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Letter

The Bocharova–Bronnikov–Melnikov–Bekenstein black hole's exact quasibound states and Hawking radiation

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Abstract In this letter, we investigate behaviour of massive and massless scalar field that is represented by the covariant Klein–Gordon equation with Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) black hole background. We successfully solve analytically the governing relativistic wave equation and discover the exact quasibound states' wave functions and energy levels of both massive and massless cases. The corresponding quasibound states have complex-valued energy $E = E_R + iE_I$ where the real part E_R can be interpreted as the scalar's relativistic quantized energy while the imaginary part represents decay as the quasibound states tunnels through the black hole's horizon. The Hawking radiation coming out of the BBMB black hole's horizon is also discussed and calculated via the Damour–Ruffini method, i.e. by singling out the particle–anti-particle parts of the obtained exact scalar's exact wave function from where, the radiation distribution function is derived and the Hawking temperature is obtained.

1 Introduction

1.1 BBMB black hole

The BBMB black hole [1] is a static spherically symmetric solution of the modified Einstein equation. A conformally coupled scalar field is added into the Einstein–Hilbert action as follows [2–4],

$$S = \frac{1}{2\kappa}\int\left(R - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{12}R\phi^2\right)\sqrt{-g}d^4x, \tag{1}$$

where R is the Ricci scalar and ϕ is the scalar field. The equations of motion of the theory are obtained from extremizing the action (1) respect to $g^{\mu\nu}$ that is resulted in this following tensorial field equation reads as follows,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{\kappa}{1-\frac{1}{6}\phi^2}\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{6}\left(g_{\mu\nu}\nabla^\alpha\nabla_\alpha - \nabla_\mu\nabla_\nu\right)\phi^2\right], \tag{2}$$

and extremizing respect to ϕ leads to a scalar field equation as follows,

$$\nabla^\beta\nabla_\beta\phi = \frac{1}{6}R\phi, \tag{3}$$

where $\kappa = \frac{8\pi G}{c^4}$. It is also possible to define the effective gravitational constant G_{eff} as follows,

$$G_{eff}(\phi) = \frac{G}{1-\frac{1}{6}\phi^2}. \tag{4}$$

However, we can investigate the tensorial equation further by calculating its contraction as follows,

$$-R = \frac{8\pi G_{eff}}{c^4}\left[-\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{2}\left(\nabla^\alpha\nabla_\alpha\right)\phi^2\right], \tag{5}$$

where,

$$\nabla^\alpha\nabla_\alpha\phi^2 = 2\left(\nabla^\alpha\phi\nabla_\alpha\phi + \phi\nabla^\alpha\nabla_\alpha\phi\right). \tag{6}$$

Now, combining with (3), the (5) reads as follows,

$$0 = \frac{8\pi G_{eff}}{c^4}\left[1 + \frac{1}{6}\phi^2\right]R, \tag{7}$$

that implies,

$$R = 0, \tag{8}$$

$$\nabla^\alpha\nabla_\alpha\phi = 0. \tag{9}$$

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By imposing the static spherically symmetric ansatz,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{10}$$

$$g_{\mu\nu} = \begin{bmatrix} -f(r) & 0 & 0 & 0 \\ 0 & f(r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}, \tag{11}$$

where the coordinates are $x^\mu = (ct,\ r, \theta, \phi)$ the solution is obtained as follows,

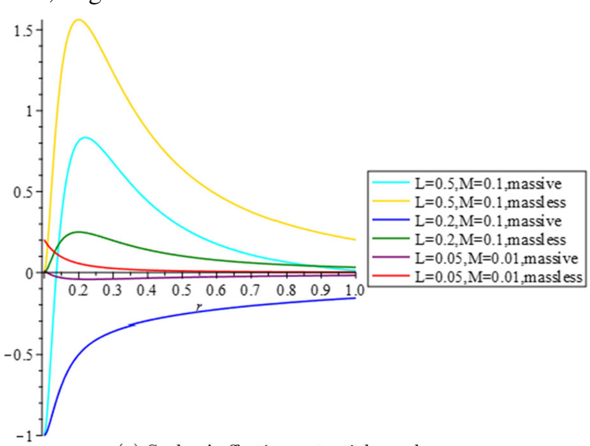
$$f(r) = \left(1 - \frac{r_H}{r}\right)^2, \quad r_H = \frac{GM}{c^2}, \tag{12}$$

where M is the black hole's mass. Notice that due to the similarity of the BBMB metric with the extremal Reissner–Nordström metric, the obtained results are applied for both cases.

