebra:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{a[c}M_{d]b} - \eta_{b[c}M_{d]a}\,,\\
[P_a, M_{bc}] &= \eta_{a[b}P_{c]}\,,\\
[K_a, M_{bc}] &= \eta_{a[b}K_{c]}\,,\\
[P_a, K_b] &= -2(\eta_{ab}D + 2M_{ab})\,,\\
[D, P_a] &= P_a\,,\\
[D, K_a] &= -K_a\,.
\end{aligned} \qquad (2.2.38)$$

In addition, we have two chiral supersymmetries in six dimensions. This is called (2,0) supersymmetry in six dimensions. When the chirality of the two supersymmetries differs, the model has (1,1) non-chiral supersymmetry. We use the notations of appendix A.3.1. The right-handed supersymmetric generators $Q^i_{\alpha'}$ are symplectic Majorana–Weyl:

$$Q^i_{\alpha'} = \tfrac{1}{2}(\mathbb{1} - \gamma_7)_{\alpha'}{}^{\beta'} Q^i_{\beta'}\,, \qquad Q^i_{\alpha'} \equiv (Q^\beta_i)^* \, (-i\gamma_0)_{\alpha'}{}^{\beta} = \mathcal{C}_{\alpha'\beta}\Omega^{ij} Q^\beta_j\,. \tag{2.2.39}$$

As already mentioned in the general discussion on conformal algebras in section 2.1.3, the commutator of $[K,Q]$ gives rise to special supersymmetries S^i_α. Also these generators are symplectic Majorana–Weyl, but they have the other chirality:

$$S^i_\alpha = \tfrac{1}{2}(\mathbb{1} + \gamma_7)_\alpha{}^\beta S^i_\beta\,. \tag{2.2.40}$$

The anticommutators of Q- and S-generators gives rise to the R-symmetry: represented by $USp(4)$-generators U_{ij} which satisfy $U_{ij} = U_{ji} = (U^{ij})^*$. Their indices are raised and lowered with the $USp(4)$-metric Ω_{ij} and $U^i{}_i = U^{ij}\Omega_{ji} = 0$. The new non-trivial (anti)commutators are:

$$\begin{aligned}
\{Q^i_{\alpha'}, \bar{Q}^\beta_j\} &= -\tfrac{1}{2}(\gamma_a)_{\alpha'}{}^\beta \delta_j{}^i P^a\,,\\
\{S^i_\alpha, \bar{S}^{\beta'}_j\} &= \tfrac{1}{2}(\gamma_a)_\alpha{}^{\beta'} \delta_j{}^i K^a\,,\\
\{Q^i_{\alpha'}, \bar{S}^{\beta'}_j\} &= \tfrac{1}{2}\delta_j{}^i \delta_{\alpha'}{}^{\beta'}\, D - \tfrac{1}{2}(\gamma_{ab})_{\alpha'}{}^{\beta'}\delta_j{}^i M^{ab} - 4\delta_{\alpha'}{}^{\beta'} U^i{}_j\,,\\
\left[M_{ab}, Q^i_{\alpha'}\right] &= -\tfrac{1}{4}(\gamma_{ab}Q)^i_{\alpha'}\,,\\
\left[M_{ab}, S^i_\alpha\right] &= -\tfrac{1}{4}(\gamma_{ab}S)^i_\alpha\,,\\
\left[U^{ij}, Q^k_{\alpha'}\right] &= \tfrac{1}{2}\Omega^{k(i}Q^{j)}_{\alpha'}\,,\\
\left[U^{ij}, S^k_\alpha\right] &= \tfrac{1}{2}\Omega^{k(i}Q^{j)}_\alpha\,,\\
\left[U_{ij}, U_{kl}\right] &= -\tfrac{1}{2}U_{il}\Omega_{jk} + \tfrac{1}{2}U_{kj}\Omega_{il}\,,\\
\left[D, Q^i_{\alpha'}\right] &= \tfrac{1}{2}Q^i_{\alpha'}\\
\left[D, S^i_\alpha\right] &= -\tfrac{1}{2}S^i_\alpha\,,\\
\left[K_a, Q^i_{\alpha'}\right] &= (\gamma_a S)^i_{\alpha'}\,,\\
\left[P_a, S^i_\alpha\right] &= -(\gamma_a Q)^i_\alpha\,.
\end{aligned} \tag{2.2.41}$$

In this algebra, we have used everywhere γ for the matrices of the Clifford algebra. From the notations in section A.3.1, it should be clear which one is meant.

Chapter 3

Methodological background

This chapter introduces two techniques that were extensively used in the development of this work. Section 3.1 reveals the essential features of superconformal tensor calculus. It explains the motivations for using this method. Further, it pays attention to its main ingredients, illustrated by the example of a massive vector. Finally, we build gravity from conformal gravity using what can be called 'conformal tensor calculus'. Some essential steps (independent from any supersymmetry) for the construction of models with local superconformal symmetry are explained here already.

In section 3.2, we will review the Batalin–Vilkovisky method, also called field-antifield formalism. This is a framework that allows studying the most general gauge theories known today in a unified framework. Both classical and quantum-mechanical aspects are treated. We will use this formalism in chapter 4 to study different gauge fixings of the Lorentz-covariant action of the self-dual two-tensor in six dimensions.

3.1 Superconformal tensor calculus

3.1.1 Motivation for superconformal tensor calculus

One of the techniques used in this work to study supergravity models in four and six dimensions is superconformal tensor calculus. Good reviews can be found in [33, 34, 35, 36]. This method enables the construction of actions or equations of motion for pure supergravity or for the coupling of different matter multiplets to Poincaré supergravity. During most of the intermediate steps of the construction, the models under consideration are invariant under superconformal symmetry, but this is *not* the final symmetry group. The main reason for breaking this conformal invariance is that it is only possible for massless fields. Conformal transformations leave invariant the light-cone. This invariance is spoiled by dimensional parameters. Since making contact with the standard model, which contains massive particles (and no conformal symmetry), is our ultimate goal, we are not always interested in models with conformal symmetry.

Why do we make this detour, if the ultimate goal is not a conformally invariant model? The main motivation to use superconformal methods is that they enable a more elegant construction of Poincaré invariant models. It also allows to see more clear the connection between different models with Poincaré symmetry. In this way, no separate tensor calculi need to be set up to study supergravity theories with different auxiliary fields. Another reason is that the origin of certain terms of the intricate expressions of supergravity models are clarified. The higher degree of symmetry makes the transformation rules simpler. Breaking to Poincaré symmetry at the end of the construction leads to certain non-linear terms in the Poincaré theory that were obscure before.

All the results are presented in component form. Also other formulations are possible, e.g., a superspace formulation, where, by using a geometric construction, the spacetime coordinates get accompanied by fermionic coordinates [37].

After all, it is not a surprise that a bigger gauge symmetry can be used that is gauge fixed afterwards. A theory with a gauge symmetry is a formulation of a physical model with more symmetry in the description than in the model itself. This redundancy in the description enables a more elegant formulation. In our approach, the redundancy is the superconformal symmetry that is extra when compared to ordinary supersymmetry. The dilatational, special conformal, and special supersymmetry are not symmetries of the physical models under consideration. Including these symmetries facilitates a more elegant construction.

Even if it seems that conformal symmetry is not a physical symmetry in the study of high energy physics, it can be interesting to study[1]. The last couple of years, conformally invariant theories have attracted more attention after the formulation of the adS/CFT-correspondence.

Superconformal tensor calculus has been used successfully in four dimensions with $N = 1$ [38], $N = 2$ [39] and $N = 4$ [40] supersymmetry. In six dimensions, it has been developed in [41] for one chiral supersymmetry and in [42] for two chiral supersymmetries. Also in five dimensions the construction of conformal gravity is possible, but this is not yet achieved.

[1] Up to now, supersymmetry also seems not to be physical, but it still is interesting to study supersymmetric theories for reasons discussed in the introduction.

3.1.2 The method in general

Gauge equivalence: the essential steps

The aim of superconformal tensor calculus is to construct models with super-Poincaré invariance. The superconformal group is introduced only as a tool to obtain this. The key ingredient in superconformal tensor calculus is using the procedure of *gauge equivalence*. This means that in the process of constructing the model, a bigger symmetry group (in our case the superconformal group) is introduced than the final symmetry required. The extra symmetry is used as a tool in the construction. One constructs representations of this bigger symmetry group. Some of these representations, the compensating multiplets, will contain fields that are not physical, nor auxiliary at the end of the construction. The breaking of the residual symmetry is done by imposing gauge-fixing conditions on fields of the compensating multiplets. The following steps are essential in the construction of an action:

- define extra symmetry
- choose compensators
- construct an invariant action or equations of motion
- impose gauge fixings for the unwanted symmetries
- rewrite the action or the equations of motion

This gauge equivalence approach is not exclusive for the construction of Poincaré supersymmetric actions. It can also be used for more simple models. In the remainder of this section, we will explain the essential ingredients in a simple example. Section 3.1.3 will study the case of conformal gravity. Both examples contain aspects of the models we will be studying later.

A simple example of gauge equivalence

The main ingredients of the gauge equivalence program will first be illustrated by the example of a massive vector. The Lagrangian of a massive vector V_μ can be written as

$$L = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}m^2 V_\mu V^\mu\,. \qquad (3.1.1)$$

Because of the mass term this Lagrangian is not invariant under an Abelian $U(1)$-symmetry. By following the different steps of the gauge equivalence program, one can recover the action (3.1.1) for a massive vector.

The first step consists of defining an extra symmetry. In this example, we choose the extra symmetry to be the Abelian group $U(1)$. This symmetry will be broken at the end, since we want to end up with the action for a massive vector which has no $U(1)$-invariance. The vector in the model will transform under this $U(1)$:

$$\delta V_\mu = \partial_\mu \Lambda \,. \tag{3.1.2}$$

In a second step, the compensator is introduced: a real scalar field ϕ that transforms as a shift:

$$\delta\phi = \Lambda \,. \tag{3.1.3}$$

By introducing an extra scalar and a new gauge symmetry, the number of degrees of freedom in the model is not altered.

The next step of the program is the construction of an action for these two fields. Therefore, the covariant derivative of the scalar is defined in the usual way: the partial derivative minus the gauge transformation with the parameter replaced by the corresponding gauge field[2]:

$$D_\mu\phi = \partial_\mu\phi - V_\mu \,. \tag{3.1.4}$$

This covariant derivative is gauge invariant and can be used in the invariant Lagrangian:

$$L = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}m^2 D_\mu\phi D^\mu\phi \,. \tag{3.1.5}$$

The final step in this example is the gauge fixing of the unwanted symmetry. The easiest thing to do is choosing $\phi = 0$. This gauge choice can always be reached by the shift symmetry. Imposing it gives rise to the action for a massive vector with mass m. In this case, the construction of an action that is known already for a long time is rather elaborate, but the main steps in the construction are all included. This example also makes clear that this technique is not tied to any conformal symmetry or to supersymmetry.

3.1.3 Conformal gravity

In this subsection, pure gravity in d dimensions will be constructed using conformal methods. This enables a clear explanation of the conventional constraints. Later

[2] In the six-dimensional model, we will recover a similar gauge transformation and covariant derivative for a scalar field, essential in the covariant formulation of the model.

on, we will use it to illustrate the gauge equivalence program. The extra symmetry in this example is conformal symmetry. The conformal algebra in d dimensions is $SO(d,2)$. Its commutation rules are given in (2.1.13). This algebra has d generators P_a for translations, $\frac{1}{2}d(d-1)$ Lorentz generators M_{ab}, one dilatational generator D and d generators K_a for special conformal transformations.

generators	P_a	M_{ab}	D	K_a
gauge fields	${e_\mu}^a$	${\omega_\mu}^{ab}$	b_μ	${f_\mu}^a$
parameters	ξ^a	λ^{ab}	Λ_D	Λ_K^a

Table 3.1: The generators of the algebra (2.1.13), its gauge fields and the corresponding parameters.

The gauge fields corresponding to these symmetries can be found in table 3.1. Gauge fields transform in general as follows:

$$\delta h_\mu^A = \partial_\mu \varepsilon^A + \varepsilon^C h_\mu^B f_{BC}^A \,. \tag{3.1.6}$$

The numbers f_{BC}^A are the structure constants (or structure functions) of the symmetry algebra. The infinitesimal transformation of the gauge fields of the conformal algebra (2.1.13) are:

$$\begin{aligned}
\delta e_\mu^a &= \mathcal{D}_\mu \xi^a - \Lambda_D {e_\mu}^a - \lambda^{ab} e_{\mu b} \,, \\
\delta b_\mu &= \partial_\mu \Lambda_D - 2\Lambda_K^a e_{\mu a} + 2\xi^a f_{\mu a} \,, \\
\delta {\omega_\mu}^{ab} &= \mathcal{D}_\mu \lambda^{ab} + 4\Lambda_K^{[a} {e_\mu}^{b]} - 4\xi^{[a} {f_\mu}^{b]} \,, \\
\delta {f_\mu}^a &= \mathcal{D}_\mu \Lambda_K^a + \Lambda_D {f_\mu}^a - \lambda^{ab} f_{\mu b} \,,
\end{aligned} \tag{3.1.7}$$

where $\mathcal{D}_\mu$ is the covariant derivative with respect to the dilatations and Lorentz rotations:

$$\begin{aligned}
\mathcal{D}_\mu \xi^a &= \partial_\mu \xi^a + b_\mu \xi^a + {\omega_\mu}^{ab} \xi_b \,, \\
\mathcal{D}_\mu \Lambda_K^a &= \partial_\mu \Lambda_K^a - b_\mu \Lambda_K^a + {\omega_\mu}^{ab} \Lambda_{Kb} \,, \\
\mathcal{D}_\mu \lambda^{ab} &= \partial_\mu \lambda^{ab} + 2{\omega_{\mu c}}^{[a} \lambda^{b]c} \,.
\end{aligned} \tag{3.1.8}$$

The natural way to form curvatures is:

$$R_{\mu\nu}{}^A = 2\partial_{[\mu} h_{\nu]}{}^A + h_{[\nu}{}^C h_{\mu]}{}^B f_{BC}^A \,. \tag{3.1.9}$$

This gives rise to the following curvatures:

$$R_{\mu\nu}^a(P) = 2\partial_{[\mu} e_{\nu]}{}^a + 2b_{[\mu} e_{\nu]}{}^a + 2\underline{\omega_{[\mu}{}^{ab} e_{\nu]b}} \,,$$

$$\begin{aligned} R_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b} - 8\underline{f_{[\mu}{}^{[a}e_{\nu]}{}^{b]}}\,, \\ R_{\mu\nu}(D) &= 2\partial_{[\mu}b_{\nu]} + 4\underline{f_{[\mu}{}^{a}e_{\nu]a}}\,, \\ R^{a}_{\mu\nu}(K) &= 2\partial_{[\mu}f_{\nu]}{}^{a} + 2\omega_{[\mu}{}^{ab}f_{\nu]b} - 2b_{[\mu}f_{\nu]}{}^{a}\,. \end{aligned} \quad (3.1.10)$$

Up to now, the conformal group (with the Poincaré group as its subgroup) has been treated as an internal symmetry group with no relation to any spacetime. Making contact with a description of spacetime needs a formulation in terms of general coordinate transformations. They describe reparametrizations of the base manifold spanned by the spacetime coordinates:

$$x^{\mu} \to x^{\mu} + \xi^{\mu}(x)\,. \quad (3.1.11)$$

The conversion of Poincaré transformations into transformations of spacetime can be understood by rewriting the general coordinate transformation of $e_{\mu}{}^{a}$ with a transport term and a rotation term as follows:

$$\begin{aligned} \delta_{gct}e_{\mu}{}^{a} &= \xi^{\nu}\partial_{\nu}e_{\mu}{}^{a} + \partial_{\mu}\xi^{\nu}\cdot e_{\nu}{}^{a} \\ &= \mathcal{D}_{\mu}(\xi^{\nu}e_{\nu}{}^{a}) - (\xi^{\nu}\omega_{\nu}{}^{ab})e_{\mu b} - (\xi^{\nu}b_{\nu})e_{\mu}{}^{b} - \xi^{\nu}R^{a}_{\mu\nu}(P)\,. \end{aligned} \quad (3.1.12)$$

Following equation (3.1.7), this can be rewritten as a combination of a translation with parameter $\xi^{\nu}e_{\nu}{}^{a}$, a Lorentz transformation with parameter $\xi^{\nu}\omega_{\nu}{}^{ab}$ and a dilatation with parameter $\xi^{\nu}b_{\nu}$ if the following constraint is introduced:

$$R^{a}_{\mu\nu}(P) = 0\,. \quad (3.1.13)$$

In this way, local Poincaré and general coordinate transformations are no longer independent. This is called a conventional constraint because the appearance of the (underlined) product of a vielbein[3] and $\omega_{\mu}{}^{ab}$ enables to express the spin connection $\omega_{\mu}{}^{ab}$ in terms of other fields. $\omega_{\mu}{}^{ab}$ is a composite gauge field. This formulation in terms of a spin connection is necessary to couple spinors to gravity. The solution of the constraint (3.1.13) is:

$$\omega_{\mu}{}^{ab} = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\rho[a}e^{b]\sigma}e_{\mu}{}^{c}\partial_{\rho}e_{\sigma c} + 2e_{\mu}{}^{[a}b^{b]}\,. \quad (3.1.14)$$

In conformal gravity, a second conventional constraint is necessary. The underlined term in expression (3.1.10) for the curvature $R_{\mu\nu}{}^{ab}(M)$ also contains the product of a field and the vielbein. Imposing

$$R_{\mu\nu}{}^{ab}e^{\nu}{}_{b} = 0\,, \quad (3.1.15)$$

[3]Remark that the vielbein $e_{\mu}{}^{a}$ has to be non-singular. Its inverse is $e_{a}{}^{\mu}$.

expresses $f_\mu{}^a$ also as a composite gauge field:

$$f_\mu{}^a = -\tfrac{1}{2(d-2)} R'_\mu{}^a(M) + \tfrac{1}{4(d-1)(d-2)} e_\mu{}^a R'(M)\,, \tag{3.1.16}$$

where $R'_\mu{}^a(M) = e^\nu{}_b R'_{\mu\nu}{}^{ba}$ and $R'_{\mu\nu}{}^{ab}$ is the Riemann curvature in (3.1.10) without the underlined term. In a supersymmetric model, the two conventional constraints (3.1.13) and (3.1.16) get accompanied by a constraint for the fermionic fields. As can be seen in (3.1.10), also the dilatational curvature contains a term with the product of a vielbein and the gauge field b_μ. However, there is no new constraint any more. The Bianchi identity for P_a gives a relation between $R_{\mu\nu}(D)$ and the contracted form of $R_{\mu\nu}{}^{ab}$ in (3.1.16). The two constraints (3.1.13) and (3.1.16) are the maximal number that can be imposed. They give rise to an irreducible representation of the conformal algebra.

We thus find that conformal gravity has two independent gauge fields: $e_\mu{}^a$ and b_μ. In d dimensions we can count the degrees of freedom in the following way. A priori, $e_\mu{}^a$ and b_μ have d^2+d degrees of freedom. Subtracting d degrees of freedom for general coordinate transformations, d other components for conformal boosts, $\frac{d(d-1)}{2}$ degrees of freedom for Lorentz rotations and one for dilatations, gives $\frac{d^2}{2} - \frac{d}{2} - 1$ off-shell components. This counting is called an off-shell counting, since no equations of motion are used. After this subtraction, this is called a 'massive' graviton[4]. In four dimensions, the massive graviton has five components. In 6 dimensions, this counting gives rise to a graviton with 14 off-shell degrees of freedom. These countings will be relevant in chapter 5 for six dimensions and in chapter 6 for four dimensions. Remark that there are no degrees of freedom associated anymore to the field b_μ. It is the only independent field that transforms under conformal boosts. Therefore, it can be completely gauged away. All the composite gauge fields transform under special conformal symmetry via their dependence on b_μ.

Gauge equivalence for conformal gravity

After the analysis of conformal gravity, it is possible to study the coupling of matter to conformal gravity. This will again illustrate the 'gauge equivalence' approach in an example of 'conformal tensor calculus'. We start from a bigger symmetry then we want to end up with: conformal symmetry.

[4]The word 'massive' is inspired by the resemblance with a massive vector in four dimensions with three degrees of freedom. Its number of degrees of freedom is the same as that of a massless vector where the equation of motion is not yet imposed.

The second step: the compensator we choose to gauge fix this symmetry later on is a scalar field. Consider a scalar field that is invariant under conformal boosts

$$\delta_K \phi = 0\,. \tag{3.1.17}$$

It has Weyl weight w under dilatations:

$$\delta_D \phi = w \Lambda_D \phi\,. \tag{3.1.18}$$

The covariant derivative of such a scalar is

$$D_\mu \phi = \partial_\mu \phi - w b_\mu \phi\,. \tag{3.1.19}$$

Since b_μ is not invariant under K_a-transformations, the covariant derivative of the K_a-invariant scalar will transform:

$$\delta_K (D_a \phi) = 2w \Lambda_{K\,a} \phi\,. \tag{3.1.20}$$

The dilatational weight of $D_a \phi$ is $w+1$. Its covariant derivative is

$$D_\mu (D^a \phi) = \partial_\mu D^a \phi + \omega_\mu{}^{ab} D_b \phi - (w+1) b_\mu D^a \phi - 2w f_\mu{}^a \phi\,. \tag{3.1.21}$$

The expression

$$\Box_c \phi = D_a (D^a \phi) = \partial_a D^a \phi + \omega_a{}^{ab} D_b \phi - (w+1) b_a D^a \phi - 2w f_a{}^a \phi \tag{3.1.22}$$

is called a covariant d'Alembertian. It is an easy job to derive the following transformation rules:

$$\begin{aligned}
\delta_K D_\mu D^a \phi &= (2w+2)\Lambda_K^a D_\mu \phi + (2w+2)\Lambda_{K\mu} D^a \phi - 2\Lambda_K^b D_b \phi\, e_\mu{}^a\,, \\
\delta_K D_a D^a \phi &= (4w+4-2d)\Lambda_K^b D_b \phi\,, \\
\delta_D D_a D^a \phi &= (w+2) D_a D^a \phi\,.
\end{aligned} \tag{3.1.23}$$

For $w = \frac{d-2}{2}$, it is possible to construct a conformally invariant Lagrangian for a scalar coupled to a background of conformal gravity. Such a background of conformal gravity is not the same as dynamical conformal gravity. Dynamical conformal gravity requires a higher derivative action, quadratic in the Weyl tensor [43]. The case of conformal supergravity is treated in [44]. The Lagrangian contains also a conformally invariant self-coupling with coupling constant g which will give rise to a cosmological term:

$$\begin{aligned} L &= -e\phi D_a D^a \phi + eg\phi^{\frac{2d}{d-2}} \\ &= -e\phi \left(\partial_a D^a \phi + \omega_a{}^{ab} D_b \phi - (w+1) b_a D^a \phi + \tfrac{w}{2(d-1)} R' \phi \right) \\ &\quad + eg\phi^{\frac{2d}{d-2}} , \end{aligned} \tag{3.1.24}$$

where we used that $f_a{}^a = -\frac{1}{4(d-1)} R'$ from (3.1.16). The vielbein determinant e provides invariance under dilatations and general coordinate transformations. This density finishes the third step in the gauge equivalence program. The sign of the kinetic term is 'wrong'. The reason for it will become clear at the end of the section. This finishes the construction of the action that is invariant under conformal symmetry. Now everything is ready to study the gauge equivalence of this model with general relativity.

The following step consists of making the appropriate gauge choices for the unwanted symmetries. Here, we want to illustrate the gauge equivalence program and get rid of the superfluous symmetries, in this case the special conformal symmetries and the dilatational symmetry. The only independent field that still transforms under special conformal transformations is b_μ. It is possible to gauge fix this symmetry by imposing:

$$b_\mu = 0 \, . \tag{3.1.25}$$

By choosing ϕ to be a constant, the covariant derivative of ϕ (3.1.19) is 0. This means that in the conformal d'Alembertian (3.1.22) only the term

$$-2w f_\mu{}^\mu \phi = \frac{w}{2(d-1)} R' \, \phi \tag{3.1.26}$$

survives. Choosing

$$\phi = \sqrt{\frac{2(d-1)}{d-2}} \tag{3.1.27}$$

changes (3.1.24) into

$$L' = -\tfrac{1}{2} e R' + eg \left(\frac{2(d-1)}{d-2} \right)^{\frac{d}{d-2}} . \tag{3.1.28}$$

This is the Einstein–Hilbert term in the action (3.1.24) if $g = 0$. For g different from 0, we find a cosmological term giving rise to a de Sitter space or an anti de

Sitter space. Positive g corresponds to a negative cosmological constant of an anti de Sitter space, and the other way around.

This procedure shows how to build the Einstein–Hilbert action for pure gravity in d dimensions by gauge fixing the action of a scalar coupled in a conformal gravity background. The 'wrong' sign for the kinetic term in (3.1.24) was needed to find the appropriate sign for the Einstein–Hilbert term here. This phenomenon is also encountered in the superconformal tensor calculus. Starting from a density with more matter fields (with the right kinetic term) than just the one compensating scalar will give rise to the coupling of matter to Poincaré gravity. This is of course a much too long derivation for the result it yields. It is because this example illustrates the importance of both the construction of an action that is invariant under the bigger symmetry group and the way in which the superfluous symmetries are gauge fixed that we have spent so much attention to it. We use the same mechanism in the supersymmetric case in four and six dimensions later on.

3.2 A review of the Batalin–Vilkovisky method

3.2.1 Some background on gauge theories

It is difficult to overrate the role of gauge symmetry in the formulation of elementary particle theories. The first example of a gauge theory was electrodynamics. Electric and magnetic forces are generated by the exchange of photons. Photons have spin one and are described by an Abelian vector field A_μ. Not all of its four components are dynamical. The two transversal components correspond to physical polarizations. The longitudinal degree of freedom plays a role in the exchange of virtual photons. The remaining gauge degree of freedom does not enter the theory. Non-Abelian gauge theories are the cornerstone of the standard model.

The quantization of gauge symmetries is far from trivial. In the Abelian case, the quantization was established in the 50's. Generally, the quantization of a gauge theory involves ghost fields. The ghost fields are needed to compensate for the effects of (superfluous) gauge degrees of freedom, in order to preserve unitarity. With linear gauges, ghosts decouple in electrodynamics and can be ignored. This is not the case anymore in non-Abelian theories. There, convenient gauges generically imply interacting ghosts. A systematic procedure for the gauge-fixing of non-Abelian theories was established by the Faddeev–Popov procedure [45] which introduced in a systematic way the ghost sector. Becchi, Rouet, Stora and Tyutin realized that this gauge-fixed action, where the presence of the ghosts is understood as a "measure effect", still contains a nilpotent, fermionic, global symmetry: the BRST-symmetry [46].

Later on, not only the group structures of gravity and the Virasoro algebra in string theory were studied, but also gauge theories with more sophisticated algebraic structures. The first one of these extensions were theories with soft algebras [47], where one had (field-dependent) structure functions instead of structure constants. Another generalization were models in gravity [48] and supergravity [49, 50, 51] with open algebras [50, 52]: the commutator of two transformations closes only if one uses the equations of motion. Another complication were reducible algebras: not all gauge transformations are independent. The main reason to work with such a redundancy is the same as to work with gauge theories: one wants to keep Lorentz invariance explicit. The dependency relations between the gauge transformations are called 'zero-modes'. These zero-modes can have further zero-modes, so a next stage of reducibility is reached [53, 54]. Also infinitely-reducible models [55, 56] are possible.

For each of these complications, appropriate solutions were invented. The big advantage of the field-antifield formalism is that it is the only formalism that can treat all these difficulties in a unified way, separately and in combination. It is the merit of Batalin and Vilkovisky to generalize the role of the symplectic structure, that was known already in the BRST-framework [57], to antibrackets, and the sources for BRST-transformations to antifields. Therefore, the field-antifield formalism is often called Batalin–Vilkovisky (or BV) formalism. Original references can be found in [58, 59, 60, 52, 61], some reviews in [62, 63, 64].

The two essential ingredients of the BV-formalism are the antifields and the antibrackets. For each field or ghost (or ghost for ghost, ...), there is an antifield. The antifields are like the momentum conjugates for the coordinates in the canonical formalism. Antifields are the sources for the BRST-symmetries, the symmetries of the Lagrangian after gauge fixing. Antibrackets, the second essential ingredient, introduce a symplectic form on the space of fields and antifields. They are an attractive feature of the formalism since they have the same role in the space of fields and antifields as the Poisson brackets in the phase space with coordinates and conjugate momenta in classical mechanics. Furthermore, the antibrackets enable a covariant treatment. For the Hamiltonian approach, we refer to [65, 66].

3.2.2 The classical Batalin–Vilkovisky formalism

In this section we give a brief technical account of the classical BV-procedure. A *ghost number* g is assigned to the gth generation of ghosts, where the classical fields have $g = 0$. For every field, including the ghosts, an *antifield* with opposite statistics is introduced. So, if the field Φ is bosonic, its antifield Φ^* will be fermionic and vice versa. We denote the set of classical and ghost fields by Φ^A and the set of their antifield counterparts by Φ^*_A. The antifields are assigned a ghost number such that

$$g(\Phi^A) + g(\Phi^*_A) = -1\,. \tag{3.2.1}$$

We also define an *antifield number*, which is 0 for fields and equal to $-g$ for antifields. For functionals $F(\Phi,\Phi^*)$, $G(\Phi,\Phi^*)$, we introduce the *antibracket*

$$(F,G) = F\frac{\overleftarrow{\partial}}{\partial\Phi^A}\frac{\overrightarrow{\partial}}{\partial\Phi^*_A}G - F\frac{\overleftarrow{\partial}}{\partial\Phi^*_A}\frac{\overrightarrow{\partial}}{\partial\Phi^A}G, \tag{3.2.2}$$

where summation over A includes also spacetime integration. The analogy of the fields and antifields in the antibrackets with respect to the coordinates and momenta in Poisson brackets can be illustrated in the following way:

$$\left(\phi^A,\phi^B\right) = 0, \qquad (\phi^*_A,\phi^*_B) = 0, \qquad \left(\phi^A,\phi^*_B\right) = \delta^A_B\,. \tag{3.2.3}$$

Sometimes we will add to the antibracket an index between brackets. $(F,G)_{(n)}$ means that in (3.2.2), only the terms with antifield number n are to be included.

The *minimal extended action* is defined by adding to the classical action a part containing the antifields, multiplied by the BRST-transformations. The parameters of the gauge transformations are replaced by ghost fields c^B:

$$\begin{aligned} S_{min} &= S_{cl} + S_{(1)} \\ &= S_{cl} + \Phi^*_A R^A{}_B\, c^B, \end{aligned} \tag{3.2.4}$$

where $\delta\Phi^A = R^A{}_B(\phi^C)\,\varepsilon^B$ is the infinitesimal gauge transformation with parameter ε^B. The extra term $S_{(1)}$ has antifield number 1.

If the gauge transformations of S_{cl} have zero modes, the gauge symmetry is reducible. Extra terms have to be introduced which have an analogous form: an antifield that is the antighost of the relevant gauge transformation, multiplied by the zero mode transformation, where the parameter has been replaced by a ghost for ghost. This yields a term of antifield number 2. In the case of an infinitely reducible theory, this is an infinite series of terms. This procedure ensures that for every gauge symmetry a ghost has been introduced, and for every zero mode a ghost for ghost. This can be checked by counting the number of zero modes of the Hessian

$$S_A{}^B = \frac{\overrightarrow{\partial}}{\partial\Phi^A}S\frac{\overleftarrow{\partial}}{\partial\Phi^*_B} \tag{3.2.5}$$

at the stationary surface. This is the surface where the classical fields satisfy the field equations and the ghosts and antifields are set to 0. The result should be half the number of fields and antifields. If this is the case, the *properness condition* is fulfilled.

Other terms have to be added, if the algebra of the model is open of soft. For a soft algebra, an additional term is introduced for the commutators of the gauge symmetries, of the form

$$\int dx\, \tfrac{1}{2}\, (-)^B c^*_A\, T^A_{BC}\, c^B c^C\,, \tag{3.2.6}$$

where the T^A_{BC} are the structure functions associated to the commutators of the gauge symmetries and $(-)^B$ is the fermion number of ghost c^B. For non-Abelian gauge theories, the T^A_{BC} are the structure constants. In our algebra in section 4.3, they will be field-dependent structure functions. In the case of an open algebra, the extended action contains a term

$$S_{(2)} = (-)^{i+A}\, \tfrac{1}{4}\, \phi^*_i \phi^*_j E^{ji}{}_{AB} c^B c^A\,. \tag{3.2.7}$$

This is the product of the square of the antifields and the square of the ghosts and the corresponding non-closure functions E. In this way, more terms of higher antifield number, $S_{(3)}, \ldots, S_{(n)}$, are added until the resulting action, called the *extended action*, satisfies the *classical master equation*, which is

$$(S_{ext}, S_{ext}) = 0\,. \tag{3.2.8}$$

The *classical limit* imposes that the truncation of the extended action to the antifield-independent part should be the classical action.
If the three conditions:

1. properness condition
2. classical master equation: $(S_{ext}, S_{ext}) = 0$
3. classical limit

are satisfied, S_{ext} is an extended action that can be used for studying the gauge-fixing and later the quantum-mechanical properties in the BV-formalism.

To perform the gauge-fixing, two main ingredients can be used: canonical transformations and addition of auxiliary ('non-minimal') fields.

Canonical transformations from the basis $\{\Phi^A, \Phi^*_A\}$ to another set $\{\tilde{\Phi}^A, \tilde{\Phi}^*_A\}$ are defined to leave the antibrackets invariant. The same terminology is used in the Hamiltonian formulation of classical mechanics. Often these canonical transformations can be determined by a fermionic generating function of ghost number -1, $F(\Phi, \tilde{\Phi}^*)$, such that

$$\tilde{\Phi}^A = \frac{\partial F(\Phi, \tilde{\Phi}^*)}{\partial \tilde{\Phi}^*_A}, \qquad \Phi^*_A = \frac{\partial F(\Phi, \tilde{\Phi}^*)}{\partial \Phi^A}. \tag{3.2.9}$$

The generating function is of the form

$$F(\Phi, \tilde{\Phi}^*) = \Phi^A \tilde{\Phi}^*_A + \Psi(\Phi). \tag{3.2.10}$$

The function Ψ, also a fermion of ghost number -1, is called the *gauge fermion*. If the canonical transformation is not invertible in the field-field part, then it cannot be generated by a gauge fermion. In section 4.4, we will give an example of such a canonical transformation.

To do the gauge fixing, it will sometimes be necessary to introduce *auxiliary fields* and their antifields b and b^*. They should not have any influence on the master equation, and the physical content of the action should not be changed. For example, one gives the field b a ghost number -1, so that its antifield has ghost number 0, and one adds to the action the term $S_{nm} = (b^*)^2$. Fields with negative ghost number are called *non-minimal*. There are two typical forms to introduce non-minimal extensions:

$$\begin{aligned} S^1_{nm} &= \tfrac{1}{2}\lambda^2 , \\ S^2_{nm} &= b^* \lambda . \end{aligned} \tag{3.2.11}$$

The first non-minimal extension is described by a bosonic field λ and its antifield, the second extension has two sets of a field and its antifield. Both types of non-minimal extensions will be used in section 4.4. Gauge fixing is performed by a combination of introducing non-minimal sets and canonical transformations on the set of fields and antifields. This operation should make sure that the antifield-independent part of the extended action has a Hessian that is invertible on the stationary surface. In that case, well-defined propagators can be calculated. Although it is not difficult to explain how the gauge fixing can be done, this does not mean that the gauge-fixing is not difficult. Raymond Stora's statement: 'Gauge fixing is an art' remains valid. There are plenty of possibilities to transform the fields and antifields into a new set where the Hessian of the fields is invertible, but it is far from trivial which choice has to be made to achieve this in a manageable way. The advantage of the gauge-fixing using canonical transformations is the possibility to go from one gauge-fixing to another by doing canonical transformations.

3.2.3 A short review of the quantum BV-formalism

After doing the gauge fixing, one has not obtained any quantum-mechanical result yet. One has only finished the preparation to start a quantum-mechanical calculation. We will sketch how to obtain also quantum-mechanical results using the quantum BV-formalism. Vaster literature can be found in the references of the previous section and in [67].

We already stated that gauge fixing is doing a canonical transformation to a gauge-fixed basis such that it becomes possible to compute propagators. There was an arbitrariness in this gauge fixing and we do not want the quantum theory to depend on this choice of gauge fixing. Therefore, the path integral should be invariant under a canonical transformation. The translation of this statement into the BV-language is that the *quantum master equation* should be satisfied:

$$(W, W) = 2\mathrm{i}\hbar\Delta W\,, \tag{3.2.12}$$

where W is the quantum action, an expansion in terms of $\hbar$ with the classical action as the $\hbar^0$-term:

$$W = S + \sum_{i=1}^{\infty} \hbar^i M_i\,. \tag{3.2.13}$$

The M_i's in the power expansion in $\hbar$ are counterterms. The Δ-operator is formally defined as

$$\Delta = \frac{\partial}{\partial\Phi^A} \cdot \frac{\partial}{\partial\Phi^*_A}\,. \tag{3.2.14}$$

Remark that the part of the quantum master equation (3.2.12) independent of $\hbar$ is the classical master equation (3.2.8). If the quantum theory is dependent of the chosen gauge-fixing, there is a symmetry in the classical model that is not surviving at the quantum level. The translation of this statement into the BV-language is that the classical master equation (3.2.8) is satisfied, but the quantum master is broken:

$$\mathcal{A} = \Delta W + \tfrac{\mathrm{i}}{2\hbar}\,(W, W)\,. \tag{3.2.15}$$

A rigorous computation of ΔW in (3.2.15) for a local action is proportional to $\delta(0)$ and thus ill defined. This expresses that a regularization is needed. In principle different regularization schemes are possible. The results obtained by using different regularization schemes can be related by choosing an appropriate local counterterm in the action. One could use dimensional regularization [68], but this has as disadvantage that it is hard to evaluate a path integral for non-integer dimensions. Another possibility is to use point-splitting [69] or lattice regularization, but then it becomes difficult to keep working with local field theories. Therefore, the regularization scheme mostly used is Pauli–Villars (PV) regularization[5] [70]. The most important ingredient of PV-regularization is the introduction of a massive field for each field in the theory. The Feynman diagrams are replaced by the original expression minus a similar expression where the particles that propagate in the loops are extremely massive. This renders finiteness. One gets rid of these massive PV-fields by sending their mass to infinity after the calculation of the diagram. In this way, their influence in low-energy physics disappears. For calculating only one-loop terms, it is sufficient to use this regularization scheme. Higher loop calculations require other regularization schemes.

Adding an extra counterterm $\hbar X$ to the quantum action (3.2.13), causes a shift in the anomaly. This can all be expressed in a cohomological sense [63]. Saying that $\mathcal{A} \neq 0$ then means that there is no counterterm that can shift $\mathcal{A}$ to 0. Using the PV-regularization scheme, it is possible to calculate anomalies up to one [72] or more [73] loops.

This concludes a short summary of the BV-formalism. For justification of the statements made here, and for further details, one should consult one or more of the clearly written original references [58, 59, 60, 52, 61, 67] or one of the reviews [62, 63, 64].

[5] Both point-splitting and Pauli–Villars regularization are used nowadays in the study of string theory on non-commutative spaces (spaces with constant antisymmetric NS two-form $B_{\mu\nu}$) in [71].

Chapter 4

A Lorentz-covariant action for the self-dual tensor

This chapter describes some bosonic aspects of the PST construction[1] of a covariant action for a self-dual two-tensor in six dimensions. Using PST gauge symmetries, it is possible to construct a local, non-polynomial Lorentz-covariant action for the self-dual two-tensor in six dimensions. First, some background on chiral bosons is given. In section 4.2, the different approaches to construct actions for chiral bosons in two, six and ten dimensions are reviewed. In a third section, the gauge symmetries, essential in this construction, are studied. Also some aspects of this action are stressed: its equations of motion and the counting of the three degrees of freedom of a self-dual two-tensor. In a last section, aspects of these gauge symmetries will be studied using the Batalin–Vilkovisky (BV) method. Two gauge fixings (a covariant one and a non-covariant one) are obtained and their possible use in quantum-mechanical calculations is given. This chapter is restricted to the bosonic model. The role of self-dual bosons in a supersymmetric context is postponed to the next chapter. Most of the new results in this chapter, mainly the different gauge fixings in the BV-language and their possible applications, were reported already in [6].

[1] The method to write Lorentz-covariant actions was introduced by P. Pasti, D. Sorokin and M. Tonin (PST) in [5].

4.1 Chiral bosons in different dimensions

As explained in section 2.2.3, chiral bosons are antisymmetric p-forms in $2p+2$ dimensions with real (anti)self-dual field strengths. Table 2.5 lists the possibilities to have chiral bosons. When the signature is Minkowskian, they appear in two, six and ten dimensions. For other signatures, also the other even dimensions are possible.

Chiral bosons appear in supersymmetric models in two, six and ten dimensions, but they are also relevant in physical models that have nothing to do with supersymmetry. In [74], the theory of chiral bosons in two dimensions is used in the description of the dynamical properties of the edge excitations of the fractional quantum Hall effect[2]. In superstring theory or supergravity, models with chiral bosons are frequent: chiral scalars appear in the worldsheet description of the heterotic string [77], chiral two-tensors pop up in the matter and the gravity multiplets of simple and extended chiral six-dimensional supergravity and in the worldvolume description of $M5$- and $NS5$-branes and the self-dual four-form is part of IIB supergravity [78] in ten dimensions.

When the signature is Euclidean, real (anti)self-dual field strengths are possible in four and eight dimensions. In four Euclidean dimensions, chiral bosons are of great importance. (Anti)self-dual field strengths appear in the context of instantons, finite-action solutions of Euclidean gauge field equations. These enable the derivation of important non-perturbative results in the context of the standard model as elucidated in the original papers [79, 80, 81, 82, 83] and one of the lectures of [84]. In eight Euclidean dimensions, self-dual three-forms are studied for instance in [85]. As can be found in table 2.5, other models with self-dual field strengths appear in spacetimes with the following signatures: $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, and other spacetimes, with more than one time direction, that satisfy the condition for chiral bosons. Even if these models with more than one time direction may look rather strange, some properties of them are known. Spaces with $(2,2)$ signature are called Atiyah–Ward spacetimes and supersymmetric non-linear sigma models in $(2,2)$ dimensions are studied in [86]. The last couple of years, models with $(10,2)$ signature are studied in [87, 88]. In [88], the $D3$-brane of IIB supergravity is embedded in 12 dimensions. This embedding is used to enlarge insight in the description of certain branes. This is done by introducing a self-dual five-form in this treatment of models with $(10,2)$ signature.

As was derived in section 2.2.3, real (anti)self-dual field strengths in four dimensions with Lorentz signature are not possible. The complex antiself-dual field strengths are an essential ingredient in the most general coupling of vector multi-

[2] In 1998, R. Laughlin, H. Störmer and D. Tsui [75] got the Nobel price *for their discovery of a new form of quantum fluid with fractionally charged excitations* [76].

plets to $N = 2$ supergravity in four dimensions. It is also possible to study self-dual tensors in six dimensions with Euclidean signature [41]. Since the conditions to have real self-dual field strengths is not satisfied, they have twice as many degrees of freedom as chiral bosons in six Lorentzian dimensions.

There is also a concept of 'self-duality in odd dimensions' [89]. With a 'self-dual field', one means there a massive antisymmetric tensor field in $4k-1$ dimensions that satisfies certain equations. It propagates with half as many massive modes as expected. In [90] is explained how this self-duality in odd dimensions can be found from a higher-dimensional embedding.

4.2 Actions for chiral bosons in 2, 6, and 10 dimensions

4.2.1 Why looking for a (Lorentz-covariant) action?

The description of chiral bosons in two, six and ten dimensions in terms of a Lorentz-covariant action has been a longstanding problem. The core of this problem is the first order self-duality condition. One prefers to have a Lorentz-covariant action to be able to use path integral methods to study quantum-mechanical aspects of these models, for instance using the Batalin–Vilkovisky quantization method. The absence of a Lorentz-covariant action makes the analysis of quantum-mechanical properties of these models more cumbersome.

Another reason for being interested in a Lorentz-covariant action is the following. There are two ways to look upon relativistic invariance. The first one demands that there should be *no privileged reference frame*. Technically, this requires that there should be a frame-independent prescription for how to transform a description of a physical situation in a particular inertial frame to a description in terms of another inertial frame. The set of transformations from one frame to another should obey the multiplication laws of the Poincaré group. Another, seemingly-stronger definition states that relativistic invariance means that the description of the system should be *manifestly covariant*. This ensures that the theory can be formulated fully independent of any reference frame. The question is whether this second definition of relativistic invariance is really stronger than the first one. In four dimensions, it is proven [91] that for free field theories the two definitions are equivalent: to every representation of the Poincaré group corresponds a manifestly covariant free field theory. In [92] was proven that it is not possible to construct a simple (i.e., for a free field quadratic), manifestly Lorentz-covariant Lagrangian for a chiral boson in six or ten dimensions. Extra propagating fields are needed. This seemed to indicate that the two ways to look upon relativistic invariance are not equivalent. In this section, we will sketch in which way (manifestly covariant) actions for chiral bosons were constructed. This does not prove that the two definitions of relativistic invariance are equivalent. It only says that the apparent counterexample of [92] is not a counterexample.

4.2.2 Different attempts to construct actions

The (anti)self-duality condition of chiral bosons makes it difficult to construct an action for them. Different attempts have been made to bypass this difficulty. We make a distinction between manifestly and non-manifestly covariant actions. The manifestly covariant approaches can be split into actions polynomial in auxiliary fields and the PST approach which is non-polynomial. We first stress actions for chiral bosons in two dimensions. Later we generalize to six and ten dimensions.

A first class of actions are the manifestly covariant ones. A first possibility, introduced by Siegel [93], is to impose the square of the self-duality condition through the introduction of an auxiliary scalar field as a Lagrange multiplier. The attempts [94] to quantize the action of [93] revealed that it suffers from an anomaly. Another approach [95] to deal with the self-duality constraint introduces a term in the action that is the product of an auxiliary vector field and the self-duality constraint. Although this has some defects as pointed out in [96, 97], this linear formulation strictly describes a chiral boson from the point of view of equations of motion at both the classical and quantum levels. A third way to construct a Lorentz-covariant action was achieved in [98]. To circumvent the infinite reducibility of the squared self-duality constraint of [93], the authors introduce an infinite set of auxiliary fields[3]. The quantum-mechanical analysis of this infinite set of auxiliary fields requires a lot of caution. A seemingly different formulation [99] in terms of infinitely many auxiliary fields was proven in [100] to be equivalent to the approach of [98]. All these constructions make use of terms polynomial in the auxiliary field(s) to impose the self-duality condition in a Lorentz-covariant action. By giving up explicit Lorentz invariance (imposing only the first definition of relativistic invariance), it became possible to construct new actions. In two dimensions, an action and its quantization properties were studied in [101]. Inspired by the quest for duality-invariant actions in four dimensions [102], it became possible to construct a Lorentz-invariant action with a finite number of auxiliary fields [103] for a chiral boson in two dimensions. The simplest case is that only one auxiliary scalar field is introduced. The exact role of this auxiliary scalar will be clarified in the next section. This approach gives rise to a Lorentz-covariant action where the auxiliary field appears in a non-polynomial way. The relation

[3]An auxiliary field is a field that has trivial dynamics and consists not purely of gauge degrees of freedom. The field a that will be introduced later, is *not* an auxiliary field in this sense.

of this approach to the one with infinitely many auxiliary fields of [98] has been clarified in [5]. A gauge choice for the gauge symmetry of the PST construction leads to the non-covariant action of [101].

We now discuss the generalization of these approaches to six and ten dimensions. The first approach, squaring the self-duality constraint and introducing a Lagrange multiplier, was already achieved in six and ten dimensions in the original paper. The implementation of the linear formulations of [95] in six dimensions is treated in [104]. The two-dimensional result with infinitely many auxiliary fields of [98], is generalized to higher order chiral p-forms in [105]. An extra complication is the reducibility of the gauge transformation of the chiral p-form. Take for instance the example of the reducible gauge symmetry of a two-form in six dimensions:

$$\delta B_{ab} = \partial_{[a}\Lambda_{b]} \tag{4.2.1}$$

is reducible for

$$\Lambda_a = \partial_a \Lambda\,. \tag{4.2.2}$$

For a chiral four-form in ten dimensions, the Lorentz-covariant action [106], derived from IIB closed superstring field theory, contains an infinite number of fields, because of bosonic ghost zero modes. Its relation to the approach of [98] is clarified in [107]. In [108], the method of [101], making an explicit splitting of the fields into time and space, was generalized to higher dimensions. In [109], another six-dimensional action was constructed with manifest five-dimensional Lorentz invariance only. There, one of the spatial dimensions was treated specially. The proof of six-dimensional Lorentz invariance needed a non-trivial check in both cases. Also some quantum-mechanical results were achieved: the gravitational anomaly of the chiral two-tensor in six dimensions was calculated. The implementation of the non-polynomial action of [103] to six dimensions was given in [5]. It gives rise to a Lorentz-covariant action with one extra scalar field. Its relation to the actions in [108, 109] will be discussed in section 4.4.

4.3 The gauge symmetries and action for a self-dual 2-tensor

The quantum-mechanical behavior of elementary particles, achieved during the last 40 years, is described in terms of gauge theories. When the description of a physical system is in terms of a gauge group, it means that there is a redundancy in the description. In principle, this redundancy can be eliminated, but reasons for not doing so are legion: locality of interactions, calculational convenience and manifest Lorentz covariance. This last reason is the main one to study the self-dual tensor using the PST gauge symmetries. Introducing *one* auxiliary scalar[4] a and two local gauge symmetries (redundancies in the description), dependent of the auxiliary scalar, it becomes possible to write down a manifestly covariant Lagrangian [5].

In this paragraph, the explicitly Lorentz-covariant action for a self-dual two-tensor is repeated. It satisfies the (seemingly?) stronger definition of Lorentz covariance. Further, the counting of the three degrees of freedom of a self-dual two-tensor is given and different gauge fixings are done.

