Fractional Charge in the Infinite Volume Limit

The origin of fractional fermionic charge is investigated by quantizing the system in a finite volume and examining the infinite volume limit. Special emphasis is given to the need to eliminate zero modes in order to have an unambiguous charge.

The occurrence of fractionally charged states in fermionic theories which describe integrally charged particles is of continuing interest.[1] This happens[2] when fermions move in a background field with nontrivial topology such as a scalar kink in one spatial dimension or a non-Abelian magnetic monopole in three spatial dimensions. The vacuum state in the presence of such a background is found to be a charge eigenstate[3] with fractional, even irrational,[4] charge.

Although this phenomenon has been given a solid mathematical development, the existence of eigenstates with fractional charge remains somewhat mysterious. How does it arise that a theory with integrally charged particles describes states with fractional charge? To answer this question we examine an appropriate one-dimensional fermionic theory quantized in a finite region. The advantage of quantizing in a finite volume is the discreteness of the energy spectrum; since the vacuum charge is given in terms of a sum over the single particle energy spectrum with each state contributing $+1$ (particle) or -1 (antiparticle), the vacuum charge cannot be an irrational number. In fact, for the particuar theory presented here the total vacuum charge vanishes. However, the spatial variation of the charge density is such that there is an accumulation

Comments Nucl. Part. Phys.
1985, Vol. 15, No. 1, pp. 35–40
0010-2709/85/1501-0035/$15.00/0

of charge at the spatial boundary. When the infinite volume limit is taken, this boundary charge escapes to infinity, leaving behind a fractional charge.

To be specific, we study the fermionic charge in one spatial dimension

$$Q = \int dx\, j^0(x). \tag{1}$$

The current operator's expectation value may be computed in terms of the fundamental bilinear,[5] the Green's function, G, satisfying

$$\{E - H\}\, G(E) = 1. \tag{2}$$

In the above we use an abstract operator notation with H the Dirac Hamiltonian describing the interaction of an otherwise free fermi field with a background potential. In terms of this Green's function, the expectation value of the charge density has the expression

$$\langle j^0(x)\rangle = \int \frac{dE}{2\pi i}\, \mathrm{tr}\langle x|G(E)|x\rangle, \tag{3}$$

with tr denoting a diagonal sum over any discrete indices. The specification of the state in which $j^0(x)$ is to be evaluated supplies

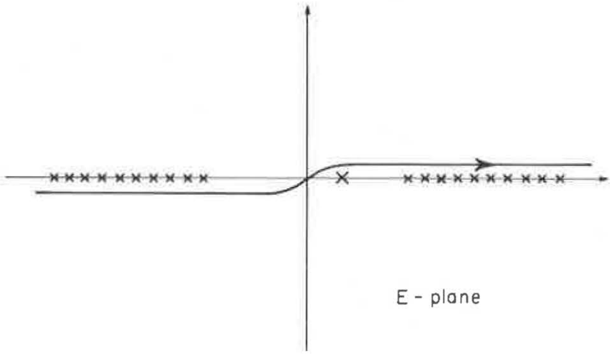

FIGURE 1 The contour of integration in the complex energy plane which defines the Green's function. Crosses denote the location of poles of $G(E)$.

the boundary condition which uniquely defines the otherwise undefined integration contour over the energy Green's function in Eq. (3). For the ground state, this is the familiar positive frequency condition,[6] which requires the contour shown in Fig. 1. This prescription uniquely defines the ground state Green's function and the vacuum charge so long as there are no normalizable zero energy bound states of the single particle Hamiltonian,

$$H\psi_0 = 0. \tag{4}$$

If such a zero mode is present, then an ambiguity arises which cannot be resolved by the usual positive frequency condition. One may define $G(E)$ by either running the contour of integration in Fig. 1 above or below this zero energy pole, corresponding to whether or not the state of ψ_0 is filled. Clearly, the two ground states so defined are degenerate in energy. Without physical motivation to pick one or the other state, the theory is ill defined and correspondingly, so is the ground state charge. It is often the case that in the presence of nontrivial background fields, the Dirac Hamiltonian does possess normalizable zero energy states. In fact, the original model for fractional fermionic charge[2] has a single normalizable zero mode. The vacuum charge was found to be $\pm 1/2$, with the ambiguity in sign due precisely to the ambiguity described above. This ambiguity must be resolved in order to make precise the notion of fractional charge. The resolution is to modify the interaction potential so as to move any normalizable bound states away from zero. With such a modification, the vacuum charge remains fractional and the original, unmodified theory is regained in a well-defined limit.[4]