Since we are dealing with a static spherically symmetric black hole solution with $g_{00} = g_{11}$, the effective potential of a scalar field under the influence of the black hole's gravitational field is given as follows,

$$V_e ff = \left(\frac{L^2}{r^2} + kc^2\right) f(r) - k^2 c^2, \tag{13}$$

where $k = 1$ is for massive scalar and $k = 0$ for massless scalar case. In the following, we present an effective potential graph for various M and L with massive and massless case for each, in geometrical unit.



(a) Scalars' effective potential graphs

Notice that in few massive case, where M is small, there exists a potential well that allows quasistable scalar bound states.

1.2 Black hole's spectroscopy

In 2015, the gravitational wave signal of a binary black hole merger was directly detected for the first time [5] that makes black hole spectroscopy a new emerging interest. The quasibound states, quasinormal modes, and shadows of black holes are among the most interesting characteristic of such an astrophysical objects in the observational measurable spectra that is generated as particles crossing into the black hole [6]. Thus, the spectrum of quasibound states have complex frequencies where the real part is associated as the scalar's energy while the imaginary part determines the stability of the system. It is possible, in principle, to extract some information about the physics of black holes as well as to validate some alternative/modified theories of gravity from these quasibound states [6]. Analogously to atomic transitions emitting photons, level transitions of axions around black holes emit gravitons [7].

The spectroscopy of black hole is also a new emerging interest in condensed matter physics as many different kinds of analogue models has been proposed after Unruh's prediction, for example Bose–Einstein condensates, electromagnetic wave guides, graphene, optical black hole, sonic black hole and ion rings [8–12]. In optical domain, [13] proposed the idea of an optical black hole for the first time in the year of 2000. The idea is that propagation of light in a moving medium resembles many features of a motion in a curved space-time background. In this letter, we will be focus on photon-fluid system, which is a nonlinear optical system that is represented by hydrodynamic equations of an interacting Bosonic gas [14]. And analogous to classical and quantum fluids, long wavelength phonon excitations in an inhomogeneous flow propagate like a scalar field on a curved spacetime [15]. Thus, it is very crucial to be able to calculate the exact quasibound states frequency analytically.

However, due to the complexity of the equations involved, especially the radial equation, analytical methods were used less often and only for certain problems. The vast majority of these studies made use of numerical techniques such as the asymptotical analysis, WKB, and continued fraction to investigate the specific task at hand. Fortunately, very recently, [16–21] successfully finds novel exact scalar quasibound state solutions respectively around charged and chargeless Lense–Thirring black hole, Reissner–Nordström black hole where the radial equations of the scalar field are successfully solved in terms of the Double Confluent Heun functions and lastly for the case of $f(R)$ theory's static spherically symmetric black hole where the radial equations of the scalar field are successfully solved in terms of the General Heun functions.

The result obtained in this research can effectively recover the well-known approximated analytical result, i.e. in small black hole limit, $M_{black\,hole} \ll \frac{m^2_{Plank} c^2}{E_0}$-where m_{Plank} is Plank mass, E_0 is the scalar particle's rest energy, and c is the speed of light,-the imaginary part of the complex valued energy is suppressed [22].

In this present work, we are going to show in detail analytical derivation of exact solutions of relativistic massive and massless quasibound states around the BBMB black hole. We successfully solve the radial equation exactly in terms of Double Confluent Heun functions and it is for the first time that the Double Confluent Heun function is used to express black hole's quasibound states exact solutions. And having the exact solutions in the hand, the complex quantized energy levels expression is obtained from the Double Confluent Heun's polynomial condition. And finally, by applying the Damour–Ruffini method, the Hawking radiation of the apparent black hole's horizon is investigated and the Hawking temperature is obtained.

This letter is organized as follows: in Sect. 2, the quasibound states solutions are derived and the wave functions and the energy levels expressions are obtained. And in the Sect. 3, by using the obtained exact wave functions and applying the Damour–Ruffini method [23], the Hawking radiation is calculated.