4.3.1 Definitions and gauge symmetries

We use the notations of appendix A.3 in six Minkowski dimensions. The following additional notations are used. The auxiliary scalar is denoted by a. Expressions containing a are[5]:

$$\begin{aligned}
&u_a = \partial_a a\,, \qquad u^2 = u^a \eta_{ab} u^b\,, \qquad v_a = \frac{u_a}{\sqrt{u^2}}\,,\\
&H = dB\,, \qquad H_{abc} = 3\partial_{[a}B_{bc]}\,,\\
&H_{ab} = \frac{u^c}{\sqrt{u^2}} H_{abc} = v^c H_{abc}\,,\\
&H^*_{ab} = \frac{u^c}{\sqrt{u^2}} H^*_{abc}\,, \qquad \text{where } H^*_{abc} = \frac{1}{6}\epsilon_{abcdef}H^{def}\,,\\
&H^\pm_{ab} = \frac{u^c}{\sqrt{u^2}} H^\pm_{abc}\,, \qquad \text{where } H^\pm_{abc} = \tfrac{1}{2}(H_{abc} \pm H^*_{abc})\,.
\end{aligned} \tag{4.3.1}$$

The action will have the following three gauge symmetries:

$$\begin{aligned}
\delta_I B_{ab} &= 2\partial_{[a}\Lambda_{b]}\,, \qquad \delta_I a = 0\,,\\
\delta_{II} B_{ab} &= 2H^-_{ab}\frac{\phi}{\sqrt{u^2}}\,, \qquad \delta_{II} a = \phi\,,\\
\delta_{III} B_{ab} &= u_{[a}\psi_{b]}\,, \qquad \delta_{III} a = 0\,.
\end{aligned} \tag{4.3.2}$$

The first symmetry is the usual gauge symmetry for a two-form. The second and the third symmetry are the PST symmetries, enabled by the introduction of the

[4]This scalar is not auxiliary in the usual sense of having trivial dynamics and being not pure gauge as in footnote 3. Here, the equation of motion of a is not trivial and a is pure gauge for one of the gauge symmetries. It is only for this scalar a that we will abuse terminology in this way. The role of the 'auxiliary scalar' a will become more clear when the equations of motion of the action are discussed in section 4.3.2

[5]To define these expressions consistently, we need to impose that $u^2 > 0$.

scalar field a. The first and the third symmetries are reducible. Together, they have three zero modes:

$$\begin{aligned} \Lambda_a &= \partial_a \Lambda\,, \psi_a = 0\,; \\ \psi_a &= u_a \psi\,, \Lambda_a = 0\,; \\ \Lambda_a &= u_a \Lambda'\,, \psi_a = 2\partial_a \Lambda'\,. \end{aligned} \tag{4.3.3}$$

The first zero mode is the usual reducible component of the two-tensor in six dimensions. The second reducible symmetry stems from the third gauge symmetry. The third one is a zero mode for a combination of the first and the third transformation. The second gauge transformation has no reducible components. These gauge transformations have non-trivial commutators with each other:

$$\begin{aligned} [\delta_{II}(\phi_2), \delta_{II}(\phi_1)] &= \delta_{III}(4\frac{H^-_{ab}}{(u^2)^{3/2}}(\phi_1\partial^b\phi_2 - \phi_2\partial^b\phi_1))\,, \\ [\delta_{II}(\phi), \delta_{III}(\psi_a)] &= \delta_I(\tfrac{1}{2}\psi_a\phi) + \delta_{III}(2\partial_{[a}\psi_{c]}\cdot u^c\frac{\phi}{u^2})\,. \end{aligned} \tag{4.3.4}$$

These transformation rules and the non-trivial commutators of the different gauge symmetries will be used in section 4.4 when studying different gauge fixings of the action of the next section using the Batalin–Vilkovisky approach.

4.3.2 The Lorentz-covariant action

In this section, the Lorentz-covariant action, its equations of motion and the way to derive the self-duality condition from this action are given. The action was first formulated in this way in [5].

The action and its equations of motion

Using the notations of paragraph 4.3.1, the action for a self-dual two-tensor is

$$\begin{aligned} S &= -\tfrac{1}{2}\int d^6x\, H^-_{ab} H^{*ab}\,, \\ &= -\tfrac{1}{2}\int d^6x\, H^-_{abc}\frac{u^c}{\sqrt{u^2}} H^{*abd}\frac{u_d}{\sqrt{u^2}}\,. \end{aligned} \tag{4.3.5}$$

Remark that this action is local and non-polynomial. The Lagrangian is a functional of only the first derivative of a and expanding the denominator would give rise to an infinite power series. Using that the variation of this action is of the form

$$\delta S = \int d^6x \left(-\tfrac{1}{4}H^*_{abc} + \frac{\varepsilon_{abcdef}}{6}v^d H^{-ef}\right)\delta H^{abc} + \frac{1}{2\sqrt{u^2}}\varepsilon^{abcdef}H^-_{ab}H^-_{cd}v_e\partial_f\delta a\,, \tag{4.3.6}$$

together with the gauge transformation rules (4.3.2) for a and B_{ab}, some calculation proves that this action is invariant under the three gauge symmetries (4.3.2). An alternative form of this action is

$$S = \int d^6x \left(-\tfrac{1}{24}H^{abc}H_{abc} + \tfrac{1}{2}H^{-ab}H^-_{ab}\right)\,. \tag{4.3.7}$$

The first term of this action is the usual kinetic term for a two-tensor. The second term ensures that this is the action for a self-dual tensor.

The equations of motion for B_{ab} and a follow from (4.3.6):

$$\begin{aligned} \epsilon^{abcdef}\partial_c\left(\frac{1}{u^2}u_d H^-_{efg}u^g\right) &= 0\,, \\ \epsilon^{abcdef}\partial_a\left(\frac{1}{\sqrt{u^2}}H^-_{bc}H^-_{de}v_f\right) &= 0\,. \end{aligned} \tag{4.3.8}$$

Using the identity

$$\frac{1}{\sqrt{u^2}}\partial_{[a}v_{b]} + v_{[a}\partial_{b]}\frac{1}{\sqrt{u^2}} = 0\,, \tag{4.3.9}$$

it is possible to prove that the equation of motion of a is implied by the equation of motion of B_{ab}. This means that it is not an independent propagating degree of freedom. a is pure gauge for symmetry II, but it has a non-trivial equation of motion. This implies that it cannot be integrated out. Remark that even if gauge symmetry II suggests that a can be shifted to an arbitrary value, choosing an arbitrary constant for a gives rise to a singular action.

The same technique with one auxiliary scalar gives rise to a Lorentz-invariant action in two and ten dimensions in [5]. The relation with the non-covariant actions is also clear: upon appropriately gauge fixing the derivative of a to a unit vector in the time direction gives rise to the models of [108]. A unit spatial vector leads to the actions in [109].

It is also possible to recover the formulation in terms of an infinite number of auxiliary fields of [105]. To this end, one has to get rid of the non-polynomiality of (4.3.5) or (4.3.7). The relation between the action (4.3.5) and the formulation with infinitely many auxiliary fields is clarified in the original paper [5].

The action (4.3.5) is the Lorentz-covariant action for a free self-dual tensor, the '$U(1)$-case' for a self-dual tensor. There is not much known about models of interacting tensors. This would be a 'Yang–Mills generalization' of the free case. In [110] is argued on geometric grounds that it is not possible to construct a non-Abelian Yang–Mills gauge field theory for extended objects coupling to non-chiral p-forms in spacetime: only the Abelian $U(1)$-group is allowed. In [111] is derived a similar result for self-dual tensors using a cohomological analysis: the only consistent interactions are (up to redefinitions) deformations that do not modify the gauge symmetries of the free theory. There are no other consistent, local deformations. This leaves the possibility of non-local deformations. Proposals in these directions exist [112]. Also for the superconformal version, some attempts have been done to study interacting tensor multiplets [113], but the mystery of interacting tensors still needs to be unraveled it seems.

The self-duality condition

Following the analysis of [5], the self-duality condition can be derived. The equation of motion of B_{ab} is

$$\epsilon^{abcdef}\partial_c(\frac{1}{u^2}u_d H^-_{efg}u^g) = 0\,. \tag{4.3.10}$$

The solution of (4.3.10) is

$$H^-_{abc}u^c = u^2\partial_{[a}\phi_{b]} + 2u_{[a}\psi_{b]}\,. \tag{4.3.11}$$

Multiplying this with u^b must be 0, so

$$0 = u^2 u^b\partial_{[a}\phi_{b]} + u_a u\cdot\psi - u^2\psi_a\,, \tag{4.3.12}$$

which leads to

$$\psi_a = u^b\partial_{[a}\phi_{b]} + u_a\lambda\,, \tag{4.3.13}$$

The last term can be neglected since it does not contribute to (4.3.11). This means that the solution of the field equation of B_{ab} can be written as

$$\begin{aligned} H^-_{abc}u^c &= u^2\partial_{[a}\phi_{b]} + u^c(\partial_c\phi_{[a})u_{b]} + u_{[a}(\partial_{b]}\phi_c)u^c \\ &= u^c 3u_{[a}\partial_b\Phi_{c]}\,. \end{aligned} \tag{4.3.14}$$

This is a gauge transformation III of H^-_{ab}. In flat space, there are no problems with global properties of spacetime to solve these equations. By using a Schouten identity, it can be proven that $H^-_{ab} = 0$ is equivalent to $H^-_{abc} = 0$. This means that it is possible to find the self-duality of B_{ab} by picking a gauge choice for the third gauge transformation. So, the formulation with an auxiliary field gives rise to the self-duality equation.

4.3.3 The counting of the degrees of freedom

In this section we will argue that the self-dual tensor has three degrees of freedom. We first derive that a tensor with gauge symmetry I has six degrees of freedom. Then, we argue that introducing the PST symmetries reduces still three degrees of freedom of this on-shell tensor. This is not a fully satisfying derivation in the sense that it would be preferable to derive the three degrees of freedom from the equation of motion of the PST action. But, it is more satisfying for our approach than what was done before: deriving that the tensor has six degrees of freedom and splitting the reducible representation of the Lorentz in two parts: the self-dual tensor and the antiself-dual tensor. Here, we are able to translate the counting of the degrees of freedom into the language of the PST symmetries. We start with 16 degrees of freedom: the antisymmetric tensor B_{ab} has a priori $\frac{6 \cdot 5}{2} = 15$ degrees of freedom and the scalar a has one. We will end up with the three on-shell degrees of freedom of a self-dual tensor in six dimensions.

The tensor

We start from the action for a tensor in six dimensions

$$S = \int d^6x \left(-\tfrac{1}{24} H^{abc} H_{abc}\right) . \qquad (4.3.15)$$

The equation of motion of the tensor in momentum space is

$$k_a \left(k^a B^{bc} + k^b B^{ca} + k^c B^{ab}\right) = k^2 B^{bc} + 2k^{[b} k_a B^{c]a} \approx 0 \,, \qquad (4.3.16)$$

where an equation that is ≈ 0, is called 'weakly 0': it vanishes provided the field equations are satisfied. The gauge transformation is

$$\delta B_{ab} = 2k_{[a} \Lambda_{b]} \, . \qquad (4.3.17)$$

We can take $k_0 \neq 0$. We generalize the temporal gauge on every gauge orbit:

$$B_{0i} = 0\,. \tag{4.3.18}$$

This reduces five degrees of freedom. The equation of motion (4.3.16) for $b = 0$ becomes

$$k^2 B^{0i} + k^0 k_a B^{ia} - k^i k_a B^{0a} \approx 0\,. \tag{4.3.19}$$

Using (4.3.18), we find that

$$k_j B^{ij} \approx 0\,. \tag{4.3.20}$$

This condition again eliminates four degrees of freedom. It has five components, but the product with k^j gives zero, so it has four independent components. The equations of (4.3.16) with $b \neq 0 \neq c$ give rise to

$$k^2 B^{ij} \approx 0\,. \tag{4.3.21}$$

This leads to a massless tensor: $k^2 = 0$. So, the action leads to a massless tensor with six degrees of freedom.

The PST symmetries

We introduce a scalar field a and the gauge symmetries II and III. This means that we start from seven degrees of freedom and that we have to gauge fix still two symmetries.

The first PST symmetry shifts the field a. We want to gauge fix it such that the derivative of a becomes a unit vector n_a:

$$a = n^a x_a\,. \tag{4.3.22}$$

This means that the degree of freedom of a will not survive on-shell.

We impose the gauge condition

$$n^a B_{ab} = 0\,, \tag{4.3.23}$$

where B_{ab} contains the six degrees of freedom left after the constraints of the previous section:

$$\begin{aligned} B_{0i} &= 0\,, \\ k^i B_{ij} &= 0\,. \end{aligned} \tag{4.3.24}$$

If we choose the tensor of the previous section to move in the fifth direction, the equation of motion gives that

$$k_a = (k_0, 0, 0, 0, 0, k_0)\,. \tag{4.3.25}$$

Choose now

$$n_a = (0, 1, 0, 0, 0, 0)\,. \tag{4.3.26}$$

Then the gauge fixing condition (4.3.23) becomes

$$B_{1j} = 0\,. \tag{4.3.27}$$

We have to check how many of these conditions are new ones. First of all, $B_{01} = 0$ does not lead to a new condition. In addition, the equation of motion and the second equation of (4.3.24) gives that

$$B_{01} = B_{51} = 0\,. \tag{4.3.28}$$

So, there remain three extra conditions from (4.3.23): the gauge fixing reduces three degrees of freedom. This argues that the self-dual tensor has three on-shell degrees of freedom. A full proof would require that we do the derivation as we did for the tensor and start with the action for the self-dual tensor. We were not able to do that analysis.

4.4 Two gauge fixings in the BV-language

In this section we will use the Batalin–Vilkovisky scheme, shortly reviewed in section 3.2, to construct two gauge-fixed actions of the self-dual tensor with the PST symmetries. To this end, we first construct the extended action using the procedure of section 3.2.2. Then, we will gauge fix this action in two different ways: a non-covariant and a covariant one. For each of these gauge fixings we will comment on some applications or possibilities to obtain quantum-mechanical information. Most of this work is reported in [6].

4.4.1 The extended action for the chiral two-form

Using the gauge symmetries (4.3.2) and their reducibility conditions (4.3.3) together with the procedure sketched in section 3.2, we will build the extended action of the chiral two-form. This is constructed as an expansion in antifield-number. The first step to achieve an extended action is the introduction of antifields B^*_{ab} and a^* for the classical fields, and ghosts c_a, c and c'_b, associated respectively to symmetries I, II and III. The ghost number and statistics of the different fields and antifields can be found in table 4.1. The gauge symmetries yield the following

Φ	B_{ab}	a	c_a	c	c'_a	d_1	d_2	d_3
$g(\Phi)$	0	0	1	1	1	2	2	2
stat(Φ)	+	+	-	-	-	+	+	+
Φ^*	B^*_{ab}	a^*	c^*_a	c^*	c'^*_a	d^*_1	d^*_2	d^*_3
$g(\Phi^*)$	-1	-1	-2	-2	-2	-3	-3	-3
stat(Φ^*)	-	-	+	+	+	-	-	-

Table 4.1: Ghost number $g(\Phi)$ and statistics stat(Φ) of the minimal fields and their antifields.

contribution to the extended action at antifield number 1, given by (3.2.4):

$$S_{(1)} = \int d^6x \left(B^{*ab}(2\partial_a c_b + 2H^-_{ab}\frac{c}{\sqrt{u^2}} + u_a c'_b) + a^* c \right) . \qquad (4.4.1)$$

A contribution at antifield number two, comes from the ghosts for ghosts d_1, d_2 and d_3, associated to the three reducible symmetries (4.3.3). This gives

$$S'_{(2)} = \int d^6x \, (c^*_a(\partial^a d_1 + u^a d_3) + c'^*_a(u^a d_2 + 2\partial_a)d_3) \; , \qquad (4.4.2)$$

where c^*_a and c'^*_a are the antifields associated to the ghosts c_a and c'_a. At antifield number two, we also have to include a term related to the commutators of the symmetries as indicated in (3.2.6). These commutators are given in (4.3.4) and give rise to the following terms in the action

$$S''_{(2)} = \int d^6x \left(\tfrac{1}{2} c^*_a c'^a c - c'^*_a \partial^{[a} c'^{b]} \cdot \frac{u_b}{u^2} c + 4c'^*_a \frac{H^{-ab}}{(u^2)^{3/2}} c\partial_b c \right) . \qquad (4.4.3)$$

It is straightforward to check that the action

$$S = S_{cl} + S_{(1)} + S_{(2)}, \qquad (4.4.4)$$

with $S_{(2)} = S'_{(2)} + S''_{(2)}$, satisfies the properness condition, i.e., that the Hessian

$$S_A{}^B = \frac{\overrightarrow{\partial}}{\delta\Phi^A} S \frac{\overleftarrow{\partial}}{\delta\Phi^*_B} \tag{4.4.5}$$

at the stationary surface has a number of zero modes that is exactly half its dimension. To check it, it is convenient to choose a point on the surface where $H^-_{abc} = 0$. The number of zero modes of the Hessian cannot vary along the stationary surface. Continuous deformations in the space of fields that avoid singularities in the action cannot cause a jump in the number of degrees of freedom. The properness condition is one of the three conditions that an extended action has to satisfy to be an action that can be used to start gauge fixing.

As far as antibrackets are concerned, one has

$$(S_{cl}, S_{(1)}) = 0\,. \tag{4.4.6}$$

This is simply a consequence of the gauge invariance of the classical action. However, the classical master equation is not yet satisfied; in particular, at antifield number two, one has the antibracket

$$\begin{aligned} 2(S_{(1)}, S_{(2)})_{(2)} + (S_{(2)}, S_{(2)})_{(2)} &= 8c'^*_a \frac{H^-_{bcd}}{(u^2)^3} u^a u^d \partial^c c \cdot c\partial^b c + 2c^*_a \partial^a (d_3 c) \\ &\quad - c^*_a u^a d_2 c + 2c'^*_a \frac{u^a u^b}{(u^2)^2} \partial_{[c} c'_{b]} \cdot c\partial^c c \\ &\quad + 2c'^*_a \left(\partial^a (d_2 c) - \frac{u^a u^b}{u^2} \partial_b d_2 \cdot c \right)\,. \end{aligned} \tag{4.4.7}$$

Since this is not 0, we have to add terms to the action at antifield number three. The choice that works is

$$\begin{aligned} S_{(3)} &= -d^*_1 d_3 c \\ &\quad + d^*_2 \left(-4 \frac{H^-_{ab}}{(u^2)^{5/2}} \partial^b c \cdot c\partial^a c + 2\frac{u^a}{(u^2)^2} \partial_{[a} c'_{b]} \cdot c\partial^b c + \frac{u^a}{u^2} \partial_a d_2 \cdot c \right) \\ &\quad + d^*_3 d_2 c\,. \end{aligned} \tag{4.4.8}$$

Then

$$\begin{aligned} 2(S_{(2)}, S_{(3)})_{(2)} - 2(S_{(1)}, S_{(2)})_{(2)} + (S_{(2)}, S_{(2)})_{(2)} &= 0\,, \\ 2(S_{(1)}, S_{(3)})_{(3)} + 2(S_{(2)}, S_{(3)})_{(3)} + (S_{(3)}, S_{(3)})_{(3)} &= 0\,. \end{aligned} \tag{4.4.9}$$

This means that all the terms in the antifield number expansion of the classical master equation are 0, so the classical master equation, (3.2.8), is satisfied. So we end up with an action that is the sum of (4.3.5), (4.4.1), (4.4.2), (4.4.3) and (4.4.8). This action satisfies the three conditions required: the classical master equation (3.2.8) is satisfied, the properness condition is obeyed and the classical limit (deleting all terms with non-zero ghost number in the action) gives the classical action (4.3.5). So, all the conditions to have a good extended action are fulfilled.

4.4.2 A covariant gauge fixing

In this section we will present a covariant gauge fixing of the extended action using gauge fermions in the BV-formalism. At the end, we will comment on some possible quantum-mechanical calculations. We start by gauge fixing the three gauge symmetries (4.3.2) of the classical action. We will use the gauges

$$\partial_a B^{ab} = 0\,, \tag{4.4.10}$$
$$u^2 = 1\,, \tag{4.4.11}$$
$$u_a B^{ab} = 0\,. \tag{4.4.12}$$

(4.4.11) is a Lorentz-invariant gauge fixing for symmetry II. The gauge (4.4.12) fixes symmetry III and is the analogue of the Lorentz gauge; using a transformation III, one can remove the component of B_{ab} that is parallel to the vector u^a.

For each of the three gauge fixings, one introduces a new non-minimal set of fermionic fields and their antifields and adds to the action a term quadratic in the antifields. In table 4.2, the non-minimal fields are denoted by b_a, b'_a and b.

The gauge fixing is done by introducing a gauge fermion for each symmetry:

$$\Psi_1 = b_a \partial_b B^{ab}\,, \tag{4.4.13}$$
$$\Psi_2 = b'_a u_b B^{ab}\,, \tag{4.4.14}$$
$$\Psi_3 = b(u^2-1)\,, \tag{4.4.15}$$

and adding to the action the non-minimal terms

$$S^1_{nm} = -\tfrac{1}{4} b^*_a b^{*a} - \tfrac{1}{4} b'^*_a b'^{*a} + b^{*2}\,. \tag{4.4.16}$$

This corresponds to the first way of introducing non-minimal fields in (3.2.11).

The gauge symmetries of the classical action also had three zero modes. Their gauge fixing is done in general by introducing two extra sets of bosonic fields and

	$\underset{(0,-1)}{B_{ab}, a}$	
	$\swarrow$	
$\underset{(-1,0)}{b_a, b'_a, b}$		$\underset{(1,-2)}{c^a, c'^a, c}$
$\swarrow$		$\swarrow$
$\underset{(0,-1)}{b'^{a_1} = \{l, n, q\}}$ $\overset{*}{\longleftrightarrow}$	$\underset{(-2,1)}{b_{a_1} = \{k, m, p\}}$	$\underset{(2,-3)}{d_1, d_2, d_3}$

Table 4.2: The fields for the gauge fixing of theories with a reducible gauge algebra, with (ghost number, ghost number of antifield) and a schematic indication of non-degeneracy conditions of the gauge fixing and the connected antifields in the non-minimal extended action.

their antifields for each zero mode. This corresponds to the second procedure to add non-minimal fields of (3.2.11). In table 4.2, they are denoted by b'^{a_1} and b_{a_1}. The arrow between them is used to indicate that one adds a non-minimal term to the action that is a product of their antifields. For the gauge fixing of these three reducible gauge symmetries, one introduces the bosons k, m, p and l, n, q. The following gauge fermions can be used:

$$\Psi_4 = k\partial_b c^b + l\partial_a b^a\,, \tag{4.4.17}$$
$$\Psi_5 = m u_a c'^a + n u_a b'^a\,, \tag{4.4.18}$$
$$\Psi_6 = p(u_a c^a - 2\partial_a c'^a) + q(u_a b^a - 2\partial_a b'^a)\,. \tag{4.4.19}$$

The following non-minimal terms are added to the action:

$$S^2_{nm} = k^* l^* + \tfrac{1}{2} m^* n^* + p^* q^*\,. \tag{4.4.20}$$

The antifield-independent part of the action becomes

$$\begin{aligned} S = \int d^6x \Big[& \tfrac{1}{8} B_{ab} \Box B_{ab} - \tfrac{1}{2} H^{-ab} H^-_{ab} - \tfrac{1}{4} q^2 u^2 - 2q\, u^a \partial_a\, l \\ & - l \Box l + (u^2-1)^2 + q \Box q + n\, u^a \partial_a\, q - \tfrac{1}{4} n^2 u^2 - \tfrac{1}{4} u_b B^{ab}\, u^c B_{ac} \\ & - b'^a \partial_a u_b \cdot c^b - b'^b\, u^a \partial_a\, c_b - \tfrac{1}{2} b'^b u^2 c'_b - b^a \Box c_a \\ & + 2\partial^a b^b \cdot H^-_{ab} \frac{c}{\sqrt{u^2}} - \tfrac{1}{2} c'_b\, u^a \partial_a\, b^b + \tfrac{5}{2} u_a b^a\, \partial^b c'_b + \tfrac{1}{2} b^a \partial_b u_a \cdot c'^b \\ & u_a c^a\, u_b b^b + 4 \partial_a c'^a \cdot \partial_b b'^b + (q b^a + 2 b u^a + n b'^a)\, \partial_a c \\ & + p\, u^a \partial_a\, d_1 + k \Box d_1 + p\, u^2\, d_3 + d_3\, u^a \partial_a\, k + \tfrac{1}{2} p\, u_a c'^a c + \tfrac{1}{2} k\, \partial^a (c'^a c) \end{aligned}$$

$$-2d_2\, u^a \partial_a\, p + m\, u^2\, d_2 + 4p\Box d_3 + 2m\, u^a \partial_a\, d_3 + 2\partial_a p \cdot \partial^{[a} c'^{b]} \cdot \frac{u_b}{u^2} c$$
$$-8\partial_a p \cdot \frac{H^{-ab}}{(u^2)^{3/2}}\, c\partial_b c + (pc^a + mc'^a)\, \partial_a c\Big] \,. \qquad (4.4.21)$$

This is a *covariant* gauge-fixed action for the self-dual two-form in six dimensions.

Comments on quantum-mechanical applications

In principle, (4.4.21) can be used to derive quantum-mechanical results. A first possibility would be to calculate gravitational anomalies[6] as in [114, 115], but the presence of the auxiliary scalar in the propagators makes an analysis of anomalies using this action very hard. The non-covariant gauge fixing of the next section can be used for this.

Another possibility to obtain a quantum-mechanical result would be to use this gauge fixing to calculate the conformal anomaly for the self-dual tensor. This would give rise to an interesting check of the adS/CFT-conjecture since the calculation at the supergravity side is already done [116]. For this purpose, a gravitational formulation of this action is needed. This will be the subject of chapter 5.

Further, this action could be used to repeat the calculation of [117]. They computed the ratio of the three- and two-point correlation functions of the stress-energy tensors of the rigid $(2,0)$ self-dual tensor multiplet. They need the propagator of the self-dual two-form in terms of its field strengths to calculate the ratio of the three-point and the two-point correlation functions of the stress-energy tensor of the self-dual tensor multiplet in six dimensions. They choose to calculate the propagator for an ordinary tensor and then use the same projection method as [114] to end up with the propagator for the self-dual tensor. After getting more familiar with this gauge-fixed action (4.4.21), maybe one could show how to re-obtain the results of [117], starting from this action and without using a projector.

Although this gauge-fixed action is too difficult to obtain some quantum-mechanical results for the moment, it could be useful for further analysis of quantum properties of chiral two-forms in the future, especially in application to the quantization of the $M5$-brane, the solitonic object of 11-dimensional supergravity.

[6] Gravitational anomalies are violations at the quantum level of the classical conservation of the energy-momentum tensor.

4.4.3 A non-covariant gauge fixing

The non-covariant gauge fixing of the second symmetry in this section corresponds to $a = x^5$ in [109] or $a = x^0$ in [108]. The second gauge symmetry can be fixed by a canonical transformation that cannot be generated by a gauge fermion:

$$
\begin{aligned}
a &\rightarrow -b^* + n_a x^a \,, &(4.4.22)\\
nonumber a^* &\rightarrow b \,. &(4.4.23)
\end{aligned}
$$

This type of canonical transformation is the analogue of changing a momentum into a coordinate and vice-versa in classical mechanics. This means that

$$
\begin{aligned}
u_c &\rightarrow -\partial_c b^* + n_c \,, &(4.4.24)\\
nonumber u^2 &\rightarrow 1 - 2n^a \partial_a b^* + (\partial b^*)^2 \,. &(4.4.25)
\end{aligned}
$$

Using the gauge fermions

$$
\begin{aligned}
\psi_1 &= b_a \partial_b B^{ab} \,, &(4.4.26)\\
\psi_3 &= b'_a n_b B^{ab} \,, &(4.4.27)\\
\psi_4 &= k \partial_b c^b + l \partial_b b^b \,, &(4.4.28)\\
\psi_5 &= m n_a c'^a + n n_a b'^a \,, &(4.4.29)\\
\psi_6 &= p\,(n_a c^a + 2\partial_a c'^a) + q\,(n_a b^a + 2\partial_a b'^a) \,, &(4.4.30)
\end{aligned}
$$

for the other symmetries, and the corresponding non-minimal terms in the action

$$
S_{nm} = -\frac{1}{4} b^{*a} b^*_a + \frac{1}{2} b'^{*a} b'^*_a - k^* l^* + \frac{1}{2} m^* n^* - \frac{1}{2} p^* q^* \,,
$$

the other symmetries of the action are gauge fixed. The antifield-independent part of the action is:

$$
\begin{aligned}
S = \int d^6x \Big[& \tfrac{1}{8} B_{ab} \Box B_{ab} - \tfrac{1}{2} H^{-abc} n_c\, H^-_{abd} n^d - b^a \Box c_a - b'^a\, n^b \partial_b\, c_a \\
& + 2b^a \partial^b H^-_{abc} \cdot n^c c + \tfrac{3}{2} n^a b_a\, \partial^b c'_b - \tfrac{1}{2} b^a\, n^b \partial_b\, c'_a - \tfrac{1}{2} b'^a c'_a + bc \\
& + k \Box d_1 + k\, n^a \partial_a\, d_3 + \tfrac{1}{2} k \partial^a (c'_a c) + m d_2 + 2m\, n^a \partial_a\, d_3 + \tfrac{1}{2} q\, n^a \partial_a\, l + \tfrac{1}{4} l \Box l \\
& + \tfrac{1}{2} n_b B^{ab}\, n^c B_{ac} + \tfrac{1}{2} n^2 - 2n\, n_a \partial^a\, q - 2q \Box q - \tfrac{1}{2} n^a c_a\, n^b b_b - 2\partial_a c'^a \cdot \partial_b b'^b \\
& + p\, n^a \partial_a\, d_1 + p d_3 + \tfrac{1}{2} p\, n^a c'_a c - 2d_2\, n^a \partial_a\, p + 4p \Box d_3 + 2\partial^a p \cdot \partial_{[a} c'_{b]} \cdot n^b c \\
& - 8\partial^a p \cdot H^-_{abc} n^c\, c \partial^b c \Big] \,. \qquad (4.4.31)
\end{aligned}
$$

This action is much more simple than (4.4.21) because the vector n_a is a constant vector here. All the terms in the denominators drop. The same procedure can be used to retrieve the non-covariant actions of [108] and [109], where the six spacetime coordinates are split into a time coordinate and five space coordinates and into five spacetime coordinates and one special *space* coordinate. In this sense, we can understand that the covariant action (4.3.5) of [5] is more general than the action of [109] and [108]: they are both gauge-fixed versions of (4.3.5).

Comments on quantum-mechanical results

The gravitational anomalies for the self-dual two-tensor in six dimensions were already calculated in [114]. Alvarez-Gaumé and Witten argued that the only bosonic field that can give rise to gravitational anomalies is the (anti)self-dual $2k$-form in $4k+2$ dimensions. Other bosonic fields have standard kinetic terms and can always be regularized. In (4.3.7) can be seen that the kinetic term in the action for a chiral two-form is not quadratic in derivatives of bosonic fields and therefore, it is not obvious how to regularize such a theory. At the time Alvarez-Gaumé and Witten wrote their article, no action for the chiral two-form was known. They were pragmatic and invented suitable Feynman rules. They first studied the energy-momentum tensor of ordinary two-tensors and argued that it was enough to calculate the propagator of the gauge-invariant field strength (H_{abc} in our case). To deal with the self-dual field, they assumed that it was correct to use the field strength propagator without modification, while modifying the energy-momentum tensor using the self-dual tensor instead of the tensor in it.

In [115], the second gauge symmetry of the action (4.3.5) was fixed by imposing $a = n_a x^a$ where n_a is an arbitrary unit vector. Using the Faddeev–Popov approach, it was proven that this gauge fixing gives rise to the propagators postulated and used in [114] to calculate the gravitational anomalies of a chiral two-form in six dimensions. This result supports the quantum reliability of the action (4.3.5) at the perturbative level. In [118] was already proven that the non-covariant action of [108] gives rise to different Feynman rules, but to the same gravitational anomalies as [114].

A last remark has to be made about the two gauge-fixed actions. It is not known whether the PST gauge symmetries are good symmetries at the quantum level or whether they develop a kind of gauge anomaly.

Chapter 5

The self-dual tensor in conformal supergravity

In chapter 4, we studied different bosonic aspects of the chiral two-tensor in six dimensions. Self-dual tensors also appear in a supersymmetric context if the supersymmetry algebra is chiral. So, there are no (anti)self-dual tensors for $(1,1)$ supersymmetry in six dimensions. Depending on the amount of chiral supersymmetry in six dimensions, it is sometimes possible to construct supersymmetric matter multiplets that contain a self-dual tensor. Self-dual tensors appear in all the gravity multiplets of chiral supersymmetry in six dimensions.

This chapter tries to describe the coupling of a self-dual tensor multiplet to a $(2,0)$ conformal supergravity background in terms of a Lorentz-covariant action. This work is originally reported in [7]. The coupling of this multiplet to conformal supergravity in terms of field equations was already studied in [42]. There are two main motivations for studying the coupling of this matter multiplet to this background.

- A first motivation is the completion of the superconformal tensor calculus program. As explained in section 3.1, this method is used to study the coupling of different matter multiplets to Poincaré supergravity in different dimensions. For this purpose, first the coupling to conformal supergravity is constructed. In this coupling of matter to superconformal gravity, the superfluous symmetries are broken later to Poincaré supersymmetry.
- Another, more recent, motivation comes from the adS/CFT-correspondence. The last couple of years, people have been trying to understand and prove

> the conjectured correspondence between certain theories with rigid conformal symmetry and string/M-theory on adS spaces times compact manifolds. The local symmetries in the bulk induce local superconformal symmetries on the boundary. Thus, it is natural to formulate the boundary field theory in a conformal supergravity background. So, we wish to study the coupling of the self-dual tensor multiplet to six-dimensional conformal supergravity, which is induced on the six-dimensional boundary of adS_7.

By using superconformal techniques, summarized in section 3.1, the coupling of self-dual tensors to a conformal supergravity background is studied before in terms of field equations, both for the case of $(1,0)$ in [41] and for extended chiral supersymmetry in [42]. What was still lacking is the construction of a covariant action for self-dual tensors in this background using the tools of chapter 4. This is what will be done in section 5.4. The consistency of the action is checked by the derivation of the field equations of [42]. We give the transformation rules of the tensor multiplet that do not make use of the self-duality condition to realize the algebra. These supersymmetry transformation rules are understood better by looking upon the first PST symmetry (gauge symmetry II) as a gauge symmetry with the derivative of the auxiliary scalar as its gauge field. We also indicate how the algebra is changed when transformation rules for the tensor multiplet are used that take into account the PST gauge symmetries. The transformation rules of the equations of motion are given. There is also sketched how the action with $(1,0)$ conformal supersymmetry can be found and how the breaking to models with Poincaré supersymmetry can be done.

Most of this chapter will be spent to the self-dual tensor in matter multiplets. Section 5.1 gives some background on chiral bosons and their importance in supersymmetric theories in six dimensions. In section 5.2, we will build the action with rigid supersymmetry for the $(2,0)$ self-dual tensor multiplet and sketch the importance of the superconformal algebra $OSp(8^*|4)$. We will use the ingredients (one auxiliary scalar and two new gauge symmetries) of the construction of the action for the bosonic model of section 4.3. This model is also suited to clarify the role of the auxiliary scalar in supersymmetric models, both as a representation with only one bosonic field and in supersymmetry transformation rules which do not need the self-duality condition of the tensor. Section 5.3 gives four arguments for the field content of the $(2,0)$ Weyl multiplet, since it is not yet proven that the Weyl multiplet with matter fields is the unique possibility for constructing a conformal supergravity background. This Weyl multiplet is used in the last section of this chapter to derive the coupling of the $(2,0)$ tensor multiplet to a conformal supergravity background. Most of the results of this chapter can be found in [7].

5.1 Self-dual tensors in 6 dimensions

5.1.1 Self-dual tensors in matter multiplets

For one and two rigid chiral supersymmetries in six dimensions, it is possible to have a supersymmetric matter multiplet that contains an (anti)self-dual tensor. For $(1,0)$ and $(2,0)$ chiral supersymmetry, the self-dual tensor multiplet contains a self-dual tensor, one or two Weyl spinors and one or five scalars. For models with $(1,0)$ supersymmetry, also other, more familiar, matter multiplets are allowed: vector multiplets (possibly with a non-Abelian Yang–Mills group), hyper multiplets, ... as studied in [119]. For $(2,0)$ supersymmetry, the tensor multiplet is the only matter multiplet allowed. Also for $(3,0)$ chiral supersymmetry, there exists a matter representation which contains six self-dual tensors [120], but this multiplet also contains a gravitino. A dynamical description of multiplets that contain a gravitino requires supergravity [121]. The gravitino then becomes part of the gravity multiplet and so it is no (rigid) matter multiplet any more.

5.1.2 Chiral 2-forms in (conformal) supergravity

In this section we will explain how chiral theories in six dimensions arise from certain string compactifications, where (anti)self-dual tensors appear in six-dimensional chiral supergravities and comment on gravitational anomalies in these theories.

Chiral theories from string compactifications

One of the main reasons why we are interested in these six-dimensional models is that some chiral supergravities can be found as string compactifications. Here we will try to explain how chiral spinors can be found from $K3$-compactifications. One starts with a Majorana–Weyl spinor in ten dimensions, an irreducible representation of $SO(1,9)$. In ten dimensions there are two possibilities: a lefthanded spinor (denoted $\mathbf{16}$) and a righthanded spinor (denoted $\overline{\mathbf{16}}$). Compactifying to six dimensions means that the spacetime $\mathbb{R}^{1,9}$ is replaced by $\mathbb{R}^{1,5} \times X$, for some compact four-dimensional manifold X. To see what happens to the supersymmetries, we need to consider how a spinor of $SO(1,9)$ decomposes under the maximal subalgebra:

$$SO(1,9) \supset SO(1,5) \oplus SO(4) \cong SO(1,5) \oplus SU(2) \oplus SU(2) \qquad (5.1.1)$$

The ten-dimensional spinors decompose as

$$\begin{aligned}\mathbf{16} &\rightarrow (\mathbf{4},\mathbf{2})\oplus(\bar{\mathbf{4}},\bar{\mathbf{2}})\cong(\mathbf{4},\mathbf{2},\mathbf{1})\oplus(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})\,,\\ \overline{\mathbf{16}} &\rightarrow (\mathbf{4},\bar{\mathbf{2}})\oplus(\bar{\mathbf{4}},\mathbf{2})\equiv(\mathbf{4},\mathbf{1},\mathbf{2})\oplus(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})\,.\end{aligned} \tag{5.1.2}$$

We want to study different backgrounds and more specifically how much supersymmetry these backgrounds allow. The condition for unbroken supersymmetry is that the variations of the Fermi fields are zero. These variations contain a covariant derivative of the supersymmetry parameter, so we need that there is a solution to the massless Dirac equation in the four compact dimensions. If the background is T^4, this is always possible and compactification gives rise to a non-chiral theory. No supersymmetries are broken.

For other backgrounds, the possible spinors are determined by the holonomy. The holonomy describes the transformation of a spinor around a closed loop under parallel transport. This parallel transport is expressed using the covariant derivative. So a vanishing covariant derivative will give rise to the original spinor. Therefore, solutions of the Dirac equation in the internal dimensions must be singlets for the holonomy of the internal manifolds. The torus has trivial holonomy, so every representation survives. This means that both parts of the decomposed spinor survive and that both chiralities appear. This shows that toroidal compactifications can never give rise to chiral six-dimensional theories.

A next example will be a compactification on a $K3$ surface. A $K3$ surface is a complex, Ricci-flat Kähler manifold of complex dimension two. It has $SU(2)$-holonomy[1]. Let us first consider the case of an $N=1$ theory in ten dimensions compactified down to six dimensions on a smooth $K3$ surface. We choose the last of the $SU(2)$ factors in the algebra as the holonomy. This means that only the $(\mathbf{4},\mathbf{2},\mathbf{1})$ spinor of the decomposed $\mathbf{16}$ is allowed. In the $K3$ background, the $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$ is not possible. It may look that we have two spinors, but the $\mathbf{2}$ is a representation of an internal $SU(2)$, the R-symmetry group. So $(\mathbf{4},\mathbf{2},\mathbf{1})$ is the symplectic Majorana–Weyl spinor doublet of $(1,0)$ supersymmetry in six dimensions. So, we conclude that $N=1$ supergravity in ten dimensions compactified on a smooth $K3$ surface will give an $N=1$ theory in six dimensions. The general rule is that compactification on a smooth $K3$ surface will preserve half of the supersymmetry. The compactifications of string theory with one supersymmetry in ten dimensions that give rise to $(1,0)$ supersymmetry in six dimensions are type I on $K3$ in [124] and the heterotic string on K_3 in [125]. Also M-theory on $K_3\times S^1/\mathbb{Z}_2$ [126] and F-theory[2] on Calabi–Yau threefolds [128] have $(1,0)$ supersymmetry in six dimensions.

[1] For more mathematical background on Calabi–Yau manifolds, see [122]. $K3$'s role in string theory is treated in [123].

[2] F-theory is a 12-dimensional theory that is defined such that its toroidal compactification to 10 dimensions gives rise to IIB theory. It was first formulated by Vafa in [127].

There are also two models with two supersymmetries in ten dimensions. The type IIA superstring in ten dimensions yields, in the low-energy limit, a theory of ten-dimensional supergravity with $N = (1,1)$. If we compactify this theory on a $K3$ surface then each of the spinors has to be decomposed. Choosing the second $SU(2)$ again as the holonomy of the $K3$ surface implies that the $(\mathbf{4},\mathbf{2},\mathbf{1})$ and the $(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$ survive. So one ends up with $N = (1,1)$ supersymmetry in six dimensions. So there is no $K3$ compactification of type IIA supergravity or string theory which gives rise to a chiral theory in six dimensions.

The other theory with maximal supersymmetry in ten dimensions is IIB string theory. IIB string theory and its low-energy limit IIB supergravity have $(2,0)$ supersymmetry in ten dimensions, so this is a chiral theory in ten dimensions. Decomposing the spinor and taking into account the $K3$ holonomy gives rise to two $(\mathbf{4},\mathbf{2},\mathbf{1})$ spinors. So there is $(2,0)$ supersymmetry in six dimensions. The two spinors can be combined into a quartet of Majorana–Weyl spinors, transforming in the $\mathbf{4}$ representation of the $USp(4)$ R-symmetry group. The theories with more chiral supersymmetry in six dimensions cannot be found from string theory compactifications.

Chiral 2-forms in supergravity

The (anti)self-dual tensors appear in two ways in gravitational theories, the low energy approximations of certain superstring compactifications. First, they can be one of the components of the gravitational multiplet in Poincaré [129, 130] supergravities. Also the conformal chiral supergravity multiplets [41, 42] and the gravitational multiplets of $(3,0)$ and $(4,0)$ and other six-dimensional supergravities contain self-dual tensors[3]. The gravitational multiplet of $(3,0)$ chiral supergravity contains 15 self-dual tensors and that of $(4,0)$ has 27 of them [120]. The gravitational multiplet is the only massless multiplet that can be used for a dynamical model with this amount of supersymmetry.

The self-dual tensor multiplets with rigid $(1,0)$ or $(2,0)$ chiral supersymmetry can also be realized as matter multiplets in a supergravity background. The coupling of matter multiplets in chiral theories with one supersymmetry can be found in [129, 131]. In these articles, the only models for which an action can be constructed, have one self-dual tensor multiplet. Together with the antiself-dual tensor of the gravity multiplet, the tensors combine into an ordinary tensor, for which of course an action exists. For multiple tensor multiplets, no satisfying action was found. They only gave the equations of motion and an action for the

[3]Also the non-chiral theories contain (anti)self-dual tensors, but they always have a combination of self-dual and antiself-dual tensors. For $(1,1)$ supersymmetry, there is one self-dual and one antiself-dual tensor which can be combined into an 'ordinary' tensor.

tensors where the self-duality is imposed only after the derivation of the equations of motion. In [132, 133, 134, 135], the Poincaré actions for the multiplets with self-dual tensors and their couplings to other matter multiplets were constructed. Also for $(2, 0)$, the coupling to supergravity is constructed in [130, 136]. This required a generalization of the actions with only rigid supersymmetry.

An alternative approach to write down an action is possible [137]. First, an action is written down with a non-chiral tensor. The variation of this action must be proportional to the antiself-dual field strength. So, this action gives rise to the exact equations of motion provided the self-duality is imposed by hand.

Gravitational anomalies

In chapter 4, we argued that the coupling of action (4.3.5) to gravity gives rise to the gravitational anomalies calculated in [114]. In that article, not only the anomalies for the self-dual tensor in six and ten dimensions are calculated (cf. chapter 4), but also the anomalies of the chiral spin-1/2 and spin-3/2 fermions. The conditions to have anomaly freedom are calculated for different dimensions. The most striking conclusion of [114] was that in ten dimensions there is a unique chiral theory with extended supersymmetry that is anomaly free: type IIB supergravity.

In six dimensions, other possibilities exist. For the case of $(2, 0)$ supersymmetry, there are only two multiplets that can change the anomaly: the gravitational multiplet and the tensor multiplet. The anomaly polynomials for gravitational anomalies in six dimensions give rise to an algebraic equation:

$$21\hat{I}_{1/2} - \hat{I}_{3/2} + 8\hat{I}_A = 0\,, \tag{5.1.3}$$

where $\hat{I}_{1/2}$, $\hat{I}_{3/2}$ and $\hat{I}_A$ are the anomaly polynomials for respectively a positive chirality spin-1/2 field, a negative chirality spin-3/2 field and a self-dual tensor. The (Poincaré) $(2, 0)$ gravitational multiplet in six dimensions [120] contains five antiself-dual tensors and two gravitini which contribute to the anomaly. The tensor multiplet contains one self-dual tensor and two positive chirality spin-1/2 fields. This implies that the coupling of 21 tensor multiplets to $(2, 0)$ chiral supergravity is the only possibility in $(2, 0)$ chiral supergravity in 6 dimensions [138] free of anomalies.

For $(1, 0)$ supergravity, there are much more matter configurations which have no gravitational anomalies. A minimal condition [139] to have no gravitational anomalies is that

$$n_H - n_v + 29n_T = 273\,, \tag{5.1.4}$$

where the n_i indicates the number of hyper, vector and tensor multiplets. Different models with vanishing (gauge, gravitational and mixed) anomalies can be found in chapter 4 of [140]. More recent results can be found in [141].

5.1.3 Self-dual tensors on the worldvolume of extended objects

String theory contains several extended objects which break half of the supersymmetry. Several string theories contain objects with five spatial dimensions that are the magnetic duals of the fundamental string. These NS5-branes are solitonic objects with tensions proportional to $1/g_s^2$. One of the ways to study these NS5-branes is to look for solutions of the low-energy supergravity action (or to the solution of the β-functions of the massless fields) in terms of a harmonic function [142]. The small fluctuations of this brane around the classical solution can be described by a six-dimensional quantum field theory. Strong restrictions come from the many symmetries that this action must respect. World-brane symmetries descend to the world-brane from the spacetime symmetries. In [143] is argued that the $\mathrm{NS5}_B$, the type IIB NS5-brane, has $(1,1)$ supersymmetry on its worldvolume while the $\mathrm{NS5}_A$, the type IIA NS5-brane, has chiral supersymmetry. This means that the worldvolume description of the $\mathrm{NS5}_A$ contains a self-dual tensor multiplet.

Also the fundamental strings in type I, the type I' string[4] and the heterotic string theories with gauge groups $SO(32)$ and $E_8 \times E_8$ have magnetic duals. The NS5-branes of type I' and of the heterotic string with gauge group $E_8 \times E_8$ also have a self-dual tensor multiplet (with $(1,0)$ supersymmetry this time) in the description of the small fluctuations of the worldvolume [145].

D-branes are another class of non-perturbative objects in string theory. These extended objects carry charge under the different RR-potentials of IIA or IIB supergravity. Their tension is proportional to $1/g_s$ and open strings can end on them. So, at weak coupling, D-branes are lighter objects than the NS5-branes. The worldvolume description of D-branes requires a vector field. Multiple D-branes on top of each other give rise to a non-Abelian Yang–Mills gauge group [2]. So, D-brane worldvolumes contain no self-dual tensors.

A last class of extended objects that appears in this context are the extended objects in 11-dimensional theories. At low energies, the description of M-theory can be done in terms of 11-dimensional supergravity [146]. Much is known about the classical solutions of this low-energy supergravity limit in terms of harmonic functions. For 11-dimensional supergravity, the $M2$-brane is an extended object

[4] Type I' string theory is an orbifold projection of IIA string theory which keeps 1 supersymmetry in ten dimensions [144].

with 2 spatial dimensions. Its 3-dimensional solution of 11-dimensional supergravity in terms of a harmonic function is in [147]. The $M5$-brane, the magnetic partner of the electric $M2$, has five spatial dimensions with a classical solution in terms of a harmonic function [148]. The worldvolume of the $M5$ is described by a six-dimensional quantum field theory. The gravitational anomalies of the $M5$-brane were calculated in [149].

One $M5$-brane breaks half of the 32 supersymmetries of 11-dimensional supergravity. This implies that the worldvolume description of the $M5$-brane will be a supersymmetric quantum field theory with 16 supercharges. In [150] was argued that this is an $(2,0)$ superconformal field theory in six dimensions. The only multiplet possible in this theory is the self-dual tensor multiplet. The five scalars can be understood as the five transversal fluctuations of the $M5$-brane. The description of this worldvolume theory of the $M5$-brane is far from trivial. The equations of motion were derived in [151]. The quest for an action was harder. A non-covariant version was achieved in [152]. A bosonic covariant action using the ingredients of section 4 was derived in [153], a supersymmetric version in [154]. The complete Wess–Zumino term was found in [155]. There was also proven that the worldvolume of the $M5$ is invariant under non-linear supersymmetry transformations and that the quadratic approximation of the worldvolume action gives rise to the action for a free self-dual tensor multiplet with the full rigid superconformal group as symmetry group.

