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Introduction

a) Why string theory ?

In order to grasp the questions string theory is primarily trying to answer, it is useful to take a look back to the state of theoretical physics in the second half of the XXth century. During the first half, two major breakthroughs had revolutionised our understanding of the laws of nature and granted us access to new scales where classical physics would fail.

The first of these breakthroughs was Quantum Mechanics. Quantum mechanics (QM) are necessary to understand physical systems below a given length scale, related to Planck's constant $\hbar$ and typically of the order of the nanometre. They state that nature may not be understood through deterministic paradigms as small systems are inherently probabilistic ; it is no longer possible to predict exactly how a system in an initial state will evolve in time - at least in general - and one has to accept that the only accessible theoretical data are probabilities the system has to be in a given state. Unlike in classical physics, where probabilistic features came into play as a palliative to the shortfall of either our experimental setups or our computational power - or both - and where access to unlimited data and infinitely fast computers would allow to solve exactly any problem in theory, the probabilistic nature of QM is not merely a reflection of our own technological weaknesses but a genuine and inherent property of nature. Another new feature compared to previous theories is that QM tells us that an observer cannot remain a purely external spectator anymore as the action of measuring any quantity has an effect on the system of interest. In other terms, to the famous question "If a tree falls in a forest and no one is around to hear it, does it make a sound ?", QM would answer that the tree would not have fallen in the first place if it had not been observed ; instead, it would have been in a mixed state, both fallen and still standing up in a sense, until a look is taken at it, *projecting* it into one of the two above states and sealing its fate. Some renowned experiments are, for instance, the Davisson–Germer double-slit experiment which exhibited the wave-like nature of matter and the Stern–Gerlach experiment which showed the angular momentum quantification.

The second breakthrough was Einstein's General Relativity. General relativity (GR) is a theory of gravity and allows to grasp highly energetic systems, the appropriate scale being there the speed of light c. GR states that space and time are intrinsically related and should be described as a whole - that is as space-time - instead as independently of one another. In contrast with Newton's laws of physics where time was an absolute and unalterable quantity, its perception is generically different for two observers in Einstein's theory. Moreover, there exists a finite limiting speed c - meaning in practice that it would take an infinite amount of energy to exceed c - which is strongly believed to be the speed of light from experimental considerations. In addition to these novelties, space-time itself is no longer considered as some immutable object within which the universe evolves ; instead, it may be deformed by matter - or more generally by the presence of *energy* - and should be considered dynamical. According to GR, gravity is actually a consequence of these local deformations as it states that free falling particles follow space-time geodesics, which are not straight lines when space-time is not a Minkowski space. Experimental supports for GR are the fact that it does explain the perihelion precession of Mercury and the notable Eddington's experiment which showed that light is deflected by massive systems - namely by the Sun in this case.

Besides from never having been proven wrong so far, both of the above theoretical frameworks have granted physicists access to a new range of phenomena, from particle physics to cosmology, and are pillars of modern physics. However, while QM and GR have answered a lot of fundamental questions, they have also raised new ones. Taking both special relativity, which is the restriction of GR in Minkowski space or equivalently GR where gravity is neglected, and QM into account leads to Quantum Field Theories (QFT) and allows to describe particle physics. This way, three out of the four known fundamental interactions of nature have been understood in a unified paradigm, namely the electromagnetic, weak and strong interactions. Yet, the fourth one - gravity - is a lot harder to incorporate into a quantum theory. If one tries to naively quantise GR, the resulting theory is nonrenormalisable, meaning that one would need an infinite amount of experimental data in order to be able to make a computation, taking away the predictive power of the theory ; as such, it may at most be understood as an effective field theory whose UV-completion remains out of reach. Phrased differently, naive quantisation of GR does not grant information on the UV sector, that is on what happens at length scales small compared to the Plank length, which is of the order of $10^{-35}m$. Another difficulty compared to the three other fundamental interactions is that gravity is by far the weakest interaction and, as such, far from the reach of the current particle accelerators [1]. Therefore, unlike most problems encountered in physics so far, the road toward a quantum mechanical description of gravity may hardly be paved with experimental cobblestones.

Along the years, several scenarios have emerged in order to give a quantum mechanical meaning to gravity, none of which having undoubtedly discarded the others for the moment. It turns out that the candidate for a quantum theory of gravity we will deal with in the following has not historically been developed for this matter. Instead, in the late 1960's, the foundations of *string theory* have been laid as an attempt to understand the dynamics of hadronic particles. The discovery of quantum chromodynamics supplanted it as a theory of strong interactions due to various theoretical difficulties inherent to string theory, such as the presence of a massless particle of spin 2, in contradiction with experiment, and of a tachyon, revealing the instability of the theory, in the spectrum. Nonetheless, after a brief lost of interest, string theory became even more appealing as before when Schwarz, Scherk and Yoneya realised that the aforementioned massless spin 2 particle could be interpreted as a graviton [2, 3], thereby providing a candidate for a quantum theory of gravity whose low-energy limit identifies to Einstein's GR.

b) The dimension issue

Now that the reasons for being interested in finding a quantum description of gravity have been clarified, we can take some time to explain roughly what string theories are and some

1. Gravity is roughly 10^{25}, 10^{36} and 10^{38} times weaker than the weak, electromagnetic and strong interactions respectively. The explicit value of the corresponding coupling constants may be found, *e.g.*, in [1].

of the questions they raise. The standard description of particles in quantum field theory is that of point-like objects whose trajectory in space-time spans a 0+1-dimensional manifold referred to as the *world-line*. In contrast, strings are one-dimensional objects and sweep, as such, a 1+1-dimensional *world-sheet* instead. The simplest string theory is a 2-dimensional non-linear sigma model (to be defined in section II.2.2.b)) on the world-sheet taking values in a D-dimensional target space interpreted as space-time ; it is known as bosonic string theory. Although bosonic string theory has been historically invented first, superstring theories were soon developed in order to take care of some phenomenologically unpleasant features of bosonic string theory - namely the fact that its spectrum only contains bosonic excitations, explaining the name of the theory, and a tachyon, reflecting its instability. Superstring adds to bosonic theories superpartners to the world-sheet bosonic fields, leading to tachyon-free theories [2] whose spectrum contains bosons and fermions.

Both bosonic and superstring theories exhibit local conformal symmetry at the classical level. In order to quantise these theories in a consistent way, one must require that no anomaly comes into play and spoils this invariance. The corresponding anomaly is usually known as the Weyl or trace anomaly, the latter being due to its relation with the trace of the stress-energy tensor. Indeed, conformal invariance classically forces this tensor to be traceless ; one may show (see, *e.g.*, [5]) that local conformal invariance is preserved in the quantum theory if and only if the vacuum expectation value (VEV) of the trace of the stress-energy tensor vanishes as well. Moreover, this VEV may be computed to read

$$\langle T^a{}_a \rangle = -\frac{c}{12} R$$

with T the stress-energy tensor, R the world-sheet Ricci scalar and c the *central charge* of the theory. In other words, this means that a string theory may only be consistent if its central charge vanishes.

In order to grasp the consequences of this claim, let us list the various contributions to the central charge of a string theory. Each free world-sheet boson contributes as 1 to c and each free world-sheet fermion as 1/2. BRST quantisation also includes ghosts in the theory which add a negative contribution to the overall central charge ; one computes that $c_{\text{ghosts}} = -26$ in the bosonic case and that $c_{\text{ghosts}} = -15$ in the superstring one. This leads to a common major phenomenological flaw of both bosonic and superstring theories - or at least what would seem like it at first glance : both theories are only consistent if the target-space, that is space-time, has strictly more than 4 dimensions. The corresponding dimension is called the *critical dimension* ; the critical dimensions for the bosonic and superstring theories are $D_{\text{bosonic}} = 26$ and $D_{\text{super}} = 10$ respectively [3]. Historically, the peculiarity of the $D = 26$ version of bosonic string theory has first been noticed in [6], where it was computed that only in this case would branch cuts cancel out in propagators to be replaced by poles as they should.

This raises an obvious question : how to make sense of a theory claiming that space-time is not four-dimensional ? We will see in chapter II what theoretical tools may be used to do so and even how higher-than-four-dimensional theories may be attractive on their own right. The simplest answer to this problem would be to postulate that space-time may be expressed

2. This is actually only true if one keeps only some of the states of the theory in a consistent way ; the corresponding procedure is the GSO projection developed in [4].

3. There also exists strings theories known as *non-critical* which are obtained by finding a way to increase the central charge without adding new bosons (and fermions in the superstring case). A common example is the case of a linear dilaton background, see [5] for more details.

as the direct product of the more familiar Minkowski space and some compact space $\mathcal{K}$ small enough to have escaped detection up to this day. However, as the only restriction on what we will refer to as the *internal space* $\mathcal{K}$ is its dimension, the number of available geometries is enormous if even finite. We will see however that the details of these geometries lead to different laws of physics from our four-dimensional perspective ; this has allowed string theorists, if not to identify a unique possible geometry, to classify them in terms of their phenomenological relevance. Some features will be of particular interest in this respect : the first one is the amount of space-time supersymmetry in the four-dimensional theory. Indeed, the only known consistent string theories naturally incorporate (world-sheet) supersymmetry ; phenomenologically speaking, $\mathcal{N} > 1$-supersymmetric theories prevents chiral interactions and may as such not describe the world we live in. However, with great amount of supersymmetry comes great control explaining why models with $\mathcal{N} > 1$ are still very popular. The second interesting feature descend from the fact that the *conformal field theory* with target space $\mathcal{K}$ may typically be continuously deformed into equally valid models ; renormalisation considerations show that some of these deformations - known as *truly marginal deformations* - preserve the local conformal invariance at the quantum level and therefore lead to equally valid models. To each of these deformations corresponds a massless scalar field in the low-energy dimensional theory which is known as a *modulus* of the theory ; in quantum field theoretical parlance, the moduli correspond to flat directions in the scalar potential. From an ideological point of view, these massless scalars are not desirable ; indeed, their vacuum expectation values are unconstrained by definition and may as such only be fixed by experimental data instead of theoretical predictions. From a phenomenological point of view, they are even worse : it would be hard to explain how these massless scalars could have stayed hidden from the current particle physics experiments, and the more moduli the harder. For all these reasons, a common concern when studying reductions of string theories down to four dimensions is *moduli stabilisation*, that is strategies to decrease the number of massless scalar fields present in the low-dimensional model.

Space-time supersymmetry constraints will be shown in section II.1 to restrict the *holonomy group* of the internal space, which represents roughly how an object transforms when travelling along loops on a manifold. Several techniques have also been found in order to deal with the typically great amount of moduli in the four-dimensional theories such as *flux compactifications*, which are models in which electromagnetic field strengths (or higher generalisations thereof) are given a non-trivial VEV. Another possible (and flux-compatible) approach is given by noticing that our usual notion of geometry may break down at the level of strings. This is a reflection of the fact that the geometries probed by strings at scales below the Planck length are very different from those probed by point particles. A very common consequence of this, which we shall analyse and generalise in section I.3, is the fact that string theories see no difference between a circle of radius R and another one of radius $1/R$ (in appropriate units). This means in particular that the geometric tools we have been mentioning so far, although quite useful, may hide part - and probably most - of the picture. Instead of considering an internal space $\mathcal{K}$, one could indeed impose the weaker - but more fundamental - vanishing constraint of the central charges of the two decoupled CFTs, leading to more general setups. We will refer to dimensional reductions which do not admit a geometric interpretation as *non-geometric models*. Such constructions may preserve any amount of space-time supersymmetry - or more precisely any amount allowed by quantum field theory considerations - unlike their geometric counterparts while typically exhibiting less moduli. Moreover, unlike flux compactifications, they have the advantage of admitting exactly solvable CFTs.

Non-geometric constructions constitute the main interest of the present thesis. We will mostly restrict to models admitting 8 supercharges - that is $\mathcal{N} = 2$ supersymmetry in four dimensions - for various reasons. First, while theories with $\mathcal{N} > 2$ are well-understood, they are too constrained to exhibit interesting features ; on the other hand, models with $\mathcal{N} < 2$ are a lot richer but also a lot harder to handle. $\mathcal{N} = 2$ theories lie in between these two extreme cases, offering both interesting and manageable theories. Then, dual pairs of such models play a key role in string theory as they typically grant access to non-perturbative features through, for instance, BPS states - which are protected states related to the existence of central charges in the supersymmetry algebra - or non-renormalisation theorems. The models we present here provide a new class of non-geometric $\mathcal{N} = 2$ pairs of dual theories ; we will also show how the conjectured duality allows to make non-perturbative predictions for theories living in a non-geometric background.

This work is organised as follows : we will start by reviewing notions which are present between the lines all along this thesis. More precisely, we will review various dimensional reduction techniques and some of their features in chapter I, from a quantum field theory point of view in sections I.1 and I.2 and from a conformal field theory perspective in I.3 and I.4. In a second phase, we specify the analysis to the reduction of superstring theories which naturally lead to space-time supersymmetry considerations as we mentioned above.

In chapter II, we therefore explain how geometry of the internal manifold and supersymmetry are correlated before specialising to the case of geometric constructions preserving 16 supercharges ; this allows us in particular to introduce the well-known duality between the type IIA and heterotic string with a K3 surface and a four-torus as respective internal spaces which will for our purposes. We then turn to reductions of closed strings on backgrounds preserving $\mathcal{N} = 2$ supersymmetry in four dimensions, which is also the case of the models to be considered later. The idea here is also to present some of the key features of such geometric and flux-free models in order to enlighten the difference with the constructions presented in chapter III. Finally, we give a brief taste of non-geometric constructions introducing in particular the notions of gauged supergravities, T-folds and mirror-folds of which the models of the following chapter are examples.

In chapter III, we have first chosen to review the family of mirror-folds of the type IIA string developed in [7] despite the fact that the author of the present thesis was not involved in their construction. This is motivated by the fact that an important part of this thesis has been devoted to understand these models by identifying dual heterotic theories thereof. The rest of this chapter is dedicated to presenting the original results corresponding to the first publication [8] of this thesis. The dual models of the non-geometric type IIA models are constructed in section III.2 and section III.3 introduces the perturbative BPS states arising in the heterotic picture and computes the corresponding protected indices. Sections III.4 and III.5 are devoted to an in-depth analysis of the duality consistency conditions and to deriving additional dual frames for these models.

In chapter IV, we take some time to review generic features of $\mathcal{N} = 2$ supergravities in four dimensions. This will allow us to introduce tools useful in deriving the perturbative corrections to the part of the moduli space spanned by scalars living in vector multiplets. In particular, we show how perturbative dualities of the theory translate to modular constraints on the associated prepotential.

In chapter V, we present the results constituting the second paper [9] produced during this thesis which is devoted to understanding the moduli space of the models introduced in chapter III. While the heterotic description allows for an exact derivation of the hypermultiplet moduli space both in the string tension α' and in the string coupling g_s, determining quantum corrections to the vector moduli space requires more work. However, we show that the method introduced in [10, 11] which exploited the modular properties of the prepotential may be used in our case to give access, at least theoretically, to the corrections to the corresponding factor of the moduli space. An explicit computation example is given for the simplest model at our disposal, namely the $p = 2$ case with the notations of chapter III.

Chapitre I

Dimensional reductions

I.1 Kaluza-Klein reduction

The interest of theorists for physical models in more than four dimensions may be traced back to the early XXth century with the seminal work of Kaluza [12]. When considering such theories, an immediate problem would be to understand how extra dimensions could have escaped detection for so long. According to Kaluza, there may be a simple explanation to this : if these additional dimensions are compact and small enough, our current particle accelerators may not be able to see them. This may be illustrated by the example of a free real massless scalar field living in a five-dimensional space-time with action

$$S = -\frac{1}{2}\int_{\mathcal{M}_5} \mathrm{d}^5\mathbf{X}\,(\partial_M\Phi)^2 \tag{I.1}$$

in natural units. In (I.1), $\mathcal{M}_5$ is an arbitrary five-dimensional manifold with coordinates $\left\{X^M\right\}$ and signature $(-++++)$; the associated equation of motion is

$$\partial_M\partial^M\Phi = 0.$$

Let us now assume that $\mathcal{M}_5$ may be expressed as a product space containing the more familiar four-dimensional Minkowski space $\mathcal{M}^{1,3}$ as a factor. Requiring the remaining bit of $\mathcal{M}_5$ to be compact lets few freedom as any compact one-dimensional manifold is either topologically equivalent to a circle or to a closed interval. In order to avoid boundary issues, we will only consider the former case meaning that $\mathcal{M}_5$ should be expressible as

$$\mathcal{M}_5 = \mathcal{M}^{1,3}\times S^1$$

with S^1 a circle of radius R. We decompose the $\mathcal{M}_5$ coordinates as $\{\mathbf{X}\} =: \{\mathbf{x}, y\}$ so that $\{\mathbf{x}\}$ and y span $\mathcal{M}^{1,3}$ and S^1 respectively. In particular, y must be identified to $y + 2\pi R$.

Classically, one may deduce an action expressed as the integral of a Lagrangian density over $\mathcal{M}^{1,3}$ by performing the integration over y in (I.1), the resulting four-dimensional action being equivalent to the original one. In order to do so, it is useful to expand $\Phi(\mathbf{x}, y)$ over a basis of functions of y. A convenient basis is given by the eigenfunctions of the internal Laplacian ∂_y^2 as we now illustrate. These are expressed as

$$\psi_n(y) := \frac{1}{\sqrt{2\pi R}}e^{\frac{in}{R}y}$$

as compatibility with the $y \cong y + 2\pi R$ identification requires ; the normalisation is chosen so that $\int_{S^1} \mathrm{d}y \psi_n \psi_m = \delta_{n,-m}$. The five-dimensional field $\Phi(\mathbf{x}, y)$ may be expanded as

$$\Phi(\mathbf{x}, y) = \sum_{n\in\mathbb{Z}} \phi_n(\mathbf{x}) e^{\frac{in}{R} y}, \tag{I.2}$$

with the reality of Φ implying $\overline{\phi}_n(\mathbf{x}) = \phi_{-n}(\mathbf{x})$. The Fourier modes $\phi_n(\mathbf{x})$ may then be seen as fields in the four-dimensional theory. Integrating over the circle variable y in (I.1) leads straightforwardly to the four-dimensional action

$$S = -\frac{1}{2} \int \mathrm{d}^4\mathbf{x} \sum_{n\in\mathbb{Z}} \left(|\partial_\mu \phi_n|^2 + \frac{n^2}{R^2} |\phi_n|^2 \right). \tag{I.3}$$

In summary, starting from a single massless scalar field in five dimensions, we end up with an infinite number of scalar fields in four. Moreover, almost all these fields are massive as ϕ_n has mass $m_n^2 := n^2/R^2$; only the zero-mode ϕ_0 remains massless. This illustrates how a small enough extra dimension may not be visible neither in everyday life nor in experiments. For R small enough, all non-zero modes of Φ would have very large masses and would therefore be out of reach from our current particle accelerators. In the range of energies accessible to us, only the zero mode ϕ_0 would be visible and its higher-dimensional origin far from being obvious. This is usually what is assumed in string theory as one expects massive particles to have masses of the order of Planck's mass.

Even though the actions (I.1) and (I.3) are classically equivalent, dealing with an infinite tower of fields is not a very appealing feature ; however, considering that all massive fields are too heavy to be observed, an effective field theory may be derived by retaining only a finite number of light fields. Performing a dimensional reduction therefore also implies to keep only a part of the spectrum in a way which is consistent with the original theory. This procedure is called a *truncation* and is a rather subtle subject in general as setting all massive fields to zero does not lead to a theory equivalent to the higher-dimensional one in general. Consistency of a truncation may either be checked explicitly or argued, typically using symmetry arguments [1]. In the following however, we will always assume that truncation to the massless sector is a consistent procedure.

Let us slightly generalise our above example by reducing a $(D+1)$-dimensional theory defined on $\mathcal{M}_{D+1} := \mathcal{M}_D \times S^1$ for some D-dimensional manifold $\mathcal{M}_D$ and for S^1 of radius $1/(2\pi)$ (the generalisation to an arbitrary radius being straightforward). We start by considering pure gravity following [12]. Keeping only the massless modes amounts here to requiring all derivatives with respect to the circle coordinate to vanish ; this condition is usually referred to as the *cylinder condition.* In natural units, the pure gravity action is given by

$$S = \int \mathrm{d}^{D+1}\mathbf{X} \sqrt{-G} \hat{\mathsf{R}}$$

with G_{MN} the metric on $\mathcal{M}_{D+1}$ and $\hat{\mathsf{R}}$ the associated Ricci scalar. As before, we split the coordinates on $\mathcal{M}_{D+1}$ as $\left\{X^M\right\} =: \{x^\mu, y\}$ with $\mathbf{x}$ and y the coordinates on $\mathcal{M}_D$ and S^1 respectively. We follow the common convention in which M (μ) labels tangent indices on $\mathcal{M}_{D+1}$ $(\mathcal{M}_D)$. The metric G_{MN}, which does not depend on y by the cylinder condition, may easily be decomposed into representations of the D-dimensional Lorentz group. From

1. For a more detailed discussion, see *e.g.* [13].

a D-dimensional point of view, $G_{\mu\nu}$, $G_{\mu y}$ and G_{yy} indeed transform as a metric, a vector and a scalar field under Lorentz transformations. However, it turns out to be more natural to consider different combinations of the above degrees of freedom in order to define the associated D-dimensional field. We parametrise the line element as

$$\mathrm{d}\hat{s}^2 := e^{2\alpha\phi}\,\mathrm{d}s^2 + e^{2\beta\phi}(\mathrm{d}y + A)^2 \tag{I.4}$$

instead, with $\mathrm{d}s^2$ the line element on $\mathcal{M}_D$, A the Kaluza-Klein one-form, ϕ the radion and α, β two constants. The Ricci scalar $\hat{\mathsf{R}}$ may then be computed in terms of the metric on $\mathcal{M}_D$, A and ϕ (see *e.g.* [14] for more details about the derivation) ; after having performed the integration over y, the result reads, up to boundary terms,

$$S = \int \mathrm{d}^D\mathbf{x}\, e^{(\beta+(D-2)\alpha)\phi}\sqrt{-g}\left[\mathsf{R} - \Big(\alpha^2(D-1)(D-2) + \beta(\beta+\alpha(D+2))\Big)(\partial\phi)^2 \right.$$
$$\left. - \frac{1}{4}e^{2(\beta-\alpha)\phi}F_{\mu\nu}F^{\mu\nu}\right]$$

with R the Ricci scalar associated to the metric on $\mathcal{M}_D$, $F := \mathrm{d}A$ the field strength associated to the Kaluza-Klein one-form A. Going back to the Einstein frame - in which the Lagrangian is $\mathcal{L} = \sqrt{-g}\mathsf{R} + \ldots$ - may be done by setting [2] $\beta = -(D-2)\alpha$. Requiring in addition the field ϕ to be properly normalised fixes α (up to a sign) and one obtains the D-dimensional form of the action

$$S = \int \mathrm{d}^D\mathbf{x}\,\sqrt{-g}\Big(\mathsf{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}F_{\mu\nu}F^{\mu\nu}\Big),$$
$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \quad \beta = -(D-2)\alpha.$$

In summary, starting from pure gravity in $(D+1)$ dimensions, one obtains a D-dimensional theory incorporating gravity, a massless scalar field and a $U(1)$ gauge field. Moreover, the radion ϕ plays a role similar to a coupling constant for the gauge field A. Although simple, this example is a nice illustration of why theories with more than four dimensions may be interesting on their own. Indeed, they allow to unify in a rather straightforward way gravity and (so far, abelian) gauge theories which seems like an appealing feature.

Since we are ultimately interested in theories with local supersymmetry, we should also be concerned with what happens to differential forms of arbitrary rank. Let then $\hat{A}^{(p)}$ be a p-form defined on $\mathcal{M}_{D+1}$. The ansatz (I.4) suggests a natural choice of vielbein basis $\left\{\hat{\theta}^A\right\}$ with $A = 1, \ldots, D+1$ as

$$\hat{\theta}^a = e^{\alpha\phi}\theta^a,\ a = 1, \ldots, d,$$
$$\hat{\theta}^{D+1} := \hat{\theta}^y = e^{\beta\phi}(\mathrm{d}y + A) \tag{I.6}$$

with $\{\theta^a\}$ a vielbein basis for the metric defined on the lower-dimensional manifold $\mathcal{M}_D$ and with α, β as before. The field strength $\hat{F}^{(p+1)} := \mathrm{d}A^{(p)}$ may then be decomposed as

2. As $e^{2\beta\phi} = G_{yy}$, this may only be done if the theory is reduced down to $D \neq 2$ dimensions.

$$\begin{aligned}\hat{F}^{(p+1)} &= \frac{1}{(p+1)!}\hat{F}^{(p+1)}_{A_1\ldots A_{p+1}}\hat{\theta}^{A_1}\wedge\ldots\wedge\hat{\theta}^{A_{p+1}}\\ &= \frac{e^{\alpha(p+1)\phi}}{(p+1)!}\hat{F}^{(p+1)}_{a_1\ldots a_{p+1}}\theta^{a_1}\wedge\ldots\wedge\theta^{a_p} + \frac{e^{(\alpha p+\beta)\phi}}{p!}\hat{F}^{(p+1)}_{a_1\ldots a_p y}\theta^{a_1}\wedge\ldots\wedge\theta^{a_p}(\mathrm{d}y+A).\end{aligned}$$

It is therefore natural to decompose $\hat{F}^{(p+1)}$ as $F^{(p+1)}+F^{(p)}$ with $F^{(p+1)}$, $F^{(p)}$ two differential forms defined on $\mathcal{M}_D$ with components being given in the local frame basis by

$$\begin{aligned}F^{(p+1)}_{a_1\ldots a_{p+1}} &:= e^{\alpha(p+1)\phi}\hat{F}^{(p+1)}_{a_1\ldots a_{p+1}},\\ F^{(p)}_{a_1\ldots a_p} &:= e^{\alpha(p+2-D)\phi}\hat{F}^{(p+1)}_{a_1\ldots a_p y}\end{aligned}$$

where we have used the $\beta=-(D-2)\alpha$ condition from (I.5). The kinetic term associated to $F^{(p+1)}$ will therefore contribute to the D-dimensional Lagrangian as

$$\frac{1}{2}\int \mathrm{d}y\,\hat{F}^{(p+1)}\wedge\star F^{(p+1)} = \frac{1}{2}e^{-2\alpha p\phi}F^{(p+1)}\wedge\star F^{(p+1)} + \frac{1}{2}e^{2\alpha(D-p-1)\phi}F^{(p)}\wedge\star F^{(p)}$$

where the Hodge star operator $\star$ is understood to be defined with respect to G_{MN} ($g_{\mu\nu}$) on the left-(right-)hand side of the above equation.

In practice, one may be interested in dimensional reduction on manifolds with more than one dimension ; this is for instance the case when trying to understand either bosonic or superstring theories in four dimensions. We then turn to discussing how to reduce a D-dimensional theory to a d-dimensional one, with $d<D$. We already know how to do so when the internal space is a product of $(D-d)$ circles - that is a $(D-d)$-torus - by applying the above procedure repeatedly. Each iteration produces a new Kaluza-Klein one-form, a new radion as well as new scalars from the already existing Kaluza-Klein one-forms known as axions. In addition, every other p-form splits into a p-form and a $(p-1)$-form as we saw.

For various reasons we will mention later, including supersymmetry considerations as we will detail in II.1, one may want to perform dimensional reductions on more complex manifolds. Let us consider a theory defined on a D-dimensional manifold $\mathcal{M}_D$ and assume we want to interpret it as a theory defined on the d-dimensional manifold $\mathcal{M}_d$. This is only possible if $\mathcal{M}_D$ looks, at least locally, like a product space $\mathcal{M}_D\times\mathcal{K}_{D-d}$. We will restrict here to the case where $\mathcal{M}_D=\mathcal{M}_d\times\mathcal{K}_{D-d}$ globally but it may be worth noting that different setups - such as flux compactifications to be discussed in II.4.2 - have been studied in the literature. In order to perform the reduction, we make the following ansatz for the background metric :

$$\langle G_{MN}(\mathbf{x},\mathbf{y})\rangle = \begin{pmatrix} g_{\mu\nu}(\mathbf{x}) & 0\\ 0 & \tilde{g}_{\alpha\beta}(\mathbf{y})\end{pmatrix} \tag{I.7}$$

with x^μ and $g_{\mu\nu}$ (respectively y^α and $\tilde{g}_{\alpha\beta}$) the coordinates and metric on $\mathcal{M}_d$ (respectively $\mathcal{K}_{D-d}$). Dimensional reduction is then obtained by expanding G_{MN} around its vacuum expectation value. This gives a prescription for computing the d-dimensional Ricci scalar appearing in the Einstein-Hilbert part of the action, either exactly or perturbatively depending on what is possible.

As in the S^1 reduction case, the resulting theory has an infinite tower of states almost none of which are massless. Indeed, let $\chi(\mathbf{x}, \mathbf{y})$ be a free massless field in the original D-dimensional theory. Its equation of motion reads

$$\mathcal{D}_{\text{D}}\chi = 0$$

for some differential operator $\mathcal{D}_{\text{D}}$ defined on $\mathcal{M}_D$ of order 2 (1) if χ is bosonic (fermionic). Using the splitting (I.7), the operator $\mathcal{D}_{\text{D}}$ may be expressed as a sum of two operators acting only on $\mathcal{M}_d$ and $\mathcal{K}_{D-d}$ respectively so that the equation of motion of χ may be rewritten as

$$\mathcal{D}_d\chi + \mathcal{D}_{\text{int}}\chi = 0. \tag{I.8}$$

Similarly to what we have done in the S^1 example, we would like to expand $\chi(\mathbf{x}, \mathbf{y})$ in a basis of functions on $\mathcal{K}_{D-d}$ so as to integrate to original action over the internal manifold ; the above equation of motion suggests to expand χ over the eigenfunctions Y^a of the internal differential operator $\mathcal{D}_{\text{int}}$ as

$$\chi(\mathbf{x}, \mathbf{y}) = \chi_a(\mathbf{x})Y^a(\mathbf{y}). \tag{I.9}$$

The χ_a's may be seen as a generalisation of the Fourier modes in equation (I.2). It is clear from (I.8) that the mass of χ_a will be given by the eigenvalue of $\mathcal{D}_{\text{int}}$ associated to Y^a. Assuming that a truncation to the massless sector is consistent, the Kaluza-Klein reduction on $\mathcal{K}_{D-d}$ may then be performed by keeping only the zero modes of $\mathcal{D}_{\text{int}}$ in the d-dimensional theory. As before, the equivalent action in d-dimensions is obtained by plugging (I.9) into the action before integrating over $\mathcal{K}_{D-d}$.

We conclude this discussion by analysing the global symmetries of Kaluza-Klein reduced theories. It is well known that the pure gravity theory in $D+1$ dimensions is invariant under general coordinate transformations parametrised by

$$\delta X^M = -\xi^M, \quad \delta G_{MN} = (\mathcal{L}_\xi G)_{MN} \tag{I.10}$$

with $\mathcal{L}_\xi G$ the Lie derivative of the metric G along the vector field ξ. However, not all transformations (I.10) preserve the cylinder condition ; imposing that δG_{MN} remains y-independent may only be done if the transformation parameter ξ satisfies

$$\partial_y\xi^\mu = 0, \qquad \partial_y\partial_M\xi^y = 0. \tag{I.11}$$

The most general solution to (I.11) is given by

$$\xi^\mu(\mathbf{x}, y) = \xi^\mu(\mathbf{x}), \quad \xi^i(\mathbf{x}, y) = cy + \lambda(\mathbf{x}) \tag{I.12}$$

with c constant. In addition to the invariance under D-dimensional general coordinate transformations, the lower-dimensional theory also inherits a $U(1)$ gauge symmetry, parametrised by $\lambda(\mathbf{x})$ and associated to the gauge one-form A, and a $\mathbb{R}$ global symmetry parametrised by c. In the more general case of reduction on a d-dimensional torus, the cylinder condition imposes all fields to be independent of the coordinates $\{y^i, i = 1, ..., d\}$ of the torus. The generalisation of (I.13) is therefore

$$\xi^\mu(\mathbf{x}, \mathbf{y}) = \xi^\mu(\mathbf{x}), \quad \xi^i(\mathbf{x}, \mathbf{y}) = c^i{}_j y^j + \lambda^i(\mathbf{x}). \tag{I.13}$$

As in the circle case, invariance under general coordinate transformation of the non-compact coordinates is retained ; in addition, there is a local $U(1)^n$ invariance and a global $GL(d,\mathbb{R})$ symmetry encoded by the parameters $\lambda^i(\mathbf{x})$ and $c^i{}_j$ respectively. The $GL(d,\mathbb{R})$ group obtained here may be understood as the combination of an *internal* $SL(d,\mathbb{R})$ symmetry - that is which leaves $g_{\mu\nu}$ invariant - inherent to torus reductions and of a scaling symmetry whose presence in the d-dimensional theory depends on the matter content of the D-dimensional one. For the pure gravity model, this is the end of the story ; however, the field content of supergravities also contains differential p-forms $\hat{A}^{(p)}$. The theory reduced on T^d with $d \geq p$ then contains scalars coming from the reduction of $\hat{A}^{(p)}$, in addition to those previously considered. This generically has the effect on enhancing the global symmetry group from $SL(d,\mathbb{R})$ to a continuous group G ; since the full duality group of the UV-completion should leave *a fortiori* the low-energy supergravity invariant as well, it is usually conjectured to be a (discrete) subgroup of G. In the literature, this duality group is known as the *U-duality group*.

I.2 Scherk-Schwarz reduction

Dimensional reduction on arbitrary manifolds may become really hard to perform, depending on the internal space of interest. As we have seen in section I.1, Kaluza-Klein reductions on tori are quite easy to handle and are therefore attractive in this respect. However, they suffer some major flaws : since we truncate all massive modes, we loose the ability of generating masses for the gauge particles. Moreover, there is no scalar potential so that all scalar fields are moduli. Another important feature is the amount of retained space-time supersymmetry : phenomenologically, supersymmetric extensions of the standard model of particle physics may have at most $\mathcal{N} = 1$ supersymmetry as chiral interactions would be forbidden otherwise. As we will discuss in more details in section II.1, the geometry of the internal space determines the amount of preserved supersymmetry in Kaluza-Klein models. In a series of two papers [15, 16], Scherk and Schwarz introduced an alternative to the Kaluza-Klein ansatz. The main motivation was to find a way to generate mass terms for the gravitini, thereby spontaneously breaking supersymmetry. Another interesting consequence is that, due to the higher-dimensional model supersymmetry, the procedure also generates a scalar potential allowing to stabilise moduli.

Let us consider the reduction of a $(D+1)$-dimensional theory on a circle containing a field $\Phi(\mathbf{x}, y)$. If the circle has radius R, then y must be identified to $y + 2\pi R$; in particular, the action must be invariant under $y \mapsto y + 2\pi R$. With the Kaluza-Klein ansatz, $\Phi(\mathbf{x}, y)$ was actually independent of y so that $\Phi(\mathbf{x}, y+2\pi R) = \Phi(\mathbf{x}, y)$ was trivially satisfied ; consequently, the action was indeed invariant under a shift of $2\pi R$ of the y coordinate. Let us now assume that our system admits a global symmetry group G. Then, if $\Phi(\mathbf{x}, y)$ is such that

$$\Phi(\mathbf{x}, y + 2\pi R) = h \cdot \Phi(\mathbf{x}, y)$$

for some element $h \in G$, the action is still invariant under $y \mapsto y + 2\pi R$ and the system remains consistent [3]. This may be achieved by setting

$$\Phi(\mathbf{x}, y) = g(y) \cdot \phi(\mathbf{x}) \tag{I.14}$$

3. To be precise, $\Phi(\mathbf{x}, y + 2\pi R) = \rho(h) \cdot \Phi(\mathbf{x}, y)$ for some representation ρ of the group G. In order to lighten the notation, we assume the difference between a group element and its image under ρ is clear from the context.

for some y-dependant element of G satisfying

$$g(y+2\pi R)\cdot g(y)^{-1}=h$$

for all values of y. The simplest way to fulfil this condition is to choose

$$g(y)=\exp\left(\frac{yM}{2\pi R}\right) \tag{I.15}$$

for some element M in the Lie algebra $\mathfrak{g}$ of G [4]. Then, the field Φ is no longer periodic around the circle but instead has *monodromy*

$$\mathcal{M}(g):=e^{M}.$$

Inserting the ansatz (I.14) into the equation of motion of Φ shows that the mass matrix is related to M as we will illustrate below ; more precisely, the resulting mass terms are either linear or quadratic in M depending on whether Φ is fermionic or bosonic. Moreover, the one-dimensional subgroup of G generated by M becomes a local invariance of the reduced theory under which the scalar fields are charged. This may be seen by considering the dimensional reduction of $\mathrm{d}\Phi$ for instance. Indeed, in terms of the vielbein (I.6), it reads

$$\mathrm{d}\Phi=\exp\left(\frac{yM}{2\pi R}\right)\left[\mathrm{D}\phi+\frac{1}{2\pi R}M\phi\hat{\theta}^{y}\right]$$

with $\mathrm{D}\phi:=\mathrm{d}\phi-\frac{1}{2\pi R}AM\phi$ the gauge covariant derivative in the reduced theory and A the Kaluza-Klein one-form coming from the reduction of the metric on the circle. The Scherk-Schwarz construction therefore naturally leads to *gauged supergravities*, to be described in more details in section II.4.1.

Let us see in more details how a scalar potential emerge in the common case where the scalar fields in $D+1$ dimensions take value in a coset space G/H. Typically, G is a non-compact Lie group and H its maximal compact subgroup. The scalars may be conveniently represented by a vielbein $\hat{\mathcal{V}}(\mathbf{x},y)$ transforming under rigid G and local H transformations as $\hat{\mathcal{V}}\mapsto g\hat{\mathcal{V}}h^{-1}(\mathbf{x},y)$ [5]. The kinetic term may be shown to read [17]

$$\mathcal{L}_{\mathrm{kin}}=-\frac{\sqrt{-\hat{G}}}{2}\,\mathrm{Tr}\left(\hat{\mathcal{V}}^{-1}\mathcal{D}_{M}\hat{\mathcal{V}}\hat{\mathcal{V}}^{-1}\mathcal{D}^{M}\hat{\mathcal{V}}\right) \tag{I.16}$$

with $\mathcal{D}$ a covariant derivative with respect to a H-connection and with $\hat{G}$ and M the metric and indices corresponding to the $(D+1)$-dimensional space-time as before. Let η be a H-invariant metric, that is such that $h\eta h^{T}=\eta$ for all element h of H and let us define the H-invariant field $\hat{\mathcal{H}}:=\hat{\mathcal{V}}\eta\hat{\mathcal{V}}^{T}$. The unphysical degrees of freedom of the vielbein $\hat{\mathcal{V}}$ may be removed from the Lagrangian (I.17) by applying the local H-transformation $h(\mathbf{x},y)=\eta\hat{\mathcal{V}}^{T}(\mathbf{x},y)$. Using the fact

4. This may always be done whether G is simply connected or not. Indeed, if it is not, it is always possible to go from a connected component of G to the one connected to the identity element by multiplication by some (constant) element h_0 of G. Since the theory admits G as a global symmetry, any choice for $g(y)$ is physically equivalent to a theory where $g(y)$ is replaced by $h_0\cdot g(y)$ with $h_0\cdot g(y)$ in the component of G connected to the identity.

5. We emphasise that the local H invariance is not a gauge symmetry (in the sense that it is not associated to any propagating gauge field) but is only included as a way to take care of the redundancy of the vielbein parametrisation.

that the covariant derivative acting on $\hat{\mathcal{H}}$ reduces to a simple derivative, the Lagrangian (I.17) becomes

$$\mathcal{L}_{\text{kin}} = \frac{\sqrt{-\hat{G}}}{2} \operatorname{Tr}\left(\partial_M \hat{\mathcal{H}}^{-1} \partial^M \hat{\mathcal{H}}\right). \tag{I.17}$$

It is now easy to see that the reduction on S^1 using the Scherk-Schwarz ansatz $\hat{\mathcal{V}}(\mathbf{x}, y) = \exp\left(\frac{yM}{2\pi R}\right)\mathcal{V}(\mathbf{x})$ produces a scalar potential proportional to

$$V(\phi, \mathcal{V}) = e^{a\phi} \operatorname{Tr}\left(M^2 + M\mathcal{H}M^T\mathcal{H}^{-1}\right) = e^{a\phi} \operatorname{Tr}\left(M^2 + \mathcal{V}^{-1}M\mathcal{V}\eta\left(\mathcal{V}^{-1}M\mathcal{V}\right)^T\eta^{-1}\right) \tag{I.18}$$

with a a constant depending on the choice of normalisation for the coefficients α and β defined in (I.4). The vacua of the theory correspond to minima of the potential (I.19) ; since V depends on the dilaton ϕ through an exponential factor, it may only be stationary with respect to ϕ variations for field configurations $\mathcal{V}_0$ satisfying $V(\phi, \mathcal{V}_0) = 0$. In the usual case where H is compact, one may always choose a basis such that η is the identity matrix ; in this case, the potential may be rewritten as

$$V(\phi, \mathcal{V}) = e^{a\phi} \operatorname{Tr}\left(Y^2\right) \tag{I.19}$$

with $Y := \mathcal{V}^{-1}M\mathcal{V} + \left(\mathcal{V}^{-1}M\mathcal{V}\right)^T$. Since Y is a real matrix, the vacua are therefore located to the points where $Y = 0$ that is where the matrix $\mathcal{V}^{-1}M\mathcal{V}$ is antisymmetric.

As before, this may be further generalised to reduction on a torus by applying the same procedure repeatedly ; the corresponding ansatz is then

$$\Phi(\mathbf{x}, \mathbf{y}) = (g_1(y_1)...g_d(y_d)) \cdot \phi(x).$$

for y_i-dependants elements g_i of G. In order for the monodromies along each cycles of T^d to be well-defined - that is in particular constant - one needs to require that all the g_i's commute. The gauge group of the resulting theory will then be the direct product of the groups generated by the matrices M_i defined as in (I.15).

Let us emphasise that different choices for $g(y)$ do not necessarily lead to physically inequivalent theories. In particular, under conjugation of $g(y)$ by a constant element $k \in G$, the mass matrix transforms as

$$g(y) \mapsto h \cdot g(y) \cdot h^{-1}, \quad M \mapsto h \cdot M \cdot h^{-1}.$$

Even though this may seem like a different reduced theory at first sight, it is related to the original one by redefining all the fields ϕ as $\phi \mapsto h \cdot \phi$ as may be seen explicitly in the above example. Therefore, all elements $g(y)$ in the same conjugacy class lead to physically equivalent theories.

Finally, we note that taking quantum corrections into account generically breaks the global symmetry group G into a discrete subgroup $G(\mathbb{Z}) \subset G$. In the quantum theory, the monodromy must therefore belong to $G(\mathbb{Z})$; the mass matrix will then typically have only integer entries in some appropriate basis.

I.3 Toroidal reductions in conformal field theory

I.3.1 The Narain lattice

We now turn to reviewing a specific type of compactification which has the advantage of admitting a solvable conformal field theory (CFT) description, namely compactification on a torus. We have already outlined the corresponding dimensional reduction from the quantum field theory point of view in section I.1 and we will see that a world-sheet description leads to additional features due to the extended structure of the strings. The models studied in this thesis and developed in chapter III strongly rely on such constructions and in particular on toroidal orbifolds which will be reviewed in section I.4.

The simplest example of string theory is a CFT of free bosons (plus their fermionic counterparts in the superstring case), each of which are interpreted as a direction in space-time. Let us consider as an example the case of critical bosonic string theory where the free bosons are X^M for $M = 0, ..., 25$. Without any further constraint, the corresponding target space is the 26-dimensional Minkowski space. Let us assume that one would like to compactify the theory on S^1 so that, say, X^{25} is the coordinate on a circle of radius R. Then, one should impose the identification

$$X^{25} \cong X^{25} + 2\pi R.$$

This induces a *global* difference with the non-compact case, as the topology of the target space is no longer the same. However, from a *local* point of view nothing has changed and the Polyakov action may be quantised in the usual way.

This will also be the case when compactification on an arbitrary-dimensional torus is considered. Indeed, any (real) torus T^d may be seen as the quotient $\mathbb{R}^d/(2\pi\Lambda)$ for some rank d lattice Λ. In other words, setting $\widetilde{X}^I$, $I = 1, ..., d$ the coordinates on T^d, the associated CFT is obtained from the free boson case by imposing

$$\widetilde{X}^I \cong \widetilde{X}^I + 2\pi\sqrt{\alpha'}\lambda^I \tag{I.20}$$

for all vectors λ in the quotienting lattice Λ. This also restricts the allowed momenta along the torus direction. This may be seen easily by considering the string vertex operator

$$e^{i\widetilde{P}_I\widetilde{X}^I}$$

which must be single-valued for the theory to be consistent. This means in particular that $\sqrt{\alpha'}\widetilde{P}_I\lambda^I$ must be an integer for all $\lambda \in \Lambda$, meaning that the dimensionless momentum $\widetilde{K} := \sqrt{\alpha'}\widetilde{P}$ must belong to the dual lattice Λ^*.

Parametrising the two-dimensional world-sheet by two real coordinates (τ, σ) so that $\sigma \cong \sigma + 2\pi$ for the closed string, the most general action is given by [5]

$$S = -\frac{1}{4\pi\alpha'}\int_0^{2\pi} \mathrm{d}\sigma \int \mathrm{d}\tau \Big(\widetilde{G}_{IJ}\eta^{\alpha\beta} - \widetilde{B}_{IJ}\epsilon^{\alpha\beta}\Big)\partial_\alpha\widetilde{X}^I\partial_\beta\widetilde{X}^J \tag{I.21}$$

at leading order in the Regge slope α' and in conformal gauge. In (I.21), $\eta^{\alpha\beta}$ has signature $(-,+)$ in the (τ, σ) basis and $\widetilde{G}_{IJ}$, $\widetilde{B}_{IJ}$ are constant background couplings. It turns out to be convenient to parametrise the theory in a slightly different way by noticing that, Λ being a lattice, there must exist a (not uniquely defined) rank-d matrix E such that any vector $\lambda \in \Lambda$ may be expressed as

$$\lambda = EW$$

for some integer-valued vector W. Since Λ has maximal rank by assumption, one may show that this implies that the momentum $\widetilde{P}$ may be expressed as

$$\widetilde{P} = \frac{1}{\sqrt{\alpha'}} E^{-t} K$$

for some integer-valued vector K. In particular, setting $X := E^{-1}\widetilde{X}$, the identification (I.20) translates to

$$X^I \cong X^I + 2\pi\sqrt{\alpha'}W^I. \tag{I.22}$$

Setting $G := E^t\widetilde{G}E$ and $B := E^tBE$, the action (I.21) becomes

$$S = -\frac{1}{4\pi\alpha'}\int_0^{2\pi} \mathrm{d}\sigma \int \mathrm{d}\tau \Big(G_{IJ}\eta^{\alpha\beta} - B_{IJ}\epsilon^{\alpha\beta}\Big)\partial_\alpha X^I \partial_\beta X^J. \tag{I.23}$$

Even though this action is very similar to (I.21), it has the advantage of having all informations about the geometry of the torus encoded in the background metric G_{IJ}.