2 The Klein–Gordon equation

In this section, we are going to consider the Klein–Gordon equation that represents the behaviour of both massive and massless scalar field with BBMB black hole background. We start with writing that relativistic wave equation [24],

$$\left\{\frac{1}{2}\hat{p}_\mu g^{\mu\nu}\hat{p}_\nu + \frac{1}{2}k^2c^2\right\}\psi = 0, \tag{14}$$

$$\hat{p}_\mu g^{\mu\nu}\hat{p}_\nu = -\hbar^2\nabla^2 = -\hbar^2\left[\frac{1}{\sqrt{|g|}}\partial_\mu\sqrt{|g|}g^{\mu\nu}\partial_\nu\right], \tag{15}$$

where the massive and massless cases are determined by the particle's rest energy per unit mass, $E_0 = kc^2$, where $k = 1$ for massive particles and $k = 0$ for massless particles.

Using (11), the Klein–Gordon equation can be rewritten as follows,

$$0 = \frac{E_0^2}{c^2\hbar^2} - \frac{1}{f}\partial_{ct}^2 + \frac{1}{r^2}\partial_r\left(fr^2\partial_r\right) + \frac{1}{r^2}\nabla_{\Omega_2}^2. \tag{16}$$

Considering the spherical symmetry of the equation, we apply this following ansatz of separation of variables,

$$\psi(t, r, \theta, \phi) = e^{-i\frac{E}{\hbar c}ct}R(r)Y_l^{m_l}(\theta, \phi), \tag{17}$$

where $Y_l^{m_l}(\theta, \phi)$ is the orthonormal spherical harmonics function which is the solution of this following spherical harmonics differential equation [25],

$$\nabla_{\Omega_2}^2 Y_l^{m_l}(\theta, \phi) = -l(l+1)Y_l^{m_l}(\theta, \phi). \tag{18}$$

Substituting the angular eigenvalue into the relativistic wave equation and multiplying the whole with $\frac{r^2}{\psi(t,r,\theta,\phi)}$, we get this following radial equation,

$$\left(\frac{E}{\hbar c}\right)^2\frac{r^2}{f} + \frac{1}{R}\partial_r\left(fr^2\partial_r R\right) - \frac{l(l+1)}{r^2}r^2 - \left(\frac{E_0}{\hbar c}\right)^2 r^2 = 0. \tag{19}$$

Substituting the explicit expression of the $f(r)$ and decomposing the first term followed by multiplying the whole equation with $\frac{1}{(r-r_H)^2}$, we get,

$$\partial_r^2 R + \left[\frac{2}{(r-r_H)}\right]\partial_r R + \left[\frac{E^2}{\hbar^2c^2}\frac{r^4}{(r-r_H)^4} - \frac{l(l+1)}{(r-r_H)^2} - \frac{E_0^2}{\hbar^2c^2}\frac{r^2}{(r-r_H)^2}\right]R = 0. \tag{20}$$

Since the region of interest is outside the horizon $r \geq r_H$, let us define this following new radial variable,

$$r - r_H = r_H y \rightarrow dr = r_H dy, \tag{21}$$

$$\partial_r = r_H^{-1}\partial_y, \tag{22}$$

that shifts the region of interest to be $y \geq 0$, we also define these following dimensionless energy parameters,

$$\Omega = \frac{Er_H}{\hbar c} \quad \Omega_0 = \frac{E_0 r_H}{\hbar c}. \tag{23}$$

In the new radial variable y, the radial equation is obtained as follows,

$$\partial_y^2 R + \left[\frac{2}{y}\right]\partial_y R + \left[\underbrace{\frac{(y+1)^2}{y^2}}_{A(y)}\left\{\Omega^2\frac{(y+1)^2}{y^2} - \Omega_0^2\right\} - \frac{l(l+1)}{y^2}\right]R = 0. \tag{24}$$

The $A(y)$ function can be expanded as follows,

$$\frac{(y+1)^2}{y^2} = 1 + \frac{2}{y} + \frac{1}{y^2}. \tag{25}$$

Substituting back to the radial equation, we can express everything in terms of $\frac{1}{y}$ as follows,

$$\partial_y^2 R + p\left(y\right) \partial_y R + q\left(y\right) R = 0, \tag{26}$$

where,

$$p(y) = \frac{2}{y}, \tag{27}$$

and,

$$q(y) = \left[\Omega^2 - \Omega_0^2\right] + \left[4\Omega^2 - 2\Omega_0^2\right]\frac{1}{y} + \left[6\Omega^2 - \Omega_0^2 - l(l+1)\right]\frac{1}{y^2} + \left[4\Omega^2\right]\frac{1}{y^3} + \left[\Omega^2\right]\frac{1}{y^4}. \tag{28}$$