The appearance of the different self-dual tensors in the NS5-branes can be understood from M-theory: compactifying one of the 11 dimensions on a circle orthogonal to the $M5$-worldvolume gives rise to the IIA NS5-brane with also a self-dual tensor multiplet on its worldvolume. In [156] is argued that the heterotic $E_8 \times E_8$ superstring is equivalent to M-theory on $S^1/\mathbb{Z}_2$. If the $M5$-brane is again chosen in a direction orthogonal to the line segment, one keeps a self-dual tensor on the worldvolume of the heterotic NS5-brane.

An alternative approach to construct an action was also applied for the $M5$-brane in [137]: first write down an action whose transformation is proportional to the antiself-dual field strength and afterwards impose the self-duality by hand.

Even if the worldvolume action for a single $M5$-brane is found in this way, it is far from clear how to describe a set of n $M5$-branes on top of each other. It is precisely this description in terms of a large number of parallel $M5$-branes that is needed to check the adS_7/CFT_6-correspondence. A stack of n $D5$-branes gives rise to a $U(n)$ Yang–Mills theory, but it remains unknown what a stack of n $M5$-branes will look like. The same problem arises for the description of n NS5_A-branes, since they also have a self-dual tensor in their worldvolume. Partial results on theories with n tensor multiplets have been reported in [157, 113, 158, 117]. The proof of [111] that the description of the model will not be a local interacting quantum field theory remains valid in this supersymmetric setting.

5.2 Chiral bosons in rigid $(2,0)$-models

In this section, we study the self-dual tensor multiplet in a context with rigid supersymmetry. This allows a clear analysis of certain aspects of the role of the auxiliary scalar and the PST gauge transformations with respect to the supersymmetry algebra.

The superconformal algebra $OSp(8^*|4)$, explicitly given in (2.2.41), allows for one matter representation in six dimensions. Both rigid and local parameters are allowed. Here, the case of global parameters is studied, the local case is the subject of study of the next two sections. The tensor multiplet contains eight bosonic and eight fermionic on-shell degrees of freedom: the self-dual tensor has three degrees of freedom, as argued in section 4.3.3, the two Weyl tensors have eight on-shell degrees of freedom and in addition there are five scalar fields. For the two-form B_{ab}, we use the same notations as in section 4.3. The spinors are described by symplectic Majorana–Weyl spinors ψ^i $(i = 1, \ldots, 4)$ which transform in the **4** representation of $USp(4)$. The five real scalars ϕ^{ij} transform as a 5-plet of $USp(4)$. The properties of these gauge fields are summarized in table 5.1. The $USp(4)$ notations, definitions of traces for different tensors, and $USp(4)$-Schouten identities can be found in appendix A.3.

Field	Type	Restrictions	USp(4)	w
B_{ab}	boson	real antisymmetric tensor gauge field	1	2
ψ^i	fermion	$\gamma_7\psi^i = -\psi^i$	4	$\frac{5}{2}$
ϕ^{ij}	boson	$\phi^{ij} = -\phi^{ji} \quad \Omega_{ij}\phi^{ij} = 0$	5	2
a	boson		1	0

Table 5.1: The fields of the rigid $(2,0)$ tensor multiplet and the auxiliary scalar a with the various algebraic restrictions on the fields, their USp(4) representations assignments and the Weyl weights w.

In [155] is proven that it is more economical to use the x-dependent supersymmetry parameter $\varepsilon(x)$

$$\varepsilon(x) = \varepsilon + \gamma^\mu x_\mu \eta \,, \tag{5.2.1}$$

where ε and η are the rigid parameters for supersymmetry and special supersymmetry. The rigid supersymmetry transformation rules that respect the algebra (2.2.41) of the self-dual tensor multiplet and the auxiliary scalar are then

$$\begin{aligned} \delta B_{ab} &= -\bar{\varepsilon}(x)\gamma_{ab}\psi \,, \\ \delta\psi^i &= \tfrac{1}{48} H^+_{abc}\gamma^{abc}\varepsilon^i(x) + \tfrac{1}{4}\, \partial\!\!\!/ \phi^{ij}\varepsilon(x)_j - \phi^{ij}\eta_j \,, \\ \delta\phi^{ij} &= -4\bar{\varepsilon}(x)^{[i}\psi^{j]} - \Omega^{ij}\bar{\varepsilon}(x)\psi \,. \end{aligned} \tag{5.2.2}$$

Keep in mind that $\gamma_{abc} = \gamma_{[a}\gamma_b\gamma_{c]}$. We want that the commutator of two supersymmetry transformations gives rise to a translation. Using the transformation rules (5.2.2) in

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]\, B_{ab} = \xi^c \partial_c B_{ab} \,, \tag{5.2.3}$$

illustrates that we are obliged to use the self-duality condition,

$$H^-_{abc} = 0 \,, \tag{5.2.4}$$

to find a translation. Therefore, these are called *on-shell* transformation rules. The aims are to write down a covariant action for the tensor multiplet and to introduce other supersymmetry transformation rules such that the self-duality condition (5.2.4) is not needed anymore to realize the algebra. The auxiliary field a and the two new gauge symmetries of chapter 4 will be a cornerstone to achieve this. Therefore, the transformation of this auxiliary field under (special) supersymmetry is chosen:

$$\delta a = 0 \,. \tag{5.2.5}$$

The auxiliary field a is inert under both supersymmetry and special supersymmetry. For the tensor B_{ab} and the scalar a, we use the same gauge symmetries as in the bosonic model. We choose the same notations as in section 4.3. The fermions ψ^i and the scalars ϕ^{ij} are inert for these gauge symmetries. Using gauge transformations and the auxiliary scalar it is possible to define a field strength which is automatically self-dual:

$$h^+_{abc} = \tfrac{1}{4} H_{abc} - \tfrac{3}{2} v_{[a} H^-_{bc]} \,. \tag{5.2.6}$$

This h^+_{abc} was already introduced in [133]. It reduces automatically to $\frac{1}{4}H^+_{abc}$ when the self-duality condition (5.2.4) is imposed. The transformation rule for the spinors can be changed into

$$\delta\psi^i = \tfrac{1}{12} h^+_{abc}\gamma^{abc}\varepsilon^i(x) + \tfrac{1}{4}\,\partial\!\!\!/\phi^{ij}\varepsilon(x)_j - \phi^{ij}\eta_j\,. \tag{5.2.7}$$

Upon imposing the self-duality condition (5.2.4), this gives rise to the old rule in (5.2.2).

When checking the commutator of two pure supersymmetries on B_{ab} we find:

$$\begin{aligned}
[\delta(\varepsilon_1),\delta(\varepsilon_2)]B_{ab} &= \tfrac{1}{2}(\bar\varepsilon_2\gamma^c\varepsilon_1)\left(\partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab} - 2u_c\frac{H^-_{ab}}{\sqrt{u^2}} - 4u_{[a}\frac{H^-_{b]c}}{\sqrt{u^2}}\right)\\
&= \tfrac{1}{2}(\bar\varepsilon_2\gamma^c\varepsilon_1)\left(\partial_c B_{ab} - 2u_c\frac{H^-_{ab}}{\sqrt{u^2}}\right) + 2\partial_{[a}(\tfrac{1}{2}\bar\varepsilon_2\gamma^c\varepsilon_1 B_{b]c})\\
&\quad -2(\bar\varepsilon_2\gamma^c\varepsilon_1)\left(u_{[a}\frac{H^-_{b]c}}{\sqrt{u^2}}\right)\\
&= \tfrac{1}{2}(\bar\varepsilon_2\gamma^c\varepsilon_1)\mathcal{D}_c B_{ab} - \delta_I\left(\tfrac{1}{2}(\bar\varepsilon_2\gamma^c\varepsilon_1)B_{cd}\right)B_{ab}\\
&\quad -\delta_{III}\left(2(\bar\varepsilon_2\gamma^c\varepsilon_1)\tfrac{H^-_{cd}}{\sqrt{u^2}}\right)B_{ab}\,,
\end{aligned} \tag{5.2.8}$$

without imposing the self-duality condition (5.2.4).

Some comments can be made about the terms in the algebra, each of them clarifying the role of one of the gauge symmetries in the model:

- the $\mathcal{D}_c$ in the first term of (5.2.8) is a covariant derivative which contains also a term for gauge transformation II

$$\mathcal{D}_c B_{ab} = \partial_c B_{ab} - u_c\left(2\frac{H^-_{ab}}{\sqrt{u^2}}\right)\,. \tag{5.2.9}$$

  This asks for some explanation. We define a covariant derivative as a partial derivative minus the gauge transformations where the parameters are replaced by the gauge fields. The second gauge symmetry has also a field that can be considered as a gauge field. The derivative of the scalar $u_a = \partial_a a$ transforms into the derivative of the parameter of the second gauge symmetry:

$$\delta_{II}u_a = \partial_a\phi\,. \tag{5.2.10}$$

  Using this definition, the covariant derivative of B_{ab} appears in (5.2.8).

- One sees that there is a field-dependent gauge transformation I. In the local case, this will be part of the general coordinate transformations.
- The algebra also contains a field-dependent term which is a gauge symmetry III. In the local case, this term will be absorbed into the covariant general coordinate transformation.

In this way, each of the three gauge symmetries plays a specific role in the realization of the algebra. This will also be the case in a slightly different way in the local superconformal case.

We can also look to the commutator of two supersymmetries on the scalar field a. The supersymmetry transformation rule (5.2.5) gives trivially that this commutator should be 0, checking the algebra gives

$$\begin{aligned}
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]\, a &= \tfrac{1}{2}\bar{\varepsilon}_2\gamma^c\varepsilon_1\, \mathcal{D}_c a + \delta_I\left(\tfrac{1}{2}(\bar{\varepsilon}_2\gamma^c\varepsilon_1)B_{bc}\right) a \\
&\quad +\delta_{III}\left(2(\bar{\varepsilon}_2\gamma^c\varepsilon_1)\frac{H^-_{bc}}{\sqrt{u^2}}\right) a \\
&= \tfrac{1}{2}\bar{\varepsilon}_2\gamma^c\varepsilon_1\, (\partial_c a - u_c) \\
&= 0\,.
\end{aligned} \tag{5.2.11}$$

The second line is the covariant derivative $\mathcal{D}_c$ of the field[5] a. This shows that the auxiliary field a is a fermionic singlet [159]. A fermionic singlet is a representation of the supersymmetry algebra which contains only one (bosonic or fermionic) component. Usually, in supersymmetric theories all fields have superpartners. This is immediately related to the supersymmetry algebra $[Q,Q] = P$, where two supersymmetries produce a translation. Therefore, the existence of singlets $Q\Phi = 0$ in general contradicts the supersymmetry algebra, since always $P\Phi \neq 0$. In general, one has $Q\Phi = \varepsilon\Psi$, where Ψ is the '-ino'partner of Φ. However, the scalar a here is an example of such a fermionic singlet. In all cases, there is an extra symmetry which cancels the general coordinate transformation. In this way, the right-hand side of the $[Q,Q]$-commutator is not invertible any more. In this case, it is the second gauge symmetry that makes the scalar field a pure gauge for this symmetry and which enables to build a fermionic singlet with it. Other examples in rigid, local and κ supersymmetry are given in [159]. This fermionic singlet is not in contradiction with the theorem of section 2.2. That theorem states that the number of fermionic and bosonic on-shell components should equal in an on-shell realization of the supersymmetry algebra and that is the case here with 0 on-shell degrees of freedom. Keep in mind that there is no off-shell realization of the self-dual tensor multiplet yet.

[5] This covariant derivative resembles the one of the scalar in section 3.1.2.

The role of this scalar a is to enable the two PST gauge symmetries. These gauge symmetries allow the self-dual tensor h^+_{abc}. With h^+_{abc}, supersymmetry transformation rules can be constructed that do not need the self-duality condition any more to realize the algebra. We were not only looking for such transformation The final aim is to construct an action that is explicitly Lorentz invariant, just as in the bosonic model. It is possible to define such an action for these fields:

$$S = \int d^6x \left(-H^-_{ab} H^{*ab} - 4\bar{\psi}^i \, \partial\!\!\!/ \psi_i + \tfrac{1}{4}\phi^{ij} \Box \phi_{ij} \right) . \tag{5.2.12}$$

In [155] is proven in detail that this action for the free rigid $(2,0)$ tensor multiplet is invariant under all the rigid superconformal symmetries. In our case the (special) supersymmetry transformation rules are (5.2.5), the first and the third of (5.2.2) and (5.2.7) for the transformation of the spinor. The self-duality of the tensor appears in the same way as in chapter 4 after deriving the equation of motion for the tensor and imposing the suitable gauge fixing for symmetry III.

This treatment is purely algebraic: starting with the algebra, looking for representations and building an action. The geometrical superspace[6] treatment of the self-dual tensor multiplet is possible in terms of a real scalar superfield satisfying an appropriate constraint [160] or equivalently in terms of a super two-form [161].

So, we clarified the role of each of the gauge transformations in the realization of the algebra when using the appropriate supersymmetry transformation rules. Further, it is possible to extend the Lorentz-covariant action for the self-dual tensor to the self-dual tensor multiplet with rigid $(2,0)$ superconformal symmetry.

5.3 The Weyl multiplet in $(2,0)$ supergravity

5.3.1 The components of the Weyl multiplet

The goal of this chapter is to write down an action for the $(2,0)$ superconformal tensor multiplet, coupled to a superconformal gravitational background. This asks for a good understanding of this background. The $(2,0)$ superconformal gravitational multiplet is a representation of $OSp(8^*|4)$, whose (anti)commutation rules are in (2.2.41). This superalgebra has generators

$$T_A = P_a \ , Q^i_{\alpha'} \ , U_{ij} \ , M_{ab} \ , K_a \ , S^i_\alpha \ , D \ , \tag{5.3.1}$$

[6]A superspace is a space where, for mathematical convenience, the (bosonic) spacetime coordinates get fermionic superpartners. This approach is very fruitful in certain calculations, e.g., in diagrammatic calculations in supersymmetric theories. A good reference is [37].

for translations, supersymmetry, $USp(4)$ R-symmetry, Lorentz rotations, special conformal transformations, special supersymmetry, and dilatations. The gauge fields corresponding to the generators of each of these symmetries are

$$e_\mu{}^a \ , \ \psi_\mu^i \ , \ V_\mu^{ij} \ , \ \omega_\mu{}^{ab} \ , \ f_\mu{}^a \ , \ \phi_\mu^i \ , \ b_\mu \ . \tag{5.3.2}$$

Some properties of these gauge fields are described in Table 5.2. They contain

Field	Type	Restrictions	USp(4)	w
$e_\mu{}^a$	boson	sechsbein	1	-1
ψ_μ^i	fermion	$\gamma_7\psi_\mu^i = +\psi_\mu^i$	4	-$\frac{1}{2}$
V_μ^{ij}	boson	$V_\mu^{ij} = V_\mu^{ji}$	10	0
$\omega_\mu{}^{ab}$	boson		1	0
$f_\mu{}^a$	boson		1	0
ϕ_μ^i	fermion	$\gamma_7\phi_\mu^i = -\phi_\mu^i$	4	$\frac{1}{2}$
b_μ	boson		1	1

Table 5.2: The gauge fields of the $OSp(8^*|4)$ algebra. We have indicated the various algebraic restrictions on the fields, their USp(4) representation assignments, and the Weyl weights w. A field ϕ of weight w transforms under dilatations as $\delta_D\phi = w\Lambda_D\phi$.

190 bosonic and 160 fermionic off-shell degrees of freedom. In the bosonic case, $\omega_\mu{}^{ab}$ is expressed in terms of (derivatives of) $e_\mu{}^a$ and other fields to replace local translations and local Lorentz rotations by general coordinate transformations as explained in section 3.1.3. In a local superconformal theory, we need constraints for $\omega_\mu{}^{ab}$, $f_\mu{}^a$, and ϕ_μ^i in terms of the other gauge fields to obtain the smallest irreducible representation with spin 2. These conventional constraints can be written using the linearized curvatures[7]:

$$R_{\mu\nu}{}^a(P) \quad = \quad 0\,,$$

[7] The fully covariant constraints follow later in (5.3.14).

$$
\begin{aligned}
R_{\mu\nu}{}^{ab}(M)e^{\nu}{}_{b} &= 0\,, \\
\gamma^{\mu}R^{i}_{\mu\nu}(Q) &= 0\,.
\end{aligned}
\tag{5.3.3}
$$

The first two equations reduce $90+36$ bosonic degrees of freedom. The third constraint eliminates 96 fermionic degrees of freedom. This implies that the multiplet still contains $64+64$ degrees of freedom. Later on, we will give the full (covariantized, matter-corrected) conventional constraints, but they reduce the same number of degrees of freedom as these linearized versions. For the analysis we aim to do now, these linearized expressions are sufficient.

In all the other constructions of Weyl multiplets (in other dimensions or with other amounts of supersymmetry), there is a discrepancy between the number of bosonic and fermionic degrees of freedom after imposing the three conventional constraints. If that is the case, it is clear that this cannot be the full Weyl multiplet. This counting argument is then used to add matter fields to the remaining gauge fields until there is a matching between the number of bosonic and fermionic components. Also here, the matter fields T^{ij}_{abc}, χ^{ij}_{k}, and $D^{ij,kl}$ are introduced to obtain a Weyl representation with $128+128$ degrees of freedom. Several properties of these matter fields are summarized in Table 5.3. The last condition on $D^{ij,kl}$ can be expressed in two equivalent ways. The last one, expressing that the fully antisymmetric part of $D^{ij,kl}$ is 0, is new (but equivalent) with respect to [42]. There is no proof that it is not possible to formulate a supergravity multiplet without introducing these matter fields. Although the counting of the degrees of freedom is in this derivation no argument to add them, some strong arguments are given to introduce them. We list four indications in favor of the matter fields:

- The $N=4$ Weyl multiplet in four dimensions should be found by compactifying the $(2,0)$ theory in $d=6$ to four dimensions [162]. The gravity multiplet for $N=4$ in $d=4$ contains $128+128$ degrees of freedom [163] and the type of matter fields that one finds by doing the compactification.

- Another argument to introduce $64+64$ matter degrees of freedom is the existence of a current multiplet in six dimensions with the same number of components and the same tensorial structure [42]. This enables the coupling of the gravitational multiplet to this current multiplet.

- Moreover, the breaking to $(1,0)$ should give the Weyl multiplet of [41] and this contains also matter fields that cannot be found in another way.

- The adS/CFT-correspondence leads to a last strong argument in favor of the matter fields. As explained in the introduction, it conjectures a correspondence between the large n limit of certain conformal field theories in d dimensions to M-theory or string theory compactified to $(d+1)$-dimensional adS

Field	Type	Restrictions	USp(4)	w
T^{ij}_{abc}	boson	$T^{ij}_{abc} = -T^{ji}_{abc}$	5	1
		$\Omega_{ij}T^{ij}_{abc} = 0$		
		$T^{ij}_{abc} = -\frac{1}{6}\varepsilon_{abcdef}T^{ij}_{def}$		
χ^{ij}_k	fermion	$\chi^{ij}_k = -\chi^{ji}_k$	16	$\frac{3}{2}$
		$\Omega_{ij}\chi^{ij}_k = 0$		
		$\chi^{ij}_i = 0$		
		$\gamma_7\chi^{ij}_k = +\chi^{ij}_k$		
$D^{ij,k\ell}$	boson	$D^{ij,k\ell} = -D^{ji,k\ell} = -D^{ij,\ell k}$	14	2
		$D^{ij,k\ell} = D^{k\ell,ij}$		
		$\Omega_{ij}D^{ij,k\ell} = \Omega_{k\ell}D^{ij,k\ell} = 0$		
		$\Omega_{ik}\Omega_{j\ell}D^{ij,k\ell} = 0$ or $D^{[ij,kl]} = 0$		

Table 5.3: *Matter fields of* $(2,0)$ *conformal supergravity* The various algebraic restrictions, the USp(4) representation assignments and the Weyl weights w are indicated.

spacetimes (times spheres). Supposing that the adS/CFT-correspondence is right, the field content of the Weyl multiplet with matter fields is derivable in the following way. The starting point is the field content of the maximal adS supergravity in 7 dimensions [164, 90]: a vielbein, 4 gravitinos, 5 real, third-rank, antisymmetric tensors, 10 $SO(5)$ gauge fields, 16 spin 1/2 fields and 14 scalars. There exists an action for these fields which is invariant under general coordinate transformations, local Lorentz rotations, local $SO(5)_g$, global $SO(5)_c$, and local supersymmetry. It is possible to study the implications of these local symmetries on the boundary of the adS space. This is done in [165] by partially gauge fixing the local symmetries. The fields on the boundary transform under the residual symmetry transformations after the gauge fixing. These symmetry transformations appear to be the local superconformal transformations of $OSp(8^*|4)$ in six dimensions. So, the seven-dimensional local symmetries give rise to local superconformal symmetry on the six-dimensional boundary of adS_7. The boundary values of the gravitational multiplet in adS_7 form the Weyl supermultiplet in six dimensions with matter fields. In [166], all the unitary irreducible representations of $OSp(8^*|4)$ in seven dimensions were derived. There is no possibility in seven dimensions for a gravitational multiplet that gives rise to a superconformal gravity multiplet without matter fields in six dimensions. So, trusting the adS/CFT-correspondence and using the possible representations of [166] is another argument that the six-dimensional Weyl multiplet must contain the matter fields T^{ij}_{abc}, χ^{ij}_k, and $D^{ij,kl}$. The same connection between adS_{d+1} and the gravitational multiplet in d dimensions is treated earlier for $d=4$ [167] and $d=2$ [168].

5.3.2 The curvatures and transformation rules

In this section, we will construct the full non-linear transformation rules, the matter-corrected curvatures, and the appropriate conventional constraints for the $(2,0)$ Weyl multiplet in six dimensions. To do so, we will use the iterative procedure of [41]. The method to construct the transformation rules, the curvatures, and the constraints is given there in general and applied to simple conformal supergravity in six dimensions. We start with the matter fields (T^{ij}_{abc}, χ^{ij}_k, and $D^{ij,kl}$ in this case) and the linear transformation rules of the gauge fields. These linear transformation rules can be derived from the algebra using the basic rule:

$$\delta h_\mu^A = \partial_\mu \varepsilon^A + \varepsilon^C h_\mu^B f_{BC}{}^A \,. \tag{5.3.4}$$

For the gauge fields of the bosonic algebra, these transformation rules were already given in (3.1.7). Using the algebra (2.2.41), this gives rise to the following linearized transformation rules:

$$\begin{aligned}
\delta &= \bar\varepsilon Q + \bar\eta S + \Lambda_D D + \lambda^{ab} M_{ab} + \Lambda_K^a K_a + \Lambda^{ij} U_{ij}\,,\\
\delta e_\mu{}^a &= \tfrac12 \bar\varepsilon\gamma^a\psi_\mu - \Lambda_D e_\mu{}^a - \lambda^{ab} e_{\mu b}\,,\\
\delta\psi_\mu^i &= \mathcal{D}_\mu\varepsilon^i + \gamma_\mu\eta^i - \tfrac12\Lambda_D\psi_\mu^i + \Lambda^i{}_j\psi_\mu^j - \tfrac14\lambda^{ab}\gamma_{ab}\psi_\mu^i\,,\\
\delta b_\mu &= \partial_\mu\Lambda_D - \tfrac12\bar\varepsilon\phi_\mu + \tfrac12\bar\eta\psi_\mu - 2\Lambda_{K\mu}\,,\\
\delta\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab} + 2\omega_{\mu c}{}^{[a}\lambda^{b]c} - \tfrac12\bar\varepsilon\gamma^{ab}\phi_\mu - \tfrac12\bar\eta\gamma^{ab}\psi_\mu + 4\Lambda_K{}^{[a}e_\mu{}^{b]}\,,\\
\delta V_\mu^{ij} &= \partial_\mu\Lambda^{ij} + \Lambda^{(i}{}_k V_\mu^{j)k} - 4\bar\varepsilon^{(i}\phi_\mu^{j)} - 4\bar\eta^{(i}\psi_\mu^{j)}\,,\\
\delta\phi_\mu^i &= \mathcal{D}_\mu\eta^i - f_\mu{}^a\gamma_a\varepsilon^i + \Lambda_K^a\gamma_a\psi_\mu^i + \tfrac12\Lambda^i{}_j\phi_\mu^j + \tfrac12\Lambda_D\phi_\mu^i - \tfrac14\lambda^{ab}\gamma_{ab}\phi_\mu^i\,,
\end{aligned}$$

$$\delta f_\mu{}^a = \mathcal{D}_\mu \Lambda_K^a + \Lambda_D f_\mu{}^a - \tfrac{1}{2}\bar{\eta}\gamma^a\phi_\mu - \lambda^{ab} f_{\mu b}\,. \tag{5.3.5}$$

The derivatives $\mathcal{D}_\mu$ are covariant with respect to dilatations, Lorentz rotations, and $USp(4)$ R-symmetry symmetries:

$$\begin{aligned}
\mathcal{D}_\mu \varepsilon^i &= \partial\varepsilon^i + \tfrac{1}{2} b_\mu \varepsilon^i + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\varepsilon^i - \tfrac{1}{2} V_{\mu}{}^i{}_j \varepsilon^j\,,\\
\mathcal{D}_\mu \eta^i &= \partial_\mu\eta^i - \tfrac{1}{2} b_\mu \eta^i + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\eta^i - \tfrac{1}{2} V_{\mu}{}^i{}_j \eta^j\,,\\
\mathcal{D}_\mu \Lambda_K^a &= \partial_\mu \Lambda_K^a - b_\mu \Lambda_K^a + \omega_\mu{}^{ab}\Lambda_{Kb}\,.
\end{aligned} \tag{5.3.6}$$

The spinors ε^i and η^i have positive and negative chirality. Using

$$R_{\mu\nu}{}^A = 2\partial_{[\mu} h_{\nu]}{}^A + h_\nu^C h_\mu^B f_{BC}{}^A\,, \tag{5.3.7}$$

and the structure constants of the algebra allows for the calculation of the different curvatures:

$$\begin{aligned}
R_{\mu\nu}{}^a(P) &= 2\partial_{[\mu} e_{\nu]}^a + 2b_{[\mu} e_{\nu]}^a + \underline{2\omega_{[\mu}{}^{ab} e_{\nu]b}} - \tfrac{1}{2}\bar{\psi}_\mu\gamma^a\psi_\nu\,,\\
R_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 8 f_{[\mu}{}^{[a} e_{\nu]}{}^{b]} + \bar{\psi}_{[\mu}\gamma^{ab}\phi_{\nu]}\,,\\
R_{\mu\nu}^i(Q) &= 2\partial_{[\mu}\psi_{\nu]}^i + b_{[\mu}\psi_{\nu]}^i + \tfrac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]}^i - V_{[\mu}^i{}_j \psi_{\nu]}^j + 2\gamma_{[\mu}\phi_{\nu]}^i\,,\\
R_{\mu\nu}(D) &= 2\partial_{[\mu} b_{\nu]} + 4 f_{[\mu}{}^a e_{\nu]a} + \bar{\psi}_{[\mu}\phi_{\nu]}\,,\\
R_{\mu\nu}{}^{ij}(V) &= 2\partial_{[\mu} V_{\nu]}{}^{ij} + V_{[\mu}{}^{k(i} V_{\nu]}{}^{j)}{}_k + 8\bar{\psi}_{[\mu}{}^{(i}\phi_{\nu]}{}^{j)}\,,\\
R_{\mu\nu}{}^i(S) &= 2\mathcal{D}_{[\mu}\phi_{\nu]}^i - 2\gamma_a f_{[\mu}{}^a\psi_{\nu]}^i\,,\\
R_{\mu\nu}{}^a(K) &= 2\mathcal{D}_{[\mu} f_{\nu]}{}^a + \tfrac{1}{2}\bar{\phi}_\mu\gamma^a\phi_\nu\,.
\end{aligned} \tag{5.3.8}$$

The method consists of the following steps:

1. The first step comprises the derivation of the *linearized* Q-transformation rules. For the gauge fields, this has been done already in (5.3.5). For the matter fields, this requires some work. One writes down the most general possibility with arbitrary constants for the moment. If derivatives of gauge fields appear somewhere, they are replaced by the corresponding linearized curvatures of (5.3.8). The coefficients are fixed by calculating the commutator of two supersymmetries and imposing that this should give rise to a translation. This leads to:

$$\begin{aligned}
\delta T_{abc}^{ij} &= \tfrac{1}{8}\bar{\varepsilon}^{[i}\left(\gamma^{\mu\nu}\gamma_{abc} + \tfrac{1}{5}\gamma_{abc}\gamma^{\mu\nu}\right) R_{\mu\nu}^{j]}(Q) - \tfrac{1}{15}\bar{\varepsilon}^k\gamma_{abc}\chi_k^{ij} - (\text{trace})\,,\\
\delta\chi_k^{ij} &= \tfrac{5}{32}\left(\partial_\mu T_{abc}^{ij}\right)\gamma^{abc}\gamma^\mu\varepsilon_k - \tfrac{15}{8}\gamma^{\mu\nu} R_{\mu\nu k}{}^{[i}(V)\,\varepsilon^{j]} - \tfrac{1}{4} D^{ij}{}_{kl}\varepsilon^l\\
&\quad -(\text{traces})\,,\\
\delta D^{ij,kl} &= -2\bar{\varepsilon}^{[i}\,\partial\!\!\!/\chi^{j],kl} - 2\bar{\varepsilon}^{[k}\,\partial\!\!\!/\chi^{l],ij} + (ij \leftrightarrow kl) - (\text{trace})\,.
\end{aligned} \tag{5.3.9}$$

Also the transformations of the gauge fields get extra matter contributions, for instance

$$\begin{aligned}\delta\psi_\mu^i &= \tfrac{1}{24}T^{ij}_{abc}\gamma^{abc}\gamma_\mu\varepsilon_j\,,\\ \delta V_\mu^{ij} &= -\tfrac{4}{15}\bar\varepsilon_k\gamma_\mu\chi^{(i,j)k}\,.\end{aligned} \tag{5.3.10}$$

2. The second step leads to the *bosonic* transformation rules of the matter fields. The Lorentz transformations follow immediately from the index structure of the fields. The Weyl weight of the matter fields follows from the Weyl weights of the gauge fields (derived from the algebra and given in (5.3.5)) and the linearized supersymmetry rules of step 1. These bosonic transformation rules of the matter fields are summarized in table 5.3.

3. Replace in the Q-transformation rules the ordinary derivatives (coming from imposing in step 1. that the commutator of two supersymmetries should give rise to a translation) by fully covariant derivatives $\hat{\mathcal{D}}_\mu$. This is defined as the partial derivative minus the gauge transformations with the parameters replaced by the corresponding gauge fields for each superconformal symmetry, except the translations. Since the transformations of the gauge fields have changed in step 1., also the curvatures will change. This will also be done when the transformation rules will be changed in one of the next steps. We find the following corrected curvatures:

$$\begin{aligned}R_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 8\underline{f_{[\mu}{}^{[a}e_{\nu]}{}^{b]}} + \bar\psi_{[\mu}\gamma^{ab}\phi_{\nu]}\\ &\quad +\bar\psi_{[\mu}\gamma^{[a}R_{\nu]}{}^{b]}(Q) + \tfrac12\bar\psi_{[\mu}\gamma_{\nu]}R^{ab}(Q) + \tfrac12\bar\psi_{\mu,i}\gamma_c\psi_{\nu,j}T^{abc,ij}\,,\\ R_{\mu\nu}{}^{ij}(V) &= 2\partial_{[\mu}V_{\nu]}{}^{ij} + V_{[\mu}{}^{k(i}V_{\nu]}{}^{j)}{}_k + 8\bar\psi_{[\mu}{}^{(i}\phi_{\nu]}{}^{j)} + \tfrac{8}{15}\bar\psi_{[\mu,k}\gamma_{\nu]}\chi^{(i,j)k}\,,\\ R^i_{\mu\nu}(Q) &= 2\partial_{[\mu}\psi^i_{\nu]} + b_{[\mu}\psi^i_{\nu]} + \tfrac12\omega_{[\mu}{}^{ab}\gamma_{ab}\psi^i_{\nu]} - V^i_{[\mu j}\psi^j_{\nu]} + 2\underline{\gamma_{[\mu}\phi^i_{\nu]}}\\ &\quad +\tfrac{1}{12}T^{ij}_{abc}\gamma^{abc}\gamma_{[\mu}\psi_{\nu]j}\,.\end{aligned} \tag{5.3.11}$$

The curvature $R_{\mu\nu}{}^a(P)$ and $R_{\mu\nu}(D)$ get no corrections. The other curvatures are not needed in the transformation rules since their gauge fields will become composite fields after imposing the conventional constraints.

4. The fourth step gives rise to the special supersymmetry transformations of the different fields. They can be found by calculating the $[Q,K]$-commutator:

$$[\delta_K(\Lambda_K^a),\delta_Q(\varepsilon)] = \delta_S(-\Lambda_K^a\gamma_a\varepsilon)\,. \tag{5.3.12}$$

This gives rise to the following special supersymmetry transformation rules:

$$\delta_S\psi_\mu^i = \gamma_\mu\eta^i\,,$$

$$\begin{aligned}
\delta_S b_\mu &= \tfrac{1}{2}\bar\eta\psi_\mu\,,\\
\delta_S V_\mu^{ij} &= -4\bar\eta^{(i}\psi_\mu^{j)}\,,\\
\delta_S \chi_k^{ij} &= \tfrac{5}{8}T_{abc}^{ij}\gamma^{abd}\eta_k-(\text{traces})\,,\\
\delta_S D^{ij,kl} &= 4\bar\eta^{[i}\chi^{j],kl}-(\text{trace})\,.
\end{aligned}\tag{5.3.13}$$

5. The last but one step changes the conventional constraints (5.3.3). First the curvatures are replaced by their covariant curvatures as defined in step 3. Their exact form is not important as long as they enable a solution of ${\omega_\mu}^{ab}$, ${f_\mu}^a$, and ϕ_μ^i in terms of the matter and other gauge fields. The form that allows to transform the conventional constraints into each other by Q-supersymmetry and also allows for a fermionic constraint invariant under S-supersymmetry (and thus leaving invariant the S-transformation of ϕ_μ^i) is:

$$\begin{aligned}
{R_{\mu\nu}}^a(P) &= 0\,,\\
{R_{\mu\nu}}^{ab}(M){e^\nu}_b+\tfrac{1}{4}T_{\mu bc}^{ij}T_{ij}^{abc} &= 0\,,\\
\gamma^\mu R_{\mu\nu}^i(Q) &= 0\,.
\end{aligned}\tag{5.3.14}$$

6. The last step allows for the construction of the full non-linear (special) supersymmetry transformation rules. As required in the first step, the commutator of two Q-transformations leads to a translation. The matter term (5.3.10) in the transformation for ψ_μ^i gives rise to a Lorentz rotation that depends on the matter field T_{abc}^{ij} in the commutator applied on ${e_\mu}^a$. This implies that the algebra is changed and that this matter-dependent Lorentz rotation should be reproduced also for the $[Q,Q]$-commutator on other fields. This requires other covariantization terms in the transformation rules, driven by an iterative procedure. In principle, this can lead also to field-dependent terms in the commutator $[Q,S]$ or $[S,S]$, but this is not the case here.

This terminates the construction of the full non-linear Weyl multiplet. This procedure leads to the following results. The solution of the constraints (5.3.14) is possible since each of the constraints contains a (underlined) term which is the product of a gauge field and an invertible sechsbein. They express ${\omega_\mu}^{ab}$, ϕ_μ^i, and ${f_\mu}^a$ in terms of other fields:

$$\begin{aligned}
{\omega_\mu}^{ab} &= 2e^{\nu[a}\partial_{[\mu}{e_{\nu]}}^{b]}-e^{\rho[a}e^{b]\sigma}{e_\mu}^c\partial_\rho e_{\sigma c}\\
&\quad+2{e_\mu}^{[a}b^{b]}+\tfrac{1}{2}\bar\psi_\mu\gamma^{[a}\psi^{b]}+\tfrac{1}{4}\bar\psi^a\gamma_\mu\psi^b\,,\\
{f_\mu}^a &= -\tfrac{1}{8}{R'_\mu}^a(M)+\tfrac{1}{80}{e_\mu}^aR'(M)+\tfrac{1}{32}T_{\mu cd}^{ij}T_{ij}^{acd}\,,\\
\phi_\mu^i &= -\tfrac{1}{16}\left(\gamma^{ab}\gamma_\mu-\tfrac{3}{5}\gamma_\mu\gamma^{ab}\right){R'_{ab}}^i(Q)\,.
\end{aligned}\tag{5.3.15}$$

The curvatures R' are the ones of (5.3.11) without the underlined term. The full non-linear (special) supersymmetry transformation rules are:

$$\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\bar{\varepsilon}\gamma^a\psi_\mu\,, \\
\delta b_\mu &= -\tfrac{1}{2}\bar{\varepsilon}\phi_\mu + \tfrac{1}{2}\bar{\eta}\psi_\mu\,, \\
\delta\psi_\mu^i &= \mathcal{D}_\mu\varepsilon^i + \tfrac{1}{24}T^{ij}_{abc}\gamma^{abc}\gamma_\mu\varepsilon_j + \gamma_\mu\eta^i\,, \\
\delta V_\mu^{ij} &= -4\bar{\varepsilon}^{(i}\phi_\mu^{j)} - \tfrac{4}{15}\bar{\varepsilon}_k\gamma_\mu\chi^{(i,j)k} - 4\bar{\eta}^{(i}\psi_\mu^{j)}\,, \\
\delta T^{ij}_{abc} &= \tfrac{1}{8}\bar{\varepsilon}^{[i}\gamma^{de}\gamma_{abc}R^{j]}_{de}(Q) - \tfrac{1}{15}\bar{\varepsilon}^k\gamma_{abc}\chi^{ij}_k - (\text{trace})\,, \\
\delta\chi^{ij}_k &= \tfrac{5}{32}\left(\mathcal{D}_\mu T^{ij}_{abc}\right)\gamma^{abc}\gamma^\mu\varepsilon_k - \tfrac{15}{16}\gamma^{\mu\nu}R_{\mu\nu k}{}^{[i}(V)\varepsilon^{j]} - \tfrac{1}{4}D^{ij}{}_{kl}\varepsilon^l \\
&\quad + \tfrac{5}{8}T^{ij}_{abc}\gamma^{abc}\eta_k - (\text{traces})\,, \\
\delta D^{ij,kl} &= -2\bar{\varepsilon}^{[i}\,\not{\mathcal{D}}\chi^{j],kl} + 4\bar{\eta}^{[i}\chi^{j],kl} + (ij \leftrightarrow kl) - (\text{trace})\,.
\end{aligned} \qquad (5.3.16)$$

The transformation rules of the composite gauge fields can be derived from these rules. The algebra takes the following form:

$$\begin{aligned}
[\delta_Q(\varepsilon_1),\delta_Q(\varepsilon_2)] &= \tfrac{1}{2}(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)\,\hat{\mathcal{D}}_\mu + \delta_M(-\tfrac{1}{2}\bar{\varepsilon}_2^i T^{abc}_{ij}\gamma_c\varepsilon_1^j) \\
&\quad + \delta_S(-\tfrac{2}{45}\bar{\varepsilon}_{1[k}\gamma_a\varepsilon_{2j]}\gamma^a\chi^{(i,j)k}) + \delta_K(-\tfrac{1}{8}\bar{\varepsilon}_2^i\gamma_b\varepsilon_1^j\,\mathcal{D}_c T^{abc}_{ij})\,, \\
[\delta_S(\eta),\delta_Q(\varepsilon)] &= \delta_D(-\tfrac{1}{2}\bar{\varepsilon}\eta) + \delta_M(-\tfrac{1}{2}\bar{\varepsilon}\gamma^{ab}\eta) + \delta_{USp(4)}(-4\bar{\varepsilon}^{(i}\eta^{j)})\,, \\
[\delta_S(\eta_1),\delta_S(\eta_2)] &= \delta_K(-\tfrac{1}{2}\bar{\eta}_2\gamma^a\eta_1)\,.
\end{aligned} \qquad (5.3.17)$$

This finishes the discussion on the Weyl multiplet for conformal supergravity. We end up with $128+128$ degrees of freedom, $64+64$ of them being matter fields. This multiplet of $(2,0)$ conformal supergravity will be used in the next section to study the coupling of a $(2,0)$ self-dual tensor multiplet to a conformal supergravity background. To construct the action for this multiplet, we will need the auxiliary scalar and the gauge symmetries of section 4.3.

5.4 An action in a conformal supergravity background

In this section, we will construct the main result of this chapter: the Lorentz-covariant action for the $(2,0)$ self-dual tensor multiplet in a conformal supergravity background. This work was reported in [7]. We will combine the experience of the construction of the rigid-supersymmetric action in section 5.2 with the field content and properties of the conformal supergravity background of section 5.3. We will retrieve the equations of motion, that were known already before. We end with some comments on the possible role of this action in presently developing subjects.

5.4.1 The algebraic aspects in the conformal background

Placing the self-dual tensor multiplet in a conformal background implies interactions between the matter multiplet and the gravitational background. We will sketch how this can be done. By generalizing the rigid transformation rules (5.2.2), the following on-shell (special) supersymmetry transformation rules are compatible with the algebra:

$$\begin{aligned} \delta B_{\mu\nu} &= -\bar{\varepsilon}\gamma_{\mu\nu}\psi + \bar{\varepsilon}^i\gamma_{[\mu}\psi^j_{\nu]}\phi_{ij} \ , && (5.4.1)\\ \delta\psi^i &= \tfrac{1}{48}H^+_{\mu\nu\rho}\gamma^{\mu\nu\rho}\varepsilon^i + \tfrac{1}{4}\not{\mathcal{D}}\phi^{ij}\varepsilon_j - \phi^{ij}\eta_j \ , && (5.4.2)\\ \delta\phi^{ij} &= -4\bar{\varepsilon}^{[i}\psi^{j]} - (\text{trace}) \ . && (5.4.3) \end{aligned}$$

In (5.4.2), the first term contains the self-dual part of the covariant field strength for the tensor:

$$\begin{aligned} H_{\mu\nu\rho} &\equiv 3\partial_{[\mu}B_{\nu\rho]} + 3\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\psi - \tfrac{3}{2}\bar{\psi}^i_{[\mu}\gamma_\nu\psi^j_{\rho]}\phi_{ij} - \tfrac{1}{2}\phi_{ij}T^{ij}_{\mu\nu\rho}\\ &= 3\partial_{[\mu}B_{\nu\rho]} + C_{\mu\nu\rho} \ . && (5.4.4) \end{aligned}$$

Using (5.4.1), this field strength $H_{\mu\nu\rho}$ is a covariantized version of the rigid field strength. Remark that we define a matter term in the covariant field strength. This matter term does not appear in the transformation rule (5.4.2) since the self-dual part of $H_{\mu\nu\rho}$ does not contain the antiself-dual matter field. We introduce this matter field in the field strength because this allows an easier and shorter analysis later on. The self-duality condition in a conformal supergravity background becomes

$$H^-_{\mu\nu\rho} = 0 \,. \qquad (5.4.5)$$

Analogous to the analysis with rigid supersymmetry, equation (5.4.5) must be used to realize the algebra (2.2.41) with the supersymmetry transformation rules (5.4.1)–(5.4.3) and (5.3.16). As could be seen from the covariant field strength already, the Weyl weight of the tensor $B_{\mu\nu}$ is 0. In the rigid case (cf. table 5.1), the conformal weight of the tensor was 2, but B_{ab} is multiplied twice with a vielbein with each time conformal weight -1 (cf. table 5.2).

Just as in the rigid case, it is again possible to construct supersymmetry transformation rules that do not need the self-duality condition (5.4.5). To this end, we again introduce the auxiliary scalar a as in the bosonic or as in the rigid supersymmetric case. Since the metric is dynamical here, the definition of u^2 and v_μ becomes

$$\begin{aligned} u^2 &\equiv u_\mu g^{\mu\nu} u_\nu \, , \\ v_\mu &\equiv \frac{u_\mu}{\sqrt{u_\nu g^{\nu\rho} u_\rho}} \, . \end{aligned} \tag{5.4.6}$$

The contractions of the field strength with the vector v_μ are defined as in the bosonic equations (4.3.1). The first and the third gauge symmetry are completely the same as in the bosonic case. In the gauge transformation II of the tensor,

$$\delta_{II} B_{\mu\nu} = 2 H^-_{\mu\nu} \frac{\phi}{\sqrt{u^2}} \, , \tag{5.4.7}$$

the full covariantized field strength of (5.4.4) is used, so this includes here *also the term with the matter field.*
The vector u_μ again transforms into the derivative of the parameter of the second gauge symmetry and as such can be treated as a gauge field for this gauge symmetry.

Making use of the gauge symmetries (with a suitable transformation II), it is again possible to write down supersymmetry transformation rules that realize the algebra without using the self-duality condition (5.4.5). Therefore, the self-dual component of the field strength in (5.4.2) will be replaced by a self-dual tensor, such that it gives the old transformation rule upon imposing the self-duality condition (5.4.5):

$$h^+_{\mu\nu\rho} \equiv \tfrac{1}{4} H_{\mu\nu\rho} - \tfrac{3}{2} v_{[\mu} H^-_{\nu\rho]} \, . \tag{5.4.8}$$

The terms with v_μ in (5.4.8) can again be seen as the product of u_μ, the gauge field of the second symmetry, and the II-transformation of $B_{\mu\nu}$. Therefore, $h^+_{\mu\nu\rho}$ can, just as in the rigid model (5.2.6), be considered as the fully covariant field strength of $B_{\mu\nu}$, where fully means *also with respect to gauge symmetry II*. $h^+_{\mu\nu\rho}$ is automatically self-dual in this way. The new (special) supersymmetry transformation rule is then:

$$\delta\psi^i = \frac{1}{12} h^+_{\mu\nu\rho} \gamma^{\mu\nu\rho} \varepsilon^i + \frac{1}{4} \not{D} \phi^{ij} \varepsilon_j - \phi^{ij} \eta_j \, . \tag{5.4.9}$$

This transformation rule depends on the matter term in $h^+_{\mu\nu\rho}$. This is necessary later to realize the algebra (5.4.14) and the invariance of the action.

In the rigid action (5.2.12) appear the kinetic terms of the spinors and the scalars. In the action with local symmetries, this terms will be generalized to the covariant derivative of ψ^i and the covariant d'Alembertian of ϕ^{ij}. From the transformation rules (5.4.1), (5.4.9), (5.4.3) and (5.3.16), the covariant derivative of ψ^i and ϕ^{ij} are:

$$\begin{aligned}\mathcal{D}_\mu\psi^i &= \left(\partial_\mu - \tfrac{5}{2}b_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\psi^i - \tfrac{1}{2}V^i_{\mu j}\psi^j \\ &\quad -\tfrac{1}{12}h^+_{abc}\gamma^{abc}\psi^i_\mu - \tfrac{1}{4}\left(\not{\mathcal{D}}\phi^{ij}\right)\psi_{\mu j} + \phi^{ij}\phi_{\mu j}\ , \\ \mathcal{D}_\mu\phi^{ij} &= \left(\partial_\mu - 2b_\mu\right)\phi^{ij} + V_\mu{}^{[i}{}_k\phi^{j]k} + 4\left(\bar\psi^{[i}_\mu\psi^{j]} - \text{trace}\right)\ . \end{aligned} \tag{5.4.10}$$

For the supercovariant d'Alembertian of the scalars, we need the superconformal transformation properties of $\mathcal{D}_a\phi^{ij}$:

$$\begin{aligned}\delta\mathcal{D}_a\phi^{ij} &= 3\Lambda_D\mathcal{D}_a\phi^{ij} + \Lambda^{[i}{}_k\mathcal{D}_a\phi^{j]k} - \Lambda_a{}^b\mathcal{D}_b\phi^{ij} + 4\Lambda_{Ka}\phi^{ij} \\ &\quad -4\bar\varepsilon^{[i}\mathcal{D}_a\psi^{j]} + \tfrac{2}{15}\left(\bar\varepsilon_l\gamma_a\chi^{(i,k)l}\phi^j{}_k - i\leftrightarrow j\right) \\ &\quad -\tfrac{1}{6}\bar\varepsilon_k\gamma_a\gamma^{bcd}T^{k[i}_{bcd}\psi^{j]} - 4\bar\eta^{[i}\gamma_a\psi^{j]} - (\text{trace})\ . \end{aligned} \tag{5.4.11}$$

This gives rise to the following supercovariant d'Alembertian:

$$\begin{aligned}\mathcal{D}^a\mathcal{D}_a\phi^{ij} &= \partial^a\mathcal{D}_a\phi^{ij} - 3b^a\mathcal{D}_a\phi^{ij} - V^{[i}_a{}_k\mathcal{D}^a\phi^{j]k} + \omega_a{}^{ab}\mathcal{D}_b\phi^{ij} - 4f_a{}^a\phi^{ij} \\ &\quad +4\bar\psi^{[i}_a\mathcal{D}^a\psi^{j]} - \tfrac{2}{15}\left(\bar\psi^a_l\gamma_a\chi^{(i,k)l}\phi^j{}_k - i\leftrightarrow j\right) \\ &\quad -\tfrac{1}{6}\bar\psi^a_k\gamma_a\gamma^{bcd}T^{k[i}_{bcd}\psi^{j]} - 4\bar\phi^{[i}_a\gamma^a\psi^{j]} - (\text{trace})\,. \end{aligned} \tag{5.4.12}$$

The Riemann scalar curvature occurs in the equation of motion for the scalar fields through the term $-4f_a{}^a\phi^{ij}$, using the solution of the second constraint for $f_\mu{}^a$ in (5.3.15):

$$f_a{}^a = -\tfrac{1}{20}R'(M) + \ldots \tag{5.4.13}$$

Upon gauge fixing the conformal symmetries (with the superconformal tensor calculus program in the back of the mind), this term gives rise to the standard Einstein term in the super-Poincaré action.