An appropriate Hamiltonian in one spatial dimension is

$$H = \alpha p + \beta m\, e^{i\alpha\theta(x)}. \tag{5}$$

The matrices α and β are the Dirac matrices satisfying

$$\{\alpha, \beta\} = 0, \tag{6}$$

$$\alpha^2 = 1 = \beta^2, \tag{7}$$

while the constant m is the mass of the fermion and p is the single particle momentum operator. The background potential $\theta(x)$ is given by a time-independent step function

$$\theta(x) = \begin{cases} \theta_+, & x > 0, \\ \theta_-, & x < 0, \end{cases} \tag{8}$$

$$\Delta\theta = \theta_+ - \theta_-, \quad 0 < \Delta\theta < \pi. \tag{9}$$

This configuration is the simplest with the topology of a kink, and is sufficient to illustrate the main features of fractionization. The particular form of the Hamiltonian is motivated by allowing the mass m to vary spatially and writing

$$m\, e^{i\alpha\theta} = \phi_0 + i\alpha\phi_1, \tag{10}$$

where

$$\phi_0 = m \cos\theta; \quad \phi_1 = m \sin\theta. \tag{11}$$

Upon setting ϕ_0 to zero, the Hamiltonian becomes

$$H' = \alpha + i\beta\alpha\phi_1, \tag{12}$$

with ϕ_1 also having a kink behavior. This Hamiltonian is the original model for charge fractionization,[2] and as discussed earlier is ill defined due to the presence of a zero energy bound state. The addition of the nonzero background, $\phi_0 \neq 0$, serves to move this state away from zero giving a unique ground state with an unambiguous charge.

For the system quantized in a finite region, $-L \leq x \leq L$, boundary conditions must be adjoined which maintain the Hermiticity of the Hamiltonian and the charge operator. For simplicity, periodic boundary conditions are taken, although the results presented here are not specific to this choice. It is not difficult to calculate the ground state Green's function. Rotating the integration contour to run along the imaginary axis, we find that

$$\langle j^0(x)\rangle = -\frac{m^2}{2}\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{(\sin\Delta\theta)\sinh\kappa(L - 2|x|)}{(\sinh\kappa L)(m^2\cos^2(\Delta\theta/2) + \epsilon^2)}, \tag{13}$$

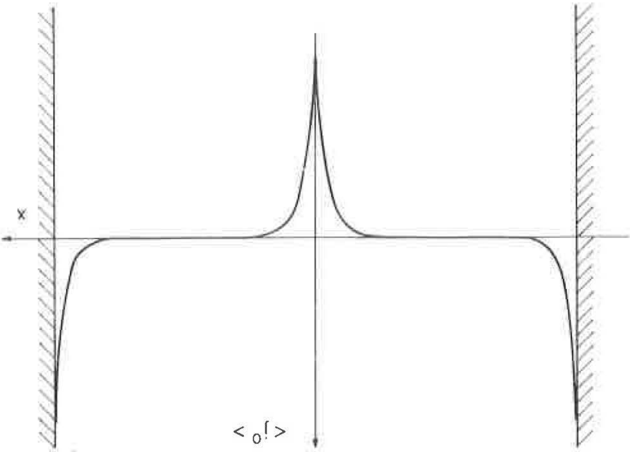

FIGURE 2 The fermionic charge density for a box with $mL = 10$ and $\Delta\theta = \pi/2$. The charge at the kink is $-1/4$.

where $\kappa = \sqrt{m^2 + \epsilon^2}$. For the physically relevant case of the box much larger than the Compton wavelength of the particle, $mL >> 1$,

$$\frac{\sinh \kappa(L - 2|x|)}{\sinh \kappa L} \approx e^{-2\kappa|x|} - \epsilon^{-2\kappa(L-|x|)}, \tag{14}$$

and one sees that the charge density has the behavior shown in Fig. 2. The charge at the boundaries ($x = \pm L$) is opposite in sign to the charge at the kink. The occurrence of an accumulation of charge at the location of the kink is expected, while the accumulaton of charge at $x = \pm L$ is explained by the boundary conditions—with periodic boundary conditions there is an effective kink at the boundary of equal magnitude but opposite sign to that at the center. By integrating the expression for $\langle j^0(x)\rangle$ over the full spatial domain, the total charge is easily seen to vanish.

One is tempted to conclude that the total charge also vanishes in the infinite box limit, $L \rightarrow \infty$. However, this limit is nonuniform. By taking the boundaries to infinity before spatially integrating over the charge density, one finds instead of zero,

$$\langle Q \rangle_{L=\infty} = \int_{-\infty}^{+\infty} dx \lim_{L \to \infty} \langle j^0(x) \rangle_L = -\frac{\Delta\theta}{2\pi}, \tag{15}$$

which is the usual result for the infinite length system.[4] This demonstrates that fractional charge is regained in the $L \to \infty$ limit by having the compensating charge at the spatial boundaries escape to infinity.[7] These boundary charges decouple from the theory as verified from a calculation of the charge density–charge density correlation function. The correlation between charge at the boundaries and that near the kink vanishes as the system size grows.[3]

One's intuition, that since each energy level contributes an integer (± 1) to the ground state charge, then that charge must be a natural number, is now seen to be correct. In a finite volume, the spectrum is discrete and this intuition applies. It is only in the infinite volume limit, where the spectrum is no longer discrete, that this intuition fails and an irrational charge can appear.

Acknowledgment

This work was supported in part by the U.S. Department of Energy, contract number DE-AC06-81ER-40048.

MARC D. BERNSTEIN and LOWELL S. BROWN
University of Washington,
Seattle, Washington 98195

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