We are going to transform the linear ordinary second order differential equation above into its normal form by following Appendix A as follows,

$$-\frac{1}{2}\partial_y p\left(y\right) = \frac{1}{y^2}, \tag{29}$$

$$-\frac{1}{4}p^2\left(z\right) = -\frac{1}{y^2}, \tag{30}$$

and finally obtain the normal form as follows,

$$\partial_y^2 Y\left(y\right) + K\left(y\right) Y\left(y\right) = 0, \tag{31}$$

$$Y\left(y\right) = yR(y), \tag{32}$$

where,

$$K\left(y\right) = -\frac{1}{2}\partial_y p\left(y\right) - \frac{1}{4}p^2\left(y\right) + q\left(y\right) = \left[\Omega^2 - \Omega_0^2\right] + \left[4\Omega^2 - 2\Omega_0^2\right]\frac{1}{y} + \left[6\Omega^2 - \Omega_0^2 - l(l+1)\right]\frac{1}{y^2} + \left[4\Omega^2\right]\frac{1}{y^3} + \left[\Omega^2\right]\frac{1}{y^4}. \tag{33}$$

2.1 The radial solution

Comparing the normal form above with the normal form of Double Confluent Heun's differential equation (see Appendix A), we can conclude as follows,

$$y = x, \tag{34}$$

$$\epsilon = 2\sqrt{\Omega_0^2 - \Omega^2}, \tag{35}$$

$$\gamma = 2i\Omega, \tag{36}$$

$$\delta = 2\left[1 + 2i\Omega\right], \tag{37}$$

$$\alpha = 4\Omega^2 - 2\Omega_0^2 + 2\sqrt{\Omega_0^2 - \Omega^2}\left[1 + 2i\Omega\right]. \tag{38}$$

Here, we have successfully obtained all of the Double Confluent Heun function's parameters. So, the novel exact solution of the Klein–Gordon equation in BBMB space-time can be written as follows,

$$\psi = \psi_0 e^{i\frac{E}{\hbar c}ct} Y_l^{m_l}\left(\theta, \phi\right)\Bigg[Ay^{\frac{1}{2}(\delta-2)}e^{\frac{1}{2}\left(\epsilon y - \frac{\gamma}{y}\right)} \times \text{HeunD}\left(\beta, \alpha, \gamma, \delta, \epsilon, y\right) + By^{-\frac{1}{2}(\delta-2)}e^{-\frac{1}{2}\left(\epsilon y - \frac{\gamma}{y}\right)} \times \text{HeunD}\left(-2 + \beta + \delta, \alpha - 2\epsilon, -\gamma, 4 - \delta, -\epsilon, y\right)\Bigg]. \tag{39}$$

2.2 Energy quantization

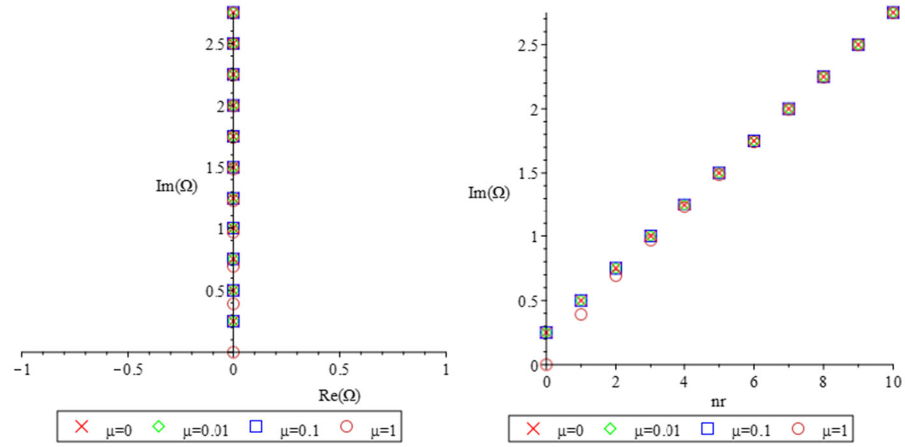
Now, consider the polynomial condition of the Double Confluent Heun see Appendix A Eq. (63). By substituting the values of the parameters explicitly, we obtain this following exact novel quantized energy expression that can only be discovered from the exact radial solution,

$$\frac{2\Omega^2 - \Omega_0^2}{\sqrt{\Omega_0^2 - \Omega^2}} + \left[1 + 2i\Omega\right] = -n_r. \tag{40}$$