Just as in the rigid case, calculating the algebra of Q- and S-transformations gives rise to field-dependent transformations. When using the transformation rules for the tensor multiplet, (5.4.1), (5.4.9) and (5.4.3), also a field-dependent term for gauge symmetry I appears in this algebra. The combination of (5.2.8) and (5.3.17) leads to

$$\begin{aligned}[\delta_Q(\varepsilon_1),\delta_Q(\varepsilon_2)] &= \tfrac{1}{2}\bar\varepsilon_2\gamma^\mu\varepsilon_1\,\hat{\mathcal{D}}_\mu + \delta_M(-\tfrac{1}{2}\bar\varepsilon^i_2T^{abc}_{ij}\gamma_c\varepsilon^j_1) \\ &\quad +\delta_S(-\tfrac{2}{45}\bar\varepsilon_{1[k}\gamma_a\varepsilon_{2j]}\gamma^a\chi^{(i,j)k}) + \delta_K(-\tfrac{1}{8}\bar\varepsilon^i_2\gamma_b\varepsilon^j_1\,\mathcal{D}_cT^{abc}_{ij}) \\ &\quad +\delta_I(-\tfrac{1}{2}\bar\varepsilon^i_2\gamma_\mu\varepsilon^j_1\phi_{ij})\,, \end{aligned} \tag{5.4.14}$$

$$\begin{aligned}
[\delta_S(\eta), \delta_Q(\varepsilon)] &= \delta_D(-\tfrac{1}{2}\bar{\varepsilon}\eta) + \delta_M(-\tfrac{1}{2}\bar{\varepsilon}\gamma^{ab}\eta) + \delta_{USp(4)}(-4\bar{\varepsilon}^{(i}\eta^{j)})\,, && (5.4.15)\\
[\delta_S(\eta_1), \delta_S(\eta_2)] &= \delta_K(-\tfrac{1}{2}\bar{\eta}_2\gamma^a\eta_1)\,. && (5.4.16)
\end{aligned}$$

Some comments can be made about the terms in the algebra:

- The $\hat{\mathcal{D}}_\mu$ in the first term of (5.4.14) stands for a covariant general coordinate transformation [35]. The general form of a general coordinate transformation of a tensor field $B_{\mu\nu}$ is:

  $$\delta_{gct}(\xi^\rho)B_{\mu\nu} = \xi^\rho\partial_\rho B_{\mu\nu} + 2\partial_{[\mu}\xi^\rho \cdot B_{\nu]\rho}\,. \qquad (5.4.17)$$

  In (5.4.14), $\xi^\nu = \frac{1}{2}\bar{\varepsilon}_2\gamma^\nu\varepsilon_1$. In a covariant general coordinate transformation, this partial derivative is generalized by field-dependent gauge transformations. For symmetry I, the covariant general coordinate transformation becomes:

  $$\begin{aligned}
  \delta_{cgct}(\xi^\rho)B_{\mu\nu} &= \xi^\rho\partial_\rho B_{\mu\nu} + 2\partial_{[\mu}\xi^\rho \cdot B_{\nu]\rho} - \delta_I(\xi^\sigma B_{\sigma\rho})B_{\mu\nu}\,, && (5.4.18)\\
  &= \xi^\rho\left(\partial_\rho B_{\mu\nu} + \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu}\right)\,, && (5.4.19)\\
  &= \xi^\rho H_{\rho\mu\nu}\,. && (5.4.20)
  \end{aligned}$$

  In the model with rigid supersymmetry, we saw that there were terms with gauge transformation I and gauge transformation III in the algebra, when we considered the second gauge symmetry as a symmetry with gauge field u_μ. In this covariant general coordinate transformation, the term with gauge symmetry I is absorbed into the field strength.

- When introducing the covariant derivative with respect to the second gauge symmetry, also the terms that are a gauge transformation III, will be absorbed in the same way as for symmetry I. So, the covariant derivative $\hat{\mathcal{D}}_\mu$ in (5.2.8) includes all the superconformal transformations, and *also gauge symmetry* II. There is no natural gauge field for symmetry III and it does not appear in the covariant general coordinate transformation.

- The algebra gets a new term proportional to a symmetry I transformation. This is not dependent on the amount of supersymmetry and was found already in [41].

So, the three gauge symmetries each play their specific role in the algebra.

With u_μ as the gauge field for symmetry II, the same reasoning can be made as in the rigid model: a is a supersymmetric singlet, also for local supersymmetry.

In [41] is given a prescription to go from extended to simple chiral superconformal symmetry. Using this prescription immediately gives rise to the Lorentz rotation and the gauge symmetry I in the algebra. The breaking to $(1,0)$ supersymmetry requires a shift in ϕ_μ^i and $f_\mu{}^a$, because otherwise the $(2,0)$ supersymmetry transformation rules and conventional constraints do not lead to those of $(1,0)$ used in [41]. This shift implies that some work has to be done to achieve the special conformal transformation and special supersymmetry transformation in the algebra.

Finally, remark that the algebra is not realized off-shell. In the transformation of $H^-_{\mu\nu\rho}$, (5.4.28), will still appear the equation of motion of the spinor. Therefore, also in the commutator of two supersymmetries on ψ^i appears an equation of motion. Also the commutator of symmetry II with supersymmetry contains terms with equations of motion.

5.4.2 The action and the equations of motion

In [155] was already proven that the rigid model has superconformal invariance. Here we give the action for the self-dual tensor with local $(2,0)$ superconformal symmetry. We also see that the action, as expected, gives rise to the equations of motion of [42].

The following action is invariant under local superconformal transformations and under the 3 bosonic symmetries in (4.3.2).

$$\begin{aligned} S \;=\; & \int d^6x\sqrt{g}\,\big[-H^-_{\mu\nu}H^{*\mu\nu} - \tfrac{1}{6}H^*_{\mu\nu\rho}C^{\mu\nu\rho} \\ & -4\bar\psi^i\,\not{\mathcal{D}}'\psi_i - \bar\psi^i_\mu\gamma^\mu\,\not{\mathcal{D}}'\psi^j\phi_{ij} \\ & +\tfrac{1}{4}\phi^{ij}\left(\partial^a\mathcal{D}_a\phi_{ij} - 3b^a\mathcal{D}_a\phi_{ij} + V^a_{ki}\mathcal{D}_a\phi_j{}^k + \omega_a{}^{ab}\mathcal{D}_b\phi_{ij} - 4f_a{}^a\phi_{ij}\right. \\ & \left.+4\bar\psi_{ai}\mathcal{D}'^a\psi_j + 4\bar\phi_{ai}\gamma^a\psi_j\right) \\ & -\tfrac{8}{15}\bar\psi^i\chi_i^{kl}\phi_{kl} + \tfrac{1}{60}D^{ij,kl}\phi_{ij}\phi_{kl} + \tfrac{1}{3}\bar\psi_i T^{ij}\cdot\gamma\psi_j \\ & +\tfrac{1}{15}\bar\chi^i_{kl}\phi^{kl}\phi_{ij}\gamma^\mu\psi^j_\mu + \tfrac{1}{24}\bar\psi^i_\mu\gamma^\mu\phi_{ij}T^{jk}\cdot\gamma\psi_k - \tfrac{1}{48}\bar\psi^i_\mu\gamma^\mu\phi_{jk}T^{jk}\cdot\gamma\psi_i \\ & -\tfrac{1}{288}\bar\psi^i_\mu\gamma^\mu T^{jk}\cdot\gamma\gamma^\nu\psi^l_\nu\phi_{ij}\phi_{kl} + \tfrac{1}{192}\bar\psi^i_\mu\gamma^\mu T^{jk}\cdot\gamma\gamma^\nu\psi^l_\nu\phi_{jk}\phi_{il} \\ & -\tfrac{1}{720}\bar\psi^i_\mu\gamma^\mu T_{ij}\cdot\gamma\gamma^\nu\psi^j_\nu\phi_{kl}\phi^{kl}\big]\;. \end{aligned} \tag{5.4.21}$$

In (5.4.21) appears the covariant derivative of ψ^i without the $h^+_{\mu\nu\rho}$-term:

$$\begin{aligned} \mathcal{D}'_\mu\psi^i \;\equiv\; & \left(\partial_\mu - \tfrac{5}{2}b_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\psi^i - \tfrac{1}{2}V^i_{\mu j}\psi^j \\ & -\tfrac{1}{4}\left(\not{\mathcal{D}}\phi^{ij}\right)\psi_{\mu j} + \phi^{ij}\phi_{\mu j}\,. \end{aligned} \tag{5.4.22}$$

Further, $T_{ij} \cdot \gamma = T_{ij}^{abc}\gamma_{abc}$.

This action describes the coupling of a self-dual tensor multiplet in six dimensions to a conformal supergravity background. The first gauge symmetry imposes that $B_{\mu\nu}$ only appears in a field strength in the action. The second and the third gauge symmetries impose the form of the action as given in (5.4.21) for terms that transform with respect to one of these symmetries. The second term contains only covariantization terms of (5.4.4). This term is imposed by the third gauge symmetry and can also be found in the local super-Poincaré actions for self-dual tensors [133, 134]. It is absent for free chiral bosons. All other coefficients of the action are fixed by imposing invariance under supersymmetry and special supersymmetry.

The action (5.4.21) gives rise to the following field equations for $B_{\mu\nu}$, a, ψ_i, and ϕ^{ij}:

$$\mathcal{G}^{\mu\nu} = \partial_\rho \left(4\, e\, h^{+\mu\nu\rho} - \tfrac{1}{6}\varepsilon^{\mu\nu\rho\sigma\tau\phi} C_{\sigma\tau\phi}\right) , \tag{5.4.23}$$

$$\mathcal{A} = \partial_\phi \left(\frac{1}{\sqrt{u^2}}\, \epsilon^{\mu\nu\rho\sigma\tau\phi} H^-_{\mu\nu} H^-_{\rho\sigma} v_\tau \right) , \tag{5.4.24}$$

$$\Gamma^i = \not{\mathcal{D}}\psi^i - \frac{1}{15}\phi^{kl}\chi^i_{kl} - \frac{1}{12} T^{ij}_{abc}\gamma^{abc}\psi_j \, , \tag{5.4.25}$$

$$C_{ij} = \mathcal{D}^a \mathcal{D}_a \phi_{ij} - \tfrac{1}{15} D^{kl}_{ij}\phi_{kl} + \tfrac{1}{3} h^+_{abc} T^{abc}_{ij} + \tfrac{16}{15}\bar{\chi}^k_{ij}\psi_k \, . \tag{5.4.26}$$

The equations (5.4.25) and (5.4.26) are the equations of motion as derived in [42]. Also the self-duality condition (5.4.5) is identical as in [42]. The only difference is the one but last term in (5.4.26), but that is a term proportional to the self-duality equation which is put to zero in [42]. Rewriting (5.4.23) gives:

$$\mathcal{G}^{\mu\nu} = -\varepsilon^{\mu\nu\rho\sigma\tau\phi}\partial_\rho \left(v_\sigma H^-_{\tau\phi} \right) . \tag{5.4.27}$$

Analogous to the rigid case, the self-duality condition (5.4.5) can be found by a gauge choice of symmetry III for the most general solution of this equation of motion, if there are no global obstructions. Global aspects of chiral bosons are studied in [169, 170]. So, this action describes a self-dual two-form.

The description using this action is more satisfactory than the one of [136]. There, first the action for an ordinary tensor is written down and the equation of motion is derived. Only then, the self-duality condition is imposed. Here the self-duality is automatically incorporated in the action and follows from the equation of motion.

The equations of motion and the self-duality condition transform in the following way into each other under (special) supersymmetry:

$$\delta H^-_{abc} = -\tfrac{1}{2}\bar{\epsilon}\gamma_{abc}\Gamma \, ,$$

$$
\begin{aligned}
\delta\mathcal{A} &= -\partial_\phi \left(\frac{1}{\sqrt{u^2}} \epsilon^{\mu\nu\rho\sigma\tau\phi} H^-_{\mu\nu} \bar\epsilon \gamma_{\rho\sigma\upsilon} \Gamma v^\upsilon v_\tau \right) , \\
\delta\mathcal{G}_{ab} &= -\bar\epsilon \gamma_{abc} \mathcal{D}^c \Gamma , \\
\delta\Gamma^i &= \tfrac{1}{4} C^{ij} \epsilon_j + \tfrac{1}{8} \gamma^{ab} \epsilon^i \mathcal{G}_{ab} - \tfrac{1}{8} \gamma^\mu \gamma_a \Gamma^i \bar\epsilon \gamma^a \psi_\mu , \\
\delta C^{ij} &= -4 \bar\epsilon^{[i} \not{\mathcal{D}} \Gamma^{j]} + \tfrac{1}{8} \bar\psi^{[i}_\mu \gamma_a \Gamma^{j]} \bar\epsilon \gamma^a \psi^\mu - 8 \bar\eta^{[i} \Gamma^{j]} - (\text{trace}) . \qquad (5.4.28)
\end{aligned}
$$

5.4.3 Comments and discussion

The action (5.4.21) leads to three other actions for self-dual tensor multiplets coupled to a (conformal) supergravity background. There are two ways to end up with actions with less symmetry. A first one is lowering the number of supersymmetries. Another possibility is to break the superconformal symmetry to Poincaré supersymmetry. Using the prescription of [42], the Poincaré action for n self-dual tensor multiplets can be derived from this superconformal formulation. One starts with $n+5$ tensor multiplets in the vector representation of $SO(n,5)$. Imposing the suitable geometrical constraints that gauge fix the appropriate superconformal symmetries (dilatations, special conformal symmetry, and special supersymmetry) yields the coupling of n self-dual tensor multiplets to $(2,0)$ Poincaré supergravity. The $5n$ scalars of these n tensor multiplets parameterize the coset $\frac{SO(n,5)}{SO(n)\times SO(5)}$ [130, 42]. The signature of $SO(n,5)$ leads to the kinetic terms for n physical and five compensating multiplets. The compensating scalar field in section 3.1.3 also had the 'wrong' sign for the kinetic energy. Filling in the constraints and the solutions for the matter fields gives rise to a Lorentz-covariant action with extended chiral Poincaré supersymmetry for n self-dual tensor multiplets.

Using the procedure given in [42] to go from $(2,0)$ to $(1,0)$, one discovers the action for a self-dual tensor multiplet in a $(1,0)$ superconformal gravity background. This procedure essentially breaks the R-symmetry group $USp(4)$ to $SU(2)$, puts equal to zero half of the (special) supersymmetry parameter, and gives a prescription for the components of the matter fields of the Weyl multiplet, such that they become the matter fields of the $(1,0)$ Weyl multiplet. This approach also requires a subtle shift in the gauge fields ${f_\mu}^a$ and ϕ^i_μ in order to be able to use the conventional constraints and transformation rules of [41].

For the case with $(1,0)$ superconformal symmetry, it is also possible to go to a Poincaré description. Starting from $n+1$ tensor multiplets in the vector representation of $SO(n,1)$ and imposing the appropriate constraints will break the superconformal symmetry. This should give the actions of [133, 134]. The scalars in the Poincaré theory are in $\frac{SO(n,1)}{SO(n)}$ [130]. Again, the signature of the scalars resembles this of section 3.1.3.

An interesting project would be to calculate explicitly the conformal anomaly for the self-dual tensor, e.g., by following the approach in [171] and using the ghost sector of the action for the self-dual tensor of chapter 4 or [6]. This calculation would lead to a check of the adS/CFT-conjecture for the adS_7/CFT_6 case. In [171], the calculation of the Weyl anomaly for the ordinary tensor required the calculation of the propagator $\langle B_{\mu\nu} B_{\rho\sigma} \rangle$. The complete ghost sector decoupled in this calculation. To repeat this calculation for the self-dual tensor is far from trivial. One needs to take into account the ghost-sector for the gauge symmetries, so one needs to use one of the gauge-fixed actions: (4.4.31) or (4.4.21). It is preferable to use the covariant gauge fixing, but the big disadvantage of this gauge fixing is that the corresponding gauge-fixed action (4.4.21) is much more difficult to handle. To this end, another approach could be to first calculate the propagator for the ordinary field strength $\langle H_{\mu\nu\rho} H_{\sigma\tau\phi} \rangle$. Then, one should project to the self-dual field strengths, just like in the calculation of gravitational anomalies of chiral two-tensors in six dimensions [114].

The conformal anomaly of the self-dual tensor multiplet is calculated in another way in [172]. The anomaly can be split into three parts: a first term that is proportional to the Euler density (i.e., a total derivative of a non-covariant expression), a second term, containing terms with independent Weyl invariants (i.e., Weyl tensor contractions with extra conformal derivative operators), and a last term that is a total derivative of a covariant expression. This follows the classification of the conformal anomalies, studied earlier in [173] and references therein. The weakness of this calculation is that it is not an explicit calculation of the conformal anomaly of the self-dual tensor. The authors impose that it is half of the conformal anomaly of a non-chiral tensor which was calculated in [171]. They calculate the anomaly for a stack of $4n^3$ free self-dual tensor multiplets. This factor of $4n^3$ is found [174] by comparing the absorption cross-sections of longitudinally polarized gravitons by n $M5$-branes in $d = 11$ supergravity and the $M5$ brane worldvolume calculation. The comparison between the result in [172] at small coupling and the result of the supergravity calculation at strong coupling reveals that there is a discrepancy for one term of the conformal anomaly, the one proportional to the Euler density. This signals a problem with the adS_7/CFT_6-version of the Maldacena conjecture.

This difference between the weak-coupling calculation here and the strong-coupling result in [116] can still have three origins.

- A first possibility is that the calculation of the conformal anomaly in [172] is not right. They suppose that the conformal anomaly of a self-dual tensor is half of the conformal anomaly of a non-chiral tensor. This might be wrong and can hopefully be clarified by doing the calculation, suggested in this concluding section.

- Another possibility would be that large n effects have to be taken into account. The calculation of [116] is done for a large number of $M5$-branes, while the calculation of [172] is done for only one brane.

- A third possibility is that there is no non-renormalization theorem for this model. In the case of $adS_5 \times S^5$, there is a non-renormalization theorem [175] which ensures that there are no corrections to the conformal anomaly when changing the coupling from weak to strong. In [172] is even argued that there cannot be such a non-renormalization theorem, that there must be a renormalization when changing the coupling.

Chapter 6

$N = 2$ vector multiplets in 4 dimensions

6.1 Introduction

The understanding of the most general coupling of vector multiplets in $N = 2$ supersymmetry or supergravity is important in very different contexts. Four-dimensional theories with two supersymmetries are interesting because their possible quantum corrections have nice properties. In [176] is proven that there are only one-loop perturbative corrections in rigid $N = 2$ theories. Theories with $N = 1$ get quantum corrections at all loops and are therefore more difficult to control. Theories with $N = 4$ on the other hand, have no quantum corrections. Since 1994, also the non-perturbative corrections of the theories with rigid $N = 2$ are known. In [177, 178], the non-perturbative corrections of the low energy effective actions of supersymmetric gauge theories and their coupling to hypermultiplets were derived.

Supergravity theories with $N = 2$ in four dimensions are an essential element in the compactification of type IIA or IIB string theory on Calabi–Yau manifolds [179, 180, 181]. This leads to a geometrical understanding of the matter content of these theories. This thesis will only consider a description of the vector multiplets. Hypermultiplet couplings were studied in [39, 182]. The role of the vector-tensor multiplet (arising in compactifications of the heterotic string on $K3 \times T^2$) is elaborated in [31, 32, 10].

The general vector multiplet coupling has been studied for the supergravity case in [183, 184] and has been given the name special geometry [185]. The similar coupling in rigid supersymmetry was obtained in [186] and is referred to as 'rigid special geometry'.

Historically, the coupling of several $N = 2$ matter multiplets to $N = 2$ supergravity in four dimensions was found using superconformal tensor calculus [187, 188, 183, 184, 39] as explained in section 3.1. First, representations of a larger algebra, $SU(2,2|2)$, concretely given in equations (2.2.37) and (2.2.34), are constructed. Then, superconformal actions are build and two compensating multiplets (one vector multiplet to give rise to the graviphoton in the Poincaré gravity multiplet and a hyper-, a linear, or a non-linear multiplet) are introduced. Finally, three conditions are imposed to gauge fix the residual symmetries and to end up with an $N = 2$ Poincaré supergravity theory coupled to $N = 2$ matter multiplets. Here we will confine ourselves to the coupling of vector multiplets to $N = 2$ supergravity in four dimensions, where, after breaking the superconformal symmetry, the complex scalars of the vector multiplets form a special Kähler manifold.

6.1.1 Symplectic transformations

Electric-magnetic duality transformations in four dimensions manifest themselves by symplectic transformations [189], as explained in chapter 1. Symplectic transformations in a special Kähler manifold have been studied in [184, 190]. A symplectic matrix

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2m, \mathbb{R}) \tag{6.1.1}$$

is defined to obey

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \qquad \text{with} \qquad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \tag{6.1.2}$$

This implies that the components satisfy

$$A^T C - C^T A = 0 \,, \quad B^T D - D^T B = 0 \,, \quad A^T D - C^T B = \mathbb{1} \,. \tag{6.1.3}$$

The kinetic action for a set of $(n+1)$ Abelian vector multiplets is

$$\mathcal{L}_1 = \tfrac{1}{4}(\mathrm{Im}\,\mathcal{N}_{IJ})\mathcal{F}^I_{\mu\nu}\mathcal{F}^{\mu\nu J} - \tfrac{\mathrm{i}}{8}(\mathrm{Re}\,\mathcal{N}_{IJ})\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\rho\sigma} \,, \tag{6.1.4}$$

where $I, J = 1, ..., n+1$, the symmetric matrix $\mathcal{N}_{IJ}$ may depend on the scalars, and $\mathcal{F}^I_{\mu\nu}$ are the field strengths of the vector fields. We choose $n+1$ vectors since we need *one* compensating vector multiplet to break the superconformal symmetry in the superconformal tensor calculus program and we want to study the coupling of n vector multiplets to supergravity. The field equations for these vectors are

$$\partial_\mu \mathrm{Im}\, \mathcal{G}_I^{+\mu\nu} = 0 \,\text{where}\, \mathcal{G}_I^{+\mu\nu} = 2\mathrm{i}\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\nu}^{+I}} \,\text{and}\, \mathcal{G}_I^{-\mu\nu} = -2\mathrm{i}\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\nu}^{-I}} \,. \tag{6.1.5}$$

From (6.1.4) one finds that

$$\mathcal{G}_I^{+\mu\nu} = \mathcal{N}_{IJ}\mathcal{F}^{+J\,\mu\nu} \ ; \qquad \mathcal{G}_I^{-\mu\nu} = \bar{\mathcal{N}}_{IJ}\mathcal{F}^{-J\,\mu\nu} \,, \tag{6.1.6}$$

where $\mathcal{F}^{\pm}_{\mu\nu} = \frac{1}{2}\left(\mathcal{F}_{\mu\nu} \pm \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma}\right)$ is used. This gives only equations of motion for the field strengths and not for the potentials. The existence of the potentials follows from the Bianchi identities:

$$\partial_\mu \mathrm{Im}\, \mathcal{F}^{+I\mu\nu} = 0 \,. \tag{6.1.7}$$

The set of $2(n+1)$ equations in (6.1.5) and (6.1.7) is invariant under symplectic transformations

$$\begin{pmatrix} \tilde{\mathcal{F}}^+ \\ \tilde{\mathcal{G}}^+ \end{pmatrix} = \mathcal{S} \begin{pmatrix} \mathcal{F}^+ \\ \mathcal{G}^+ \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^+ \\ \mathcal{G}^+ \end{pmatrix} , \tag{6.1.8}$$

if at the same time is imposed that the form of the relation (6.1.6) between $\mathcal{F}$ and $\mathcal{G}$ is preserved. To this end, $\mathcal{N}$ has to transform under symplectic transformations into

$$\tilde{\mathcal{N}} = (C + D\mathcal{N})\,(A + B\mathcal{N})^{-1} \ . \tag{6.1.9}$$

If the role of electric and magnetic excitations is interchanged, the role of $\mathcal{F}$ and $\mathcal{G}$ is also interchanged. The equations for $\mathcal{G}$ then become Bianchi identities and therefore, the tensors $\mathcal{G}_{\mu\nu}$ are called magnetic field strengths.

The action (6.1.4) is not invariant under general symplectic transformations. It is invariant under the classical subgroup of $Sp(2(n+1), \mathbb{R})$:

$$S_{\rm cl} = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} . \tag{6.1.10}$$

These transformations map respectively the electric and magnetic field strengths into each other. Another subgroup is formed by the perturbative symplectic matrices:

$$S_{\rm pert} = \begin{pmatrix} A & 0 \\ C & (A^T)^{-1} \end{pmatrix} . \tag{6.1.11}$$

These transformations mix the electric field strengths among one another and give rise to new magnetic field strengths which are linear combinations of magnetic and electric field strengths. It remains clear which are the elementary excitations. These transformations leave the action invariant up to a total derivative. This term does not influence the equations of motion. However, it becomes relevant quantum mechanically. This total derivative term corresponds to a shift in the second term of (6.1.4), whose coefficient is called the theta-angle. This yields a symmetry of the path integral as long as the added term is an integer multiple of $2\pi\hbar$. These transformations will also play a role in section 6.5 where we will clarify some aspects of the coupling of *non-Abelian* vector multiplets to $N = 2$ supergravity.

Generic elements of the symplectic group (with $B \neq 0$) are sometimes called non-perturbative. They mix the electric (elementary) field strengths with magnetic (solitonic) ones. These transformations are not a symmetry of the complete action, but only of the equations of motion. In particular, the prepotential is not an invariant of the symplectic transformations. On the other hand, for a coordinate-free formulation of special geometry [191], the symplectic symmetry is an essential ingredient. The symplectic setup also clarified the link to Calabi–Yau manifolds [179].

Before the role of the symplectic transformations was clarified, a prepotential, a holomorphic function of second order, was an essential ingredient to construct the theory. In [191, 192, 193], other approaches were used to describe the coupling of vector multiplets to supergravity. In [194], vector multiplet couplings to supergravity were constructed for which no prepotential existed, by performing a symplectic transformation of an action based on a prepotential. These models are relevant in the partial breaking of $N = 2$ to $N = 1$ supersymmetry [195]. The resulting action was thus not based on a prepotential. One of the purposes of this chapter is to obtain a symplectic covariant formulation of the coupling of vector multiplets to $N = 2$ supergravity which at the same time uses superconformal tensor calculus. In particular, it should thus contain the coupling of [194]. To obtain an action in superconformal tensor calculus, one needs a prepotential, and hence one is obliged to give up the symplectic covariance. The combination of superconformal and symplectic covariance will, however, be possible if we only construct equations of motions without an action.

6.1.2 Definition(s) of a special Kähler manifold

The various possible actions and geometric formulations were compared in [8], and one has arrived at a new definition of special geometry. Remarkably, it was also noticed that one part of the definition, expressed by differential constraints, can be formulated in two different ways. These two forms are equivalent when more than one vector multiplet is coupled to supergravity, but *inequivalent if only one vector multiplet is coupled*. The presentation in [8] contained the constraints such that for one vector multiplet the coupling known previously (e.g., from superconformal tensor calculus) is obtained. But it was noted that another form of the constraints is possible which is also symplectic covariant. Obvious physical arguments could not exclude the existence of hitherto unknown couplings of one vector multiplet to supergravity that obey the weaker constraint, and not the stronger one.

To be more explicit, we repeat here one of the formulations of the three equivalent definitions of a special Kähler manifold of [8][1]. We choose this definition in terms of symplectic products, since it makes the symplectic structure more explicit than the other definitions.

Take a complex manifold $\mathcal{M}$. Suppose we have in every chart a $2(n+1)$-component vector $V(z^\alpha,\bar z^{\bar\alpha})$ such that on overlap regions there are transition functions of the form

$$e^{\frac{1}{2}(f(z^\alpha)-\bar f(\bar z^{\bar\alpha}))}\,S\,, \tag{6.1.12}$$

with f a holomorphic function and S a constant $Sp(2(n+1),\mathbb{R})$ matrix. (These transition functions have to satisfy the cocycle condition.) Take a $U(1)$ connection of the form $\kappa_\alpha\,dz^\alpha+\kappa_{\bar\alpha}\,d\bar z^{\bar\alpha}$ with

$$\kappa_{\bar\alpha}=-\overline{\kappa_\alpha}\,, \tag{6.1.13}$$

under which $\bar V$ has opposite weight as V. Denote the covariant derivative by $\mathcal{D}$:

$$\begin{array}{ll} U_\alpha\equiv\mathcal{D}_\alpha V\equiv\partial_\alpha V+\kappa_\alpha V\,, & \mathcal{D}_{\bar\alpha}V\equiv\partial_{\bar\alpha}V+\kappa_{\bar\alpha}V\,,\\ \bar U_{\bar\alpha}\equiv\mathcal{D}_{\bar\alpha}\bar V\equiv\partial_{\bar\alpha}\bar V-\kappa_{\bar\alpha}\bar V\,, & \mathcal{D}_\alpha\bar V\equiv\partial_\alpha\bar V-\kappa_\alpha\bar V\,. \end{array} \tag{6.1.14}$$

We impose the following conditions:

$$1.\quad \langle V,\bar V\rangle=\mathrm{i}\,, \tag{6.1.15}$$

$$2.\quad \mathcal{D}_{\bar\alpha}V=0\,, \tag{6.1.16}$$

$$3.\quad \mathcal{D}_{[\alpha}U_{\beta]}=0\,, \tag{6.1.17}$$

$$4.\quad \langle V,U_\alpha\rangle=0\,, \tag{6.1.18}$$

where $\langle\cdot,\cdot\rangle$ denotes the symplectic inner product, e.g., $\langle V,\bar V\rangle=V^T\Omega\bar V$, with an antisymmetric matrix Ω, which has as standard form

$$\Omega_{st}=\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}. \tag{6.1.19}$$

[1]The first one is not explicitly symplectic covariant, but we could as well have discussed here definition 2, where the constraint relevant for the discussion below was formulated as $\langle v,\partial_\alpha v\rangle=0$. The alternative form is then $\langle\partial_\alpha v,\partial_\beta v\rangle=0$.

Define

$$g_{\alpha\bar{\beta}} \equiv \mathrm{i}\langle U_\alpha, \bar{U}_{\bar{\beta}}\rangle \,, \tag{6.1.20}$$

where $\bar{U}_{\bar{\beta}}$ denotes the complex conjugate of U_α. If this is a positive-definite metric, $\mathcal{M}$ is called a special Kähler manifold.

It can then be shown that locally a function K' exists such that

$$\kappa_\alpha = \tfrac{1}{2}\partial_\alpha K' \,, \qquad \kappa_{\bar{\alpha}} = -\overline{\kappa_\alpha} = -\tfrac{1}{2}\partial_{\bar{\alpha}}\bar{K}' \,. \tag{6.1.21}$$

The real part of K' is the Kähler potential K. If there is an imaginary part Im K', then

$$V' = e^{\mathrm{iIm}\ K'/2}V \,, \tag{6.1.22}$$

satisfies the same constraints with K' replaced by the real K.

As discussed at the end of section 4.2.2 in [8], the constraints have a clear physical interpretation, related to the positivity of kinetic terms in the action. However, as suggested there already, the fourth constraint (6.1.18) could be replaced by

$$4'. \ \ \langle U_\alpha, U_\beta\rangle = 0 \,, \tag{6.1.23}$$

without violating the physical arguments. The constraint 4 implies $4'$, by taking a covariant derivative and antisymmetrizing, and with $4'$ it was shown that 4 follows when $n>1$. However, for $n=1$, equation $4'$ is empty. Taking $4'$ as constraint thus allows $\langle V, U_z\rangle \neq 0$. Such $N=2$ models would be new, and this possibility is worked out in section 6.4.3. It will be called 'the special case'. The case with $n>1$ or $\langle V, U_\alpha\rangle = 0$ will be called 'the generic case'.

In appendix C of [8], two $n=1$ examples are given where the condition (6.1.18) is not fulfilled. In these examples it was shown that the relaxation of that constraint leads to models not allowed by other definitions of special geometry. Here we will first give further evidence of the non-triviality of (6.1.18). The main result will be that indeed models which violate (6.1.18), still allow an $N=2$ supersymmetric formulation. Section 4.3 of [8] makes contact between the geometry of the moduli spaces of Calabi–Yau manifolds and the scalars of vector multiplets. The moduli spaces of Calabi–Yau threefolds are natural candidates for special Kähler manifolds. Using the definitions (6.1.15)–(6.1.18), the connection can be made between symplectic vectors $\{V, U_\alpha, \bar{V}, \bar{U}_{\bar{\alpha}}\}$ and integrals of (p,q) forms over a canonical basis of 3-cycles. The question arises whether *all* models with vector multiplets can be found as some string compactification or whether there exist consistent models that cannot come from a compactification of string theory. The answer to this question is in section 6.4.3. There will be given a consistent four-dimensional model that can *not* be found from a Calabi–Yau compactification of type II string theory.

6.1.3 The plan of the construction

The scalars of the special Kähler manifolds are the lowest components of chiral multiplets of $N = 2$ supersymmetry. A chiral multiplet is a reducible representation of $N = 2$ supersymmetry. After imposing suitable *reducibility constraints*, it gives a vector multiplet. In rigid supersymmetry these constraints can be written in superspace for a symplectic section of superfields or, in components, as a linear multiplet of constraints of symplectic sections [8, 196]. With a standard symplectic metric (6.1.19), the rigid *special Kähler constraints* can be used to write the lower part of the symplectic sections V in terms of the upper one. The reducibility constraints for the lower parts of the sections then give rise to the field equations of the fields in the upper parts. We want to use this approach to construct the field equations of vector multiplets, coupled to $N = 2$ supergravity.

Chiral and vector multiplets can also be defined as representations of the local superconformal algebra $SU(2, 2|2)$ of equations (2.2.37) and (2.2.34). Then the fields of the gauge multiplet of the superconformal gauge invariances, which is the Weyl multiplet, enter in the transformation rules of the multiplets [187, 188]. To describe the coupling of n on-shell vector multiplets to supergravity, we will start from $2n + 2$ chiral multiplets. The linear multiplet of constraints, which reduce these chiral multiplets to vector multiplets in supergravity, will contain additional terms with fields of the Weyl multiplet [187].

The equations that follow by supersymmetry from this weak definition of special Kähler geometry are derived for the complete set of $2n+2$ chiral multiplets. The constraints defining special Kähler geometry involve a breaking of dilatations and the $U(1)$ transformations in the superconformal group. We also choose a symplectic fermionic constraint as the gauge choice for S-supersymmetry. Special conformal symmetry is broken by a choice for the dilatation gauge field as in previous approaches. So, finally this leads to the breaking of superconformal to super-Poincaré spacetime symmetry with a residual internal $SU(2)$ in a consistent way, without relying on a prepotential or an action. Combining the reducibility constraints with the constraints of special Kähler geometry we find n on-shell vector multiplets, coupled to 24+24 supergravity components, remnants of the Weyl multiplet.

These 24+24 components reside in a 'current multiplet', which we identify as a reduced chiral self-dual superfield. The full supergravity equations, however, would rely on a second compensating multiplet, which is independent of the symplectic formulation. For these aspects we refer to the three known constructions of auxiliary field formulations [197, 187].

In section 6.2, the building blocks of the construction, the Weyl multiplet and the chiral multiplet, are given. Their supersymmetry transformation rules and the constraint to make a vector multiplet out of a chiral multiplet are recapitulated. In section 6.3 the special Kähler constraints and the supersymmetric relatives are treated for the most general case. In section 6.4 we combine the constraints imposed on the chiral multiplets and those found in section 6.3 to find on-shell vector multiplets. We comment on the remaining off-shell components of supergravity and their field equations. We give a concrete example of the special case and comment on the impossibility to find this model from a Calabi–Yau compactification. Section 6.5 gives the role of extra terms in the action for non-Abelian vector multiplets coupled to supergravity of [39] in terms of symplectic transformations. Most of the results in this chapter were reported in [9]. Some other new results are the discussion on the connection to Calabi-Yau compactifications of string theory and the possible extension to non-Abelian gaugings of Abelian models and the role of symplectic transformations.

6.2 The building blocks of the construction

In this section we review the Weyl multiplet, i.e., the gauge multiplet of the $N = 2$ superconformal symmetry, and the superconformal chiral multiplet, coupled to the Weyl multiplet. They are the two relevant representations of $SU(2,2|2)$ that are needed in this chapter, since there exist constraints which reduce the chiral to the vector multiplets. In the spirit of superconformal tensor calculus (cf. section 3.1), we start from representations of the superconformal algebra. Most of the material presented here is well known (see e.g., [188, 31])[2]. We use the notations of appendix A.2 and the algebra given in (2.2.37) and (2.2.34).

6.2.1 The Weyl multiplet

The Weyl multiplet is the gravitational multiplet of $N = 2$ superconformal gravity. We start to build this representation of $SU(2,2|2)$ from the gauge fields $e_\mu{}^a, \omega_\mu^{ab}, b_\mu, f_\mu{}^a, \mathcal{V}_\mu{}^i{}_j, A_\mu, \psi_\mu^i$ and ϕ_μ^i. They are, respectively, gauge fields of general coordinate transformations, Lorentz rotations, dilatations, special conformal boosts, chiral $SU(2)$ and $U(1)$, supersymmetry, and special supersymmetry. The $SU(2)$ gauge field is anti-Hermitian. It satisfies the following condition

$$(\mathcal{V}_{\mu i}{}^j) \equiv (\mathcal{V}_\mu{}^i{}_j)^* = -\mathcal{V}_\mu{}^j{}_i\,, \tag{6.2.1}$$

[2] However, here we use different normalizations, more suited for a manifestly symplectic formulation of the theory. We use the notations of [198]. So the old supersymmetry parameters are $\frac{1}{\sqrt{2}}$ the new ones and the old fermionic fields are $\sqrt{2}$ the new ones. Also keep in mind that $\varepsilon^{0123} = \mathrm{i}$.

where * means complex conjugation. The representation is completed by the Lorentz tensor T_{ab}^{ij}, antisymmetric in $[ij]$, the spinor χ^i, and the scalar D. Here, in contrast with the six-dimensional $(2,0)$ Weyl multiplet, the counting of the bosonic and fermionic degrees of freedom already imposes that the Weyl multiplet should contain auxiliary fields. Note that T_{ab}^{ij} is an antiself-dual tensor, and its complex conjugate T_{abij} is self-dual. The spin connection and the gauge fields for the special conformal transformations and special supersymmetry are composite gauge fields, derived from the three conventional constraints and given by

$$\begin{aligned}
\omega_\mu^{ab} &= -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\sigma e_\nu{}^c - 2e_\mu{}^{[a}e^{b]\nu}b_\nu \\
&\quad -\tfrac{1}{2}(2\bar\psi_\mu^i\gamma^{[a}\psi_i^{b]} + \bar\psi^{ai}\gamma_\mu\psi_i^b + \text{h.c.})\,, \\
\phi_\mu^i &= (\sigma^{\rho\sigma}\gamma_\mu - \frac{1}{3}\gamma_\mu\sigma^{\rho\sigma})\left(\mathcal{D}_\rho\psi_\sigma^i - \tfrac{1}{8}\sigma\cdot T^{ij}\gamma_\rho\psi_{\sigma j}\right) + \tfrac{1}{2}\gamma_\mu\chi^i\,, \\
f_\mu{}^a &= \tfrac{1}{2}\mathcal{R}_\mu^a - \tfrac{1}{2}e_\mu{}^a f_\nu{}^\nu - \tfrac{\mathrm{i}}{4}e^{a\nu}\varepsilon_{\mu\nu}{}^{\rho\sigma}\hat R_{\rho\sigma}(\mathrm{U}(1)) + \tfrac{1}{16}T_{ij}^{ab}T_{\mu b}^{ij} - \tfrac{3}{4}e_\mu{}^a D \\
&\quad +\left(\bar\psi^i_{[\mu}\sigma^{ab}\phi_{\nu]i} + \tfrac{1}{2}\bar\psi^i_{[\mu}T_{ij}^{ab}\psi^j_{\nu]} - \tfrac{3}{2}\bar\psi^i_{[\mu}\gamma_{\nu]}\sigma^{ab}\chi_i - \bar\psi^i_{[\mu}\gamma_{\nu]}\hat R^{ab}(\mathrm{Q})_i + \text{h.c.}\right)e_b{}^\nu\,.
\end{aligned} \tag{6.2.2}$$

The following expressions are used in $f_\mu{}^a$:

$$\begin{aligned}
f_\mu{}^\mu &= \tfrac{1}{6}\mathcal{R} - D \\
&\quad -\left(\tfrac{1}{6}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i} - \tfrac{1}{6}\bar\psi_\mu^i\psi_\nu^j T_{ij}^{\mu\nu} - \tfrac{1}{2}\bar\psi_\mu^i\gamma^\mu\chi_i + \text{h.c.}\right), \\
\hat R_{\rho\sigma}(\mathrm{U}(1)) &= 2\partial_{[\mu}A_{\nu]} - \mathrm{i}\left(2\bar\psi^i_{[\mu}\phi_{\nu]i} + \tfrac{3}{2}\bar\psi^i_{[\mu}\gamma_{\nu]}\chi_i + \text{h.c.}\right)\,, \\
\hat R^{ab}(\mathrm{Q})^i &= 2\mathcal{D}_{[\mu}\psi^i_{\nu]} - \gamma_{[\mu}\phi^i_{\nu]} - \tfrac{1}{4}\sigma\cdot T^{ij}\gamma_{[\mu}\psi_{\nu]j}\,.
\end{aligned} \tag{6.2.3}$$

Also, $\mathcal{D}_\mu$ is covariant with respect to the Lorentz transformations, dilatations, $U(1)$, and $SU(2)$, i.e.,

$$\mathcal{D}_\mu\psi_\nu^i = \left(\partial_\mu - \tfrac{1}{2}\omega_\mu^{ab}\sigma_{ab} + \tfrac{1}{2}b_\mu + \tfrac{\mathrm{i}}{2}A_\mu\right)\psi_\nu^i + \tfrac{1}{2}\mathcal{V}_\mu{}^i{}_j\psi_\nu^j\,. \tag{6.2.4}$$

Furthermore, $\mathcal{R} = e_a^\mu e_b^\nu\mathcal{R}_{\mu\nu}{}^{ab}$ is the Ricci scalar derived from the Riemann tensor

$$\mathcal{R}_{\mu\nu}{}^{ab} = 2\partial_{[\mu}\omega_{\nu]}^{ab} - 2\omega_{[\mu}^{ac}\omega_{\nu]c}{}^b \tag{6.2.5}$$

and

$$\mathcal{R}_\mu{}^a = e_b^\nu\mathcal{R}_{\mu\nu}{}^{ab}\,. \tag{6.2.6}$$

The transformation rules of the independent fields of the Weyl multiplet under supersymmetry, special supersymmetry and special conformal transformations (with

	Weyl multiplet										
field	${e_\mu}^a$	ψ_μ^i	b_μ	A_μ	${\mathcal{V}_\mu}^i{}_j$	T_{ab}^{ij}	χ^i	D	ω_μ^{ab}	${f_\mu}^a$	ϕ_μ^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$
c	0	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$
γ_5		$+$					$+$				$-$

Table 6.1: Weyl and chiral weights (w and c, respectively) and fermion chirality (γ_5) of the Weyl multiplet component fields and of the supersymmetry transformation parameters. The parameter for (special) supersymmetry has the same Weyl and chiral weights and fermion chirality as its corresponding gauge field.

parameters ε^i, η^i and Λ_K^a) are

$$\begin{aligned}
\delta {e_\mu}^a &= \bar\varepsilon^i\gamma^a\psi_{\mu i} + \text{h.c.}\,, \\
\delta\psi_\mu^i &= \mathcal{D}_\mu\varepsilon^i - \tfrac{1}{8}\sigma\cdot T^{ij}\gamma_\mu\varepsilon_j - \tfrac{1}{2}\gamma_\mu\eta^i\,, \\
\delta b_\mu &= \tfrac{1}{2}\bar\varepsilon^i\phi_{\mu i} - \tfrac{3}{4}\bar\varepsilon^i\gamma_\mu\chi_i - \tfrac{1}{2}\bar\eta^i\psi_{\mu i} + \text{h.c.} + \Lambda_K^a\, e_{\mu\, a}\,, \\
\delta A_\mu &= \tfrac{1}{2}\mathrm{i}\bar\varepsilon^i\phi_{\mu i} + \tfrac{3}{4}\mathrm{i}\bar\varepsilon^i\gamma_\mu\chi_i + \tfrac{1}{2}\mathrm{i}\bar\eta^i\psi_{\mu i} + \text{h.c.}\,, \\
\delta{\mathcal{V}_\mu}^i{}_j &= 2\bar\varepsilon_j\phi_\mu^i - 3\bar\varepsilon_j\gamma_\mu\chi^i + 2\bar\eta_j\psi_\mu^i - (\text{h.c.}\,;\,\text{traceless})\,, \\
\delta T_{ab}^{ij} &= 8\bar\varepsilon^{[i}\hat R_{ab}(Q)^{j]}\,, \\
\delta\chi^i &= -\tfrac{1}{12}\sigma\cdot \not{D} T^{ij}\varepsilon_j + \tfrac{1}{6}\hat R(\text{SU}(2))^i{}_j\cdot\sigma\varepsilon^j - \tfrac{1}{3}\mathrm{i}\hat R(\text{U}(1))\cdot\sigma\varepsilon^i \\
&\quad + \tfrac{1}{2}D\,\varepsilon^i + \tfrac{1}{12}\sigma\cdot T^{ij}\eta_j\,, \\
\delta D &= \bar\varepsilon^i \not{D}\chi_i + \text{h.c.}\,.
\end{aligned} \tag{6.2.7}$$

The other transformation rules can be derived from table 6.1. The modified $SU(2)$-curvature is defined as

$$\hat R(\text{SU}(2))_{\mu\nu}{}^i{}_j = R(\text{SU}(2))_{\mu\nu}{}^i{}_j + 6\left(\bar\psi^i_{[\mu}\gamma_{\nu]}\chi_j - (\text{h.c.};\text{traceless})\right)\,. \tag{6.2.8}$$

6.2.2 The chiral multiplet

A chiral multiplet is a reducible representation of the superconformal algebra [188]. By imposing a linear multiplet of constraints it becomes a vector multiplet. This is an irreducible representation of the superconformal algebra. The constraints are called the generalized Bianchi identities, because they contain a Bianchi identity for the tensor in the chiral multiplet.

Later we want to couple vector multiplets to conformal supergravity. The scalars of these vector multiplets form a symplectic section. They are the lowest components of a multiplet. Therefore, all the components of the multiplets have to form such a symplectic section. This is the reason to start from a $(2n+2)$-dimensional section of chiral multiplets:

$$\begin{aligned}\tilde{\Phi} &= V+\bar{\theta}^i\tilde{\Omega}_i+\tfrac{1}{4}\bar{\theta}^i\theta^j\tilde{Y}_{ij}+\tfrac{1}{4}\varepsilon_{ij}\bar{\theta}^i\sigma\cdot\tilde{\mathcal{F}}^-\theta^j\\ &\quad+\tfrac{1}{6}\varepsilon_{ij}(\bar{\theta}^i\sigma_{ab}\theta^j)\bar{\theta}^k\sigma^{ab}\tilde{\Lambda}_k+\tfrac{1}{48}(\varepsilon_{ij}\bar{\theta}^i\sigma_{ab}\theta^j)^2\tilde{C}\,. \end{aligned} \tag{6.2.9}$$

The components of the section are denoted by

$$\tilde{\Phi}=\begin{pmatrix}\phi^I\\ \phi_{F,I}\end{pmatrix}, \tag{6.2.10}$$

with $I=0,\ldots,n$, which gives for the components of the chiral superfield

$$\begin{aligned} V &= \begin{pmatrix}X^I\\ F_I\end{pmatrix}, \qquad \tilde{\Omega}_i=\begin{pmatrix}\Omega_i^I\\ \Omega_{F,Ii}\end{pmatrix}, \qquad \tilde{Y}_{ij}=\begin{pmatrix}Y_{ij}^I\\ Y_{F,Iij}\end{pmatrix},\\ \tilde{\mathcal{F}}^-_{ab} &= \begin{pmatrix}\mathcal{F}^{I-}_{ab}\\ \mathcal{G}^-_{F,Iab}\end{pmatrix}, \qquad \tilde{\Lambda}_i=\begin{pmatrix}\Lambda_i^I\\ \Lambda_{F,Ii}\end{pmatrix}, \qquad \tilde{C}=\begin{pmatrix}C^I\\ C_{F,I}\end{pmatrix}. \end{aligned}\tag{6.2.11}$$

This section of multiplets is independent of the existence of a prepotential. The multiplets starting with F_I are on an equal footing with the ones starting with X^I. As long as we do not impose the constraints to obtain a vector superfield or the special Kähler constraints, these are $2n+2$ independent chiral multiplets. The full superconformal transformation rules are given by

$$\begin{aligned}
\delta V &= \bar{\varepsilon}^i\tilde{\Omega}_i+(\Lambda_D-\mathrm{i}\Lambda_A)V\,,\\
\delta\tilde{\Omega}_i &= \not{D}V\varepsilon_i+\tfrac{1}{2}\tilde{Y}_{ij}\varepsilon^j+\tfrac{1}{2}\sigma\cdot\tilde{\mathcal{F}}^-\varepsilon_{ij}\varepsilon^j+V\eta_i+(\tfrac{3}{2}\Lambda_D-\tfrac{\mathrm{i}}{2}\Lambda_A)\tilde{\Omega}_i+\Lambda_{SU(2)}{}_i{}^j\tilde{\Omega}_j\,,\\
\delta\tilde{Y}_{ij} &= 2\bar{\varepsilon}_{(i}\not{D}\tilde{\Omega}_{j)}-2\bar{\varepsilon}^k\tilde{\Lambda}_{(i}\varepsilon_{j)k}+2\Lambda_D\tilde{Y}_{ij}+2\Lambda_{SU(2)\,(i}{}^k\tilde{Y}_{j)k}\,,\\
\delta\tilde{\mathcal{F}}^-_{ab} &= \varepsilon^{ij}\bar{\varepsilon}_i\not{D}\sigma_{ab}\tilde{\Omega}_j+\bar{\varepsilon}^i\sigma_{ab}\tilde{\Lambda}_i-2\varepsilon^{ij}\bar{\eta}_i\sigma_{ab}\tilde{\Omega}_j+2\Lambda_D\tilde{\mathcal{F}}^-_{ab}\,,\\
\delta\tilde{\Lambda}_i &= -\tfrac{1}{2}\sigma\cdot\tilde{\mathcal{F}}^-\overleftarrow{\not{D}}\varepsilon_i-\tfrac{1}{2}\not{D}\tilde{Y}_{ij}\varepsilon_k\varepsilon^{jk}+\tfrac{1}{2}\tilde{C}\varepsilon^j\varepsilon_{ij}\\
&\quad-\tfrac{1}{8}\not{D}(V\varepsilon^{jk}T_{jk}\cdot\sigma)\varepsilon_i-\tfrac{3}{2}(\bar{\chi}_{[i}\gamma_a\tilde{\Omega}_{j]})\gamma^a\varepsilon_k\varepsilon^{jk}\\
&\quad-\tilde{Y}_{ij}\varepsilon^{jk}\eta_k+\tfrac{5}{2}\Lambda_D\tilde{\Lambda}_i+\tfrac{\mathrm{i}}{2}\Lambda_A\tilde{\Lambda}_i+\Lambda_{SU(2)}{}_i{}^j\tilde{\Lambda}_j\,,\\
\delta\tilde{C} &= -2\varepsilon^{ij}\bar{\varepsilon}_i\not{D}\tilde{\Lambda}_j-6\bar{\varepsilon}_i\chi_j\tilde{Y}_{kl}\varepsilon^{ik}\varepsilon^{jl}+\tfrac{1}{2}\bar{\varepsilon}_i\sigma\cdot T_{jk}\not{D}\tilde{\Omega}_l\varepsilon^{ij}\varepsilon^{kl}+2\varepsilon^{ij}\bar{\eta}_i\tilde{\Lambda}_j\\
&\quad+3\Lambda_D\tilde{C}+\mathrm{i}\Lambda_A\tilde{C}\,.
\end{aligned}\tag{6.2.12}$$

This superconformal chiral superfield can be reduced to a vector superfield with the constraints

$$0 = \tilde{Y}_{ij}-\varepsilon_{ik}\varepsilon_{jl}\tilde{Y}^{kl}\,, \tag{6.2.13}$$

$$0 = \not{D}\tilde{\Omega}^i - \varepsilon^{ij}\tilde{\Lambda}_j\,, \tag{6.2.14}$$

$$0 = D^a(\tilde{\mathcal{F}}^+_{ab} - \tilde{\mathcal{F}}^-_{ab} + \frac{V}{4}T_{ab\,ij}\varepsilon^{ij} - \frac{\bar{V}}{4}T^{ij}_{ab}\varepsilon_{ij}) - \tfrac{3}{2}(\varepsilon^{ij}\bar{\chi}_i\gamma_b\tilde{\Omega}_j - \text{h.c.})\,, \tag{6.2.15}$$

$$0 = -2\Box\bar{V} - \tfrac{1}{4}\tilde{\mathcal{F}}^+_{\mu\nu}T^{\mu\nu}_{ij}\varepsilon^{ij} - 6\bar{\chi}_i\tilde{\Omega}^i - \tilde{C}\,. \tag{6.2.16}$$

The symplectic vector of chiral multiplets with these constraints defines $2n+2$ vector multiplets in superconformal gravity. The special Kähler constraints will relate them such that one ends up with $n+1$ vectors and n complex scalars and spinors obeying field equations.