Let us now work out the spectrum of the theory (I.23). Since G_{IJ} and B_{IJ} are constant, the equation of motion reads

$$\partial_+\partial_- X^I = 0$$

with $\partial_\pm$ the derivation with respect to $\sigma_\pm := \tau \pm \sigma$. The usual splitting $X(\tau,\sigma) = X_L(\sigma_+) + X_R(\sigma_-)$ thus still holds. However, because of the change in the target space topology with respect to the non-compact case, the usual closed string condition must be changed. Indeed, when seen as a function of $\mathbb{R}$, X^I is no longer single-valued ; requiring that the string closes then means that under $\sigma \mapsto \sigma + 2\pi$, the representative of the X coordinate under the identification (I.22) should not change. X itself therefore satisfies

$$X^I(\tau, \sigma + 2\pi) = X^I(\tau,\sigma) + 2\pi\sqrt{\alpha'}W^I$$

for some integers W^I. These integers count the number of times the string winds around each cycle of the torus and are therefore known as *winding numbers*. The mode expansion of X_L and X_R are then given by

$$X_L^I(\sigma_+) = x_L^I + \frac{\alpha'}{2}p_L^I\sigma_+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}a_n^I e^{-in\sigma_+} \ ,$$
$$X_R^I(\sigma_-) = x_R^I + \frac{\alpha'}{2}p_R^I\sigma_- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{a}_n^I e^{-in\sigma_-}$$

with the left- and right-moving momenta satisfying

$$p_L^I - p_R^I = \frac{2}{\sqrt{\alpha'}}W^I. \tag{I.24}$$

The total momentum K may be obtained as usual by integrating the corresponding charge density over σ ; moreover, K must be an integer-valued vector so that vertex operators remain single-valued as we argued above. One then has

$$K_I = \sqrt{\alpha'} \int_0^{2\pi} \mathrm{d}\sigma \frac{\delta S}{\delta \partial_\tau X^I} = \frac{\sqrt{\alpha'}}{2} \Big[G_{IJ} \Big(p_L^J + p_R^J \Big) + B_{IJ} (p_L^J - p_R^J) \Big] \tag{I.25}$$

Combining (I.24) and (I.25) finally allows to express p_L and p_R as

$$\begin{aligned} p_L^I &= \frac{1}{\sqrt{\alpha'}} G^{IJ} \Big(K_J + (G_{JK} - B_{JK}) W^K \Big), \\ p_R^I &= \frac{1}{\sqrt{\alpha'}} G^{IJ} \Big(K_J - (G_{JK} + B_{JK}) W^K \Big) \end{aligned} \tag{I.26}$$

with, once again, K_I and W^I integers corresponding to the momentum and winding numbers of the string state respectively. This shows in particular that even though the B-field contribution to the action is a topological term with no local effect, it modifies the string spectrum. Moreover, introducing the dimensionless left-moving and right-moving momenta $l_{L,R} := (\alpha'/2)^{1/2} p_{L,R}$ one may give the set

$$\Gamma_{d,d} = \Big\{ (l_L(K,W), l_R(K,W)) \ , \ K,W \in \mathbb{Z}^d \Big\}$$

a lattice structure by equipping it with the scalar product defined by

$$\langle (l_L, l_R)(K,W) | (l'_L, l'_R)(K', W') \rangle := l_L^I G_{IJ} l_L'^J - l_R^I G_{IJ} l_R'^J = K_I W'^I + K'_I W^I. \tag{I.27}$$

The lattice $\Gamma_{d,d}$ is known as the *Narain lattice* in the literature. It turns out that this lattice is very constrained by (I.26) and by consistency requirements. First, equation (I.26) shows that the scalar product between any two vectors of $\Gamma_{d,d}$ is integer, showing that $\Gamma_{d,d} \subset \Gamma_{d,d}^*$ with $\Gamma_{d,d}^*$ the dual of $\Gamma_{d,d}$. In addition, $\Gamma_{d,d}$ is an *even* lattice, meaning that all its vectors have even squared norm as it is clear by applying (I.27) to the $(K', W') = (K, W)$ case. A final constraint is given by demanding modular invariance of the theory. This may be done in the following way : compared to the non-compact case, the toroidal compactification only modifies the bosonic zero modes. Consequently, the $(4\pi\alpha'\tau_2)^{-d}$ contribution from integrating over the momentum of the d formerly non-compact bosons gets replaced by $Z_{\Gamma_{d,d}}(\tau)$, with τ the modular parameter of the world-sheet two-torus and

$$Z_{\Gamma_{d,d}}(\tau) := \mathrm{Tr}\Big(q^{L_0} \overline{q}^{\overline{L}_0} \Big) = \sum_{(l_L, l_R) \in \Gamma_{d,d}} \exp\Big(i\pi\tau l_L^I G_{IJ} l_L^J - i\pi\overline{\tau} l_R^I G_{IJ} l_R^J \Big).$$

The evenness of $\Gamma_{d,d}$ ensures the invariance of $Z_{\Gamma_{d,d}}(\tau)$ under the $\tau \mapsto \tau + 1$ generator of $SL_2(\mathbb{Z})$. Its transformation under $\tau \mapsto -1/\tau$ may be deduced by applying Poisson summation formula [6]. One finds that

6. A way to state Poisson formula is the following : let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function and Λ a lattice. Then

$$\sum_{\lambda \in \Lambda} f(\lambda) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} \hat{f}(k) e^{2i\pi k \cdot x}$$

with $|\Lambda|$ the volume of the unit cell of Λ and Λ^* its dual lattice.

$$Z_{\Gamma_{d,d}}\left(-\frac{1}{\tau}\right) = \frac{1}{|\Gamma_{d,d}|}|\tau|^d Z_{\Gamma^*_{d,d}}(\tau). \tag{I.28}$$

Provided that the non-compact theory is consistent by itself, its toroidal compactification version is therefore modular anomaly free if and only if $\Gamma_{d,d} = \Gamma^*_{d,d}$, that is if $\Gamma_{d,d}$ is unimodular [7].

In summary, the Narain lattice $\Gamma_{d,d}$ must be both even and unimodular. Lattice theory teaches us that those are such stringent constraints that $\Gamma_{d,d}$ is actually determined up to isometries. More precisely, the following theorem holds true :

Theorem I.3.1 ([18])**.** *An even unimodular lattice of signature* (p, q) *exists if and only if* $p = q \mod 8$*. Moreover, it is isomorphic to*

$$E_8(1)^{\oplus n} \oplus U^{\oplus q}$$

where $p - q := 8n$ *is assumed to be positive, where* $E_8(1)$ *is the positive definite root lattice of the exceptional Lie group* E_8 *and where* U *is the hyperbolic lattice whose bilinear form is given by*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $p - q$ *is negative, the corresponding result is obtained by changing the sign of the bilinear form.*

In our case, this shows that $\Gamma_{d,d} \cong U^{\oplus d}$. However, this is not sufficient to fully characterise the theory. This may be seen for instance in the following way : imposing level-matching, one shows that the mass of an arbitrary state of the theory with winding and momentum quantum numbers W^I and K_I reads

$$M^2 = \frac{1}{\alpha'}\begin{pmatrix} W & K \end{pmatrix}\begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}\begin{pmatrix} W \\ K \end{pmatrix} =: \frac{1}{\alpha'}\begin{pmatrix} W & K \end{pmatrix}\mathcal{G}^{-1}\begin{pmatrix} W \\ K \end{pmatrix} \tag{I.29}$$

up to integer multiples of $4/\alpha'$ due to the oscillator modes. In (I.29), the final equation defines the matrix $\mathcal{G}$. This shows that two inequivalent backgrounds lead generically to spectra of states with different sets of masses, that is necessarily to different theories. In summary, even if $\Gamma_{d,d}$ is constrained to be isomorphic to $U(1)^{\oplus d}$, the details of the theory lie in the background fields, that is in the details of the isomorphism - or the *embedding* of $\Gamma_{d,d}$ into $\mathbb{R}^{2d}$ - itself.

Even though different backgrounds lead to different theories in general, there exists transformations which preserve all physical observables and which may therefore really be seen as connecting different formulations of a single theory. Those are known as *dualities* and we now show how to derive the duality group of toroidal reduction of string theories. It is worth noticing that the Hilbert space $\mathcal{H}^{(d)}_{\text{tor}}$ corresponding to the compactification of a string theory on a d-torus may be represented as the tensorial product

$$\mathcal{H}^{(d)}_{\text{tor}} \cong \mathcal{H} \otimes \mathcal{H}_{\Gamma_{d,d}}$$

7. This implies in turn that the unit cell volume is $|\Gamma_{d,d}| = 1$ so that (I.28) shows that $Z_{\Gamma_{d,d}}$ transforms covariantly under $\tau \mapsto -1/\tau$ as it should.

where $\mathcal{H}_{\Gamma_{d,d}} = \{|p_L, p_R\rangle\}$ corresponds to the winding and momentum states along the torus and where $\mathcal{H}$ is roughly what is left from the Hilbert space of the original non-compact theory. In particular, $\mathcal{H}$ does not contain any $\Gamma_{d,d}$ dependency. Any transformation preserving $\Gamma_{d,d}$ therefore leads to the same Hilbert space $\mathcal{H}_{\Gamma_{d,d}}$ and therefore to physically identical theories. The *duality group* then contains as a factor the automorphism group [8]

$$O(\Gamma_{d,d}) := \left\{ g \in GL_{2D}(\mathbb{R}) \middle| g \cdot \Gamma_{d,d} = \Gamma_{d,d}, \langle g \cdot x | g \cdot y \rangle = \langle x, y | x, y \rangle \ \ \forall x, y \in \Gamma_{d,d} \right\}$$

of $\Gamma_{d,d}$, with $\langle \cdot | \cdot \rangle$ the symmetric bilinear form defined by (I.27). $O(\Gamma_{d,d})$ is isomorphic to $O(d,d;\mathbb{Z})$, the subgroup of integer-valued matrices of $O(d,d)$. This may be seen by considering the action of an element of $O(\Gamma_{d,d})$ on the quantum numbers (W^I, K_I). Let us then assume that those transform as

$$\begin{pmatrix} W \\ K \end{pmatrix} \mapsto A \begin{pmatrix} W \\ K \end{pmatrix}$$

for some matrix $A \in GL_{2D}(\mathbb{R})$. Then, the scalar product (I.27) is preserved if and only if

$$A^t \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix} A = \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}$$

that is if and only if $A \in O(d,d;\mathbb{R})$. In order for A to also preserve the lattice $\Gamma_{d,d}$, the image and the preimage of (W, K) must both be integer-valued for all values of W^I and K_I. This may only be true if both A and A^{-1} are integer-valued as well, showing that $A \in O(d,d;\mathbb{Z})$ as claimed. The isomorphism between $O(\Gamma_{d,d})$ and $O(d,d;\mathbb{Z})$ is given by

$$\begin{array}{ccc} O(d,d;\mathbb{Z}) & \xrightarrow{\sim} & O(\Gamma_{d,d}) \\ A & \mapsto & EAE^{-1} \end{array}$$

with $E = E(G,B)$ defined by the relation $(l_L, l_R)^t =: E(W,K)^t$ and readable directly from (I.26).

The action of the duality group on the background field is most easily expressed in terms of the $O(d,d;\mathbb{Z})$ matrix A and may be inferred directly from (I.29) by requiring the mass of a state to be preserved under the duality. One then obtains the transformation law

$$\mathcal{G} \mapsto A \mathcal{G} A^T \tag{I.30}$$

where (the inverse of)

$$\mathcal{G} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}$$

has been defined in the mass formula (I.29). The duality group $O(\Gamma_{d,d})$ is called the *T-duality group*. When $d = 1$ - that is when compactifying on a circle - it is generated by the circle inversion $y \mapsto -y$ and by the renowned duality which maps the theories reduced on circles of radius R and $1/R$.

8. The duality group may also contain other elements coming from transformations acting on the other parts of the CFT, but $O(\Gamma_{d,d})$ must always be a factor in the total (perturbative) duality group.

We now have enough knowledge to derive the *moduli space* of the theory, that is the space of parameters leading to physically inequivalent theories. As we have seen, the spectrum depends on the background fields G_{IJ} and B_{IJ} with $I, J = 1, ..., d$. Since the former is symmetric and the latter antisymmetric, there are d^2 parameters in total. The only restriction on these parameters is that G_{IJ}, which represents the metric on the d-dimensional torus, must be positive definite. From a physical point of view, this is equivalent to requiring the kinetic term of the scalar fields X^I to have the correct sign. We have seen that an equivalent way of parametrising the theory was to think of it as descending from a specific embedding of the Narain lattice $\Gamma_{d,d}$ in $\mathbb{R}^{2d}$. Since we know from theorem I.3.1 that there exists an $O(d,d)$ rotation which maps $\Gamma_{d,d}$ to $U^{\oplus d}$, selecting an embedding of $\Gamma_{d,d}$ amounts to choosing a d-dimensional, say, space-like space corresponding to the space spanned by the p_L basis. The corresponding parameter space is by definition a *Grassmannian* and may be expressed as

$$\mathcal{T}_d := O(d,d)/(O(d) \times O(d)). \tag{I.31}$$

This may be understood as follows : we are trying to look for an orthogonal basis ($e_i, f_i, i = 1, ..., d$) of $\mathbb{R}^{2d}$ such that all e_i's (f_i's) are space-like (time-like). By definition, the matrix $(e_1, ..., e_d, f_1, ..., f_d)$ is then an element of $O(d,d)$. However, not any two elements of $O(d,d)$ parametrise different subspaces in $\mathbb{R}^{2d}$ as any rotation among the space-like basis vectors e_i does not lead to a new d-dimensional subspace (and similarly for the f_i's). All elements of $O(d,d)$ related by a $O(d) \times O(d)$ rotation, with the first factor acting on the e_i's and the second on the f_i's, then parametrise identical d-dimensional subspaces of $\mathbb{R}^{2d}$ and should be identified, leading to the expression (I.31). One may compute that the dimension of $\mathcal{T}_d$ is d^2, in accordance with the number of parameters coming from the background fields.

The space $\mathcal{T}_d$ is therefore the space parametrising the different embedding of $\Gamma_{d,d}$ in R^{2D} - or equivalently the different values allowed for the background fields G_{IJ} and B_{IJ} - and is usually referred to as the *Teichmüller space* in this context. As we have seen, not any two points in $\mathcal{T}_d$ parametrise different theories ; instead, the moduli space $\mathcal{M}_d$ of the theory is obtained by identifying points in $\mathcal{T}_d$ which are related by T-duality. The moduli space of conformal field theories on a d-dimensional torus is therefore isomorphic to

$$\mathcal{M}_d \cong O(\Gamma_{d,d}) \backslash O(d,d)/(O(d) \times O(d)). \tag{I.32}$$

An important example for the rest of this thesis is the case of the compactification on a two-torus. One may conveniently describe the corresponding moduli space in terms of two complex scalar fields T and U which correspond to the Kähler and complex structure of the T^2 ; the torus metric and B-field may then be parametrised as

$$G_{IJ} = \frac{T_2}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}, \quad B_{IJ} = \begin{pmatrix} 0 & T_1 \\ -T_1 & 0 \end{pmatrix}$$

with the subscripts 1 and 2 denoting respectively the real and imaginary parts of T and U. It turns out that the duality group $O(\Gamma_{2,2})$ takes in this case the simpler form [19]

$$O(\Gamma_{2,2}) = (SL_2(\mathbb{Z})_T \times SL_2(\mathbb{Z})_U) \rtimes \mathbb{Z}_2$$

where $SL_2(\mathbb{Z})_T$ and $SL_2(\mathbb{Z})_U$ act on T and U as

$$(T,U) \mapsto \left(\frac{aT+b}{cT+d}, \frac{a'U+b'}{c'U+d'}\right), \ ad-bc = a'd'-b'c' = 1$$

with all coefficients integers and with unprimed and primed coefficients correspond to the first and second $SL_2(\mathbb{Z})$ factors respectively. The Z_2 element corresponds to T-duality along one of the torus cycles and exchanges T and U as well as the two $SL_2(\mathbb{Z})$ factors in (II.22).

I.3.2 Enhanced symmetry

The BRST quantisation of the bosonic string theory shows that physical fields correspond to CFT operators of conformal dimension 1 [20]. With our notations, the Virasoro generators L_0 and $\overline{L}_0$ corresponding to zero modes of the stress-energy tensor acts on an arbitrary state as

$$\begin{aligned} L_0 &= -\frac{\alpha' M^2}{4} + \frac{1}{2} l_L^2 := -\frac{\alpha' M^2}{4} + l_L^I G_{IJ} l_L^J + N_L \ , \\ \overline{L}_0 &= -\frac{\alpha' M^2}{4} + \frac{1}{2} l_R^2 := -\frac{\alpha' M^2}{4} + l_R^I G_{IJ} l_R^J + N_R \ , \end{aligned} \tag{I.33}$$

with $(l_L, l_R) \in \Gamma_{d,d}$ and with $N_{L,R}$ regrouping all contributions from the oscillators. As $l_L^2 := l_L^I G_{IJ} l_L^J$ and $l_R^2 := l_R^I G_{IJ} l_R^J$ depend on the real-valued moduli G_{IJ} and B_{IJ}, they are not integer-valued in general. The massless spectrum therefore contains generically no other states that those satisfying $N_L = N_R = 1$ and $(l_L, l_R) = (0, 0)$. However, at some points in the moduli space, the equations $L_0 = \overline{L}_0 = 1$ admit solutions with non-trivial momentum and/or winding numbers.

Of particular interest will be the chiral and anti-chiral currents appearing at such points, that is operators with conformal dimensions $(1, 0)$ and $(0, 1)$ respectively. Focusing on the former case - as the discussion is identical in the latter - the chiral currents with non-trivial momentum/winding numbers satisfy [9]

$$\begin{aligned} l_L^2 &:= l_L^I G_{IJ} l_L^J = 2, \\ l_R^2 &:= l_R^I G_{IJ} l_R^J = 0, \\ N_L &= N_R = 0. \end{aligned} \tag{I.34}$$

In particular, since G_{IJ} is non-degenerate, the second line of (I.34) implies that $l_R = 0$.

Whenever (I.34) admits solutions, one may define currents of the CFT as follows

$$\begin{aligned} j^I(z) &:= \frac{2i}{\alpha'} \partial X^I(z), \\ g^{(l_L)} &:= e^{i\sqrt{\frac{2}{\alpha'}} l_L^I G_{IJ} X^J}(z), \end{aligned}$$

for $I = 1, ..., d$ and for all l_L satisfying (I.34). It is well-known that currents form algebras in CFT known as a *Kac-Moody algebra* (see *e.g.* [21]). In particular, their zero modes satisfy a Lie algebra ; the commutators may be computed using the currents operator product expansions and read

9. States with both l_L and l_R different from 0 may also appear but they cannot be currents from equation (I.33).

$$\left[j_0^I, j_0^J\right] = 0,$$
$$\left[j_0^I, g_0^{(l_L)}\right] = l_L^I g^{(l_L)},$$
$$\left[g_0^{(l_L)}, g_0^{(l'_L)}\right] = \begin{cases} g_0^{(l_L+l'_L)} & \text{if } l_L^I G_{IJ} l_L'^J = -1 \\ 0 & \text{else} \end{cases}$$

In particular, deriving the last commutator in (I.35) requires to notice that the scalar product $l_L^I G_{IJ} l_L'^J$ may only take very few values. Indeed, since $l_R = l'_R = 0$, $l_L^I G_{IJ} l_L'^J$ must be an integer ; from the positivity of the metric and the fact that l_L, l'_L both have squared norm 2, this integer should be equal to $2\cos\theta$ with θ the angle between the two vectors. Therefore $l_L^I G_{IJ} l_L'^J \in \{0, \pm 1, \pm 2\}$. Moreover, one may notice that $l_L + l'_L$ is solution to (I.34) if and only if both l_L and l'_L are and if $l_L^I G_{IJ} l_L'^J = -1$.

At a generic point in the moduli space (I.32), only the j^I's are currents and they realise a $U(1)^d$ algebra with level 1. However, at the points where (I.34) admits solutions, the commutators (I.35) show that the abelian gauge-symmetry is enhanced to a non-abelian one with the additional currents given by the $g^{(l_L)}$'s. In fact, one may see $\left\{j^I\right\}$ as associated to the Cartan subalgebra and $\left\{g^{(l_L)}\right\}$ to the root system of the corresponding Lie group. This is a crucial difference with the point particle point of view where toroidal reductions cannot give rise to non-abelian gauge theories ; this may be explained by the fact that the states associated with the extra currents have necessarily non-trivial winding numbers which may only occur thanks to the extended nature of the strings.

Such considerations play a special role in the bosonic description of the heterotic string which, in its simpler form, consists of 10 world-sheet bosons of both chiralities and 16 chiral ones. In order to give a geometric meaning to the model, the latter must be compactified leading to quantised winding and momenta numbers as above. However, the corresponding lattice Γ_{16} is Euclidean with signature, say, $(0, 16)$ instead of (d, d) as before. According to theorem I.3.1, such lattices are extremely constrained and would not have existed if the number of chiral bosons would not have been a multiple of 8. In fact, there are exactly two such lattices up to isomorphisms, namely the $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ root lattices [22]. This implies that currents corresponding to either one of the above Lie groups appear in the world-sheet theory, explaining why there exists two ten-dimensional heterotic string theories.

The points in moduli space where new states become massless are very special and will be essential in computing moduli spaces related quantities in chapter IV. We can already sketch briefly a peculiarity of the vicinity of such points regarding the associated Wilsonian effective field theory (EFT). Deriving an EFT from a UV-complete theory is generically done by integrating any mode over a given cut-off Λ. Depending on the choice of Λ, it may be possible to split the fields into two sets, namely the fields ϕ_i which are considered to be light enough to be still observed and the fields Φ_I which, on the contrary, are considered too heavy to be produced. The light fields ϕ_i may then be further divided into contributions with low and high momentum with respect to Λ as $\phi_i = \phi_i^{(l)} + \phi_i^{(h)}$. A correlation function is computed by performing the path integral

$$\left\langle \hat{\mathcal{O}}\left(\hat{\phi}_i, \hat{\Phi}_I\right)\right\rangle = \int \mathcal{D}\phi_i^{(l)} \mathcal{D}\phi_i^{(h)} \mathcal{D}\Phi_I \mathcal{O}\left(\phi_i^{(l)} + \phi_i^{(h)}, \Phi_I\right) e^{iS[\phi_i, \Phi_I]}$$

for any field-dependant operator $\hat{\mathcal{O}}\left(\hat{\phi}_i, \hat{\Phi}_I\right)$. Therefore, at least theoretically, one could obtain an equivalent theory containing only the low modes of the light fields $\phi_i^{(l)}$ by performing the path integral over $\phi_i^{(h)}$ and Φ_I, leading to modified action and operators ; this procedure is called *integrating out* the heavy fields. The effective couplings between the light fields are in particular computed by taking into accounts Feynman diagrams involving the heavy fields only as virtual states, that is only occurring in loops. Let us now go back to our case of a string theory compactified on a d-dimensional torus. One may want to derive the corresponding low-energy theory ; in order to do so, it is usually assumed that all massive fields have a mass of the order of Planck mass and may therefore be considered as heavy fields and integrated out. This procedure is perfectly fine almost everywhere in the moduli space, but let us now think about what happens near a points where gauge enhancement occurs. Let us assume that a scalar field Φ becomes massless for a given background (G_0, B_0). Almost everywhere in the moduli space, Φ is heavy and integrated out but near (G_0, B_0), Φ becomes lighter and lighter. At the critical point where the mass of Φ vanishes, the effective couplings involve massless scalars running in loops and therefore develop logarithmic singularities in four dimensions. Our analysis so far is of course very qualitative but the main idea is the following : if the moduli space of the aforementioned theories contains points of enhanced symmetry, then a Wilsonian EFT may only be defined locally. In chapter V, we will exploit the fact that this failure of global definiteness is reflected in effective couplings singularities in order to derive constraints on the low-energy theory from complex analysis considerations.

I.4 Orbifolds

Finally, we turn to a useful extension of the toroidal compactification which is compactifications on toroidal orbifolds. In general, an orbifold is defined similarly as a manifold except for a point : instead of looking locally like $\mathbb{R}^n$ for some integer n, they are allowed to look locally like a quotient of $\mathbb{R}^n$ by the action of a finite group. In practice, we will only deal with orbifolds which may be written as global quotients of a manifold. In order to set the notations, we will then define $\mathcal{O} := \mathcal{M}/G$ the orbifold obtained by the quotient of the manifold $\mathcal{M}$ under the action of the finite group G. If G is freely acting, $\mathcal{O}$ is a manifold itself ; else, it has conical singularities at the fixed points known as *orbifold singularities*[10].

String theories living on orbifolds of torus also admit a CFT description. Indeed, similarly to what has been done in the previous section, one may impose the identification

$$X \cong \Theta(X) \quad \forall \Theta \in G. \tag{I.36}$$

One could notice that, up to here, our construction is very similar to the simple toroidal compactification reviewed in I.3 as we had noticed that a torus could be seen as the quotient of the real space by a lattice, that is by an abelian group. This procedure may be generalised here by quotienting by a finite subgroup of the Euclidean group, that is by allowing an identification of points related either by a translation, a rotation or a combination of both.

10. This may be understood as the following : let U be an open subset of $\mathcal{M}$ containing no fixed point. It is then always possible to choose U small enough so that $g(U) \cap U = \emptyset$ for all elements $g \in G$. Then, under the projection $\pi : \mathcal{M} \to \mathcal{O}$, one sees that $\pi(U) \cong U$; since $\mathcal{M}$ is a manifold, $\pi(U)$ is then diffeomorphic to a local subset of $\mathbb{R}^n$ by hypothesis. However, if U contains a point p_0 which is fixed under the action of G, this is no longer the case and $\pi(U)$ will not look like $\mathbb{R}^n$ no matter how small U is. The image $\pi(p_0)$ of the fixed point will then be a conical singularity as claimed.

This finite subgroup is often referred to as the *space group* **S**, which contains elements of the form (Θ, λ_Θ) acting on the fields as

$$(\Theta, \lambda_\Theta) \cdot X := \Theta \cdot X + 2\pi\sqrt{\alpha'}\lambda_\Theta \tag{I.37}$$

with Θ a rotation of the $2d$ compact fields and λ a $2d$-dimensional vector. For convenience, we also define the *point group* **P** as

$$\mathbf{P} := \left\{\Theta \in O(d,d) \middle| \exists \lambda_\Theta : (\Theta, \lambda_\Theta) \in \mathbf{S}\right\}$$

Importantly, Θ should not mix left- and right-moving fields, so as to be consistent with the conformal symmetry, and therefore actually belongs to $O(d) \times O(d)$. In particular, splitting Θ as $\text{diag}(\Theta_L, \Theta_R)$ with $\Theta_L(\Theta_R) \in O(d)$ acting on the left (right) movers, if there exists a base in which the two matrices Θ_L and Θ_R are identical, then the orbifold is said to be *symmetric*. This means in particular that the internal manifold may be understood geometrically as $\mathbb{R}^d/G$ for some group G. If the orbifold is *asymmetric*, such a geometrical interpretation is no longer available. An interesting example is the following : it is well-known that a lattice of rank d may only admit automorphisms of order p for p such that

$$\phi(p) \leq d \tag{I.38}$$

with ϕ the Euler totient function [23]. Therefore, a geometric orbifold of, say, a four-torus may only be of order $p \in \{2, 3, 4, 5, 6, 8, 10, 12\}$. However, since a string actually sees two copies of a torus, non-geometric orbifolds are not constrained by the bound (I.38) and may consistently be defined on orbifolds of higher rank. As an example, an orbifold of the four-torus with point group generated by (Θ_L, Θ_R) with Θ_L and Θ_R or order 3 and 5 respectively will be of order 15. We will give explicit realisations of this in chapter III.

From (I.37), one deduces that the identity element is given by $(\mathbb{1}, 0)$ and that the group multiplication and inverse laws read

$$\begin{aligned}(\Theta, \lambda_\Theta)(\Theta', \lambda'_{\Theta'}) &= (\Theta\Theta', \Theta\lambda'_{\Theta'} + \lambda_\Theta), \\ (\Theta, \lambda_\Theta)^{-1} &= \left(\Theta^{-1}, -\Theta^{-1}\lambda_\Theta\right).\end{aligned}$$

An important subgroup of **S** is defined as

$$\Gamma := \left\{\lambda \in R^{2d} \middle| (\mathbb{1}, \lambda) \in \mathbf{S}\right\}.$$

Indeed, one may consider a two-steps construction : taking first the quotient of $\mathbb{R}^d$ by Γ before quotienting again the resulting theory by the space group, or more precisely by the group obtained when identifying elements of **S** related by a Γ translation. The intermediate theory is then nothing but the toroidal compactification we have analysed in the previous section, implying that Γ is constrained to be a Narain lattice - that is even and unimodular. In other words, the space group **S** may only lead to a consistent theory if its pure translation subgroup is a Narain lattice. Consequently, one may consider directly orbifolds of the toroidal theory with no loss of generality.

An additional remark may be done about the allowed elements of **S**. Indeed, since **S** is finite by assumption, for any $(\Theta, \lambda_\Theta) \in \mathbf{S}$ there must exist a (Θ-dependent) integer p called the *order* of (Θ, λ_Θ) such that $\Theta^p = \mathbb{1}$. This implies in particular that

$$(\Theta, \lambda_\Theta)^p = \left(\mathbb{1}, \sum_{k=0}^{p-1} \Theta^k \lambda_\Theta\right) \in \Gamma$$

where (I.37) has been used and by definition of the subgroup Γ. Then, one gets that

$$\frac{1}{p}\sum_{k=0}^{p-1} \Theta^k \lambda_\Theta := \mathcal{P}_\Theta \lambda_\Theta \in \frac{1}{p}\Gamma$$

with $\mathcal{P}_\Theta$ the projector onto Θ-invariant states. Therefore, λ_Θ is not arbitrary either but quantised in the directions left invariant by Θ. Moreover, consistency requires

$$(\Theta, \lambda_\Theta)(1, \lambda)(\Theta, \lambda_\Theta)^{-1} = (1, \Theta\lambda) \in \mathbf{S}$$

for all $(\Theta, \lambda_\Theta) \in \mathbf{S}$ and all $\lambda \in \Gamma$. This implies in particular that $\Theta\Gamma = \Gamma$ for all $\Theta \in \mathbf{P}$, that is that $\mathbf{P}$ must be a subgroup of the automorphism group of the Narain lattice. Physically, this is the reason why orbifold constructions may freeze some or all moduli. Indeed, one may choose a point group $\mathbf{P}$ which leaves only subspaces of the moduli space with *enhanced discrete symmetry* invariant ; quotienting by $\mathbf{P}$ therefore forces the theory to remain in this subspace. As a simple example, consider the two-torus case. In a geometric construction, the associated rank two lattice always has the order 2 symmetry $-\mathbb{1}$, but only the square and hexagonal lattices have an order 4 and 3 symmetry respectively. Identifying points related by such a transformation may then only be done to the corresponding points in the moduli space. However, as we discuss below, some other sectors of the spectrum absent from the original theory must appear in orbifold constructions and generically add new directions to the moduli space.

Toroidal compactifications may then be considered as the simplest example of orbifold construction where the point group $\mathbf{P}$ is trivial. We now show how to derive the spectrum of such theories. For convenience, we consider the toroidal theory quotiented by the point group $\mathbf{P}$ instead of the non-compact case quotiented by the space group $\mathbf{S}$, since both constructions are equivalent. As before, the identification (I.36) shows that the closed string condition may be generalised to

$$X(\tau, \sigma + 2\pi) = \Theta \cdot X(\tau, \sigma) + 2\pi\sqrt{\alpha'}\lambda_\Theta, \ (\Theta, \lambda_\Theta) \in \mathbf{S}. \tag{I.39}$$

States satisfying (I.39) with $g = \mathbb{1}$ are similar to the states considered in the previous section and are known as *untwisted states*. Conversely, the *twisted states* correspond to states which are not closed in the toroidal theory. In fact, consistency of the theory imposes to take such states into account as we now show. Since two states related by a point group transformation are equivalent, what we are really doing here is gauging the discrete group $\mathbf{P}$. Consequently, the orbifold construction only keeps the subspace of the original Hilbert space which is invariant under $\mathbf{P}$. A convenient way of selecting all $\mathbf{P}$-invariant states is to insert the projector

$$\mathcal{P}_\mathbf{P} := \frac{1}{|\mathbf{P}|}\sum_{\Theta\in\mathbf{P}} \Theta, \tag{I.40}$$

with $|\mathbf{P}|$ the order of $\mathbf{P}$, into any correlation function so that evaluating it over the whole spectrum of the original theory is equivalent to do so over only the surviving states of the orbifold theory. Let us then try to understand the generic correlators

$$\left\langle \hat{\mathcal{O}} \right\rangle \begin{bmatrix} h \\ g \end{bmatrix} := \int_{\phi_h} \mathcal{D}\phi_h \, e^{-S[\phi_h]} g\mathcal{O} \tag{I.41}$$

with $\mathcal{O}$ and operator of the theory and g, h elements of the point group and ϕ_h field configurations satisfying the periodicity condition (I.39) with $\Theta = h$. We consider one-loop contributions to (I.41) corresponding to world-sheets of genus 1 - or tori. We have already analysed such contributions to expectation values in I.3 in order to impose the unimodularity of the Narain lattice without digging too much into the details of CFTs defined on tori. However, a remainder of how such theories are constructed may be useful at this point ; more complete reviews are available in the literature, see *e.g.* [21] or [24]. Constructing CFTs defined on torus may easily be done in the following way : let w be the complex coordinate on the world-sheet torus. Then, there must exist two complex numbers λ_1 and λ_2 such that w is identified to $w + 2\pi\lambda_{1,2}$ and such that $\arg(\lambda_1) \neq \arg(\lambda_2)$. By rotating and rescaling the torus, one usually defines the complex parameter $\tau := \lambda_2/\lambda_1$ so that $z \cong w + 2\pi \cong w + 2\pi\tau$. Since λ_1 and λ_2 may be interchanged freely, one may always choose to set $\text{Im}(\tau) > 0$. Elementary complex analysis then teaches us that two complex parameters τ and τ' related by a $SL_2(\mathbb{Z})$ transformation acting as

$$\tau' = \frac{a\tau + b}{c\tau + d}, \;\; a, b, c, d \in \mathbb{Z} | ad - bc = 1 \tag{I.42}$$

lead to equivalent complex structures. The moduli space of inequivalent complex structures on the torus is therefore given by

$$\mathcal{H}/SL_2(\mathbb{Z})$$

with $\mathcal{H}$ the complex upper-half plane defined by $\text{Im}(\tau) > 0$. We recall that elementary considerations of the expectation values in quantum mechanics allow to relate traces over the Hilbert space to path integrals as

$$\text{Tr}\left(\hat{\mathcal{O}} e^{-\beta \hat{H}}\right) = \int_{\text{PBC}} \mathcal{D}x \, \mathcal{O}(x) e^{iS[x]} \tag{I.43}$$

for any operator $\hat{\mathcal{O}}$, with $\int_{\text{PBC}}$ the integral over periodic field configurations - that is fields with Periodic Boundary Conditions. The path integrals in CFTs defined on tori may be similarly computed as follow. Consider the CFT defined on a cylinder of circumference 1 and of length $2\pi\tau_2$. Twisting one end of the cylinder by $2\pi\tau_1$ and gluing the two ends together then gives a torus of complex structure τ. Similarly as in (I.43), the path integral may then be evaluated as follows : going from one end of the cylinder the other amounts as a translation in (world-sheet Euclidean) time of $2\pi\tau_2$. Twisting one end of the cylinder may then be done by shifting the world-sheet space coordinate by $2\pi\tau_1$ and gluing the two ends is equivalent to taking the trace over the Hilbert space of the theory. Since translation in world-sheet time and space are generated respectively by the Hamiltonian $H := (L_0)_{\text{cyl}} + \left(\overline{L}_0\right)_{\text{cyl}}$ and momentum $P := (L_0)_{\text{cyl}} - \left(\overline{L}_0\right)_{\text{cyl}}$ operators, correlation functions for CFTs defined on tori may be evaluated as

$$\left\langle \hat{\mathcal{O}} \right\rangle = \text{Tr}\left(\hat{\mathcal{O}} e^{-2\pi\tau_2 H + 2i\pi\tau_1 P}\right) = \text{Tr}\left(\hat{\mathcal{O}} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24}\right)$$

where L_0 and $\overline{L}_0$ are the Virasoro operators on the sphere and where $q := \exp(2i\pi\tau)$. Similar considerations show that the correlation function (I.41) may then be rewritten as

$$\left\langle \hat{\mathcal{O}} \right\rangle \begin{bmatrix} h \\ g \end{bmatrix} (\tau) = \mathrm{Tr}_h \left(g\mathcal{O} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} \right) \tag{I.44}$$

where the dependence to the modular parameter τ has been made explicit and with Tr_h the trace over the subspace of the Hilbert space with states X twisted by h, that is satisfying

$$X(w + 2\pi) = h \cdot X(w) \tag{I.45}$$

in accordance with (I.39). We refer to this subspace as the h-twisted sector. According to what we have just said, the trace over states satisfying $X(z + 2\pi\tau)$ with an insertion of the point group element g may equivalently be seen as a trace without insertion but over states satisfying the modified boundary condition

$$X(w + 2\pi\tau) = g \cdot X(w). \tag{I.46}$$

We notice that (I.45) and (I.46) may only be compatible if h and g commute. This allows to compute the behaviour of the correlation functions (I.41) under modular transformations. For clarity, let us restore momentarily the $\lambda_{1,2}$ formalism so that the torus coordinates satisfies $Z \cong z + 2\pi\lambda_1 \cong Z + 2\pi\lambda_2$. Equations (I.45) and (I.46) then becomes

$$\begin{aligned} X(w + 2\pi\lambda_1) &= h \cdot X(w), \\ X(w + 2\pi\lambda_2) &= g \cdot X(w). \end{aligned} \tag{I.47}$$

Let us now assume that $\lambda_{1,2}$ transform under the modular $SL_2(\mathbb{Z})$ action as

$$\begin{aligned} \lambda_1' &= d\lambda_1 + c\lambda_2 \\ \lambda_2' &= b\lambda_1 + a\lambda_2 \end{aligned}$$

so that the modular parameter τ' transforms as in (I.42). Then, the field periodicity conditions (I.47) become

$$\begin{aligned} X(w + 2\pi\lambda_1') &= h' \cdot X(w), \\ X(w + 2\pi\lambda_2') &= g' \cdot X(w) \end{aligned}$$

with $(h', g') := (h^d g^c, h^b g^a)$. This implies that

$$\left\langle \hat{\mathcal{O}} \right\rangle \begin{bmatrix} h \\ g \end{bmatrix} (\tau) = \left\langle \hat{\mathcal{O}} \right\rangle \begin{bmatrix} h^d g^c \\ h^b g^a \end{bmatrix} \left(\frac{a\tau + b}{c\tau + d} \right),$$

that is that

$$\left\langle \hat{\mathcal{O}} \right\rangle \begin{bmatrix} h \\ g \end{bmatrix} \left(\frac{a\tau + b}{c\tau + d} \right) = \left\langle \hat{\mathcal{O}} \right\rangle \begin{bmatrix} h^a g^{-c} \\ h^{-b} g^d \end{bmatrix} (\tau). \tag{I.48}$$

We are now ready to see why consistency of orbifold constructions require states belonging to twisted sectors to be present in the theory. As we already discussed, correlation functions in the orbifolded theory may be evaluated in terms of correlation functions in the original theory with insertion of the projector (I.40). One then obtains

$$\left\langle \hat{\mathcal{O}} \right\rangle_{\text{orb}}^{\text{untw}} (\tau) = \frac{1}{|\mathbf{P}|} \sum_{g\in\mathbf{P}} \mathrm{Tr}_{\mathbb{1}} \left(g\mathcal{O} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} \right) = \frac{1}{|\mathbf{P}|} \sum_{g\in\mathbf{P}} \left\langle \hat{\mathcal{O}} \right\rangle \begin{bmatrix} \mathbb{1} \\ g \end{bmatrix} (\tau)$$

for the contribution of the untwisted sector to correlation function $\left\langle \hat{\mathcal{O}} \right\rangle_{\text{orb}}^{\text{untw}} (\tau)$ in the orbifold CFT. Equation (I.48) then shows that $\left\langle \hat{\mathcal{O}} \right\rangle_{\text{orb}}^{\text{untw}} (\tau)$ transforms under the action of the modular group as

$$\left\langle \hat{\mathcal{O}} \right\rangle_{\text{orb}}^{\text{untw}} \left(\frac{a\tau + b}{c\tau + d} \right) = \frac{1}{|\mathbf{P}|} \sum_{g\in\mathbf{P}} \left\langle \hat{\mathcal{O}} \right\rangle \begin{bmatrix} g^{-c} \\ g^{d} \end{bmatrix} (\tau).$$

Therefore, modular transformation mixes contributions coming from the untwisted and twisted sectors so that no modular invariant theory may be built from the untwisted sector alone.

We now turn to deriving the spectrum of such theories. As we said, the story goes as in the simpler toroidal case in the untwisted sector provided that one inserts the projector (I.40) onto **P**-invariant states. In order to give a taste of what happens in the twisted sectors, we consider fields satisfying the closed string condition (I.45) with $h \in \mathbf{P}$ of order p. Since $h \in O(d) \times O(d)$ by assumption, one may always go to a basis in which h is diagonal. More precisely, let X_1 and X_2 be two, say, left-moving real fields on which h acts as a rotation of angle $2\pi k/p$ for some non-vanishing integer k. Then, defining Z as

$$Z(z) := \frac{X^1 + iX^2}{\sqrt{2}},$$

h acts on Z as

$$h \cdot Z = e^{\frac{2i\pi k}{p}} Z.$$

On the cylinder, the periodicity condition (I.45) then becomes $Z(w + 2\pi) = e^{2i\pi k/p} Z(w)$. The mode expansion of Z and its conjugate $\overline{Z}$ on the plane therefore read

$$Z(z) = z_0 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \frac{1}{n + \frac{k}{p}} \frac{a_{n+k/p}}{z^{n+k/p}}, \tag{I.49}$$
$$\overline{Z}(z) = \overline{z}_0 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \frac{1}{n - \frac{k}{p}} \frac{\overline{a}_{n-k/p}}{z^{n-k/p}}$$

with $\left\{ a_{n+k/p} \right\}$ and $\left\{ \overline{a}_{n-k/p} \right\}$ independent operators with commutation relations

$$\left[a_{m+k/p}, a_{m+k/p} \right] = \left[\overline{a}_{m-k/p}, \overline{a}_{n-k/p} \right] = 0$$
$$\left[a_{m+k/p}, \overline{a}_{n-k/p} \right] = \left(m + \frac{k}{p} \right) \delta_{m+n,0}.$$

Unlike in the usual case where the oscillation number is integer-valued, the orbifold grants access to fields with rational oscillation quanta. Moreover, in (I.49), selecting **P**-invariant states forces the constant term z_0 to be a fixed point of the point group ; in other words, it must satisfy

$$\left(1 - e^{\frac{2i\pi k}{p}}\right) z_0 \in 2\pi\sqrt{\alpha'}\Gamma.$$

This means that states in the twisted sectors must be localised at the fixed points of **P** ; to each fixed point then corresponds a vacuum of the theory.

One may check that the Virasoro generators (or, more precisely, the contribution of Z and $\overline{Z}$ thereof) are given by

$$L_m = \sum_{n\in\mathbb{Z}} {}^{\circ}_{\circ} a_{n+k/p} \overline{a}_{m-n-k/p} {}^{\circ}_{\circ} + \frac{k}{2p}\left(1 - \frac{k}{p}\right)\delta_{m,0} \tag{I.50}$$

where ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ is the creation-annihilation normal ordering, meaning that the annihilation operators $\left\{a_{n+k/p}, \overline{a}_{n+1-k/p}, n \in \mathbb{N}\right\}$ are always placed to the right of the creation ones. This allows to make an important remark. From (I.48), we see that a necessary condition for the theory to be modular invariant is that the contribution to any correlation function of the h-twisted sector should be invariant under $\tau \mapsto \tau + p$, with p the order of h. In particular, the explicit dependence of the correlation function in τ read from (I.44) implies that any state should satisfy

$$p\left(L_0 - \frac{c}{24} - \overline{L}_0 + \frac{c}{24}\right) = 0 \mod 1. \tag{I.51}$$

In particular, let us consider now the full theory with h splitting as $h = \mathrm{diag}(h_L, h_R)$, h_L and h_R acting on the left- and right-moving fields respectively. We denote k_L^i and k_R^i the associated angles. Then, equation (I.50) shows that (I.51) becomes

$$\sum_i \frac{\left(k_L^i\right)^2 - \left(k_R^i\right)^2}{2p} = 0 \mod 1 \tag{I.52}$$

in the bosonic string case [11], where we have used the fact that the oscillation numbers are integer multiples of $1/p$. This has first been noticed in [25, 26] and will be used in section III.2.4 to find the values of k_L^i and k_R^i leading to consistent orbifold constructions in a particular setup to be defined in chapter III.

Of particular interest are freely acting orbifolds. Indeed, in such models, there is generically no massless states coming from the twisted states which is an attractive feature for moduli stabilisation purposes. We now explain an interesting connection between such orbifolds and the field theoretical Scherk-Schwarz construction from section I.2 which has long been noticed [27, 28]. Let us assume that the space group is generated by the group element acting on $T^d \cong T^{d-1} \times S^1$ as

$$(X, Y) \mapsto \left(\Theta \cdot X, Y + 2\pi\frac{R}{p}\right)$$

11. In superstring theories, the world-sheet fermions also contribute non-trivially to (I.52) and must be taken into account.

with X and Y coordinates on T^{d-1} and S^1 respectively, with p the order of Θ and with R the radius of the circle factor. Then, the construction is precisely the same than the Scherk-Schwarz reduction on a circle of radius R/n with monodromy Θ. It was moreover shown in [29] that the minima of the scalar potential - that is the vacua of the theory - are precisely located at the subspace of the toroidal CFT moduli space where identification by the point group is possible. Turning things around, the minima of the scalar potential in the Scherk-Schwarz construction form then precisely the space in which a CFT description is available ; the latter reduction also has the advantage of being applicable anywhere in moduli space thereby granting access to more generic theories.