Now, let us consider massless scalar case with rest energy parameter $\Omega_0 = 0$. Here we obtain a purely imaginary energy level as follows,

$$\Omega = i\frac{n_r + 1}{4}. \tag{41}$$

Here we visualize of the scalar's quantized spectrum in a complex plane for various scalar mass with radial quantum number $n = 0, 1, \ldots, 10$,



(a) Values of Ω for $\tilde{\mu} = 0, 0.01, 0.1, 1$ with $n_r = 0 \ldots 10$

(b) Values of $Im\Omega$ for $\tilde{\mu} = 0, 0.01, 0.1, 1$ with $n_r = 0 \ldots 10$

3 Hawking radiation

In this section, the BBMB black hole's Hawking radiation will be investigated. We are going it make use the successfully obtained exact solution of the radial wave function in order to derive the Hawking radiation distribution function that comes out of the apparent horizon of the BBMB black hole. Let us start with rewriting the complete solution of the wave function as follows,

$$\psi = \psi_0 e^{i\frac{E}{\hbar c}ct} Y_l^{m_l}(\theta,\phi)\Big[A y^{\frac{1}{2}(\delta-2)} e^{\frac{1}{2}\left(\epsilon y - \frac{\gamma}{y}\right)} \times \mathrm{HeunD}(\beta,\alpha,\gamma,\delta,\epsilon,y) + B y^{-\frac{1}{2}(\delta-2)} e^{-\frac{1}{2}\left(\epsilon y - \frac{\gamma}{y}\right)} \times \mathrm{HeunD}(-2+\beta+\delta,\alpha-2\epsilon,-\gamma,4-\delta,-\epsilon,y)\Big]. \tag{42}$$

Approaching the horizon r_H, we could expand the wave function in the lowest order of y where we also impose,

$$e^{\frac{1}{2}\left(-\frac{\gamma}{y\to 0}\right)}\,\mathrm{HeunD}(\beta,\alpha,\gamma,\delta,\epsilon,y\to 0) = C_1, \tag{43}$$

$$e^{\frac{1}{2}\left(\frac{\gamma}{y\to 0}\right)}\,\mathrm{HeunD}(-2+\beta+\delta,\alpha-2\epsilon,-\gamma,4-\delta,-\epsilon,y\to 0) = C_2 \to \infty. \tag{44}$$

Thus, the near horizon approximation can be rewritten in a complete wave function as follows,

$$\psi = \psi_0 e^{i\frac{E}{\hbar c}ct} Y_l^{m_l}(\theta,\phi)\left[AC_1 y^{\frac{1}{2}(\delta-2)} + BC_2 y^{-\frac{1}{2}(\delta-2)}\right]. \tag{45}$$

The wave function consists of two parts as follows,

$$\psi = \begin{cases} \psi_{+in} = AC_1 e^{i\frac{E}{c}ct} Y_\ell^{m_\ell}(\theta,\phi)\, y^{\frac{1}{2}(\delta-2)} \\ \psi_{+out} = BC_2 e^{i\frac{E}{c}ct} Y_\ell^{m_\ell}(\theta,\phi)\, y^{-\frac{1}{2}(\delta-2)} \end{cases}, \tag{46}$$

where ψ_{+in} is the ingoing wave and ψ_{+out} is the outgoing wave.