6.3 Gauge choices and special Kähler constraints

To obtain a Poincaré supergravity theory of n vector multiplets, we start from the assumption that the components in the symplectic sections V are the lowest components of reduced chiral multiplets, as is the case in previous constructions of matter couplings in $N = 2$ supergravity. To achieve that, we have to impose the reducibility constraints (6.2.13)–(6.2.16) on the chiral multiplets and suitable constraints that impose restrictions on the sections such that the resulting theory contains n physical vector multiplets and the gravity multiplet. The superfluous symmetries of the superconformal construction need to be broken by suitable gauge choices. The symplectic section V can be seen as a function of n scalars z^α and their complex conjugates $\bar{z}^{\bar{\alpha}}$ ($\alpha = 1, ..., n$). These scalars can be interpreted as the coordinates of a special Kähler manifold.

Having introduced K' in (6.1.21), we have exhausted constraint (6.1.17). The remaining relevant constraints are then (6.1.15), (6.1.16), and we will take the formulation with (6.1.23). Condition (6.1.15) gauge fixes the dilatations, choosing the canonical kinetic term for the graviton. Equation (6.1.16) imposes the holomorphicity of the scalar fields. For the symmetry of the kinetic matrix of the vectors, one needs another constraint, which is (6.1.23). In all previous papers on special geometry, one imposed instead (6.1.18), which is equivalent for $n > 1$, but not for $n = 1$ as mentioned in the introduction. There is no physical argument known to demand (6.1.18), but up to now, no physical applications have been found not fulfilling (6.1.18).

We have thus seen that we can look upon equations (6.1.15) and (6.1.16) in two ways. They are the defining equations of special geometry, as well, they can be considered as gauge choices for the dilatations and chiral $U(1)$ transformations present in the superconformal algebra. As we will see below, a supersymmetric extension of these constraints will include the gauge choice of S-supersymmetry.

From (6.1.14) follows

$$g_{\alpha\bar\beta} \equiv \partial_\alpha \partial_{\bar\beta} K = \mathrm{i}\langle U_\alpha, \bar U_{\bar\beta}\rangle \,. \tag{6.3.1}$$

Furthermore, we impose the 'physical' condition (positivity of the kinetic energy terms of the vectors [8]) that[3]

$$\begin{array}{rlll} \det g_{\alpha\bar\beta} > 0 & & \text{if} & \langle V, U_\alpha\rangle = 0\,, \\ g'_{z\bar z} \equiv g_{z\bar z} - Z_z \bar Z_{\bar z} & > 0 & \text{for} & Z_z \equiv \langle V, U_z\rangle\,. \end{array} \tag{6.3.2}$$

Using the constraints it can be shown that

$$\mathcal{W} = (V, U_\alpha, \bar V, \bar U_{\bar\alpha}) \tag{6.3.3}$$

forms for every $z, \bar z$ a basis for symplectic vectors. More information about the expansion coefficients can be found in appendix A.2.2. This expansion will be used in the derivation of the supersymmetric extension of the special Kähler constraints and of the field equations.

6.3.1 The constraint on the curvature

Covariant derivatives involve the Kähler connection as in (6.1.14), and after choosing a real Kähler potential one may define a Kähler weight[4] p for a symplectic section W, such that

$$\mathcal{D}_\alpha W = \left(\partial_\alpha + \tfrac{p}{2}(\partial_\alpha K)\right) W\,, \qquad \mathcal{D}_{\bar\alpha} W = \left(\partial_{\bar\alpha} - \tfrac{p}{2}(\partial_{\bar\alpha} K)\right) W\,. \tag{6.3.4}$$

If W carries indices α or $\bar\alpha$ there is a further metric connection, defined such that $\mathcal{D}_\alpha g_{\beta\bar\gamma} = 0$. The curvature of the special Kähler manifold is then defined by

$$[\mathcal{D}_\alpha, \mathcal{D}_{\bar\beta}] X_\gamma = -p g_{\alpha\bar\beta} X_\gamma - R_{\alpha\bar\beta\gamma}{}^{\delta} X_\delta\,, \tag{6.3.5}$$

where X_α is a generic vector with Kähler weight p. Applying this for X_α replaced by U_α and taking the symplectic inner product with $\bar U_{\bar\delta}$ one finds

$$R_{\alpha\bar\beta\gamma\bar\delta} \equiv g_{\delta\bar\delta} R_{\alpha\bar\beta\gamma}{}^{\delta} = -2 g_{(\alpha|\bar\beta|} g_{\gamma)\bar\delta} - \mathrm{i}\langle \mathcal{D}_\alpha U_\gamma, \mathcal{D}_{\bar\beta} \bar U_{\bar\delta}\rangle\,. \tag{6.3.6}$$

[3]Keep in mind that for $n > 1$ one always has $Z_z = 0$.

[4]V and U_α have weight 1, while Z_z has weight 2, and for their complex conjugates respectively -1 and -2.

If we introduce a symmetric tensor

$$C_{\alpha\beta\gamma} = \langle U_\alpha, \mathcal{D}_\beta U_\gamma \rangle \,, \tag{6.3.7}$$

and expand the last term of (6.3.6) in the basis $\mathcal{W}$ according to appendix A.2.2 we obtain the following two cases:
1. The generic case :

$$\mathcal{D}_\alpha U_\beta = \mathrm{i} C_{\alpha\beta\gamma} g^{\gamma\bar{\gamma}} \bar{U}_{\bar{\gamma}} \ , \tag{6.3.8}$$

and the curvature is constrained to

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} \equiv g_{\beta\bar{\beta}} R^{\beta}_{\alpha\gamma\bar{\delta}} = -2 g_{(\alpha|\bar{\beta}|} g_{\gamma)\bar{\delta}} + C_{\alpha\gamma\varepsilon} g^{\varepsilon\bar{\varepsilon}} \bar{C}_{\bar{\beta}\bar{\delta}\bar{\varepsilon}}. \tag{6.3.9}$$

2. The special case :
In a similar way one finds that in this case

$$\mathcal{D}_z U_z = \mathrm{i} g'^{z\bar{z}} (C_{zzz} \bar{U}_{\bar{z}} - g_{z\bar{z}} \mathcal{D}_z Z_z \bar{V}') \,. \tag{6.3.10}$$

The curvature becomes

$$R_{z\bar{z}z\bar{z}} = -2 g^2_{z\bar{z}} - g_{z\bar{z}} (\mathcal{D}_z Z_z) g'^{z\bar{z}} (\mathcal{D}_{\bar{z}} \bar{Z}_{\bar{z}}) + C_{zzz} g'^{z\bar{z}} \bar{C}_{\bar{z}\bar{z}\bar{z}} \,. \tag{6.3.11}$$

6.3.2 An adapted basis and metric for the special case

When $Z_z \neq 0$, one may diagonalize the matrix of symplectic products between $V, \bar{V}, U_z$ and $\bar{U}_{\bar{z}}$ by defining

$$U'_z = U_z + \mathrm{i} Z_z \bar{V} \ ; \qquad \bar{U}'_{\bar{z}} = \bar{U}_{\bar{z}} - \mathrm{i} \bar{Z}_{\bar{z}} V \,. \tag{6.3.12}$$

We then have symplectic products

$$\begin{aligned} &\langle V, \bar{V} \rangle = \mathrm{i} \,, \qquad \langle V, U'_z \rangle = \langle V, \bar{U}'_{\bar{z}} \rangle = \langle \bar{V}, U'_z \rangle = \langle \bar{V}, \bar{U}'_{\bar{z}} \rangle = 0 \,, \\ &\langle \bar{U}'_{\bar{z}}, U'_z \rangle = \mathrm{i} (g_{z\bar{z}} - Z_z \bar{Z}_{\bar{z}}) = \mathrm{i} g'_{z\bar{z}} \,. \end{aligned} \tag{6.3.13}$$

In this way we find the Hermitian metric $g'_{z\bar{z}}$ which is invertible because of (6.3.2), but is not the second derivative of the Kähler potential K, used to define the covariant derivatives in (6.3.4). With this definition, covariant derivatives on the above equations lead to

$$\langle \mathcal{D}_z U'_z, \bar{V} \rangle = \langle \mathcal{D}_z U'_z, V \rangle = 0 \,. \tag{6.3.14}$$

The defining expressions for U_z and Z_z imply

$$\mathcal{D}_z \bar{U}_{\bar{z}} = g_{z\bar{z}} \bar{V} \,, \qquad \mathcal{D}_{\bar{z}} U_z = g_{z\bar{z}} V \,, \qquad \mathcal{D}_z \bar{Z}_{\bar{z}} = \mathcal{D}_{\bar{z}} Z_z = 0 \,, \tag{6.3.15}$$

which in the new basis give

$$\mathcal{D}_z \bar{U}'_{\bar{z}} = g'_{z\bar{z}} \bar{V} - \mathrm{i} \bar{Z}_{\bar{z}} U'_z \,, \qquad \mathcal{D}_{\bar{z}} U'_z = g'_{z\bar{z}} V + \mathrm{i} Z_z \bar{U}'_{\bar{z}} \,. \tag{6.3.16}$$

When we define $\mathcal{D}'$ with metric connection such that $\mathcal{D}'_z g'_{z\bar{z}} = 0$, all the above relations remain valid for $\mathcal{D}'$, as the non-zero connections are just Γ^z_{zz} and $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$. The new definition now implies

$$\langle \bar{U}'_{\bar{z}}, \mathcal{D}'_z U'_z \rangle = 0 \,. \tag{6.3.17}$$

The analogue of (6.3.7) is then the definition

$$C'_{zzz} \equiv \langle U'_z, \mathcal{D}'_z U'_z \rangle = C_{zzz} \,. \tag{6.3.18}$$

This leads again to

$$\mathcal{D}'_z U'_z = \mathrm{i} C_{zzz} g'^{z\bar{z}} \bar{U}'_{\bar{z}} \,. \tag{6.3.19}$$

We define then the curvature based on the metric g' by

$$[\mathcal{D}'_z, \mathcal{D}'_{\bar{z}}] X_z = -p g_{z\bar{z}} X_z - R'_{z\bar{z}z}{}^z X_z \,. \tag{6.3.20}$$

Observe that the first term has g and not g' as this is the Kähler curvature. Calculating as before the curvature R' by replacing X_z with U', and an inner product with $\bar{U}'$, the last terms in (6.3.16) lead to extra terms such that we find

$$R'_{z\bar{z}z\bar{z}} \equiv R'_{z\bar{z}z}{}^z g'_{z\bar{z}} = -2 g_{z\bar{z}} g'_{z\bar{z}} + C_{zzz} g'^{z\bar{z}} \bar{C}_{\bar{z}\bar{z}\bar{z}} \,. \tag{6.3.21}$$

Rephrasing as much as possible in terms of the metric $g'_{z\bar{z}}$, we thus recover another geometry than for other special Kähler models. There is an essential difference in the product of metrics in (6.3.6) and here. We tried to extend our analysis in the basis $\mathcal{W}' = (V, U'_z, \bar{V}, \bar{U}'_{\bar{z}})$, but ran into problems with the transformation rules because we want V to be the lowest component of a chiral multiplet, as demanded at the beginning of section 6.3. So, it is not possible to get rid of $Z_z \neq 0$ by choosing another basis while keeping a section of chiral multiplets. The model with $Z_z \neq 0$ is really another model compared to those studied in the past.

For some calculations below, it is also useful to introduce another basis with symplectic vectors orthogonal to U. That is, we introduce

$$\begin{aligned}
V' &= V + \mathrm{i} Z_z g^{z\bar{z}} \bar{U}_{\bar{z}} \,, \qquad \bar{V}' = \bar{V} - \mathrm{i} \bar{Z}_{\bar{z}} g^{z\bar{z}} U_z \,, \\
\langle V', \bar{V}' \rangle &= \mathrm{i}(1 - g^{z\bar{z}} Z_z \bar{Z}_{\bar{z}}) \,, \qquad \langle \bar{U}_{\bar{z}}, U_z \rangle = \mathrm{i} g_{z\bar{z}} \,, \\
\langle V', U_z \rangle &= \langle V', \bar{U}_{\bar{z}} \rangle = \langle \bar{V}', U_z \rangle = \langle \bar{V}', \bar{U}_{\bar{z}} \rangle = 0 \,.
\end{aligned} \tag{6.3.22}$$

6.3.3 Supersymmetric extension of special Kähler constraints

It is clear that the constraint (6.1.15) breaks the superconformal symmetry. The constraints and their supersymmetric partners therefore play the role of gauge conditions for some of the superconformal symmetries. The residual symmetry should then still contain the symmetries of Poincaré supergravity. In this subsection we will derive the supersymmetric partners of the constraints (6.1.15), (6.1.16), and (6.1.23), and compute the decomposition rule for the resulting supergravity, i.e., the rule which gives the remaining symmetry as a linear combination of original, superconformal, symmetries.

Gauge choices and decomposition rule

Before we go to these constraints, we *break the special conformal symmetry* by imposing a constraint on b_μ:

$$K\text{-gauge:} \quad b_\mu = 0\,. \tag{6.3.23}$$

This does not alter the number of degrees of freedom as b_μ is pure gauge in the Weyl multiplet (cf. table 6.2).

The decomposition rule for the special conformal symmetry is

$$\Lambda_K^a = -e^{\mu a}\left(\tfrac{1}{2}\bar\varepsilon^i\phi_{\mu i} - \tfrac{3}{4}\bar\varepsilon^i\gamma_\mu\chi_i - \tfrac{1}{2}\bar\eta^i\psi_{\mu i} + \text{h.c.}\right)\,. \tag{6.3.24}$$

Constraint (6.1.15) *breaks the dilatations.* Indeed, the superconformal transformation of (6.1.15) gives

$$\langle \bar V, \bar\varepsilon^i\tilde\Omega_i\rangle - \langle V, \bar\varepsilon_i\tilde\Omega^i\rangle + 2\mathrm{i}\Lambda_D = 0\,, \tag{6.3.25}$$

and the dilatations are now a combination of other symmetries. We choose as S-gauge

$$S\text{-gauge:} \quad \langle \bar V, \tilde\Omega_i\rangle = 0 \quad \text{and} \quad \langle V, \tilde\Omega^i\rangle = 0\,. \tag{6.3.26}$$

Remark that after this gauge choice the decomposition rule (6.3.25) simplifies to

$$\Lambda_D = 0\,, \tag{6.3.27}$$

such that we can forget about the original dilatations completely. Demanding that the sections V depend on z^α and $\bar z^{\bar\alpha}$ in the way described in (6.1.14)–(6.1.16), is

fields	d.o.f.	comments
The Weyl multiplet (24+24)		
$e_\mu{}^a$	5	16 - 4(translation.) - 6(Lorentz) - 1(dilatation)
b_μ	0	4 - 4(special conformal.)
A_μ	3	4 - 1($U(1)$)
$\mathcal{V}_\mu{}^i{}_j$	9	12 - 3($SU(2)$)
ψ_μ^i	16	32 - 8(Q-supersymmetry) - 8(S-supersymmetry)
T_{ab}^{ij}	6	complex antiself-dual
χ_i	8	
D	1	real scalar
symplectic section of chiral multiplets $(16(2n+2)+16(2n+2))$		
V	2(2n+2)	
$\tilde{\Omega}_i$	8(2n+2)	
$\tilde{Y}_{ij}$	6(2n+2)	
$\tilde{\mathcal{F}}_{ab}^-$	6(2n+2)	
$\tilde{\Lambda}_i$	8(2n+2)	
$\tilde{C}$	2(2n+2)	

Table 6.2: Degrees of freedom in the model before the constraints.

a *gauge choice for the chiral $U(1)$-transformations*. In fact, consider the transformation of the first line of (6.2.12) using these equations:

$$\delta V = U_\alpha \delta z^\alpha - \tfrac{1}{2}(\partial_\alpha K' \delta z^\alpha - \partial_{\bar\alpha} K' \delta \bar z^{\bar\alpha}) V\,. \tag{6.3.28}$$

An inner product with $\bar V$ gives (using (6.1.15) and its covariant derivative) a decomposition rule for the $U(1)$-transformations, i.e.,

$$\Lambda_A = \mathrm{Im}\,(\partial_\alpha K' \delta z^\alpha)\,, \tag{6.3.29}$$

where we have already used (6.1.21).

The decomposition rule for $\delta_S(\eta_i)$ follows from the variation of the S-gauge:

$$\begin{aligned}\eta_i \quad = \quad & -\mathrm{i}\langle \bar V, \not{D} V\rangle \varepsilon_i - \tfrac{\mathrm{i}}{2}\langle \bar V, \tilde Y_{ij}\rangle \varepsilon^j \\ & -\tfrac{\mathrm{i}}{2}\langle \bar V, \tilde{\mathcal{F}}^-_{ab}\rangle \varepsilon_{ij}\sigma^{ab}\varepsilon^j - \mathrm{i}\langle \bar\varepsilon_j \tilde\Omega^j, \tilde\Omega_i\rangle\,.\end{aligned} \tag{6.3.30}$$

From now on, we only request that the constraints are invariant under the resulting Poincaré supersymmetry

$$\delta(\varepsilon_i) = \delta_Q(\varepsilon_i) + \delta_S(\eta_i) + \delta_A(\Lambda_A) + \delta_K(\Lambda_K)\,, \tag{6.3.31}$$

with Λ_K, Λ_A, and η_i defined in (6.3.24), (6.3.29), and (6.3.30).

Having the symplectic sections as functions of z and $\bar{z}$, we can consider the transformations of the bosonic constraints (6.1.15)–(6.1.17) and (6.1.23). The variation of the first one determined the breaking of dilatations. The constraints (6.1.16) and (6.1.17) are used to determine the $z, \bar{z}$ dependences of V, U, and K and their supersymmetry transformations are thus trivial if we compute them in terms of δz and $\delta\bar{z}$. The constraint (6.1.23) is only non-trivial for $n > 1$. Its variation is

$$\delta\langle U_\alpha, U_\beta\rangle = 2\langle \mathcal{D}_\gamma U_{[\alpha}, U_{\beta]}\rangle \delta z^\gamma \,, \tag{6.3.32}$$

which is 0 due to the symmetry of (6.3.7). This finishes the supersymmetry variations of the bosonic special Kähler constraints.

Physical fermions and fermionic constraints

The first line of (6.2.12), using (6.3.27) and (6.3.29), is in terms of δz:

$$\bar{\epsilon}^i \Omega_i = U_\alpha \delta z^\alpha \,. \tag{6.3.33}$$

Therefore, the supersymmetry transformation of z is chiral, and we define λ_i^α as

$$\bar{\varepsilon}^i \lambda_i^\alpha \equiv \delta z^\alpha \,, \tag{6.3.34}$$

leading to

$$\tilde{\Omega}_i = U_\alpha \lambda_i^\alpha \,, \tag{6.3.35}$$

compatible with the S-gauge. The relation (6.3.35) can be inverted to

$$\lambda_i^\alpha = -\mathrm{i} g^{\alpha\bar{\alpha}} \langle \bar{U}_{\bar{\alpha}}, \tilde{\Omega}_i\rangle \,. \tag{6.3.36}$$

That $\tilde{\Omega}_i$ has only components in the U direction implies the constraints (the primes here and below are irrelevant for $n > 1$ or $Z_z = 0$)

$$\langle V, \tilde{\Omega}_i\rangle = Z_z \lambda_i^\alpha \;\text{ or }\; \langle V', \tilde{\Omega}_i\rangle = 0 \,, \qquad \langle U_\alpha, \tilde{\Omega}_i\rangle = 0 \,. \tag{6.3.37}$$

The transformation rules for z^α and λ_i^α are[5]

$$\begin{aligned} \delta z^\alpha &= \bar{\varepsilon}^i \lambda_i^\alpha \,, \\ \delta\lambda_i^\alpha &= -\Gamma^\alpha_{\beta\gamma} \lambda_i^\beta \delta z^\gamma + \tfrac{1}{4}(\partial_\beta K \delta z^\beta - h.c.)\lambda_i^\alpha \\ &\quad + \not{\nabla} z^\alpha \varepsilon_i - \tfrac{\mathrm{i}}{2} g^{\alpha\bar{\alpha}} \langle \bar{U}_{\bar{\alpha}}, \tilde{Y}_{ij}\rangle \varepsilon^j - \tfrac{\mathrm{i}}{2} g^{\alpha\bar{\alpha}} \langle \bar{U}_{\bar{\alpha}}, \tilde{\mathcal{F}}^-_{ab}\rangle \varepsilon_{ij} \sigma^{ab} \varepsilon^j \,, \end{aligned} \tag{6.3.38}$$

[5]In the transformation laws below, there is still the $SU(2)$ transformation which is not gauge fixed and thus independent of the other transformations. We will not indicate these transformations explicitly, as they follow from the position of the i indices.

where

$$\nabla_\mu z^\alpha = \partial_\mu z^\alpha - \bar\psi^i_\mu \lambda^\alpha_i \,. \tag{6.3.39}$$

Further variations of constraints in the generic case

At the first fermionic level we have imposed the gauge choice (6.3.26), and found in addition the constraints (6.3.37), leaving n physical fermions as shown in (6.3.35) and (6.3.36). The variation of the S-gauge leads to the decomposition rule. Here we will determine the further constraints on the $2(n+1)$ chiral multiplets in the symplectic vector. We first perform this analysis for the generic case where $\langle V, U_\alpha\rangle = 0$, and treat the case $n = 1$ separately afterwards.

The Poincaré transformations on (6.3.37) give

$$\begin{aligned}
\langle V, \tilde Y_{ij}\rangle &= 0\,,\\
\langle U_\alpha, \tilde Y_{ij}\rangle &= -C_{\alpha\beta\gamma}\bar\lambda^\beta_i\lambda^\gamma_j\,,\\
\langle V, \tilde{\mathcal F}^-_{ab}\rangle &= 0\,,\\
\langle U_\alpha, \tilde{\mathcal F}^-_{ab}\rangle &= -\tfrac12 C_{\alpha\beta\gamma}\varepsilon^{ij}(\bar\lambda^\beta_i\sigma_{ab}\lambda^\gamma_j)\,.
\end{aligned} \tag{6.3.40}$$

To analyze the content of these equations, we make use of lemma B.1 of [8]. This says that the $2(n+1)\times(n+1)$ matrix (V, U_α) has rank $(n+1)$. Thus we can solve (6.3.40) for half of the components of $\tilde Y_{ij}$ and $\tilde{\mathcal F}^-_{ab}$.

Straightforward variation of these two equations under Poincaré supersymmetry yields a set of new constraints:

$$\langle V, \tilde\Lambda_i\rangle = -\tfrac16 C_{\alpha\beta\gamma}\varepsilon^{kl}(\bar\lambda^\alpha_k\sigma^{ab}\lambda^\beta_l)\sigma_{ab}\lambda^\gamma_i\,, \tag{6.3.41}$$

$$\begin{aligned}
\langle U_\alpha, \tilde\Lambda_i\rangle &= \tfrac{\mathrm i}{2} C_{\alpha\beta\gamma}g^{\beta\bar\beta}\left(\langle \bar U_{\bar\beta}, \tilde Y_{ij}\rangle\varepsilon^{jk}\lambda^\gamma_k + \langle \bar U_{\bar\beta}, \sigma\cdot\tilde{\mathcal F}^-\rangle\lambda^\gamma_i\right)\\
&\quad +\tfrac16 \mathcal D_\alpha C_{\beta\gamma\delta}\cdot\varepsilon^{kl}(\bar\lambda^\beta_k\sigma_{ab}\lambda^\gamma_l)\sigma^{ab}\lambda^\delta_i\,.
\end{aligned} \tag{6.3.42}$$

Varying constraint (6.3.41) yields

$$\begin{aligned}
\langle V, \tilde C\rangle &= \tfrac{\mathrm i}{2}\varepsilon^{ik}\varepsilon^{jl}C_{\alpha\beta\gamma}g^{\alpha\bar\alpha}\langle \bar U_{\bar\alpha}, \tilde Y_{ij}\rangle\bar\lambda^\beta_k\lambda^\gamma_l\\
&\quad -\tfrac{\mathrm i}{2}C_{\alpha\beta\gamma}g^{\alpha\bar\alpha}\langle \bar U_{\bar\alpha}, \tilde{\mathcal F}^-_{ab}\rangle\varepsilon^{kl}(\bar\lambda^\beta_k\sigma^{ab}\lambda^\gamma_l)\\
&\quad -\tfrac16 \mathcal D_\alpha C_{\beta\gamma\delta}\cdot\varepsilon^{ij}(\bar\lambda^\alpha_i\sigma_{ab}\lambda^\beta_j)\varepsilon^{kl}(\bar\lambda^\gamma_k\sigma^{ab}\lambda^\delta_l)\,.
\end{aligned} \tag{6.3.43}$$

The variation of (6.3.42) gives

$$\langle U_\alpha, \tilde C\rangle = \tfrac14 C_{\alpha\beta\gamma}g^{\beta\bar\beta}g^{\gamma\bar\gamma}\varepsilon^{ik}\varepsilon^{jl}\langle \bar U_{\bar\beta}, \tilde Y_{ij}\rangle\langle \bar U_{\bar\gamma}, \tilde Y_{kl}\rangle$$

$$
\begin{aligned}
&-\tfrac{1}{2}C_{\alpha\beta\gamma}g^{\beta\bar\beta}g^{\gamma\bar\gamma}\langle\bar U_{\bar\beta},\tilde{\mathcal F}^-_{ab}\rangle\langle\bar U_{\bar\gamma},\tilde{\mathcal F}^{-ab}\rangle\\
&-\tfrac{\mathrm i}{2}\varepsilon^{ik}\varepsilon^{jl}[C_{\alpha\beta\gamma}\langle\bar V,\tilde Y_{ij}\rangle+\mathcal D_\alpha C_{\beta\gamma\delta}\cdot g^{\delta\bar\delta}\langle\bar U_{\bar\delta},\tilde Y_{ij}\rangle]\bar\lambda^\beta_k\lambda^\gamma_l\\
&+\tfrac{\mathrm i}{2}[C_{\alpha\beta\gamma}\langle\bar V,\tilde{\mathcal F}^-_{ab}\rangle+\mathcal D_\alpha C_{\beta\gamma\delta}\cdot g^{\delta\bar\delta}\langle\bar U_{\bar\delta},\tilde{\mathcal F}^-_{ab}\rangle]\varepsilon^{ij}(\bar\lambda^\beta_i\sigma^{ab}\lambda^\gamma_j)\\
&-2\mathrm i C_{\alpha\beta\gamma}g^{\beta\bar\beta}\varepsilon^{ij}\bar\lambda^\gamma_i\langle\bar U_{\bar\beta},\tilde\Lambda_j\rangle\\
&+\tfrac{1}{12}\mathcal D_\alpha\mathcal D_\beta C_{\gamma\delta\varepsilon}\cdot\varepsilon^{ij}(\bar\lambda^\beta_i\sigma_{ab}\lambda^\gamma_j)\,\varepsilon^{kl}(\bar\lambda^\delta_k\sigma^{ab}\lambda^\varepsilon_l)\,. \qquad (6.3.44)
\end{aligned}
$$

These are all the possible 'Kähler' constraints on the sections. Let us review the degrees of freedom. Before imposing the constraints, we have the degrees of freedom as in table 6.2. First of all there is the Weyl multiplet with 24+24 degrees of freedom. The gauge invariances have been used to determine the counting. Indeed, the dilatation invariance can be seen as removing the trace of the vierbein e^μ_μ and $\gamma^\mu\psi^i_\mu$ is pure gauge under special supersymmetry. Similarly the vectors A_μ and $\mathcal V_\mu{}^i{}_j$ loose a degree of freedom because of their gauge transformations. Secondly, we have the symplectic vectors of $2n+2$ chiral multiplets, which altogether consist of $(2n+2)16+(2n+2)16$ degrees of freedom.

Then we have imposed the constraints (6.1.15)–(6.1.17), (6.1.23) and their supersymmetry partners. The new counting is in table 6.3. All the symplectic sections are first reduced to $(n+1)$ rather than $(2n+2)$ degrees of freedom, as inner products with V and with U_α are removed by the constraints. The symplectic vector V is further reduced to n complex variables z^α, by constraints which we have interpreted as gauge choices of dilatations and chiral $U(1)$. These invariances have thus disappeared, and in the upper part of the table we should thus no longer subtract from degrees of freedom of the vierbein and of A_μ. Similarly, the constraint $\langle\bar V,\tilde\Omega_i\rangle=0$ removed a spinor doublet from the degrees of freedom of $\tilde\Omega_i$, but this breaks the S-symmetry, and thus the gravitino still has 24 degrees of freedom. As a result, the superconformal invariance is reduced to super-Poincaré. The super-Poincaré multiplet contains the graviphoton, which resides in $\langle\bar V,\tilde{\mathcal F}^-_{ab}\rangle$. Similarly the other internal products with $\bar V$ can be seen as auxiliary fields of the 40+40 off-shell super-Poincaré multiplet. In other formulations [197, 187] they are expressed in terms of another compensating multiplet. This compensating multiplet is then also used to gauge fix the $SU(2)$ invariance which we have not broken here.

Further variations of constraints in the special case

Now we continue the analysis of the supersymmetry transformations on special Kähler constraints for supergravity theories with $Z_z=\langle V,U_z\rangle\neq 0$. This can only happen for $n=1$, because that is the only case where this condition is not

fields	d.o.f.	comments
The gravity multiplet (40+40)		
$e_\mu{}^a$	6	16 - 4(translation) - 6(Lorentz)
A_μ	4	gauge vector $\rightarrow$ vector
$\mathcal{V}_\mu{}^i{}_j$	9	12 - 3($SU(2)$)
ψ^i_μ	24	32 - 8(Q-supersymmetry)
T^{ij}_{ab}	6	complex antiself-dual
χ_i	8	
D	1	real scalar
$\langle \bar{V}, \tilde{Y}_{ij} \rangle$	6	
$\langle \bar{V}, \tilde{\mathcal{F}}^-_{ab} \rangle$	6	
$\langle \bar{V}, \tilde{\Lambda}_i \rangle$	8	
$\langle \bar{V}, \tilde{C} \rangle$	2	
Symplectic section of constrained chiral multiplets $(16\,n + 16\,n)$		
z^α	2n	(d.o.f. of V)/2 - 2 (=trace vierbein + extra comp. of A_μ)
λ^α_i	8n	(d.o.f. of $\tilde{\Omega}_i$)/2 - 8 (=$\gamma^\mu \psi^i_\mu$)
$\langle \bar{U}_{\bar{\alpha}}, \tilde{Y}_{ij} \rangle$	6n	
$\langle \bar{U}_{\bar{\alpha}}, \tilde{\mathcal{F}}^-_{ab} \rangle$	6n	
$\langle \bar{U}_{\bar{\alpha}}, \tilde{\Lambda}_i \rangle$	8n	
$\langle \bar{U}_{\bar{\alpha}}, \tilde{C} \rangle$	2n	

Table 6.3: Degrees of freedom in the model after the special Kähler constraints

equivalent with (6.1.23). Because $n = 1$, U_α and $\mathcal{D}_\alpha$ can be replaced by U_z and $\mathcal{D}_z$.

The computation of the special Kähler constraint goes along the same track as for the generic case, but extra terms appear because of the weaker constraint. The new contributions appear for the first time after the supersymmetry variation of (6.3.37):

$$\begin{aligned}
\langle V', \tilde{Y}_{ij} \rangle &= -\mathcal{D}_z Z_z \cdot \bar{\lambda}^z_i \lambda^z_j \,, \\
\langle U'_z, \tilde{Y}_{ij} \rangle &= -C_{zzz} \bar{\lambda}^z_i \lambda^z_j \,, \\
\langle V', \tilde{\mathcal{F}}^-_{ab} \rangle &= -\tfrac{1}{2} \mathcal{D}_z Z_z \cdot \varepsilon^{ij} (\bar{\lambda}^z_i \sigma_{ab} \lambda^z_j) \,, \\
\langle U'_z, \tilde{\mathcal{F}}^-_{ab} \rangle &= -\tfrac{1}{2} C_{zzz} \varepsilon^{ij} (\bar{\lambda}^z_i \sigma_{ab} \lambda^z_j) \,.
\end{aligned} \tag{6.3.45}$$

Define a new vector V'' [6]

$$V'' \equiv V' - \frac{g^{z\bar{z}}\mathcal{D}_{\bar{z}}C_{zzz}\cdot U'_z}{C_{zzz}} = V' - \frac{\mathcal{D}_z Z_z \cdot U'_z}{C_{zzz}}\,. \tag{6.3.46}$$

In terms of V'' the constraints will have the same form as before:

$$\langle V'', \tilde{Y}_{ij}\rangle = \langle V'', \tilde{\mathcal{F}}^-_{ab}\rangle = 0\,. \tag{6.3.47}$$

Then, one finds

$$\begin{aligned}
\langle V', \tilde{\Lambda}_i\rangle &= \tfrac{\mathrm{i}}{2}(\mathcal{D}_z Z_z) g^{z\bar{z}}(\langle \bar{U}_{\bar{z}}, \tilde{Y}_{ij}\rangle \varepsilon^{jk}\lambda_k^z + \langle \bar{U}_{\bar{z}}, \sigma\cdot\tilde{\mathcal{F}}^-\rangle\lambda_i^z) \\
&\quad -\tfrac{1}{6}(C_{zzz} - \mathcal{D}_z\mathcal{D}_z Z_z)\,\varepsilon^{kl}\bar{\lambda}_k^z\sigma^{ab}\lambda_l^z\sigma_{ab}\lambda_i^z\,, &(6.3.48)\\
\langle U'_z, \tilde{\Lambda}_i\rangle &= \tfrac{\mathrm{i}}{2}C_{zzz}g^{z\bar{z}}(\langle \bar{U}_{\bar{z}}, \tilde{Y}_{ij}\rangle\varepsilon^{jk}\lambda_k^z + \langle \bar{U}_{\bar{z}}, \sigma\cdot\tilde{\mathcal{F}}^-\rangle\lambda_i^z) \\
&\quad +\tfrac{1}{6}\mathcal{D}_z C_{zzz}\cdot\varepsilon^{kl}(\bar{\lambda}_k^z\sigma_{ab}\lambda_l^z)\sigma^{ab}\lambda_i^z\,, &(6.3.49)
\end{aligned}$$

where we have used that

$$\langle V, \mathcal{D}_z\mathcal{D}_z U_z\rangle = -C_{zzz} + \mathcal{D}_z\mathcal{D}_z Z_z\,. \tag{6.3.50}$$

Note that these are the analogues of (6.3.42) and (6.3.41). It is possible to replace (6.3.48) by

$$\langle V'', \tilde{\Lambda}_i\rangle = \tfrac{1}{6}\,\varepsilon^{kl}\bar{\lambda}_k^z\sigma_{ab}\lambda_l^z\sigma^{ab}\lambda_i^z\,\frac{1}{C_{zzz}}((-C_{zzz} + \mathcal{D}_z\mathcal{D}_z Z_z)\,C_{zzz} - \mathcal{D}_z Z_z\cdot\mathcal{D}_z C_{zzz})\,. \tag{6.3.51}$$

Using the notation

$$\mathcal{O}_{zzzzz} \equiv g'^{z\bar{z}}\bar{Z}_{\bar{z}}\left[(-C_{zzz} + \mathcal{D}_z\mathcal{D}_z Z_z)\,C_{zzz} - \mathcal{D}_z Z_z\cdot\mathcal{D}_z C_{zzz}\right]\,, \tag{6.3.52}$$

the variation of (6.3.49) now gives

$$\begin{aligned}
\langle U'_z, \tilde{C}\rangle &= \tfrac{1}{4}C_{zzz}g^{z\bar{z}}g^{z\bar{z}}\varepsilon^{ik}\varepsilon^{jl}\langle \bar{U}_{\bar{z}}, \tilde{Y}_{ij}\rangle\langle \bar{U}_{\bar{z}}, \tilde{Y}_{kl}\rangle \\
&\quad -\tfrac{1}{2}C_{zzz}g^{z\bar{z}}g^{z\bar{z}}\langle \bar{U}_{\bar{z}}, \tilde{\mathcal{F}}^-_{ab}\rangle\langle \bar{U}_{\bar{z}}, \tilde{\mathcal{F}}^{-ab}\rangle \\
&\quad -\tfrac{\mathrm{i}}{2}\varepsilon^{ik}\varepsilon^{jl}(C_{zzz}\langle \bar{V}, \tilde{Y}_{ij}\rangle + \mathcal{D}_z C_{zzz}\cdot g^{z\bar{z}}\langle \bar{U}_{\bar{z}}, \tilde{Y}_{ij}\rangle)\bar{\lambda}_k^z\lambda_l^z \\
&\quad +\tfrac{\mathrm{i}}{2}(C_{zzz}\langle \bar{V}, \tilde{\mathcal{F}}^-_{ab}\rangle + \mathcal{D}_z C_{zzz}\cdot g^{z\bar{z}}\langle \bar{U}_{\bar{z}}, \tilde{\mathcal{F}}^-_{ab}\rangle)\,\varepsilon^{kl}(\bar{\lambda}_k^z\sigma^{ab}\lambda_l^z) \\
&\quad -2\mathrm{i}C_{zzz}g^{z\bar{z}}\varepsilon^{ij}\bar{\lambda}_i^z\langle \bar{U}_{\bar{z}}, \tilde{\Lambda}_j\rangle \\
&\quad +\tfrac{1}{12}(\mathcal{D}_z\mathcal{D}_z C_{zzz} + \tfrac{1}{2}\mathcal{O}_{zzzzz})\,\varepsilon^{ij}(\bar{\lambda}_i^z\sigma_{ab}\lambda_j^z)\varepsilon^{kl}(\bar{\lambda}_k^z\sigma^{ab}\lambda_l^z)\,. &(6.3.53)
\end{aligned}$$

[6] Note that $g_{z\bar{z}}\mathcal{D}_z Z_z = \mathcal{D}_{\bar{z}}C_{zzz}$ is not necessarily 0 in this case.

Straightforward variation of constraint (6.3.48) gives

$$\begin{aligned}
\langle V', \tilde{C}\rangle \quad = \quad & \tfrac{1}{4}\mathcal{D}_z Z_z \cdot g^{z\bar{z}} g^{z\bar{z}} \varepsilon^{ik}\varepsilon^{jl}\langle \bar{U}_{\bar{z}}, \tilde{Y}_{ij}\rangle\langle \bar{U}_{\bar{z}}, \tilde{Y}_{kl}\rangle \\
& -\tfrac{1}{2}\mathcal{D}_z Z_z \cdot g^{z\bar{z}} g^{z\bar{z}}\langle \bar{U}_{\bar{z}}, \tilde{\mathcal{F}}^-_{ab}\rangle\langle \bar{U}_{\bar{z}}, \tilde{\mathcal{F}}^{-ab}\rangle \\
& +\tfrac{\mathrm{i}}{2}\varepsilon^{ik}\varepsilon^{jl}\Big((C_{zzz} - \mathcal{D}_z\mathcal{D}_z Z_z) g^{z\bar{z}}\langle \bar{U}_{\bar{z}}, \tilde{Y}_{ij}\rangle - \mathcal{D}_z Z_z \cdot \langle \bar{V}, \tilde{Y}_{ij}\rangle\Big)\bar{\lambda}^z_k \lambda^z_l \\
& -\tfrac{\mathrm{i}}{2}\Big((C_{zzz} - \mathcal{D}_z\mathcal{D}_z Z_z)\, g^{z\bar{z}}\langle \bar{U}_{\bar{z}}, \tilde{\mathcal{F}}^-_{ab}\rangle - \mathcal{D}_z Z_z \cdot \langle \bar{V}, \tilde{\mathcal{F}}^-_{ab}\rangle\Big)\varepsilon^{kl}(\bar{\lambda}^z_k \sigma^{ab}\lambda^z_l) \\
& -2\mathrm{i}\mathcal{D}_z Z_z \cdot g^{z\bar{z}}\varepsilon^{ij}\bar{\lambda}^z_i\langle \bar{U}_{\bar{z}}, \tilde{\Lambda}_j\rangle \\
& -\tfrac{1}{12}(2\mathcal{D}_z C_{zzz} - \mathcal{D}_z\mathcal{D}_z\mathcal{D}_z Z_z)\varepsilon^{ij}(\bar{\lambda}^z_i \sigma_{ab}\lambda^z_j)\varepsilon^{kl}(\bar{\lambda}^z_k \sigma^{ab}\lambda^z_l)\,. \qquad (6.3.54)
\end{aligned}$$

This can be rewritten in

$$\begin{aligned}
\langle V'', \tilde{C}\rangle \quad = \quad & \frac{\mathrm{i} g^{z\bar{z}}}{2C_{zzz}}\left[(C_{zzz} - \mathcal{D}_z\mathcal{D}_z Z_z)\, C_{zzz} + \mathcal{D}_z C_{zzz} \cdot \mathcal{D}_z Z_z\right] \cdot \\
& \Big[\varepsilon^{ik}\varepsilon^{jl}\langle \bar{U}_{\bar{z}}, \tilde{Y}_{ij}\rangle\bar{\lambda}^z_k\lambda^z_l - \langle \bar{U}_{\bar{z}}, \tilde{\mathcal{F}}_{ab}\rangle\varepsilon^{kl}(\bar{\lambda}^z_k\sigma^{ab}\lambda^z_l)\Big] \\
& -\tfrac{1}{12}\left[(2\mathcal{D}_z C_{zzz} - \mathcal{D}_z\mathcal{D}_z\mathcal{D}_z Z_z) + \frac{\mathcal{D}_z Z_z}{C_{zzz}}\left(\mathcal{D}_z\mathcal{D}_z C_{zzz} + \tfrac{1}{2}\mathcal{O}_{zzzzz}\right)\right] \cdot \\
& \varepsilon^{ij}(\bar{\lambda}^z_i\sigma_{ab}\lambda^z_j)\varepsilon^{kl}(\bar{\lambda}^z_k\sigma^{ab}\lambda^z_l)\,. \qquad (6.3.55)
\end{aligned}$$

6.4 The generalized Bianchi identities combined with the special Kähler constraints

In this section, we start by imposing the reduction constraints on the chiral multiplets. Because the constraints on the field strengths are Bianchi identities, this linear multiplet of constraints is called the generalized Bianchi identities. We combine these constraints with the special Kähler constraints of section 6.3. Together they give the field equations of n vector multiplets and expressions for the auxiliary fields χ_i and D. We derive this first for the generic case $\langle V, U_\alpha\rangle = 0$. We comment on the supergravity equations of motion in this generic case. Finally, we give the equations for the special case where $\langle V, U_\alpha\rangle \neq 0$, we give a concrete example and comment on the connection to string theory compactifications.

6.4.1 The field equations for the generic case

To see what follows from equations (6.2.13)–(6.2.16), we take the symplectic inner product of these equations with the basis $\mathcal{W}$. The four components of equation (A.2.17) give four equations for each constraint.

From the first identity we learn that the section $\tilde{Y}_{ij}$ is totally constrained, as it should be, because it is auxiliary. We have

$$\tilde{Y}_{ij} = -\mathrm{i}g^{\alpha\bar{\alpha}}\left[C_{\alpha\beta\gamma}\bar{\lambda}_i^{\beta}\lambda_j^{\gamma}\bar{U}_{\bar{\alpha}} - \bar{C}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\varepsilon_{ik}\varepsilon_{jl}\bar{\lambda}^{\bar{\beta}k}\lambda^{\bar{\gamma}l}U_{\alpha}\right]. \tag{6.4.1}$$

It is more interesting to take a look at (6.2.14). Taking the symplectic inner product of this equation with all components of the basis $\mathcal{W}$ (6.3.3), and using the special Kähler constraints of section 6.3.3 and (6.4.1) gives:

$$\begin{aligned}
\langle\bar{V},\tilde{\Lambda}_i\rangle &= 0\,,\\
\langle\bar{U}_{\bar{\alpha}},\tilde{\Lambda}_i\rangle &= -\varepsilon_{ij}\bar{C}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\not{\nabla}\bar{z}^{\bar{\beta}}\cdot\lambda^{\bar{\gamma}j}\,,\\
0 &= \chi^i - \tfrac{2}{3}\sigma^{\mu\nu}(\mathcal{D}_\mu\psi_\nu^i - \tfrac{1}{8}\sigma\cdot T^{ij}\gamma_\mu\psi_{\nu j}) - \tfrac{1}{2}g_{\alpha\bar{\beta}}\not{\partial}z^{\alpha}\cdot\lambda^{i\bar{\beta}}\\
&\quad+\tfrac{\mathrm{i}}{2}\gamma^\mu(\not{\mathcal{Q}}-\not{A})\psi_\mu^i - \tfrac{\mathrm{i}}{4}\varepsilon^{ij}\gamma^\mu\langle V,\sigma\cdot\tilde{\mathcal{F}}^+\rangle\psi_{\mu j}\\
&\quad+\tfrac{\mathrm{i}}{12}C_{\alpha\beta\gamma}\varepsilon^{ij}\varepsilon^{kl}(\bar{\lambda}_k^{\alpha}\sigma^{ab}\lambda_l^{\beta})\sigma_{ab}\lambda_j^{\gamma}\,,\\
0 &= \mathrm{i}g_{\alpha\bar{\beta}}\left(\not{\nabla}\lambda^{i\bar{\beta}} + \tfrac{\mathrm{i}}{2}(\not{\mathcal{Q}}-\not{A})\lambda^{i\bar{\beta}}\right) + \tfrac{\mathrm{i}}{2}\varepsilon^{ij}C_{\alpha\beta\gamma}g^{\beta\bar{\beta}}\langle\bar{U}_{\bar{\beta}},\sigma\cdot\tilde{\mathcal{F}}^-\rangle\lambda_j^{\gamma}\\
&\quad+\tfrac{\mathrm{i}}{2}C_{\alpha\beta\gamma}\bar{C}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}g^{\gamma\bar{\gamma}}(\bar{\lambda}^{i\bar{\alpha}}\lambda^{j\bar{\beta}})\lambda_j^{\beta} + \tfrac{1}{6}\mathcal{D}_\alpha C_{\beta\gamma\delta}\cdot\varepsilon^{ij}\varepsilon^{kl}(\bar{\lambda}_k^{\beta}\sigma^{ab}\lambda_l^{\gamma})\sigma_{ab}\lambda_j^{\delta}\,,
\end{aligned} \tag{6.4.2}$$

where

$$\begin{aligned}
\nabla_\mu\lambda_i^\alpha &= \partial_\mu\lambda_i^\alpha - \tfrac{1}{2}\omega_\mu^{ab}\sigma_{ab}\lambda_i^\alpha + \tfrac{1}{2}\mathcal{V}_{\mu i}{}^{j}\lambda_j^\alpha + \Gamma^\alpha_{\beta\gamma}\partial_\mu z^\beta\cdot\lambda_i^\gamma - \tfrac{\mathrm{i}}{2}\mathcal{Q}_\mu\lambda_i^\alpha\\
&\quad-\not{\nabla}z^\alpha\cdot\psi_{\mu i} + \tfrac{\mathrm{i}}{2}g^{\alpha\bar{\alpha}}\left[\langle\bar{U}_{\bar{\alpha}},\tilde{Y}_{ij}\rangle + \varepsilon_{ij}\langle\bar{U}_{\bar{\alpha}},\sigma\cdot\tilde{\mathcal{F}}^-\rangle\right]\psi_\mu^j\,,
\end{aligned} \tag{6.4.3}$$

and

$$\mathcal{Q}_\mu = -\tfrac{\mathrm{i}}{2}(\partial_\alpha K\cdot\partial_\mu z^\alpha - \text{h.c.}) \tag{6.4.4}$$

is the Kähler one-form.