So far, we have barely made any mention to the presence or absence of world-sheet supersymmetry. We now turn to a the common concern of analysing the amount of space-time supersymmetry preserved by orbifold constructions in world-sheet supersymmetric theories. We consider a theory with left-moving fermions $\psi(z)$, the discussion for right-moving ones in type II theories following uneventfully. In order for the theory to make sense, the world-sheet supercurrent

$$T_F := i\sqrt{\frac{2}{\alpha'}}\psi \cdot X$$

must be invariant under the point group action. If X transform as $X \mapsto h_L X$ under the point group, with $h_L \in O(d)$, this imposes that the fermions should transform as $\psi \mapsto h_L \psi$ as well. The Ramond ground states therefore transform as

$$|s_1, s_2, s_3, s_4\rangle \mapsto e^{\frac{2i\pi}{p}\mathbf{s}.\mathbf{k_L}} |s_1, s_2, s_3, s_4\rangle \tag{I.53}$$

where $s_i \in \left\{\pm\frac{1}{2}\right\}$ are the associated spins and $\mathbf{k_L}$ the vector corresponding to the angles of the rotation h_L as before [12]. The amount of preserved supersymmetry may then easily be understood by counting the number of gravitini surviving the orbifold projection. Since a gravitino will be obtained by tensoring a Ramond ground state with a state with opposite chirality and carrying a vector index, massless gravitini are in one-to-one correspondence with the Ramond ground states invariant under (I.53). The number of preserved supercharges in the reduced theory is then given by the number of independent solutions to

$$\mathbf{s}.\mathbf{k_L} = 0 \mod p.$$

As we have seen, the fixed points of the point group $\mathbf{P}$ play a special role as they label vacua in the twisted sectors. Moreover, we have just said that orbifold constructions could also break some to all space-time supersymmetry. When considering manifolds instead of orbifolds, preserving some supersymmetry is a feature of Calabi-Yau manifolds as we will detail in II.1. It turns out that there may be a geometric connection between all these concepts, that is if the orbifold admits a geometric interpretation. If it were not for the fixed points, an orbifold $\mathcal{O}$ would be locally diffeomorphic to the real space everywhere and would then be a manifold. There exists a geometrical construction called *blowing up* the fixed points which amounts to

12. This shows that for an orbifold to have order p, one should impose

$$\sum_i k_L^i = 0 \mod 2.$$

If it is not the case, the theory may still be consistent but will really be an orbifold of order $2p$ instead.

"smooth out" the neighbourhood of the fixed points, turning the orbifold $\mathcal{O}$ into a manifold $\mathcal{M}_{\mathcal{O}}$. Since this procedure does not change the global properties of the orbifold, from what we have just said $\mathcal{M}_{\mathcal{O}}$ must be Calabi-Yau if compactification on $\mathcal{O}$ preserves some space-time supersymmetry. Apart from the toroidal case, string theories with Calabi-Yau target spaces generically do not admit a CFT description. However, turning the above discussion upside down shows that there exists points in the corresponding moduli spaces where the Calabi-Yau manifold tend to an orbifold. At these points, the string theory does admit a CFT description. There are other points where the world-sheet theory may be understood in terms of a Gepner model [20]. In particular, such points have played a key role in the construction of the models introduced in [7] and reviewed in III.1.

We conclude by illustrating the orbifold procedure with a simple example. Let us assume that one would like to construct a geometric orbifold starting from the two-dimensional torus. We assume that the space group G may be generated by a single element g which acts on the periodic world-sheet bosons as $X \mapsto g \cdot X$. From section I.3.1, we know that G must be a finite subgroup of $O(\Gamma_{2,2})(SL_2(\mathbb{Z})_T \times SL_2(\mathbb{Z})_U) \rtimes \mathbb{Z}_2$. As we will discuss in more details in II.4, choosing $g \in SL_2(\mathbb{Z})_U$ leads to a geometric action on the torus. Since g generates a subgroup of $SL_2(\mathbb{Z})$ of finite order, few possibilities remain at this point : indeed, g must be either be the identity matrix or in the $SL_2(\mathbb{Z})$ conjugacy class of one of the following matrices

$$g_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g_3 := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^{\pm 1}, \quad g_4 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\pm 1}, \quad g_6 := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{\pm 1}$$

leading to space groups of order 2, 3, 4 or 6 respectively (see *e.g.* [30] for more details about $SL_2(\mathbb{Z})$ subgroups of finite order). From the background fields transformation rules (I.30), one deduces that the orbifold puts constraints on the allowed values for U. More precisely, it selects lattices parametrised by U which admit g as an automorphism ; U is therefore unconstrained in the $\mathbb{Z}_2$ case (as any lattice admits g_2 as an automorphism), $U \sim i$ in the $\mathbb{Z}_4$ case and $U \sim \exp\left(\frac{2i\pi}{3}\right)$ in the $\mathbb{Z}_3$ and $\mathbb{Z}_6$ cases where $\sim$ means equality up to a $SL_2(\mathbb{Z})_U$ transformation. This shows explicitly how moduli (coming from the untwisted sector) may be frozen in orbifold constructions ; as we have seen, moduli coming the twisted sectors also generically arise but these may easily given a mass by considering freely acting orbifolds.

Chapitre II

Superstring compactifications

II.1 Space-time supersymmetry

As we already emphasised, a key tool in string theory is supersymmetry. Usually, supersymmetric theories are indeed easier to analyse than their non-supersymmetric counterparts and typically satisfy non-renormalisation theorems. Through mathematical tools such as localisation techniques in QFT or dualities in string theories, supersymmetry may also grant access to some non-perturbative features which are in particular missed by a world-sheet approach. The number of supercharges in a theory is also an interesting parameter for classification purposes. An essential problem in studying geometric compactifications of string theories is therefore to understand how supersymmetry may get broken in dimensional reductions. We now show how to extract this information from the geometrical details of the internal manifold. Since a world-sheet description of dimensionally reduced string theories is generically not available, we will focus on the corresponding low-energy effective theories which are supergravities. The strategy here is then to start from the ten-dimensional theory with either 16 or 32 supercharges, for the type I and heterotic theories or for the type II ones respectively, and to study their dimensional reduction as in chapter I.

In a quantum theory, a symmetry is unbroken if and only if the associated generator Q annihilates the vacuum $|\mathrm{vac}\rangle$. This is equivalent to requiring that $\left\langle \mathrm{vac} \middle| \left[Q, \hat{\mathcal{O}}\right] \middle| \mathrm{vac} \right\rangle = 0$ or $\left\langle \mathrm{vac} \middle| \left\{Q, \hat{\mathcal{O}}\right\} \middle| \mathrm{vac} \right\rangle = 0$ for all operators $\hat{\mathcal{O}}$, depending on whether Q is bosonic or fermionic. In other words, the expected value of the variation of any operator under the symmetry generated by Q should vanish. If Q is a supercharge, there are then two cases one should consider. The first one is when $\hat{\mathcal{O}}$ is bosonic, in which case $\left\{Q, \hat{\mathcal{O}}\right\}$ is fermionic ; its VEV then automatically vanish as a consequence of Lorentz invariance. The only constraints one gets from requiring supercharges to be conserved therefore comes from the second case in which $\hat{\mathcal{O}}$ is fermionic. The fermionic content of the ten-dimensional superstring theories depend on the model of interest. The type II string spectrum contain two gravitini and a dilatino ; the type I and heterotic models both have a gravitino, a dilatino and 1 and 496 gauginos respectively.

The supersymmetry transformation laws depend on the background fields in general. However, generic backgrounds do not lead to conformally invariant theories on the world-sheet [31, 32]. The simplest consistent configuration, at leading order, is the case of a theory without fluxes and with a constant dilaton. In this case, the supersymmetry of the dilatino vanishes while those of the gravitini read [33]

$$\delta_\epsilon \psi_M = \nabla_M \epsilon$$

with ϵ the infinitesimal supersymmetry parameter and ∇_M the covariant derivative with respect to the connection on the ten-dimensional space-time. Consequently, requiring unbroken supersymmetry implies the existence of a ten-dimensional metric admitting covariantly constant spinors. For heterotic theories, the vanishing of the variation of the gluinos also imposes constraints on the gauge bundle which must in particular be holomorphic [33].

For the sake of simplicity, let us now be more specific and consider the case of a reduction down to four dimensions - the other cases being analysed in a similar way. We assume that space-time may be split as $\mathcal{M}^{1,3} \times \mathcal{K}_6$ for some six-dimensional compact manifold $\mathcal{K}_6$. Then, the supersymmetry parameter ϵ may be split into the tensor product of four- and six-dimensional spinors as

$$\epsilon := \xi_4 \otimes \eta_6 + \text{h.c.}. \tag{II.1}$$

The connection on Minkowski space vanishes and the covariant derivative identifies to the usual derivative ; requiring $\nabla_\mu \epsilon = 0$ then amounts to impose $\partial_\mu \xi_4 = 0$, $\mu = 0, ..., 3$ being indices tangent to $\mathcal{M}^{1,3}$, that is that the spinor ξ_4 must be constant. The remaining constraints are therefore given by

$$\nabla_m \eta_6 = 0, \quad m = 4, ..., 9 \tag{II.2}$$

that is that the six-dimensional spinor η should be covariantly constant by itself. In summary, this shows that under the assumption that the four-dimensional space-time is Minkowski, the amount of preserved supersymmetries is directly related to purely geometric considerations - at least as long as geometric compactifications are considered. In addition, the above analysis grants access to the amount of conserved supercharges as each independent spinor of $\mathcal{K}_6$ satisfying (II.2) may be used as a supersymmetry generator.

We recall that the covariant derivative - or the affine connection in a more geometrical description - may be seen as a way of defining how to transport a vector from the tangent space of a point in a manifold to the other. Indeed, parallel transport of a field ψ along the flow curve associated to a vector field X is defined through the differential equation

$$\nabla_X \psi := X^m \nabla_m \psi = 0. \tag{II.3}$$

The value of the field ψ is then completely determined by its initial value and the connection ∇. In particular, when fields are transported around a closed curve, they do not go back to their original value but are transformed in general. Defining an appropriate composition law for the closed curves, one may show that such transformations form a group known as the *holonomy group*. However, a covariantly constant field trivially satisfies (II.3) for any vector field X ; it therefore remains invariant under parallel transport around closed loops. In other words, it transforms in a singlet representation of the holonomy group. A generic orientable Riemannian manifold of dimension d has holonomy group $SO(d)$ which means that spinors of $\mathcal{K}_6$ transform under $SO(6)$ or a subgroup thereof. Depending on its chirality, a spinor lives either in the fundamental or anti-fundamental representation of $SU(4) \cong SO(6)$. This means that $\mathcal{K}_6$ may only admit a covariantly constant spinor if its holonomy group H is strictly contained in $SU(4)$ in such a way that the decomposition of, say, the **4** of $SU(4)$ contains a H-singlet. In particular, if $H \subseteq SU(3)$, then the $\mathbf{4}$ of $SU(4)$ contains at least a singlet as required. In this case, compactification over $\mathcal{K}_6$ preserves at least $\mathcal{N} = 1$ supersymmetry in four dimensions. A $2d$-dimensional manifold which may be equipped with a connection leading to a holonomy group contained in $SU(d)$ is known as a *Calabi-Yau manifold*[1]. More details about these manifolds are given in appendix A.

In the following sections II.2 and II.3, we will be interested in geometric reductions preserving a given amount of supercharges. The above discussion will then allow us to determine straightforwardly which internal manifold to choose so as to obtain such four-dimensional theories, depending on the superstring theory of interest. In the following, we analyse geometric compactifications of the heterotic and type II theories down to six dimensions and preserving sixteen supercharges.

II.2 Theories with 16 supercharges

II.2.1 Reduction of the heterotic string

We begin by considering the heterotic superstring theories. Those are constructed from the bosonic string by adjoining chiral superpartners to the world-sheet bosons, by opposition to the type II theories where there are two sets of superpartners, one for each chirality. As a result, the heterotic string in ten dimensions has $\mathcal{N} = (1, 0)$ supersymmetry. We are looking for space-times factorising as a product between the six-dimensional Minkowski space $\mathcal{M}^{1,5}$ and a four-dimensional compact space $\mathcal{K}_4$. In addition, we want the reduced theory to preserve sixteen supercharges.

In the non-compact theory, there is only one supersymmetry generator ϵ. Spinorial irreducible representations of the ten-dimensional Lorentz group $SO(1, 9)$ are Majorana-Weyl fermions with 16 real degrees of freedom and a fixed chirality. If one wants to obtain a six-dimensional theory with 16 supercharges, then all degrees of freedom of ϵ should be kept. In light of what we have said in section II.1, this means that the holonomy group of internal manifold $\mathcal{K}_4$ should be trivial. This may only be achieved for a vanishing connection, that is for a flat space ; this narrows down the range of possibilities for $\mathcal{K}_4$ to only one, namely to the four-torus T^4. Following our analysis, splitting ϵ as in (II.1), one has the decomposition

$$\begin{array}{ccc} SO(1,9) & \longrightarrow & SO(1,5) \times SO(4) \\ \mathbf{16}_+ & \longrightarrow & (\mathbf{4}_+, \mathbf{2}_+) \oplus (\mathbf{4}_-, \mathbf{2}_-) \end{array} \tag{II.4}$$

with the $\pm$ index indicating the chirality of the spinor. Each of the two terms $(\mathbf{4}_\pm, \mathbf{2}_\pm)$ are independent supersymmetry generators in the six-dimensional theory ; since the two generators have opposite chiralities, the resulting model has $\mathcal{N} = (1, 1)$ supersymmetry.

Since we already analysed toroidal compactifications of string theories in some details in I.3, we will not spend too much time doing this again here. The heterotic string may be seen as the tensor product of, say, a left-moving supersymmetric sector and a right-moving bosonic one with critical dimensions 10 and 26 respectively. In order to reduce the theory down to 6 dimensions, one may therefore take 4 left-moving bosons and 20 right-moving ones to be periodic and study the corresponding CFT along the same lines as in I.3. This will lead

1. In the literature, and in particular in physics, it is also common to encounter the slightly different definition that a d-dimensional Calabi-Yau manifold has the whole $SU(d)$ as a holonomy group instead of a subgroup thereof.

to the introduction of a shift lattice $\Gamma_{4,20}$ of signature $(4,20)$ which is restricted, by modular invariance, to be even and unimodular. Theorem I.3.1 then implies that

$$\Gamma_{4,20} \cong U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$$

with $E_8(-1)$ the negative definite Cartan matrix associated to the root system of the exceptional Lie group E_8. As before, the parameters spanning the moduli space may be seen as reflecting the choice of an embedding of $\Gamma_{4,20}$ into $\mathbb{R}^{4,20}$; the same considerations as before show that these correspond to the choice of a - say - four-dimensional space-like subspace of $\mathbb{R}^{4,20}$. We have seen that the corresponding parameter space is a Grassmannian ; in addition, one should identify points related by a duality, that is by an automorphism of $\Gamma_{4,20}$. In summary, the moduli space $\mathcal{M}_{\text{het}}$ of heterotic conformal field theories with target space a four-torus is given by

$$\mathcal{M}_{\text{het}}^{(\text{CFT})} \cong O(\Gamma_{4,20}) \backslash O(4,20)/(O(4) \times O(20)).$$

In addition to the four-torus metric and B-field, which account in total for $4^2 = 16$ parameters, there are $4 \times 16 = 64$ flat directions in $\mathcal{M}_{\text{het}}$ corresponding to Wilson lines ; from the worldsheet theory, these correspond to couplings between chiral and non-chiral bosons. Setting all Wilson lines to zero leads generically to either a $E_8 \times E_8$ or $\text{Spin}(32)/\mathbb{Z}_2$ gauge group ; turning them on breaks this gauge group to a smaller one and allows to interpolate continuously between the two heterotic string theories.

Finally, the moduli space of the total NLSM should also include the dilaton field. It is believed that the moduli space of the whole theory factorises as [34]

$$\mathcal{M}_{\text{het}} = \mathcal{M}_{\text{het}}^{(\text{CFT})} \times \mathbb{R}$$

with $\mathbb{R}$ the contribution from the dilaton. In the following, we will assume this is the case indeed.

II.2.2 Reduction of the type II string

We now turn to the type II string theories, both of which admit 32 independent supercharges in ten dimensions. More precisely, the type IIA (IIB) supergravity has $\mathcal{N} = (1,1)$ ($\mathcal{N} = (2,0)$) supersymmetry before dimensional reduction. We are looking for compactifications down to six dimensions preserving half the supercharges present in the mother theory ; as in (II.4), the generators of the ten dimensional supersymmetries decompose as

$$\begin{array}{ccc} SO(1,9) & \longrightarrow & SO(1,5) \times SO(4) \\ \mathbf{16}_+ & \longrightarrow & (\mathbf{4}_+, \mathbf{2}_+) \oplus (\mathbf{4}_-, \mathbf{2}_-) \\ \mathbf{16}_\mp & \longrightarrow & (\mathbf{4}_+, \mathbf{2}_\mp) \oplus (\mathbf{4}_-, \mathbf{2}_\mp) \end{array} \tag{II.5}$$

where, in the second line, the upper (lower) index corresponds to the type IIA (type IIB) chirality. Preserving half the supercharges may therefore be done by considering internal spaces $\mathcal{K}_4$ with holonomy preserving only the $\mathbf{2}_+$ (or, equivalently, only the $\mathbf{2}_-$) spinorial representation of $SO(4)$. This is easily done by noticing that

$$SO(4) \cong (SU(2) \times SU(2))/\{\pm \text{Id}\};$$

indeed, thinking of $\mathbb{R}^4$ as the quaternion space, the action of $SO(4)$ on a quaternion h may be written as

$$h \mapsto q_L h q_R^{-1}$$

with $q_L, q_R \in SU(2)$ [2]. Requiring $\mathcal{K}_4$ to have only one of the two $SU(2)$ factors as holonomy group therefore preserves only one of the two $SO(4)$ representations $\mathbf{2}_+$ and $\mathbf{2}_-$ in (II.5), leading to 16 supercharges in the six-dimensional theory as expected. This is enough to identify $\mathcal{K}_4$ as a *K3 surface*, that is a Calabi-Yau space of (complex) dimension 2. In particular, reduction of the type IIA (type IIB) string on a K3 surface leads to a six-dimensional theory with $\mathcal{N} = (1,1)$ ($\mathcal{N} = (2,0)$) supersymmetry.

As we discussed at the end of section I.4, apart from special points in the moduli space which may be understood in terms as toroidal orbifolds or Gepner models, string theories with a K3 surface as target space generically do not admit a CFT description. We now analyse the moduli space of such theories. Before doing so, we recall that conformal invariance on the world-sheet may be achieved at the quantum level by demanding that G is Ricci-flat, B is closed and ϕ is constant. While the Ricci-flatness condition is fulfilled for K3 backgrounds, we will assume in the following that the other requirements are also met.

In order to compare theories with sixteen space-time supercharges descending from type II and heterotic string theories, we will derive the associated moduli spaces. As a first step toward understanding the moduli space of Non-Linear Sigma Models (NLSM) - to be defined in II.2.2.b) - with target space K3, we will start by deriving the purely geometric moduli space of Ricci-flat metrics defined on a K3 surface [3].

II.2.2.a) Moduli space of Ricci-flat metrics on a K3 surface

The moduli space $\mathcal{M}_{\text{RICCI}}$ we are trying to identify here may be understood as a parameter space in which any two points correspond to different Ricci-flat metrics on a K3 surface. This may seem like a particularly ambitious task as, up to this day, not even one such Ricci-flat metric has been explicitly found (although some progress has been made in this direction, see *e.g.* [35]). However, Yau's theorem [36] is of great help here. In substance, it states here that given a complex structure J on a K3 surface X and a Kähler form Ω, there exists a unique Ricci-flat metric. This means in particular that determining $\mathcal{M}_{\text{RICCI}}$ may be done without any actual reference to the K3 metric by focusing on J and Ω instead.

We begin by fixing an even unimodular lattice of signature (3,19) $\Gamma_{3,19} \subset \mathbb{R}^{3,19}$. As we have seen before, any two such lattices are isomorphic to each other ; therefore, if X is a K3 surface, there exists an isomorphism $\alpha : H^2(X,\mathbb{Z}) \xrightarrow{\sim} \Gamma_{3,19}$. The choice of such an isometry is called a *marking* of the K3 surface and the pair (X, α) is usually referred to as a marked K3 surface. It turns out that a marking also defines a Hodge structure of weight 2. Indeed, let $\omega \in \Gamma_{3,19}^{\mathbb{C}} := \Gamma_{3,19} \otimes \mathbb{C}$ be the image of the generator of $H^{(2,0)}(X,\mathbb{C})$ which is defined up to multiplication by a constant factor. Then, using the scalar product defined on $\Gamma_{3,19}$, one has

$$\langle \omega, \omega \rangle = \int_{\alpha(X)} \omega \wedge \omega = 0$$

2. Since $(-q_L, -q_R)$ is mapped to the same $SO(4)$ element as (q_L, q_R), one should indeed identify $SU(2) \times SU(2)$ under the action of $\{\pm \text{Id}\}$ as claimed.

3. In the following, we give only a sketch of the proofs and arguments used in deriving the moduli space we are looking for ; the reader interested in a more complete study may find more details in [34].

as integration of a (p,q)-form over a complex manifold M with $\dim_{\mathbb{C}} M = m$ may only be non-zero if $p = q = m$. Moreover, $\langle \omega, \overline{\omega} \rangle > 0$ since this scalar product is the integral of a non-negative integrand and since the $\langle \omega, \overline{\omega} \rangle = 0$ case would imply $\omega = 0$. The aforementioned Hodge structure of weight 2 may then be defined as :

$$
\begin{aligned}
&H^{(2,0)}(\omega) := \mathbb{C}\omega \in \Gamma^{\mathbb{C}}_{3,19} \\
&H^{(0,2)}(\omega) := \mathbb{C}\overline{\omega} \in \Gamma^{\mathbb{C}}_{3,19} \\
&H^{(1,1)}(\omega) := \left[H^{(2,0)}(\omega) \oplus H^{(0,2)}(\omega)\right]^{\perp} \in \Gamma^{\mathbb{C}}_{3,19}
\end{aligned}
$$

with $\left[H^{(2,0)}(\omega) \oplus H^{(0,2)}(\omega)\right]^{\perp}$ the orthogonal complement of $H^{(2,0)}(\omega) \oplus H^{(0,2)}(\omega)$ in $\Gamma^{\mathbb{C}}_{3,19}$. What we have done so far is essentially assigning to the complex structure of a K3 surface a point ω in the parameter space

$$\left\{\omega \in \Gamma^{\mathbb{C}}_{3,19} \middle| \langle \omega, \omega \rangle = 0, \langle \omega, \overline{\omega} \rangle > 0\right\} \Big/ \mathbb{C}^*. \tag{II.6}$$

One may actually show that this parameter space is enough to completely characterise the complex structure of a marked K3 surface [37]. We may now decompose ω into elements of $\alpha^* H^2(X, \mathbb{R})$ by setting

$$\omega := x + iy.$$

Since ω is an element of the space (II.6), one must have $\langle x, x \rangle = \langle y, y \rangle > 0$ and $\langle x, y \rangle = 0$. In particular, this means that x and y must be linearly independent and space-like elements of $\alpha^* H^2(X, \mathbb{R})$.

Let us now consider the image Ω of the Kähler form of X under the isometry α. As it is the case for any Hermitian manifold of complex dimension m, the volume form of $\alpha(X)$ may be taken to be $\Omega^{\wedge m} = \Omega \wedge \Omega$ setting $m = 2$ [38]. This implies in particular that $\langle \Omega, \Omega \rangle$ is strictly positive so that Ω defines a space-like element of $\alpha^* H^2(X, \mathbb{R})$. Moreover, since the Kähler form of a K3 surface is of bidegree $(1,1)$ by construction, Ω must belong to $H^{(1,1)}(\omega)$. As before, this means that the integral of $\Omega \wedge \omega$ over $\alpha(X)$ vanishes, that is that $\langle \Omega, \omega \rangle = 0$. In terms of x and y, this means that Ω, x and y are three linearly independent and space-like elements of $\alpha^* H^2(X, \mathbb{R})$. Moreover, unlike that of x or y, the norm of Ω under the $\Gamma_{3,19}$ inner product is important as it is related to the volume of X.

In summary, we have just seen that the choice of a complex structure and of a Kähler form on a marked K3 surface X defines a three-dimensional space-like plane Σ in the ambient space $\mathbb{R}^{3,19}$ of $\Gamma_{3,19}$. Moreover, (x, y, Ω) define a natural orientation of the three-plane. The associated parameter space $\mathcal{T}_{\text{RICCI}}$ is then the Grassmannian of oriented space-like three-planes in $\mathbb{R}^{3,19}$ times a $\mathbb{R}^+$ factor which denotes the volume of the K3 surface. $\mathcal{T}_{\text{RICCI}}$ then reads

$$\mathcal{T}_{\text{RICCI}} = O(3,19)^+ / [SO(3) \times O(19)] \times \mathbb{R}^+.$$

Finally, by construction $\mathcal{T}_{\text{RICCI}}$ is the moduli space of Ricci-flat metrics on K3 surfaces but what we really want is to get rid of the marking α. Let us assume that α and α' are two markings for a K3 surface X, that is such that

$$\begin{array}{ccc} H^2(X,\mathbb{Z}) & \xrightarrow[\sim]{\alpha} & \Gamma_{3,19} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ H^2(X,\mathbb{Z}) & \xrightarrow[\alpha']{\sim} & \Gamma_{3,19} \end{array}$$

is a commutative diagram for a global diffeomorphism f of X and for some function g. Clearly, g is an isometry of $\Gamma_{3,19}$ which must preserve its scalar product. Let $O(\Gamma_{3,19})$ be the group of such isometries. It was shown in [39] that the elements of $O(\Gamma_{3,19})$ which could be induced by diffeomorphisms of K3 surfaces were precisely the subgroup $O(\Gamma_{3,19})^+$ of $O(\Gamma_{3,19})$ preserving the orientation of the space-like directions. Consequently, the moduli space of Ricci-flat metrics on a K3 surface reads

$$\mathcal{M}_{\text{RICCI}} = O(\Gamma_{3,19})^+ \backslash O(3,19)^+ / [SO(3) \times O(19)] \times \mathbb{R}^+. \tag{II.7}$$

There is a last remark one could make here which is not necessary but which will allow for an easier comparison with the moduli space of NLSM on a K3 surface. The indefinite orthogonal group satisfies $O(3,19) = O(3,19)^+ \times \{\pm\mathbb{1}\}$. Enhancing the former to the latter may therefore be done by introducing the extra generator $-\mathbb{1}$ which lies in the center of $O(3,19)$. This may therefore be done without affecting the left-right quotient (II.7) so that one could equivalently write

$$\mathcal{M}_{\text{RICCI}} = O(\Gamma_{3,19}) \backslash O(3,19) / [O(3) \times O(19)] \times \mathbb{R}^+.$$

II.2.2.b) Moduli space of NSLMs on K3

The moduli space we have just derived only takes into account the deformations associated to the Ricci-flat metric of K3 surfaces and while it will play a role in the discussion to come it misses part of the information we need here. From the world-sheet point of view, string theories may be expressed as models mapping a two-dimensional manifold Σ to a D-dimensional target space $\mathcal{M}$. For historical reasons, such theories are referred to as *non-linear sigma models*. The most general form of the action is given by

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma \Big(G_{ab}(X)\mathrm{d}X^a \wedge \star\mathrm{d}X^b + iB_{ab}(X)\mathrm{d}X^a \wedge \mathrm{d}X^b + \alpha' R^{(2)} \phi(X) \star 1 \Big) + \ldots, \tag{II.8}$$

where we have ignored potential fermionic terms. G, B and ϕ are interpreted respectively as a background metric, B-field and dilaton. As we already noted, a necessary condition for a string theory to be well-defined is local conformal invariance on the world-sheet to hold at the quantum level [5]. This is equivalent to requiring the associated beta functions, computed as expansions in powers of the Regge slope α', to vanish. At leading order, a sufficient condition for conformal invariance to be preserved at the quantum level is to require the metric G to be Ricci-flat, the B-field to be closed and the dilaton to be constant [34]. Defining a string theory with target space a K3 surface therefore amounts to specify, in addition to a Ricci-flat metric on K3, a closed B-field and a constant dilaton ; consequently, the moduli space we are looking for should also take the latter parameters into account.

There is a rather natural way of embedding the deformations of the metric and B-field in such a way that the derivation of the moduli space looks familiar to what we have seen in II.2.2.a). We begin by fixing an even unimodular lattice $\Gamma_{4,20} \subset \mathbb{R}^{4,20}$ of signature $(4,20)$ which is, according to theorem I.3.1, unique up to isomorphisms as was $\Gamma_{3,19}$. We also define a space-like four-plane $\Pi \subset \mathbb{R}^{4,20}$. A convenient way of embedding $\Gamma_{3,19}$ into $\Gamma_{4,20}$ is to take a null vector $w \in \Gamma_{4,20}$, that is such that $\langle w, w\rangle = 0$, and to define $w^\perp$ as the space of $\mathbb{R}^{4,20}$ vectors whose scalar product with w vanishes. Since $w \in w^\perp$ by construction, one may define the subspace $w^\perp/w$ obtained by identifying any two vectors of the form x and $x + \lambda w$ for $x \in \mathbb{R}^{4,20}$ and $\lambda \in \mathbb{R}$. Let us define

$$\Lambda_{3,19} := \Gamma_{4,20} \cap \frac{w^\perp}{w}.$$

Then, since $w \in \Gamma_{4,20}$, $\Lambda_{3,19}$ is a lattice of signature $(3,19)$. Moreover, $\Lambda_{3,19}$ inherits the evenness and unimodularity properties of $\Gamma_{4,20}$, so that we conclude that $\Lambda_{3,19} \cong \Gamma_{3,19}$. The choice of a null vector w may then be mapped to a choice of embedding of $\Gamma_{3,19}$ into $\Gamma_{4,20}$.

Now, let us pick a space-like four-plane Π into $\mathbb{R}^{4,20}$. A convenient way to identify a particular three-plane contained in Π is to use the w vector we have already defined ; indeed, $\Sigma' := \Pi \cap w^\perp$ may be projected onto $w^\perp/w$ to give a space-like three-plane which we identify with the three-plane Σ spanned by x, y and Ω from the previous section. In summary, so far we have made nothing more than embedding our previous setup into a higher-dimensional one. However, the advantage of doing so is that we may construct the orthogonal complement B' of Σ' in Π ; projecting B' onto $w^\perp/w$ gives a $R^{3,19} \cong H^2(X,\mathbb{R})$ vector which may be identified with the B-field. Finally, normalising B' by requiring $B'.w = 1$, one may show that $B'.B'$ is the volume of the K3 surface X [34].

In summary, the space of parameters of an NLSM with target space a K3 surface may be understood as the choice of a particular space-like four-plane in $R^{4,20}$; as before, the corresponding space $\mathcal{T}_\sigma$ is a Grassmannian which reads [4]

$$\mathcal{T}_\sigma = O(4,20)/[O(4) \times O(20)].$$

As in the previous section, one should still have to identify points in $\mathcal{T}_\sigma$ which lead to equivalent NLSMs. As we have seen before, the global diffeomorphisms of X induce a $O(\Gamma_{3,19})^+$ group of transformations on the moduli space which should still be present in the corresponding non linear sigma model. In addition, one sees from (II.8) that any shift of the B-field by an element of $X^2(X,\mathbb{Z})$ - that is a shift of the B vector defined above by a vector of $\Gamma_{3,19}$ - would amount to an addition of a integral multiple of $2\pi i$ to the action ; any such transformation then relates physically equivalent theories. Finally, complex conjugation of the action provides the $-\mathbb{1}$ generator necessary to enhance $O(\Gamma_{3,19})^+$ to $O(\Gamma_{3,19})$ as well as reversing the sign of B. We conclude that the moduli space we are looking for is the quotient of $\mathcal{T}_\sigma$ by a discrete group G_σ containing the semi-direct product $O(\Gamma_{3,19}) \ltimes \Gamma_{3,19}$, that is the group of rotations and translations of $\Gamma_{3,19}$.

However, that is not the end of the story. As in the toroidal case, conformal theories with distinct target spaces may be physically equivalent and therefore correspond to a same point in the moduli space. The relevant dualities here are known as *mirror symmetries* ; we will give more details about them in section III.1.1. For the moment, all we need to know is that these mirror symmetries are enough to generate, together with $O(\Gamma_{3,19}) \ltimes \Gamma_{3,19}$, the full $O(\Gamma_{4,20})$

4. As we have discussed before how considering only orientation-preserving transformations or not led to the same space in the end, we drop the "+" superscripts from the beginning.

group[40]. We conclude that the moduli space $\mathcal{M}_\sigma$ of NLSMs with target space a K3 surface is

$$\mathcal{M}_\sigma = O(\Gamma_{4,20})\backslash O(4,20)/[O(4)\times O(20)]. \tag{II.9}$$

Mathematically, the $\Gamma_{4,20}$ lattice may be interpreted as the whole cohomology ring $H^*(X,\mathbb{Z}) := H^0(X,\mathbb{Z})\oplus H^2(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})$ equipped with a bilinear form known as the Mukai pairing. Let $u_{(i)}$ be the formal sum $u^0_{(i)}+u^2_{(i)}+u^4_{(i)}$ for $i=1,2$, with $u^n_{(i)}\in H^n(X,\mathbb{Z})$. Then, the Mukai pairing $(u_{(1)},u_{(2)})$ is defined to be

$$(u_{(1)},u_{(2)}) := u^0_{(1)}\cdot u^4_{(2)} + u^4_{(1)}\cdot u^0_{(2)} + u^2_{(1)}\cdot u^2_{(2)}$$

with, as before, $\cdot$ being given by the cup product.

II.2.2.c) Moduli spaces of type II theories

We now turn to analysing the moduli spaces of type II theories. In addition to the Neuveu-Schwarz Neuveu-Schwarz (NS-NS) moduli analysed above, one should also take massless states coming from the Ramond-Ramond (RR) sector. In the type IIA and IIB string cases, these states correspond to differential forms with odd and even rank respectively. As we have seen in section I.1, massless modes originating from differential forms are in one-to-one correspondence with the harmonic forms of the internal space. Let then $\widetilde{C}_{(p)}$ be a p-form in the ten-dimensional theory ; using Künneth theorem, $\widetilde{C}_{(p)}$ may be rewritten as

$$\widetilde{C}_{(p)} = \sum_{i=0}^{p} C^A_{(p-i)}\omega^{(i)}_A$$

with $\omega^{(i)}_A \in H^i(\mathcal{K}_4)$ and $C^A_{(p-i)}$ a $(p-i)$ differential form defined on $\mathcal{M}_6$. If $\mathcal{K}_4$ is a K3 surface, its Hodge diamond is given by (A.2). In particular, since there is no non-trivial cohomology element with odd rank the above discussion shows that no massless scalars - that is no moduli - come from the RR sector in the type IIA case. In weak-string coupling limit, that is the limit in which the dilaton $\phi\to-\infty$, the type IIA moduli space is therefore given by (II.9). It is actually believed that the full moduli space of the theory may be given as the direct product of the moduli space (II.9) and of the dilaton, as argued in [34]. Assuming this is true, the moduli space of the type IIA string compactified on a K3 surface then reads

$$\mathcal{M}_{\text{IIA}} = (O(\Gamma_{4,20})\backslash O(4,20)/[O(4)\times O(20)])\times\mathbb{R}$$

with the $\mathbb{R}$ factor corresponding to the space spanned by the dilaton.

In the following, we will be less interested in the type IIB case and will therefore not spend too much time deriving the corresponding moduli space ; instead, we sketch a possible derivation and refer to [34] for more details. Unlike what happens the type IIA case considered above, there are massless scalars coming from the RR sector in the type IIB theory. More precisely, there are 1, 22 and 1 moduli coming from the 0-form, 2-form and self-dual 4-form of the type IIB spectrum, as $b_0(\text{K3}) = b_4(\text{K3}) = 1$ and $b_2(\text{K3}) = 22$. These may be combined and interpreted as a vector living in the ambient space of the intersection lattice $\Gamma_{4,20}$ of the K3 surface. Counting all corresponding degrees of freedom, holonomy arguments allow to conclude that the moduli space of the type IIB theory should be given as the Grassmannian $O(5,21)/(O(5)\times O(21))$ up to duality identifications. One may also show that the duality group may here be understood as the automorphism group of the lattice $\Gamma_{5,21}$ defined as before ; the moduli space of the type IIB theory therefore reads

$$\mathcal{M}_{\text{IIB}} = O(\Gamma_{5,21})\backslash O(5,21)/(O(5)\times O(21)).$$

II.2.3 Heterotic - type IIA duality

From the above analysis, one may notice that the theories obtained from compactifying either the heterotic string on a four-torus or the type IIA string on a K3 surface share some features : not only do they both lead to $\mathcal{N}=(1,1)$ supersymmetric theories, they also lead to the same moduli space. Moreover, as noticed in [41], their low-energy limits are equivalent up to the field redefinition

$$\begin{aligned} \phi_{\text{het}} &= -\phi_{\text{IIA}}, \\ g_{\text{het}} &= e^{-2\phi_{\text{IIA}}} g_{\text{IIA}}, \\ \mathrm{d}B_{\text{het}} &= e^{-2\phi_{\text{IIA}}} \star \mathrm{d}B_{\text{IIA}}, \\ A_{\text{het}} &= A_{\text{IIA}}, \end{aligned} \tag{II.10}$$

with ϕ the dilaton, g the six-dimensional metric, B the Kalb-Ramond B-field and A any gauge one-form coming from reduction on the torus in the heterotic theory and from the RR states wrapping non-trivial cycles of the K3 surface in the type IIA one.

This leads to a duality conjecture, namely that the heterotic and type IIA strings compactified on a four-torus and on a K3 surface respectively are actually different formulations of the same theory. One should stress here that this is not in contradiction with the fact that the corresponding world-sheet theories seem to be incompatible. Indeed, the mapping (II.10) shows that the sign of the dilaton gets reversed in the duality : in particular, both theories may not be analysed from perturbation theory at the same point in moduli space. This must be contrasted to the related four-dimensional duality discussed in II.2.4.

It is worthwhile considering the gauge fields in the six-dimensional theory. In the type IIA model, the conformal field theory approach shows that there are 24 $U(1)$ gauge fields : 1 comes directly from the type IIA one-form, 22 from wrapping the three-form around two-cycles of the K3 and 1 more from dualising it. The gauge group of the theory is therefore $U(1)^{24}$ and one may show that no extra non-abelian contribution can come from the CFT approach [34]. However, from the heterotic perspective while the same conclusion generically holds true, we know from II.2.1 that there are points in the CFT moduli space where the $U(1)^{24}$ gauge group gets enhanced to a non-abelian one. More precisely, the set of all states with left and right momentum p_L and p_R satisfying

$$\begin{aligned} p_L^2 &= 0, \\ p_R^2 &= 2, \end{aligned} \tag{II.11}$$

if any, form the root lattice of the associated gauge group. Notably, all roots in (II.11) have equal length showing that the associated Lie group must be *simply laced* - that is part of the *A*-*D*-*E* classification. A particular subspace of the heterotic moduli space with non-trivial gauge group is given by the subspace with vanishing Wilson lines, in which case the Narain lattice factorises as $\Gamma_{4,20} = \Gamma_{4,4} \oplus E_8(-1)^{\oplus 2}$, proving that (II.11) does admit solutions. This may indicate two things : either the conjectured duality is wrong or the corresponding type IIA gauge group enhancement remains invisible from the world-sheet point of view.

A detailed analysis of the interpretation of the Narain lattice $\Gamma_{4,20}$ from both perspectives shows that the points where states satisfying (II.11) exist correspond to orbifold limits of the K3 surface from the type IIA perspective. This is no coincidence ; indeed, it was shown in [41] that a type IIA string compactified on an orbifold could exhibit a non-abelian gauge group such that the A-D-E classification of the orbifold singularities coincides with that of the gauge group. Moreover, a $\Gamma_{4,20}$ vector α satisfying (II.11) is orthogonal to the space-like four-plane Π introduced in II.2.2.b), that is to both the space-like three-plane Σ and to the $\mathbb{R}^{4,20}$ vector B' which encode respectively the complex and Kähler structure of the K3 surface and the B-field. Orthogonality with Σ is actually equivalent to saying that the K3 surface has orbifold singularities [34]. The orthogonality of B' and α however, leads to an additional requirement, namely that the component of the B-field along the direction dual to α must be zero.

From the type IIA point of view, gauge symmetry enhancement come from non-perturbative effects which may be understood intuitively following [42]. In the type IIA theory, the D-branes are solitons coming from the RR sector. They may in particular wrap around non-trivial cycles of the target space, the mass of the associated states being related to the volume of the cycles and to the B-field. When coming closer to an orbifold limit of a K3 surface and provided that the B-field vanishes along the appropriate directions, at least some of its cycles shrink down leading to light D-branes ; at the orbifold point, these become massless and should be kept in the truncation to the massless sector. These extra massless states may be precisely mapped to the extra winding and momentum states satisfying (II.11) in the heterotic theory by supersymmetry arguments [5]. In summary, since D-branes are solitonic, the gauge group in the type IIA theory gets enhanced indeed but only through non-perturbative effects, explaining the inability of the conformal field theory to grasp them.

Up to this day, nothing has disproved the above conjectured duality ; we will assume in the following that it holds true, which will turn out to be essential to the analysis of chapter III. There, the duality will allow us to probe non-perturbative features of the type IIA model first described in [7] and reviewed in III.1 by constructing a heterotic dual model ; indeed, since the duality described above interpolates between the weak and strong coupling regimes of each theory, it allows to go beyond perturbative considerations as the above enhanced gauge symmetry points analysis illustrated.

II.2.4 Four-dimensional theories

We now turn to four-dimensional theories with sixteen supercharges, that is with $\mathcal{N} = 4$ extended supersymmetry. Using once again the arguments from section II.1, we find that requiring space-time to factorise globally as $\mathcal{M}^{1,9} = \mathcal{M}^{1,3} \times \mathcal{K}_6$ imposes the internal space $\mathcal{K}_6$ to have $SU(2)$ holonomy in the type II theory. While a four-dimensional manifold with $SU(2)$ holonomy may only be a K3 surface, this is no longer the case for a six-dimensional one. However, cohomology considerations show that the universal cover of $\mathcal{K}_6$ must be isomorphic

5. A complete dictionary may be found *e.g.*in [43, 44] ; we will give such a dictionary for four-dimensional theories in the next section.

to K3 $\times T^2$ [34] ; consequently, any geometric compactification of the type II string down to four dimensions preserving $\mathcal{N} = 4$ supersymmetry has an internal space of the form

$$\mathcal{K}_6 \cong \left(\text{K3} \times T^2\right)/G$$

for some group G. Moreover, as detailed in A.3, the action of G on the K3 piece of $\mathcal{K}_6$ must be symplectic in order to preserve $\mathcal{N} = 4$ supersymmetry. Such an action necessarily has fixed points on K3 [45] ; in order to avoid quotient singularities, any non-trivial action of G on K3 should therefore be combined with a translation on T^2, which imposes in particular that the action of G on K3 must be abelian. In summary, our requirements put a lot of constraints on the quotienting group G and its possible actions on K3 have been fully classified in [46]. Such models are usually referred to as *CHL orbifolds* in the literature.

A fair question would be to wonder whether the corresponding type IIA theory would still admit a heterotic dual constructed as above. In order to answer this question, we start by noticing that any such theory may be obtained in two steps, by first compactifying the type IIA string on K3$\times T^2$ and then taking the quotient by the discrete group G. The moduli space of $\mathcal{N} = 4$ theories in 4 dimensions is quite constrained ; indeed, holonomy arguments show that the "Teichmüller space" - that is the space of parameters obtained before identifying duality related points - must factorise as [47]

$$\mathcal{T}_{\text{4D}}^{\mathcal{N}=4} = \frac{SL(2)}{U(1)} \times \frac{O(6,m)}{O(6) \times O(m)} \tag{II.12}$$

with m the number of vector multiplets of the theory. Counting the number of moduli coming from compactification of the type II string shows that $m = 22$ for both type IIA and type IIB theories [6]. Assuming that the type IIA string compactified on K3 is dual to the heterotic string compactified on T^4, it seems natural to assume that the duality still holds when compactifying both theories over an additional two-torus. Therefore, a natural guess for a potential heterotic dual is the theory obtained by its reduction on $T^4 \times T^2 \cong T^6$. In this case, from section I.3 one also finds that the Teichmüller space is given by (II.12) with $m = 22$. In this case, the $SL(2)/U(1)$ part corresponds to the axio-dilaton modulus and the $O(6,22)/(O(6) \times O(22))$ one to the moduli space of conformal field theories on the torus with 6 and 22 left- and right-moving dimensions. Using (II.10), one may also show that the $SL(2)/U(1)$ factor corresponds to the Kähler moduli of T^2 in the type IIA picture. Moreover, since T-duality exchanges type IIA and type IIB theories, one could also derive a new type IIB dual theory by dualising along a cycle of T^2 ; in this case, the $SL(2)/U(1)$ factor in (II.12) would correspond to the complex structure of the T^2 as T-duality exchanges Kähler and complex moduli of the two-torus. This means in particular that the perturbative limit $\phi \to \infty$ in the heterotic string picture may be perturbatively understood from both type II perspectives, as it corresponds to a large volume and degenerate torus limit for the type IIA and IIB theories respectively.

One may now get access to the duality group of the theory and then to the full moduli space. From the type IIB analysis, where the $SL(2)/U(1)$ factor corresponds to the complex modulus of T^2, one gets that any two points related by a $SL_2(\mathbb{Z})$ transformation are equivalent ; from the heterotic point of view, the $O(6,22)/(O(6) \times O(22))$ part corresponds to the conformal field theory of 6 chiral and 22 anti-chiral periodic bosons with momenta and

6. In counting the degrees of freedom coming from the RR sector in the type IIB case, one should recall that the four-form is self-dual.