The Damour–Ruffini method works as follows. Suppose we have an ingoing wave hitting the apparent horizon at r_H and inducing a particle–antiparticle pair where the particle will enhance the reflected wave and the antiparticle will become the transmitted wave. Analytical continuation of the wave function $\psi\left(\frac{r-r_H}{r_H}\right)$ can be calculated as follows,

$$(y)^\lambda \to [y+i\epsilon]^\lambda = \begin{cases} y^\lambda, & r > r_H \\ |y|^\lambda e^{i\lambda\pi}, & r < r_H \end{cases}. \tag{47}$$

We can get the $\psi_{-out} = \psi_{+out}\left(y \to y e^{i\pi}\right)$ simply by changing $y \to -y = y e^{i\pi}$ as follows,

$$\psi_{-out} = BC_2 e^{i\frac{E}{c}ct} Y_\ell^{m_\ell}(\theta,\phi)\, y^{-\frac{1}{2}(\delta-2)}, \quad = \psi_{+out} e^{-\frac{1}{2}i\pi(\delta-2)} \tag{48}$$

$$\left|\frac{\psi_{-out}}{\psi_{+in}}\right|^2 = \left|\frac{\psi_{+out}}{\psi_{+in}}\right|^2 e^{-i2\pi(\delta-2)}, \tag{49}$$

and for spherical wave $l = 0$,

$$\left|\frac{\psi_{-out}}{\psi_{+in}}\right|^2 = \left|\frac{\psi_{+out}}{\psi_{+in}}\right|^2 e^{\frac{4\pi}{r_H}\left[\frac{E}{\hbar c} r_H^2\right]}. \tag{50}$$

Using the fact that total probability of the particle wave going out from the horizon and the antiparticle wave going in must be equal to 1, we obtain this following distribution function,

$$\left\langle \frac{\psi_{out}}{\psi_{in}} \middle| \frac{\psi_{out}}{\psi_{in}} \right\rangle = 1 = \left|\frac{BC_2}{AC_1}\right|^2 \left|1 - e^{\frac{4\pi}{r_H}\left[\frac{E}{\hbar c} r_H^2\right]}\right|, \tag{51}$$

$$\left|\frac{B}{A}\right|^2 = \left|\frac{C_1}{C_2}\right|^2 \frac{1}{e^{\frac{4\pi}{r_H}\left[\frac{E}{\hbar c} r_H^2\right]} - 1} \to 0. \tag{52}$$

Now, let us write the general black hole's apparent horizon's radiation distribution function,

$$\zeta(T_H) = \frac{1}{e^{\frac{\hbar\omega}{k_B T_H}} - 1}, \tag{53}$$

where k_B is the Boltzmann constant and $T_H = \frac{c\hbar}{4\pi k_B r_H}$.

By comparing (52) and (53), we can conclude that the radiation distribution function of the BBMB black hole's horizon possesses $\zeta(T_H) \to 0$ and it implies $T_H \to 0$. This means that the black hole's horizon has a zero Hawking temperature, thus, does not radiate. The zero temperature the black hole's horizon are also found in the case of extremal black holes where the space-time metric resembles the metric (11) in [26].

4 Conclusions

In this work, exact analytical massive and massless quasi-bound state's quantized energy levels and their wave functions in BMBB black hole background are presented in detail. The exact angular solution is found in terms of pure harmonics functions while the radial exact solutions are discovered in terms of Double Confluent Heun functions (39). The obtained exact radial solution is valid for all region of interest, i.e. $r_H \leq r < \infty$, a significant improvement of any asymptotical method that can only solves for either region very close to the horizon of very far away from the horizon which was presented in [27].

Using the polynomial condition of the Double Confluent Heun function (63), the quantized energy level expression is obtained (40). We also have presented the visualization of the quantized spectrum in a complex plane for various scalar mass with radial quantum number $n = 0, 1, \ldots, 10$.

The last section is dedicated to investigate the Hawking radiation via the Damour–Ruffini method [23] by making use of the priceless exact wave function. The same scenario as Klein pair production, where pair production is induced by an incoming particle then pair production occurs at the horizon, is used. The particle goes to infinity while the antiparticle goes inside the black hole and its negative energy will reduce the black hole mass itself. The probability function of Hawking radiation is derived from the modulus square of the ratio between particle and antiparticle wave functions. Finally, by comparing the radiation distribution function with the Boltzmann distribution, we obtain the fact that the BBMB black hole's horizon has a zero Hawking temperature, thus, it does not radiate (52).

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Code Availability Statement My manuscript has no associated code/software. [Author's comment: This manuscript has no associated code/software.]