The first two equations in (6.4.2) imply with (6.3.42) and (6.3.41) that all components of $\tilde{\Lambda}_i$ are expressed in terms of other fields, and thus they contain no independent degrees of freedom. The third expresses χ^i in terms of other fields. In a superconformal calculation using a Lagrangian, this expression for χ^i can be found from the equation of motion of the fermion of the compensating vector multiplet. The fourth equation is the field equation for n fermion doublets λ_i^{β}.

We now proceed with the analysis of (6.2.15). We first repeat that (6.3.40) implies that there are $(n+1)$ independent antisymmetric tensors in the symplectic vector $\tilde{\mathcal{F}}_{ab}$. Apart from these there is another antiself-dual tensor in the Weyl multiplet T^{ij}_{ab}. A few definitions make (6.2.15) more transparent. First define the combination in brackets as

$$\tilde{\hat{F}}_{ab} = \tilde{\mathcal{F}}_{ab} + \tfrac{1}{4} V T_{ab\,ij}\varepsilon^{ij} + \tfrac{1}{4}\bar{V} T^{ij}_{ab}\varepsilon_{ij}\,. \tag{6.4.5}$$

Then we take out covariantization terms:

$$\tilde{F}_{ab} = \tilde{\hat{F}}_{ab} - 2(\bar{\tilde{\Omega}}^i\gamma_{[a}\psi^j_{b]}\varepsilon_{ij} + \bar{V}\bar{\psi}^i_a\psi^j_b\varepsilon_{ij} + \mathrm{h.c.})\,. \tag{6.4.6}$$

This is chosen such that covariant derivatives in (6.2.15) can be rewritten as ordinary derivatives, and the equation reduces to

$$\partial_\mu \varepsilon^{\mu\nu\rho\sigma}\tilde{F}_{\rho\sigma} = 0\,. \tag{6.4.7}$$

Applying this on the $(n+1)$ independent components of $\tilde{F}_{\mu\nu}$, implies that they can be expressed in terms of $(n+1)$ vectors. The other $(n+1)$ equations of (6.4.7) are the equivalent of field equations for these vectors. Here, it is clear how our formulation keeps the symplectic covariance. Only in the interpretation, we do make a distinction between one half of the equations as Bianchi identities and the others as field equations. These could have been interchanged giving the 'magnetic dual formulation'. Also the fact of whether or not a prepotential exists is hidden here. The difference is seen only when breaking the symplectic formulation in finding an explicit solution of equations (6.3.40). If the $(n+1)\times(n+1)$-matrix, formed by the upper part of $(V\,, U_\alpha)$, is invertible, then (6.3.40) expresses the $(n+1)$ lower components of $\mathcal{F}^-_{ab}$ in terms of the upper ones. This is the case where there is a prepotential. When this matrix is not invertible[7], then one can still solve (6.3.40) for other $(n+1)$ components of $\tilde{\mathcal{F}}^-_{ab}$.

We thus conclude that we have $n+1$ on-shell vectors and their field equations also depend on the 6 degrees of freedom of the tensor T^{ij}_{ab} of the Weyl multiplet.

Now let us have a look at (6.2.16). It involves the covariant Laplacian,

$$\begin{aligned}
\Box\bar{V} &\equiv \eta^{mn}D_mD_n\bar{V}\\
&= e^{-1}\partial_\mu(eD^\mu\bar{V}) + (b^\mu - \mathrm{i}A^\mu)D_\mu\bar{V} + f^\mu_\mu\bar{V} + 2\bar{\psi}^{i[\mu}\gamma_\mu\psi^{\nu]}_i D_\nu\bar{V}\\
&\quad -\bar{\psi}^\mu_i D_\mu\tilde{\Omega}^i + \tfrac{1}{2}\bar{\phi}^\mu_i\gamma_\mu\tilde{\Omega}^i + \tfrac{1}{8}\bar{\psi}^i_\mu\gamma^\mu\sigma\cdot T_{ij}\tilde{\Omega}^j - \tfrac{3}{2}\bar{\psi}^i_\mu\gamma^\mu\chi_i\bar{V}\,.
\end{aligned} \tag{6.4.8}$$

[7]As proven in [8], it is only the matrix $\left(f^I_\alpha\bar{X}^I\right)$ that is always invertible, where f^I_α are the first $n+1$ components of U_α.

To derive this expression, we used the theorem on covariant derivatives of [35]. We can again take the symplectic inner product of (6.2.16) with $\mathcal{W}$:

$$
\begin{aligned}
\langle \bar V, \tilde C\rangle &= 0\,,\\
\langle \bar U_{\bar\alpha}, \tilde C\rangle &= -2\bar C_{\bar\alpha\bar\beta\bar\gamma}\nabla_\mu \bar z^{\bar\beta}\cdot\nabla^\mu \bar z^{\bar\gamma} + \tfrac14 \bar C_{\bar\alpha\bar\beta\bar\gamma}(\bar\lambda^{k\bar\beta}\sigma\cdot T_{kl}\lambda^{l\bar\gamma})\,,\\
0 &= -2e^{-1}\partial_\mu\left(e(\mathcal{Q}^\mu - A^\mu)\right) + 2\mathrm{i} g_{\alpha\bar\alpha}\partial_\mu z^\alpha\cdot\nabla^\mu\bar z^{\bar\alpha} - 2\mathrm{i} g_{\alpha\bar\alpha}\partial_\mu z^\alpha\cdot(\bar\psi_i^\mu\lambda^{i\bar\alpha})\\
&\quad -2\mathrm{i}(\mathcal{Q}^\mu - A^\mu)(\mathcal{Q}_\mu - A_\mu) + 2\mathrm{i} f_\mu^\mu - 4\bar\psi^{i[\mu}\gamma_\mu\psi_i^{\nu]}(\mathcal{Q}_\nu - A_\nu)\\
&\quad +\bar\psi_i^\mu\langle V,\sigma\cdot\tilde{\mathcal{F}}^+\rangle\varepsilon^{ij}\psi_{\mu j} + 2\mathrm{i}\bar\psi_i^\mu\phi_\mu^i - 2\bar\psi_i^\mu(\not{\mathcal{Q}} - \not{A})\psi_\mu^i\\
&\quad -3\mathrm{i}\bar\psi_\mu^i\gamma^\mu\chi_i + \tfrac14\langle V,\tilde{\mathcal{F}}^+_{\mu\nu}\rangle T_{ij}^{\mu\nu}\varepsilon^{ij} - \tfrac{\mathrm{i}}{2} C_{\alpha\beta\gamma} g^{\alpha\bar\alpha}\bar C_{\bar\alpha\bar\beta\bar\gamma}(\bar\lambda^{\bar\beta k}\lambda^{\bar\gamma l})(\bar\lambda_k^\beta\lambda_l^\gamma)\\
&\quad -\tfrac{\mathrm{i}}{2} C_{\alpha\beta\gamma} g^{\alpha\bar\alpha}\langle\bar U_{\bar\alpha},\tilde{\mathcal{F}}^-_{ab}\rangle\varepsilon^{kl}(\bar\lambda_k^\beta\sigma^{ab}\lambda_l^\gamma)\\
&\quad -\tfrac16\mathcal{D}_\alpha C_{\beta\gamma\delta}\cdot\varepsilon^{ij}(\bar\lambda_i^\alpha\sigma_{ab}\lambda_j^\beta)\,\varepsilon^{kl}(\bar\lambda_k^\gamma\sigma^{ab}\lambda_l^\delta)\,,\\
0 &= -2\mathrm{i}e^{-1} g_{\alpha\bar\alpha}\partial_\mu(e\nabla^\mu\bar z^{\bar\alpha}) + 4(\mathcal{Q}_\mu - A_\mu) g_{\alpha\bar\alpha}\nabla^\mu\bar z^{\bar\alpha}\\
&\quad + g_{\alpha\bar\alpha}(\mathcal{Q}_\mu - A_\mu)(\bar\psi_i^\mu\lambda^{i\bar\alpha}) - 2\mathrm{i} g_{\alpha\bar\alpha}\Gamma^{\bar\alpha}_{\bar\beta\bar\gamma}\partial_\mu\bar z^{\bar\beta}\cdot\nabla^\mu\bar z^{\bar\gamma}\\
&\quad -4\mathrm{i} g_{\alpha\bar\alpha}\bar\psi^{i[\mu}\gamma_\mu\psi_i^{\nu]}\nabla_\nu\bar z^{\bar\alpha} + 2\mathrm{i} g_{\alpha\bar\alpha}\bar\psi_i^\mu\nabla_\mu\lambda^{i\bar\alpha} - \mathrm{i} g_{\alpha\bar\alpha}\bar\phi_i^\mu\gamma_\mu\lambda^{i\bar\alpha}\\
&\quad +2C_{\alpha\beta\gamma}\varepsilon^{ik}\varepsilon^{jl}(\bar\lambda_k^\beta\lambda_l^\gamma)(\bar\psi_i^\mu\psi_{j\mu}) + 2\bar\psi_i^\mu\langle U_\alpha,\sigma\cdot\tilde{\mathcal{F}}^+\varepsilon^{ij}\rangle\psi_{\mu j}\\
&\quad -\tfrac{\mathrm{i}}{4} g_{\alpha\bar\alpha}\bar\psi_\mu^i\gamma^\mu\sigma\cdot T_{ij}\lambda^{j\bar\alpha} - 6\mathrm{i} g_{\alpha\bar\alpha}\bar\chi_i\lambda^{i\bar\alpha}\\
&\quad +\tfrac14\langle U_\alpha,\tilde{\mathcal{F}}^+_{\mu\nu}\rangle T_{ij}^{\mu\nu}\varepsilon^{ij} + \tfrac14 C_{\alpha\beta\gamma} g^{\beta\bar\beta} g^{\gamma\bar\gamma}\varepsilon_{ik}\varepsilon_{jl}\bar C_{\bar\beta\bar\delta\bar\varepsilon}(\bar\lambda^{i\bar\delta}\lambda^{j\bar\varepsilon})\bar C_{\bar\gamma\bar\alpha\bar\zeta}(\bar\lambda^{k\bar\alpha}\lambda^{l\bar\zeta})\\
&\quad -\tfrac12 C_{\alpha\beta\gamma} g^{\beta\bar\beta} g^{\gamma\bar\gamma}\langle\bar U_{\bar\beta},\tilde{\mathcal{F}}^-_{ab}\rangle\langle\bar U_{\bar\gamma},\tilde{\mathcal{F}}^{-ab}\rangle\\
&\quad -\tfrac{\mathrm{i}}{2}\mathcal{D}_\alpha C_{\beta\gamma\delta}\cdot g^{\delta\bar\delta}\bar C_{\bar\delta\bar\beta\bar\gamma}(\bar\lambda^{k\bar\beta}\lambda^{l\bar\gamma})(\bar\lambda_k^\beta\lambda_l^\gamma)\\
&\quad +\tfrac{\mathrm{i}}{2}\left[C_{\alpha\beta\gamma}\langle\bar V,\tilde{\mathcal{F}}^-_{ab}\rangle + \mathcal{D}_\alpha C_{\beta\gamma\delta}\cdot g^{\delta\bar\delta}\langle\bar U_{\bar\delta},\tilde{\mathcal{F}}^-_{ab}\rangle\right]\varepsilon^{ij}(\bar\lambda_i^\beta\sigma^{ab}\lambda_j^\gamma)\\
&\quad +\tfrac{1}{12}\mathcal{D}_\alpha\mathcal{D}_\beta C_{\gamma\delta\varepsilon}\cdot\varepsilon^{ij}(\bar\lambda_i^\beta\sigma_{ab}\lambda_j^\gamma)\varepsilon^{kl}(\bar\lambda_k^\delta\sigma^{ab}\lambda_l^\varepsilon)\\
&\quad -2\mathrm{i} C_{\alpha\beta\gamma} g^{\beta\bar\beta}\bar C_{\bar\beta\bar\gamma\bar\delta}\bar\lambda_i^\gamma\not{\nabla}\bar z^{\bar\gamma}\cdot\lambda^{i\bar\delta}\,.
\end{aligned}
\tag{6.4.9}
$$

The first two equations, together with (6.3.44) and (6.3.43) (which can be simplified using the second equations in (6.4.2)), determine that $\tilde C$ is completely determined in terms of other fields. The real and imaginary part of the third equation have both to be 0. The real part constrains the divergence of $\mathcal{Q}_\mu - A_\mu$, and the imaginary part gives an expression for the D-field of the Weyl multiplet. (D is hidden in the $f_\mu{}^\mu$-term in the second line by using the first equation of (6.2.3).) The fourth equation of the expansion in terms of $\mathcal{W}$ gives the field equations for n complex scalars. So we find the same structure in the equations as for the fermions: $n+1$ equations express $\tilde C$ in terms of other fields, while the $n+1$ other equations give the field equations for n complex scalars z^α, an expression for D, and a constraint for $(\mathcal{Q}_\mu - A_\mu)$. The degrees of freedom are described in table 6.4.

fields	d.o.f.	comments
The Gravity Multiplet		
$e_\mu{}^a$	6	16 - 4(translation) - 6(Lorentz)
A_μ	3	$\partial^\mu A_\mu$ constrained
$\mathcal{V}_\mu{}^j{}_i$	9	12 - 3($SU(2)$)
ψ_μ^i	24	32 - 8(Q-supersymmetry)
T_{ab}^{ij}	6	n on-shell and 2 off-shell vectors
χ_i	0	expressed in terms of other fields
D	0	expressed in terms of other fields

Table 6.4: Off-shell degrees of freedom after the special Kähler constraints and the generalized Bianchi identities: $24+24$ d.o.f.. All other variables are expressed in terms of these fields or have a field equation (for n complex scalars, n doublet spinors, and $n+1$ vectors).

6.4.2 Comments on the supergravity equations

Using the results of the previous section, we comment on the appearance of equations of motion for the remaining 24+24 components of table 6.4 from one symplectic invariant constraint

$$\langle V, \tilde{\mathcal{F}}_{ab}^{+} \rangle \approx 0 \,, \tag{6.4.10}$$

which gives rise to a 24+24 'current' multiplet. The $\approx$-sign is used to denote that we only expose the linear terms. This already shows the essential features of this symplectic covariant formulation. With the linearized approximation we mean that we keep terms with an arbitrary power of undifferentiated scalar fields or metric, but only linear in other fields. In a full treatment of $N=2$ supergravity couplings, the r.h.s. of (6.4.10) would, for example, contain an additional coupling to hypermultiplets. Here we only want to see how far we can go by using the symplectic structure that allowed already the determination of the equations of motion of the vector multiplets.

To discuss the supersymmetry partners of (6.4.10), we derive a new $N=2$ multiplet with 24 + 24 components. The multiplet starting with the symplectic expression $\langle V, \tilde{\mathcal{F}}_{ab}^{+} \rangle$ is a supergravity realization of this multiplet. As shown below the supermultiplet of constraints derived from (6.4.10) is only equivalent to the supergravity equations of motion, up to integration 'constants'. These 8+8 remaining unknowns can be determined when one of the three possibilities of a second compensating multiplet is introduced as in [197, 187, 39]. In our approach this is the place where the second compensating multiplet, which is also needed for consistency in the Lagrangian formulation, comes into play.

A restricted chiral self-dual tensor multiplet

The supermultiplet structure of the 'current' multiplet from (6.4.10) is that of a chiral self-dual tensor multiplet

$$\begin{aligned} W^+_{ab} &= A^+_{ab} + \bar\tau^i\psi_{abi} + \tfrac14\bar\tau^i\tau^j B_{abij} + \tfrac14\varepsilon_{ij}\bar\tau^i\sigma_{cd}\tau^j F_{ab}{}^{cd} \\ &\quad + \tfrac16\varepsilon_{ij}(\bar\tau^i\sigma_{cd}\tau^j)\bar\tau^k\sigma^{cd}\chi_{abk} + \tfrac{1}{48}(\varepsilon_{ij}\bar\tau^i\sigma_{cd}\tau^j)^2 C^+_{ab}\,. \end{aligned} \tag{6.4.11}$$

It has the following field content. A^+_{ab} is a self-dual complex tensor with 6 degrees of freedom. ψ_{abi} has 24 left-handed fermionic components. The tensor B_{abij} has 18 components. The tensor $F_{ab}{}^{cd}$ is self-dual in its first and antiself-dual in its second pair of indices, leading to 9 complex components. It also satisfies the following properties:

$$\begin{aligned} F_{ab,cd} + F_{cd,ab} &= \tfrac12\varepsilon_{ab}{}^{ef}\left(F_{ef,cd} - F_{cd,ef}\right)\,, \\ F_{ab,cd} - F_{cd,ab} &= \varepsilon_{abe[d}\left(F^e{}_{c]} + F_{c]}{}^e\right)\,, \\ F_{ab,cd} &= \delta_{a[c}F_{d]b} - \delta_{b[c}F_{d]a} - \varepsilon_{abe[c}F^e{}_{d]}\,, \\ F_{[ab]} &= 0\,, \\ F^a{}_a &= 0\,, \end{aligned} \tag{6.4.12}$$

where

$$F_a{}^c = F_{ab}{}^{cd}\delta^b_d\,. \tag{6.4.13}$$

A general component of this self-dual–antiself-dual tensor $F_{ab}{}^{cd}$ can thus be written in terms of the traceless symmetric part $F_{(ab)}$ with 9 components. The fermion χ_{abi} has again 24 left-handed components and C^+_{ab} has 6. So, this is a chiral multiplet with $48+48$ components.

The transformation rules of this multiplet are the same as for a chiral multiplet with a complex scalar as lowest component, but with the components replaced straightforwardly:

$$\begin{aligned} \delta A^+_{ab} &= \bar\varepsilon^i\psi_{abi}\,, \\ \delta\psi_{abi} &= \not{\partial} A^+_{ab}\epsilon_i + \tfrac12 B_{abij}\epsilon^j + \tfrac12\sigma_{cd}F_{ab}{}^{cd}\varepsilon_{ij}\epsilon^j\,, \\ \delta B_{abij} &= 2\bar\varepsilon_{(i}\not{\partial}\psi_{abj)} - 2\bar\varepsilon^k\chi_{ab(i}\varepsilon_{j)k}\,, \\ \delta F_{ab}{}^{cd} &= \varepsilon^{ij}\bar\varepsilon_i\not{\partial}\sigma^{cd}\psi_{abj} + \bar\varepsilon^i\sigma^{cd}\chi_{abi}\,, \end{aligned}$$

$$
\begin{array}{rcl}
\delta\chi_{abi} &=& -\frac{1}{2}\sigma_{cd}F_{ab}^{cd}\overleftarrow{\not\partial}\epsilon_i - \frac{1}{2}\not\partial B_{abij}\varepsilon^{jk}\epsilon_k + \frac{1}{2}C^+_{ab}\varepsilon_{ij}\epsilon^j\,,\\
\delta C^+_{ab} &=& -2\varepsilon^{ij}\bar{\varepsilon}_i\not\partial\chi_{abj}\,.
\end{array}
\qquad (6.4.14)
$$

Since we have broken superconformal symmetry to super-Poincaré and $SU(2)$, we only need a super-Poincaré version of this multiplet. Note that it cannot be extended to a superconformal one. The commutator of a supersymmetry and a special supersymmetry has to give a Lorentz transformation that can never be realized because of the duality and chirality properties of the spinors. For this reason, it is only possible to construct an antiself-dual chiral tensor multiplet, realizing the superconformal algebra, as given in [163].

To study the field equations of the fields of table 6.4, we need a multiplet with $24+24$ components. A suitable multiplet of constraints is:

$$0 = \partial^a(B_{abij} + \varepsilon_{ik}\varepsilon_{jl}\bar{B}^{kl}_{ab})\,, \qquad (6.4.15)$$

$$0 = \partial^a(\chi^i_{ab} - \varepsilon^{ij}\not\partial\psi_{abj})\,, \qquad (6.4.16)$$

$$0 = \partial^a(C^-_{ab} - \Box A^+_{ab})\,, \qquad (6.4.17)$$

$$0 = \partial^a\partial_c(F_{ab}{}^{cd} + \bar{F}_{ab}{}^{cd})\,. \qquad (6.4.18)$$

These are the analogues of the constraints (5.4) in [163]. This set contains $(9+6+9)+24$ equations. The constraint for $F_{ab}{}^{cd}$ splits up in a part symmetric in (bd) (6 independent equations) and an antisymmetric part in $[bd]$ (3 independent equations), which correspond to the real and imaginary part of F_{ac}:

$$
\begin{array}{rcl}
0 &=& -\partial^c\left(\partial_{(b}(F_{d)c} + \bar{F}_{d)c})\right) + \frac{1}{2}\delta_{bd}\partial^a\partial^c(F_{ac}+\bar{F}_{ac}) + \frac{1}{2}\Box(F_{bd}+\bar{F}_{bd})\\
&& +\frac{1}{2}\varepsilon_{bdae}\partial^a\partial^c(F^e{}_c - \bar{F}^e{}_c)\,.
\end{array}
\qquad (6.4.19)
$$

As far as we know, this reduced multiplet is a new representation of the rigid $N=2$ algebra.

An explicit supergravity realization of this reduced multiplet is given by

$$
\begin{array}{rcl}
A^+_{ab} &=& \langle V, \tilde{\mathcal{F}}^+_{ab}\rangle\,,\\
\psi_{abi} &\approx& -\mathrm{i}\varepsilon_{ij}\gamma^\rho\sigma_{ab}\phi^j_\rho\,,\\
B_{abik} &\approx& 2\mathrm{i}\varepsilon_{ij}R(\mathrm{SU(2)})^+_{ab}{}^j{}_k\,,\\
F_{ab}{}^{cd} &\approx& 2\delta^{[c}_{[a}\Big(\partial_{b]}(\mathcal{Q}^{d]} - A^{d]}) + (\partial^{d]}(\mathcal{Q}_{b]} - A_{b]})) - 2\mathrm{i}\mathcal{R}_{b]}{}^{d]} + \frac{\mathrm{i}}{2}\delta^{d]}_{b]}\mathcal{R}\Big)\\
&& -\varepsilon^{cd}{}_{ef}\delta^{[e}_{[a}\Big(\partial_{b]}(\mathcal{Q}^{f]} - A^{f]}) + (\partial^{f]}(\mathcal{Q}_{b]} - A_{b]})) - 2\mathrm{i}\mathcal{R}_{b]}{}^{f]} + \frac{\mathrm{i}}{2}\delta^{f]}_{b]}\mathcal{R}\Big)\,.
\end{array}
\qquad (6.4.20)
$$

In deriving this multiplet we used the constraints of sections 6.3 and 6.4. The expression for B_{abij} satisfies constraint (6.4.15), which is a Bianchi identity that expresses the existence of $SU(2)$-vectors. The expression for $F_{ab}{}^{cd}$ fulfils (6.4.12). It also satisfies (6.4.18) when the third equation of (6.4.9) for $(\mathcal{Q}_\mu - A_\mu)$ is used. Therefore, the multiplet derived from $\langle V, \tilde{\mathcal{F}}^+_{ab}\rangle$ has 24 + 24 components.

Some comments on the multiplet of equations from $\langle V, \tilde{\mathcal{F}}^+_{ab}\rangle \approx 0$

Putting the ‘current’ multiplet (6.4.21) to 0, will give rise to some supergravity field equations. These are 24 + 24 equations for the 24 + 24 remaining degrees of freedom of table 6.4. The counting in this table subtracts the gauge degrees of freedom. The multiplet here is a multiplet of curvatures and the counting is equivalent if we take into account the Bianchi identities.

However, our equations are not equivalent to the complete supergravity equations of motion. They differ modulo ‘integration constants’. These can be determined when a second compensating multiplet is coupled [187, 197]. Since this step is independent of the symplectic formulation of the coupling of vector multiplets to supergravity, we do not treat it here.

Let us give a brief discussion of the content of the equations following from (6.4.10). Equation (6.4.10) reduces six degrees of freedom. It expresses the ‘graviphoton’ field strength T_{abij} as a combination of the $n+1$ on-shell vectors obtained above. It is the symplectic expression for the algebraic equation of motion that one finds in the Lagrangian approach, (4.11) in [39].

Using (6.4.2) in (6.2.2) with

$$R^{\mu i} \equiv e^{-1}\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\left(\mathcal{D}_\rho\psi^i_\sigma - \tfrac{1}{8}\sigma\cdot T^{ij}\gamma_\rho\psi_{\sigma j}\right) \tag{6.4.21}$$

in the second component of the current multiplet, gives that

$$\phi^i_\rho = R^i_\rho - \tfrac{1}{4}\gamma_\rho\gamma\cdot R^i_\rho \approx 0\,, \tag{6.4.22}$$

the traceless part of the field equation of the gravitini. Therefore, this equation cannot determine the trace-part $\gamma\cdot R^i$. However, combining (6.4.22) with the Bianchi identity for the gravitino field strength $\partial^\mu R^i_\mu \approx 0$, yields

$$\not{\partial}\gamma\cdot R^i \approx 0\,, \tag{6.4.23}$$

which determines $\gamma\cdot R^i$ in terms of eight ‘integration constants’.

The B_{abij} component yields

$$R(\mathrm{SU}(2))_{ab}{}^i{}_j \approx 0\,. \tag{6.4.24}$$

Together with the Bianchi identity for the $SU(2)$ curvature it states that the gauge fields $\mathcal{V}_\mu{}^i{}_j$ are pure gauge, i.e.,

$$\mathcal{V}_\mu{}^i{}_j = (\varphi^{-1}\partial_\mu\varphi)^i{}_j\,, \tag{6.4.25}$$

where φ is a group element of $SU(2)$. The three local parameters defining φ are left undetermined.

$F_{ab}{}^{cd}$ has its components in the traceless part of $F_{(ac)}$. From $F_{ab}{}^{cd} \approx 0$ follows

$$F_{ac} = 2\partial_{(a}(\mathcal{Q}_{c)} - A_{c)}) - 2\mathrm{i}\mathcal{R}_{ac} + \tfrac{\mathrm{i}}{2} g_{ac}\mathcal{R} \approx 0\,. \qquad (6.4.26)$$

The imaginary part is the traceless part of the Einstein equation. Again we cannot determine the scalar curvature $\mathcal{R}$ from this equation. However, combining this equation with the Bianchi identity for the Einstein tensor

$$\partial^a(\mathcal{R}_{ab} - \tfrac{1}{2} g_{ab}\mathcal{R}) = 0\,, \qquad (6.4.27)$$

gives

$$\partial^a \mathcal{R} \approx 0 \qquad (6.4.28)$$

and again $\mathcal{R}$ is determined up to a constant. The real part of the F-component gives that $A_\mu \approx \mathcal{Q}_\mu$ up to a constant vector. Also in the Lagrangian approach [39], one finds

$$A_\mu \approx \mathcal{Q}_\mu\,. \qquad (6.4.29)$$

The additional 8+8 remaining unknowns can be determined through the field equations of a second compensating multiplet. This concludes the short discussion of the supergravity equations of motion.

6.4.3 The field equations for the special case

In this subsection, the expressions of section 6.4.1 are generalized for the case $n = 1$ where further $Z_z = \langle V, U_z\rangle \neq 0$. This is the case that was excluded by the former definitions and where our less restrictive definitions becomes relevant. The equations are found by expanding the constraints in terms of the basis of symplectic vectors using appendix A.2.2.

The section $\tilde{Y}_{ij}$ remains totally constrained:

$$\begin{aligned}\tilde{Y}_{ij} \;=\; & g'^{z\bar{z}}\Big(-\mathrm{i}\varepsilon_{ik}\varepsilon_{jl} g_{z\bar{z}} \mathcal{D}_{\bar{z}}\bar{Z}_{\bar{z}} \cdot \bar{\lambda}^{k\bar{z}}\lambda^{l\bar{z}} V + \mathrm{i} g_{z\bar{z}}\mathcal{D}_z Z_z \cdot \bar{\lambda}^z_i \lambda^z_j \bar{V} \\ & + \mathrm{i}\varepsilon_{ik}\varepsilon_{jl}\bar{C}_{\bar{z}\bar{z}\bar{z}}\bar{\lambda}^{k\bar{z}}\lambda^{l\bar{z}} U_z - \mathrm{i} C_{zzz}\bar{\lambda}^z_i\lambda^z_j \bar{U}_{\bar{z}}\Big)\,. \end{aligned} \qquad (6.4.30)$$

This equation reduces to the former equation (6.4.1) when $\langle V, U_z\rangle = 0$.

The equations that can be derived from the constraints for the fermions are the following ones:

$$\begin{aligned}
\langle \bar V', \tilde\Lambda_i\rangle &= -\mathcal{D}_{\bar z}\bar Z_{\bar z}\cdot \varepsilon_{ij}\not\nabla \bar z\cdot\lambda^{j\bar z}\,,\\
\langle \bar U'_{\bar z}, \Lambda_i\rangle &= -\varepsilon_{ij}\bar C_{\bar z\bar z\bar z}\not\nabla\bar z\cdot\lambda^{j\bar z}+\tfrac12\gamma^\mu\varepsilon_{ij}\left(\langle \bar U'_{\bar z},\tilde Y^{jk}+\sigma\cdot\tilde{\mathcal F}^+\varepsilon^{jk}\rangle\right)\psi_{\mu k}\,,\\
0 &= \chi^i-\tfrac23\sigma^{\mu\nu}(\mathcal{D}_\mu\psi^i_\nu-\tfrac18\sigma\cdot T^{ij}\gamma_\mu\psi_{\nu j})-\tfrac12\left(g_{z\bar z}\not\partial z\cdot\lambda^{i\bar z}-\mathrm{i}\gamma^\mu(\not{\mathcal Q}-\not A)\right)\psi^i_\mu\\
&\quad -g'^{z\bar z}\Big[\tfrac12 Z_z\bar C_{\bar z\bar z\bar z}\not\partial\bar z\cdot\lambda^{i\bar z}+\tfrac{\mathrm i}{4}\gamma^\mu(g_{z\bar z}\langle V',\tilde Y^{ij}+\sigma\cdot\tilde{\mathcal F}^+\varepsilon^{ij}\rangle\psi_{\mu j}\\
&\quad -\tfrac14\varepsilon^{ij}\mathcal{D}_z Z_z\cdot(\langle\bar U_{\bar z},\tilde Y_{jk}\rangle\varepsilon^{kl}\lambda^z_l+\langle\bar U_{\bar z},\sigma\cdot\tilde{\mathcal F}^-\rangle\lambda^z_j)\\
&\quad -\tfrac{\mathrm i}{12}\varepsilon^{ij}g_{z\bar z}(C_{zzz}-\mathcal{D}_z\mathcal{D}_z Z_z)\,\varepsilon^{kl}(\bar\lambda^z_k\sigma_{ab}\lambda^z_l)\sigma^{ab}\lambda^z_j\Big]\ ,\\
0 &= \mathrm{i}g'_{z\bar z}\left(\not\nabla\lambda^{i\bar z}+\tfrac{\mathrm i}{2}(\not{\mathcal Q}-\not A)\,\lambda^{i\bar z}\right)\\
&\quad +\tfrac{\mathrm i}{2}\varepsilon^{ij}C_{zzz}g^{z\bar z}\left(\langle\bar U_{\bar z},\tilde Y_{jk}\rangle\varepsilon^{kl}\lambda^z_l+\langle\bar U_{\bar z},\sigma\cdot\tilde{\mathcal F}^-\rangle\lambda^z_j\right)\\
&\quad +\tfrac16\mathcal{D}_zC_{zzz}\cdot\varepsilon^{ij}\varepsilon^{kl}(\bar\lambda^z_k\sigma^{ab}\lambda^z_l)\sigma_{ab}\lambda^z_j-\mathrm{i}Z_z\mathcal{D}_{\bar z}\bar Z_{\bar z}\cdot\not\nabla\bar z\cdot\lambda^{i\bar z}\,. \qquad (6.4.31)
\end{aligned}$$

Also here, the equations reduce to those that we have found for the generic case where $\langle V, U_z\rangle = 0$. The same comments as in section 6.4.1 are valid here.

Repeating the analysis for the equations of the vectors, it appears that there is no information used about Z_z. This means that the analysis of the equations for the vectors of section 6.4.1 remains valid. This is no surprise because the equations for the vectors are a symplectic section of equations. All the other equations are singlets for the symplectic group and can therefore be written as symplectic invariant equations.

Also the last constraint can be decomposed with respect to the symplectic basis. Then the equations become:

$$\begin{aligned}
\langle\bar V',\tilde C\rangle &= -2\bar\psi^\mu_i\langle\bar V,\tilde Y^{ij}+\sigma\cdot\tilde{\mathcal F}^+\varepsilon^{ij}\rangle\psi_{\mu j}-2\partial_\mu\bar z\cdot\mathcal{D}_{\bar z}\bar Z_{\bar z}\cdot\left(\nabla^\mu\bar z-\bar\psi^\mu_i\lambda^{i\bar z}\right)\\
&\quad +\tfrac14\mathcal{D}_{\bar z}\bar Z_{\bar z}\bar\lambda^{i\bar z}\sigma\cdot T_{ij}\lambda^{j\bar z}\,,\\
\langle\bar U'_{\bar z},\tilde C\rangle &= -2\bar C_{\bar z\bar z\bar z}\nabla_\mu\bar z\cdot\nabla^\mu\bar z+\tfrac14\bar C_{\bar z\bar z\bar z}(\bar\lambda^{k\bar z}\sigma\cdot T_{kl}\lambda^{l\bar z})\,,\\
0 &= g^{z\bar z}g'_{z\bar z}\Big(2e^{-1}\partial_\mu\left(e(\mathcal Q^\mu-A^\mu)\right)+2\mathrm{i}(\mathcal Q^\mu-A^\mu)(\mathcal Q_\mu-A_\mu)-2\mathrm{i}f_\mu{}^\mu\\
&\quad +3\mathrm{i}\bar\psi^i_\mu\gamma^\mu\chi_i-2\mathrm{i}g_{z\bar z}\partial_\mu z\cdot(\nabla^\mu\bar z-\bar\psi^\mu_i\lambda^{i\bar z})\\
&\quad +4\bar\psi^{i[\mu}\gamma_\mu\psi^{\nu]}_i(\mathcal Q_\nu-A_\nu)+2\bar\psi^\mu_i(\not{\mathcal Q}-\not A)\psi^i_\mu-2\mathrm{i}\bar\psi^\mu_i\phi^i_\mu\Big)\\
&\quad +\bar\psi^\mu_i\psi_{\mu j}\varepsilon^{ik}\varepsilon^{jl}\mathcal{D}_zZ_z\cdot\bar\lambda^z_k\lambda^z_l-\bar\psi^\mu_i\langle V',\sigma\cdot\tilde{\mathcal F}^+\rangle\varepsilon^{ij}\psi_{\mu j}
\end{aligned}$$

$$
\begin{aligned}
& -\tfrac{1}{4}g^{z\bar z}g_{z\bar z}\langle V',\tilde{\mathcal F}^+_{\mu\nu}\rangle T^{\mu\nu}_{ij}\varepsilon^{ij}\\
& +2\mathrm{i}g^{z\bar z}Z_z\left(\mathrm{i}\mathcal D_{\bar z}\bar Z_{\bar z}\cdot\partial_\mu\bar z\cdot(\mathcal Q_\mu-A_\mu)+\bar\psi^\mu_i\bar C_{\bar z\bar z\bar z}\partial_\mu\bar z\cdot\lambda^{i\bar z}\right)\\
& +\tfrac{1}{2}g^{z\bar z}\mathcal D_zZ_z\cdot g^{z\bar z}\left(\tfrac{1}{2}\varepsilon^{ik}\varepsilon^{jl}\langle\bar U_{\bar z},\tilde Y_{ij}\rangle\langle\bar U_{\bar z},\tilde Y_{kl}\rangle-\langle\bar U_{\bar z},\tilde{\mathcal F}^-_{ab}\rangle\langle\bar U_{\bar z},\tilde{\mathcal F}^{-ab}\rangle\right)\\
& -\tfrac{\mathrm{i}}{2}\varepsilon^{ik}\varepsilon^{jl}\left(\mathcal D_zZ_z\cdot\langle\bar V,\tilde Y^{ij}\rangle+(-C_{zzz}+\mathcal D_z\mathcal D_zZ_z)\,g^{z\bar z}\langle\bar U_{\bar z},\tilde Y_{ij}\rangle\right)\bar\lambda^z_k\lambda^z_l\\
& -2\mathrm{i}g^{z\bar z}\mathcal D_zZ_z\cdot\varepsilon^{ij}\bar\lambda^z_i\langle\bar U_{\bar z},\tilde\Lambda_j\rangle\\
& -\tfrac{1}{12}\left(\mathcal D_z\mathcal D_z\mathcal D_zZ_z-2\mathcal D_zC_{zzz}\right)\varepsilon^{ij}(\bar\lambda^z_i\sigma^{ab}\lambda^z_j)\,\varepsilon^{kl}(\bar\lambda^z_k\sigma_{ab}\lambda^z_l)\\
& -\tfrac{\mathrm{i}}{2}\mathcal D_zZ_z\cdot\langle\bar V,\tilde{\mathcal F}^-_{ab}\rangle\varepsilon^{kl}(\bar\lambda^z_k\sigma^{ab}\lambda^z_l)\\
& -\tfrac{\mathrm{i}}{2}(-C_{zzz}+\mathcal D_z\mathcal D_zZ_z)\,g^{z\bar z}\langle\bar U_{\bar z},\tilde{\mathcal F}^-_{ab}\rangle\varepsilon^{kl}(\bar\lambda^z_k\sigma^{ab}\lambda^z_l)\,,\\
0 \;=\;& -2\mathrm{i}g'_{z\bar z}\Big[e^{-1}\partial_\mu(e\nabla^\mu\bar z)+2\mathrm{i}(\mathcal Q_\mu-A_\mu)\nabla^\mu\bar z\\
& +\tfrac{\mathrm{i}}{2}(\mathcal Q_\mu-A_\mu)(\bar\psi^\mu_i\lambda^{i\bar z})+2\bar\psi^{i[\mu}\gamma_\mu\psi^{\nu]}_i\nabla_\nu\bar z\\
& +3\bar\chi_i\lambda^{i\bar z}-\tfrac{1}{2}\bar\lambda^{i\bar z}\gamma_\mu\phi^\mu_i+\Gamma^{\bar z}_{\bar z\bar z}\partial_\mu\bar z\cdot\nabla^\mu\bar z+\tfrac{1}{8}\bar\psi^i_\mu\gamma^\mu\sigma\cdot T_{ij}\lambda^{j\bar z}\\
& -\bar\psi^\mu_i\left(\nabla_\mu\lambda^{i\bar z}+\tfrac{\mathrm{i}}{2}(\mathcal Q_\mu-A_\mu)\lambda^{i\bar z}+\tfrac{\mathrm{i}}{2}g^{z\bar z}\langle U_z,\tilde Y^{ij}+\sigma\cdot\tilde{\mathcal F}^+\varepsilon^{ij}\rangle\psi_{\mu j}\right)\Big]\\
& +2\mathrm{i}Z_z\left(\partial_\mu\bar z\cdot\mathcal D_{\bar z}\bar Z_{\bar z}(\nabla^\mu\bar z-\bar\psi^\mu_i\lambda^{i\bar z})+\bar\psi^\mu_i\langle\bar V,\tilde Y^{ij}+\sigma\cdot\tilde{\mathcal F}^+\varepsilon^{ij}\rangle\psi_{\mu j}\right)\\
& +\tfrac{1}{4}C_{zzz}g^{z\bar z}g^{z\bar z}\varepsilon^{ik}\varepsilon^{jl}\langle\bar U_{\bar z},\tilde Y_{ij}\rangle\langle\bar U_{\bar z},\tilde Y_{kl}\rangle\\
& -\tfrac{1}{2}C_{zzz}g^{z\bar z}g^{z\bar z}\langle\bar U_{\bar z},\tilde{\mathcal F}^-_{ab}\rangle\langle\bar U_{\bar z},\tilde{\mathcal F}^{-ab}\rangle+\tfrac{1}{4}\langle U'_z,\tilde{\mathcal F}^+_{\mu\nu}\rangle T^{\mu\nu}_{ij}\varepsilon^{ij}\\
& -\tfrac{\mathrm{i}}{2}\varepsilon^{ik}\varepsilon^{jl}\left(C_{zzz}\langle\bar V,\tilde Y_{ij}\rangle+\mathcal D_zC_{zzz}\cdot g^{z\bar z}\langle\bar U_{\bar z},\tilde Y_{ij}\rangle\right)\bar\lambda^z_k\lambda^z_l\\
& +\tfrac{\mathrm{i}}{2}\left(C_{zzz}\langle\bar V,\tilde{\mathcal F}^-_{ab}\rangle+\mathcal D_zC_{zzz}\cdot g^{z\bar z}\langle\bar U_{\bar z},\tilde{\mathcal F}^-_{ab}\rangle\right)\varepsilon^{ij}(\bar\lambda^z_i\sigma^{ab}\lambda^z_j)\\
& +\tfrac{1}{12}\left(\mathcal D_z\mathcal D_zC_{zzz}+\tfrac{1}{2}\mathcal O_{zzzzz}\right)\varepsilon^{ij}(\bar\lambda^z_i\sigma_{ab}\lambda^z_j)\varepsilon^{kl}(\bar\lambda^z_k\sigma_{ab}\lambda^z_l)\\
& -2\mathrm{i}C_{zzz}g^{z\bar z}\varepsilon^{ij}\bar\lambda^z_i\langle\bar U_{\bar z},\tilde\Lambda_j\rangle\,.
\end{aligned}
\tag{6.4.32}
$$

The metric in front of the kinetic term of the scalar in the fourth equation is positive because of the physical condition (6.3.2). Again, all these equations reduce to the equations of section 6.4.1 if $Z_z=0$ and the same conclusions can be drawn as in section 6.4.1. For this purpose, we conclude at this point that the ‘special case’ is a valid alternative for a theory with $N=2$ supergravity and one vector multiplet.

A concrete example

This is the simplest example of a very special Kähler manifold. We start from the case of a prepotential

$$F = \frac{(X^1)^3}{X^0}\,, \tag{6.4.33}$$

but we add a parameter a in the section and check its consequences. This is also very reminiscent to the case of appendix C.2 of [8], but here we take another prepotential, that is less trivial and should give non-vanishing C_{zzz}. This gives the following results:

$$V = \begin{pmatrix} 1 \\ az \\ -z^3 \\ 3z^2 \end{pmatrix} e^{K/2}\,. \tag{6.4.34}$$

Imposing $\langle V, \bar{V}\rangle = i$, one finds

$$K(z,\bar{z}) = -\log(i(\bar{z}-z)(z^2 + (1-3a)z\bar{z} + \bar{z}^2))\,. \tag{6.4.35}$$

From this Kähler potential, the Kähler metric can be derived. Straightforward calculation gives that

$$\langle V, U_z\rangle = \frac{3i(a-1)z^2}{z^3 - 3az^2\bar{z} + 3az\bar{z}^2 - \bar{z}^3}\,. \tag{6.4.36}$$

The new metric,

$$g'_{z\bar{z}} = g_{z\bar{z}} - \langle V, U_z\rangle\langle \bar{V}, \bar{U}_{\bar{z}}\rangle\,, \tag{6.4.37}$$

becomes for this example

$$g'_{z\bar{z}} = \frac{-3a(z-\bar{z})^2}{(z^2 + (1-3a)z\bar{z} + \bar{z}^2)^2} \tag{6.4.38}$$

and it has a well-defined positivity domain. Further calculation gives that

$$\langle V, \mathcal{D}_z U_z\rangle = \frac{-6\mathrm{i}(a-1)az(z-\bar{z})^2(z+\bar{z})}{(z^2 + (1-3a)z\bar{z} + \bar{z}^2)(a(z-\bar{z})^4 + 3(1-a)^2 z^2\bar{z}^2)}\,. \tag{6.4.39}$$

One also finds that

$$C_{zzz} = \frac{6\mathrm{i}a(z-\bar{z})}{(z^2+(1-3a)z\bar{z}+\bar{z}^2)^2}\,. \tag{6.4.40}$$

For $a=1$, this model obeys the 'strong definition' of special Kähler geometry: $\langle V, U_z\rangle = 0$.

It is also possible to find other 'special' models, starting from the prepotential

$$F = \frac{(X^1)^n}{(X^0)^{n-2}} \tag{6.4.41}$$

and introducing again a parameter a in the section in the similar way as in the previous example. These are also examples of models where $\langle V, U_\alpha\rangle = 0$ is not satisfied, but the formulas look less attractive than the ones of the model worked out here.

(No) connection to Calabi–Yau compactifications

We clarify why it is not possible to construct models with $Z_z \neq 0$ from Calabi–Yau compactifications of type II string theory. It is possible to find Calabi–Yau compactifications with one vector multiplet. This is a Calabi–Yau compactification of type IIA string theory on a manifold with one Kähler modulus or of type IIB with one complex structure modulus. An example of this is the quintic [199] for IIA or its mirror symmetric Calabi–Yau in a IIB compactification.

A Calabi–Yau n-fold is a compact n complex dimensional Kähler manifold with vanishing first Chern class. Good reviews can be found in [180, 123, 179]. The Hodge numbers of a Calabi–Yau threefold are of a very restricted form:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{6.4.42}$$

Yau proved Calabi's theorem which states that a given Kähler manifold with associated Kähler form J_0 has a unique Ricci-flat Kähler metric whose Kähler form J is in the same cohomology class as J_0. The possible deformations of this Ricci-flat metric into another Ricci-flat metric form two distinct sets: the variations of the complex structure are associated to $H^{2,1}(X)$ with dimension $h^{2,1}$ and the moduli space of Kähler deformations with dimension $h^{1,1}$.

The scalars of the vector multiplets span the moduli space $\mathcal{M}_V$. The compactification of IIA string theory on a Calabi–Yau threefold X gives rise to $\mathcal{M}_V$ which is spanned by the deformation of the complexified Kähler form and has complex dimension $h^{1,1}(X)$. For type IIB compactified on a Calabi–Yau threefold Y we have that $\mathcal{M}_V$ is spanned by the deformation of the complex structure of Y and has complex dimension $h^{2,1}(Y)$.

In [185, 179], the connection is made between elements of special Kähler geometry and the period matrix of integrals of $(3,0),(2,1),(1,2)$ and $(0,3)$ forms over 3-cycles. There are $2(h^{2,1}+1)$ homologically different 3-cycles in the Calabi–Yau manifold. One chooses a canonical basis $\{A^I, B_I\}$ with intersection numbers

$$A^I \cap A^J = 0\,, \qquad B_I \cap B_J = 0\,, \qquad A^I \cap B_J = -B_J \cap A^I = \delta^I_J\,. \tag{6.4.43}$$

Symplectic rotations, corresponding to changes of this canonical homology basis, leave the intersection numbers invariant. The following identification can be made:

$$V = e^{K/2} \int \Omega^{(3,0)}\,, \qquad \text{or} \qquad v = \begin{pmatrix} \int_{A^I} \Omega \\ \int_{B_I} \Omega \end{pmatrix}. \tag{6.4.44}$$

where $\Omega^{(3,0)}$ is the unique holomorphic 3-form and $V = e^{K/2} v$. The small variations of a (p,q) form can give at most $(p \mp 1, q \pm 1)$-forms. Applied to $\Omega^{(3,0)}$, this leads to

$$\partial_\alpha \Omega = \Omega_\alpha - k_\alpha \Omega\,, \tag{6.4.45}$$

where Ω_α are $(2,1)$-forms. Integrals of 3-forms over 3-cycles on the Calabi–Yau are defined as follows:

$$\sum_I \left[\int_{A^I} \omega \int_{B_I} \chi - \int_{B_I} \omega \int_{A^I} \chi \right] = \int\!\!\int \omega \wedge \chi\,. \tag{6.4.46}$$

This allows to connect the symplectic product of symplectic vectors, formed by integrals over 3-cycles of 3-forms, with integrals of the exterior product of forms over the Calabi–Yau:

$$-\mathrm{i}\langle v, \bar{v}\rangle = -\mathrm{i} \int_{CY} \Omega \wedge \bar{\Omega} > 0\,. \tag{6.4.47}$$

The integral formula (6.4.46) implies that only integrals of $(p,3-p)$ with $(3-p,p)$ forms can be non-vanishing. The following identifications can be obtained

$$U_\alpha = e^{K/2}\int \Omega^{(2,1)}_\alpha \; ; \; \bar{U}_{\bar\alpha} = e^{K/2}\int \Omega^{(1,2)}_{\bar\alpha} \; ; \; \bar{V} = e^{K/2}\int \Omega^{(0,3)} \, . \tag{6.4.48}$$

From this identification immediately follows that for Calabi–Yau manifolds

$$\langle V, U_\alpha\rangle = \langle U_\alpha, U_\beta\rangle = 0 \, . \tag{6.4.49}$$

This implies that the special case, developed in this section, can *never* be found from a type II Calabi–Yau compactification, since always $\langle V, U_z\rangle = 0$. It is not excluded that there be other compactifications which keep $N=2$ in four dimensions that allow these models, but then there will not be such a nice geometrical interpretation as in the case of a Calabi–Yau compactification. References and comments on these non-geometrical compactifications (e.g., Gepner or Landau-Ginzburg models) are in chapter 19 of the second volume of [2].

6.5 Perturbative duality transformations and non-Abelian gauge symmetry

In this section we will try to find out which actions for Abelian vector multiplets allow for non-Abelian gaugings. We will first situate the problem and clarify the role of perturbative symplectic transformations in it. Then we will explain how a part of the problem can be analyzed in terms of algebraic cohomology. We will end with the extension of the action that is needed and give some concrete examples that illustrate the possible (non-)existence of a non-Abelian gauging.