Heterotic string on T^6		Type IIA string on K3 $\times T^2$	
$K/1$	$NS5/\hat{1}$	$D0$	$D2/T^2$
$K/2,3,4$	$NS5/\hat{2},\hat{3},\hat{4}$	$D2/\omega_a$	$D4/(T^2\times\omega_a)$
$K/5,6$	$NS5/\hat{6},\hat{5}$	$K/5,6$	$W/6,5$
$W/1$	$KKM/\hat{1}$	$D4/\mathrm{K3}$	$D6/(\mathrm{K3}\times T^2)$
$W/2,3,4$	$KKM/\hat{2},\hat{3},\hat{4}$	$D2/\omega_a$	$D4/(T^2\times\omega_a)$
$W/5,6$	$KKM/\hat{5},\hat{6}$	$NS5/\hat{6},\hat{5}$	$KKM/\hat{5},\hat{6}$
$Q_{1,\dots,16}$	$HM_{1,\dots,16}$	$D2/\omega_a$	$D4/(T_2\times\omega_a)$

Table II.1 – Dictionary between BPS states in the heterotic on T^6 and type IIA on K3 $\times\, T^2$ models given in [48]. Even and odd columns are related by electric-magnetic duality. The legend goes as follows : K/i is a momentum state along direction i, $i=1,\dots,6$. W/i is a winding state along direction i. $NS5/\hat{\imath}$ is an NS5 brane wrapping all internal directions but i. $KKM/\hat{\imath}$ is a Kaluza-Klein monopole localised in direction i. ω_a is a basis of 2 cycles of K3. $D0$, $D2$, $D4$ and $D4$ are 0, 2, 4 and 6 branes ; $D2/\omega_a$ is a 2-brane wrapping ω_a and so on. Q_I is a charge of the rank 16 heterotic gauge group. HM_I is the corresponding H-monopole.

winding living in a Narain lattice $\Gamma_{6,22}$. Any two points related by a $\Gamma_{6,22}$ automorphism are therefore equivalent, leading to the moduli space

$$\mathcal{M}_{4\mathrm{D}}^{\mathcal{N}=4}=(SL_2(\mathbb{Z})\backslash SL(2)/U(1))\times(O(\Gamma_{6,22})\backslash O(6,m)/(O(6)\times O(m))).$$

This allows in particular to understand the action of a quotienting group on each side of the duality as an automorphism of $\Gamma_{6,22}\cong\Gamma_{4,20}\oplus\Gamma_{2,2}$ with the action on $\Gamma_{4,20}$ being interpreted as an automorphism of K3 and of the Narain lattice in the type IIA and heterotic pictures respectively and with a shift along vectors of $\Gamma_{2,2}$. Importantly, the quotient may only admit a geometric interpretation from the type IIA perspective if the orbifold group acts non-trivially on a sub-lattice of $\Gamma_{3,19}\subset\Gamma_{4,20}$ only as $\Gamma_{3,19}$ may then be understood as the intersection lattice of K3. Moreover, special states known as *BPS states* are protected by supersymmetry and should be present in both type IIA and heterotic string, constituting both a non-trivial test of the conjectured duality and better understanding of the non-perturbative sector of each theory. Comparing their respective $\Gamma_{6,22}$ charges allows to derive the dictionary given in table II.1. In particular, one may notice that the fundamental string winded around a cycle of the T^2 in one theory in understood in terms of a NS5-brane wrapping all other directions of the internal space in the other ; winding modes of the heterotic string around cycles of the T^4 are by contrast mapped to D-branes wrapping cycles of the K3.

Now that the type II/heterotic duality has been understood for the unquotiented theory, it is time to consider generic orbifolds as well, the argument being the same for both CHL and non-CHL constructions. It is not obvious to see whether the duality still holds after the orbifold procedure ; in fact, explicit examples where it does not have been found in the literature. In [49], for example, it has been shown that the orbifold of the type IIA string compactified on K3 by the $\mathbb{Z}_2$ group generated by $(-1)^{F_L}$, F_L being the left-moving fermionic number on the world-sheet, is isomorphic to the type IIB string compactified on K3. On the other hand, quotienting the originally dual heterotic theory by the same automorphism leads to an anomalous model. A key point here to build dual models from orbifolds is the adiabatic argument of [49] that we now review. Consider the type IIA theory compactified on K3 $\times S^1$ with S^1 a circle of radius R and a discrete group G acting freely on S^1. Since the action of G on S^1 is free, the quotient $(\mathrm{K3} \times S^1)/G$ has no singularities ; for R large enough, a low energy observer will then not be able to tell the difference between S^1 and a non-compact direction. One may therefore get locally a dual description in terms of the heterotic string by applying the duality fiberwise. Once again, the main difference here with the duality breaking example considered above is the free action of the quotient group on the circle, which removes any singularity and allows to consider a smooth decompactification limit. Of course, this argument alone is not sufficient to identify a heterotic dual as the large R limit masks some features to a low energy observer who may in particular not observe non-trivial winding modes in this case. In order to unambiguously identify a dual theory, one then typically has to rely on consistency requirements such as modular invariance of the theory. We will explicitly study such an example in chapter III, where we consider models which may be viewed as non-geometric generalisations of CHL orbifolds using non-symplectic automorphisms of K3 surfaces instead of symplectic ones.

II.3 Theories with 8 supercharges

II.3.1 Reduction of the type II string

We now turn to the analysis of four-dimensional theories with 8 supercharges, that is with $\mathcal{N} = 2$ supersymmetry, coming from geometric compactification of string theories. We first focus on reductions of type II models. As in chapter II.2, requiring a preserved amount of space-time supersymmetry puts constraints on the holonomy group of the internal space, to which we keep referring as $\mathcal{K}_6$. With similar considerations as in section II.2.2, one shows that in the type II case, the holonomy group of the internal manifold must be restricted to $SU(3)$ [33], forcing $\mathcal{K}_6$ to be a Calabi-Yau three-fold.

Four-dimensional $\mathcal{N} = 2$ supergravities are quite constrained as we will review in IV. In particular, two types of multiplets may contain massless scalar fields : vector and hypermultiplets, which contain respectively two and four real scalars. Holonomy arguments show that the moduli space of such theories split as

$$\mathcal{M}_{\mathrm{4D}}^{\mathcal{N}=2} = \mathcal{M}_{\mathrm{V}} \times \mathcal{M}_{\mathrm{H}} \tag{II.13}$$

with $\mathcal{M}_{\mathrm{V}}$ a special Kähler manifold and $\mathcal{M}_{\mathrm{H}}$ a quaternionic Kähler manifold spanned by the vector and hypermultiplets respectively [50]. Both $\mathcal{M}_{\mathrm{V}}$ and $\mathcal{M}_{\mathrm{H}}$ generically receive quantum corrections expanded in terms of two parameters : the first one is the string coupling g_S, related the the dilaton VEV. From a space-time perspective, this may be understood as a loop expansion corresponding to summation over world-sheet topologies. The second parameter is the string tension α', related to the Planck mass. Its appearance is due to the fact that a generic interacting string theory is not solvable. The usual strategy is therefore to express the world-sheet fields as $X(\tau, \sigma) = X_0 + \sqrt{\alpha'}\delta X(\tau, \sigma)$ and to expand the background parameters around X_0. Such an expansion is therefore necessary only for models which do not admit a CFT description. It is then particularly useful to determine whether the dilaton and Kähler modulus lie in a vector or in a hypermultiplet ; the splitting (II.13) implies that either $\mathcal{M}_{\mathrm{V}}$ or $\mathcal{M}_{\mathrm{H}}$ may be evaluated exactly in the perturbative limit $\phi \to \infty$ - that is exactly in g_s and/or in the large volume limit - that is exactly in α'. Of course, this comes at the cost of the ability of evaluating the other factor in (II.13) in perturbation theory.

We start by looking at the NS-NS sector which is identical for both type II theories. We consider a compactification without flux, meaning that the B-field must be closed, the dilaton constant and the metric Ricci-flat. This last requirement deserves some special attention as it constraints the allowed deformations in a non-trivial way. The idea here is to fix a given Calabi-Yau background around which to expand the metric, that is to set

$$g_{mn} = g^{(0)}_{mn} + \delta g_{mn}$$

with $g^{(0)}_{mn}$ the background metric. One may use diffeomorphism invariance and tracelessness of the metric to impose

$$g^{(0)mn}\delta g_{mn} = \nabla^m \delta g_{mn} = 0. \tag{II.14}$$

with ∇ the covariant derivative with respect to the background connection. We emphasise here that (II.14) should not be seen as physical constraints on δg_{mn} as much as a gauge choice. By contrast, one should impose the internal space to remain Ricci-flat in order to ensure the absence of conformal anomaly on the world-sheet. In the "gauge" (II.14), this gives at leading order the Lichnerowicz equation

$$\nabla^o \nabla_o \delta g_{mn} + 2R^{(0)}{}_{m\ n}^{\ o\ p}\delta g_{op} = 0 \tag{II.15}$$

where we have used the Ricci-flatness of the background metric and where $R^{(0)}_{klmn}$ is the Riemann tensor with respect to the background connection. In particular, equation (II.15) does not intertwine purely (anti-)holomorphic and mixed components of the metric. One may therefore see (II.15) as two sets of equations, one for the δg_{ij}'s and the other one for the $\delta g_{i\bar{\jmath}}$'s with i, $j = 1, ..., 3$ holomorphic indices. The latter may easily be embedded into a differential form, namely into

$$i\delta g_{i\bar{\jmath}}\, \mathrm{d}y^i \wedge \mathrm{d}\bar{y}^{\bar{\jmath}} = \delta J \tag{II.16}$$

with J the Kähler form. It turns out that one may similarly construct a differential form from δg_{ij} ; a way to accommodate the symmetry between its indices is to introduce the holomorphic 3-form Ω so as to define the $(2,1)$-form

$$\Omega_{ij}{}^{\bar{l}}\delta g_{\bar{k}\bar{l}}\, \mathrm{d}y^i \wedge \mathrm{d}y^j \wedge \mathrm{d}\bar{y}^{\bar{k}}\,. \tag{II.17}$$

One may then show that (II.15) is satisfied if and only if both forms (II.16) and (II.17) are harmonic [51]. The 10-dimensional metric may therefore be expanded as

$$\begin{aligned} g_{mn}(x,y) &= g^{(0)}_{mn}(x,y) + \delta g_{mn}(x,y), \\ \delta g_{i\bar{\jmath}}(x,y) &= i v^a(x)(\omega_a)_{i\bar{\jmath}}(y), \\ \delta g_{ij}(x,y) &= i\bar{z}^k \left(\frac{\Omega_i{}^{\bar{l}\bar{m}}(\overline{\chi}_k)_{\bar{l}\bar{m}j}}{|\Omega|^2} \right)(y) \end{aligned} \tag{II.18}$$

with x and y coordinates on the Minkowski space-time and on the internal Calabi-Yau manifold and with $\{\omega_a\}$ and $\{\chi_k\}$ bases of the $(1,1)$ and $(2,1)$ Dolbeault cohomology group of the background Calabi-Yau manifold respectively [7]. From (II.16), it is clear that the $h^{(1,1)}$ mixed deformations of the metric correspond to the deformations of the Kähler form. On the other hand, since the internal manifold is Calabi-Yau, it must always admit a Hermitean metric ; the $h^{(2,1)}$ purely (anti-)holomorphic deformations of the metric must therefore be related to deformations of the complex structure, that is of the splitting between holomorphic and anti-holomorphic indices.

The remaining fields coming from the NS-NS sector are the dilaton ϕ and the Kalb-Ramond B-field B. They may be expanded in terms of four-dimensional fields as

$$\begin{aligned}\phi(x,y) &= \phi(x),\\ B(x,y) &= B(x) + b^a(x)\omega_a(y).\end{aligned}$$

The fields coming from the RR sector are different in each type II theory ; in the type IIA case, they may be expanded as

$$\begin{aligned}C_1(x,y) &= c_1(x),\\ C_3(x,y) &= c_1^a(x)\omega_a(y) + \xi^K(x)\alpha_K(y) - \widetilde{\xi}_K(x)\beta^K(y)\end{aligned}$$

while in the type IIB model, the expansion reads [8]

$$\begin{aligned}C_0(x,y) &= c_0(x),\\ C_2(x,y) &= c_2(x) + c_0^a(x)\omega_a(y),\\ C_4(x,y) &= V_1^K(x)\alpha_K(y) + \rho_a(x)\widetilde{\omega}^a(y).\end{aligned} \tag{II.19}$$

From there, one may insert the above expansions into the ten-dimensional type II supergravity and integrate over the Calabi-Yau basis in order to get the four-dimensional action. While the computation is quite straightforward, it is somewhat cumbersome and not especially enlightening for the purposes of this thesis. Since what we really care about here is the repartition of the four-dimensional fields into $\mathcal{N} = 2$ multiplets, which may be understood from deriving the four-dimensional effective supergravity, we simply state the result in tables II.2 and II.3, referring *e.g.*to [52] for more details.

Gravity multiplet	$(g_{\mu\nu}, c_1)$
Vector multiplets	(c_1^a, b^a, v^a)
Hypermultiplets	$\left(z^k, \xi^k, \widetilde{\xi}_k\right)$
Tensor multiplet	$\left(B, \phi, \xi^0, \widetilde{\xi}_0\right)$

TABLE II.2 – Bosonic part of the $\mathcal{N} = 2$ multiplets of the type IIA string.

7. More details about the notations may be found in appendix A.1.
8. In (II.19), the self-duality of $\mathrm{d}C_4$ in ten dimensions has been used ; without this constraint, terms involving β^K and ω_a should have been present in the expansion as well .

Gravity multiplet	$(g_{\mu\nu}, V_1^0)$
Vector multiplets	$\left(V_1^k, z^k\right)$
Hypermultiplets	(v^a, b^a, c^a, ρ_a)
Tensor multiplet	(B, ϕ, c_2, c_0)

Table II.3 – Bosonic part of the $\mathcal{N} = 2$ multiplets of the type IIB string.

In both cases, the tensor multiplet, which contains the dilaton, may be dualised into a hypermultiplet. In summary, there are $h^{(1,1)}$ ($h^{(2,1)}$) vector multiplets and $h^{(2,1)}+1$ ($h^{(1,1)}+1$) hypermultiplets in the spectrum of the type IIA (IIB) string effective action. One may notice that the spectra of both type II theories are very similar to each other, up to an exchange of the Hodge numbers $h^{(1,1)}$ and $h^{(2,1)}$. This suggests some kind of symmetry between Calabi-Yau three-folds ; more precisely, if the type IIA string compactified on X is equivalent to the type IIB string compactified on X' with X and $\widetilde{X}$ Calabi-Yau three-folds, then X is defined to be the *mirror* of $\widetilde{X}$. Of course, from what we have seen above, the Hodge numbers of X and $\widetilde{X}$ are exchanged, meaning that

$$\begin{aligned} h^{(1,1)}(X) &= h^{(2,1)}(\widetilde{X}), \\ h^{(2,1)}(X) &= h^{(1,1)}(\widetilde{X}). \end{aligned}$$

We will see two explicit mirror constructions in section III.1, one specific to K3 surfaces and the other one allowing to obtain mirror pairs of Calabi-Yau manifolds of any dimension.

We have seen that deformations of the Kähler form were encoded in the fields v^a from (II.18) ; we deduce from tables II.2 and II.3 that the Kähler modulus lies respectively in a vector and hypermultiplet in the type IIA and type IIB theories. Moreover, the dilaton lies in a hypermultiplet in both theories ; from our discussion above, this means that the part $\mathcal{M}_{\text{V}}$ of the moduli space spanned by scalars living in vector multiplets may be evaluated directly by going to the $\phi \to \infty$ limit. Similarly, $\mathcal{M}_{\text{H}}$ ($\mathcal{M}_{\text{V}}$) may be evaluated exactly in α' by going to the large volume limit in the type IIA (IIB) theory. The conclusions about the dilaton could actually have been made directly from the world-sheet theory. Indeed, in this case the superconformal algebra has enhanced $\mathcal{N} = (2, 2)$ supersymmetry and one may show that this is enough to constrain the dilaton to live in a hypermultiplet [49]. In this case, space-time supersymmetry comes from both left- and right-movers on the world-sheet. Another possibility is the case where all supersymmetry comes from, say, the left-movers ; in this case, the world-sheet supersymmetry is enhanced to $\mathcal{N} = (4, 1)$. Since this kind of compactifications does not treat both world-sheet chiralities on an equal footing, it may not admit a geometric interpretation in the usual sense. Along the same lines as in the previous case, one then shows that the dilaton must lie in a vector multiplet [49]. This is in particular what happens in the family of models reviewed in section III.1 and which we will be primarily be interested about. Unlike the common case of Calabi-Yau reductions, this will grant us access to an exact form for the hypermultiplets moduli space in chapter V ; in addition, we will be able to use in the same chapter mathematical constraints to derive the full perturbative corrections to the vector multiplets moduli space.

II.3.2 Reduction of the heterotic string

As in section II.2.4, geometric compactifications of the heterotic string preserving $\mathcal{N} = 2$ supersymmetry in four dimensions require an internal space of the form

$$\mathcal{K}_6 \cong \left(K3 \times T^2\right)/G$$

for some discrete group G whose elements act on $K3$ as a symplectic automorphism thereof and on T^2 as a translation. A difference compared to the type II case is the non-abelian gauge group naturally present in the ten-dimensional theory. Compactification of the heterotic string is then achieved in general by specifying a vector bundle $E \to \mathcal{K}$ over the internal space with structure group contained either in $E_8 \times E_8$ or in $\text{Spin}(32)/\mathbb{Z}_2$, depending on the heterotic model of interest. In the toroidal case, this is exactly what happened : far from the enhanced symmetry points, the gauge group was broken to the Cartan subgroup $U(1)^{16}$ of the above Lie groups. The only data needed to characterise the bundle connection was then contained in the $(16 \times d)$-dimensional matrix formed by the Wilson lines. Conformal invariance was ensured to leading order in α' by requiring A to be constant, implying that the gauge bundle E was flat.

As we briefly mentioned in section II.1, requiring the variation of the gluinos to vanish under supersymmetry transformations amounts to imposing the gauge bundle to be holomorphic in a flux-free model. Moreover, anomaly cancellation requires [53]

$$c_2(E) = c_2(T_{\mathcal{K}_6})$$

with $T_{\mathcal{K}_6}$ the tangent bundle of $\mathcal{K}_6$. Therefore, in the current case where $\mathcal{K}_6 = \text{K3} \times T^2$, the gauge bundle may no longer be trivial. Since the main point of interest of this thesis is to study models obtained as toroidal orbifolds of the heterotic string, we will not spend more time discussing these models for which the analysis of the gauge bundle would be much harder. Before closing this section, we briefly mention two facts : first, all space-time supersymmetry comes from world-sheet fields of the same chirality by construction of the heterotic string. As we emphasised in section II.3.1, this implies that the dilaton lies in a vector multiplet whether the construction admits a geometric interpretation or not ; the Kähler modulus, as for it, may be shown to belong to a hypermultiplet [54]. This motivates in particular a research of heterotic/type II dual theories preserving $\mathcal{N} = 2$ supersymmetry, as we have seen before that the dilaton belongs to a hypermultiplet in the reduction of the type II string on a Calabi-Yau three-fold. Indeed, one could then theoretically have access to both parts of the whole moduli space (II.13) using perturbation theory only (this is summarised in table II.4). A second point concerns the potential dual theories. A common conjecture is that if a compactification of the heterotic string on $(\text{K3} \times T^2)/G$ for some discrete group G as above is dual to a reduction of the type IIA string on a Calabi-Yau three-fold X and if there exists a region in the moduli space where both perturbative theories converge, then X must be constructed as a K3-fibration over a $\mathbb{P}^1$ basis [34]. Viewing the $(\text{K3} \times T^2)/G$ internal space of the heterotic string as a T^4-fibration over $\mathbb{P}^1$, this statement may be seen as a fiber-wise application of the duality conjectured in section II.2.3.

Explicit examples of dualities between four-dimensional $\mathcal{N} = 2$ theories descending from the duality described in II.2.3 have been found in the literature. An especially relevant construction for this thesis is the FHSV model introduced in [55]. The type IIA model internal Calabi-Yau space is obtained as a freely acting $\mathbb{Z}_2$ orbifold of $\text{K3} \times T^2$ generated by

	$\mathcal{M}_{\text{V}}$	$\mathcal{M}_{\text{H}}$
Type IIA string on CY_3	Exact in g_S Expansion in α'	Expansion in g_S Exact in α'
Type IIB string on CY_3	Exact in g_S Exact in α'	Expansion in g_S Expansion in α'
Heterotic string on $(\text{K3} \times T^2)/G$	Expansion in g_S Exact in α'	Exact in g_S Expansion in α'

Table II.4 – Consequence of the nature of the dilaton $\mathcal{N} = 2$ multiplet on the control over the moduli space. For the type II string, ϕ lies in a hypermultiplet ; $\mathcal{M}_{\text{H}}$ therefore receives corrections from all orders in perturbation theory while $\mathcal{M}_{\text{V}}$ may be evaluated in the perturbative limit $\phi \to \infty$.

$g \in O(\Gamma_{6,22})$. The action of g on the K3 intersection lattice $\Gamma_{3,19} \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ is to exchange two copies of $(E_8(-1) \oplus U)$ and to act as $-\mathbb{1}$ on the last U factor while it acts as an inversion on T^2. The resulting manifold is known as an Enriques surface and is self-dual under mirror symmetry. This description of g in terms of its action on the charge lattice $\Gamma_{6,22} \cong \Gamma_{3,19} \oplus U \oplus \Gamma_{2,2}$ is almost enough to determine the action of g in the heterotic picture. The only freedom left lies in the action of g on the remaining U factor in the above $\Gamma_{6,22}$ decomposition ; from the type IIA perspective, only Ramond-Ramond states are charged under this U lattice which has then no effect at the perturbative level. From the heterotic point of view however, this U corresponds to winding and momenta charges and its behaviour under g must be understood. It was shown in [55] that g should act as a $\mathbb{Z}_2$ shift with non-vanishing norm in order to have a modular invariant theory. As we will see, there are interesting similarities as well as differences between the FHSV model and the models to be analysed in chapter III ; we will therefore return to these constructions in more details in subsection III.4.3.

II.4 Non-geometric constructions

II.4.1 Gauged supergravities

So far, we have mainly focused on geometric backgrounds in order to understand some of their features and to emphasise the differences with non-geometric constructions. Actually, even though they may seem more intuitive at first sight, there is no physical reason why string theory should admit a classical geometric interpretation. Indeed, let us go back to the reasons mentioned in the introduction in order to justify the high number of dimensions required by string theories. We have seen that a necessary and sufficient condition for the local conformal invariance on the world-sheet to hold at the quantum level is that the central charges of the two decoupled CFTs associated to the left and right sectors vanishes. The geometric interpretation associated with the dimension of space-time only comes in a second step when bosons of opposite chiralities are paired and interpreted as directions in the target space. However, more exotic CFTs such as Wess-Zumino-Witten models would lead to equivalently receivable theories. This kind of considerations may also be refined in order to take into account space-time supersymmetry : for instance, the internal SCFT of a heterotic string theory preserving $\mathcal{N}=2$ supersymmetry in four dimensions may be shown to split into two pieces with $c=6$ and $\mathcal{N}=4$ supersymmetry and $c=3$ and $\mathcal{N}=2$ supersymmetry [56]. Once again, no reference is made to any geometrical interpretation in general.

A very important framework in considering generic compactifications of string theories is supergravity. This is due to several reasons : first and maybe most importantly, there are a number of reasons to believe that supergravity models may describe accurately our world (at low energy with respect to the Planck mass) independently from any string theoretical considerations. Besides from the usual arguments in favour of supersymmetry [9], invariance under *local* supersymmetric transformations allows for higher mass degeneracy between the fields of the standard model and their expected superpartners and are as such more in line with experiment [17]. Moreover, supergravity theories typically exhibit rich global symmetry structures and, as we mentioned at the end of section I.1, the full U-duality group of the UV-complete theory is generally assumed to be a discrete subgroup of the low-energy global symmetry group. Supergravity may also describe non-perturbative solutions such as branes which are necessarily missed by a world-sheet approach. When considering non-geometric constructions, a low-energy point of view also circumvents some of the difficulties as the details of the string geometry may not be probed below the Planck mass.

Reductions of either ten-dimensional type II or the unique eleven-dimensional supergravities on Ricci-flat manifolds lead to theories with an abelian gauge group under which all fields are neutral, as stems from section I.1 for the toroidal case. Such supergravities are known as *ungauged*. Ungauged supergravities are quite simple to derive but not very satisfying from a phenomenological point of view as they generically preserve too much space-time supersymmetry and exhibit high-dimensional moduli spaces, as for the constructions described in the previous parts of this chapter. Either from a string theory or supergravity point of view, most relevant models are given by *gauged supergravities* which allow for realistic gauge groups and to generate a scalar potential and mass terms for the gravitini, thereby reducing the number of moduli of the theory and of preserved supercharges. Indeed, the low-energy limit of most flux and/or non-geometric compactifications of superstring theories come with non-abelian gauge groups under which the matter fields are charged, motivating our interest for such supergravities. A convenient way of deriving gauged supergravity is by considering first the ungauged theory obtained by toroidal reduction before promoting a subgroup of the corresponding global symmetry group to a local gauge invariance and coupling it to the Kaluza-Klein one-forms.

We now see in more details how such models are constructed. We consider the (half-)maximal ungauged supergravity case where scalar fields live on a coset space G/H. As in I.2, the scalar fields may conveniently be described by a vielbein $\mathcal{V}$ transforming under global G and local H transformations as $\mathcal{V}\mapsto g\mathcal{V}h^{-1}(x)$; once again, H is not a gauge group but is instead a convenient way of taking care of the redundancy of the degrees of freedom of $\mathcal{V}$. In addition to the scalar fields, there are in general n_v vector fields A_μ^M, $M=1,...,n_v$ granting the theory a $U(1)^{n_v}$ gauge group. As described above, we wish to promote a subgroup $G_0\subset G$ into a genuine gauge group. In order to fix the notations, we fix a basis of generators $\{t_\alpha,\alpha=1,...,\dim G\}$ of G with structure constants

$$[t_\alpha,t_\beta]=:f_{\alpha\beta}{}^\gamma t_\gamma.$$

9. We recall that incorporating supersymmetry in a gravity model necessarily leads to a supergravity theory [17].

Selecting a subgroup G_0 then amounts to choosing a set of generators $\{X_M, M = 1, ..., n_v\}$ in the Lie algebra $\mathfrak{g}$ of G. A convenient way of parametrising this in a manifestly G-covariant way is to introduce a constant $(n_v \times \dim G)$ matrix ${\Theta_M}^\alpha$ as

$$X_M =: {\Theta_M}^\alpha t_\alpha. \tag{II.20}$$

The dimension of the subgroup G_0 generated by the X_M's is then given by the rank of ${\Theta_M}^\alpha$; since it encodes the way G_0 is embedded in G, this matrix is known as the *embedding tensor*. As we are going to see, not any set of generators - and therefore not any embedding tensor - lead to consistent theories. In particular, a straightforward requirement is that the X_M's must form a Lie subalgebra of $\mathfrak{g}$, meaning that there must exist a set of constants ${X_{MN}}^P$ such that

$$[X_M, X_N] = {X_{MN}}^P X_P.$$

As we will see however, the numbers ${X_{MN}}^P$ may not be interpreted as the structure constants of a Lie group in general ; in particular, they are not anti-symmetric in M and N in general [57].

In view of the action of G on the scalars in the ungauged theory, the vielbein $\mathcal{V}$ should transform as

$$\delta\mathcal{V} = g\Lambda^M(x)X_M\mathcal{V} \tag{II.21}$$

under an infinitesimal G_0 transformation with local parameter $\Lambda^M(x)$ (in the rest of the discussion, we will drop the explicit space-time dependence of Λ). In (II.21), we have introduced a gauge coupling constant g. The most common procedure of promoting a global symmetry to a local invariance is to replace all derivatives by covariant derivatives involving the gauge vector field, namely to make the substitution

$$\partial_\mu \to D_\mu := \partial_\mu - gA_\mu^M X_M.$$

The covariant derivative of the vielbein $\mathcal{V}$ then transforms covariantly under the action of G_0 as expected provided that the vector fields transform as

$$\delta A_\mu^M = \partial_\mu\Lambda^M + g\Lambda^P A_\mu^N {X_{PN}}^M.$$

The embedding tensor is not invariant under the global symmetry group G ; however, one must demand its invariance under the gauge group G_0 for the theory to make sense. From the definition (II.20), one show that this translates into

$$\delta{\Theta_M}^\alpha = \Lambda^N{\Theta_N}^\beta\Big[{(t_\beta)_M}^P{\theta_P}^\alpha + {\Theta_M}^\gamma{f_{\beta\gamma}}^\alpha\Big] = 0$$

which is equivalent, up to t_α contraction, to

$${X_{MN}}^P = {\Theta_M}^\alpha{(t_\alpha)_N}^P.$$

This constraint is quadratic in Θ. There is an additional requirement the embedding tensor should satisfy as we now explain. From its definition, Θ must live in a $\mathbf{R}^*\otimes\mathbf{R}_{\text{adj}}$ representation of the global symmetry group G, $\mathbf{R}^*$ being the representation dual to those in which the scalars $\mathcal{V}$ transform and $\mathbf{R}_{\text{adj}}$ the adjoint representation. Such a representation may be expressed in terms of irreducible representations as

$$\mathbf{R}^* \otimes \mathbf{R}_{\text{adj}} = \bigoplus_i \mathbf{R}_i .$$

for some representations $\mathbf{R}_i$ of G. As a matter of fact, one may show that some of the $\mathbf{R}_i$'s are incompatible with the local supersymmetry algebra ; the additional constraint, which is linear in Θ, therefore has the effect of selecting only the allowed representations for Θ [58].

Any choice of Θ satisfying both linear and quadratic constraint describes a valid gauging and may be used to obtain a gauged supergravity. It is not obvious to see from this discussion only where mass terms for the scalar fields could come from. Actually, the bare replacement of all derivatives by covariant derivatives alone is not enough to promote G_0 to a gauge group and moreover destroys supersymmetry. The former is due to the differential p-forms in the action and is taken care of by modifying the associated field strengths and adding a topological term to the action ; in order to restore the latter, one must add fermionic mass terms and a scalar potential to the action (in addition to imposing the already mentioned linear constraint on Θ) [58].

It may be worthwhile noting that we have already encountered gauged supergravities when discussing Scherk-Schwarz reductions in I.2. The type IIA model to be reviewed in section III.1 may also be seen as such a twisted reduction ; this explains why the $\mathcal{N} = 4$ supersymmetry present in the original theory - namely the type IIA string compactified on K3 $\times T^2$ - is broken down to $\mathcal{N} = 2$ as the duality twist generates a mass term for half the gravitini (more details about the supergravity aspects of these models may be found in the original work of [7]).

II.4.2 T-folds and mirror-folds

In the following subsection, we give some examples of non-geometric constructions discussed in the literature. We begin by emphasising that non-geometric constructions may sometimes be related to geometric ones by duality. In order to illustrate this, we will very briefly consider the example of the CFT of two scalar fields taking value in a two-torus. In this case, we recall from section I.3.1 that the moduli space may be spanned by two complex scalar fields T and U interpreted as the Kähler and complex moduli of the T^2 respectively and that the automorphism subgroup of the corresponding Narain lattice factorises as

$$O(\Gamma_{2,2}) = (SL_2(\mathbb{Z})_T \times SL_2(\mathbb{Z})_U) \rtimes \mathbb{Z}_2 \tag{II.22}$$

where $SL_2(\mathbb{Z})_T$ and $SL_2(\mathbb{Z})_U$ act on T and U respectively. Geometric orbifolds may be obtained from a space group generated by $SL_2(\mathbb{Z})_U$ transformations which corresponds to the automorphism group of the torus. In contrast, identifying points related by a $SL_2(\mathbb{Z})_T$ transformation- that is points corresponding to tori with different sizes - lead to non-geometric constructions ; however, as T-duality exchange T and U, these models are related to the aforementioned geometric ones. One may therefore argue that such constructions should be considered as geometric in the sense that they do admit a geometric interpretation in a dual frame. By contrast, quotienting simultaneously by non-trivial elements of both $SL_2(\mathbb{Z})$ factors prevents from finding any T-dual theory in which the model may be understood as geometric and may therefore be regarded as “truly” non-geometric in the above sense. More details about these constructions from a Scherk-Schwarz reduction perspective may be found in [29]. As a side remark, the models studied in III and which constitute the core of this thesis fall in the last category as we will show in due time.

Twisted torus

We now discuss one of the simplest constructions from which one may obtain a non-geometric "background" [10], namely the case of a three-dimensional rectangular torus with H-flux turned on and in a constant dilaton background. Non-closed B-fields may be conveniently taken into account by using Stokes theorem to make the substitution

$$\frac{i}{2\pi\alpha'}\int_\Sigma B \longrightarrow \frac{i}{2\pi\alpha'}\int_\Xi H$$

into the NLSM action (II.8), Ξ being any three-dimensional manifold with boundary $\partial\Xi = \Sigma$. One may show that the action may only be independent of the specific choice for Ξ if the above integral is quantised as [59]

$$\frac{1}{2\pi\alpha'}\int_\Xi H \in 2\pi\mathbb{Z}.$$

We fix a one-form basis $\{\mathrm{d}y^i, i = 1,2,3\}$ normalised such that $y^i \cong y^i + 2\pi\sqrt{\alpha'}$ and in which the background fields read

$$\mathrm{d}s^2 = \frac{1}{\alpha'}\sum_{i=1}^{3} R_i^2\left(\mathrm{d}y^i\right)^2, \quad H = \frac{h}{2\pi\sqrt{\alpha'}}\mathrm{d}y^1\wedge\mathrm{d}y^2\wedge\mathrm{d}y^3$$

for some integer h and some positive real numbers R_i, $i = 1,2,3$, corresponding to the radii of the T^3 cycles. While this construction is geometric, one may construct a non-geometric background from it by looking at its image under T-duality transformations. This is analogous to the situation we described at the beginning of this section where non-geometric orbifolds with space group $G \in SL_2(\mathbb{Z})_T$ were related to geometric ones. T-duality may be understood for curved backgrounds from a path integral approach ; the background fields $\left(\hat{G},\hat{B},\hat{\phi}\right)$ of the corresponding dual theory may be computed using a set of equations known as the *Buscher rules* detailed in appendix B

The above background metric admits three Killing vectors $k_i := \partial_i$ for $i = 1,2,3$. We may therefore T-dualise our system in the, say, k_1 direction. Equations (B.23) show that the background fields of the dual theory may be expressed as

$$\begin{aligned}\mathrm{d}\hat{s}^2 &= \frac{\alpha'}{R_1^2}\xi^2 + \frac{R_2^2}{\alpha'}\left(\mathrm{d}y^2\right)^2 + \frac{R_3^2}{\alpha'}\left(\mathrm{d}y^3\right)^2,\\ \hat{H} &= 0,\\ \hat{\phi} &= \phi - \frac{1}{2}\log\left(\frac{R_1^2}{\alpha'}\right)\end{aligned} \tag{II.23}$$

where we have defined

$$\xi := \mathrm{d}\hat{y}^1 - \frac{h}{2\pi\sqrt{\alpha'}}y^3\,\mathrm{d}y^2\,,$$

with $\hat{y}^1$ the coordinate dual to y^1. In addition, $\hat{y}^1$ must be periodic with periodicity $2\pi\sqrt{\alpha'}$. We then see that from the dual theory perspective, there is no H-flux in contrast with the

10. It should be emphasised here that while this example is interesting in its own right, it may not be used as a string theory background as it spoils the world-sheet conformal invariance at the quantum level.

original theory ; the cost for this is that the metric now has non-trivial off-diagonal entries. Moreover, while $\hat{y}^1$ and y^2 are still $2\pi\sqrt{\alpha'}$-periodic, this is no longer the case for y^3 ; instead, equation (II.23) shows that its periodicity condition must be modified as

$$\left(\hat{y}^1, y^3\right) \sim \left(\hat{y}^1 + \frac{h}{\sqrt{\alpha'}} y^2, y^3 + 2\pi\right).$$

The resulting theory may therefore not be interpreted as a three-torus but instead as a two-torus - along the $\hat{y}^1$ and y^2 directions - non-trivially fibered over a circle spanned by y^3. This is known as a *twisted torus* in the literature. The torsion-free spin connection associated to the dual metric may be computed using Cartan's structure equation : defining the vielbein as

$$\hat{e}^1 := \frac{\sqrt{\alpha'}}{R_1}\xi, \quad \hat{e}^2 := \frac{R_2}{\sqrt{\alpha'}}\mathrm{d}y^2, \quad \hat{e}^3 := \frac{R_3}{\sqrt{\alpha'}}\mathrm{d}y^3,$$

one easily computes that the only independent and non-vanishing component of the spin connection $\hat{\omega}^a := \frac{1}{2} f_{bc}{}^a \hat{e}^b \hat{e}^c$ reads

$$f_{23}{}^1 = \frac{\alpha'}{2\pi} \frac{h}{R_1 R_2 R_3}.$$

The H-flux of the original theory may therefore be interpreted in the dual model as a contribution to the spin connection : for this reason, $f_{23}{}^1$ is known as a *geometric flux* is the literature.

T-fold

As we mentioned earlier, our original three-torus geometry admits three Killing vectors. We have seen that T-dualising along any one of them led to a twisted torus geometry ; we now analyse what happens when applying T-duality along two directions, say along ∂_1 and ∂_2. Applying once again the Buscher rules (B.23) to the geometry (II.23) shows that the background fields $\left(\tilde{G}, \tilde{H}, \tilde{\phi}\right)$ of the doubly dualised theory read

$$\begin{aligned} \mathrm{d}\tilde{s}^2 &= \frac{1}{\rho}\left(R_2^2\left(\mathrm{d}\tilde{y}^1\right)^2 + R_1^2\left(\mathrm{d}\tilde{y}^2\right)^2\right) + \frac{R_3^2}{\sqrt{\alpha'}}\left(\mathrm{d}y^3\right)^2, \\ \tilde{H} &= -\frac{1}{2\pi\sqrt{\alpha'}}\frac{h}{\rho^2}\left[\frac{R_1^2 R_2^2}{\alpha'^2} - \left(\frac{h}{2\pi\sqrt{\alpha'}}y^3\right)^2\right] \mathrm{d}\tilde{y}^1 \wedge \mathrm{d}\tilde{y}^2 \wedge \mathrm{d}y^3, \\ \tilde{\phi} &= \phi - \frac{1}{2}\log(\rho) \end{aligned} \tag{II.24}$$

where we have defined

$$\rho := \frac{R_1^2 R_2^2}{\alpha'^2} + \left(\frac{h}{2\pi\sqrt{\alpha'}}y^3\right)^2$$

and where $\tilde{y}^{1,2}$ is the coordinate dual to $y^{1,2}$. As before, both $\tilde{y}^1$ and $\tilde{y}^2$ are still $2\pi\sqrt{\alpha'}$-periodic and therefore span a two-torus while y^3 is no longer periodic by itself. One may wish to supplement the map $y^3 \mapsto y^3 + 2\pi\sqrt{\alpha'}$ with a diffeomorphism so as to preserve $\mathrm{d}\tilde{s}^2$ as we did in the twisted torus case ; however, no such diffeomorphism exist [11]. This implies that the background (II.24) *does not admit a geometric description*, meaning that we have obtained a non-geometric background as desired. A way to understand (II.24) in terms of geometric-like concepts has been proposed in [61] : since this background admits a local description and since we understand the global structure of its T-dual backgrounds analysed above, it may be described in terms of a two-torus fibration over a circle with transition functions incorporating T-duality transformations. Such objects are known as *T-folds*. As before, one may show that the original H-flux is now encoded in terms of another flux ${Q_3}^{12}$ which reads [60]

$$ {Q_3}^{12} = \frac{\alpha'}{2\pi}\frac{h}{R_1 R_2 R_3} $$

and which is a combination of geometric quantities and of the H-field ; consequently, this dual frame is an example of reduction with a non-geometric flux.

It is tempting to generalise the above construction in order to construct non-geometric backgrounds understandable as fibrations with transition functions involving various duality transformations. As first noticed in [29], the Scherk-Schwarz construction reviewed in I.2 provides a perfect framework to do so, granting direct access to the corresponding gauged supergravities where the U-duality group is contained in the global symmetry group. In particular, one might want to define *mirror-folds* as backgrounds constructed from twists involving mirror symmetry. Although there is nothing wrong in trying to do so, finding examples of such theories is made particularly hard by the fact that, unlike T-duality, mirror symmetry is not a self-duality *a priori*. Mirror-folds obtained from K3 fibrations have been considered in [62] but were shown to be unstable. The first explicit construction of such background was introduced in [7] and has played a central role in this thesis as we already mentioned.

The three fluxes considered above H_{123}, ${f_{23}}^1$ and ${Q_3}^{12}$ correspond to gauged supergravities and therefore to non-trivial solutions to the embedding tensor linear and quadratic constraints. One may then derive a connection between the aforementioned fluxes and the structure constants of the corresponding embedding tensor. A convenient way to do so is to think once again in terms of representations of the global symmetry group G of the ungauged supergravity. As we have seen, Θ must satisfy a linear constraint which selects allowed representations of the U-duality group G ; comparing this with the representation a given flux lives in then allows to identify the components of the embedding tensor it corresponds to. In particular, one may notice that not all available gaugings may be related to geometric fluxes ; consequently, one expects from the gauged supergravity point of view the possibility of turning on non-geometric fluxes.

II.4.3 Heterotic/type II theories and non-geometric backgrounds

We now generalise the above considerations to toroidal compactifications of the heterotic string which are richer and more consistent from a string theoretical point of view than the T-fold model presented in the previous section. With the type IIA/heterotic duality reviewed in II.2.3 in mind, we will focus to the case where the internal space is a four-torus as we would not need to explicitly mention the torus dimension in the following discussion anyway. As we have seen in section I.4, quotienting by a discrete group G may only be done consistently if

11. A simply way of seeing this is to compute the Ricci scalar $\tilde{\mathsf{R}}$ associated to $\mathrm{d}\tilde{s}$; indeed, since $\tilde{\mathsf{R}}$ is diffeomorphism invariant, it must be invariant under $y^3 \mapsto y^3 + 2\pi\sqrt{\alpha'}$. Direct computation shows that $\tilde{\mathsf{R}}$ is a degree two polynomial in y^3 and consequently that no such diffeomorphism may exist as claimed [60].

G is a subgroup of $O(\Gamma_{4,20})$, the automorphism group of the Narain lattice. The geometrical data of the original toroidal reduction is encoded into the metric and B-field on the T^4 as well as into either the $E_8 \times E_8$ or the Spin(32)/$\mathbb{Z}_2$ gauge bundle. Any geometric quotient must therefore correspond to an identification under symmetries of these objects. One may show that the corresponding subgroup of $O(\Gamma_{4,20})$ may be generated by changes of bases for the target space and for the gauge degrees of freedom as well as by B-field and Wilson line discrete shifts [63]. The remaining part of $O(\Gamma_{4,20})$ may only be generated by including a T-duality involution generator and therefore correspond to symmetries of the underlying CFT but not of the geometric setup itself in the usual sense. However, as we noticed in the T^2 reduction example from II.4.2, some of the models obtained by using such automorphisms as generators of the space groups may admit geometric descriptions in dual frames though. In general however, the resulting backgrounds would be understood as T-folds as we explained in the previous subsection.

It may be fruitful to also consider the originally dual picture of the type IIA string compactified on a K3 surface. We recall that even if this conjectured duality holds true, it is generically broken by an orbifolding procedure ; while motivated by duality considerations, the following discussion does not deal with models dual to the heterotic T-folds previously considered. Once again, the space group must be contained in the duality group $O(\Gamma_{4,20})$ which has however a very different meaning in the type IIA perspective. We recall from II.2.2.b) that only transformations in the subgroup $G_{\text{geom}} := O(\Gamma_{3,19}) \ltimes \Gamma_{3,19}$, with the first factor corresponding to rotations of $\Gamma_{3,19}$ and the second one to B-field shifts, admit a geometric interpretations. Indeed, any element of $O(\Gamma_{4,20})$ not contained in G_{geom} must involve a mirror involution in the sense developed in subsection III.1.1.a). Similarly as above, an orbifold with space group generated by such element would therefore correspond to a mirror-fold background.

After this introductory chapter reviewing some important features with regard to this thesis, we now turn to more recent outcomes. We will first spend a fair amount of time reviewing the mirror-fold construction developed in [7] ; although these models have been constructed prior to this thesis, understanding their heterotic dual features has been one of the main goals of this project. As we will see, these constructions also correspond to gauged supergravities which exhibit unusual characteristics ; for instance, the minimum of the potential correspond to $\mathcal{N} = 2$ STU supergravities, to be discussed in the next chapter, with no massless hypermultiplets. In a second phase, we will review some of the original results obtained in the last three years.

Chapitre III

Non-geometric constructions and duality

III.1 Review of the type IIA model

After this review of compactifications of string theories, we now present a family of non-geometric models we will be interested in in the following. This section is exclusively a review of the models introduced in [7] and does not contain any original result obtained by the author of the present thesis.

III.1.1 Two kinds of mirror symmetry

Mirror symmetry is a duality between two models which in particular exchanges supercharges on the underlying theories. It turns out that in the case of K3 surfaces, two such kinds of constructions have been found. The first one is the notion of *lattice-polarised* mirror symmetry which has first been introduced by Pinkham [64] and independently by Dolgachev and Nikulin [65, 66, 67]. The second one is closest from the mirror symmetry physicists are used to and is the Berglund-Hübsch construction [68] which relates mirror Landau-Ginzburg orbifolds. In the following, we give some more details about each of these constructions in order to use them in building new models.

III.1.1.a) Lattice-polarised mirror symmetry

We start by defining lattice-polarised (LP) mirror symmetry. In order to do so, we first need to define a few concepts.

Definition III.1.1 (Primitive lattice)**.** Let L be a lattice and $M \subset L$ a sublattice of L. M is said to be a *primitive* sublattice if the quotient L/M viewed as an abelian group is torsion-free. Similarly, a lattice embedding $\imath : M \hookrightarrow L$ is said to be primitive if $\imath(M)$ is a primitive sublattice of L.

Definition III.1.2 (Lattice-polarised K3 surface)**.** Let X be a K3 surface and L be an even lattice of signature $(1, \rho - 1)$ admitting a primitive embedding $\imath : L \hookrightarrow \Gamma_{3,19}$. If there exists a primitive embedding $\jmath : L \hookrightarrow S(X)$ with $S(X)$ the Picard lattice of X, X is said to be an L-polarised K3 surface.

The lattice polarised mirror symmetry construction may now be defined as follows.

Definition III.1.3 (Lattice-polarised mirror symmetry)**.** Let L be an even lattice of signature $(1, t-1)$ with $t \leq 18$ and with primitive embedding $\imath \hookrightarrow \Gamma_{3,19}$. If the orthogonal complement of $\imath(L)$ in $\Gamma_{3,19}$ admits a primitive embedding $\imath' : U \hookrightarrow \imath(L)^{\perp}$, with U the hyperbolic lattice bilinear form defined in theorem I.3.1, then the *mirror lattice* $L^{\vee}$ of L is defined by

$$\imath(L)^{\perp} = \imath'(U) \oplus L^{\vee}.$$

Two K3 surfaces X and $X^{\vee}$ are said to be lattice-polarised mirror of each other if X is L-polarised and $X^{\vee}$ is $L^{\vee}$-polarised.

LP mirror symmetry therefore mixes the Picard and transcendental lattices of K3 surfaces, defined in appendix A.2. In the original derivation of (II.9) in [40], the mirror symmetry element necessary to enlarge the geometric duality group $O(\Gamma_{3,19}) \ltimes \Gamma_{3,19}$ to the whole $O(\Gamma_{4,20})$ may actually be seen as a particular example of LP mirror symmetry. In this example, the K3 surface X is polarised by the whole Picard lattice $S(X)$; by construction, the transcendental lattice $T(X)$ is then identified to $U \oplus S(X^{\vee})$ with $X^{\vee}$ the LP mirror of X. Defining the quantum Picard lattice

$$S^{\mathfrak{Q}}(X) := S(X),$$

the LP mirror symmetry exchanges T and $S^{\mathfrak{Q}}$ in this specific case.

III.1.1.b) Berglund-Hübsch mirror symmetry

In this section, we define a construction first introduced in [68] and which may be applied to Calabi-Yau manifolds of any dimension which are realised as the minimal resolution of a hypersurface in a weighted projective space, which is a generalisation of a projective space we know define.