Appendix A: Normal form

There is a trick to simplify a second order linear differential equation which is called normal form. With this method, we do not have to consider the far and near behaviour as we usually do when we deal with the ordinary differential equation. The solution for the aforementioned limit will present by itself in the solution. Let us consider this trick first before continue the calculation. Suppose we have a second order linear differential equation as follows,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \tag{54}$$

Now, let us define a new function as follows,

$$y = Y(x)e^{-\frac{1}{2}\int p(x)dx}, \tag{55}$$

$$\frac{dy}{dx} = \frac{dY}{dx}e^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}Ype^{-\frac{1}{2}\int p(x)dx}, \tag{56}$$

$$\frac{d^2y}{dx^2} = \frac{d^2Y}{dx^2}e^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}\frac{dY}{dx}pe^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}Y\frac{dp}{dx}e^{-\frac{1}{2}\int p(x)dx} + \frac{1}{4}Yp^2e^{-\frac{1}{2}\int p(x)dx}. \tag{57}$$

Substituting the expressions to (54), a lot of things cancel out and we obtain this following equation without the first order derivative term,

$$\frac{d^2Y}{dx^2} + \left(-\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q\right)Y = 0, \tag{58}$$

$$Y = ye^{\frac{1}{2}\int p(x)dx}. \tag{59}$$

Now, let us consider the Double Confluent Heun differential equation [28,29],

$$\frac{d^2y}{dx^2}+\left(\frac{\gamma}{x^2}+\frac{\delta}{x}+\epsilon\right)\frac{dy}{dx}+\left(\frac{\alpha}{x}-\frac{\beta}{x^2}\right)y=0, \quad (60)$$

$$y=A\,\text{HeunD}\,(\beta,\alpha,\gamma,\delta,\epsilon,x)+Be^{\frac{\gamma}{x}-\epsilon x}x^{2-\delta}$$
$$\times\,\text{HeunD}\,(-2+\beta+\delta,\alpha-2\epsilon,-\gamma,4-\delta,-\epsilon,x)\,, \quad (61)$$

but, for the sake of notation simplicity, we will write the solution using this following way,

$$y=HeunD(x)=A\,\text{HeunD}(x)+Be^{\frac{\gamma}{x}-\epsilon x}x^{2-\delta}\,\text{HeunD}'(x), \quad (62)$$

$$\frac{\alpha}{\epsilon}=-n_r,\quad n_r=0,1,2,\ldots. \quad (63)$$

Now, let us express Double Confluent Heun's differential equation in its the normal form by recognizing p and q function (see Appendix A). First, we recognize,

$$p=\frac{\gamma}{x^2}+\frac{\delta}{x}+\epsilon, \quad (64)$$

$$q=\frac{\alpha}{x}-\frac{\beta}{x^2}, \quad (65)$$

$$y=\text{HeunD}=Y(x)e^{-\frac{1}{2}\left(\epsilon x-\frac{\gamma}{x}\right)}x^{-\frac{\delta}{2}}, \quad (66)$$

and this leads to,

$$-\frac{1}{2}\frac{dp}{dx}=\frac{1}{x^2}\left(\frac{\delta}{2}\right)+\frac{\gamma}{x^3}, \quad (67)$$

$$-\frac{1}{4}p^2=-\frac{\epsilon^2}{4}-\frac{1}{x}\frac{\epsilon\delta}{2}-\frac{1}{x^2}\left(\frac{\delta^2+2\epsilon\gamma}{4}\right)$$
$$-\frac{1}{x^3}\frac{\delta\gamma}{2}-\frac{1}{x^4}\frac{\gamma^2}{4} \quad (68)$$

$$K(x)=-\frac{1}{2}\frac{dp}{dx}-\frac{1}{4}p^2+q=-\frac{\epsilon^2}{4}+\frac{1}{x}\left(-\frac{\epsilon\delta}{2}+\alpha\right)$$
$$+\frac{1}{x^2}\left(\frac{\delta}{2}-\frac{\delta^2+2\epsilon\gamma}{4}-\beta\right)+\frac{1}{x^3}\left(\gamma-\frac{\delta\gamma}{2}\right)$$
$$+\frac{1}{x^4}\left(-\frac{\gamma^2}{4}\right). \quad (69)$$

Combining everything, we get the Double Confluent Heun equation's normal form,

$$\frac{d^2Y}{dx^2}+K(x)Y=0, \quad (70)$$

$$Y(x)=\text{HeunD}\,e^{\frac{1}{2}\left(\epsilon x-\frac{\gamma}{x}\right)}x^{\frac{\delta}{2}}. \quad (71)$$

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