6.5.1 Introduction

The action for an arbitrary number of vector multiplets coupled to supergravity was constructed in [39] using a prepotential F. In most circumstances, models with Abelian vector multiplets are studied as they arise in the low-energy effective descriptions of compactifications of string theories. These models with Abelian vector multiplets are endowed with a symplectic structure, as revealed in section 6.4. In section 6.1.1 was explained already that the action for Abelian vector multiplets is invariant under the classical subgroup of the symplectic group. These are matrices of the form

$$S_{\rm cl} = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} . \tag{6.5.1}$$

Further, there was explained that the perturbative symplectic matrices, matrices of the form

$$S_{\rm pert} = \begin{pmatrix} A & 0 \\ C & (A^T)^{-1} \end{pmatrix} , \tag{6.5.2}$$

leave invariant the action up to a total derivative.

In this section we want to clarify some properties of models with *non-Abelian gauge symmetry*. We want to study under which conditions the action for n Abelian vector multiplets can be made invariant under a local non-Abelian gauge group G with dim $G \leq n$. We will explain how these non-Abelian gaugings give rise to very specific perturbative symplectic transformations that leave the action invariant.

Introducing a non-Abelian Lie algebra implies that all the fields of the multiplet take values in it. The gauge transformations also act on the scalars X^I of the vector multiplets:

$$\delta_G X^I = g f^I_{JK} \Lambda^J X^K . \tag{6.5.3}$$

In general, the introduction of an arbitrary non-Abelian gauge symmetry implies that the action needs extra terms to remain invariant under the non-Abelian gauge symmetry. These extra terms imply sometimes that the symplectic structure of the Abelian model is not appropriate any more.

Here we would like to tackle the following problems: which non-Abelian gaugings are possible such that the action is/can be made gauge invariant and is there still a role for (a subset of) the symplectic transformations in this? These questions were answered partially in [39] in a context were the prepotential was the backbone of the construction. In section 6.4, we have worked out the most general case of special Kähler geometry by starting from the symplectic structure. In this approach the prepotential is a derived concept. Here we will try to reformulate the problem of the possible gaugings of Abelian vector multiplets as much as possible in a symplectic context.

The bosonic part of the Lagrangian for Abelian vector multiplets can be written as

$$\mathcal{L} = \tfrac{1}{4} (\mathrm{Im}\,\mathcal{N}_{IJ})\, \mathcal{F}^I_{\mu\nu} \mathcal{F}^{\mu\nu J} - \tfrac{\mathrm{i}}{8} (\mathrm{Re}\,\mathcal{N}_{IJ})\, \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}^I_{\mu\nu} \mathcal{F}^J_{\rho\sigma} , \tag{6.5.4}$$

where I, J counts the number of vector multiplets and $\mathcal{N}_{IJ}$ depends on the scalars. This is only the kinetic part for the vectors of the $N = 2$ supergravity Lagrangian. The matrix $\mathcal{N}_{IJ}$ relates the scalars X^I with the F_I of the section V:

$$F_I = \mathcal{N}_{IJ} X^J \,. \tag{6.5.5}$$

If there is a prepotential, the following relations are valid:

$$\begin{aligned} \mathcal{N}_{IJ} &= \bar{F}_{IJ} + 2\mathrm{i} \frac{(\mathrm{Im} F_{IK})(\mathrm{Im} F_{JL}) X^K X^L}{(\mathrm{Im} F_{KL}) X^K X^L} \,, \\ F_I &= F_{IJ} X^J \,, \end{aligned} \tag{6.5.6}$$

where F_{IJ} is the second derivative of the prepotential with respect to the scalars X^I.

In a first step of the construction of non-Abelian actions in [39] was imposed that the prepotential F has to be gauge invariant. Since $F = \frac{1}{2} X^I F_I$, it follows that

$$\delta_G F_I = -g f_{KI}^J \Lambda^K F_J \,. \tag{6.5.7}$$

This implies that the non-Abelian gauge transformation of the symplectic vector of scalars can be written as a classical symplectic matrix:

$$\delta_G \begin{pmatrix} X^I \\ F_I \end{pmatrix} = g \Lambda^K \begin{pmatrix} f_{KJ}^I & 0 \\ 0 & -f_{KI}^J \end{pmatrix} \begin{pmatrix} X^J \\ F_J \end{pmatrix} . \tag{6.5.8}$$

So, the possible gauge transformations of the symplectic vector of scalars can be expressed as classical symplectic transformations with the structure constants as non-trivial entries. The condition that the prepotential has to be gauge invariant, which was used in the construction with a prepotential, is translated into:

$$\delta_G \mathcal{N}_{IJ} = 2 \mathcal{N}_{K(I} f_{J)L}^K \Lambda^L \,. \tag{6.5.9}$$

This expresses that the (scalar-dependent) kinetic matrix has to transform covariantly under gauge transformations. This condition does not rely on the existence of a prepotential. Using (6.5.5), the same classical symplectic transformations express the allowed gauge transformations.

In [39] was further argued that it is possible to relax the condition that the prepotential has to be gauge invariant. There was proven that this is possible for a prepotential that transforms under non-Abelian gauge transformations as

$$\begin{aligned}\delta_G F &= gF_J f^J_{KL}\Lambda^K X^L = gX^I F_{IJ} f^J_{KL}\Lambda^K X^L \\ &= \tfrac{g}{2} C_{I,JK}\Lambda^I X^J X^K\,,\end{aligned} \tag{6.5.10}$$

for real $C_{I,JK}$'s that are symmetric in the last two indices and satisfy

$$C_{(I,JK)} = 0\,. \tag{6.5.11}$$

This remains a strong condition on the prepotential: the gauge variation of a homogeneous function of the second degree has to be quadratic. This type of transformation still allows for the construction of a gauge-invariant action. If there is a prepotential, it is possible to prove that

$$C_{I,JK} = 2F_{L(J} f^L_{|I|K)} + F_{LJK} f^L_{IM} X^M\,, \tag{6.5.12}$$

by using that the gauge transformation of the prepotential can be written in two ways as in (6.5.10).

If one constructs the action for a prepotential of the form

$$F = A_{IJ} X^I X^J, \tag{6.5.13}$$

for a real symmetric matrix A_{IJ}, this reduces the action to a total derivative, as can easily be seen for the kinetic terms from (6.5.6). So, such quadratic terms do not contribute to the action. We will meet these quadratic terms later in the cohomological analysis of the concrete examples. The gauge variation of the prepotential, (6.5.10), is almost of the form (6.5.13). The spacetime-dependent gauge parameter Λ^I causes that this term will not be a total derivative in the variation of the action. In [39] is proven that it is possible to regain gauge invariance by introducing an additional term that depends on the constants $C_{I,JK}$.

Now we want to allow for a similar relaxed condition for the gauge transformation of the matrix $\mathcal{N}_{IJ}$ as was done for the prepotential. The advantage of imposing the condition on $\mathcal{N}$ is again that we do not require the existence of a prepotential. We will relax (6.5.9) to

$$\delta_G \mathcal{N}_{IJ} = (2\mathcal{N}_{K(I} f^K_{J)L} + C_{L,IJ})\Lambda^L\,. \tag{6.5.14}$$

The (a priori complex) constants $C_{I,JK}$ are symmetric in the last two indices as imposed by the symmetry of $\mathcal{N}$ and have to satisfy $C_{(I,JK)} = 0$. From (6.5.5) follows that

$$\delta_G F_I = g\Lambda^K (C_{K,IJ} X^J - f^J_{KI} F_J)\,. \tag{6.5.15}$$

Demanding gauge invariance of the symplectic condition $\langle V, \bar{V}\rangle = \mathrm{i}$, implies that the constants $C_{I,JK}$ have to be real. This means that these $C_{I,JK}$'s are exactly the same ones as in the case with a prepotential. So, one finds that this more general transformation of $\mathcal{N}$ leads to a gauge transformation of the symplectic section V that can be written in a symplectic form:

$$\delta_G \begin{pmatrix} X^I \\ F_I \end{pmatrix} = g\Lambda^K \begin{pmatrix} f^I_{KJ} & 0 \\ C_{K,IJ} & -f^J_{KI} \end{pmatrix} \begin{pmatrix} X^J \\ F_J \end{pmatrix} . \qquad (6.5.16)$$

This type of symplectic transformations was called a perturbative symplectic transformation in section 6.1.1. As explained there, they are called perturbative because they mix the electric field strengths among one another and give rise to new magnetic field strengths which are a linear combination of 'old' electric and magnetic field strengths. The invariance of the electric field strengths means that the elementary, electrically-charged objects remain the fundamental excitations in the theory. That is the reason for calling these transformations perturbative. These more general gauge transformations form symplectic transformations with the structure constants of the non-Abelian group on the diagonal and the constants $C_{I,JK}$ in the lower-diagonal entries. In the remainder of this section, we will study the additional term that gives rise to a non-Abelian action in a cohomological setting. We also give some concrete examples of possible gauge transformations that can be expressed as perturbative symplectic transformations.

6.5.2 $C_{I,JK}$ and algebraic cohomology

We have derived in the introduction of section 6.5 that more prepotentials allow for a non-Abelian gauge group if the gauge transformation of the prepotential or $\mathcal{N}$ transforms into specific terms with real constants $C_{I,JK}$, symmetric in the last two indices and obeying $C_{(I,JK)} = 0$. These constants $C_{I,JK}$'s can be formulated in a cohomological setup following [200]. To introduce the notion of algebraic cohomology, consider first a group manifold with right invariant one-forms e^I corresponding to the components of $dg \cdot g^{-1}$ in some basis labeled by $I\,(I = 1, \ldots, \mathrm{dim}G)$. The commutation relations of the non-Abelian group are equivalent to the Maurer–Cartan equations $de^I = f^I_{JK} e^J e^K$. Any n-form $C = C_{I_1\cdots I_n} e^{I_1} \cdots e^{I_n}$, antisymmetric in $I_1 \cdots I_n$, with constant coefficients then satisfies:

$$dC = (\mathcal{D}C)_{I_0 \ldots I_n}\, e^{I_0} \cdots e^{I_n} , \qquad (6.5.17)$$

with

$$(\mathcal{D}C)_{I_0 \ldots I_n} = \tfrac{n}{2} C_{J[I_2 \cdots I_n} f^J_{I_0 I_1]} . \qquad (6.5.18)$$

This defines an algebraic operation on the coefficients: $C_{I_1\ldots I_n}$ is transformed into $(\mathcal{D}C)$ with $n+1$ indices. As $d^2=0$, also $\mathcal{D}^2$ is nilpotent, which is easily verified using the Jacobi identities.

This gives rise to a notion of cohomology. This notion can further be refined for C's which transform under some non-trivial representation:

$$C(\theta,\xi)=C_{I_1\ldots I_n,\alpha}\theta^{I_1}\cdots\theta^{I_n}\xi^\alpha \tag{6.5.19}$$

for anticommuting θ^I and ξ^α, transforming under G as

$$\begin{aligned}\delta_G\theta^I &= g f^I_{JK}\Lambda^J\theta^K\,,\\ \delta_G\xi^\alpha &= (T_I)^\alpha{}_\beta\Lambda^I\xi^\beta\,,\end{aligned} \tag{6.5.20}$$

where the generators T_I satisfy $[T_I,T_J]=f^K_{IJ}T_K$. Define ∂^*, a generalization of the exterior derivative, as:

$$\partial^*\theta^I=\tfrac12 f^I_{JK}\theta^J\theta^K\,,\ \partial^*(\theta^I\theta^J)=\partial^*\theta^I\cdot\theta^J-\theta^I\partial^*\theta^J\,,\ \partial^*\xi^\alpha=(T_I)^\alpha{}_\beta\theta^I\xi^\beta\,. \tag{6.5.21}$$

This ∂^* induces an operation on the coefficients C:

$$\begin{aligned}\partial^*C(\theta,\xi) &= C_{I_1\ldots I_n,\alpha}\partial^*(\theta^{I_1}\cdots\theta^{I_n}\xi^a)\\ &= C_{I_1\ldots I_n,\alpha}\left(\tfrac{n}{2}f^{I_1}_{JK}\theta^J\theta^K\theta^{I_2}\cdots\theta^{I_n}\xi^\alpha+(-)^n\theta^{I_1}\cdots\theta^{I_n}(T_J)^\alpha{}_\beta\theta^J\xi^\beta\right)\\ &= (\mathcal{D}^*C)_{I_0\ldots I_n,\alpha}(\theta^{I_0}\cdots\theta^{I_n}\xi^a)\,,\end{aligned} \tag{6.5.22}$$

with

$$(\mathcal{D}^*C)_{I_0\ldots I_n,\alpha}=\tfrac{n}{2}C_{K[I_2\cdots I_n,\alpha}f^K_{I_0I_1]}+(T_{[I_0})^\beta{}_\alpha C_{I_1\cdots I_n],\beta}\,. \tag{6.5.23}$$

The Jacobi identity again implies that $(\partial^*)^2=0$, which leads to $(\mathcal{D}^*)^2=0$. This defines cohomology classes for the coefficients C. If C is $\mathcal{D}^*$-closed: $\mathcal{D}^*C=0$, there is an equivalence class of all $C'=C+\mathcal{D}^*\Lambda$. The trivial cohomology class contains $C=\mathcal{D}^*\Omega$.

For semisimple (not necessarily compact) groups, the Cartan–Killing metric $g_{IJ}=-f^L_{IK}f^K_{LJ}$ is invertible. This allows to introduce an operation $\mathcal{I}$ that lowers the number of Lie algebra indices:

$$(\mathcal{I}C)_{I_1\ldots I_n,\alpha}\equiv(n+1)C_{JI_1\ldots I_n,\beta}(T^J)^\beta{}_\alpha\,, \tag{6.5.24}$$

where $T^J=g^{JI}T_I$, which is only possible if the Cartan–Killing metric is invertible. One then finds that

$$(\mathcal{I}\mathcal{D}^*+\mathcal{D}^*\mathcal{I})C_{I_1\ldots I_n,\alpha}=C_{I_1\ldots I_n,\beta}C_2(T)^\beta{}_\alpha\,, \tag{6.5.25}$$

where the Casimir operator $C_2(T)^\beta{}_\alpha = g^{IJ}(T_I)^\beta{}_\gamma(T_J)^\gamma{}_\alpha$. This implies that any $\mathcal{D}^*$-closed C is $\mathcal{D}^*$-exact for a semisimple gauge group. So, for semisimple Lie algebras, there are *no* non-trivial cohomology classes taking their values in a non-trivial representation.

The $C_{I,JK}$'s in the symplectic transformations (6.5.16) fit in this context. The I of $C_{I,JK}$ corresponds to $n = 1$ in (6.5.22), while the JK of $C_{I,JK}$ can be considered as two copies of commuting u^J's transforming in the symmetric product of two copies of the Lie algebra. This means that the matrices T_I are given in terms of the structure constants. So, we find that equations (6.5.19), (6.5.22), and (6.5.23) become

$$\begin{aligned} C(\theta,\xi) &= C_{I,JK}\theta^I u^J u^K\,, \\ \partial^* C(\theta,\xi) &= C_{I,JK}\partial^*(\theta^I u^J u^K) \\ &\equiv (\mathcal{D}^*C)_{I_0 I,JK}(\theta^{I_0}\theta^I u^J u^K)\,, \\ (\mathcal{D}^*C)_{I_0 I,JK} &= \tfrac{1}{2}C_{L,JK}f^L_{I_0 I} - 2f^L_{J[I_0}C_{I],LK}\,. \end{aligned} \tag{6.5.26}$$

Generalizing the operation ∂^* to $\hat{\partial}^*$ where

$$\hat{\partial}^*\theta^I - \tfrac{1}{2}f^I_{JK}\theta^J\theta^K = \alpha u^I\,, \tag{6.5.27}$$

with α some constant, it is possible to derive cohomology classes as before if furthermore

$$C_{(I,JK)} = 0 \tag{6.5.28}$$

is imposed. Now it is possible to make the connection to non-Abelian vectors. The θ^I, u^J and the operator $\hat{\partial}^*$ can be realized as the gauge potential A^I, the field strength $\alpha^{-1}F^I$ and the exterior derivative d.

6.5.3 An extension of the action

The cohomological approach of the previous section allows the construction of an action that is invariant under non-Abelian gauge transformations for $\mathcal{N}_{IJ}$ obeying (6.5.14). The Jacobi identity and the closure of the algebra impose that $C_{I,JK}$ must be $\mathcal{D}^*$-closed:

$$(\mathcal{D}^*C)_{LI,JK} = \tfrac{1}{2}f^M_{LI}C_{M,JK} + C_{L,M(J}f^M_{K)I} - C_{I,M(J}f^M_{K)L} = 0\,. \tag{6.5.29}$$

Equation (6.5.28) and (6.5.29) were derived already in [39] in the context of a prepotential. Here, we have repeated the derivation which did not rely on the existence of a prepotential. In [39, 200] is derived for both the approaches that the action (6.5.4) becomes gauge invariant by adding the following term

$$S_{\text{new}} = \int d^4x \, C_{K,IJ} A^K A^I (dA^J - \tfrac{3}{8} f_{LM}^J A^L A^M) \,, \tag{6.5.30}$$

if the prepotential or $\mathcal{N}_{IJ}$ transforms as imposed by (6.5.10), respectivily (6.5.14). So, adding this term to the action allows for a non-Abelian extension of an Abelian action.

In [201], the same model was studied in the context of gauged non-linear σ-models. They proved that a simplification of (6.5.30) is possible by introducing yet another cohomology. Also the analysis in terms of Killing vectors and Killing prepotentials as in [191, 193] relies on the algebraic cohomology[8].

In [200] is argued that shifting $\mathrm{Re}\mathcal{N}_{IJ}$ by constants corresponds to shifting C's by an exact piece. This shifting can be done as follows. Start from a model with a prepotential F. Consider for simplicity that the gauge transformation of the section V giving rise to this prepotential, is of the form (6.5.8). Adding to this prepotential a term $(X^0)^2$ gives rise to

$$\begin{aligned} F' &= F + (X^0)^2 \,, \\ \delta_G F' &= \delta_G F + 2X^0 g f_{IJ}^0 \Lambda^I X^J \\ &= \delta_G F + 2g\tilde{C}_{I,0J} \Lambda^I X^0 X^J \,. \end{aligned} \tag{6.5.31}$$

As argued earlier, these quadratic terms with real constants in the prepotential do not influence the equations of motion. They only give rise to terms in the action that are total derivatives. The gauge transformation of this prepotential can be interpreted in two different ways. A first possibility is to say that V will change into a new section V':

$$V' = \begin{pmatrix} X^I \\ F_0 + 2X^0 \\ F_{I>0} \end{pmatrix} \tag{6.5.32}$$

and that the symplectic form of the gauge transformation remains the same

$$\delta_G V' = g\Lambda^K \begin{pmatrix} f_{KJ}^I & 0 \\ 0 & -f_{KI}^J \end{pmatrix} V' \,. \tag{6.5.33}$$

Another possibility is to say that the gauge transformation of the original section

[8] We thank Pietro Fré for a fruitful discussion on this subject.

V has changed into

$$\delta_G \begin{pmatrix} X^I \\ F_I \end{pmatrix} = g\Lambda^K \begin{pmatrix} f^I_{KJ} & 0 \\ \tilde{C}_{K,0J} & -f^J_{KI} \end{pmatrix} \begin{pmatrix} X^J \\ F_J \end{pmatrix} . \qquad (6.5.34)$$

So, adding quadratic terms to the prepotential is equivalent to shifts in the lower-diagonal terms in the gauge transformations if these gauge transformations are written as perturbative symplectic matrices. Both prepotentials (with or without $(X^0)^2$) are in the same cohomology class.

We will give some examples of prepotentials which allow or do not allow an action that is invariant under non-Abelian gauge transformations. The prepotential

$$F = (X^0)^2 + (X^1)^2 + (X^2)^2 \qquad (6.5.35)$$

allows the gauge group $SU(2)$, because it is gauge invariant.
On the other hand, the prepotential

$$F = X^0X^1 + X^1X^2 + X^2X^0 \qquad (6.5.36)$$

with the canonical section

$$V = \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^1 + X^2 \\ X^0 + X^2 \\ X^0 + X^1 \end{pmatrix} \qquad (6.5.37)$$

does not allow for gauge group $SU(2)$. The gauge group $SU(2)$ is also not possible for

$$F = \frac{X^1(X^2)^2}{2X^0} . \qquad (6.5.38)$$

The $SU(2)$ gauge transformation of this function is not a quadratic function. Yet, it is possible to make a model with this prepotential and another non-Abelian gauge group [39]. Choose a gauge group with three generators and commutation rules

$$\begin{aligned} [t_0, t_1] &= -2t_1 , \\ [t_0, t_2] &= t_2 , \\ [t_1, t_2] &= 0 . \end{aligned} \qquad (6.5.39)$$

This gives rise to the following non-vanishing $C_{I,JK}$'s:

$$C_{1,22} = 2, \qquad C_{2,12} = C_{2,21} = -1\,, \tag{6.5.40}$$

such that (6.5.28) is satisfied. This can be derived from (6.5.12). The gauge algebra in the example (6.5.39) is not semisimple: $[G,G] = G^{(1)}$ with $G^{(1)} = \{t_1, t_2\}$ in this case. Since $[t_1, t_2] = 0$, the derived series of the algebra ends with an empty set, such that the algebra is solvable. This means that the Cartan–Killing metric, which is used in the analysis of the cohomology classes, is not invertible. So, there are non-trivial cohomology classis. The $C_{I,JK}$ in (6.5.40) are in a non-trivial cohomology class of the algebraic cohomology and cannot be removed by adding quadratic terms to the prepotential. The prepotential (6.5.38) is an example of a very special Kähler manifold. These are characterized by a prepotential of the form

$$F = d_{ABC} \frac{X^A X^B X^C}{X^0}\,, \tag{6.5.41}$$

with the real constants d_{ABC} fully symmetric. These very special Kähler manifolds are related to $N = 2$ supergravity in five dimensions coupled to Abelian vector multiplets [202]. The question whether the extra terms in the action (6.5.30) are related to five-dimensional models (via the Chern-Simons term there or via models with very special real manifolds [203]) is still under consideration.

Another question that one can ask, is whether it is possible to do symplectic transformation of certain sections (or prepotentials) that still allow a gauge transformation of the form (6.5.16) after the symplectic transformation. This is certainly not a general property. Start from the prepotential (6.5.38) with the gauge group (6.5.39) and do the symplectic transformation

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.5.42}$$

This symplectic transformation rotates the section V into

$$V' = \begin{pmatrix} X^1 \\ X^2 \\ \frac{-X^1 (X^2)^2}{2(X^0)^2} \\ \frac{(X^2)^2}{2X^0} \\ \frac{X^1 X^2}{X^0} \\ -X^0 \end{pmatrix}. \tag{6.5.43}$$

This section can be expressed in new coordinates $\tilde{X}^I$such that the three first elements of the section are the coordinates and the following three elements of the section can be seen as the derivatives of the associated prepotential

$$\tilde{F} = \sqrt{\tilde{X}^0 \tilde{X}^2} \tilde{X}^1 \,. \tag{6.5.44}$$

It is clear that this prepotential will not allow the same gauge group as the original prepotential and its associated section. We also were not able to derive another gauge group that gave rise to an invariant prepotential or a gauge transformation of it proportional to quadratic terms. This is caused by the fact that the upper right part of the matrix S in (6.5.42) is not 0. It is a *non-perturbative symplectic transformation*. If one uses a perturbative symplectic transformation

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} . \tag{6.5.45}$$

The transformed section is

$$\begin{pmatrix} A^I{}_J X^J \\ C_{IJ} X^J + D_I{}^J F_J \end{pmatrix} . \tag{6.5.46}$$

The prepotential F transforms into

$$F + (A^T C)_{JK} X^J X^K \,, \tag{6.5.47}$$

which corresponds to a shifting of the C's in the perturbative symplectic gauge transformations of the original section (together with some linear combination of the structure constants if $A \neq \mathbb{1}$). This transformed model still allows for a gauge-invariant formulation in terms of a symplectic transformation, while this is in general not possible for non-perturbative symplectic transformations.

So, we can conclude that we were able to derive a condition for the possible gauging of a set of Abelian vector multiplets coupled to $N = 2$ supergravity. This condition does not rely on the existence of a prepotential. To achieve more general gaugings, an additional term has to be added to the action. The condition on these constants can be analyzed using algebraic cohomology. The gauge symmetry gives rise to a very specific subgroup of the symplectic group that remains a symmetry of the action. We gave some concrete examples and speculated on a possible connection to five dimensions.

Appendix A

Notations and Conventions

A.1 Notations and conventions in d dimensions

These notations and conventions are used in chapter 2. We work in spaces of arbitrary signature. We call t the number of time directions and s the number of space directions. The total number of dimensions is $d = t + s$. Indices are denoted by $a, b, \ldots$. (Anti)symmetrization is always done with weight one: $A_{[ab]} = \frac{1}{2!}\left(A_{ab} - A_{ba}\right)$ and $A_{(ab)} = \frac{1}{2!}\left(A_{ab} + A_{ba}\right)$ and the generalization of this for more indices. In a space with an arbitrary signature, the antisymmetric Levi–Civita tensor is chosen to be

$$\varepsilon_{1\ldots d} = 1 \,, \qquad \varepsilon^{1\ldots d} = (-)^t \,. \tag{A.1.1}$$

The contraction of the Levi–Civita tensors is

$$\varepsilon^{a_1 a_2 \cdots a_p b_1 \cdots b_{d-p}} \varepsilon_{a_1 a_2 \cdots a_p c_1 \cdots c_{d-p}} = (-)^t \, p! \, (d-p)! \, \delta^{b_1}_{[c_1} \cdots \delta^{b_{d-p}}_{c_{d-p}]} \,. \tag{A.1.2}$$

The metric signature for Minkowski space is chosen to be $(-+\cdots+)$. We choose

$$\varepsilon_{0\ldots(d-1)} = 1 \,, \qquad \varepsilon^{01\ldots(d-1)} = -1 \,. \tag{A.1.3}$$

A.2 Notations and conventions in 4 dimensions

These notations and conventions will be used in chapter 6. They differ slightly from the ones in the previous section to allow to make contact with the literature.

The Levi–Civita tensor with curved indices is defined as

$$\varepsilon^{\mu\nu\rho\sigma} = \sqrt{-g} e_a^\mu e_b^\nu e_c^\rho e_d^\sigma \varepsilon^{abcd} ; \qquad \varepsilon^{0123} = \mathrm{i} , \tag{A.2.1}$$

where the former implies that the latter is true for flat as well as for curved indices. (Anti)self-dual tensors F and G are introduced:

$$F^{\pm}_{ab} = \tfrac{1}{2}(F_{ab} \pm \tilde{F}_{ab}) \qquad \text{with} \qquad \tilde{F}_{ab} = \tfrac{1}{2}\varepsilon_{abcd} F^{cd} . \tag{A.2.2}$$

Since ε is imaginary, the following properties hold:

$$F^{+}_{ab} G^{-ab} = 0 , \qquad (F^{+}_{ab} G^{+ab})^{\dagger} = F^{-}_{ab} G^{-ab} , \tag{A.2.3}$$

where † means Hermitian conjugation.

A.2.1 γ-matrices and spinors

The gamma and sigma matrices are defined by

$$\gamma_a \gamma_b = \eta_{ab} + 2\sigma_{ab} , \qquad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 , \tag{A.2.4}$$

which implies that $\frac{1}{2}\epsilon^{abcd}\sigma_{cd} = -\gamma^5\sigma^{ab}$. The following realization generates the two-component formalism:

$$\gamma_0 = \begin{pmatrix} 0 & \mathrm{i}\mathbb{1}_2 \\ \mathrm{i}\mathbb{1}_2 & 0 \end{pmatrix} ; \qquad \gamma_\alpha = \begin{pmatrix} 0 & -\mathrm{i}\sigma_\alpha \\ \mathrm{i}\sigma_\alpha & 0 \end{pmatrix} ; \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{A.2.5}$$

where $\alpha = 1,2,3$. The matrices γ_α and γ_5 are Hermitian, while γ_0 is anti-Hermitian. The following identities are useful in the calculations:

$$\begin{aligned}
\gamma_\mu\gamma^\mu &= 4 , & \gamma^\nu\gamma_\mu\gamma_\nu &= -2\gamma_\mu , \\
\sigma_{\mu\nu} &= \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) , & \sigma_{\mu\nu} &= -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\sigma^{\lambda\sigma}\gamma_5 , \\
\sigma^{\mu\nu}\sigma_{\mu\nu} &= -3 , & \sigma^{\lambda\sigma}\sigma_{\mu\nu}\sigma_{\lambda\sigma} &= \sigma_{\mu\nu} , \\
\gamma_\mu\sigma^{\nu\rho}\gamma^\mu &= 0 , & \sigma^{\nu\rho}\gamma_\mu\sigma_{\nu\rho} &= 0 , \\
[\gamma^\lambda, \sigma_{\mu\nu}] &= 2\delta_{[\mu}{}^\lambda\gamma_{\nu]} , & \{\gamma^\lambda, \sigma_{\mu\nu}\} &= \varepsilon_{\mu\nu}{}^{\rho\sigma}\gamma_5\gamma_\sigma , \\
[\sigma_{\mu\nu}, \sigma^{\rho\sigma}] &= -4\delta_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]} , & \{\sigma_{\mu\nu}, \sigma^{\rho\sigma}\} &= -\delta_{[\mu}{}^\rho\delta_{\nu]}{}^\sigma + \tfrac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\gamma_5 .
\end{aligned} \tag{A.2.6}$$

There is a charge conjugation matrix $\mathcal{C}$ such that

$$\mathcal{C}^T = -\mathcal{C} ; \qquad \mathcal{C}\gamma_a\mathcal{C}^{-1} = -\gamma_a^T . \tag{A.2.7}$$

This fixes $\mathcal{C}$ up to a (complex) constant. One can fix the proportionality constant (up to a phase) by demanding $\mathcal{C}$ to be unitary, so that $\mathcal{C}^* = -\mathcal{C}^{-1}$. In the representation (A.2.5) we can choose

$$\mathcal{C} = \begin{pmatrix} \epsilon_{AB} & 0 \\ 0 & \epsilon^{\dot{A}\dot{B}} \end{pmatrix} , \tag{A.2.8}$$

where ϵ is the antisymmetric symbol with $\epsilon_{AB} = -\epsilon^{\dot{A}\dot{B}} = 1$ for $A = 1$, $B = 2$.

Majorana spinors satisfy the 'reality condition' which says that their 'Majorana conjugate' is equal to the 'Dirac conjugate':

$$\bar{\chi} \equiv \chi^T \mathcal{C} = -\mathrm{i}\chi^\dagger \gamma_0 \equiv \bar{\chi}^C \qquad \text{or} \qquad \chi^C \equiv -\mathrm{i}\gamma_0 \mathcal{C}^{-1}\chi^* = \chi \, . \tag{A.2.9}$$

The factor $-\mathrm{i}$ is just a conventional choice as the phase of $\mathcal{C}$ is arbitrary. Majorana spinors can thus be thought as spinors $\chi_1 + \mathrm{i}\chi_2$, where χ_1 and χ_2 have real components, but these are related by the above condition. There exists a Majorana representation where the matrices γ_μ are real, and with a convenient choice of the phase factor of $\mathcal{C}$ the Majorana spinors are just real. We often use Weyl spinors, where the left and right chiral spinors are defined as

$$\chi_L = \tfrac{1}{2}(1 + \gamma_5)\chi \; ; \qquad \chi_R = \tfrac{1}{2}(1 - \gamma_5)\chi \; . \tag{A.2.10}$$

The chirality is indicated by the position of the i, j index (index running over $1, 2$ for $N = 2$ supergravity). The choice of chirality for the spinor with an upper (lower) index can change for each spinor. Note that these Weyl spinors are not Majorana. A spinor which is Majorana and Weyl with the above definitions would only be possible in $d = 2$ mod 8. The following Fierz identities will also be used in the calculations:

$$\begin{aligned} \lambda_L \bar{\chi}_L &= -\tfrac{1}{2}\left(\bar{\chi}_L \lambda_L\right) \mathbb{1} + \tfrac{1}{2}\sigma^{ab}\left(\bar{\chi}_L \sigma_{ab} \lambda_L\right) , \\ \lambda_L \bar{\chi}_R &= -\tfrac{1}{2}\gamma^a \left(\bar{\chi}_R \gamma_a \lambda_L\right) . \end{aligned} \tag{A.2.11}$$

A.2.2 A basis for symplectic vectors

In this appendix we show that

$$\mathcal{W} = (V, U_\alpha, \bar{V}, \bar{U}_{\bar{\alpha}}) \tag{A.2.12}$$

is a basis for symplectic vectors. Since we are dealing with a $2(n+1)$ dimensional vector space we only have to show that these vectors are independent.
Proof: Suppose

$$\lambda^0 V + \lambda^{\bar{0}} \bar{V} + \lambda^\alpha U_\alpha + \lambda^{\bar{\alpha}} \bar{U}_{\bar{\alpha}} = 0 \, , \tag{A.2.13}$$

then it follows that all $\lambda^i = 0$ if and only if the determinant obtained by left symplectic inner products with, respectively, $\bar{V}$, V, $\bar{U}_{\bar{\beta}}$ and U_β, is non–zero:

$$\det \begin{pmatrix} -\mathrm{i} & 0 & 0 & \langle \bar{V}, \bar{U}_{\bar{\alpha}} \rangle \\ 0 & \mathrm{i} & \langle V, U_\alpha \rangle & 0 \\ 0 & \langle \bar{U}_{\bar{\beta}}, \bar{V} \rangle & \mathrm{i} g_{\alpha\bar{\beta}} & 0 \\ \langle U_\beta, V \rangle & 0 & 0 & -\mathrm{i} g_{\alpha\bar{\beta}} \end{pmatrix} \neq 0\,. \tag{A.2.14}$$

We can split this up in two cases:
1. <u>The generic case :</u>
Then $\langle V, U_\alpha \rangle = 0$, and (A.2.14) is

$$(\det g_{\alpha\bar{\beta}})^2 > 0\,, \tag{A.2.15}$$

which is satisfied by the metric.
2. <u>The special case :</u>
Then we define $Z_z = \langle V, U_z \rangle$ and the determinant equation leads to

$$(g_{z\bar{z}} - Z_z \bar{Z}_{\bar{z}})^2 \neq 0\,. \tag{A.2.16}$$

However this follows from the "physical" condition on the sections that leads to the right signs for the kinetic energy of the scalars and the vectors, cf. (6.3.2).

Now that we have a basis, we can expand every symplectic vector in this basis. Take a generic symplectic vector X_A, where the index A denotes a generic index. It is again useful to separate two cases.
1. <u>The generic case :</u>
This leads to

$$\begin{aligned} X_A &= \mathrm{i}\langle \bar{V}, X_A \rangle V - \mathrm{i}\langle V, X_A \rangle \bar{V} \\ &\quad + \mathrm{i} g^{\alpha\bar{\alpha}} \left(\langle U_\alpha, X_A \rangle \bar{U}_{\bar{\alpha}} - \langle \bar{U}_{\bar{\alpha}}, X_A \rangle U_\alpha \right)\,. \end{aligned} \tag{A.2.17}$$

2. <u>The special case :</u>
In the basis $\mathcal{W}$, the expansion becomes

$$\begin{aligned} X_A = -\mathrm{i} g'^{z\bar{z}} \Big(& (-g_{z\bar{z}} \langle \bar{V}, X_A \rangle + \mathrm{i}\bar{Z} \langle U_z, X_A \rangle) V \\ & + (g_{z\bar{z}} \langle V, X_A \rangle + \mathrm{i} Z \langle \bar{U}_{\bar{z}}, X_A \rangle) \bar{V} \\ & + (-\mathrm{i}\bar{Z} \langle V, X_A \rangle + \langle \bar{U}_{\bar{z}}, X_A \rangle) U_z \\ & - (\mathrm{i} Z \langle \bar{V}, X_A \rangle + \langle U_z, X_A \rangle) \bar{U}_{\bar{z}} \Big)\,. \end{aligned} \tag{A.2.18}$$

In this case we better use the basis

$$\mathcal{W}' = (V, U'_z, \bar{V}, \bar{U}'_{\bar{z}})\,. \tag{A.2.19}$$

The same formulas hold as above, when replacing $g_{\alpha\bar{\beta}}$ with $g'_{z\bar{z}}$.

A.3 Notations and Conventions in 6 dimensions

A.3.1 Γ-matrices and their properties

The Clifford algebra in six spacetime dimensions has dimension $2^{d/2} = 8$. The Γ-matrices satisfy the property

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = 2\eta_{ab}\,. \tag{A.3.1}$$

Since we will be using Weyl spinors, it is useful to work in a basis where

$$\gamma_7 = \Gamma^0 \cdots \Gamma^5 = -\Gamma_0 \cdots \Gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{A.3.2}$$

We denote a spinors λ^i as:

$$\lambda^i_{\hat{\alpha}} = \begin{pmatrix} \lambda^i_\alpha \\ \lambda^i_{\alpha'} \end{pmatrix}, \tag{A.3.3}$$

$$\lambda_L = \frac{(1+\gamma_7)}{2}\lambda_L = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}; \qquad \lambda_R = \frac{(1-\gamma_7)}{2}\lambda_R = \begin{pmatrix} 0 \\ \lambda_{\alpha'} \end{pmatrix}. \tag{A.3.4}$$

where the index α is chiral, α' is antichiral. Γ-matrices have the form

$$(\Gamma_a)_{\hat{\alpha}}{}^{\hat{\beta}} = \begin{pmatrix} 0 & (\gamma_a)_\alpha{}^{\beta'} \\ (\tilde{\gamma}_a)_{\alpha'}{}^{\beta} & 0 \end{pmatrix}. \tag{A.3.5}$$

We use matrices γ_a and $\tilde{\gamma}_a$ to indicate that they have different chirality indices.

The charge conjugation matrix $\mathcal{C}$ can be defined as:

$$\mathcal{C} = \begin{pmatrix} 0 & c \\ c^T & 0 \end{pmatrix} \tag{A.3.6}$$

or

$$\mathcal{C}^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & c^{\alpha\beta'} \\ c^{\alpha'\beta} & 0 \end{pmatrix} , \tag{A.3.7}$$

where c is the four-dimensional charge conjugation matrix:

$$c = \begin{pmatrix} \varepsilon_{ab} & 0 \\ 0 & \varepsilon^{a'b'} \end{pmatrix} , \tag{A.3.8}$$

where $\varepsilon_{12} = -\varepsilon^{12} = 1$. Using this definition, one finds that

$$\Gamma_a^T = -\mathcal{C}\,\Gamma_a\,\mathcal{C}^{-1} , \qquad \mathcal{C}\,\mathcal{C}^\dagger = \mathbb{1} , \qquad \mathcal{C}^T = \mathcal{C} . \tag{A.3.9}$$

This means that the upper signs of table 2.3 in six dimensions are valid. It is possible to express $\gamma^{(n)}$ in terms of $\gamma^{(6-n)}$ using the duality relation

$$\gamma^{a_1\cdots a_n} = \frac{S_n}{(6-n)!}\varepsilon^{a_1\cdots a_n b_1\cdots b_{6-n}}\gamma_{b_1\cdots b_{6-n}}\gamma_7 , \qquad S_n = \begin{cases} +1 \text{ for } n = 0,1,4,5 \\ -1 \text{ for } n = 2,3,6 \end{cases} . \tag{A.3.10}$$

This is an alternative form of (2.2.10). In the calculations with γ-matrices is often made use of the following identity for contractions of γ-matrices in products:

$$\gamma^{(n)}\gamma^{(m)}\gamma_{(n)} = n!C(n,m)\gamma^{(m)} . \tag{A.3.11}$$

The coefficients for $C(n,m)$ for $n, m \leq 3$ are given in table A.3.1.

m\n	0	1	2	3
0	1	6	−15	−20
1	1	−4	−5	0
2	2	1	1	4
3	1	0	3	0

Table A.1: The coefficients $C(n,m)$ defined in (A.3.11)

Products of γ-matrices are given by:

$$\gamma_{a_1\cdots a_n}\gamma^{b_1\cdots b_m} = \gamma_{a_1\cdots a_n}{}^{b_1\cdots b_m} + (-)^{n+1}\begin{pmatrix} n \\ 1 \end{pmatrix}\begin{pmatrix} m \\ 1 \end{pmatrix} 1!\,\delta^{[b_1}_{[a_1}\gamma_{a_2\cdots a_n]}{}^{b_2\cdots b_m]}$$

$$
\begin{aligned}
&+(-1)^{(2n+1)} \begin{pmatrix} n \\ 2 \end{pmatrix} \begin{pmatrix} m \\ 2 \end{pmatrix} 2!\, \delta^{[b_1}_{[a_1} \delta^{b_2}_{a_2} \gamma_{a_3 \cdots a_n]}{}^{b_3 \cdots b_m]} \\
&+(-1)^{n} \begin{pmatrix} n \\ 3 \end{pmatrix} \begin{pmatrix} m \\ 3 \end{pmatrix} 3!\, \delta^{[b_1}_{[a_1} \delta^{b_2}_{a_2} \delta_{a_3} \delta^{b_3} \gamma_{a_4 \cdots a_n]}{}^{b_4 \cdots b_m]} \\
&+\cdots
\end{aligned}
\tag{A.3.12}
$$

A.3.2 Conventions and properties of spinors and tensors

Bosons

As explained above, for Minkowski space we use indices from 0 to 5 with signature $(-+\cdots+)$. Therefore the Levi–Civita tensor is

$$\varepsilon_{012345} = 1 = -\varepsilon^{012345} . \tag{A.3.13}$$

The ε's with curved indices are defined as follows:

$$
\begin{aligned}
\varepsilon_{\mu\nu\rho\sigma\tau\phi} &= e^{-1} e^a_\mu e^b_\nu e^c_\rho e^d_\sigma e^e_\tau e^f_\phi \varepsilon_{abcdef} , \\
\varepsilon^{\mu\nu\rho\sigma\tau\phi} &= e e^\mu_a e^\nu_b e^\rho_c e^\sigma_d e^\tau_e e^\phi_f \varepsilon^{abcdef} .
\end{aligned}
\tag{A.3.14}
$$

The contraction of the Levi–Civita tensors is

$$\varepsilon^{a_1 a_2 \cdots a_p b_1 \cdots b_{d-p}} \varepsilon_{a_1 a_2 \cdots a_p c_1 \cdots c_{d-p}} = -\, p!\, (d-p)!\, \delta^{b_1}_{[c_1} \cdots \delta^{b_{d-p}}_{c_{d-p}]} \tag{A.3.15}$$

The essential formula defining duality is:

$$\gamma_{abc} \gamma_7 = -\tilde{\gamma}_{abc} , \tag{A.3.16}$$

where the dual is defined as:

$$\tilde{H}_{abc} \equiv \tfrac{1}{3!} \varepsilon_{abcdef} H^{def} , \tag{A.3.17}$$

and

$$\gamma_{abc} = \gamma_{[a} \gamma_b \gamma_{c]} . \tag{A.3.18}$$

Some other properties and definitions of (anti) self-dual tensors are

$$
\begin{aligned}
H^{\pm}_{abc} &= \tfrac{1}{2} (H_{abc} \pm \tilde{H}_{abc}) , \\
H^{+}_{abc} H^{-abc} &= \tfrac{1}{2} H_{abc} H^{abc} , \\
H^{+}_{abc} H^{+abc} &= 0 .
\end{aligned}
\tag{A.3.19}
$$

Fermions

We raise and lower $USp(4)$ indices with Ω^{ij} using NW-SE contraction:

$$\lambda^i = \Omega^{ij}\lambda_j \ , \qquad \lambda_i = \lambda^j\Omega_{ji} \ . \tag{A.3.20}$$

Contracted spinors with $USp(4)$ indices omitted are defined as

$$\bar\lambda\gamma^{(n)}\psi = \bar\lambda^i\gamma^{(n)}\psi_i \ . \tag{A.3.21}$$

We use symplectic Majorana–Weyl spinors. The symplectic Majorana condition is imposed: the Dirac conjugate must be equal to the symplectic Majorana conjugate:

$$\bar\lambda^i \equiv (\lambda_i)^\dagger\,(-\mathrm{i}\Gamma_0) = \left(\lambda^i\right)^T\mathcal{C}\,, \tag{A.3.22}$$

or, using spinor indices

$$\bar\lambda^{i\alpha'} \equiv (\lambda_{i\alpha})^\dagger\,(-i\Gamma_0)_\alpha{}^{\alpha'} = \lambda_{j\beta}\Omega^{ij}c^{\beta\alpha'} \ . \tag{A.3.23}$$

The same equation with all indices α and α' interchanged is also valid. The conjugate of a chiral spinor with index α has index α'. This implies for left- and righthanded spinors λ_L and λ_R that

$$\bar\lambda_L\Gamma_a\chi_L = \bar\lambda_L^{\alpha'}\,(\tilde\gamma_a)_{\alpha'}{}^\alpha\,\chi_{\alpha L} = \bar\lambda_L\tilde\gamma_a\chi_L\,, \qquad \bar\lambda_R\Gamma_a\chi_R = \bar\lambda_R\gamma_a\chi_R\,. \tag{A.3.24}$$

Changing the order of spinors in a bilinear leads to the following signs

$$\bar\psi^{(1)}\gamma^{(n)}\chi^{(2)} = t_n\;\bar\chi^{(2)}\gamma^{(n)}\psi^{(1)} \ , \qquad \begin{cases} t_n = -1 \text{ for } n = 0,3,4 \\ t_n = 1 \text{ for } n = 1,2,5,6 \end{cases} \tag{A.3.25}$$

where the labels (1) and (2) denote any $USp(4)$ representation, e.g., $(1) = i$ and $(2) = [jk]$.

We frequently use the following Fierz rearrangement formula:

$$\psi_j\bar\psi^i = -\tfrac14\left(\bar\psi^i\gamma_a\psi_j\gamma^a - \tfrac{1}{12}(\bar\psi^i\gamma_{abc}\psi_j)\gamma^{abc}\right)\tfrac12(1-\gamma_7)\,, \tag{A.3.26}$$

for the case where both fermions are left-handed and

$$\psi_j \bar{\lambda}^i = -\tfrac{1}{4}\left(\bar{\lambda}^i \psi_j + \tfrac{1}{2}(\bar{\lambda}^i \gamma_{ab} \psi_j)\gamma^{ab}\right)\tfrac{1}{2}(1-\gamma_7)\,, \tag{A.3.27}$$

for left-handed λ and right-handed ψ.

The notation "- (trace)" denotes terms that are proportional to either Ω^{ij} or δ^i_j (with "free" indices). We use the notation "-(traces)" if both invariant tensors occur. For the convenience of the reader we give below the explicit expressions of some trace terms:

$$\begin{aligned}
X^{ij} - (\text{trace}) &= X^{ij} + \tfrac{1}{4}\Omega^{ij} X^k{}_k \;, \\
A^{ij} X_k - (\text{traces}) &= A^{ij} X_k + \tfrac{4}{5} A^{\ell[i} X_\ell \delta^{j]}_k - \tfrac{1}{5}\Omega^{ij} A_{k\ell} X^\ell \;, \\
S_k{}^{[i} X^{j]} - (\text{traces}) &= S_k{}^{[i} X^{j]} - \tfrac{1}{5}\delta^{[i}_k S^{j]\ell} X_\ell \; + \tfrac{1}{5}\Omega^{ij} S_k{}^\ell X_\ell \;.
\end{aligned} \tag{A.3.28}$$

where X^i and X^{ij} are arbitrary $USp(4)$ tensors, while A^{ij} is an antisymmetric traceless and S^{ij} a symmetric tensor. In the calculations, the following $USp(4)$ Schouten identities are very useful:

$$\begin{aligned}
&2\bar{\varepsilon}^{[i}\phi^{j]k}\psi_k + 2\bar{\varepsilon}^k \phi_k{}^{[i}\psi^{j]} + \phi^{ij}\bar{\varepsilon}\psi - (\text{trace}) = 0 \;, \\
&\gamma^{abc}\varepsilon_k\, \bar{\psi}^i \gamma_{abc}\psi^j + 2\gamma^{abc}\varepsilon^{[i}\, \bar{\psi}^{j]}\gamma_{abc}\psi_k - (\text{traces}) = 0 \;.
\end{aligned} \tag{A.3.29}$$

Appendix B

Samenvatting

B.1 Algemene inleiding

Het standaardmodel en algemene relativiteitstheorie vatten de huidige kennis van het gedrag van elementaire deeltjes samen door middel van vier fundamentele krachten. In de 19de eeuw waren alleen de gravitatiekracht en de elektromagnetische kracht bekend. In die tijd werd gravitatie beschreven door de wetten van Newton en de klassieke Maxwell vergelijkingen verklaarden de elektromagnetische fenomenen. In het begin van de 20ste eeuw werd duidelijk dat kwantummechanica nodig is voor een adequate beschrijving van het gedrag van elementaire deeltjes op kleine afstanden.

Op dit moment verklaart het standaardmodel alle experimenten in deeltjesversnellers. Het standaardmodel is een renormaliseerbare kwantumveldentheorie met ijkgroep $SU(3)_c \times SU(2)_L \times U(1)$. Een andere grote verwezenlijking van de afgelopen honderd jaar is een betere beschrijving van de gravitationele kracht door middel van algemene relativiteitstheorie. Deze meetkundige beschrijving van ruimte en tijd vervangt de wetten van Newton. Ze is compatibel met al de huidige experimentele testen. Merkwaardig genoeg is de gravitationele kracht slechts getest tot op afstanden van de orde van centimeters. Deze kracht is zo zwak in vergelijking met de andere drie krachten dat de klassieke geometrische beschrijving volstaat. Er is geen kwantummechanische beschrijving nodig om de tot op heden waargenomen fenomenen te verklaren. Zelfs dan blijft er de principiële vraag naar een kwantummechanische beschrijving van ook deze kracht. De laatste jaren hebben astrofysici steeds meer aanwijzingen gevonden voor het bestaan van zwarte gaten. Aan de horizon van deze klassieke singulariteiten van de ruimtetijd is de gravitatiekracht veel sterker en is een kwantummechanische beschrijving vereist. De aanpak om de andere krachten te kwantiseren, namelijk door regularisatie en renormalisatie, kan niet gebruikt worden in het geval van de gravitatiekracht. De gravitationele koppelingsconstante is niet dimensieloos en het renormalisatieprogramma is gedoemd om te mislukken. Dit betekent dat een andere aanpak vereist is om een kwantummechanische formulering van gravitatie te realiseren. Ook het standaardmodel bevredigt niet volledig. Het vereist bijna 20 willekeurige parameters, de verschillen in massa van de elementaire deeltjes blijken moeilijk te verklaren, het is niet duidelijk waarom er drie generaties leptonen zijn, ... Het sterkste argument om te zoeken naar een meer fundamentele theorie is het feit dat gravitatie niet in het standaardmodel vervat zit. De zoektocht naar een geünificeerde theorie voor de vier fundamentele krachten begon reeds jaren geleden. Gedurende verschillende decennia al trachten mensen deze 'theorie van alles' te ontrafelen.