Definition III.1.4 (Weighted projective space)**.** Let V be a $(n+1)$-dimensional vector space over a field $\mathbb{K}$. The weighted projective space $\mathbb{P}^n V_{[w_0,...,w_n]}$ is defined as the quotient $V/\sim$ by the equivalence relation

$$(x_0, ..., x_n) \sim (\lambda^{w_0} x_0, ..., \lambda^{w_n} x_n)$$

for any vector $(x_0, ..., x_n) \in V$ and for any $\lambda \in K^*$. The coefficients w_i are called the *weights* of $\mathbb{P}^n V_{[w_0,...,w_n]}$.

In the following, we will only consider the $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^{n+1}$ case, so that we set $\mathbb{P}^n \mathbb{C}^{n+1}_{[w_0,...,w_n]} := \mathbb{P}^n_{[w_0,...,w_n]}$. The usual projective space may be seen as a trivial weighted projective space with every weights equal to 1. Moreover, one may show that for any choice of weights $(w_0, ..., w_n)$, there exists integers $(\tilde{w}_0, ..., \tilde{w}_n)$ such that the weighted projective space $\mathbb{P}^n_{[w_0,...,w_n]}$ is isomorphic to $\mathbb{P}^n_{[\tilde{w}_0,...,\tilde{w}_n]}$ and such that any n of the $n+1$ weights $\tilde{w}_0, ..., \tilde{w}_n$ are coprime [69]. In the following, we will then assume with no loss of generality that this is the case.

Similarly to the usual projective spaces, sections of the k-th power of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n_{[w_0,...,w_n]}}(k)$ are given by degree k polynomials ; however, whereas these polynomials had to be homogeneous in the projective space case, compatibility with the equivalence relation in definition III.1.4 requires the corresponding polynomials to be *quasi-homogeneous* of weight $[w_0, ..., w_n]$. We now give the definition of such polynomials.

Definition III.1.5 (Quasi-homogeneous polynomial)**.** Let $W_d : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial. W_d is said to be quasi-homogeneous of degree d if there exists integers $w_0, ..., w_n$ such that

$$W_d(\lambda^{w_0} x_0, ..., \lambda^{w_n} x_n) = \lambda^d W_d(x_0, ..., x_n)$$

for all $\lambda \in \mathbb{C}^*$. The integers $w_0, ..., w_n$ are called the weights of $x_0, ..., x_n$.

Now let us consider a hypersurface X_W defined as the zero-locus of a non-degenerate quasi-homogeneous polynomial $W_d \in \mathcal{O}_{\mathbb{P}^n_{[w_0,...,w_n]}}$ in a weighted projective space $\mathbb{P}^n_{[w_0,...,w_n]}$. The non-degeneracy condition means that one requires that W has an isolated singularity at the origin and that all fractional weights $w_0/d, ..., w_n/d$ may be uniquely determined from W_d. We may compute the Chern class of X_W using the adjunction formula, which implies in this case that[70]

$$c(T_{X_W}) = \frac{\prod_{i=1}^n (1 + w_i \xi)}{1 + d\xi}.$$

with $\xi := c_1(\mathcal{O}_{\mathbb{P}^1})$ the fundamental generator of $H^2(\mathbb{P}^1, \mathbb{Z})$. In particular, the first Chern class of X_W is given by

$$c_1(T_{X_W}) = \left(\sum_{i=1}^n w_i - d\right)\xi \tag{III.1}$$

so that X_W, or at least a minimal resolution thereof, is a Calabi-Yau manifold if and only if $d = \sum_{i=1}^n w_i$.

We consider only special polynomials W_d for which the number of monomials is the same as the number of variables ; such polynomials are said to be *invertible*. In this case, W_d may be written as

$$W_d(x_0, ..., x_n) = \sum_{i=0}^n \prod_{j=0}^n x_j^{a_i^{\ j}} \tag{III.2}$$

with $A_W := (a_i^{\ j})$ a matrix of positive coefficients. In this case, the non-degeneracy condition on W_d implies that A_W is invertible over $\mathbb{Q}$. We also define the abelian group G_W of diagonal symmetries as

$$G_W := \left\{(\mu_0, ..., \mu_n) \in (\mathbb{C}^*)^{n+1} \middle| W_d(\mu_0 x_0, ..., \mu_n x_n) = W_d(x_0, ..., x_n)\right\}.$$

and SL_W the subgroup of G_W consisting of elements of the form

$$SL_W := \left\{(\mu_0, ..., \mu_n) = \left(e^{2i\pi g_0}, ..., e^{2i\pi g_n}\right) \in G_W \middle| \sum_{i=0}^n g_i \in \mathbb{Z}\right\}.$$

From a physical point of view, SL_W corresponds to the subgroup of G_W corresponding to supersymmetry-preserving transformations. G_W always contain a subgroup J_W (referred to as the quantum symmetry group in the seminal paper of [68]) generated by the element $j_w := \left(e^{2i\pi w_0/d}, ..., e^{2i\pi w_n/d}\right)$ by definition of W_d.

We now turn to the $n = 3$ case which corresponds to Calabi-Yau manifolds of complex dimension 2, that is to K3 surfaces. Let us define the quotient group $\widetilde{G} := G/J_W$, with G a subgroup of G_W satisfying $J_W \subseteq G \subseteq SL_W$; since $\widetilde{G}$ is a group of supersymmetry-preserving transformations, the minimal resolution $X_{W,G}$ of the orbifold $X_W/\widetilde{G}$ is also a K3 surface. The Berglund-Hübsch (BH) mirror symmetry is then defined as follow.

Definition III.1.6. Let $X_{W,G}$ the minimal resolution of $X_W/\widetilde{G}$. We define the quasi-homogeneous polynomial W^T as the polynomial of the form (III.2) parametrised by the matrix $A_{W^T} := (A_W)^T$, with $(A_W)^T$ the usual matrix transpose of A_W. We also define the group G^T as

$$G^T := \left\{ g \in G_{W^T} \middle| g A_W h^T \in \mathbb{Z} \forall h \in G \right\}.$$

The minimal resolution X_{W^T,G^T} of the orbifold $X_{W^T}/\widetilde{G}^T$ is defined as the BH mirror of the surface $X_{W,G}$.

III.1.2 Construction of the type IIA model

We now turn to the main point of interest of this section which is building the models introduced in [7]. There is a continuous action of the group $O(4,20)$ on the moduli space (II.9) and hence on the type IIA string compactified on K3. The type IIA string compactified on K3 can be further compactified on T^2 with duality twists through an ansatz in which the dependence of all fields on the toroidal coordinates y^1, y^2 is given by a y^i-dependent $O(4,20)$ transformation :

$$g_1(y^1) = e^{N_1 y^1} \;\; , \quad g_2(y^2) = e^{N_2 y^2} \; .$$

for two commuting Lie algebra generators N_1, N_2. Then the monodromies are

$$\gamma_1 = g_1(0)^{-1} g_1(2\pi R_1) = e^{2\pi R_1 \, N_1}, \qquad \gamma_2 = g_2(0)^{-1} g_2(2\pi R_2) = e^{2\pi R_2 \, N_2}$$

This compactification has a low energy effective action given by a gauged $\mathcal{N}=4$ supergravity theory in four dimensions [71, 7]. The scalar potential will have a global minimum with zero energy, giving a Minkowski space vacuum, if the monodromies are elliptic, i.e. they are $O(4,20)$-conjugate to elements of the compact subgroup $O(4) \times O(20)$. Then each monodromy γ_i is an element of the discrete group $O(\Gamma_{4,20}) \subset O(4,20)$ such that there are $U_i \in O(4,20)$ and $M_i \in O(4) \times O(20)$ with

$$\gamma_i = U_i M_i U_i^{-1} \; ; \qquad U_i \in O(4,20) \; , \qquad M_i \in O(4) \times O(20) \; , \qquad i = 1, 2.$$

Each monodromy is of finite order, i.e. there are integers p_1, p_2 such that

$$\gamma_1^{p_1} = \gamma_2^{p_2} = 1 \, ,$$

and each will have a fixed locus in the Teichmuller space $O(4,20)/O(4) \times O(20)$. We will denote the coset containing $g \in O(4,20)$ as (g). As a rotation $O(4) \times O(20)$ has a fixed point at the origin (1), the fixed loci of γ_1, γ_2 will spanned by $(U_1), (U_2)$ with U_i satifying (III.1.2).

For there to be a global minimum of the potential, the intersection of the two fixed loci should be non-empty, and we will take $U_1 = U_2 \equiv U$ inside the intersection. Then there will be a minimum of the potential be at (U) and each monodromy γ_i is an $O(\Gamma_{4,20})$ transformation conjugate to a rotation $M_i \in O(4) \times O(20)$. Conjugating both monodromies by the same element V of $O(\Gamma_{4,20})$ then takes

$$\gamma_i \mapsto \gamma_i' = V \gamma_i V^{-1}$$

with a point of the fixed locus now at (VU). In this way, one can always arrange for an element of the fixed locus to be in any given fundamental domain of the Teichmuller space.

Regarding g as a 24×24 matrix acting in the fundamental representation of $O(4,20)$, the left coset $O(4,20)/O(4)\times O(20)$ can be parameterised by the 'generalised metric'

$$\mathcal{H}(g) = g^t g\,.$$

The group $O(4,20)$ acts on this by

$$\mathcal{H} \mapsto k^t \mathcal{H} k,\ \ k\in O(4,20)\,.$$

Then the stabiliser of a point $(g_0)\in O(4,20)/O(4)\times O(20)$ is the subgroup $H_0\subset O(4,20)$ preserving $\mathcal{H}_0 = \mathcal{H}(g_0)$ given by

$$H_0 = \{g\in O(4,20) : g^t\mathcal{H}_0 g = \mathcal{H}_0\}\,.$$

At the identity, $g_0 = \mathbb{1}$, $\mathcal{H}_0 = \mathbb{1}$ and H_0 is the standard $O(4)\times O(20)$ subgroup

$$H(\mathbb{1}) = \{g\in O(4,20) : g^t g = \mathbb{1}\}$$

while for general g_0, H_0 is a conjugate $O(4)\times O(20)$ subgroup

$$H_0 = g_0^{-1}H(\mathbb{1})g_0 = \{g\in O(4,20) : g = g_0^{-1}kg_0,\ \ k\in H(\mathbb{1})\}\,.$$

We will write this as

$$H_0 = O(4)_0\times O(20)_0$$

where $O(4)_0$ is conjugate to the standard $O(4)$ and $O(20)_0$ is conjugate to the usual $O(20)$:

$$\begin{aligned} O(4)_0 &= \{g\in O(4,20) : g = g_0^{-1}kg_0,\ \ k\in O(4)\subset H(\mathbb{1})\},\\ O(20)_0 &= \{g\in O(4,20) : g = g_0^{-1}kg_0,\ \ k\in O(20)\subset H(\mathbb{1})\}\,. \end{aligned}$$

As a result, any automorphism at (g_0) must be in the $O(4)\times O(20)$ subgroup H_0, and so the monodromies γ_1,γ_2 must be in

$$O(\Gamma_{4,20})\cap H_0$$

and we see that (III.1.2) is satisfied with

$$U^{-1} = g_0$$

for an $O(4)\times O(20)$ matrix $M\in H(\mathbb{1})$.

The models that we consider should furthermore preserve eight supercharges in four dimensions. Taking the $O(4)$ part of the rotation M to be in

$$SO(4)\sim \frac{SU(2)_L\times SU(2)_R}{\mathbb{Z}_2}\,,$$

the condition for the reduction to preserve 8 of the 16 supersymmetries and so to give $\mathcal{N}=2$ supersymmetry in four dimensions is that the rotation M is in $SU(2)_L\times O(20)$ or $SU(2)_R\times O(20)$.

Then the twisted reduction giving an $\mathcal{N}=2$ supersymmetric Minkowski vacuum in four dimensions consists of a duality twist with monodromy γ_1 of order p_1 on the y^1 circle and a twist of γ_2 of order p_2 on the y^2 circle with

$$\gamma_i = UM_iU^{-1}; \qquad U \in O(4,20), \qquad M_i \in SU(2) \times O(20)$$

At some fixed point in moduli space, the reduction becomes an orbifold by transformations (γ_1, t_1), (γ_2, t_2) where t_i is a shift on the i'th circle of order p_i

$$t_i : y^i \to y^i + 2\pi/p_i$$

and the twisted reduction reduces to a freely-acting asymmetric $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ orbifold of the $K3 \times T^2$ compactification.

An interesting class of models is that in which one of the monodromies is trivial, $\gamma_2 = 1$. Then we have a twisted reduction on one circle with monodromy γ_1 and a standard (untwisted) reduction on the other circle. This is sufficient to break the supersymmetry to $\mathcal{N}=2$ and gives a simple class of models that captures many of the features we want to study. We will focus on the implications of a single twist here ; the second twist would be treated similarly and doesn't qualitatively change the physics, but leads to a more general mass spectrum, as discussed in [7].

For a single twist γ conjugate to a rotation $M \in SO(4) \times SO(20)$ by (III.1.2), the $SO(4)$ rotation is characterised by two angles s_1, s_2 and the $SO(20)$ rotation is characterised by ten angles $r_1, \dots r_{10}$. For supersymmetry, the $SO(4)$ angles must satisfy $s_1 = \pm s_2$ [7]. For any admissible twist, γ satisfying our conditions, $V\gamma_i V^{-1}$ will also be an admissible twist for all $V \in O(\Gamma_{4,20})$. Changing from γ to $V\gamma_i V^{-1}$ will move a fixed point in Teichmuller space from (U) to (VU). As the volume of the K3 is one of the moduli, this change of representative can change the volume of the K3 ; all such theories related in this way are physically equivalent as they are related by dualities.

It is a rather non-trivial problem to find 24×24 integer-valued matrices representing elements of $O(\Gamma_{4,20})$ that are conjugate to $SU(2) \times O(20)$ rotations. In [7], an explicit construction was given. The starting point was finding a special algebraic K3 surface with a geometric automorphism σ of order p, and then constructing from this an automorphism $\hat{\sigma}$ of the K3 conformal field theory whose action $\hat{\sigma}^*$ on the lattice $\Gamma_{4,20}$ satisfied all the conditions above, and so taking $\gamma = \hat{\sigma}^*$ gives the construction of our non-geometric Calabi-Yau space. These are the mirrored automorphisms and their construction which we now review.

Let us consider *p-cyclic* K3 surfaces, that is surfaces X_p defined as the zero-locus of a non-degenerate invertible potential W_p of the form

$$W_p(x_0, x_1, x_2, x_3) = x_0^p + f(x_1, x_2, x_3)$$

in a weighted projective space, with f a quasi-homogeneous polynomial. Such surfaces admit in particular the automorphism $\sigma_p : x_0 \mapsto \zeta_p x_0$ with $\zeta_p := e^{2i\pi/p}$ a primitive p-root of unity. Such an automorphism does not belong to the supersymmetry-preserving transformation group G_W and acts therefore non-trivially on the holomorphic two-form $\omega(X_p)$.

As reviewed in appendix A.3, the automorphism σ_p is therefore non-symplectic. By construction, the BH mirror of X_p is constructed from the polynomial

$$W^T(\widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = \widetilde{x}_0^p + \widetilde{f}(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$$

and admits consequently another non-symplectic automorphism or order p defined by σ_p^T : $\widetilde{x}_0 \mapsto \zeta_p \widetilde{x}_0$. In the following, we define $S(\sigma_p)$ ($S(\sigma_p^T)$) the sub-lattice of the Picard lattice invariant under the action of σ_p (σ_p^T). We will need the following theorem which has been proven for $p = 2$ in [72], for $p \in 3, 5, 7, 13$ in [73] and for any other p satisfying $\phi(p)|12$ in [74, 75], with ϕ the Euler totient function.

Theorem III.1.1. *Let $X_{W,G}$ be a $S(\sigma_p)$ p-cyclic polarised K3 surface and X_{W^T,G^T} its Berglund-Hübsch mirror polarised by $S(\sigma_p^T)$. Then, $X_{W,G}$ and X_{W^T,G^T} also form a mirror pair in the sense of lattice-polarised mirror symmetry.*

Let us define $S^{\Omega}(\sigma_p) := S(\sigma_p) \oplus U$ similarly as we defined the quantum Picard lattice and $T(\sigma_p)$ the orthogonal complement of $S(\sigma_p)$ in $\Gamma_{3,19}$ or, equivalently, the orthogonal complement of $S^{\Omega}(\sigma_p)$ in $\Gamma_{4,20}$. We have seen in III.1.1.a) that LP mirror symmetry exchanges the role of S^{Ω} and T; therefore, the above theorem implies that $T(\sigma_p^T)$ is the orthogonal complement of $T(\sigma_p)$ in $\Gamma_{4,20}$, that is that

$$\Gamma_{4,20}^{\mathbb{R}} = T(\sigma_p)^{\mathbb{R}} \oplus T(\sigma_p^T)^{\mathbb{R}} \tag{III.4}$$

with, as before, $L^{\mathbb{R}} := L \otimes \mathbb{R}$ for any lattice L.

However, equation (III.4) does not imply that the lattice $\Gamma_{4,20}$ may be identified to the direct sum $T(\sigma_p) \oplus T(\sigma_p^T)$ as this is not the case in general. As was shown in [7], the diagonal action of σ_p and σ_p^T may still be extended to an action over the whole $\Gamma_{4,20}$ lattice. We may therefore define a *mirrored automorphism* $\hat{\sigma}_p$ as

$$\hat{\sigma}_p := \mu^{-1} \circ \sigma_p^T \circ \mu \circ \sigma_p$$

with μ the BP/LH mirror involution which maps the K3 surface to its mirror. The automorphism $\hat{\sigma}_p$ is an element of $O(\Gamma_{4,20})$ which is not contained in the geometric subgroup $O(\Gamma_{3,19}) \ltimes \Gamma_{3,19}$ and is therefore non-geometrical by construction.

We now give an explicit example of the above construction. Consider the hypersurface X defined by

$$w^2 + x^3 + y^8 + z^{24} = 0 \subset \mathbb{P}^3_{[12,8,3,1]}. \tag{III.5}$$

As we have seen in (III.1), the first Chern class of X vanishes implying that X is a K3 surface. X admits an order 3 non-symplectic automorphism acting as $\sigma_3 : x \mapsto e^{2i\pi/3}x$. The invariant sublattice of $\Gamma_{4,20}$ w.r.t. the action of σ_3 and its orthogonal complement are given in this case by

$$S^{\Omega}(\sigma_3) = E_6 \oplus U \oplus U \ , \quad T(\sigma_3) = E_8 \oplus A_2 \oplus U \oplus U \, .$$

where E_6 and A_2 are the negative-definite lattices associated with the corresponding Dynkin diagrams. The action of σ_3 on the vector space $T(\sigma_3) \otimes \mathbb{R}$ corresponds to an element of the orthogonal group $O(T(\sigma_3))$ that can be found explicitly in [7].

The mirror of the $K3$ surface (III.5), using the Greene-Plesser map [76], is given by an orbifold of a similar hypersurface

$$\tilde{w}^2 + \tilde{x}^3 + \tilde{y}^8 + \tilde{z}^{24} = 0 \ \subset \ \mathbb{P}_{[12,8,3,1]}$$

by a discrete symmetry group $G \cong \mathbb{Z}_2$ generated by

$$g : \begin{cases} \tilde{w} & \mapsto & -\tilde{w} \\ \tilde{y} & \mapsto & -\tilde{y} \end{cases}$$

This mirror surface also admits an order three automorphism σ_3^T, which acts in a similar same way to σ_3, with $\sigma_3^T \; : \; \tilde{x} \mapsto e^{2i\pi/3}\,\tilde{x}$. However, the invariant sublattice for σ_3^T and its orthogonal complement in $\Gamma_{4,20}$ are now

$$S^{\Omega}(\sigma_3^T) = E_8 \oplus A_2 \oplus U \oplus U \;, \quad T(\sigma_3^T) = E_6 \oplus U \oplus U \,.$$

Comparing with the corresponding sublattices (III.1.2) for the original surface (III.5), we see that the two sublattices have been interchanged. In particular, this means that $T(\sigma_3)$ and $T(\sigma_3^T)$ are orthogonal complements to each other in $\Gamma_{4,20}$ as claimed, allowing one to define an associated mirrored automorphism.

An important remark is the following : we started our construction by considering non-symplectic automorphisms of K3 surfaces. In a string theory context, compactification on an orbifold by such an automorphism breaks all supersymmetry. However, as was discussed in [77, 7], mirrored automorphisms preserve all space-time supercharges coming from the left-movers on the worldsheet, while projecting out all that come from the right-movers. This would not be possible with geometric automorphisms, emphasising the non-geometric nature of the construction presented here.

The fixed points of mirrored automorphisms, *i.e.* the K3 CFTs that are invariant under both the automorphism σ_p and the automorphism σ_p^T of a mirror pair, can be orbifolded by the automorphism. Of particular interest are certain models for which there is a duality frame in which the K3 surface has small volume in string units. These give Landau-Ginzburg (LG) orbifolds [78] which are special points in the moduli space of non-linear models on algebraic K3 surfaces at small volume ; when the polynomial defining the surface is of the Fermat type, as in (III.5), they can be described as Gepner models [79] and are explicitly solvable CFTs. In this framework, the cyclic group generated by the automorphism σ_p^T of the mirror surface is an order p subgroup of the 'quantum symmetry' of LG orbifolds and the diagonal action of (σ_p, σ_p^T) corresponds to an order p orbifold with a specific discrete torsion, see [77, 7] for details.

Here, as in [80, 7], we will focus on freely acting orbifolds of the type IIA superstring compactified on $K3 \times T^2$. We supplement the action of the mirrored automorphism $\hat{\sigma}_p$ on the $K3$ CFT with an order p translation along a one-cycle of the two-torus.

The orbifold CFT gives in space-time a four-dimensional theory with $\mathcal{N} = 2$ supersymmetry. Unlike compactifications on CY 3-folds, all space-time supersymmetry comes from the left-movers, signaling the non-geometric nature of the compactification as we mentioned. An important consequence is that, from the point of view of the low-energy 4d theory, the dilaton lies in a vector multiplet and not in a hypermultiplet. Furthermore, a large part, if not all, of the $K3$ moduli are lifted, see [80] for a detailed analysis of the massless spectra and for one-loop partition functions of the models.

To summarise, mirrored automorphisms are non-geometric symmetries of $K3$ CFTs in the Landau-Ginzburg regime, and are associated with isometries of the total cohomology lattice $\Gamma_{4,20}$ that have no invariant sublattices. Freely-acting orbifolds constructed from the action of a mirrored automorphism on a $K3$ Gepner model together with a translation along the two-torus give rise to interesting $\mathcal{N} = 2$ non-geometric compactifications of type IIA superstrings, providing explicit examples of the general construction outlined in this section. The heterotic duals of these models will be found in the next section.

We now turn to one of the main goals of this thesis which was to build and understand a heterotic dual to the models described in this section. Perturbative type IIA vacua preserving $\mathcal{N} = 2$ supersymmetry in four dimensions can be obtained from compactifications on Calabi-Yau three-folds (CY_3), or from orbifold or Gepner-point limits of these. In these cases the underlying worldsheet $(c, \bar{c}) = (9, 9)$ conformal field theory (CFT) has an extended $(2, 2)$ superconformal symmetry and both left- and right-moving R-charges are integer-valued. From the worldsheet perspective, four of the eight space-time supercharges come from the left-movers and the other four from the right-movers.

There exists, however, another possibility, in which *all eight* of the supercharges come from, say, the left-movers. This happens in a $(2, 2)$ superconformal field theory (SCFT) with integer-valued R-charges from the left-movers but no integer-valued R-charges from the right-movers; examples of this arise in free-fermion constructions or asymmetric toroidal orbifolds (see *e.g.* [81]). A large class of non-geometric models with all supercharges arising from left-movers based on Calabi-Yau compactifications in the Landau-Ginsburg regime were recently studied in [80, 77, 82], related to older works [78, 83]. These models have a volume modulus for the target space which is fixed by the construction, so that one cannot continuously take a large-volume limit, and are intrinsically non-geometric with the number of massless moduli typically being very small.

In [7], the new class of "compactifications" of type II strings to four dimensions we reviewed in III.1 was found, based on the work on Landau-Ginzburg models mentioned above, in which all eight supersymmetries come from the left-moving sector. As we saw, the starting point is type IIA string theory compactified on K3 with duality symmetry $O(\Gamma_{4,20})$, which is the group of isometries of the charge lattice $\Gamma_{4,20}$. This is then followed by a duality-twisted compactification on T^2 with an $O(\Gamma_{4,20})$ monodromy round each circle. The conditions on the monodromies for this to give a stable four-dimensional Minkowski vacuum preserving $\mathcal{N} = 2$ supersymmetry were found and an explicit algebraic geometric construction of such monodromies was given. These give non-geometric backgrounds of type II string theory giving four-dimensional Minkowski vacua preserving the same amount of supersymmetry as Calabi-Yau compactifications and, for this reason, such backgrounds were referred to as non-geometric Calabi-Yau spaces. As the monodromies involve mirror transformations, the non-geometric internal spaces are mirror-folds [61].

The two monodromies $\gamma_1, \gamma_2 \in O(\Gamma_{4,20})$ satisfying the conditions for $\mathcal{N} = 2$ Minkowski vacua are necessarily of finite order, p_1, p_2, with $\gamma_1^{p_1} = \gamma_2^{p_2} = 1$ for some integers p_1, p_2. Each duality γ satisfying the conditions above has a fixed locus (i.e. locus of fixed points) in the K3 moduli space [7], so that γ is an automorphism of the K3 CFT at any point in moduli space that is on the fixed locus. Moreover, the K3 surface X admitting the automorphism γ is an algebraic surface, with a mirror algebraic K3 surface $\tilde{X}$, such that the action of γ on X can be understood as the composition of four transformations : a diffeomorphism of X followed by the mirror map to $\tilde{X}$, then a diffeomorphism of $\tilde{X}$ followed by the inverse mirror map back to X. In the K3 CFT at a Landau-Ginzburg orbifold point, the diffeomorphism of $\tilde{X}$ appears as a discrete torsion. In [7], such automorphisms were referred to as *mirrorred automorphisms*; it is striking that they involve transformations of both the K3 surface and its mirror.

For the twisted reduction, γ_1, γ_2 must commute and, if there is to be a Minkowski vacuum, the intersection of their fixed loci must be non-empty. The orbifold is by transformations (γ_1, t_1), (γ_2, t_2) where the automorphism γ_i of degree p_i is combined with a shift t_i on the i'th circle of the T^2 by $2\pi/p_i$ $(i = 1, 2)$. Then at a fixed point the twisted reduction reduces to a freely-acting asymmetric orbifold of the $K3 \times T^2$ compactification, resulting in the simplest cases in the asymmetric Gepner models of [80].

Here we will focus on the heterotic duals of these constructions, using the duality between type IIA string theory compactified on K3 and heterotic string theory compactified on T^4 [84]. Starting from the $\mathcal{N} = 4$ duality in four dimensions between the type IIA string on $K3 \times T^2$ and the heterotic string on T^6, one can reach $\mathcal{N} = 2$ dual pairs through (freely-acting) orbifolds preserving half of the supersymmetry. An important example, the construction of Ferrara, Harvey, Strominger and Vafa (FHSV) [55], relates the type IIA string compactified on the Enriques Calabi-Yau three-fold to an asymmetric toroidal orbifold of the heterotic string. More generally, it is expected that type IIA compactified on a $K3$-fibred CY_3 with a compatible elliptic fibration is non-perturbatively dual to a heterotic string compactification on $K3 \times T^2$; see [34] for a review. Our models extend this to non-geometric dual constructions.

In the six-dimensional heterotic/type IIA duality, the $O(\Gamma_{4,20})$ duality symmetry group of the type IIA string compactified on $K3$ is identified as the $O(\Gamma_{4,20})$ T-duality symmetry group of the heterotic string, for which $\Gamma_{4,20}$ is the Narain lattice. Then the duality-twisted reduction on T^2 with monodromies $\gamma_1, \gamma_2 \in O(\Gamma_{4,20})$ has a heterotic realisation as a T-fold [61, 29, 85] with T-duality monodromies – it is a "compactification" of the heterotic string on a non-geometric space that has a fibration of T^4 CFTs over a T^2 base, with T-duality transition functions. Then the heterotic/type IIA duality maps the non-geometric Calabi-Yau mirror-fold reduction of type IIA to a T-fold reduction of the heterotic string. At a fixed point in moduli space (a point preserved by both γ_1, γ_2), the heterotic T-fold reduces to an asymmetric orbifold of the heterotic string on T^6 by the transformations (γ_1, t_1), (γ_2, t_2) consisting of $O(\Gamma_{4,20})$ T-duality transformations on T^4 combined with shifts on T^2. The K3 CFT at the fixed point in moduli space gives no enhanced gauge symmetry, so the corresponding T^4 heterotic compactification also has no enhanced gauge symmetry – instead it has enhanced discrete symmetry as in [86].

In a recent article [86], Harvey and Moore made the following point : *"It is not, a priori, obvious that heterotic/type II duality should apply to asymmetric orbifolds of the heterotic string"*. Indeed, while the FHSV model provides an example of such a dual to an asymmetric heterotic orbifold, no general statement appears to have been made so far. It is usually assumed that the type IIA side of such a duality should involve a Calabi-Yau three-fold. Here we show that in many cases an asymmetric orbifold of the heterotic string has a type IIA dual that is a non-geometric compactification, an orbifold of the type IIA string on $K3 \times T^2$ by non-geometric symmetries.

Remarkably, while the IIA construction is a consistent construction for the perturbative type IIA string, the naive heterotic dual is not perturbatively consistent – it is not modular invariant. The perturbative heterotic construction can be modified to obtain modular invariance, but via duality this then introduces non-perturbative modifications to the type IIA construction. This complies with the adiabatic argument put forward in [49] to relate non-perturbative dualities with different amounts of supersymmetry. Such non-perturbative modifications were also seen in the FHSV model. As we shall see, for the asymmetric orbifolds discussed here, modular invariance of the heterotic models is only obtained if the shifts on the two-torus are combined with shifts on the T-dual torus, corresponding to introducing phases dependent on the string winding numbers on the two-torus. Under heterotic/type II duality, the fundamental heterotic string is mapped to a IIA NS5-brane wrapping K3, so that new heterotic phases are mapped to phases dependent on NS5 wrapping numbers in the

IIA string. These NS5-brane contributions give non-perturbative modifications to the non-geometric Calabi-Yau construction. The corresponding non-perturbative corrections to the prepotential governing the vector moduli space geometry in the low-energy type IIA effective action will be analyzed in V.

We will mostly focus here on the case when the second twist γ_2 and shift t_2 are trivial. Having one non-trivial twist γ_1 is sufficient to break supersymmetry to $\mathcal{N} = 2$ and give an interesting class of models. This can be thought of as a duality-twisted reduction on a circle to five dimensions with monodromy γ_1 followed by a conventional (untwisted) compactification on a further circle. This is sufficient for most of our purposes ; the generalisation to two twists is straightforward and will be discussed briefly.

Once the heterotic dual has been found, non-perturbative aspects of the theory can be probed. We will study the perturbative heterotic BPS states that are dual to type IIA bound states of NS5-branes (wrapping a one-cycle of the two-torus and the $K3$ fibre) and momentum states on the T^2 by computing the generating function for the helicity supertraces. The map between BPS states is, in a way, easier to understand than in standard cases of $\mathcal{N} = 2$ heterotic/type II dualities as there are no D-branes bound-states to take into account in the present context.

The plan of this chapter is as follows. In section III.2 we find the heterotic dual of the non-geometric Calabi-Yau type IIA models. Section III.3 discusses some of the BPS states that arise and calculates the corresponding indices. In section III.4 we present the consequences of perturbative heterotic consistency in the type IIA duality frame. Section III.5 is devoted to a duality-covariant analysis of our models in four and five dimensions and of the FHSV model, allowing us to construct further dual forms of these models. Finally conclusions are presented in section III.6.

III.2 Heterotic Duals of Non-Geometric Type II Compactifications

The remarkable string theory duality between the type IIA superstring theory compactified on a K3 surface and heterotic string theory compactified on a four-torus [84, 41] is non-perturbative, in the sense that it maps the strongly-coupled regime of the heterotic string to the weakly-coupled limit of the type IIA string and *vice versa* (for a review, see [34]). From this fundamental duality one can infer numerous other connections between string theories with lower dimensionality.

The duality-twisted reduction on a further T^2 of the IIA string on K3 reviewed in sections 2 and 3 should then be dual to a duality-twisted reduction on a further T^2 of the heterotic string on T^4. At a fixed point, the orbifold of the IIA string on $K3 \times T^2$ should then be dual to an orbifold of the heterotic string on T^6. On the IIA side, the orbifold is by the symmetry (γ, t) where γ is a mirrored automorphism of K3 and t is a shift on T^2. On the heterotic side, $O(\Gamma_{4,20})$ is the heterotic T-duality group, suggesting that the heterotic dual could be the asymmetric orbifold of the heterotic string on T^6 by (γ, t), where γ is a heterotic T-duality and t is the same shift on T^2 as before. However, duality and orbifolding do not necessarily commute in general, so this conjectured dualisation needs further examination.

The general idea behind heterotic/type II duality in four dimensions is to apply fibre-wise the duality between a K3 fibration over some base B on the type IIA side and a T^4 fibration over B on the heterotic side [87]. Our construction has a base $B = T^2$ and does not constrain the size of the T^2 part of the (type IIA) internal space, so that one could go to the decompactification limit of the T^2 base ; moreover, the action of the automorphism (γ, t) is free, so that the quotient does not develop singularities. Under these two conditions the adiabatic argument of [49] holds and, at least in the limit of a large T^2 base which allows to perform the dualtiy 'locally' on the fibre, the heterotic dual should be the asymmetric orbifold of the heterotic string on T^6 by (γ, t).

We shall show that this correspondence must be refined for small T^2, with heterotic string winding mode contributions modifying the orbifold (this type of contribution to heterotic/type II dual pairs was anticipated already in [49]). Specifically, the automorphism (γ, t) must be supplemented by an order p shift in the T-dual circle conjugate to winding number, so that the full orbifold is by (γ, t) where now t is a shift on both the original T^2 and the T-dual T^2. This modification of the heterotic orbifold in turn implies a non-perturbative modification of the type IIA orbifold.

Our construction has some similarities with the model of Ferrara, Harvey, Strominger and Vafa (FHSV) [55] which relates type IIA compactified on the Enriques Calabi-Yau 3-fold, which is a freely-acting orbifold of $K3 \times T^2$, to heterotic strings compactified on a freely-acting, asymmetric orbifold of T^6. In the FHSV construction, the automorphism acts freely on the $K3$ surface and has fixed points on the base. In our case it is the opposite : the automorphism acts freely on the two-torus and has fixed points on the $K3$ surface. The comparison between these two classes of models will be further developed in sections III.4 and III.5.

III.2.1 Type IIA - Heterotic Duality in Six Dimensions

The type IIA string compactified on a $K3$ surface gives $(1, 1)$ supergravity in six dimensions at low energies. The moduli space is given by

$$\mathscr{M}_{6\text{D}} \cong O(\Gamma_{4,20})\backslash O(4, 20)/(O(4) \times O(20)) \times \mathbb{R}$$

where the first factor is the moduli space (II.9) of non-linear sigma-models on $K3$ and the second is the dilaton zero-mode. BPS states arise from branes wrapping cycles of the K3, and include a BPS solitonic string obtained by wrapping an NS5-brane around the K3 surface.

In the duality between the Type IIA string compactified on K3 and heterotic strings compactified on T^4, the six-dimensional dilatons and metrics are related by $\phi_{\text{HET}} = -\phi_{\text{IIA}}$ and $g_{\text{HET}} = \exp(2\phi_{\text{HET}})\, g_{\text{IIA}}$. The heterotic moduli space is again (III.2.1), but now with the first factor being interpreted as the moduli space of Narain lattices of signature $(4, 20)$ [88, 89]. Geometrically, it describes the moduli space of flat metrics and constant B-fields on T^4 and of $U(1)^{16}$ Wilson lines on T^4. While the $O(4)$ factor rotates the left-moving bosons of the free CFT with T^4 target space, the $O(20)$ factor mixes together the right-moving bosons of the T^4 CFT with the 16 bosons describing the gauge sector. The duality group $O(\Gamma_{4,20})$ is the heterotic T-duality group of the heterotic string on T^4. The second term in (III.2.1) is now the heterotic dilaton zero-mode.

The heterotic theory has a BPS solitonic string arising from an NS5-brane wrapping the T^4 ; under the duality, it is mapped to the type IIA fundamental string, while the type IIA NS5-brane wrapped on K3 maps to the heterotic fundamental string [44]. Such a correspondance will be useful in the analysis of BPS states in section III.3.

III.2.2 Construction of the Heterotic Dual

The starting point of our construction is a point in the moduli space (II.9) that is invariant under a $\mathbb{Z}_p$ automorphism generated by an element $\gamma \in O(\Gamma_{4,20})$. Viewing this as a type IIA construction, this is the type IIA string compactified on $K3 \times T^2$, where the K3 is chosen to be at a Gepner point in the K3 moduli space so that the corresponding CFT is given by a Gepner model described in [7] (*e.g.* the K3 at the Gepner point is (III.5) in the example given in section III.1.2). The automorphism γ acts on the K3 as a mirrored automorphism $\hat{\sigma}$.

In the dual heterotic interpretation, the moduli space (II.9) is viewed as the moduli space of Narain lattices of signature $(4,20)$, acted on by the heterotic T-duality group $O(\Gamma_{4,20})$. The special point in moduli space corresponds to a Narain lattice with enhanced discrete symmetry but without enhanced non-Abelian gauge symmetry. [1]

Then γ is an element of the discrete group $O(\Gamma_{4,20})$ that is $O(4,20)$-conjugate to an element M of the compact subgroup $O(4) \times (20)$, i.e. there is a $U \in O(4,20)$ and $M \in O(4) \times (20)$ so that (III.1.2) holds. The transformation M in $O(4) \times (20)$ is specified by two angles characterising a rotation in $O(4)$ and 10 angles characterising a rotation in $O(20)$. As it satisfies $M^p = 1$, the angles in $O(4)$ are $2\pi s_1/p, 2\pi s_2/p$ for integers s_1, s_2 and the angles in $O(20)$ are $2\pi r_1/p, \dots 2\pi r_{10}/p$ for integers $r_1, \dots, r_{10}$. An important result from [7] is that for a mirrored automorphism none of the angles is zero, so that no directions are left invariant by the rotation.

Recall from section III.1.2 that at a point in the moduli space (II.9) given by the coset representative (g_0), the stabilizer is

$$H_0 = O(4)_0 \times O(20)_0 \,,$$

where $O(4)_0$ and $O(20)_0$ are the subgroups of $O(4,20)$ defined in eqns. (III.3,III.3). An important point is that, in the heterotic string realisation, at the point (g_0) in the moduli space $O(4)_0$ acts only on the left-movers of the heterotic string and $O(20)_{20}$ acts only on the right-movers. In particular, the vector Π at (g_0) encoding the heterotic momenta and winding numbers and taking values in the lattice $\Gamma_{4,20}$ decomposes into a 4-component momentum Π_L with contributions only from left-moving degrees of freedom and transforming under $O(4)_0$ but not $O(20)_0$ together with a 20-component momentum Π_R with contributions only from right-moving degrees of freedom and transforming under $O(20)_0$ but not $O(4)_0$. Taking (g_0) to be the special point in moduli space, the twist γ is in $O(4)_0 \times O(20)_0$.

If γ corresponds in the dual type IIA picture to a mirrored automorphism $\hat{\sigma}_p$, a lemma from [7] shows that the matrix M representing it can be diagonalised over the complex numbers to give

$$M = \begin{pmatrix} \zeta_p \mathbb{I}_q & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \zeta_p^k \mathbb{I}_q & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \zeta_p^{p-1} \mathbb{I}_q \end{pmatrix} , \qquad \text{(III.6)}$$

where ζ_p is a primitive p'th root of unity and k takes all values from 1 to $p-1$ satisfying $\gcd(k,p) = 1$; put differently, γ has an eigenspace of dimension q for each primitive p'th

1. For instance, the heterotic lattice associated with the dual type IIA Gepner model for the K3 surface (III.5) has a discrete symmetry $[(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8 \times \mathbb{Z}_{24})/\mathbb{Z}_{24}] \times \mathbb{Z}_{24}$.

root of unity. The dimension q is therefore equal to $24/\varphi(p)$, where $\varphi(p)$ is the Euler totient function (that is the number of integers k with $k \leq p$ satisfying $\gcd(k,p) = 1$). For prime orders $p \in \{2,3,5,7,13\}$, the eigenspaces of γ are then all $24/(p-1)$-dimensional.

The type IIA construction is an orbifold of the IIA string on $T^2 \times K3$ by the $\mathbb{Z}_p$ symmetry generated by (γ, t) where γ is a mirrored automorphism and t is a shift of order p on one of the circles. On the heterotic side, we have a $\mathbb{Z}_p$ orbifold by the twist γ acting as a T-duality automorphism of the Narain lattice together with a shift t. In the large volume limit of the T^2 base, the shift should agree with that on the type II side, but as discussed above, we will need to consider more general shifts here. As explained *e.g.* in [63], any component of a shift vector along directions in which the twist acts non-trivially may be absorbed by a redefinition of the origin of the coordinates, so that without loss of generality one may consider a shift only along the directions in which γ acts trivially. In other words, decomposing the full Narain lattice of winding and momenta as

$$\Gamma_{6,22} \cong \Gamma_{4,20} \oplus \Gamma_{2,2},$$

we quotient by an order p twist γ in $O(\Gamma_{4,20})$, so acting non-trivially on the $\Gamma_{4,20}$ lattice only, together with a shift t defined by a lattice vector δ such that $p\,\delta \in \Gamma_{2,2}$ but $\delta \notin \Gamma_{2,2}$.

It is easy to check that $\mathcal{N} = 2$ supersymmetry is preserved by the heterotic orbifold in this picture. Indeed, the action of γ on the world-sheet fermions is deduced from its action on the left-moving bosons, as usual, by requiring world-sheet supersymmetry to be preserved. The group $SO(4)_0$ acts on the left-handed Ramond ground states as a spinor, transforming as a $(2,1)+(1,2)$ under $Spin(4) \sim SU(2) \times SU(2)$. If $s_1 = \pm s_2$, the twist is in just one of the two $SU(2)$ subgroups and so half the spinor degrees of freedom remain massless.

III.2.3 Geometric and Non-geometric Twists

The twist γ is in the intersection of the $O(\Gamma_{4,20})$ and $O(4)_0 \times O(20)_0$ subgroups of $O(4,20)$. If the $O(20)_0$ part of the twist is in fact in an $O(4)_0 \times O(16)_0$ subgroup, then the twist can be regarded as an $O(4)_0 \times O(4)_0 \subset O(4,4)$ transformation acting on the T^4 CFT and an $O(16)_0$ transformation acting on the gauge degrees of freedom. If moreover it is in the diagonal subgroup $O(4)_{diag} \subset O(4)_0 \times O(4)_0$, then it is a geometric transformation (rotation) on the T^4 and the orbifold is a conventional (not asymmetric) orbifold of T^4 combined with an (unconventional) orbifold action on the gauge sector. This of course requires choosing an $O(4)$ subgroup of $O(20)$, and acting with $O(\Gamma_{4,20})$ can change such a 'geometric' orbifold to a non-geometric one. However, the twist in $O(16)_0$ may not be related to the orbifold limit of an ordinary vector bundle. This will be discussed further in section III.5.

Bearing this in mind, we now address the question of whether a given theory is dual to a geometric orbifold via an $O(\Gamma_{4,20})$ transformation. The answer turns out to depend strongly on the order of the orbifold. For simplicity we discuss only the cases with p prime below.

In the $p = 2$ case, $\Gamma_{4,20}$ is quotiented by the involution which flips all directions of the lattice ; therefore, as the twist γ may be represented here by $-\mathbb{1}$, it does not mix space-time and gauge degrees of freedom, so that its restriction to the four-torus admits a standard geometric interpretation (as the same involution that gives a $T^4/\mathbb{Z}_2$ orbifold). Moreover, γ therefore remains the same under $O(\Gamma_{4,20})$ conjugation so that the resulting theory always has a geometric interpretation. [2]

2. According to [49], this action on the $\Gamma_{4,20}$ lattice is the same as the action of $(-1)^{F_L}$ on type IIA compactified on K3.

In the $p = 3$ case, looking for a representative of the conjugacy class of a twist which belongs to $O(4)_{diag} \times O(16)_0$ is not straightforward in general. However, it is possible to show that the explicit example of an order 3 twist given in [7] may be understood as having a geometric action (this may be seen using for instance the parametrisation of $O(\Gamma_{4,20})$ of [63]). Therefore, there exist models in the $p = 3$ case which are equivalent to geometric theories from the heterotic point of view.

The $p = 5$ case is more tricky, as no explicit matrix representation of $\hat{\sigma}_5$ is known by the authors. It is known that there are no supersymmetric T^4 orbifolds of order five, see *e.g.* [90]. A simple argument given in [91] rules out the possibility of a left-right symmetric action of the orbifold on the T^4. Let us assume first that there is an order 5 twist γ with a geometric action, that is such that $\gamma \in O(4)_{diag} \times O(16)_0$; then, looking at the action on T^4, $\mathcal{N} = 2$ supersymmetry imposes the trace of its matrix representation to be equal to $8\cos\left(\pm\frac{2\pi}{5}\right) = 2(\sqrt{5} - 1)$ through the $|s_1| = |s_2|$ condition derived in section III.1.2. This is incompatible with the requirement that the twist belongs to the duality group of the lattice, as this forces in particular the trace of its matrix representation to be integer-valued. Therefore, although there exist rank-4 Euclidean lattices admitting an order five symmetry, it is not possible to find a twist whose action would admit a geometric interpretation in the $p = 5$ case.

In the $p = 7$ and $p = 13$ cases, the orbifold must be asymmetric by construction. Indeed, such a construction could only be obtained if the twist were acting as an automorphism in $O(4)_{diag}$ (together with an action on the gauge degrees of freedom). A result from lattice theory states that there exist euclidean lattices Λ admitting an order N symmetry if and only if rank $\Lambda \geq \varphi(N)$, φ being Euler's totient function as before [23]. It is then immediate that no rank-4 lattice may admit an order-p symmetry when $p = 7$ or 13. The asymmetry of the construction between left- and right-movers on the T^4 is even more striking in the $p = 13$ case in which there are exactly two angles of absolute value $2\pi k/13$ for each k between 1 and 6 ; therefore, the $\mathcal{N} = 2$ supersymmetry condition $s_1 = \pm s_2$ ensures us that there may be no angle r_I equal to any of the s_i's, making the asymmetric nature of the model obvious in this case. Once again, we can therefore conclude that this construction does not admit a standard geometric interpretation in the heterotic framework either.

III.2.4 Modular Invariance and Restrictions on the Shift Vector

We now turn to the choice of shift vector in the heterotic orbifold. The twist γ is to be accompanied by a translation by a shift vector δ with $p\delta \in \Gamma_{2,2}$. The choice of group action $\mathbb{Z}_p \subset U(1)^2 \cong T^2$ on the type IIA side of the duality fixes the momentum associated with this shift vector , *i.e.* the generator of translations along the corresponding one-cycle, but not its action on states with winding number.

It has long been known that in order to preserve modular invariance in orbifold models, there must be a relation between twists and shift vectors [25, 26]. For later convenience, we define the charge vector $\Delta \in \Gamma_{2,2}\backslash(p\,\Gamma_{2,2})$ so that the shift vector δ satisfies $\Delta = p\,\delta$. As discussed in section III.1.2, preserving $\mathcal{N} = 2$ space-time supersymmetry imposes $s_1 = \pm s_2$. Using this, it is possible to show that the necessary and sufficient condition for modular invariance of our theory is given by [3] [25, 26]

$$\Delta^2 + \sum_{I=1}^{10} r_I^2 = 0 \qquad \text{mod } fp \qquad \text{where } f = \begin{cases} 1 \text{ if } p \text{ is odd} \\ 2 \text{ if } p \text{ is even} \end{cases} \tag{III.7a}$$

Furthermore, the spectrum of γ is completely fixed; indeed, equation (III.6) shows that, of the 12 angles, there are exactly $\frac{12}{\varphi(p)}$ angles equal to $\frac{k}{p}$ mod 1 for each value of k between 1 and $p-1$ such that $\gcd(k,p) = 1$. Then, as shown in appendix C, the quantity $\sum_{I=1}^{10} r_I^2 + \sum_{i=1}^{2} s_i^2$ may be explicitly computed for any p, so that equation (III.7a) may be simplified to become

$$\Delta^2 = 2\Psi_p \qquad \text{mod } fp \tag{III.8}$$

where

$$\Psi_p := s^2 - \prod_{\substack{q|p \\ q\,\text{prime}}} (-q)$$

with the product running over the distinct prime divisors q of p. We parametrise the shift vector as $\delta = (\alpha^i, \beta_i)$ so that $\Delta = p(\alpha^i, \beta_i)$ is a lattice vector and the shift is generated by $\alpha^i k_i + \beta_i w^i$ where k_i and w^i are respectively the momentum and winding charges; the constraint (III.8) then becomes the condition

$$p^2 \alpha^i \beta_i = \Psi_p \quad \text{mod } p\,, \tag{III.9}$$

which prevents β from vanishing, as Ψ_p may never be vanishing modulo p (since any q in the above product is a divisor of p and as such is not invertible over $\mathbb{Z}_p$, unlike $s^2 \in \mathbb{Z}_p^\times$). This translates into a non-perturbative modification of the orbifold from the type IIA perspective that will be discussed in section III.4. As a side remark, we may note that whenever p is square-free, the condition (III.8) simplifies to

$$\Delta^2 = 2s^2 \text{ mod } fp\,; \tag{III.10}$$

in particular, equation (III.10) holds for p prime. One can further simplify this condition by choosing $s = 1$, *i.e.* that the rotation in $O(4)_0$ corresponds to the angles $2\pi/p$ and $\pm 2\pi/p$ (any other choice is related to this one by relabelling the orbifold sectors), so that

$$\Delta^2 = 2 \text{ mod } fp\,.$$

Let us now derive the partition function of the theory in order to check explicitly the relations (III.7). As usual with conformal field theories defined on orbifolds, the partition function of the model may be expressed as a sum

$$Z(\tau,\bar\tau) = \frac{1}{p} \sum_{k,l=0}^{p-1} Z\begin{bmatrix} k \\ l \end{bmatrix}(\tau,\bar\tau)$$

3. To be precise, there are two additional constraints in the case of even p, namely that $s_1 + s_2 = 0$ mod 2 and $(v, \gamma^{p/2} v) = 0$ mod 2 for all $v \in \Gamma_{6,22}$; however, the first condition must be fulfilled as a consequence of the $\mathcal{N} = 2$ supersymmetry preserved by the orbifold. Moreover, as $\gamma^{p/2}$ acts as minus the identity operator on $\Gamma_{4,20}$ and trivially on $\Gamma_{2,2}$ for any even p, the second condition is taken care of by the fact that $\Gamma_{6,22}$ is an even lattice.

over all allowed boundary conditions (that is, twisted or untwisted), with the contribution from the (k,l) sector defined as

$$Z\begin{bmatrix}k\\l\end{bmatrix}(\tau,\bar\tau) := \mathrm{Tr}_{\mathcal{H}_k}\left(g^l q^{L_0-\frac{c}{24}}\bar q^{\bar L_0-\frac{\bar c}{24}}\right),$$

where g is the element of the point group whose action combines a twist by γ and a shift by t and where $\mathrm{Tr}_{\mathcal{H}_k}$ stands for the trace over states in the sector twisted by g^k. The various blocks of the partition function are then computed in the usual way to give [4]

$$Z\begin{bmatrix}0\\0\end{bmatrix} = \frac{1}{2\tau_2}\frac{\Theta_{\Gamma_{4,20}}(\tau,\bar\tau)\times\Theta_{\Gamma_{2,2}}(\tau,\bar\tau)}{\eta^{12}(\tau)\bar\eta^{24}(\bar\tau)}\sum_{\alpha,\beta=0}^{1}(-1)^{\alpha+\beta+\alpha\beta}\vartheta^4\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\tau|0)$$

$$Z\begin{bmatrix}k\\l\end{bmatrix} = \frac{\kappa\left[{k\atop l}\right]}{2\tau_2}\exp\left\{\frac{i\pi kl}{p^2}\Big(2\Psi_p-\Delta^2\Big)\right\}\frac{\Gamma\left[{k\atop l}\right](\tau)}{|\eta(\tau)|^{12}\,\bar F\left[{k\atop l}\right](\bar\tau)}$$
$$\times\sum_{\alpha,\beta=0}^{1}(-1)^{\alpha+\beta+\alpha\beta}\vartheta^2\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\tau|0)\prod_{i=1}^{2}\frac{\vartheta\left[{\alpha+2ks_i/p\atop \beta+2ls_i/p}\right](\tau|0)\times\bar\vartheta\left[{1+2ks_i/p\atop 1+2ls_i/p}\right](\tau|0)}{\vartheta\left[{1+2ks_i/p\atop 1+2ls_i/p}\right](\tau|0)}\,, \quad \text{(III.11a)}$$

where the second equation is only valid for $(k,l)\neq(0,0)$. In the above equations, we have defined Θ_Λ as the sum over charges lying in the lattice Λ, the sum over the lattice $\Gamma\left[{k\atop l}\right]$ as

$$\Gamma\begin{bmatrix}k\\l\end{bmatrix}(\tau) := \sum_{Q\in\Gamma_{2,2}+k\delta} q^{Q_L^2/2}\bar q^{Q_R^2/2}e^{2i\pi\langle Q,\delta\rangle},$$

the function $F\left[{k\atop l}\right]$ as

$$F\begin{bmatrix}k\\l\end{bmatrix}(\tau) := \frac{1}{\eta^{12}(\tau)}\prod_{I=1}^{10}\vartheta\begin{bmatrix}1+2kr_I/p\\1+2lr_I/p\end{bmatrix}(\tau|0)\prod_{i=1}^{2}\vartheta\begin{bmatrix}1+2ks_i/p\\1+2ls_i/p\end{bmatrix}(\tau|0)$$

and the "degeneracy factor" $\kappa\left[{k\atop l}\right]$ as

$$\kappa\begin{bmatrix}k\\l\end{bmatrix} := \prod_{d|p}\left(\frac{d}{\gcd(k,l,d)}\right)^{\frac{12}{\varphi(p)}\gcd(k,l,d)\mu\left(\frac{p}{d}\right)}\,.$$

One may notice that the phase factor in the partition function block (III.11a) may be set to one by choosing an appropriate representative of the shift vector $\delta\in\frac{1}{p}\Gamma_{2,2}/\Gamma_{2,2}$, as shown in appendix C. As discussed in the introduction, the Narain lattice $\Gamma_{4,20}$ appearing in the $(0,0)$ sector lies at a point in moduli space corresponding, on the type IIA side, to a Gepner model admitting a mirrored automorphism of order p.

Anticipating the following section, we emphasise here that no sum over the charge lattice $\Gamma_{4,20}$ appears (except for the term with $(k,l)=(0,0)$), which is due to the fact that the twist γ acts non trivially on the whole lattice ; hence any state with non-vanishing momentum in $\Gamma_{4,20}$ is projected out in the orbifolding procedure (see *e.g.* [25] for an extensive discussion). On the

4. Conventions and properties of the Jacobi ϑ-functions are collected in appendix C.

type IIA side of the duality, it means that there is no lattice of BPS D-brane charges, which is easy to understand as that theory has no massless Ramond-Ramond ground states [80]. There is no non-abelian gauge group enhancement coming from $\Gamma_{4,20}$ on the heterotic theory as we are at a point in the moduli space corresponding to a non-singular K3 CFT.

The full partition function of the heterotic orbifold CFT, given by the sum (III.2.4) over all sectors, is therefore modular invariant provided that

$$Z\begin{bmatrix} k+p \\ l \end{bmatrix} = Z\begin{bmatrix} k \\ l+p \end{bmatrix} = Z\begin{bmatrix} k \\ l \end{bmatrix} , \quad \forall\, k, l\,.$$

This is precisely what is ensured by the equations (III.7) which are therefore interpreted, in the heterotic picture, as necessary constraints on the shift vector to obtain a (perturbatively) well-defined string vacuum. In short, a non-vanishing winding shift is imposed by the modular invariance constraints.

III.3 BPS States

BPS states have dual interpretations in the two dual theories, the type IIA theory on $K3 \times T^2$ and the heterotic string on $T^4 \times T^2$ [84] (see *e.g.* Table 1 of [48]). In particular, winding and momenta along one-cycles of the four-torus in the heterotic theory correspond to D-branes wrapping cycles of K3 in the type IIA description of the theory, while momentum and winding states on T^2 in the heterotic picture are respectively understood as momentum states on T^2 and as NS5 branes wrapping $K3 \times S^1 \subset K3 \times T^2$ from the type IIA perspective. On the type IIA side, after the quotient by the mirrored K3 automorphism, no D-brane states remain ; this is due to the fact that space-time supersymmetry is entirely carried by left-movers so that there are no massless Ramond-Ramond p-forms hence no BPS Dp-branes. In the heterotic dual, this corresponds to the fact that fundamental strings with momentum and/or winding on the 4-torus are projected out, as the automorphism used in the quotient leaves no cycle of T^4 invariant. Fundamental heterotic strings with winding around a one-cycle of the T^2 are dual to the type IIA NS5-brane wrapping the same one-cycle of the base together with the $K3$ fibre ; on taking the quotient, this descends to what can be thought of as an NS5-brane wrapping a 'cycle' of the non-geometric Calabi-Yau background. [5]

In this section, we shall study the BPS states that arise in the perturbative spectrum of the heterotic string orbifold. The type IIA duals of these states will in general be non-perturbative states carrying NS5-brane charge.

III.3.1 Helicity Supertraces

In practice, a powerful tool in studying BPS states is the computation of helicity supertraces, that are protected quantities which do not change when the string coupling is increased ; however, in four-dimensional theories with $\mathcal{N} = 2$ supersymmetry, they can jump across walls of marginal stability in the moduli space. It can be shown (see *e.g.* [19] for a review of helicity supertraces properties and references therein) that in $\mathcal{N} = 2$ theories the only non-vanishing helicity supertrace is

$$\Omega_2(\mathfrak{R}) := \mathrm{Tr}_{\mathfrak{R}} \left[(-1)^{2J_3} J_3^2 \right] ,$$

5. For a discussion of branes in non-geometric backgrounds, see e.g. [92] and references therein.

for any representation $\mathfrak{R}$ of the $\mathcal{N}=2$ algebra, with J_3 the space-time helicity operator. Ω_2 vanishes for any (long) massive representation of $\mathcal{N}=2$ supersymmetry while it is unchanged under recombinations of two BPS multiplets into a long multiplet or *vice versa*, making it a well-defined quantity on the moduli space.

In the heterotic frame, it will receive contributions from the perturbative Dabholkar-Harvey (DH) half-supersymmetric BPS states [93] that are heterotic fundamental strings in their left-moving superconformal ground state characterized by their winding and momentum charges on the torus. It is possible to extract Ω_2 from the partition function by introducing a chemical potential for the helicity ; more precisely, defining

$$Z(\tau,\bar\tau|v,\bar v) := \mathrm{Tr}_{\mathcal{H}}\left[(-1)^{2J_3}e^{2i\pi v J_3^{(L)}+2i\pi\bar v J_3^{(R)}}q^{L_0}\bar q^{\bar L_0}\right],$$

with $\mathrm{Tr}_{\mathcal{H}}$ the trace over the whole Hilbert space of the theory and with $J_3^{(L)}$ and $J_3^{(R)}$ the left and right moving components of the helicity respectively, Ω_2 is generated by the function B_2 defined as

$$B_2(\tau,\bar\tau) := \left(\frac{1}{2i\pi}\frac{\partial}{\partial v}+\frac{1}{2i\pi}\frac{\partial}{\partial \bar v}\right)^2 Z(\tau,\bar\tau|v,\bar v)\Bigg|_{v=\bar v=0} = \sum_{Q\in\Lambda}\Omega_2(Q)q^{L_0}\bar q^{\bar L_0}. \tag{III.12}$$

where Λ stands for the lattice of electric charges of the orbifolded theory, given here by

$$\Lambda = \bigoplus_{k=0}^{p-1}(\Gamma_{2,2}+k\delta)\,.$$

Using the results from section III.2 and identities from appendix C, one may then show that the modified partition function reads

$$Z(\tau,\bar\tau|v,\bar v) = \frac{1}{p}\sum_{k,l=0}^{p-1} Z\begin{bmatrix}k\\l\end{bmatrix}(\tau,\bar\tau|v,\bar v),$$

where the orbifold blocks are now

$$Z\begin{bmatrix}0\\0\end{bmatrix}(\tau,\bar\tau|v,\bar v) = \frac{1}{\tau_2}\frac{\Theta_{\Gamma_{4,20}}(\tau,\bar\tau)\times\Theta_{\Gamma_{2,2}}(\tau,\bar\tau)}{\eta^{12}(\tau)\bar\eta^{24}(\bar\tau)}\xi(\tau|v)\bar\xi(\bar\tau|\bar v)\vartheta_1^4\left(\tau|\frac{v}{2}\right)$$

$$Z\begin{bmatrix}k\\l\end{bmatrix}(\tau,\bar\tau|v,\bar v) = \frac{\kappa\left[{k\atop l}\right]}{\tau_2}\exp\left[\frac{i\pi kl}{p^2}\left(2\Psi_p-\Delta^2\right)\right]\frac{\Gamma\left[{k\atop l}\right](\tau)}{|\eta(\tau)|^{12}\,\bar F\left[{k\atop l}\right](\bar\tau)}$$
$$\times\,\xi(\tau|v)\bar\xi(\bar\tau|\bar v)\vartheta_1^2\left(\tau|\frac{v}{2}\right)\prod_{i=1}^{2}\frac{\vartheta\left[{1+2ks_i/p\atop 1+2ls_i/p}\right]\left(\tau|\frac{v}{2}\right)\times\bar\vartheta\left[{1+2ks_i/p\atop 1+2ls_i/p}\right](\tau|0)}{\vartheta\left[{1+2ks_i/p\atop 1+2ls_i/p}\right](\tau|0)}\,.$$

Here, $\xi(\tau|v)$ is the usual space-time transverse bosons helicity generating function defined as

$$\xi(\tau|v) := \prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^ne^{2i\pi v})(1-q^ne^{-2i\pi v})}.$$

Differentiating $Z(v,\bar{v})$ with respect to v and $\bar{v}$ gives the index B_2 :

$$B_2 \quad = \quad -\frac{1}{2p\tau_2}\sum_{k,l}{}' \exp\left[\frac{i\pi kl}{p^2}\left(2\Psi_p-\Delta^2\right)\right]\frac{\kappa\left[{k \atop l}\right]\Gamma\left[{k \atop l}\right](\tau)}{\bar{\eta}^6(\bar{\tau})\bar{F}\left[{k \atop l}\right](\bar{\tau})}\prod_{i=1}^{2}\bar{\vartheta}\begin{bmatrix}1+2ks_i/p\\1+2ls_i/p\end{bmatrix}(\tau|0) \qquad \text{(III.14)}$$

where the primed sum $\sum'_{k,l}$ stands for the sum running over all values of (k,l) in $\mathbb{Z}_p\times\mathbb{Z}_p$ except $(0,0)$. Note that the term with $(k,l)=(0,0)$, *i.e.* the untwisted sector contribution with no quotienting group element insertion, does not contribute to the index. This illustrates once more the absence, in the orbifolded theory, of states with charges lying in the $\Gamma_{4,20}$ lattice.

As the automorphism generating the $\mathbb{Z}_p$ group we are quotienting by has a non-trivial action on the charge lattice, one cannot factorise the BPS index as the product of a sum over the charge lattice by a function with well-defined modular properties ; however, it is still possible to split it into smaller blocks which factorise in a similar way by expressing the charge lattice as

$$\Lambda = \bigoplus_{k,a=0}^{p-1}\Lambda^{(k,a)},$$

where we define $\Lambda^{(k,a)}$ as

$$\Lambda^{(k,a)} := \left\{v+k\delta\in\Gamma_{2,2}+k\delta\,\middle|\,\langle v,\delta\rangle=\frac{a}{p}\ \mathrm{mod}\ 1\right\}.$$

Each $\Gamma\left[{k \atop l}\right]$ may then be expressed in terms of the theta functions associated with the lattices $\Lambda^{(k,a)}$, for a between 0 and $p-1$. This allows one to extract from the BPS index (III.14) the indices for each sublattice $\Lambda^{(k,a)}$ of the charge lattice, as all charges in a given $\Lambda^{(k,a)}$ transform in the same way under the whole automorphism g_p. Defining as before $\Theta_{(k,a)}$ as

$$\Theta_{(k,a)} := \sum_{Q\in\Lambda^{(k,a)}} q^{Q_L^2/2}\bar{q}^{Q_R^2/2},$$

the whole B_2 index may be expressed as $B_2=\sum_{k,a=0}^{p-1}B_2^{(k,a)}\Theta_{(k,a)}$ with

$$B_2^{(k,a)} := -\frac{1}{2p\tau_2}\sum_{l=0}^{p-1}\exp\left[\frac{i\pi kl}{p^2}\left(2\Psi_p+\Delta^2\right)+\frac{2i\pi al}{p}\right] \times\frac{\kappa\left[{k \atop l}\right]}{\bar{\eta}^6(\bar{\tau})\bar{F}\left[{k \atop l}\right](\bar{\tau})}\prod_{i=1}^{2}\bar{\vartheta}\begin{bmatrix}1+2ks_i/p\\1+2ls_i/p\end{bmatrix}(\bar{\tau}|0) \qquad \text{(III.15)}$$

There is a subtlety to take into account here ; the definition (III.12) of B_2 implies that two different charges Q and P will contribute to the same index Ω_2 if they satisfy $Q_i^2=P_i^2$ for $i=L,R$ (Q_L and Q_R standing for the left and right components of the charge vector Q respectively, as before). In particular, this means that opposite charge vectors Q and $-Q$ always contribute to the same index $\Omega_2(Q)$; from a more physical point of view, this is a reflection of the CPT invariance of the theory which imposes that any representation of the $\mathcal{N}=2$ supersymmetry algebra must be accompanied by its CPT conjugate, which has charge $-Q$, to form a CPT-invariant multiplet. This means that the index $\Omega_2(Q)$ must be computed by taking into account not only the contributions from $B_2^{(k,a)}$ but also from the possible non-trivial degeneracies of states in the sum over the charge lattice.

III.3.2 Some Explicit Results

A straightforward check of the validity of the above indices may be obtained by evaluating the constant term in $B_2^{(0,0)}$; indeed, this will give the index of the $\mathcal{N}=2$ supersymmetry multiplets whose charge Q has vanishing norm. At a generic point in moduli space, these are the only massless multiplets of the theory so that this gives some insight about the dimension of the Coulomb and Higgs branches. More precisely, one may show (see *e.g.* [19]) that the supergravity and vector multiplets each contribute $+1$ to B_2 while hypermultiplets each contribute -1. The classical vector moduli space was shown to be that of the STU model in [7].

With three vector multiplets and one supergravity multiplet, one expects the constant term to be $4-n_H$, with n_H the number of hypermultiplets remaining in the orbifold theory. An explicit expansion of $B_2^{(0,0)}$ in power series yields results that match the analysis of the moduli space that will be given in V : one finds for instance respectively $20, 10, 4, 2$ and 0 massless hypermultiplets in the $p=$ 2, 3, 5, 7 and 13 theories. [6]

For each specific value of Q, it is also possible to extract the index $\Omega_2(Q)$ from the formulæ given above. Let us consider for simplicity the five-dimensional theory one would get from an orbifold of $T^4\times S^1$, a decompactification limit of the case we have studied so far. The charges may then be parametrised as $Q=(n,w)$, n and w being the momentum and winding numbers of the string respectively. Finding $\Omega_2(Q)$ may easily be done by identifying in which sublattice $\Lambda^{(k,a)}$ Q lies and using the level-matching condition

$$\frac{Q^2}{2}=N+\alpha_k, \tag{III.16}$$

where N is the level of the BPS state and α_k arises from the difference i ground state energy between left- and right-movers. Setting $|s_i|=1 \bmod p$ for $i=1,2$ (which amounts to choosing a generator of the cyclic group $\gamma\in\mathbb{Z}_p$), α_k may be explicitly computed and is

$$\alpha_k=\frac{k^2}{p^2}-\frac{k}{p}-\left(\frac{\gcd(k,p)}{p}\right)^2\prod_{\substack{q|p\\ q\,\text{prime}}}(-q)+\begin{cases}\frac{1}{2}-\frac{k}{p} & \text{if } 0\le k\le\frac{p}{2}\\ \frac{k}{p}-\frac{1}{2} & \text{if } \frac{p}{2}\le k\le p-1\end{cases}$$

for $1\le k\le p-1$ and, of course, $\alpha_0=-1$ as usual.

Let us take a simple example, say $Q=(1,5)$ in the above notation, and consider also a model with $p=3$; then, setting once again $\delta=(1/3,1/3)$, one has $\langle Q,\delta\rangle=0 \bmod 1$ which indicates that $Q\in\Lambda^{(0,0)}$ with the above notations. Now, as $\frac{Q^2}{2}=5$, $\Omega_2(Q)$ is simply given by the coefficient of the $\bar{q}^5$ term in the power expansion of $B_2^{(0,0)}$; computing the first terms in this expansion gives in this specific case $\Omega_2\left[Q=(1,5)\right]=176$. One should remember here that Ω_2 does not represent a degeneracy, *per se*, as contributions from integer and half-integers spin multiplets are counted respectively positively and negatively [7]. This explains for instance that for other values of Q, one may find negative values of Ω_2 (*e.g.* $\Omega_2\left[Q=(2,2)\right]=-90$).

One may also consider BPS states lying in twisted sectors, which have non-integer charges in general ; explicit computations show that $|\Omega_2|$ seems to grow faster with the level N in the

6. The numbers of hypermultiplets $20, 10, 4, 2$ and 0 are the quaternionic dimensions of the corresponding hypermultiplet moduli spaces.

7. Here, the "spin of a multiplet" is understood to be the spin of the middle state of the multiplet. This make sense as we are only considering here short multiplets, since any long multiplet has vanishing contribution to B_2 as explained earlier.

twisted sectors than in the untwisted one (*e.g.* $\Omega_2\left[Q=\left(\frac{1}{3},\frac{10}{3}\right)\right]=-236196$, while the two untwisted-sector examples considered above had higher values of N but lower values of Ω_2.). In [48], it was noted that Ω_2 is generically exponentially smaller in untwisted sectors than in twisted ones in $\mathcal{N}=2$ orbifold models ; explicit expansions of the various $B_2^{(k,a)}$ in powers of $\bar{q}$ seem to confirm this statement.

The asymptotic behaviour of $\Omega_2(Q)$ is also accessible for high values of Q^2 following the procedure described in [48] which we will briefly review here. As illustrated above with a few examples, the power expansion of the $B_2^{(k,a)}$'s gives us access to the Ω_2 indices ; more precisely, writing $B_2^{(k,a)}$ as

$$B_2^{(k,a)}(\bar{q}) := \sum_N a_N^{(k,a)} \bar{q}^{N-\alpha_k},$$

it is clear from (III.15) and from the level-matching condition (III.16) that $\Omega_2(Q)=\epsilon(k,a)a^{(k,a)}_{Q^2/2}$ for $Q\in\Lambda^{(k,a)}$, where $\epsilon(k,a)$ is a factor taking into account the fact that both $Q\in\Lambda^{(k,a)}$ and $-Q\in\Lambda^{(-k,-a)}$ contribute to $\Omega_2(Q)$ (explicitely, $\epsilon(k,a)=1$ if $\Lambda^{(-k,-a)}=\Lambda^{(k,a)}$ and 2 else). Performing an inverse Laplace transform, it is possible to compute $a_N^{(k,a)}$ to find

$$a_N^{(k,a)} = \int_{\epsilon-i\pi}^{\epsilon+i\pi} \frac{\mathrm{d}t}{2i\pi} e^{Nt} B_2^{(k,a)}\left(e^{-t}\right). \tag{III.17}$$

When N reaches high values, the imaginary exponential in (III.17) becomes rapidly oscillating so that the integral is dominated by the behaviour of the integrand around $t\sim\epsilon$. In this regime, the $\epsilon\to 0$ limit of this integral is dominated by a function of $e^{-t}\sim 1$ so that the power expansion of $B_2^{(k,a)}$ is not useful here ; however, one may use the modular properties of $B_2^{(k,a)}$ to replace it by a function of $e^{-4\pi^2/t}$ which is a small parameter when t goes to 0. The above integral becomes

$$a_N^{(k,a)} = \int_{\epsilon-i\pi}^{\epsilon+i\pi} \frac{\mathrm{d}t}{2i\pi} e^{Nt}\left(S\cdot B_2^{(k,a)}\right)\left(e^{-\frac{4\pi^2}{t}}\right),$$

so that expanding $S\cdot B_2^{(k,a)}$ instead in powers of $\bar{q}$ lead to an approximation of the asymptotic behaviour of $\Omega_2(Q)$ for large values of $\frac{Q^2}{2}$. Here, S is the usual generator of $SL(2,\mathbb{Z})$ acting on the world-sheet parameter τ as $\tau\mapsto -1/\tau$.

We consider below the models with p a prime number. Explicit computations to leading order show that the asymptotic behaviour of Ω_2 in the untwisted sector is given, up to a multiplicative constant, by

$$\Omega_2^{\text{untw}}(Q) \underset{Q^2\gg 1}{\sim} \begin{cases} -\sqrt{\frac{Q^2}{2}}\,J_1\left(\frac{4\pi}{3}\sqrt{\frac{Q^2}{2}}\right) & p=3 \\ -\sqrt{\frac{Q^2}{2}}\,I_1\left(\frac{4\pi}{5}\sqrt{\frac{Q^2}{2}}\right) & p=5 \\ -\sqrt{\frac{Q^2}{2}}\,I_1\left(\frac{4\pi\sqrt{5}}{7}\sqrt{\frac{Q^2}{2}}\right) & p=7 \\ -\sqrt{\frac{Q^2}{2}}\,I_1\left(\frac{4\pi\sqrt{29}}{13}\sqrt{\frac{Q^2}{2}}\right) & p=13 \end{cases} \tag{III.18}$$

Here, J_1 (I_1) is the (modified) Bessel function of first kind. In the twisted sectors, the asymptotic behaviour of the BPS index is surprisingly identical for any order prime p of the quotienting group ; it is then given in all these cases by

$$\Omega_2^{\text{tw}}(Q) \underset{Q^2 \gg 1}{\sim} -\sqrt{\frac{Q^2}{2}} I_1 \left(4\pi\sqrt{\frac{Q^2}{2}}\right) . \tag{III.19}$$

Replacing the Bessel functions by their asymptotic expansions in equations (III.18) and (III.19) then confirms the exponentially small growth of $|\Omega_2(Q)|$ in the untwisted sector compared to that of $|\Omega_2(Q)|$ in the twisted ones discussed in [48].

III.4 The Non-Perturbative Type IIA Construction and Duality

In section III.2 modular-invariance constraints on the heterotic duals of the type IIA non-geometric Calabi-Yau backgrounds were analyzed. It was found that perturbative consistency of the heterotic constructions leads to the constraint (III.9) on the shift vector for the two-torus and this implies the shift vector should have non-vanishing winding charge. In this section we will examine the consequences of this condition on the type IIA side of the duality, where it leads to a non-perturbative modification of the $K3 \times T^2$ orbifold. For clarity of the presentation, we will restrict ourselves here to the case in which the order p of the automorphism is a prime number.

III.4.1 Interpretation of the Shift Vector

For both the type IIA and heterotic constructions, we have an orbifold by a twist $\gamma \in O(\Gamma_{4,20})$ and a shift t on the two-torus by a vector $\delta = (\alpha^i, \beta_i)$ with $p\,\delta \in \Gamma_{2,2}$. The momentum vector k_i ($i = 1, 2$) on the 2-torus combines with the string winding charges w^i to form a generalised momentum vector $\Pi_I = (k_i, w^i) \in \Gamma_{2,2}$. The shift acts on a momentum state $|\Pi\rangle = |k, w\rangle$ with $k_i, w^i \in \mathbb{Z}$ as :

$$|\Pi\rangle \mapsto \exp\Big(2\pi i \delta^I \Pi_I\Big)|\Pi\rangle$$

so that

$$|k, w\rangle \mapsto \exp\Big(2\pi i[\alpha^i k_i + \beta_i w^i]\Big)|k, w\rangle$$

For a shift symmetry of order p, *i.e.* isomorphic to $\mathbb{Z}_p$, we take $\Delta = p\,\delta \in \Gamma_{2,2}$ to be a lattice vector, with norm

$$\Delta^2 = p^2 \delta^2 = 2p^2 \alpha^i \beta_i \,.$$

If the momenta k_i are realised on the periodic coordinates $y^i \sim y^i + 2\pi$ of the 2-torus in the usual way, $\exp(2\pi i \alpha^i k_i)$ generates the shift

$$y^i \mapsto y^i + 2\pi\alpha^i$$

If dual coordinates $\tilde{y}_i$ conjugate to the winding charge are introduced then $\exp(2\pi i \beta_i w^i)$ generates the dual shift

$$\tilde{y}_i \mapsto \tilde{y}_i + 2\pi\beta_i$$

so the shift acts on the coordinates $Y^I = (y^i, \tilde{y}_i)$ of the doubled torus as

$$Y^I \mapsto Y^I + 2\pi\delta^I \,.$$

In both type IIA and heterotic constructions with a single twist, we can take the shift to be on a single cycle of the 2-torus, so that $p\,\delta \in \Gamma_{1,1} \subset \Gamma_{2,2}$. In the perturbative type IIA construction, we had

$$\delta = (\alpha^1, 0, 0, 0), \qquad \alpha^1 = \frac{1}{p}\,,$$

giving a shift

$$y^1 \mapsto y^1 + \frac{2\pi}{p}\,.$$

For the heterotic string, the modular invariance constraint

$$\Delta^2 = 2p^2\alpha^i\beta_i = 2 \mod p$$

obtained in section III.2, eqn. (III.2.4), implies that both α and β are non-zero. Setting $\alpha^1 = 1/p$ (in order to match with the perturbative type IIA construction in the large T^2 limit) one can solve this constraint with

$$\delta = (\alpha^1, 0, \beta_1, 0), \qquad \alpha^1 = \frac{1}{p}, \qquad \beta_1 = \frac{1}{p}\,.$$

This vector generates the shifts

$$y^1 \mapsto y^1 + \frac{2\pi}{p}\,,$$
$$\tilde{y}_1 \mapsto \tilde{y}_1 + \frac{2\pi}{p}\,.$$

It was to be expected that the shift in y^1 should agree in the two pictures, but we see that there is a surprising difference in that a shift in the dual coordinate $\tilde{y}_1$ is essential for heterotic modular invariance but there was no corresponding shift on the type IIA side in our construction.

In our models, the perturbatively consistent type IIA construction determines the heterotic dual in the large volume limit of the T^2, with an orbifold by a twist $\gamma \in O(\Gamma_{4.20})$ and a shift of the coordinate y^1 of a cycle of T^2. However, away from the decompactification limit this heterotic construction is not perturbatively consistent and must be modified by winding number shifts. Then duality implies that there should be a dual modification of the type IIA theory. This modification is non-perturbative in the type IIA theory, so does not affect the perturbative consistency of the original construction. This is in accord with the discussion of [49], where it is argued that duality does not completely determine the shift vector, and consistency conditions, such as level matching and modular invariance are needed to fix the shift vector.

A similar situation was encountered in the FHSV model [55]. We will discuss further this example in subsection III.4.3, and compare it with our models. In both cases, it is natural to speculate that the modifications in the type IIA theory could arise from a condition for non-perturbative consistency of the IIA string.

III.4.2 The Non-Perturbative Type IIA Construction

A convenient way of representing the modifications to the type IIA construction is as follows. The transformation t acts on a heterotic state with momentum k and winding w by

$$|k, w\rangle \mapsto \exp\Big(2\pi i[\alpha^i k_i + \beta_i w^i]\Big)|k, w\rangle$$

The type IIA dual of the heterotic momenta k_i and winding charges w^i are some charges x_i and z^i in the $\Gamma_{6,22}$ lattice of type IIA compactified on $K3 \times T^2$. For our construction, x_i remains the momentum on the torus, so $k_i = x_i$, and z^i is the winding charge on the i'th circle for the solitonic string obtained by wrapping the IIA NS5-brane on K3, so that z^i is the NS5-brane charge for NS5-branes wrapping $K3 \times S^1$, with the S^1 being the i'th circle. (For the FHSV model, x_1 and z^1 are D0-brane and D4-brane charges, as we will discuss in the next subsection.)

Then for the models considered here, the heterotic transformation (III.4.2) becomes the type IIA transformation

$$|k, z\rangle \mapsto \exp\Big(2\pi i[\alpha^i k_i + \beta_i z^i]\Big)|k, z\rangle$$

where k_i is the momentum on the i'th circle and z^i is the winding number of the solitonic string (from the NS5-brane wrapped on $K3$) on the i'th circle. From eq. (III.4.1), consistency of the heterotic perturbative limit is satisfied with $\alpha^1 = \beta_1 = 1/p$ an $\alpha^2 = \beta_2 = 0$.

Non-perturbative type IIA states with non-zero winding number for the solitonic string around the first circle of T^2 are therefore charged under the symmetry used to obtain the non-geometric Calabi-Yau background. For perturbative states with $z = 0$, the transformation (III.4.2) is of course the same as the one used in the perturbative construction with shift vector (III.4.1).

As we have seen, the action of t on a heterotic state $|k, w\rangle$ given by (III.4.2) gives a shift of the coordinates y^i conjugate to k_i together with a shift of the dual coordinates $\tilde{y}_i$ conjugate to w^i. Similarly, for the IIA string, if we introduce coordinates $\hat{y}_i$ conjugate to z^i, then the action of t on a type IIA state (III.4.2) can be understood as a shift of the coordinates $y^i, \hat{y}_i$. In general, phase rotations of this kind dependent on brane charges can be reinterpreted as shifts of suitable dual coordinates, justifying our referring to t as a shift ; this will be discussed further in the next section.

The above discussion implies, using heterotic/type IIA duality, that non-perturbatively consistent non-geometric Calabi-Yau backgrounds in type IIA superstring theory should be defined using a shift symmetry of the form (III.4.2) that includes a non-perturbative contribution. In the FHSV construction that we will discuss below, a similar type of non-perturbative modification of the shift symmetry occurs, involving D-brane charges rather than NS5-brane charges.

III.4.3 The FHSV Model

The starting point for the FHSV construction [55] is a special K3 surface admitting a freely acting $\mathbb{Z}_2$ involution, such that the quotient of $K3$ by this is an Enriques surface. This non-symplectic K3 automorphism acts on the lattice $\Gamma_{4,20}$ of total K3 cohomology by interchanging two $E_8 \oplus U$ sublattices, acting as -1 on one sublattice U and leaving the final U invariant. [8] This is then combined with the reflection $y^i \mapsto -y^i$ on the coordinates of T^2 to give a freely acting automorphism γ of $K3 \times T^2$. The quotient of $K3 \times T^2$ by this gives a Calabi-Yau manifold with Euler number zero, called the Enriques Calabi-Yau 3-fold. It is a K3 fibration over $\mathbb{P}^1$ with a monodromy around each of the four singularities of the base given by the Enriques involution.

8. This involution is a geometric automorphism, i.e. a large diffeomorphism of K3, whose action is an element of $O(\Gamma_{3,19})$. The invariant sublattice U is the lattice generated by $H^0(K3;\mathbb{Z})$ and $H^4(K3;\mathbb{Z})$, see [94] for details.

The action of γ on the charge lattice of IIA strings on $K3 \times T^2$

$$\Gamma_{6,22} \cong (E_8 \oplus U) \oplus (E_8 \oplus U) \oplus [U \oplus U \oplus U] \oplus U$$

is then to interchange the two $(E_8 \oplus U)$ terms, act as -1 on $U \oplus U \oplus U$ and to leave the final U invariant.

To find the heterotic dual of the FHSV orbifold, $\Gamma_{6,22}$ is interpreted as the Narain lattice for the heterotic string compactified on T^6, with the six sub-lattices U associated with the lattice $\Gamma_{6,6}$ of heterotic momenta and winding numbers on the six-torus. The action of γ on the charge lattice and moduli space then defines an action on the heterotic string theory (as we have done in section III.2 for our models). In particular, the involution leaves one of the six circles invariant.

However the quotient of the heterotic string theory by this involution is not modular invariant. This was remedied in [55] by supplementing the twist γ by a shift t on the circle that is invariant under the involution. The shift vector δ is such that $\Delta = 2\delta \in U$ (where this U is the last factor in (III.4.3), *i.e.* the invariant sub-lattice) and modular invariance requires $\Delta^2 = 2$, so that $\delta = (1/2, 1/2)$. Then the shift $y \to y + \pi$ on the circle is accompanied by a shift $\tilde{y} \to \tilde{y} + \pi$ on the dual circle.

While this heterotic description looks quite similar to what happens in our models, in the type IIA duality frame the physics is rather different. The identification of the heterotic and type IIA charge lattices under duality relates the heterotic momentum k and winding w on the invariant circle with the type IIA D0-brane charge x and the charge z for D4-branes wrapping K3 :

$$k = x, \qquad w = z$$

In the type IIA duality frame, the action of the 'shift' t is then given as a phase rotation of the form

$$|x, z\rangle \mapsto \exp(2\pi i[\alpha x + \beta z])|x, z\rangle = \exp(\pi i[x + z])|x, z\rangle$$

Then the IIA involution is supplemented by multiplying by the phase (III.4.3) depending on the D0-brane and D4-brane charges. That is, the involution (γ, t) consists of the geometric involution on $K3 \times T^2$ (the freely acting involution of K3 combined with the reflection on T^2) supplemented by the phase rotation (III.4.3). These modifications to the Calabi-Yau compactification are visible to D-branes but not to fundamental strings, and so will not affect the perturbative type IIA string.

III.5 Duality Covariant Formulation and New Non-Geometric Constructions

III.5.1 Dualities and Quotients

Suppose we have a theory X on a background M with a symmetry G, together with a duality map that takes this to a theory X' on a background M' with a symmetry G'. Then we can consider the quotient of X on M by G and the quotient of X' on M' by G' and ask whether they are dual, i.e. whether taking the quotient commutes with the duality transformation. As discussed in [49], in general the quotients will not be dual, but in some special cases, such as those in which the adiabatic argument applies, they can be dual. As usual, without a non-perturbative formulation of string theory the duality cannot be proved, but we can seek non-trivial tests of the duality.

We have already seen here a case where they are not dual. Taking X on M to be the IIA string on $K3 \times S^1$ and taking G to be the group $\mathbb{Z}_p$ generated by a twist of the K3 CFT (corresponding to a mirrored automorphism) and a shift in a circle coordinate, then the heterotic dual of this is not modular invariant and so not consistent. In this case we modified the heterotic symmetry G' to include a winding contribution to the shift, and then made the dual modification to the action of G, involving non-perturbative NS5-brane contributions. Then a necessary condition for the quotients to be dual is that the group G is chosen so that both are perturbatively consistent. Further duals could then give further non-perturbative constraints on the group G.

Here we are interested in two examples : our non-geometric Calabi-Yau construction and the FHSV model for the type IIA string, together with the conjectured heterotic duals that were discussed in section III.4. Consistency of the heterotic dual required modifications of the original symmetry to include D0- and D4-brane contributions in the FHSV model and NS5-brane contributions for the non-geometric Calabi-Yau construction. However, as we shall see, this is not enough to completely determine the non-perturbative action of the symmetry in each case. In our non-geometric Calabi-Yau construction, the adiabatic argument provides strong support for the duality with the heterotic T-fold.

We now turn to the action of duality transformations on our model and that of FHSV to obtain new dual constructions. For this, a duality covariant viewpoint is useful.

III.5.2 Compactifications to five dimensions

We consider first compactifications to five dimensions, in both heterotic and type II duality frames.

Symmetries and Automorphisms

The heterotic string compactified on T^5 or type IIA string compactified on $K3 \times S^1$ has, at generic points in the moduli space, a symmetry

$$[O(\Gamma_{5,21}) \ltimes U(1)^{26}] \times U(1)\,.$$

The $U(1)^{26} \times U(1)$ is a gauge symmetry associated with $26+1$ abelian vector fields, and at special points in the moduli space this is enhanced to a non-abelian group. A subgroup $U(1)^5$ arises from isometries of the heterotic five-torus. The extra $U(1)$ symmetry arises in five dimensions as the NS-NS two-form b_2 (in either the heterotic or type IIA string) can be dualised to a vector field, with a further $U(1)$ gauge symmetry that commutes with $O(\Gamma_{5,21})$. There are $26+1$ electric 0-brane charges (Z^I, K) corresponding to the gauge symmetry, with Z^I transforming as the 26-dimensional representation of $O(5,21)$.The charge K is a singlet under $O(5,21)$; the 5-dimensional supersymmetry algebra has 5+1 central charges, consisting of the 5 electric charges for the $U(1)^5$ gauge symmetry associated with the gauge fields in the supergravity multiplet and the singlet charge K.

In the heterotic string, the BPS states carrying the charge K are heterotic five-branes wrapping T^5. This charge can be thought of as the winding number on S^1 of the solitonic string obtained from wrapping the heterotic five-brane on T^4. The solitonic string of the heterotic theory is dual to the fundamental string of the type IIA theory, so in the type IIA theory the singlet charge K is the winding number of fundamental type IIA strings on the S^1 in $K3 \times S^1$. In the IIA string theory on $K3 \times S^1$ there is not a T-duality relating the winding number K to the momentum on S^1, as that T-duality is not a proper symmetry of the IIA theory, but instead maps the IIA string theory on $K3 \times S^1$ to the IIB string theory on $K3 \times S^1$.

We are interested in automorphisms that consist of a twist $\gamma \in O(\Gamma_{5,21})$ and a shift $t \in U(1)^{27}$ in which t commutes with γ. One possibility is to choose the shift t to be generated by the singlet charge K, and then any $\gamma \in O(\Gamma_{5,21})$ can in principle be used. Another is to choose a sub-lattice $\Gamma_{4,20} \oplus \Gamma_{1,1} \subset \Gamma_{5,21}$ so that the symmetry algebra has a subgroup

$$[O(\Gamma_{4,20}) \ltimes U(1)^{24}] \times [O(\Gamma_{1,1}) \ltimes U(1)^2]\,,$$

and to use a twist $\gamma \in O(\Gamma_{4,20})$ from the first factor and a shift $t \in U(1)^2$ from the second factor, and these indeed commute. The automorphisms that we used in earlier sections are of this form.

The Heterotic String Perspective

The moduli space of heterotic strings compactified on T^5 is

$$\mathscr{M}_{5\mathrm{D}} \cong O(\Gamma_{5,21})\backslash O(5,21)/(O(5) \times O(21)) \times S^1 \times \mathbb{R}\,,$$

where the extra S^1 factor corresponds to a Wilson line for the gauge field dual to b_2 and the $\mathbb{R}$ factor is the zero mode of the heterotic dilaton. A T^5 CFT has a moduli space

$$O(\Gamma_{5,5})\backslash O(5,5)/(O(5) \times O(5))$$

identified under the T-duality group $O(\Gamma_{5,5})$. Then choosing a subgroup $O(5,5) \subset O(5,21)$ with corresponding sublattice $\Gamma_{5,5} \subset \Gamma_{5,21}$ splits the heterotic degrees of freedom into degrees of freedom on T^5 described by a CFT on T^5 and the remaining right-moving modes representing the gauge degrees of freedom. This choice is not unique, and acting with the duality group $O(\Gamma_{5,21})$ will change the split into torus and gauge degrees of freedom.

For a twist $\gamma \in O(\Gamma_{4,20})$ from the first factor in (III.5.2) and a shift $t \in U(1)^2$ from the second factor in (III.5.2), it is natural to choose a torus $T^4 \times S^1$ so that the first factor of (III.5.2) acts on the heterotic string on T^4 and the second acts on the CFT on S^1. Then the heterotic momentum k and winding number w on the S^1 factor are the charges generating the $U(1)^2$ and transforming as a doublet under $O(1,1)$. A shift generated by (k,w) then gives a heterotic automorphism of the kind discussed in earlier sections. It can in principle be augmented by a shift generated by the singlet charge K. The 3 charges (k,w,K) take values in a lattice $\Gamma_{1,1} \oplus \mathbb{Z}$ and transform as a $2+1$ under $O(1,1)$, with k and w forming a doublet.

The general construction could then involve a shift vector $\delta = (\alpha, \beta, \kappa)$ with three components, so that

$$\delta \cdot \Pi = \alpha k + \beta w + \kappa K\,.$$

This would then lead to a charge-dependent phase $\exp(2\pi i \delta \cdot \Pi)$ in the automorphism. The transformation generated by K is non-perturbative and does not affect the perturbative heterotic string. Perturbative consistency requires that (α, β) satisfy some modular invariant constraints, but places no constraint on κ. For the models considered in this article, the condition (III.9) is satisfied for $\alpha\beta = 1/p^2$. As we shall see, perturbative consistency of dual forms of the theory will impose further constraints on the shift.

Acting with $O(\Gamma_{5,21})$ will transform k and w into two other linear combinations of the 26 non-singlet charges, and in particular can lead to shifts that involve charges from the gauge sector. This can also be thought of as changing the original choice of split into T^5 degrees of freedom and gauge degrees of freedom to a new choice. It will also transform the twist γ to a conjugate twist $\tilde{\gamma}$.