B.1.1 Supersnaren en supergravitatie

Waarom supersnaren introduceren?

Op dit moment is er slechts één volwaardige (en veelbelovende!) kandidaat om de (alle?) problemen met het standaardmodel en algemene relativiteitstheorie op te lossen: *snaartheorie* [1, 2]. Snaartheorie is een theorie waarbij de elementaire bouwstenen van de materie geen puntdeeltjes meer zijn, maar kleine ééndimensionale entiteiten. De trillingswijzen van deze snaren kunnen beschouwd worden als de verschillende elementaire deeltjes[1]. Net zoals deeltjes een wereldlijn maken in de ruimtetijd, zullen deze snaren een wereldoppervlak vormen in de hogerdimensionale ruimtetijd. Een Lorentz-invariante formulering van snaartheorie vereist 26 (ja, zesentwintig) ruimtetijd dimensies. Zelfs dan is de toestand van laagste energie een tachyon, een toestand met negatieve gekwadrateerde massa. Dit signaleert dat de theorie instabiel is. Er is echter een manier om deze ongemakken aan te pakken.

De invoering van het concept *supersymmetrie* maakt een stabiele, kwantummechanische formulering van snaartheorie mogelijk in tien dimensies. Supersymmetrie is een veralgemening van bosonische symmetrieën. Deze symmetrie relateert bosonen en fermionen, toestanden met verschillende spin. Het is duidelijk dat bij lage energieën supersymmetrie gebroken is. Het is echter niet duidelijk of supersymmetrie bij hoge energieën een fysische symmetrie zal blijken te zijn. Op dit moment komt de enige indicatie dat supersymmetrie bestaat, van de unificatie van

[1] Het was op deze manier dat snaartheorie geïntroduceerd werd in natuurkunde. Voor het succes van QCD om de sterke kracht te verklaren, werd snaartheorie gebruikt als een model om de mesonische resonanties te beschrijven.

de verschillende koppelingen bij hoge energie (10^{16} GeV) in de minimaal supersymmetrische versie van het standaardmodel. Deeltjesfysici hopen deze symmetrie in de nabije toekomst te vinden in LHC. Supersymmetrie is vooral nuttig bij het ontdekken van eigenschappen van snaartheorie en supergravitatie theorieën. Er zijn een aantal sterke argumenten die voor supersymmetrie pleiten. Door de invoering van supersymmetrie zijn er geen tachyonische toestanden meer in het spectrum van snaartheorie. Verder is ook gravitatie automatisch geïncorporeerd. Het spectrum bevat een massaloos deeltje met spin 2 dat met het graviton geïdentificeerd kan worden[2]. Snaartheorie laat ook een geünificeerde beschrijving van gravitatie en ijktheorie toe. Deze unificatie is een van de dromen van natuurkundigen: één theorie die alle fundamentele krachten kan beschrijven. Supersnaartheorie heeft twee eigenschappen die op het eerste zicht afschrikken: consistentie vereist tien ruimtetijd dimensies en er lijken vijf verschillende supersnaartheorieën mogelijk. Allebei deze aspecten zullen later nuttig blijken. Ten eerste is er type IIA en IIB theorie. Dit zijn theorieën die gesloten snaren bevatten en die twee ruimtetijd supersymmetrieën hebben. De heterotische snaar is een gesloten snaar met één supersymmetrie die consistente formuleringen toelaat voor de ijkgroepen $SO(32)$ en $E_8 \times E_8$. Type I snaartheorie, tenslotte, beschrijft open snaren. Bij lage energieën geven deze supersnaartheorieën aanleiding tot supergravitatie theorieën. Het onderzoek in deze thesis spitst zich toe op deze lage-energie modellen. We bestuderen verscheidene materiemultipletten in verschillende supergravitatie achtergronden.

Supergravitatie: de lage-energie limiet van supersnaren

We willen afleiden dat supergravitatie de lage-energie limiet van supersnaartheorie is. Het uitgangspunt is daarom de beschrijving van snaartheorie in een gekromde ruimtetijd. De wereldoppervlak actie van de snaar is een veralgemening van de wereldlijn actie voor een deeltje. De bosonische beschrijving gebeurt door middel van een interagerende tweedimensionale kwantumveldentheorie die de massaloze toestanden van snaartheorie bevat (bijvoorbeeld de ruimtetijd metriek $G_{\mu\nu}$, de antisymmetrische tensor $B_{\mu\nu}$ en het dilaton Φ). Deze massaloze toestanden zijn afgeleid uit een analyse in een vlakke ruimtetijd. Het fundamentele veld uit deze theorie is de plaats van de snaar $X^\mu(\sigma, \tau)$, die afhangt van de wereldoppervlakcoördinaten σ en τ. Deze velden X^μ spannen de gekromde ruimtetijd op. Later zullen we supersymmetrische partners invoeren en wordt de ruimtetijd een superruimte. Als de kinetische termen van de velden in de actie veld-afhankelijk zijn, noemen we de actie een niet-lineair σ-model.

[2]Het was net dit deeltje met spin 2 dat het gebruik van snaartheorie als een model voor de sterke interactie danig bemoeilijkte.

Als de karakteristieke lengte van de gekromde ruimtetijd R_c is, dan is er een effectieve dimensieloze koppelingsconstante $\alpha'^{1/2}R_c^{-1}$, waarbij de Regge helling α' als eenheid ruimtetijd-lengte in het kwadraat heeft. α' staat als volgt in verband met de spanning T van de snaar: $T = \frac{1}{2\pi\alpha'}$. Als R_c veel groter is dan de karakteristieke snaarlengte, is deze koppeling klein. Storingsrekenen is dan een nuttige techniek in de tweedimensionale veldentheorie. Dit verschil in lengteschalen impliceert ook dat de interne structuur van de snaar verwaarloosbaar wordt en dat een beschrijving door middel van een lage-energie effectieve veldentheorie nuttig is. Deze lage energie was al impliciet gebruikt door in de beschrijving van de wereldoppervlak actie alleen de massaloze snaartoestanden te gebruiken. De massieve snaartoestanden hebben een massa van de orde van de Planck massa, $M_P \simeq 10^{19}$ GeV en dit is veel hoger dan wat bereikbaar is (en zal zijn) in deeltjesversnellers.

De consistentie van snaartheorie vereist conforme of Weyl invariantie van de wereldoppervlak actie. Dit betekent dat het spoor van de energie-momentum tensor nul moet zijn. Dit kan ook afgeleid worden uit een effectieve ruimtetijd actie. Op deze manier geeft supersnaartheorie aanleiding tot een supergravitatie actie bij lage energieën. Elk van de vijf snaartheorieën heeft een corresponderende supergravitatie theorie. Een van de grote, moderne uitdagingen van theoretische hoge-energie fysica is een microscopische beschrijving te geven die 11-dimensionale supergravitatie als zijn lage-energielimiet heeft. De supergravitatie theorieën zijn op zich ook al interessant. Ze beschrijven gravitatie en ijktheorie tesamen. Zelfs als het geen fundamentele, microscopische theorieën zijn, vertonen ze al een heleboel interessante eigenschappen. Ze zijn een soort brug tussen het standaardmodel en algemene relativiteitstheorie aan de ene kant en een echte 'theorie van alles' aan de andere kant. Daarom zijn we geïnteresseerd om hun eigenschappen te bestuderen.

Compactificatie

Een van de twee belangrijke problemen met supersnaartheorie (en zijn lage-energie limiet supergravitatie) was de aanwezigheid van tien ruimtetijd dimensies. De oplossing van dit probleem ligt in de compactificatie van een aantal dimensies. Snaartheorie gebiedt immers niet dat al die tien dimensies oneindig uitgestrekt moeten zijn. Dit betekent dat onze wereld in een snaartheoretisch kader bestaat uit vier macroscopische, zichtbare dimensies, terwijl de zes andere dimensies op een bepaalde manier opgerold zijn. Deze compacte dimensies zijn zo klein dat we ze niet kunnen zien, of toch niet bij de energieën die we op dit moment kunnen bereiken. Afhankelijk van de structuur van de compacte variëteit, blijven verschillende hoeveelheden supersymmetrie over in de effectieve theorie in minder dimensies. In plaats van onmiddellijk het volledige probleem aan te pakken van een compactificatie naar vier dimensies, hebben theoreten ook compactificaties naar andere dimensies bestudeerd, bijvoorbeeld naar zes dimensies. Hoewel het duidelijk is dat die berekeningen geen realistische voorspellingen opleveren, kunnen ze wel bijdragen aan een beter begrijpen van compactificaties van supergravitatie theorieën. Een van de mogelijkheden om vier dimensies te compactificeren geeft aanleiding tot chirale[3] theorieën in zes dimensies. IIB theorie op een $K3$ variëteit leidt tot chirale $(2,0)$ supergravitatie in zes dimensies. De enige mogelijke materie-representatie in deze theorieën is een zelfduaal tensor multiplet. Een zelfduale tensor in zes dimensies is een twee-index tensor die een reële zelfduale veldsterkte heeft. We zullen veel aandacht besteden aan modellen met een zelfduale tensor in hoofdstuk 4 en 5.

Ook compactificaties op Calabi–Yau variëteiten zijn vaak bestudeerd. Deze zesdimensionale variëteiten voldoen aan bepaalde voorwaarden zodat een kwart van de tiendimensionale supersymmetrie bewaard blijft in vier dimensies. De Calabi–Yau compactificatie van de heterotische supersnaar leidt naar een chirale theorie met één supersymmetrie in vier dimensies die een bepaalde ijkgroep heeft. Deze theorieën komen kwalitatief dicht bij het minimale supersymmetrische standaardmodel. De Calabi–Yau compactificatie van IIA of IIB theorie geeft aanleiding tot theorieën in vier dimensies met twee supersymmetrieën. Die worden bestudeerd in hoofdstuk 6. Tot nu toe is niemand er in gelukt een realistisch supersnaarvacuüm af te leiden.

Niet-perturbatieve aspecten van veldentheorie

De oplossing van het probleem van de vijf, schijnbaar-verschillende supersnaar theorieën vereist iets extra. Tot nu toe hebben we alleen resultaten verklaard die gevonden zijn in een perturbatieve benadering van snaartheorie, net zoals de expansie in Feynman diagrammen in deeltjesfysica een perturbatieve benadering geeft van de theorie. Tot enkele jaren geleden was dit, in tegenstelling tot in veldentheorie, de enige manier die gekend was om snaartheorie te beschrijven. De laatste jaren echter is nieuw inzicht verworven in aspecten van snaartheorie die deze perturbatieve benadering overstijgen. Toch zijn we nog ver van een volledige beschrijving van de theorie. We zullen eerst een aantal niet-perturbatieve aspecten van veldentheorie uitleggen, die later relevant zullen zijn in snaartheorie. In een volgende sectie zullen we dan niet-perturbatieve aspecten van snaartheorie aansnijden. Het is ook in de context van deze niet-perturbatieve aspecten van veldentheorie dat we vector multipletten in vier dimensies zullen bestuderen.

[3] In even dimensies is het mogelijk om een spinor (en dus ook een superlading uit de supersymmetrie algebra) in twee chiraliteiten te splitsen door een projectie te gebruiken. Dit is uitgelegd in hoofdstuk 2. $(2,0)$ supersymmetrie betekent dat er twee supersymmetrieën zijn met dezelfde chiraliteit.

In veldentheorie zijn er puntdeeltjes met elektrische lading e. Ze koppelen aan een vectorveld, een één-vorm. In vier dimensies is er ook een ander object dat aan deze vorm koppelt: de magnetische monopool. Deze magnetische monopool heeft magnetische lading g. De Dirac kwantisatie conditie,

$$eg = 2\pi\hbar n\,, \tag{1}$$

legt op dat het product van de elektrische en de magnetische lading gelijk moet zijn aan een constante. Dit impliceert dat de elektrische en de magnetische lading omgekeerd evenredig zijn. Deze magnetische monopolen verschijnen in $U(1)$ ijktheorieën. In niet-Abelse ijktheorieën komen 't Hooft–Polyakov monopolen voor als er een $U(1)$-deelgroep is. De aanwezigheid van zulke topologisch stabiele, magnetisch geladen deeltjes opent de mogelijkheid voor een symmetrie tussen elektriciteit en magnetisme. Er is tot nu toe geen experimentele aanwijzing voor het bestaan van een magnetische monopool. In de reële wereld, met alleen elektrische geladen materie, is er geen elektromagnetische dualiteit. In vacuüm zijn de Maxwell vergelijkingen invariant onder de transformatie

$$\vec{E} \to \vec{B}\,, \qquad \vec{B} \to -\vec{E}\,, \tag{2}$$

van het elektrische veld $\vec{E}$ en het magnetische veld $\vec{B}$. Als er elektrisch en magnetisch geladen deeltjes zijn, worden hun ladingen verwisseld onder deze transformatie. Deze transformatie kan veralgemeend worden tot rotaties over een willekeurige hoek

$$\begin{aligned} \vec{E} &\to \cos\theta\,\vec{E} + \sin\theta\,\vec{B}\,, \\ \vec{B} &\to -\sin\theta\,\vec{E} + \cos\theta\,\vec{B}\,. \end{aligned} \tag{3}$$

Dezelfde symmetrie werkt ook in op de versie van de Maxwell vergelijkingen in functie van de veldvergelijkingen en de Bianchi identiteit[4]:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0\,, \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0\,, \end{aligned} \tag{4}$$

waarbij $F_{\mu\nu}$ de veldsterkte is voor de vector en $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. Deze vergelijkingen zijn opnieuw invariant onder rotaties over een willekeurige hoek:

$$\begin{pmatrix} F'^{\mu\nu} \\ \tilde{F}'^{\mu\nu} \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^{\mu\nu} \\ \tilde{F}^{\mu\nu} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^{\mu\nu} \\ \tilde{F}^{\mu\nu} \end{pmatrix}, \tag{5}$$

[4] De Bianchi identiteit drukt uit dat de veldsterkte lokaal afleidbaar is van een vector.

waarbij $\mathcal{S} \in GL(2,\mathbb{R})$. In de aanwezigheid van m Abelse vectorvelden veralgemeent deze symmetrie tot $GL(2m,\mathbb{R})$. De veldvergelijkingen kunnen afgeleid worden van een actie

$$\mathcal{L}_1 = \tfrac{1}{4}(\mathrm{Im}\,\mathcal{N}_{IJ})F^I_{\mu\nu}F^{\mu\nu J} - \tfrac{i}{8}(\mathrm{Re}\,\mathcal{N}_{IJ})\epsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}\ , \qquad (6)$$

waarbij $I, J = 1, ..., m$, de matrix $\mathcal{N}_{IJ}$ is symmetrisch en $F^I_{\mu\nu}$ zijn de veldsterkten voor de vectorvelden. In een theorie met ook scalairen, kan $\mathcal{N}_{IJ}$ afhangen van deze scalairen. In supersymmetrische theorieën zal deze afhankelijkheid zeer specifiek zijn.

De veldsterkten transformeren op een welbepaalde manier onder de elektromagnetische dualiteitstransformaties. Opleggen dat de veldvergelijkingen nog van een actie kunnen afgeleid worden, impliceert dat de groep van transformaties beperkt wordt tot de symplectische matrices in $Sp(2m,\mathbb{R})$:

$$\mathcal{S}^T\Omega\mathcal{S} = \Omega \qquad \text{met} \qquad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \qquad (7)$$

Dit vereist dat de componenten van $\mathcal{S}$ voldoen aan

$$A^TC - C^TA = 0\ ,\quad B^TD - D^TB = 0\ ,\quad A^TD - C^TB = \mathbb{1}\,. \qquad (8)$$

De matrix $\mathcal{N}$ transformeert als volgt:

$$\mathcal{N} \to \tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\,. \qquad (9)$$

Als zowel elektrische als magnetische ladingen toegevoegd worden aan de combinatie van veldvergelijkingen en Bianchi identiteiten, beperkt de Schwinger–Zwanziger kwantisatievoorwaarde de mogelijke ladingen tot een rooster. De ladingen van twee dyonen (q_1, g_1) en (q_2, g_2) moeten bijvoorbeeld voldoen aan

$$q_1g_2 - q_2g_1 = 2\pi n\hbar\,. \qquad (10)$$

Dit rooster beperkt de symplectische transformaties verder tot hun discrete deelgroep $Sp(2m,\mathbb{Z})$. Deze gehele symplectische transformaties zullen een essentiële rol spelen in de behandeling van vectormultipletten gekoppeld aan $N = 2$ supergravitatie[5] in vier dimensies. In hoofdstuk 6 zijn we geïnteresseerd in de koppeling van vectormultipletten aan supergravitatie. De symplectische vorm van de elektromagnetische dualiteitstransformaties zal daar een meetkundige interpretatie krijgen.

[5] We zullen het aantal supersymmetrieën in de specifieke dimensie noteren met N in deze thesis.

Een symmetrie tussen de elektrisch en magnetisch geladen objecten werd al in de late jaren '70 geconjectureerd door Montonen en Olive. In 1994 gaf Sen verdere bewijzen voor de aanwezigheid van S-dualiteit in $N = 4$ supersymmetrische Yang–Mills theorie. Deze S-dualiteit is een sterke-zwakke koppelingsdualiteit. Ze relateert een regime bij sterke koppeling met een regime van dezelfde of een andere theorie bij zwakke koppeling. Daar kunnen perturbatieve resultaten verkregen worden. De aanwezigheid van veel supersymmetrie was essentieel in zijn bewijs, aangezien supersymmetrie de mogelijke kwantumcorrecties beperkt. Relaties tussen koppelingsconstanten, die gerenormaliseerd dienen te worden, zijn inderdaad van belang. Supersymmetrie kan de relaties tussen deze koppelingsconstanten beschermen van renormalisatie effecten.

Niet-perturbatieve aspecten van snaartheorie

De ideeën uit de vorige sectie uit kwantumveldentheorie kunnen veralgemeend worden naar meer dimensies en ook naar snaartheorie. In het algemeen koppelt een object, uitgestrekt in p dimensies, elektrisch aan een $(p+1)$-vorm. Zijn magnetisch duale partner in dan uitgestrekt in $d-p-4$ ruimtelijke richtingen. Dit betekent dat een snaar in tien dimensies koppelt aan een twee-vorm, de NSNS[6] antisymmetrische vorm $B_{\mu\nu}$. Dit impliceert dat er een niet-perturbatief object bestaat dat magnetische geladen is met betrekking tot deze twee-vorm. Dit object heeft vijf ruimtelijke dimensies en wordt een NS5-braan genoemd. Elk van de vijf snaartheorieën heeft dergelijke NS5-branen. De massa van deze vijf-branen is evenredig met $1/g_s^2$ als g_s de snaarkoppelingsconstante is. Ook in 11-dimensionale supergravitatie zijn er elektrisch en magnetisch geladen objecten. Er is een drie-vorm potentiaal die suggereert dat er elektrisch geladen membranen zijn. Deze membranen hebben een magnetisch duaal object met weer vijf ruimtelijke dimensies: het $M5$-braan.

Verschillende snaartheorieën bevatten verder verschillende antisymmetrische RR-vormen. De objecten die geladen zijn onder deze vormen worden Dp-branen genoemd[7]. Type IIA/IIB snaartheorie laat Dp-branen toe voor even/oneven p. Verschillende uitgangspunten zijn geschikt om het inzicht in deze objecten te vergroten. Een eerste manier om D-branen te bestuderen is als oplossingen van de supergravitatieveldvergelijkingen. Deze oplossingen worden uitgedrukt met behulp van harmonische functies. Aan de andere kant zijn D-branen uitgestrekte objecten in snaartheorie die aan Dirichlet randvoorwaarden voldoen. De eindpunten van

[6] De bosonische velden in de ruimtetijd worden NSNS-velden (NS voor Neveu–Schwarz) genoemd als ze van wereldoppervlak spinor bilineairen komen met antiperiodische randvoorwaarden en RR-velden (R Van Ramond) als ze afkomstig zijn van periodische wereldoppervlak fermionen.

[7] De p in Dp-branen telt het aantal ruimtelijke dimensies van deze niet-perturbatieve objecten: een $D0$-braan is een deeltje, een $D1$-braan is een snaar, een $D2$-braan is een membraan ... Het standaardwerk over D-branen is [2].

open supersnaren kunnen vrij bewegen over de D-branen. Het is mogelijk om het gedrag van deze D-branen bij lage energie te beschrijven door middel van een wereldvolume theorie. De veralgemening van een wereldoppervlak beschrijving is een σ-model in $(p+1)$ dimensies waarbij de positie van het braan in de ruimtetijd een deel van de veldinhoud is. De wereldvolume theorie van een D-braan bevat verder een vectorveld en fermionen. Samen vormen deze velden een vector multiplet. Het op mekaar leggen van n verschillende branen geeft aanleiding tot een veldentheorie met niet-Abelse ijkgroep $U(n)$.

Al deze uitgebreide objecten leiden tot de conclusie dat snaartheorie meer is dan een theorie van snaren. Snaartheorie krioelt van de meerdimensionale objecten. De branen laten toe een aantal relaties tussen de op het eerste zicht verschillende snaartheorieën te verduidelijken. Die relaties zijn dualiteiten. Een eerste type dualiteit is T-dualiteit. T-dualiteit in zijn eenvoudigste vorm verbindt IIA met IIB theorie. Op die manier verklaren D-branen de relaties tussen snaartheorieën. Ook mirror-symmetrie kan beschouwd worden als T-dualiteit. Mirror-symmetrie betekent dat een type II snaartheorie op een bepaald Calabi–Yau variëteit equivalent is aan de andere type II theorie op een andere, maar gerelateerde Calabi–Yau variëteit. Dit soort T-dualiteit is relevant voor de compactificatie van snaartheorie naar vier dimensies.

Een ander soort dualiteit is de veralgemening van S-dualiteit in kwantumveldentheorie. Door de snaar koppelingsconstante g_s te laten toenemen, worden de solitonische objecten de lichtste uit de theorie. Een perturbatieve beschrijving met $1/g_s$ als koppelingsconstante wordt interessant op die manier. Een combinatie van T- en S-dualiteiten verbindt al de verschillende snaartheorieën. Een merkwaardige conclusie die de laatste jaren getrokken werd, is dat het ook mogelijk bleek een 11-dimensionale theorie terug te vinden. Type IIA supergravitatie bij sterke koppeling geeft aanleiding tot 11-dimensionale supergravitatie. Het matrixmodel [3] is een kandidaat voor de microscopische beschrijving van de theorie die 11-dimensionale supergravitatie als zijn lage-energie limiet heeft. De definitieve versie van deze microscopische beschrijving is nog niet bekend, maar kreeg alvast de naam M-theorie. De dualiteiten tussen al deze verschillende snaartheorieën weerlegt het bezwaar dat er verschillende (perturbatieve) snaartheorieën zijn. Ze zijn allemaal verbonden op het niet-perturbatieve niveau.

De Maldacena conjectuur

Als een groot aantal, n, $D3$-branen op mekaar geplaatst worden, wordt de klassieke supergravitatie oplossing voor de metriek een goede benadering voor de ruimtetijd meetkunde in snaartheorie. De limiet nemen voor grote n, waarbij de snaarlengte naar nul gestuurd wordt, leidt tot een nabije-horizon meetkunde die het produkt is van $adS_5 \times S^5$. De groep van isometrieën van deze ruimte is $SO(4,2) \times SO(6)$. Anderzijds hebben we al gezien dat n $D3$-branen op mekaar aanleiding geven tot $N=4$ Yang–Mills ijktheorie in vier dimensies met ijkgroep $U(n)$. Deze ijktheorie is conform invariant en heeft ruimtetijd symmetriegroep $SO(4,2)$. De interne R-symmetriegroep, die de verschillende supersymmetrieladingen in mekaar roteert, is $SU(4)$. Aangezien $SU(4)$ de enkelvoudig samenhangende groep van $SO(6)$ is, vinden we dat de groep van isometrieën voor beide theorieën dezelfde is. Dit was de aanleiding voor Maldacena om de adS/CFT-conjectuur te postuleren [4]: snaartheorie in een $adS_5 \times S^5$ achtergrond[8] is equivalent aan een $N=4$ superconforme Yang–Mills theorie in vier dimensies in de limiet voor grote n. Deze veldentheorie bevat geen gravitatie. De theorie is gedefinieerd op de vierdimensionale rand van de adS_5-ruimte. Exacter, de amplitudes in de IIB bulk supergravitatie zijn functies van de snaarvelden op punten op de rand van adS_5. De conjectuur schrijft voor dat deze velden op de rand de bronnen zijn voor bepaalde operatoren in de Yang–Mills theorie. De snaaramplituden kunnen geïdentificeerd worden met correlatiefuncties uit de conforme veldentheorieën. Deze correspondentie kan leiden tot het berekenen van sterke kwantumeffecten in conforme veldentheorie door een berekening in klassieke supergravitatie.

Maldacena formuleerde zijn conjectuur ook voor M-theorie op $adS_4 \times S^7$ en M-theorie op $adS_7 \times S^4$. Deze gevallen kunnen geïnterpreteerd worden als de meetkunde op de horizon van een aantal $M2$-branen of $M5$-branen op mekaar. De wereldvolume beschrijving van het $M5$-braan bevat een zelfduale tensor. Dit impliceert dat de wereldvolume beschrijving voor veel $M5$-branen een interagerende theorie voor zelfduale tensoren vereist. Het blijft één van de uitdagingen van hoge-energie theoretische natuurkunde om een theorie te vinden die dit beschrijft.

B.2 Overzicht van de thesis

Naast deze algemene inleiding bevat deze samenvatting een overzicht van de verschillende hoofdstukken in dit proefschrift. Geïnteresseerde lezers kunnen voor meer technische details de relevante delen van dit werk raadplegen.

Hoofdstuk 2

Hoofdstuk 2 bevat vooral wiskundige achtergrond. Wiskunde blijkt de taal bij uitstek om natuurkunde accuraat te beschrijven. In dit werk is het concept symmetrie uiterst belangrijk. In wiskundige termen worden deze symmetrieën beschreven met

[8] Minder symmetrische ruimtes dan perfecte sferen geven aanleiding tot situaties waarbij de correspondentie geldig is met minder supersymmetrie.

behulp van algebra's, groepen en hun representaties. Supersymmetrie en conforme symmetrie zijn de twee belangrijke hoekstenen in dit werk. Er zijn twee redenen om modellen met superconforme symmetrie te bestuderen. Een eerste motivering is dat we de koppeling van bepaalde modellen aan Poincaré supergravitatie kunnen beschrijven met behulp van superconforme tensor calculus (zie daarvoor hoofdstuk 3). We bestuderen eerst de koppeling van die multipletten aan conforme gravitatie en de conforme symmetrie wordt later gebroken tot Poincaré supersymmetrie. Een tweede reden om de koppeling van materie aan superconforme gravitatie te beschrijven komt van de adS/CFT correspondentie.

In dit hoofdstuk willen we tot een classificatie van conforme superalgebra's komen. We starten met de bosonische algebra's die verschillende ruimtetijd configuraties beschrijven, geven hun supersymmetrische veralgemening en classificeren de superconforme algebra's. Verder besteden we aandacht aan hun representaties: de verschillende spinor representaties en chirale bosonen. We beklemtonen de mogelijkheid om kleinere spinorrepresentaties te vinden: de chiraliteitsvoorwaarde leidt tot Weyl spinoren, een realiteitsvoorwaarde tot Majorana of symplectische Majorana spinoren. We leiden ook af voor welke dimensies en signaturen de twee voorwaarden samen opgelegd kunnen worden. Verder bestuderen we chirale bosonen: antisymmetrische vormen met een reële, (anti)zelfduale veldsterkte. Ze komen voor in dezelfde dimensies en signaturen waar er (symplectische) Majorana–Weyl spinoren zijn. In supersymmetrische representaties zitten ze samen in een multiplet. Symplectische Majorana–Weyl spinoren en chirale spinoren komen we later in de zesdimensionale modellen tegen.

Met behulp van de spinor representaties is het mogelijk om de conforme superalgebra's die relevant zijn in dit werk te geven: de zesdimensionale chirale, superconforme $(2,0)$ algebra $OSp(8^*|4)$ en de superconforme algebra in vier dimensies met twee supersymmetrieën $SU(2,2|2)$.

Hoofdstuk 3

Hoofdstuk 3 beschrijft de methodes die we gebruikt hebben in de analyse van de twee modellen in dit werk. We hebben voor allebei de modellen gebruik gemaakt van superconforme tensor calculus. Deze techniek wordt gebruikt om de koppeling van verschillende materie multipletten aan Poincaré supergravitatie op een elegante manier te bekomen. Het uitgangspunt is eerst modellen met te veel symmetrie te bestuderen: de verschillende representaties alsook de actie (of de veldvergelijkingen) worden opgesteld zodat ze invariant zijn onder superconforme transformaties. Compenserende multipletten worden ingevoerd om later door bepaalde algebraïsche voorwaarden de overbodige symmetrie te breken. In het geval van superconforme tensor calculus zijn die overbodige symmetrieën de dilataties, de speciale conforme symmetrie en de speciale supersymmetrie. De essentiële stappen in deze procedure wordt geschetst aan de hand van twee voorbeelden. Het belang van de conventionele beperkingen in (super)conforme tensor calculus wordt benadrukt in een voorbeeld dat algemene relativiteit afleidt uit conforme gravitatie. Dit voorbeeld bevat de gedachtegang die ook gevolgd wordt in de twee modellen die expliciet uitgewerkt zullen worden in deze thesis. De lezer die niet vertrouwd is met superconforme tensor calculus, wordt dan ook aangeraden zeker dit voorbeeld te bestuderen.

Sectie 3.2 vat de Batalin–Vilkovisky methode samen. Die maakt het mogelijk de meest ingewikkelde ijksymmetrieën te behandelen: niet-Abelse Yang–Mills theorieën, 'softe' algebra's met veldafhankelijke structuurfuncties, (oneindig) reducibele ijksymmetrieën, waarbij niet alle ijktransformaties onafhankelijk zijn, open algebra's, ... Batalin en Vilkovisky voerden in hun constructie voor elk veld een antiveld in. Een ander essentieel ingrediënt in hun constructie is de invoering van antihaakjes. Die spelen dezelfde rol als de Poissonhaken in de beschrijving van klassieke mechanica. De antihaakjes laten ook een covariante behandeling toe. We geven de relevante aspecten van deze Batalin–Vilkovisky methode die nodig zijn om de klassieke uitgebreide actie af te leiden. De vorm van die uitgebreide actie hangt af van de verschillende soorten ijksymmetrieën van het model en moet aan drie voorwaarden voldoen: De 'properness' voorwaarde, die uitdrukt dat alle reducibele symmetrieën meegenomen zijn, en de klassieke 'master' vergelijking, uitgedrukt met behulp van de antihaakjes, moeten voldaan zijn. De klassieke limiet van de uitgebreide actie, waarbij alle antivelden nul gesteld worden, moet ook leiden tot de originele actie.

Verder beschrijven we hoe ijkfixing, het vastleggen van de symmetrie ter voorbereiding van een kwantummechanische behandeling, gebeurt door een canonieke transformatie, een transformatie die de antihaakjes bewaart. Tenslotte schetsen we kort uit hoe de berekening van kwantummechanische resultaten kan gebeuren met behulp van het Batalin–Vilkovisky formalisme. In deze thesis zullen we geen kwantummechanische resultaten afleiden.

B.2.1 De zelfduale tensor in 6 dimensies

Al deze achtergrond wordt gebruikt in de analyse van bepaalde aspecten van twee modellen. We bestuderen eerst zelfduale tensoren in zes dimensies. Hoofdstuk 4 bestudeert bosonische aspecten van de Lorentz-covariante actie voor een zelfduale tensor in zes dimensies. De supersymmetrische aspecten, en meer specifiek de koppeling van het zelfduale tensor multiplet aan een achtergrond van chirale, conforme supergravitatie wordt behandeld in hoofdstuk 5.

De zelfduale tensor in zes dimensies is een chiraal boson. Het is een twee-tensor

B_{ab} waarvan de veldsterkte zelfduaal en reëel is. Chirale bosonen komen voor in de beschrijving van heterotische snaren en bij instantonen in vier Euclidische dimensies. De zelfduale vier-vorm in type IIB supergravitatie is ook een chiraal boson. Chirale bosonen worden ook gebruikt in de beschrijving van bepaalde fenomenen van het fractionele quantum Hall effect. De zelfduale twee-tensor vind je zowel in materiemultipletten als in chirale-gravitatie multipletten. Chirale supergravitaties in zes dimensies kunnen gevonden worden uit compactificaties van tiendimensionale supergravitatie op geschikte variëteiten. Type IIB supergravitatie geeft aanleiding tot modellen met $(2,0)$ supersymmetrie in zes dimensies. De wereldvolume beschrijving van het $M5$-braan en het $NS5$-braan van IIA theorie gebeurt door middel van een zelfduaal $(2,0)$ tensor multiplet. Hoewel de veldinhoud voor de beschrijving van de wereldvolume theorie van het $M5$-braan gekend is, blijft het zoeken naar een beschrijving van een interagerende theorie voor zelfduale tensoren. Die is nodig voor de beschrijving van meerdere samenvallende branen.

Hoofdstuk 4

In dit hoofdstuk willen we een aantal bosonische aspecten van de Lorentz-covariante actie voor de twee-tensor bestuderen. De kern van het probleem om die actie neer te schrijven is de zelfdualiteitsvoorwaarde. Er zijn twee redenen om zo'n actie op te schrijven. Een eerste reden is dat een covariante actie nodig is om een kwantummechanische analyse met behulp van het padintegraalformalisme mogelijk te maken. Een tweede motivering spruit voort uit de mogelijkheid om Lorentz-covariantie op twee, a priori niet-equivalente, manieren te definiëren. Een eerste mogelijkheid is te eisen dat er geen bevoorrecht referentiestelsel is. Dat betekent dat een prescriptie vereist is die oplegt hoe de overgang van het ene naar het andere referentiestelsel dient te gebeuren. De verzameling van die transformaties moet aan de vermenigvuldigingswet van de Poincarégroep voldoen. Een tweede, op het eerste zicht strengere, definitie van Lorentz-covariantie legt op dat de beschrijving van het model manifest covariant moet zijn. De vraag is of deze manier om Lorentz-covariantie te definiëren echt strenger is. De formulering van de Lorentz-covariante actie weerlegt dat de zelfduale twee-tensor een voorbeeld is dat aantoont dat de twee definities van Lorentz-covariantie niet equivalent zouden zijn.

In 1996 lukten Pasti, Sorokin en Tonin (PST) er in om een expliciet Lorentz-covariante actie neer te schrijven. Daartoe dienden zij wel één extra scalair veld a in te voeren. Dat veld maakte het mogelijk om behalve de gewone ijktransformatie voor een tensor, ook nog twee nieuwe symmetrieën te introduceren, vanaf nu PST symmetrieën genoemd. a is puur ijk voor de eerste PST symmetrie. Met behulp van die symmetrieën argumenteren we dat de zelfduale tensor drie vrijheidsgraden bevat. De nieuwe resultaten in dit hoofdstuk beschrijven twee ijkfixings van de ijksymmetrieën van de zelfduale tensor. Daarvoor bouwen we eerst de uitgebreide actie op in het Batalin–Vilkovisky formalisme. We ijkfixen deze actie op twee manieren. Een eerste manier leidt tot een covariante ijkfixing. Deze ijkfixing zou kunnen leiden tot kwantummechanische resultaten voor de $M5$-braan uit M-theorie. De andere ijkfixing is niet covariant. Die ijkfixing werd vroeger al gebruikt om de gravitationele anomalieën van de chirale tensor in zes dimensies te berekenen. Een neerslag van deze nieuwe resultaten is ook te vinden in [6].

Hoofdstuk 5

Hoofdstuk 5 behandelt de supersymmetrische aspecten van het zelfduale tensor multiplet met $(2,0)$ chirale supersymmetrie. Het enige materie multiplet dat de $(2,0)$ supersymmetrie algebra toelaat, bevat een zelfduale tensor, een kwartet symplectische Majorana–Weyl spinoren en vijf scalairen.

In sectie 5.2 geven we de beschrijving van het zelfduale tensor multiplet met $(2,0)$ rigide supersymmetrie. De on-shell transformatieregel voor de spinor bevat het zelfduale deel, H^+_{abc}, van de veldsterkte. Om dé elementaire eigenschap van een supersymmetrie algebra – de commutator van twee supersymmetrieën is een translatie – te realiseren, is het nodig de zelfdualiteitsvoorwaarde

$$H^-_{abc} = 0 \tag{1}$$

te gebruiken. We maken duidelijk dat dit kan vermeden worden door de invoering van het scalaire hulpveld a. Die a maakt het mogelijk om een transformatieregel te vinden met een a-afhankelijke, zelfduale tensor h^+_{abc} die wel aanleiding geeft tot een translatie in de commutator van twee supersymmetrieën. Die commutator bevat verder zowel een term die een ijktransformatie is voor de gewone ijksymmetrie, als één voor de tweede PST symmetrie. De eerste PST ijksymmetrie verschijnt in een covariante afgeleide in de algebra. We definiëren de covariante afgeleide van een veld als zijn partiële afgeleide min de ijktransformaties waarbij de parameter vervangen is door het ijkveld. We realiseerden ons dat de afgeleide van a fungeert als een ijkveld: de transformatie ervan is de afgeleide van de ijkparameter. Door deze nieuwe interpretatie, voor het eerst beschreven in [7], komt de eerste PST ijksymmetrie in covariante afgeleiden voor. De tensor h^+_{abc} kan dan ook als een covariante grootheid beschouwd worden.

Verder herhalen we dat a een fermionisch singlet is. Dat is een representatie van een supersymmetrie algebra die slechts één bosonische of fermionische component heeft. Het scalair veld is puur ijk voor de eerste PST symmetrie, dus a is een representatie van de supersymmetrie algebra met nul bosonische en nul fermionische vrijheidsgraden. Het is door de aanwezigheid van de PST ijksymmetrie dat het toch mogelijk is de algebra te realiseren voor dit veld. Tenslotte geven we de actie voor het $(2,0)$ zelfduale tensor multiplet. Die is invariant onder superconforme, rigide transformaties.

Sectie 5.3 beschrijft de volgende stap om tot de koppeling van het zelfduale tensor multiplet aan conforme gravitatie te komen: een accurate beschrijving van de supergravitatie achtergrond. Het Weyl multiplet voor $(2,0)$ conforme supergravitatie multiplet bevat a priori al de ijkvelden voor de symmetrieën van de superconforme algebra: translaties, Lorentz rotaties, dilataties, speciale conforme transformaties, supersymmetrie, $USp(4)$ R-symmetrie en speciale supersymmetrie. Net als in het geval van (bosonische) conforme gravitatie in hoofdstuk 3 leggen we conventionele beperkingen op: vergelijkingen die garanderen dat we irreducibele transformaties vinden. Die vergelijkingen voor de krommingen van translaties, Lorentz rotaties en supersymmetrie leiden tot vergelijkingen voor de spin connectie, het ijkveld voor speciale conforme transformaties en dat voor speciale supersymmetrie. Na die beperkingen hebben we 64 + 64 vrijheidsgraden. We voeren dan materievelden in met 64 + 64 componenten. Een telargument, dat voor alle andere constructies in superconforme tensor calculus gebruikt wordt, gaat hier niet op. Er is ook nog niet bewezen dat het niet mogelijk is om een supergravitatie multiplet te construeren zonder materievelden. Wij geven vier argumenten om ze wel in te voeren: zowel de compactificatie naar het vierdimensionale $N = 4$ conforme supergravitatie multiplet, als de koppeling aan een multiplet van stromen met 128 + 128 vrijheidsgraden en de reductie naar het Weyl multiplet van $(1,0)$ conforme supergravitatie wordt gerealiseerd vanuit het Weyl multiplet mét materievelden. Een laatste argument steunt op de adS/CFT correspondentie. De adS-zijde van de correspondentie wordt in goede benadering beschreven door supergravitatie in adS_7. Op de zesdimensionale rand van adS_7 leiden de velden van het gravitatiemultiplet na een partiële ijkfixing tot het $(2,0)$ Weyl multiplet met de materievelden erbij. In adS_7 is wel bewezen dat er geen kleiner gravitatie multiplet bestaat dan het multiplet dat aanleiding geeft tot het Weyl multiplet met de materievelden.

Met behulp van dit Weyl multiplet met 128+128 vrijheidsgraden construeren we dan de gecorrigeerde krommingen, covariante conventionele beperkingen en superconforme transformatieregels voor het hele multiplet.

De combinatie van het Weyl multiplet in sectie 5.3 en het zelfduale tensor multiplet in sectie 5.2 leidt dan tot een koppeling van het materie multiplet aan een achtergrond van superconforme gravitatie. Al de nieuwe resultaten uit deze sectie zijn ook te vinden in [7]. In de covariante veldsterkte voor de tensor (voorlopig nog niet covariant voor de eerste PST symmetrie) staat ook een term met de materievelden uit het Weyl multiplet. Die zal later nodig blijken bij het realiseren van de supersymmetrie algebra en een invariante Lorentz-covariante actie. Het hulpveld a leidt weer tot een veldsterkte $h^{+}_{\mu\nu\rho}$ die automatisch zelfduaal is en die covariant is voor de eerste PST symmetrie. In het geval van lokale supersymmetrie wordt de translatie in de algebra vervangen door een algemene coördinaten transformatie. Als er andere ijksymmetrieën in het spel zijn, wordt dat een covariante algemene coördinaten transformatie. We leiden de rol van de ijksymmetrieën in de commutator van twee supersymmetrieën af.

Het belangrijkste resultaat van dit hoofdstuk is de constructie van de Lorentz-covariante actie. Die is invariant onder lokale superconforme transformaties en geeft aanleiding tot veldvergelijkingen van deze actie, die al gekend waren uit de literatuur. In zekere zin is dit de meest algemene Lorentz-covariante actie voor het zelfduale tensor multiplet. Het is immers zowel mogelijk om supersymmetrie (gedeeltelijk) te breken als om de conforme symmetrieën te ijkfixen.

Verder becommentariëren we op welke manier de berekening van de conforme anomalie van de zelfduale tensor een rol zou kunnen spelen in een (voorlopige?) discrepantie in de adS_7/CFT_6 correspondentie.

We kunnen dus zeggen dat we er in gelukt zijn een aantal bosonische en fermionische eigenschappen van zelfduale tensor (multipletten) af te leiden. We hebben de uitgebreide actie geconstrueerd voor de zelfduale tensor en zijn ijksymmetrieën op twee manieren geijkfixt. Verder hebben we een aantal supersymmetrische eigenschappen verduidelijkt. Zowel voor globale als lokale supersymmetrie hebben we aangetoond dat we betere supersymmetrie transformatie regels vinden door een veldsterkte te definiëren die covariant is voor de eerste PST ijksymmetrie. Het ijkveld voor deze symmetrie is de afgeleide van het hulpveld a. Dat hulpveld a blijkt een fermionisch singlet te zijn. We hebben dan het Weyl multiplet voor $(2,0)$ conforme supergravitatie afgeleid: de materievelden, transformatieregels, covariante krommingen en conventionele beperkingen. Tenslotte hebben we de Lorentz-covariante actie geconstrueerd voor het zelfduale tensor multiplet in de achtergrond van het Weyl multiplet.

B.2.2 Vector multipletten in $N = 2$ in 4 dimensies

Het tweede model dat we onder de loupe nemen, zijn vector multipletten in vier dimensies, gekoppeld aan $N = 2$ supergravitatie. Die multipletten spelen een belangrijke rol in zowel supersymmetrische als supergravitatie theorieën. We streven naar een meetkundige beschrijving waarbij symplectische transformaties, de veralgemening van elektromagnetische dualiteit, een essentiële rol spelen.

Hoofdstuk 6

In een eerste sectie benadrukken we de rol van symplectische transformaties in theorieën met scalairen en vectoren. In de supersymmetrische behandeling van de vector multipletten in de rest van dit hoofdstuk zal die symplectische symmetrie immers essentieel zijn. Verder herhalen we in sectie 6.1 een definitie van speciale Kähler meetkunde die gebruik maakt van symplectische haken. Speciale Kähler meetkunde is de naam voor de meetkunde van de variëteiten die de scalairen uit de $N = 2$ vector multipletten in vier dimensies opspannen. Verder schetsen we de ambiguïteit in deze definitie: voor het geval van één fysisch vector multiplet zijn er twee definities mogelijk. Tot hiertoe werd in de literatuur altijd de sterke definitie gebruikt. Wij zullen in dit hoofdstuk modellen construeren met een vector multiplet die wel aan de zwakke definitie voldoen, maar niet aan de sterke. Voor meerdere vector multipletten zijn de twee definities equivalent. Verder schetsen we de procedure die we zullen volgen. We zijn geïnteresseerd in de veldvergelijkingen van vector multipletten gekoppeld aan Poincaré gravitatie. We vertrekken van superconforme multipletten en zullen de veldvergelijkingen afleiden die Poincaré supersymmetrisch zijn. Daarvoor gebruiken we zowel superconforme tensor calculus (cf. hoofdstuk 3) als de vergelijkingen die speciale Kähler meetkunde definiëren en de supersymmetrie partners van deze vergelijkingen.

In sectie 6.2 herhalen we de eigenschappen van de superconforme bouwblokken van de constructie. Het Weyl multiplet is het multiplet van conforme supergravitatie in $N = 2$ in vier dimensies en het superconforme chirale multiplet geeft aanleiding tot een vector multiplet als een supersymmetrisch stel beperkingen opgelegd wordt. Die beperkingen worden veralgemeende Bianchi identiteiten genoemd omdat een van de vergelijkingen de Bianchi identiteit voor de vector uit het multiplet is. Voor de koppeling van n vector multipletten aan gravitatie voeren we er $n + 1$ in. Dat extra multiplet is een compenserend multiplet.

In sectie 6.3 onderzoeken we de eerste consequenties van de zwakkere definitie van speciale Kähler meetkunde. De kromming voor dit speciale geval wordt vergeleken met de algemene kromming van een speciale Kähler variëteit. Het blijkt ook nuttig een aangepaste metriek te definiëren. Verder wordt de stap van conforme naar Poincaré gravitatie voorbereid. De definitie van speciale Kähler meetkunde blijkt de dilataties al te ijkfixen. De supersymmetrische partners van deze definiërende betrekkingen doen hetzelfde voor de speciale supersymmetrie. De ijkkeuze voor speciale conforme symmetrie breekt de derde symmetrie. Zo blijft het model invariant onder Poincaré supersymmetrie en een $SU(2)$, afkomstig van de R-symmetrie uit de superconforme algebra. Vervolgens worden alle andere supersymmetrie transformaties van de vergelijkingen van de definitie van speciale Kähler meetkunde afgeleid, zowel voor het generieke als voor het speciale geval. Die uitdrukkingen worden later nog gebruikt bij de afleiding van de veldvergelijkingen. In sectie 6.4 worden de veralgemeende Bianchi identiteiten gecombineerd met de vergelijkingen voor speciale Kähler meetkunde. Zowel het generieke geval als het speciale geval worden uitgewerkt. Die combinatie van resultaten uit sectie 6.3 en sectie 6.4 geeft aanleiding tot de veldvergelijkingen voor n, respectievelijk één vector multiplet gekoppeld aan $N = 2$ Poincaré supergravitatie. Voor het generieke geval wordt nog bestudeerd in welke mate een nieuwe symplectische vergelijking aanleiding geeft tot de gravitatie veldvergelijkingen. Tot nu toe hadden we alleen veldvergelijkingen voor de vector multipletten en nog niet voor de velden uit het gravitatie multiplet. We vinden die veldvergelijkingen op 8+8 'integratieconstanten' na. Die 'constanten' zouden gevonden worden door het invoeren van een tweede compenserend multiplet. Daardoor zouden we echter de elegante symplectische structuur verspelen. Verder geven we in deze sectie expliciet een klasse van modellen die wel aan de zwakke en niet aan de sterke definitie van speciale Kähler meetkunde voldoen. We bewijzen ook waarom de modellen uit de speciale klasse niet kunnen komen van een Calabi–Yau compactificatie van snaartheorie. De resultaten uit deze sectie, op het stuk over de relatie met Calabi–Yau compactificaties na, zijn gebaseerd op [9]. In een laatste sectie onderzoeken we welke deelgroepen van de symplectische transformaties nog mogelijk zijn als er een niet-Abelse ijkgroep is. De volledige symplectische structuur van elektromagnetische transformaties is alleen geldig voor Abelse vector multipletten. We vinden dat de actie moet uitgebreid worden met een extra term opdat de invariantie onder een deel van de symplectische groep bewaard blijft.

Een opmerking ter conclusie

Men kan zich afvragen: "Wat is nu de zin van dit alles als al deze resultaten niet lijken bij te dragen aan de finale betrachting van snaartheorie? Waarom onderzoek doen in snaartheorie (supergravitatie) als het tot nu toe onmogelijk is gebleken om contact te maken tussen het standaardmodel en algemene relativiteitstheorie aan de ene kant en een microscopische beschrijving van elementaire 'deeltjes' die al de fundamentele krachten omvat, aan de andere kant?"
Dit uitweiden van hoge-energie theoretische fysici naar problemen die er helemaal niet naar streven, laat staan er in lukken, contact te maken met het standaardmodel, heeft mijns inziens niet te maken met slechte wil. Het is eerder de overtuiging van deze groep fysici dat een dieper begrijpen nodig is van de wiskundige structuren die aan de basis liggen van snaartheorie, vooraleer betrouwbare voorspellingen gemaakt kunnen worden. Hopelijk zet deze thesis een klein, klein stapje in de goeie richting.

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