Alternatively, we can take the shift t to be generated by the singlet charge K, and take $\gamma \in O(\Gamma_{5,21})$. Then the shift is

$$\delta \cdot \Pi = \kappa K$$

for some κ. This shift does not affect the perturbative heterotic string, so the perturbative construction is simply a quotient by $\gamma \in O(\Gamma_{5,21})$. In general, this will have fixed points and will result in a non-freely acting asymmetric orbifold of the heterotic string. This then restricts γ to satisfy the constraints of [25, 26] for the asymmetric orbifold to be modular invariant.

The Type IIA String Perspective

As we have seen, there are many ways of choosing a split of the heterotic degrees of freedom into degrees of freedom on T^5 and gauge degrees of freedom. For any such choice of T^5, one can choose a $T^4 \subset T^5$ in a number of ways, and for each choice one can dualize the heterotic T^4 to a type IIA K3 . Thus there are many ways of choosing a K3 moduli space as a subspace of the five-dimensional moduli space (III.5.2) – the choices correspond to choosing an $O(\Gamma_{4,20})$ subgroup of $O(\Gamma_{5,21})$ – and acting with $O(\Gamma_{5,21})$ will change this choice. Then there is no canonical way of choosing which degrees of freedom are associated with K3 and which with S^1, and it can be changed by acting with $O(\Gamma_{5,21})$; it can result in different dual forms of a given compactification.

For a twist $\gamma \in O(\Gamma_{4,20})$, it is natural to choose a split such that the twist γ acts on K3 and the shift t on S^1, and we now investigate this choice. In the type IIA string, the NS-NS 2-form is dualised to a vector field with charge $\hat{z}$. This is the charge for NS5-branes wrapped on $K3 \times S^1$. This can also be thought of as the winding charge for the solitonic string obtained from wrapping NS5-branes on K3 and so is dual to the heterotic string winding number. In addition, there is a momentum $\hat{k}$ and a winding $\hat{w}$ of the type IIA string on the extra circle. There are again 3 charges, and duality relates these to heterotic charges : $k = \hat{k}$, $w = \hat{z}$ and $K = \hat{w}$. Thus for the type IIA string, it is $(k, \hat{z})$ that form a doublet under $O(1,1)$ and $\hat{w}$ is a singlet.

The general construction involves a shift vector $\delta = (\alpha, \beta, \kappa)$ with three components, giving the heterotic shift (III.5.2) which is realised in the type IIA string as

$$\delta \cdot \Pi = \alpha k + \beta \hat{z} + \kappa \hat{w}\,.$$

This shift leads to a charge-dependent phase $\exp(2\pi i \delta \cdot \Pi)$ in the automorphism.

The Type IIB String Perspective

T-duality on the S^1 takes the IIA string on $K3 \times S^1$ to the IIB string on $K3 \times S^1$. If the IIB string has momentum k_B and winding $\hat{w}_B$ on the S^1, and NS5-brane charge $\hat{z}_B$ for NS5-branes wrapping $K3 \times S^1$, these are related to the IIA string charges $k, \hat{w}, \hat{z}$ by

$$k_B = \hat{w}, \qquad \hat{w}_B = k, \qquad \hat{z}_B = \hat{z}\,.$$

Then the shift with shift vector $\delta = (\alpha, \beta, \kappa)$ acts on the type IIB string through

$$\delta \cdot \Pi = \alpha \hat{w}_B + \beta \hat{z}_B + \kappa k_B \, .$$

Models

The original type IIA construction reviewed in section III.1.2 had $\alpha \neq 0$. Perturbative consistency of the heterotic dual theory required $\beta \neq 0$, with $\alpha\beta = 1/p^2$. Perturbative consistency of the type IIA construction was achieved with no type IIA winding contributions, so this means it is consistent to take $\kappa = 0$. Then with $\alpha = \beta = 1/p$ and $\kappa = 0$ we obtain a theory which is modular invariant in both the perturbative heterotic and perturbative type IIA formulations. Taking $\alpha = \beta = 1/p$ but with $\kappa \neq 0$, type IIA level-matching requires $\kappa = 0$ mod p, so that the shift $\kappa\hat{w}$ is by a lattice vector and so the corresponding phase is trivial. There is then no loss of generality in taking $\kappa = 0$. In this case the perturbative IIB formulation is also consistent.

Acting with $O(\Gamma_{5,21})$ will in general take the twist γ to a conjugate transformation that acts not just on the K3 CFT but which acts on the full $K3 \times S^1$ CFT. Note that for $p = 2$ the action of the conjugate transformation on the string theory may include the world-sheet parity-reversing transformation Ω, leading to an orientifold, or $(-1)^{F_L}$. A factor of Ω is needed whenever the conjugate transformation reverses the space-time parity. At the same time, the $O(\Gamma_{5,21})$ transformation will rotate the charges $k, \hat{z}$ to other charges for the $U(1)^{26}$ symmetry. The singlet charge $K = \hat{w}$ does not change. (This can instead be viewed as changing which subsector of the theory is to be interpreted as corresponding to the K3 CFT.) For example, there is a transformation that takes k to the D0-brane charge Z_0 and $\hat{z}$ to the charge Z_4 for D4-branes wrapping K3. This would give a shift

$$\delta \cdot \Pi = \alpha Z_0 + \beta Z_4 + \kappa \hat{w}$$

which is completely non-perturbative, giving a phase rotation to any given state depending on its D0,D4 and NS5 charges. For the perturbative theory, this is simply a $\mathbb{Z}_p$ orbifold of the type IIA string on $K3 \times T^2$ by $\tilde{\gamma}$, with $\tilde{\gamma}$ now acting non-trivially on $K3 \times T^2$ (i.e. not just acting on K3). Perturbative consistency of this then does not depend at all on the parameters α, β, κ and only depends on the choice of twist γ. However, this is still dual to the heterotic construction, and perturbative consistency of the heterotic dual constrains α and β, as above. Similarly, the original IIA version sets $\kappa = 0$.

Finally, we can instead take the shift t to be generated by the singlet charge K, and take $\gamma \in O(\Gamma_{5,21})$. Then the shift becomes

$$\delta \cdot \Pi = \kappa \hat{w}$$

for the type IIA string. $\hat{w}$ is a perturbative charge for the type IIA string, but it is not constrained by IIA modular invariance since the shift vector involves a winding charge but no momentum. In this case, the only constraint is that $p\kappa\hat{w}$ is a lattice vector, so $\kappa = n/p$ for some integer $n < p$.

III.5.3 Compactifications to four dimensions

We now turn to compactifications to four dimensions, which allow more general constructions.

Symmetries and Automorphisms

The heterotic string compactified on T^6 or type IIA string compactified on $K3 \times T^2$ has, at generic points in the moduli space, a symmetry

$$[O(\Gamma_{6,22}) \times SL(2,\mathbb{Z})] \ltimes U(1)^{56}\,.$$

There is a $U(1)^{28}$ gauge symmetry associated with 28 gauge fields, and, formally, a further $U(1)^{28}$ symmetry associated with the S-dual gauge fields. In different S-duality frames, different subgroups $U(1)^{28} \subset U(1)^{56}$ will be realised as fundamental gauge symmetries. There are 28 electric and 28 magnetic charges, transforming in the $(28, 2)$ representation under $O(6, 22) \times SL(2)$.

Here we will focus on twists in $O(\Gamma_{6,22})$ and not consider twists involving S-duality. The discussion is then very similar to the 5-dimensional case above. We will consider an automorphism (γ, t) consisting of a twist $\gamma \in O(\Gamma_{6,22})$ and a shift $t \in U(1)^{56}$ where t commutes with γ.

Choosing a sub-lattice $\Gamma_{5,21} \oplus \Gamma_{1,1} \subset \Gamma_{6,22}$, the symmetry algebra of the theory has a subgroup

$$[O(\Gamma_{5,21}) \ltimes U(1)^{52}] \times [O(\Gamma_{1,1}) \ltimes U(1)^{4}]\,.$$

We can then use a twist $\gamma \in O(\Gamma_{5,21})$ from the first factor and a shift $t \in U(1)^4$ from the second factor, and these indeed commute.

We will also consider choosing a sub-lattice $\Gamma_{4,20} \oplus \Gamma_{2,2} \subset \Gamma_{6,22}$, selecting a subgroup of the symmetry algebra

$$[O(\Gamma_{4,20}) \ltimes U(1)^{48}] \times [O(\Gamma_{2,2}) \ltimes U(1)^{8}]$$

and using a twist $\gamma \in O(\Gamma_{4,20})$ from the first factor and a shift $t \in U(1)^8$ from the second factor. One class of examples arises in taking a reduction to 5 dimensions of the kind considered in the previous subsection, with a twist $\gamma \in O(\Gamma_{4,20})$ and a shift on a circle, followed by a standard reduction (no twist or shift) on a further circle ; such cases have been the main focus in this paper. We can also consider reductions by (γ_1, t_1) and (γ_2, t_2) where γ_1, γ_2 are two commuting twists in $O(\Gamma_{4,20})$ and t_1, t_2 are two shifts in $U(1)^8$ (see [7] for an analysis of models with two twists). Note that the 8-charges for $U(1)^8$ transform as a $(4, 2)$ under $O(2, 2) \times SL(2)$. Using $O(2, 2) \sim SL(2) \times SL(2)$, this is the $(2, 2, 2)$ representation of $SL(2) \times SL(2) \times SL(2)$.

The Heterotic String Perspective

Consider first the case with a twist $\gamma \in O(\Gamma_{5,21})$ from the first factor in (III.5.3) and a shift $t \in U(1)^4$ from the second factor in (III.5.3). It is natural to choose a split so that the sub-lattice $\Gamma_{5,21}$ is associated with the heterotic string compactified on T^5 and the sub-lattice $\Gamma_{1,1}$ with a further circle compactification. The charges for the $U(1)^4$ symmetry are the heterotic momentum k and winding w on the extra circle, the heterotic 5-brane charge z for heterotic 5-branes wrapping T^5 and the Kaluza-Klein (KK) monopole charge q.[9]

Then the general shift vector is given by $\delta = (\alpha, \beta, \lambda, \kappa)$ with four components, so that

$$\delta \cdot \Pi = \alpha k + \beta w + \lambda q + \kappa z\,.$$

9. The KK monopole charge arises from solutions of the form $\mathbb{R} \times ALF \times T^5$ where $\mathbb{R}$ is a timelike direction and ALF denotes an ALF gravitational instanton with charge q (so that for $q = 1$ we have self-dual Taub-NUT space). The 'extra circle' is the fibre of the ALF gravitational instanton.

The shifts involving z, q do not affect the perturbative theory. For the models of section III.2 – with $\gamma \in O(\Gamma_{4,20})$ – perturbative consistency is achieved if both α and β are non-zero with $\alpha\beta = 1/p^2$. A similar analysis can be done for the general case with arbitrary $\gamma \in O(\Gamma_{5,21})$.

The case with a twist $\gamma \in O(\Gamma_{4,20})$ from the first factor in (III.5.3) and a shift $t \in U(1)^8$ from the second factor in (III.5.3) is very similar. Choosing the natural split in which the sub-lattice $\Gamma_{4,20}$ is associated with the heterotic string compactified on T^4 and the sub-lattice $\Gamma_{2,2}$ with a further T^2 compactification, the charges for the $U(1)^8$ symmetry are the heterotic momenta k_i and windings w^i on the T^2, the heterotic 5-brane charges z_i for heterotic 5-branes wrapping T^5 and the Kaluza-Klein monopole charges q^i, where $i = 1, 2$ is a coordinate index on T^2, which has coordinates y^i. The charge z_i is for a 5-brane wrapping the y^i circle and the T^4, so it is the winding number for the solitonic string from the 5-brane wrapping T^4. The general shift is then of the form

$$\delta \cdot \Pi = \alpha^i k_i + \beta_i w^i + \lambda_i q^i + \kappa^i z_i$$

For the models of section III.2, perturbative consistency requires $\alpha^i \beta_i = 1/p^2$. Taking the only non-zero coefficients to have, say, $i = 1$ reduces this to the previous case.

The Type IIA String Perspective

For the case with a twist $\gamma \in O(\Gamma_{5,21})$ from the first factor in (III.5.3) and a shift $t \in U(1)^4$ from the second factor in (III.5.3), the natural choice of split has the lattice $\Gamma_{5,21}$ associated with the type IIA string compactified on $K3 \times S^1$ and the lattice $\Gamma_{1,1}$ associated with a further compactification on a circle with coordinate y^1. In this case the $U(1)^4$ charges are the momentum $\hat{k}$ and type IIA winding $\hat{w}$ on the y^1 circle, the charge $\hat{z}$ from an NS5-brane wrapping K3 and the y^1 circle, and the KK monopole charge $\hat{q}$ associated with the y^1 circle.

Then heterotic-type II duality relates these to heterotic charges : $k = \hat{k}$, $w = \hat{z}$ and $z = \hat{w}$, $q = \hat{q}$. The general shift vector $\delta = (\alpha, \beta, \lambda, \kappa)$ gives (III.5.3) in the heterotic picture and

$$\delta \cdot \Pi = \alpha\hat{k} + \beta\hat{z} + \kappa\hat{w} + \lambda\hat{q}$$

for type IIA. This shift again leads to a charge-dependent phase $\exp(2\pi i \delta \cdot \Pi)$ in the automorphism. Level matching of the perturbative type IIA string with $\alpha \neq 0$ leads to $\kappa = 0$, as in the five-dimensional analysis above but places no constraints on β, λ as they correspond to non-perturbative contributions for the IIA string. Requiring perturbative consistency of both the IIA and heterotic formulations is satisfied (for the models of section III.2) with $\alpha = \beta = 1/p$, $\kappa = 0$ but puts no constraints on λ. The perturbative IIB formulation gives no further constraints.

As in the five dimensional case, we can consider acting on a dual pair with a duality transformation. This will transform the charges appearing in the shift, and take the twist to a conjugate one, which for $p = 2$ might include factors of Ω or $(-1)^{F_L}$.

S-Duality

To find a constraint on the parameter λ, one could seek a duality that transforms q to a perturbative charge that would enter into the perturbative constraints in the dual theory. Such a duality is provided by the heterotic string S-duality.

The heterotic charges (k, w, z, q) transform as a $(2, 2)$ under $O(\Gamma_{1,1}) \times SL(2, \mathbb{Z})$, with (k, w) and (z, q) transforming as doublets under the T-duality $O(\Gamma_{1,1})$ and (k, q) and (w, z) transforming as doublets under the S-duality $SL(2, \mathbb{Z})$. Then acting with the $SL(2, \mathbb{Z})$ element

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

takes the shift (III.5.3) to

$$\delta \cdot \Pi = -\kappa w - \lambda k + \alpha q + \beta z \,.$$

while leaving the twist unchanged. If we were to demand perturbative consistency of this S-dual theory, this would be achieved only if both λ and κ are non-zero with $\lambda\kappa = 1/p^2$ once again for the models of section III.2. We then learn that perturbative consistency of the heterotic string and of the S-dual heterotic string would require all four components of the shift vector to be non-zero, and we could satisfy these requirements by taking

$$\alpha = \beta = \lambda = \kappa = \frac{1}{p}$$

However, in this case S-duality doesn't commute with the quotient – the strong coupling behaviour of the $\mathcal{N} = 2$ supersymmetric theory arising from the quotient is not given by the strong coupling behaviour of the original $\mathcal{N} = 4$ supersymmetric theory. Then the constraint $\lambda\kappa = 1/p^2$ should not be applied to the original theory, and we can keep $\kappa = 0$, as found above.

One can see directly why the adiabatic argument fails in this case. Heterotic S-duality corresponds, in type IIA variables, to a double T-duality on the two-torus, sending the torus area A to $(\alpha')^2/A$. The adiabatic argument holds in the limit where the T^2 base is large, hence is not compatible with this duality transformation.

The FHSV Model Revisited

The lattice $\Gamma_{5,21}$ is given by

$$\Gamma_{5,21} \cong E_8 \oplus E_8 \oplus U \oplus U \oplus U \oplus U \oplus U \,.$$

Consider then the automorphism γ given by interchanging two $E_8 \oplus U$ sublattices and acting as -1 on the remaining sublattice $U \oplus U \oplus U$. This twist $\gamma \in O(\Gamma_{5,21})$ can be associated with the first factor in (III.5.3) and combined with a shift $t \in U(1)^4$ from the second factor in (III.5.3), with shift vector $\delta = (\alpha, \beta, \lambda, \kappa)$.

The heterotic string dual of the FHSV model discussed in subsection III.4.3 is of precisely this form. With the natural choice of split in which the the sub-lattice $\Gamma_{5,21}$ is associated with the heterotic string compactified on T^5 and the sub-lattice $\Gamma_{1,1}$ with a further circle compactification, the shift is $\delta \cdot \Pi = \alpha k + \beta w + \lambda q + \kappa z$. Perturbative consistency required both α, β to be non-zero with $\alpha\beta = 1/4$ [55].

In the FHSV model, γ is not taken to act on $K3 \times T^2$ in the way we have referred to as 'natural'. In choosing the sub-lattice $\Gamma_{5,21} \oplus \Gamma_{1,1} \subset \Gamma_{6,22}$, we take the $\Gamma_{1,1}$ part of the charge lattice to be the one corresponding to D0-brane charge and D4-brane charge (for D4-branes wrapping K3). Then γ acts on K3 through the Enriques involution and on T^2 as a reflection. For the model of [55, 94], the shift was taken to be

$$\delta \cdot \Pi = \alpha Z_0 + \beta Z_4 \,,$$

where Z_0 is the D0-brane charge and Z_4 the charges of D4-branes wrapping K3, giving a phase rotation to any given state depending on its D0 and D4 charges. The general heterotic shift $\delta \cdot \Pi = \alpha k + \beta w + \lambda q + \kappa z$ would correspond to extending the FHSV construction must be extended by taking λ, κ non-zero, giving

$$\delta \cdot \Pi = \alpha Z_0 + \beta Z_4 + \kappa Z_2 + \lambda Z_6$$

where Z_2 is the charge for D2-branes wrapping T^2 and Z_6 is the charge for D6-branes wrapping $K3 \times T^2$. Perturbative consistency of the FHSV construction places no constraint on the four parameters.

However, we can instead make the following choice, giving a type IIA dual of the heterotic FHSV model which looks different from the original Enriques Calabi-Yau type IIA compactification. In choosing the sub-lattice $\Gamma_{5,21} \oplus \Gamma_{1,1} \subset \Gamma_{6,22}$, we now take $\Gamma_{5,21}$ to be the charge lattice for the IIA string on $K3 \times S^1$, so that γ acts as an involution of $K3 \times S^1$, with a fixed point locus, and $\Gamma_{1,1}$ is associated with a further circle reduction. Then the shift is

$$\delta \cdot \Pi = \alpha \hat{k} + \beta \hat{z} + \kappa \hat{w} + \lambda \hat{q}$$

where $\hat{k}$ is the IIA momentum, $\hat{z}$ is the NS5-brane charge. $\hat{w}$ is the IIA string winding number and $\hat{q}$ is the KK monopole charge. The perturbative charges are $\hat{k}, \hat{w}$.

In this case, the transformation given by this twist and shift is not quite a symmetry of the IIA string on $K3 \times T^2$. The twist involves a reflection $y \to -y$ on the circle in $K3 \times S^1$ and this must be combined with a world-sheet parity transformation Ω to give a symmetry. We then have an orientifold of the IIA string on $K3 \times T^2$ by Ω combined with the shift and twist described above. For the shift $\delta \cdot \Pi = \alpha \hat{k}$ this is an orientifold analysed in [49] as a dual to the FHSV model. We can consider generalising this by extending the shift to (III.5.3). Perturbative consistency of the IIA theory then leads to $\kappa = 0$ as before. The heterotic dual gives $\alpha\beta = 1/4$. As noted in [49], the adiabatic argument supports the duality between this orientifold and the heterotic dual, but does not apply to the duality between the FHSV model and its heterotic version.

This type IIA orientifold is non-geometric, following the analysis of section III.1.2 ; the action on the K3 CFT is in $O(\Gamma_{4,20})$ but not in $O(\Gamma_{3,19})$, and the shift corresponding to the second circle has both momentum and winding components. However through heterotic/type IIA duality it is expected to be non-perturbatively equivalent to type IIA compactified on the Enriques Calabi-Yau threefold.

III.5.4 Non-Geometric Constructions

The general class of construction we have been discussing consists of a quotient of a string theory background by a twist γ of order p in a duality group $O(\Gamma_{n,n+16})$ for $n = 4$ or $n = 5$ together with a shift t. From the discussion in section 2, when t is a simple shift $t : y \mapsto y + 2\pi/p$ of a circle coordinate y, this can be seen as a special point in the moduli space of a duality twisted reduction, with the dependence of all fields on y given by a continuous duality transformation $g(y) \in O(n, n+16)$ with monodromy γ. If the monodromy transformation acts geometrically, this constructs a bundle over a circle with fibre T^4 or T^5 or K3 or $K3 \times S^1$. For example, starting from type IIA compactified on $K3 \times S^1$ with $\gamma \in O(\Gamma_{3,19}) \subset O(\Gamma_{4,20})$ acting as a K3 diffeomorphism, the duality twisted reduction can be understood as a geometric compactification of the type IIA string on a K3 bundle over S^1. More generally, the result is non-geometric. If γ involves T-duality transformations, we have a T-fold and if it includes mirror transformations, we have a mirror-fold.

Our heterotic construction involved a shift vector $\delta = (\alpha, \beta)$ so that the shift is generated by

$$2\pi i\delta \cdot \Pi = \alpha k + \beta w$$

with $\alpha\beta = 1/p^2$. As we have seen in section III.4, this can be thought of as acting as a phase rotation on a state with momentum k and heterotic winding number w, or as giving a shift on the 2-dimensional doubled circle with coordinates $y, \tilde{y}$ with $y \to y + 2\pi\alpha$, $\tilde{y} \to \tilde{y} + 2\pi\beta$. The theory can be formulated as a double field theory with fields depending on both y and $\tilde{y}$. Then this is a special point in the moduli space of a duality twisted reduction in which the dependence of all fields on $y, \tilde{y}$ is given by a continuous duality transformation $g(y, \tilde{y}) \in O(n, n + 16)$ with monodromy γ :

$$g(y, \tilde{y})^{-1} g(y + 2\pi\alpha, \tilde{y} + 2\pi\beta) = \gamma$$

(This is a special case of a more general construction in which there could be different monodromies in the y and $\tilde{y}$ directions.) For geometric monodromy in $GL(n, \mathbb{Z})$ acting as a diffeomorphism of T^n, this constructs a T^n bundle over the doubled circle, while for a T-duality monodromy in $O(\Gamma_{n,n})$ this constructs a bundle of a 2n-dimensional doubled n-torus over the doubled circle, which gives the geometric realisation of a T-fold in the doubled formalism [61]. For a general monodromy in $O(\Gamma_{n,n+16})$, this is a bundle with fibre the heterotic doubled torus $T^{n,n+16}$ of dimension $2n + 16$ over the 2-dimensional doubled circle, which can be regarded as a configuration for heterotic double field theory.

Our general construction involved further charges Q_I, so that the shift vector was of the form $\delta = (\alpha, \beta, \lambda^I)$

$$2\pi i\delta \cdot \Pi = \alpha k + \beta w + \lambda^I Q_I$$

These too can be geometrised by going to an extended field theory with further coordinates u^I on which the charges Q_I act as translations :

$$Q_I = -i\frac{\partial}{\partial u^I}$$

Then in the extended field theory, the coordinates that the fields depend on include $y, \tilde{y}, u^I$ and the shift t acts as a translation on $y, \tilde{y}, u^I$, resulting in a generalised bundle over a base space (typically a torus) with coordinates $y, \tilde{y}, u^I$.

III.6 Summary

In this chapter, we have proposed a four-dimensional $\mathcal{N} = 2$ non-perturbative duality relating non-geometric Calabi-Yau compactifications of the type IIA superstring to T-fold compactifications of the heterotic superstring and have shown that this duality follows from the adiabatic argument. The non-geometric type II backgrounds were constructed in [7] as $K3$ fibrations over T^2 with monodromy twists associated with the action of *mirrored K3 automorphisms* on the $K3$ CFT. The K3 automorphisms are realised in the heterotic string as element of the T-duality group, and the heterotic duals are T^4 fibrations over T^2 with T-duality monodromy twists. At points in the moduli space which are fixed under the action of the monodromy automorphisms, the construction reduces to an asymmetric orbifold on the heterotic side and to an asymmetric Gepner model in type IIA. At these fixed points, there is no enhanced gauge symmetry but there is enhanced discrete symmetry.

These models preserve $\mathcal{N} = 2$ supersymmetry in four dimensions. The automorphism acts on the lattice $\Gamma_{4,20}$ by an isometry in $O(\Gamma_{4,20})$ that leaves no sublattice of $\Gamma_{4,20}$ invariant. For the heterotic string on T^4, all of the four left-moving and twenty right-moving chiral bosons transform. On the type II side of the duality, the D-brane charge lattice is $\Gamma_{4,20}$ and the fact that no sublattice is left invariant by the twist means that all BPS D-brane states are projected out by the orbifold. This is consistent with the fact that there are no Ramond-Ramond ground states in these theories, since all space-time supersymmetry comes from the left-movers.

The naive heterotic dual of the type IIA construction is not modular invariant. We found a modification of the heterotic construction that is modular invariant, and this modification led in turn to a non-perturbative modification of the type IIA model. A similar story applies to the FHSV model. For the type IIA string, the modification can be viewed as necessary for non-perturbative consistency. Although we do not have a complete non-perturbative formulation, it seems that a necessary condition for the non-perturbative consistency of a model should be that the theory is modular invariant in all possible duality frames, and in any given frame this can require non-perturbative corrections, as we have seen. Our models are perturbatively consistent in the IIA, IIB and heterotic duality frames. Acting with a duality transformation then takes us to a new perturbative theory (which can also be thought of as choosing a different modulus of the original theory as a coupling constant) and we again require consistency in this new perturbative theory.

Let us explore this further. It is believed there is a non-perturbatively consistent string solution that can be treated as a perturbation theory in terms of the IIA coupling constant, the IIB coupling constant or the heterotic coupling constant. The perturbative heterotic theory is the heterotic string compactified on T^6, while the perturbative IIA (IIB) theory is the IIA (IIB) string compactified on $K3 \times T^2$. The theory is believed to have an exact non-perturbative symmetry (III.5.3) and we are interested in taking quotients of the theory by a $\mathbb{Z}_p$ subgroup of this. The key question is which $\mathbb{Z}_p$ subgroups lead to consistent theories. We have seen that different restrictions arise from requiring perturbative consistency as a IIA, IIB or heterotic theory. Acting with a symmetry in (III.5.3) maps the $\mathbb{Z}_p$ subgroup to a conjugate $\mathbb{Z}_p$ subgroup embedded differently in the symmetry group and gives a new quotient. The new quotient will not in general be dual to the original one, but in an important class of cases, such as the ones studied here in which the adiabatic argument can be applied, this gives a new dual of the original construction. Perturbative consistency of each such dual theory gives further constraints. In this way, we find a set of necessary conditions for the consistency of the quotient. Knowing whether these are sufficient would require an understanding of the non-perturbative theory, but these conditions give us important information about the non-perturbative theory that it would be interesting to investigate further.

The $\mathbb{Z}_p$ symmetries we have been quotienting by are generated by a transformation (t, γ) consisting of a twist $\gamma \in O(\Gamma_{5,21})$ and a shift $t \in U(1)^4$ (or a twist $\gamma \in O(\Gamma_{4,20})$ and a shift $t \in U(1)^8$). The adiabatic argument led us to use the same twist in each duality frame, but we found different consistency conditions on the shift in different duality frames. In our original IIA construction, the shift was a simple order-p shift of a circle coordinate $y \mapsto y + 2\pi/p$ and this was sufficient for IIA modular invariance. For the heterotic dual, heterotic modular invariance required also shifting the T-dual coordinate $\tilde{y} \mapsto \tilde{y} + 2\pi/p$, or, equivalently, the action of t on a state with momentum k and winding w on the circle was to multiply by a phase $\exp(2\pi i(k+w)/p)$. Transforming back to the IIA theory, the heterotic winding number w is mapped to the NS5-brane wrapping number, and so action of t on the IIA string involves a phase depending on the NS5-brane charge, giving a non-perturbative modification of the theory. The general picture involves a phase depending on four charges for a shift $t \in U(1)^4$ or eight charges for a shift $t \in U(1)^8$, and acting with a duality transformation can change which charges they are. For example, the FHSV construction involved a phase depending on the D0- and D4-brane charges, while the dual we found had a phase depending on the type IIA momentum and NS5-brane charge.

We now return to Harvey and Moore's question. There are two classes of $\mathcal{N}=2$ heterotic toroidal orbifolds with a known type II dual : quotients by symmetries that preserve a D-brane charge lattice, corresponding to IIA on Calabi-Yau three-folds, and quotients that do not preserve any charge lattice, corresponding to non-geometric compactifications based on mirrored automorphisms. In general, an orbifold of the heterotic string on T^{4+n} by a symmetry G is mapped to an orbifold of the type IIA string on $K3 \times T^n$ by the dual G which now acts on the type IIA string ; this will require that the orbifold is non-perturbatively consistent, so that in particular it is modular invariant in both the heterotic and type IIA duality frames. Consider for example a general $\mathbb{Z}_p$ orbifold of the heterotic string on $T^4 \times S^1$ by (γ, t), where $\gamma \in O(\Gamma_{4,20})$ acts as a heterotic T-duality and the shift gives a phase depending on the momentum and the heterotic winding number on the S^1. This is then mapped to an orbifold of the type IIA string on $K3 \times S^1$ by the transformation (γ, t) in which γ acts as a K3 automorphism and t gives a phase depending on the momentum on the S^1 and the NS5-brane charge for NS5-branes wrapping $K3 \times S^1$. In some cases the type IIA dual is a CY compactification, but in general it will lead to a non-geometric construction. It will be interesting to explore this duality further, for example for the models of [80, 77, 82].

In the next chapter, we turn to a problem we have left untreated so far which is the determination of the moduli space of the theory. After a brief review of essential tools regarding in particular special geometry, we will derive the scalar moduli space partially from the type IIA dual of the theory, completing the results of [7], and entirely from the heterotic one.

Chapitre IV

Special geometry

The following part is devoted to derive the explicit form - or at least to give a general method to do so - for the moduli space of the theory defined in III.1.2. This is a rather natural thing to do as moduli spaces are both phenomenologically relevant and useful in testing duality conjectures, since two dual models should of course lead to the same conclusions. At the moment, no such test may be done though because the exact form of the full moduli space - and in particular the non-perturbative contributions thereto - has not been computed.

Let us start by recalling an important fact about the moduli space of four-dimensional $\mathcal{N} = 2$-supersymmetric theories. We will refer to the aforementioned moduli space as $\mathcal{M}$. When parallel transported along a path in $\mathcal{M}$, the supercharges generically transform and form therefore a priori non-trivial representations of the holonomy group of $\mathcal{M}$. One may actually show that requiring that the supersymmetry algebra is preserved under such transformations impose that the holonomy group may be factorised with the R-symmetry group as a factor [95]. So far, what we have said applies to any supersymmetric theory ; focusing now on four-dimensional theories with $\mathcal{N} = 2$ supersymmetries, the R-symmetry group is given by $U(2) \cong U(1) \times SU(2)$. It turns out that two kinds of $\mathcal{N} = 2$ supermultiplets contain massless scalars : vector multiplets and hypermultiplets, which transform trivially under $SU(2)$ and $U(1)$ respectively. In this case, a mathematical result proved by de Rham shows that $\mathcal{M}$ must factorise locally as [1]

$$\mathcal{M} = \mathcal{M}_{\mathrm{V}} \times \mathcal{M}_{\mathrm{H}}.$$

where $\mathcal{M}_{\mathrm{V}}$ and $\mathcal{M}_{\mathrm{H}}$ are spanned by scalars living in vector multiplets and hypermultiplets respectively. Moreover, in can be shown that $\mathcal{M}_{\mathrm{V}}$ is a *special Kähler manifold* and that $\mathcal{M}_{\mathrm{H}}$ is *quaternionic Kähler* (see *e.g.* [50]).

The hypermultiplets moduli space has been partially understood in [7] from the type IIA perspective and will be derived from the heterotic side in V.3. In order to study the vector multiplets moduli space, it will be convenient to introduce in the next section a few notions about the underlying effective field theory and its (special) geometry.

1. The theorem we have used here may be found together with a proof in [96], theorem 5.4. Even though this theorem is actually only valid for simply-connected manifolds, as far as we are concerned it would be sufficient to consider the universal cover of $\mathcal{M}$ in the case it does not meet this condition.

IV.1 Special Kähler geometry

In this section and in the followings, we give a brief review of chosen topics on $\mathcal{N}=2$ supergravity in four dimensions. The purpose here being to introduce some tools necessary to fully grasp the analysis of section V.2, we are not looking to be in any way exhaustive in the treatment made and refer the reader to the already existing and by far more complete reviews available in the literature (see [50] for instance).

A convenient way to derive supergravity actions is called *superconformal tensor calculus*. Supergravity theories are, by definition, theories with local super-Poincaré invariance - that is invariance under supersymmetry and Poincaré transformations. One may exploit the fact that the super-Poincaré group is a subgroup of the superconformal group as it is easier to write down actions with local invariance under the former. Supergravity theories are then recovered after having gauge-fixed the superfluous generators. This procedure has first been introduced in the context of $\mathcal{N}=1$ supergravity in [97, 98, 99]. Since an explicit computation may be somewhat lengthy, we will not go into much details regarding these sorts of computations and state the results we need directly.

From now on, we focus on a $\mathcal{N}=2$ supergravity theory with $n+1$ abelian vector superfields, one of which containing the graviphoton. We denote the scalars in these vector multiplets as X^I, $I=0,...,n$. Superconformal tensor calculus then shows that the action may be encoded into a holomorphic function $F(X)$ of the scalars. In addition, F must be homogeneous of degree 2 [50], meaning that

$$F(\lambda X)=\lambda^2 F(X)\ ,\ \forall\lambda\in\mathbb{C}^*. \tag{IV.1}$$

In particular, deriving (IV.1) with respect to λ and setting $\lambda=1$ leads to the useful identities

$$F_I X^I=2F, \qquad F_{IJ}X^J=F_I, \qquad F_{IJK}X^I=0 \tag{IV.2}$$

where we use the standard convention that $F_{I_1...I_r}$ stands for the r-th derivative of F with respect to $X^{I_1},...,X^{I_r}$. The function F is known as the *prepotential* of the theory and contains all the information about the action as we will see in the following for the bosonic sector of the theory. Superconformal tensor calculus also shows that gauge fixing the dilatations in such a way that the kinetic terms for the scalars and for the graviton decouples forces the physical scalar to span a n-dimensional hypersurface defined by the condition [100]

$$\mathrm{Im}\left(\overline{X}^I F_I\right)=\text{constant}.$$

The embedding of the n-dimensional hypersurface may be done by expressing the scalars as

$$X^I=yZ^I(z^\alpha)$$

with z^α, $\alpha=1,...,n$, coordinates on the n-dimensional surface. One may then show that the manifold spanned by the physical scalar fields is Kähler whose Kähler potential is given by

$$\mathcal{K}(z,\overline{z})=-\log\left(i\overline{Z}^I(\overline{z})\mathcal{F}_I(z)-iZ^I(z)\overline{\mathcal{F}}_I(\overline{z})\right).$$

where we have used the homogeneity of the prepotential F to define

$$F_I(X^I)=F_I(yZ^I(z)):=y^2\mathcal{F}_I(Z^I(z)).$$

We emphasize that with this definition, $\mathcal{F}_I(Z)$ may also be viewed as the derivative of $\mathcal{F}(Z)$ with respect to Z^I with $\mathcal{F}(Z)$ being formally the same function as $F(X)$. Finally, we note that a convenient choice for the functions Z^I is to set

$$Z^0(z) = 1\,, \qquad Z^\alpha(z) = z^\alpha\,,\ \alpha = 1, ..., n. \tag{IV.3}$$

This set of coordinates is called the set of *special coordinates* and may equivalently be written as

$$z^\alpha = \frac{X^\alpha}{X^0}.$$

IV.2 Gauge sector

Having defined all the above notations, we now turn to analysing the gauge field sector of the theory. We recall that in four-dimensional Minkowski space, the Hodge dual squares to minus identity on the field strengths F^I, $I = 0, ..., n$. Therefore, the associated eigenvalues are $\pm i$ and we define

$$\mathsf{F}^{\pm I} := \frac{\mathsf{F}^I \mp i \star \mathsf{F}^I}{2}$$

satisfying $\star\mathsf{F}^{\pm I} = \pm i\mathsf{F}^{\pm I}$, $I = 0, ..., n$. The Lagrangian for the $(n+1)$ abelian gauge fields reads [50]

$$\mathcal{L}_{\text{gauge}} = \frac{1}{2}\,\text{Im}\left(\mathcal{N}_{IJ}\mathsf{F}^{+I} \wedge \star\mathsf{F}^{+J}\right) \tag{IV.4}$$

with $\mathcal{N}_{IJ}$ the matrix of complexified gauge couplings, whose imaginary and real part gives the effective gauge coupling and the theta angles respectively. $\mathcal{N}$ is related to the prepotential F by the relation

$$\mathcal{N}_{IJ} := \overline{F}_{IJ} + 2i\frac{\text{Im}(F_{IK})X^K\,\text{Im}(F_{JL})X^L}{\text{Im}(F_{KL})X^K X^L}. \tag{IV.5}$$

It turns out to be useful to introduce another field G_I, which may be viewed as some sort of magnetic counterpart to the field strength F^I, using $\mathcal{N}_{IJ}$; we then define

$$\mathsf{G}_I := \mathcal{N}_{IJ}\mathsf{F}^J \tag{IV.6}$$

so that the Lagrangian (IV.4) may be rewritten as

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4}\,\text{Im}\left(\mathsf{F}^{+I} \wedge \star\mathsf{G}_I^+\right), \tag{IV.7}$$

$\mathsf{G}_I^\pm$ being defined similarly as $\mathsf{F}^{\pm I}$. From (IV.7), one may compute the gauge fields equations of motion which read, together with the Bianchi identities

$$\begin{aligned}\text{Bianchi identities :} \quad & \mathrm{d}\,\text{Re}\left(\mathsf{F}^{+I}\right) = 0,\\ \text{Equations of motion :} \quad & \mathrm{d}\,\text{Re}\left(\mathsf{G}_I^+\right) = 0.\end{aligned}$$

Having formulated the theory in terms of F^{+I} and G^+_I shows that the set of Bianchi identities and equations of motion is invariant under linear transformations of F^{+I} and G^+_I, that is under

$$\begin{aligned} \hat{\mathsf{F}}^{+I} &= A^I{}_J \mathsf{F}^{+J} + B^{IJ}\mathsf{G}^+_J, \\ \hat{\mathsf{G}}^{+I} &= C_{IJ}\mathsf{F}^{+J} + D_I{}^J \mathsf{G}^+_J \end{aligned} \tag{IV.8}$$

with A, B, C and D real matrices only constrained so far by the requirement that (IV.8) must be invertible. However, in order to recover equivalent theories, one must require that $\hat{\mathsf{F}}^{+I}$ and $\hat{\mathsf{G}}^+_I$ form dual fields in the sense of (IV.6), that is that there must exist a symmetric matrix $\hat{\mathcal{N}}$ such that $\hat{\mathsf{G}}^+_I = \hat{\mathcal{N}}_{IJ}\hat{\mathsf{F}}^{+J}$. From (IV.8), one deduces the matrix $\mathcal{N}$ transforms as

$$\hat{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}. \tag{IV.9}$$

Therefore, imposing the symmetricity of $\hat{\mathcal{N}}$ implies, up to an overall multiplication of F^{+I} and G^+_I by a real constant, that

$$\begin{pmatrix} \hat{\mathsf{F}}^{+I} \\ \hat{\mathsf{G}}^+_I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathsf{F}^{+I} \\ \mathsf{G}^+_I \end{pmatrix} =: \mathcal{S} \begin{pmatrix} \mathsf{F}^{+I} \\ \mathsf{G}^+_I \end{pmatrix}$$

with $\mathcal{S}$ a $\mathrm{Sp}(2(n+1),\mathbb{R})$ matrix, that is such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix}. \tag{IV.10}$$

The transformations (IV.8) do not preserve the Lagrangian (IV.7) in general and should therefore be viewed as dualities rather than as symmetries of the action. The transformation of the gauge kinetic term indeed shows that (IV.8) defines a symmetry of the Lagrangian only when $B = C = 0$ (implying $D = A^{-T}$ in accordance with (IV.10)). However, (IV.9) shows that under a transformation with only $B = 0$ the matrix of complexified gauge couplings transforms as

$$\hat{\mathcal{N}} = A^{-T}(\mathcal{N} + \Lambda)A^{-1}$$

where $D = A^{-T}$ and where we defined $C := A^{-T}\Lambda$ with $\Lambda = \Lambda^T$ a real symmetric matrix in virtue of (IV.10). This transformation therefore induces a shift in the theta angles ; for transformations in a discrete subgroup of $\mathrm{Sp}(2(n+1),\mathbb{R})$, the transformation with $B = 0$ and C non-vanishing therefore amounts to an integral shift in the theta angles which does not modify the path integral and may therefore be seen as a symmetry of the theory. We will see later how such shifts play a role in our analysis.

The transformations (IV.8) also induce an action on X^I and F_I. First, one may notice using (IV.2) and (IV.5) that X^I and F_I are dual in the same sense as F^{+I} and G^+_I, that is that

$$F_I = \mathcal{N}_{IJ}X^I. \tag{IV.11}$$

Having (IV.11) in mind, it is straightforward to see that the transformation

$$\hat{X}^I = A^I{}_J X^J + B^{IJ} F_J, \qquad \text{(IV.12)}$$
$$\hat{F}_I = C_{IJ} X^J + D_I{}^J F_J$$

is compatible with the transformation (IV.9) of the $\mathcal{N}$ matrix. It turns out than ax explicit computation using the $\mathcal{N} = 2$ supersymmetry shows that X^I and F_I transform as in (IV.12) indeed [11]. The dual theory may however not admit a prepotential itself, that is that there may be no function $\hat{F}(\hat{X})$ satisfying

$$\hat{F}_I = \frac{\partial \hat{F}(\hat{X})}{\partial \hat{X}^I}.$$

Indeed, a prepotential may exist (at least locally) if and only if the first line of (IV.12) is invertible so that the Jacobian $\partial \hat{X}^I / \partial X^J$ is as well. In this sense, the symplectic vector $\left(X^I, F_I\right)$ may be seen as more fundamental than the prepotential itself $F(X)$. In the next section, we apply the tools developed here in order to draw conclusions about the loop corrections to the vector multiplets modular space of a large class of theories.

IV.3 Compactifications of heterotic and type IIA superstring theories

IV.3.1 Tree-level analysis

As we are ultimately interested in compactifications of the heterotic and type IIA superstrings dual to each other, we now turn to models compatible with such theories. From the heterotic point of view, the dilaton must sit in a vector multiplet and is therefore expected to be a coordinate of $\mathcal{M}_{\text{V}}$. If all other moduli can be understood from a world-sheet perspective, then in the perturbative regime the moduli space described by the conformal field theory point of view should become exact ; in other words, we expect that in this limit, $\mathcal{M}_{\text{V}}$ factorises as $\mathcal{M}_{\text{CFT}} \times \mathcal{M}_{\text{DILATON}}$. It was shown in [101] that a special Kähler manifold could only factorise if it was locally of the form

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n-1)}{\mathrm{SO}(2)\times \mathrm{SO}(n-1)}. \qquad \text{(IV.13)}$$

In the classical limit, that is at tree-level, the potential of the $\mathcal{N} = 2$ supergravity corresponding to the effective field theory of such a compactification therefore identifies to the prepotential corresponding to (IV.13), which may be written as

$$F(X) := \frac{X^1}{X^0} X^a \eta_{ab} X^b \qquad \text{(IV.14)}$$

with $a = 2, ..., n$ and η_{ab} a constant metric of signature $(+, -^{n-2})$. Since we have not used any assumption about the heterotic compactification of interest other than the amount of preserved space-time supersymmetries, the tree-level prepotential (IV.14) is actually universal for this class of models.

It turns out that a different set of periods $\left(\hat{X}^I, \hat{F}_I\right)$ is more convenient for our purposes ; it is defined by

$$\hat{X}^1 = F_1 \,, \hat{X}^i = X^i \,, i = 0, 2, ..., n$$
$$\hat{F}_1 = -X^1 \,, \hat{F}_i = F_i \,, i = 0, 2, ..., n. \tag{IV.15}$$

Equation (IV.10), ensures that such a transformation leads to a physically equivalent theory. However, this formulation does not admit a prepotential as we anticipated at the end of section IV.2 because in this case the relation $\hat{X}^I = \hat{X}^I(X)$ is not invertible. Replacing F_1 by its explicit value in (IV.15), one gets

$$\hat{X}^1 = F_1 = \frac{X^a \eta_{ab} X^b}{X^0} = \frac{\hat{X}^a \eta_{ab} \hat{X}^b}{\hat{X}^0};$$

the new periods $\hat{X}^I$ therefore satisfy the constraint

$$\hat{X}^I \hat{\eta}_{IJ} \hat{X}^J := -\hat{X}^0 \hat{X}^1 + \hat{X}^a \eta_{ab} \hat{X}^b \tag{IV.16}$$

where the constant metric $\hat{\eta}_{IJ}$ of signature $\left(+^2, -^{n-1}\right)$ is defined by (IV.16), with $I, J = 0, ..., n$. One may furthermore explicitly check that the new periods $\left(\hat{X}^I, \hat{F}_I\right)$ satisfy the relation

$$\hat{F}_I = 2S \hat{\eta}_{IJ} \hat{X}^J \tag{IV.17}$$

where S is one of the variables $\{z^\alpha\}$ (using special coordinates, $S := z^1 = X^1/X^0$). S is actually the variable spanning the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ part of (IV.11) and may as such be identified with the axio-dilaton.

IV.3.2 Quantum corrections

Because of the universality of the tree-level prepotential in heterotic compactifications with $\mathcal{N} = 2$ space-time supersymmetries, any interesting feature really lie in the quantum corrections to the prepotential. In the following section, we will show that perturbative corrections are subjected to stringent constraints which we will be able to exploit in V.2.

Non-renormalisation theorems proven in [102, 103, 104] show all quantum corrections to the prepotential of $\mathcal{N} = 2$ supersymmetric theories come from either one-loop or non-perturbative contributions ; the exact potential of the theory may then be written as

$$F = F^{(0)} + F^{(1)} + F^{\mathrm{np}} \tag{IV.18}$$

with $F^{(0)}$ the tree-level universal prepotential (IV.14) and $F^{(1)}$ and F^{np} the one-loop and non-perturbative contributions respectively. Before moving on, we present the argument of [11] which explains very briefly why no higher-than-one-loop contributions to the prepotential may appear from a string theoretical point of view. In the following, we will not be concerned about non-perturbative corrections and therefore drop the F^{np} term from (IV.18). We start by giving the special coordinates version (of the perturbative part) of (IV.18) by defining

$$F^{(i)}(X) = F^{(i)}(yZ(z)) := \left(X^0\right)^2 f^{(i)}(z) \,, \ i = 0, 1$$

where we have used the definition (IV.3) of the special coordinates, which implies in particular that $y = X^0$. Let us set as before the axio-dilaton field $S = X^1/X^0$; since the dilaton may be seen as a loop-counting parameter in string theory, one expects $f^{(0)}$ to be proportional to S - which is the case from (IV.14) - and $f^{(1)}$ to be independant of S. Similarly, any higher loop contribution would be proportional to a strictly negative power of S. However, in string theory there is a discrete Peccei-Quinn symmetry at all orders in perturbation theories which acts as a shift of the axion, that is as a shift of the real part of S. Under such a shift, the tree-level prepotential is not invariant, but it may be seen as a symplectic transformation (IV.12) with $B = 0$; as we have argued in section IV.2, such transformations amount to an integral shift of the theta angles and do not change the path integral. Concerning the one-loop contribution to the prepotential, it is independent of S and therefore inert under the Peccei-Quinn symmetry. However, since any higher-loop contribution should contain S at a negative power, the only way they do so compatible with the Peccei-Quinn symmetry is that only $\mathrm{Im}(S)$ actually appears with a strictly negative power. Since this conflicts the requirement from supersymmetry that the prepotential must be holomorphic, this shows that no higher-than-one-loop contribution may correct the perturbative prepotential as claimed.

Let us now perform the same transformation as in (IV.15). Because $F^{(1)}$ is a one-loop contribution, it does not depend on the dilaton and therefore on X^1. Consequently,

$$\hat{X}^1 = F_1 = F_1^{(0)}$$

so that the expression of $\hat{X}^1$ in terms of $\left\{X^I, I = 0, ..., n\right\}$ does not change compared to the tree-level case ; in particular, the constraint (IV.16) still holds. However, equation (IV.17) is no longer satisfied. Noticing that $F_1^{(1)} = \partial F^{(1)}/\partial X^1 = 0$, it may nonetheless be conveniently replaced by

$$\hat{F}_I = 2S\hat{\eta}_{IJ}\hat{X}^J + F_I^{(1)}. \tag{IV.19}$$

This gives us a way to extract the one-loop correction to the prepotential $F^{(1)}$ in terms of the periods $(\hat{X}^I, \hat{F}_I)$; indeed, multiplying (IV.19) by $\hat{X}^I$ and summing over I gives

$$\hat{F}_I\hat{X}^I = 2S\hat{X}^I\hat{\eta}_{IJ}\hat{X}^J + F_I^{(1)}\hat{X}^I = 0 + F_I^{(1)}X^I = 2F^{(1)}$$

where the constraint (IV.16) and the homogeneity of $F^{(1)}$ have been used together with the fact that $F_I^{(1)}\hat{X}^I = F_I^{(1)}X^I$, since $\hat{X}^i = X^i$ for $i \neq 1$ and $F_1^{(1)} = 0$. In summary, $F^{(1)}$ may be expressed as

$$F^{(1)} = \frac{1}{2}\hat{F}_I\hat{X}^I. \tag{IV.20}$$

This gives an explicit way to check how $F^{(1)}$ transforms under the duality transformations (IV.12). In particular, let us consider one more time the special case where

$$\begin{aligned} \hat{X}^I &\mapsto A^I{}_J\hat{X}^J \\ \hat{F}_I &\mapsto \left(A^{-T}\right)_I{}^J\Lambda_{KJ}\hat{X}^J + \left(A^{-T}\right)_I{}^J\hat{F}_J \end{aligned}$$

which is the most general case of transformation of the form (IV.12) satisfying $B = 0$ and therefore, as we argued, which may directly be seen as a symmetry of the theory. Then, (IV.12) together with (IV.20) show that

$$F^{(1)} \mapsto F^{(1)} + \frac{1}{2}\Lambda_{IJ}\hat{X}^I\hat{X}^J. \tag{IV.21}$$

In particular, even if $A = \mathbb{1}$ - that is even if $\hat{X}^I$ is mapped to itself - the one-loop correction to the prepotential may undergo a shift. This is perfectly acceptable from an effective field theory point of view since the prepotential is only defined up to transformations leaving physically relevant quantities invariant ; in particular, it may be shown that the transformation (IV.21) only modifies the action by a total derivative. On the other hand, this signals that the function $F^{(1)}$ may have singularities around which closed monodromies may give rise to such shifts.

Singularities in the prepotential may be understood by thinking about how effective field theories are derived. Let us consider a given model which is described by a (UV-complete) theory ; for various reasons, one would be interested in studying instead a simpler theory (for instance, the UV-complete theory may not be known or too complicated). A solution is to consider an effective field theory, meaning a theory which is obtained by considering only low-energy excitations. In practice, this involves integrating out fields which are too massive to be observed at energies below the chosen UV cut-off. Let us now consider the situation we have here : the supergravity limit theory of string theory is obtained by truncating the spectrum in a consistent way, meaning that all massive excitations are considered to be of the order of Planck's mass and as such integrated out [2]. However, the actual mass of the fields generically depends on the moduli and it may happen that some otherwise massive fields become massless at some isolated points in the moduli space, leading for instance to enhanced gauged symmetry. In a small enough neighbourhood of such points, the corresponding fields may no longer be considered as heavy leading into a breakdown of the effective field theory. As a consequence, several effective field theory descriptions may be necessary in order to patch the whole moduli space, generically overlapping close to such problematic points. This explains the singularities of the prepotential, which are simply due to the integration of massless scalar fields.

Without going into further details for now, we conclude by giving the form of (IV.21) in special coordinates. We set

$$S := \frac{X^1}{X^0}\ ,\ T := \frac{X^2}{X^0}\ ,\ U := \frac{X^3}{X^0}\ ,\ \phi^\alpha := \frac{X^\alpha}{X^0}\ \text{for}\ \alpha = 4, ..., n.$$

As before, S is understood as the dilaton, T and U as the Kähler and complex structure of the two-torus respectively and ϕ^α the potential Wilson lines. A generic theory in superstring theory will have a duality group containing transformations of T acting as

$$T \mapsto \frac{aT+b}{cT+d} \tag{IV.22}$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}_T$$

for some discrete duality group $\mathcal{G}_T$. $\mathcal{G}_T$ is generically a congruence subgroup of $SL_2(\mathbb{Z})$, that is a group which admits $\Gamma(N)$ as a subgroup for some integer N with $\Gamma(N)$ the *principal*

2. In practice, consistent truncations may be more subtle than only throwing away all massive fields. For simplicity, we assume here that the described procedure it consistent.

congruence subgroup, that is the group of 2×2 matrices with unit determinant and equal to the identity modulo N. In special coordinates, this gives in particular

$$f^{(1)}(z) \mapsto \frac{f^{(1)}(z) + \Xi(z)}{J(\mathcal{S}, z)}, \tag{IV.23}$$

$\mathcal{S}$ being the symplectic transformation of (IV.10), with

$$\Xi(z) := \frac{1}{2}\Lambda_{IJ}\frac{\hat{X}^I}{\hat{X}^0}\frac{\hat{X}^J}{\hat{X}^0},$$
$$J(S, z) := A^0{}_I\frac{\hat{X}^I}{\hat{X}^0}A^0{}_J\frac{\hat{X}^J}{\hat{X}^0}.$$

Equation (IV.23) will turn out to be extremely useful in section V.2 where we use it to derive the expression of $f^{(1)}$ as a function of the moduli.

A simple and familiar example is the following : let us assume that a choice of periods (X^I, F_I) has been made so that the tree-level prepotential reads

$$F^{(0)} = \frac{X^1}{X^0}\left(X^2X^3 - X^\alpha\delta_{\alpha\beta}X^\beta\right)$$

with $\alpha = 4, ..., n$ (this may always be achieved by starting from (IV.14) and applying suitable transformations (IV.12)). Then, the transformation parametrised in the $(\hat{X}^I, \hat{F}_I)$ basis by

$$A = \left(\begin{array}{cccc|ccc} d & 0 & c & 0 & 0 & \dots & 0 \\ 0 & a & 0 & b & 0 & \dots & 0 \\ b & 0 & a & 0 & 0 & \dots & 0 \\ 0 & c & 0 & d & 0 & \dots & 0 \\ \hline 0 & 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & & \mathbb{1}_{n-3} & \\ 0 & 0 & 0 & 0 & & & \end{array}\right), \tag{IV.24}$$

$B = C = 0$ and $D = A^{-T}$ is a classical symmetry provided that $ad - bc = 1$. It leaves invariant the axio-dilaton field S and may therefore be a symmetry of the theory in the perturbative regime. Its action on T is given by (IV.22) while it maps U and ϕ^α to

$$U \mapsto U - \frac{\phi^\alpha\delta_{\alpha\beta}\phi^\beta}{cT + d},$$
$$\phi^\alpha \mapsto \frac{\phi^\alpha}{cT + d}.$$

The non-trivial transformation of U is a consequence of the trivial action of (IV.24) on the dilaton. According to the analysis from above, under such a transformation $f^{(1)}$ would transform as

$$f^{(1)}(T, U, \phi^\alpha) \mapsto \frac{f^{(1)}(T, U, \phi^\alpha)}{(cT + d)^2}.$$

This gives the usual $(cT + d)^{-2}$ factor we will encounter extensively in the following.

Chapitre V

Moduli spaces of non-geometric type II/heterotic dual pairs

V.1 Introduction

Non-perturbative dualities between $\mathcal{N} = 2$ compactifications to four dimensions play a pivotal role in our understanding of string theory dynamics, see [95] for a review. A classical example is the duality relating heterotic strings compactified on $K3 \times T^2$ to type IIA superstrings compactified on a Calabi–Yau three-fold that is a $K3$ fibration [105, 55] ; by applying the duality fiber-wise, as was suggested in [49], this four-dimensional duality is obtained from a more fundamental six-dimensional duality between heterotic on T^4 and type IIA on a $K3$ surface [84]. More general $\mathcal{N} = 2$ dualities can be obtained by considering, on the heterotic side, quotients of $K3 \times T^2$ by supersymmetry-preserving discrete symmetries (that were originally considered in type II [106, 107, 108] as duals of CHL compactifications [109]), see [110] for a recent work.

Deriving the quantum moduli space of $\mathcal{N} = 2$ four-dimensional compactifications is an essential quantitative test of non-perturbative dualities (see *e.g.* [105, 111, 112, 113]), as quantum corrections on one side are typically mapped to classical expressions on the other side of the duality. As we saw at the beginning of IV, by supersymmetry, the moduli space of an $\mathcal{N} = 2$ compactification splits, at least locally, into the vector multiplets moduli space and the hypermultiplets moduli space :

$$\mathcal{M} \cong \mathcal{M}_{\text{V}} \times \mathcal{M}_{\text{H}} \,, \tag{V.1}$$

where the first factor is a special Kähler manifold and the second factor a quaternionic Kähler manifold. Depending on the duality frame used, each factor may receive α' corrections (if the corresponding factor contains Kähler moduli) as well as g_s corrections (if the dilaton belongs to one of the corresponding multiplets).

For standard dualities between type IIA compactified on Calabi-Yau 3-folds and heterotic on $K3 \times T^2$, the quantum vector multiplets moduli space has been studied in great detail as, on the type II side, mirror symmetry allows to solve the problem exactly. The hypermultiplets moduli space is much less understood, as it receives worldsheet instanton corrections on the heterotic side and D-brane and NS5-brane instanton corrections on the type IIA side. [1]

1. See [114] for a review about corrections to the hypermultiplets moduli space and more specifically [115] for recent advances in understanding heterotic/type II duality in the hypermultiplets sector.

For the dual pairs described in III, the situation is quite different on the type IIA side. While for IIA compactifications on Calabi–Yau threefolds, such as in the FHSV model [55], the two space-time supersymmetries correspond respectively to a left-moving and to a right-moving worldsheet vertex operator, in the present case both space-time supersymmetries correspond to left-moving vertex operators. As a result, the action of these charges on the axio-dilaton generates a vector multiplet, rather than a hypermultiplet. Therefore, $\mathcal{M}_{\rm V}$ may receive one string-loop corrections as well as non-perturbative corrections on both sides of the duality and there is no duality frame where the problem can be solved classically. We will argue below that the one-loop corrections to the prepotential vanish on the type IIA side, and compute the corrections to the prepotential – hence to the metric on the vector multiplets moduli space – on the heterotic side, extending the method used in [10, 11].

By contrast, the hypermultiplet moduli space $\mathcal{M}_{\rm H}$ is tree-level exact in both heterotic and type II duality frames. Deriving this moduli space on the type IIA side using algebraic geometry tools is not easy as mirrored automorphisms lack, by definition, a geometrical description, see [7] for a discussion. Here, using the heterotic description as an asymmetric toroidal orbifold, we are able to derive the exact hypermultiplets moduli space (both in α' and g_s).

This chapter is organized as follows. In section V.2 we discuss the general structure of the vector multiplet moduli space as well as the duality groups appearing in the perturbative limits ; an explicit computation of the one-loop corrections to the prepotential for some of the models is presented in subsection V.2.5. In section V.3 we provide a description of the exact hypermultiplets moduli space.

V.2 One-loop corrections to the prepotential

In this section we will analyse the space $\mathcal{M}_{\rm V}$ spanned by the scalars in the vector multiplets (see equation (V.1)). As it has long been known, $\mathcal{N}=2$ supersymmetry imposes that $\mathcal{M}_{\rm V}$ is a special Kähler manifold [116], whose geometry is completely encoded in a holomorphic function f of the moduli, the prepotential, from which one can derive a Kähler metric on $\mathcal{M}_{\rm V}$.

As the axio-dilaton sits in a vector multiplet in both type IIA and heterotic perspectives, the prepotential (and consequently the Kähler metric) generically receives corrections from quantum contributions in both cases. It is well known then that, due to the Peccei-Quinn symmetry of the axio-dilaton vector multiplet, any perturbative correction to f higher than one-loop must vanish as a consequence of $\mathcal{N}=2$ supersymmetry [117, 113, 10, 11, 118] :

$$f = f^{(0)} + f^{(1)} + f^{\rm np}, \tag{V.2}$$

$f^{(0)}$, $f^{(1)}$ and $f^{\rm np}$ being the tree-level, the one-loop and the non-perturbative contributions to the prepotential respectively.

The tree-level contribution to the vector multiplets moduli space is rather easy to understand, as there are generically only three vector multiplets for all values of p, and is similar to the moduli space of more ordinary $\mathcal{N}=2$ compactifications like heterotic strings on $K3\times T^2$ without Wilson lines, see *e.g.* [11, 10]. One of them contains the axio-dilaton and will be named S in the heterotic description :

$$S = a + ie^{-\phi}\,,$$

where the scalar a is the four-dimensional dual of the NS-NS two-form. The other two are associated with the moduli of the two-torus. The moduli parametrise a "Teichmüller space" $\mathcal{T}_V$ which may be expressed as the direct product

$$\mathcal{T}_{\rm V} = \left(SL(2)/U(1)\right)_S \times \left(O(2,2)/O(2)\times O(2)\right)_{T^2},$$

where the $SL(2)/U(1)$ and the $O(2,2)/\left[O(2)\times O(2)\right]$ factors correspond to the axio-dilaton and the two-torus moduli spaces respectively. The latter may be further split to give

$$\mathcal{T}_{\rm V} = \left(SL(2)/U(1)\right)_S \times \left(SL(2)/U(1)\right)_T \times \left(SL(2)/U(1)\right)_U .$$

with the second $SL(2)/U(1)$ factor (resp. the third) corresponding in the heterotic description to the complexified Kähler (resp. complex structure) moduli space of the 2-torus, respectively T and U. This is the Teichmüller space of the STU model which has already been extensively studied in the literature. The actual classical moduli space is the quotient of this Teichmüller space by the discrete duality group that will be described in subsection V.2.2, which is a subgroup of the T-duality group (V.9) of two-torus compactifications.

The interesting piece of information accessible to a perturbative study therefore lies in the one-loop correction $f^{(1)}$; as usual with one-loop diagrams in string theory, an explicit computation would involve an integration over the worldsheet two-torus complex structure which turns out to be hard to handle technically.

In particular, as was shown in [8], for the heterotic models at hand, the integrand of the modular integral does not factorise into a product of a Narain lattice and a modular form (neither for $SL(2,\mathbb{Z})$ nor for a congruence subgroup associated with the orbifold) because the shift vector has a non-zero norm. As a consequence, the powerful procedure developed in [119, 120, 121, 122] in order to compute one-loop integrals in string theory, based on an expansion of the modular form into Niebur-Poincaré series, cannot be applied to our models. For similar reasons (non-factorisability of the integrand), the older 'unfolding trick' developed in [123] by Dixon, Kaplunovsky and Louis would also not be helpful here. This can be seen by recalling how the unfolding methods works. In order to evaluate the modular integral :

$$\mathcal{I} := \int_{\mathcal{F}} \mathrm{d}^2\mu\, F(\tau)\Gamma(\tau,\bar\tau),$$

with $\int_{\mathcal{F}} \mathrm{d}^2\mu := \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2}$ the usual integration over the $SL(2,\mathbb{Z})$ fundamental domain $\mathcal{F}$ of the upper-half plane, one introduces a function j_G which may be written as

$$j_G(\tau,\bar\tau) = \sum_{g\in G} \tilde{j}_G(g\cdot\tau, g\cdot\bar\tau)$$

for some subgroup G of $SL(2,\mathbb{Z})$ and for some "seed" $\tilde{j}_G$. The usual strategy is then to write $\mathcal{I}$ as

$$\mathcal{I} = \int_{\mathcal{F}} \mathrm{d}^2\mu\, F(\tau)\Gamma(\tau,\bar\tau)\times\frac{j_G}{j_G} = \int_{\mathcal{F}_G} \mathrm{d}^2\mu\, F(\tau)\Gamma(\tau,\bar\tau)\times\frac{\tilde{j}_G}{j_G}$$

where $\mathcal{F}_G$ is the integration domain obtained by unfolding $\mathcal{F}$ along G [2]. From there, integration may be done if j_G cancels the non-holomorphic part of the integrand of $\mathcal{I}$. This cannot be

2. For example, in the appendix of [123], the group G used is $SL(2,\mathbb{Z})/\langle T\rangle$, with $\langle T\rangle$ the group of $SL(2,\mathbb{Z})$ transformations preserving ∞. As a result, $\mathcal{F}_G$ is the usual strip $|\tau_1|\le 1/2$, $\tau_2>0$ in this case.

done in our case, as Γ contains a sum over the charge lattice of the spacetime two-torus which is not left invariant by any congruence subgroup of level p (with p the order of the orbifold) because the shift vector has non-zero norm, as has been shown in [8]. [3]

Our strategy will therefore be close in essence to the one already used in, *e.g.*, [10, 11], and consists in using modularity in the space-time moduli T and U rather than modularity in the worldsheet modulus τ. The third derivatives of the one-loop prepotential $f^{(1)}(T,U)$ are modular forms in both variables, and their behaviour under the T-duality group of the orbifolded theory, together with the localization of physical singularities related to accidental massless states, gives very stringent constraints on them. Using results from modular functions theory and physical requirements may then be enough to fix the one-loop correction to the prepotential, granting access to all perturbative corrections to the vector multiplets moduli space at once while preserving manifest T-duality covariance. We will show in this section how this strategy works in general and an explicit result for $f^{(1)}$ in the $p=2$ case will be provided.

V.2.1 One-loop correction to the vector multiplet moduli space

It has long been known that the one-loop correction to the prepotential in $\mathcal{N}=2$ theories is related to the new supersymmetric index of [124] ; as shown in [117], the one-loop correction to the Kähler potential may be explicitly written as

$$K^{(1)}(T,U) = \frac{i}{16(2\pi)^3}\int_{\mathcal{F}} \mathrm{d}^2\mu\, \bar{\eta}^{-2}\mathrm{Tr}_{\mathrm{R}}\left(J_0(-1)^{J_0}q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) \tag{V.3}$$

with J_0, L_0 and $\bar{L}_0$ the respective zero-modes of the $U(1)$ R-current and of the Virasoro generators. One can then relate the one-loop prepotential $f^{(1)}$ to the modular integral (V.3), using (see [10]) :

$$\partial_T\partial_{\bar{T}}K^{(1)} = -\frac{i}{8T_2^2}\left(\partial_T + \frac{i}{T_2}\right)\left(\partial_U + \frac{i}{U_2}\right)f^{(1)} + \text{ h.c.} \tag{V.4}$$

where we have decomposed the heterotic prepotential as :

$$f(S,T,U) = STU + f^{(1)}(T,U) + f^{\mathrm{np}}(S,T,U)\,,$$

respectively the tree-level, one-loop and non-perturbative contributions (the latter being exponentially suppressed in the limit $|S|\to\infty$) following equation (V.2).

Under perturbative symetries (*i.e.* T-dualities) the one-loop prepotential $f^{(1)}$ does not transform covariantly in general. For heterotic compactifications with a T^2 factor, it transforms as a modular function of weight $(-2,-2)$ in T and U under $PSL(2,\mathbb{Z})_T\times PSL(2;,\mathbb{Z})_U$ up to order-two polynomials due to monodromies around the singularities of the prepotential due to the appearance of additional massless states [11, 10] ; therefore $f^{(1)}$ may not be expressed in terms of modular forms. In our case the story is similar but, as we will see shortly, the duality group is different.

However, even though the n-th derivative of a modular function is generically not modular, the third derivative of a modular function of weight -2 turns out to always be a genuine

3. In very restricted cases (orthogonal torus with no B-field), one could in principle unfold along the cycle of the space-time two-torus left invariant under the action of the orbifold while canceling the j_G denominator. However, even then the integrand would remain non-holomorphic so that no good way of evaluating the corresponding modular integrals has been found by the authors.

modular function of weight 4 ; therefore, $\partial_T^3 f^{(1)}$ is a modular function of weight $(4,-2)$ in T and U respectively. It turns out that $\partial_T^3 f^{(1)}$ may be directly extracted from (V.4) as [10]

$$\partial_T^3 f^{(1)} = -\frac{16iU_2^2}{T_2^2}\partial_T T_2^2 \partial_T \partial_{\bar{U}} T_2^2 \partial_T \partial_{\bar{T}} K^{(1)}. \tag{V.5}$$

In the same way, $\partial_U^3 f^{(1)}$ is a modular function of weight $(-2,4)$ in T and U respectively. As explained in [10] (see in particuliar eqn. (2.24) there) it allows to derive the actual form of $f^{(1)}(T,U)$, up to a quadratic polynomial in T and U depending on the path of integration in the T and U planes. This ambiguity is due to the non-trivial quantum monodromies around singular points in the vector multiplet moduli space.

We are then finally ready to extract $\partial_T^3 f^{(1)}$ from (V.3). As usual in orbifold theories, traces must be taken over all (un)twisted sectors and projection onto orbifold-invariant states should be enforced, leading to summing over boundary conditions ; schematically the one-loop Kähler potential (V.3) can be decomposed as :

$$K^{(1)} = \sum_{h,g=0}^{p-1} \int_{\mathcal{F}} \mathrm{d}^2\mu\, \phi \begin{bmatrix} h \\ g \end{bmatrix} \Gamma \begin{bmatrix} h \\ g \end{bmatrix} (T,U) \tag{V.6}$$

where $\phi[^h_g](\tau)$ would be, in a standard $K3 \times T^2$ compactification without Wilson lines, a modular form of the congruence subgroup of $PSL(2,\mathbb{Z})_T \times PSL(2,\mathbb{Z})_U$ associated with the orbifold [4] and $\Gamma[^h_g]$ is the usual sum over the charge lattice of the two-torus defined as

$$\Gamma \begin{bmatrix} h \\ g \end{bmatrix} := \sum_{Q\in\Lambda_h} q^{\frac{|Q_L|^2}{2}} \bar{q}^{\frac{|Q_R|^2}{2}} e^{2i\pi g(Q,\delta)}$$

where we keep the convention from [10] for the expression of the left and right charges, namely :

$$Q_L := \frac{\mu_1\bar{U} - \mu_2 + \nu_1\bar{T} + \nu_2\bar{T}\bar{U}}{\sqrt{2T_2U_2}} \quad , \quad Q_R := \frac{\mu_1\bar{U} - \mu_2 + \nu_1 T + \nu_2 T\bar{U}}{\sqrt{2T_2U_2}} \, , \tag{V.7}$$

and where

$$(\mu_i, \nu_i) \in \mathbb{Z}^4 + h\delta$$

are the corresponding coordinates of the charges in the sub-lattice Λ_h of the Narain lattice associated with the h-th twisted sector.

Inserting the worldsheet modular integral (V.6) into the general formula (V.5) then finally gives :

$$\begin{aligned}\partial_T^3 f^{(1)} =& \frac{16i\pi^2 U_2}{T_2^2} \sum_{h,g=0}^{p-1} \int_{\mathcal{F}} \mathrm{d}^2\mu\, \phi \begin{bmatrix} h \\ g \end{bmatrix} \\ & \times \tau_2 \partial_\tau \partial_{\bar{\tau}} \tau_2^2 \partial_\tau \tau_2^2 \sum_{Q\in\Lambda_h} Q_L \bar{Q}_R^3 q^{\frac{|Q_L|^2}{2}} \bar{q}^{\frac{|Q_R|^2}{2}} e^{2i\pi g(Q,\delta)} \, .\end{aligned} \tag{V.8}$$

4. It is not the case for our non-geometric heterotic models, since the contributions from the $K3$ factor and the T^2 factor are no longer separately modular-invariant, see [8] for details.

The above expression should of course be properly renormalised in order to give a well-defined expression for $\partial_T^3 f^{(1)}$ (see *e.g.* [125]) ; however, it is already useful in the present form in order to determine the location of its poles as well as to understand the duality group of the theory. One may also verify that $\partial_T^3 f^{(1)}$ behaves as a modular function of weight $(4, -2)$ with respect to (T, U) under a transformation of the T-duality group to be derived in the following section from equation (V.8) as anticipated.

Obtaining $\partial_T^3 f^{(1)}$ from its modular and analyticity properties requires the knowledge of what happens at large distances in the vector moduli space, *i.e.* when either T or U tends to a cusp. Any such limit may be understood as a decompactification limit as we will explain below. While this is obvious for the $T \to \infty$ limit, the cases of the other cusps ($T \to s$ for $s \in \mathbb{Q}$) correspond to a two-torus of vanishing volume with a constant B-field background. It is not generically a decompactification limit of the theory of interest *per se*, but it is always possible to find another theory for which the corresponding limit is a genuine decompactification limit by acting on T with a $SL(2,\mathbb{Z})$ element. The limits obtained by taking U close to a cusp may be understood in a similar fashion by considering the dual torus instead. As argued in [126], it follows then from EFT considerations that one does not expect any pole for $\partial_T^3 f^{(1)}$ at the cusps.

In the following, we will derive the duality group – or at least a subgroup thereof – of the theory as well as its behavior when one of the moduli gets close to a cusp in both the heterotic and type IIA pictures.

V.2.2 Vector multiplets moduli space : dualities

As explained above, the actual classical moduli space is given by the quotient of the Teichmüller space (V.2) by the perturbative duality group acting on the second factor. Deriving this duality group, or at least a sufficiently large subgroup thereof, is essential in order to constrain sufficiently the modular functions $\partial_T^3 f^{(1)}(T,U)$ and $\partial_U^3 f^{(1)}(T,U)$. After some general remarks we will study first the perturbative duality group of the type IIA models, and second of their heterotic duals.

V.2.2.a) Deriving the perturbative duality group

It is a generic feature of orbifold compactifications to have a duality group different from the parent theory, as some symmetries of the latter may not be present in the daughter theory and *vice-versa*. As far as the vector multiplet moduli space is concerned, the relevant orbifold action of the models described in section III.1.2, either in type IIA and in heterotic, is the action on the two-torus that corresponds to a translation. In the following, we will call $\mathcal{G}$ the duality group acting on $\mathcal{M}_{\rm V}$.

In the parent heterotic theory, the duality group acting on the torus moduli T and U is given by $O(\Gamma_{2,2})_{T^2}$, $\Gamma_{2,2}$ being the charge lattice of the T^2. A convenient decomposition is :

$$O(\Gamma_{2,2})_{T^2} \cong P\Big[SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U\Big] \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)\,. \tag{V.9}$$

In this expression, $P[SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U]$ is the quotient of the group $SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U$ by the involution $(g,h) \mapsto (-g,-h)$, while the two $\mathbb{Z}_2$ factors correspond respectively to the exchange [5] of T and U and to $(T,U) \mapsto (-\bar{T}, -\bar{U})$.

5. In the corresponding type IIA duality group the $\mathbb{Z}_2$ factor associated with the $S \leftrightarrow U$ exchange is mirror symmetry on T^2 and maps type IIA to type IIB.

In general, a shift vector δ will break $O(\Gamma_{2,2})_{T^2}$ into a smaller subgroup. In order to understand the unbroken symmetries of the orbifold models, let us consider a one-loop correction of the schematic form

$$\left\langle f(\hat{\mathcal{Q}})\right\rangle^{(1)}(T,U)=\int_{\mathcal{F}}\mathrm{d}\mu\sum_{h,g=1}^{p}\Phi\begin{bmatrix}h\\g\end{bmatrix}(\tau)F\begin{bmatrix}h\\g\end{bmatrix}(\tau;T,U) \tag{V.10}$$

with $f(\hat{\mathcal{Q}})$ depending on the internal charge operators $\hat{\mathcal{Q}}$ of the theory, taking values in the lattice (V.7). A necessary and sufficient condition for a transformation acting on (T,U) to leave (V.10) invariant – and then to be a duality of the theory – is that it should mix the sectors (h,g) in such a way that the sum over all sectors remains invariant. This is obtained for instance by allowing (T,U) to transform as

$$F\begin{bmatrix}h\\g\end{bmatrix}(\tau;T,U)\mapsto F\begin{bmatrix}h\\g\end{bmatrix}(\tau;T',U')=F\begin{bmatrix}h'\\g'\end{bmatrix}(\tau;T,U)$$

with $\Phi\left[{h'\atop g'}\right](\tau)=\Phi\left[{h\atop g}\right](\tau)$.

The function $F\left[{h\atop g}\right]$ may be explicitly written in terms of a sum over the charge lattice and reads :

$$F\begin{bmatrix}h\\g\end{bmatrix}(\tau;T,U)=\sum_{Q\in\Lambda_h}f(Q)e^{-\pi\tau_2\mathcal{M}^2(Q;T,U)+i\pi\tau\langle Q,Q\rangle+2i\pi g\langle Q,\delta\rangle} \tag{V.11}$$

where the scalar product $\langle\cdot,\cdot\rangle$ is defined with respect to $\Gamma_{2,2}$ and where the mass function is given by :

$$\mathcal{M}^2\left(Q=\left[\begin{pmatrix}\mu_1\\\mu_2\end{pmatrix},\begin{pmatrix}\nu_1\\\nu_2\end{pmatrix}\right];T,U\right):=\frac{1}{T_2U_2}\left|\begin{pmatrix}1&T\end{pmatrix}\begin{pmatrix}\mu_1&-\mu_2\\\nu_2&\nu_1\end{pmatrix}\begin{pmatrix}U\\1\end{pmatrix}\right|^2. \tag{V.12}$$

An arbitrary transformation $\hat{g}\in\mathcal{G}$ acts on $X=T,U$ as

$$X\mapsto\rho_X(\hat{g})\cdot X$$

with ρ_X some representation of $\mathcal{G}$. It may easily be seen from equation (V.11) that $F\left[{h\atop g}\right]$ may only transform into $F\left[{h'\atop g'}\right]$ under the action of $\hat{g}$ if there exists some representation ρ of $\mathcal{G}$ such that

$$\mathcal{M}^2(Q;\rho_T(\hat{g})\cdot T,\rho_U(\hat{g})\cdot U)=\mathcal{M}^2(\rho(\hat{g})\cdot Q;T,U)$$

for any $Q\in\Lambda$. Then, setting $(T',U'):=(\rho_T(g)\cdot T,\rho_U(g)\cdot U)$ for clarity, $F\left[{h\atop g}\right](T,U)$ transforms as :

$$F\begin{bmatrix}h\\g\end{bmatrix}(\tau;T',U')=\sum_{Q\in\rho(\hat{g})\cdot\Lambda_h}f(\rho(\hat{g}^{-1})\cdot Q)e^{-\pi\tau_2\mathcal{M}^2(Q;T',U')+i\pi\tau\left\langle\rho(\hat{g}^{-1})\cdot Q,\rho(\hat{g}^{-1})\cdot Q\right\rangle+2i\pi g\left\langle\rho(\hat{g}^{-1})\cdot Q,\delta\right\rangle}$$

Therefore, a necessary condition for $F\left[{h\atop g}\right](\tau;T',U')$ to be identified to $F\left[{h'\atop g'}\right](\tau;T,U)$ for some h' and g' is to have

$$\left\langle\rho(\hat{g}^{-1})\cdot Q,\rho(\hat{g}^{-1})\cdot Q\right\rangle=\langle Q,Q\rangle\qquad\forall Q\in\Lambda_h\,.$$

Assuming furthermore that $\rho(\hat{g})$ acts on the charges Q linearly, this is equivalent to requiring that $\rho(\hat{g})$ belongs to $O(\Gamma_{2,2}\otimes\mathbb{R})$. Imposing this restriction, the above equation now reads :

$$F\begin{bmatrix}h\\g\end{bmatrix}(\tau;T',U') = \sum_{Q\in\rho(\hat{g})\cdot\Lambda_h} f(\rho(\hat{g}^{-1})\cdot Q)e^{-\pi\tau_2\mathcal{M}^2(Q;T,U)+i\pi\tau\langle Q,Q\rangle+2i\pi g\langle Q,\rho(\hat{g})\cdot\delta\rangle}\,.$$

The transformation $(T,U)\mapsto(T',U')$ may then be a duality of the theory only if it preserves the full charge lattice, that is if $\rho(\hat{g})\cdot\Lambda=\Lambda$, and if

$$f(\rho(\hat{g})^{-1}\cdot Q) = J(\hat{g};T,U)f(Q) \qquad \forall Q\in\Lambda$$

for some function J independent of the charge vector Q. The first condition will allow us in the following to identify the duality group of the theory from either type IIA and heterotic points of view while the second one is only reflecting the usual behaviour of modular covariant correlator functions.

V.2.2.b) Perturbative type IIA symmetries

The perturbative duality group acting on Teichmüller space (V.2) will be very different depending on whether one considers the theory in the type IIA or in the heterotic perturbative regime. In addition to the exchange of the T and S moduli (that follows from heterotic/type IIA duality in four dimensions), the shift vectors δ_{IIA} and δ_{HET} used in the respective perturbative limits are of different nature (light-like in the former case but not in the latter) as we have reviewed in section III.1.2.

We will start by looking at the type IIA duality frame, where S and U are the two-torus moduli and T the axio-dilaton. The model is understood as an orbifold of $K3\times T^2$ acting as an order p mirrored automorphism on the $K3$ factor and as a shift along the two-torus. In the type IIA theory, the shift vector satisfies $\delta_{\text{IIA}}^2=0$ hence may be chosen as :

$$\delta_{\text{IIA}} = \left(\frac{1}{p},0,0,0\right)\,, \tag{V.13}$$

i.e. as an order p momentum shift along one circle.

Let us first focus on the component of $O(\Gamma_{2,2}\otimes\mathbb{R})$ connected to the identity, which acts on the moduli of the torus as :

$$(S,U)\mapsto(g_S\cdot S, g_U\cdot U) := \left(\frac{aS+b}{cS+d},\frac{a'U+b'}{c'U+d'}\right)\ ,\ ad-bc=a'd'-b'c'=1.$$

The parametrisation given in (V.12) allows one to infer straightforwardly the corresponding action on the charges of the lattice :

$$\begin{pmatrix}\mu_1 & -\mu_2\\ \nu_2 & \nu_1\end{pmatrix}\mapsto\begin{pmatrix}d & b\\ c & a\end{pmatrix}\begin{pmatrix}\mu_1 & -\mu_2\\ \nu_2 & \nu_1\end{pmatrix}\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}$$

with $(\mu_i,\nu_i)\in\mathbb{Z}^4+h\delta_{\text{IIA}}$ in the h-th twisted sector. As we have just seen, a necessary condition for a transformation to give rise to a duality of the theory is that it must preserve the charge lattice Λ. As a result, g_S and g_U must have the form :

$$g_S = \frac{1}{\sqrt{e}}\begin{pmatrix}ae & b\\ cp & de\end{pmatrix}\ ,\ g_U = \frac{1}{\sqrt{e}}\begin{pmatrix}a'e & b'p\\ c' & d'e\end{pmatrix}\ ,\ e|p\ ,\ \gcd\left(e,\frac{p}{e}\right)=1\,.$$

These are the Atkin-Lehner involutions[6] already encountered in [122, 127]. Such an action on S and U is not generically a duality of the theory though as a vector Q in the h sector is mapped to another one in the h' sector, with $h' = da'he + bc'n_1\frac{p}{e}$ and n_1 the winding number of Q around the first circle of T^2. An arbitrary transformation then generically splits $\Gamma\left[{h \atop g}\right]$ into a sum of contributions coming from various sectors, so that deriving the full duality group would require a more in-depth analysis of the details of the model.

For our purposes it will be sufficient to identify only a simpler subgroup of the whole duality group, by restricting to transformations which preserve each sub-lattice Λ_h separately. This may easily be obtained from the above transformations by setting $e = 1$; the corresponding transformations all belong to $\mathcal{G}_p^{\text{IIA}}$, defined as :

$$\mathcal{G}_p^{\text{IIA}} := \left\{(g, g') \in \Gamma_0(p)_S \times \Gamma^0(p)_U \middle| g_{11} = g'_{11}, g_{22} = g'_{22} \text{ mod } p\right\}$$

with $\Gamma_0(p)$ (resp. $\Gamma^0(p)$) the group of $SL_2(\mathbb{Z})$-matrices whose lower (resp. upper) off-diagonal component vanishes modulo p. One has in particular

$$\Gamma_1(p)_S \times \Gamma^1(p)_U \subsetneq \mathcal{G}_p^{\text{IIA}} \subsetneq \Gamma_0(p)_S \times \Gamma^0(p)_U$$

with the congruence subgroups $\Gamma_1(p) = \{g \in \Gamma_0(p) | g_{11} = g_{22} = 1 \text{ mod } p\}$ and, in a similar way, $\Gamma^1(p) = \{g \in \Gamma^0(p) | g_{11} = g_{22} = 1 \text{ mod } p\}$.

We now turn to a brief analysis of the behavior of the models at the cusps of $\Gamma_0(p)$. First, the $S \to i\infty$ limit is the type IIA decompactification limit of the two-torus and, due to the freely-acting nature of the orbifold that acts as a momentum shift along T^2, it is described by a type IIA theory compactified on $K3$ (see *e.g.* [128]), thereby effectively restoring $\mathcal{N} = 4$ supersymmetry.

When S gets close to one of the other inequivalent cusps, one may analyse the situation by performing a double T-duality along the two-torus and going to the decompactification limit of this dual torus. Following [127], the type IIA worldsheet theory obtained by a double T-duality can be described as an orbifold $(\mathcal{M}/\langle\hat{\sigma}_p\rangle \times \tilde{S}^1)/G_p \times \tilde{S}^1$, where $\mathcal{M}/\langle\hat{\sigma}_p\rangle$ is the quotient of the $K3$ CFT $\mathcal{M}$ by the mirrored automorphism $\hat{\sigma}_p$ and G_p is an order p cyclic group acting on the first factor as the quantum symmetry of the orbifold $\mathcal{M}/\langle\hat{\sigma}_p\rangle$ and on the second factor as an order p shift.

A crucial property of the mirrored automorphisms is that the orbifold $\mathcal{M}/\langle\hat{\sigma}_p\rangle$ is actually isomorphic to the original $K3$ CFT $\mathcal{M}$, owing to fractional mirror symmetry [77]. Hence the theory obtained after double T-duality is exactly of the same type as the original theory, so that the same conclusions hold for the behavior at all cusps : $\mathcal{N} = 4$ is restored.

The analysis of the cusps in the U-plane is similar to what we have obtained for the behavior at the cusps in the S-plane, considering the mirror type IIB model instead of type IIA.

V.2.2.c) Perturbative heterotic symmetries

We now consider the heterotic dual of the model, *i.e.* an orbifold of $T^4 \times T^2$ acting as an automorphism of the $\Gamma_{4,20}$ Narain lattice on T^4 and as a shift along the two-torus whose Kähler moduli is T and whose complex structure moduli is U. We restrict the analysis to the

6. The term "involution" is related to the fact that the square of an Atkin-Lehner involution is in $\Gamma_0(p)$ and acts therefore trivially on the corresponding modular forms.

case of an orbifold by a group isomorphic to $\mathbb{Z}_p$; then, as shown in [8], one may choose the shift vector to have components [7]

$$\delta_{\text{HET}} = \left(\frac{1}{p}, 0, \frac{1}{p}, 0\right)$$

with no loss of generality, see eqn. (III.10).

The derivation of the perturbative duality group in this case goes along the same lines as in the type IIA case. The non-vanishing norm of the shift vector forbids in this case any sector-mixing behavior comparable to what we had observed in the type IIA case ; to be more precise, a vector in Λ_h may only be mapped to a vector in $\Lambda_{h'}$ if $h^2 = h'^2 \bmod p$. In particular, for p prime, this means that Λ_h may only be mapped to $\Lambda_{\pm h}$. Let's consider a transformation :

$$(T, U) \mapsto (g_T \cdot T, g_U \cdot U) := \left(\frac{aT+b}{cT+d}, \frac{a'U+b'}{c'U+d'}\right) \;,\; ad - bc = a'd' - b'c' = 1\,.$$

In order to avoid unnecessary complications, we will restrict from now to the cases where all coefficients in the above equations are integers, the rationale being that possibile dualities with non-integer coefficients will not be needed for the analysis of the prepotential below.

First, one realizes that the two $\mathbb{Z}_2$ factors of (V.9) from the mother theory duality group remain symmetries of the daughter theory. Indeed, though preserving one $\mathbb{Z}_2$ was expected as the orbifold leaves a one-cycle of the two-torus invariant, preserving the second one as well is somewhat more unusual. As this T-duality exchanges momentum and winding number, it may remain a symmetry of the orbifold theory only if the shift of the orbifold acts in a similar fashion on both the two-torus and its dual, which is the case with the shift vector (V.2.2.c)).

Second, imposing in addition that a duality must preserve the charge lattice Λ and keeping in mind that $\Phi\left[{h \atop g}\right]$ in equation (V.10) must be equal to $\Phi\left[{-h \atop -g}\right]$ as a result of CPT invariance, one can check that the perturbative duality group of the heterotic theory must contain :

$$\mathcal{G}_p^{\text{HET}} := \left\{(g, g') \in SL_2(\mathbb{Z})_T \times SL_2(\mathbb{Z})_U | g' = \pm\sigma_3 g \sigma_3 \bmod p\right\} \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2).$$

as a subgroup, σ_3 being the third Pauli matrix. Equivalently, one has

$$\mathcal{G}_p^{\text{HET}} = \Big(SL(2,\mathbb{Z}) \times \Gamma(p)\Big) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)\,,$$

since the condition $g' = \pm\sigma_3 g \sigma_3 \mod p$ can be solved as $g' = \pm\gamma\sigma_3 g\sigma_3$ with $g \in SL(2,\mathbb{Z})$ and $\gamma \in \Gamma(p)$, $\Gamma(p) := \{g \in SL(2,Z) | g = \mathbb{I} \bmod p\}$ being the principal congruence subgroup of level p.

Acting non-trivially on only one of the complex moduli of T^2 (that is, setting either $g = \mathbb{I}$ or $g' = \mathbb{I}$ in the above definition) gives the subgroup :

$$\Gamma(p)_T \times \Gamma(p)_U \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2) \subsetneq \mathcal{G}_p^{\text{HET}}\,. \tag{V.14}$$

For the rest of the discussion, we will focus on this subgroup and won't attempt to derive the full heterotic perturbative duality group of the theory.

The behavior of the theory when going to large distances in the moduli space may be extracted directly from *e.g.* the partition function of the model in this case along the lines

7. Here, it is understood that the shift vector is chosen so that the heterotic theory is dual to the type IIA theory with shift vector defined in (V.13).

of [128] and using its explicit form computed in [8]. It turns out that when either T or U tend to any cusp of $\Gamma(p)$, the theory may be described by a heterotic string theory on a four-torus, restoring once again $\mathcal{N}=4$ supersymmetries (for the U modulus, this is requested by heterotic/type IIA duality).

It is worth mentioning that the fact that the gravitini masses vanish in those limits, thereby restoring $\mathcal{N}=4$ supersymmetry, does not imply in general the vanishing of quantities which would vanish in a "genuine" $\mathcal{N}=4$ theory; in particular, it does not imply that the Yukawa coupling $\partial_T^3 f^{(1)}$ tends to zero when T or U tends to a cusp. [8]

As explained in [128], while the mass of the two massive gravitini tend to zero, some charged states may be lighter in this limit. Those light charged states would always keep track of the original $\mathcal{N}=2$ behavior of the theory no matter how small one makes the gravitini mass; consequently, there would be no reason to expect, say, $\partial_T^3 f^{(1)}$ to be vanishing in this limit.

We will now argue that this is the case for the models considered in this work. Let us first consider the string states corresponding to the massive gravitini, and the large volume limit $|T|\to\infty$. These states are of the form :

$$|\Psi_{\rm G}\rangle = \big(|s_0;p^\mu\rangle_{\rm R}\otimes|s';0\rangle_{\rm R}\otimes|\hat{s};P_L\rangle_{\rm R}\big)\otimes\big(\tilde{\alpha}^\mu_{-1}|p^\mu\rangle\otimes|0\rangle\otimes|P_R\rangle\big)\,,$$

where we have chosen for the T^4 CFT a Ramond ground state $|s';0\rangle_{\rm R}$ with unit charge under the action of the $\mathbb{Z}_p$ orbifold. The momentum (P_L,P_R) along the T^2 is chosen such that $|\Psi_{\rm G}\rangle$ is even under the orbifold projection associated with the shift vector (V.2.2.c)). Given the mass formula (V.12) the lightest such state has $\mu_1=1$ and $\mu_2=\nu_1=\nu_2=0$ (*i.e.* one unit of momentum along the first circle of the two-torus) and the gravitino mass is given by $M_g=|U|/\sqrt{U_2T_2}$.

Light charged states can be obtained easily from the Kaluza-Klein modes of the T and U vector multiplets. Specifically, consider a state of the form

$$|\Psi_{\rm K}\rangle = \big(|s_0;p^\mu\rangle_{\rm R}\otimes|s;0\rangle_{\rm R}\otimes|\hat{s};Q_L\rangle_{\rm R}\big)\otimes\big(|p^\mu\rangle\otimes|0\rangle\otimes\tilde{\alpha}^1_{-1}|Q_R\rangle\big)\,,$$

where the Ramond ground state $|s;0\rangle_{\rm R}$ of the T^4 CFT is neutral under the orbifold action, and where the oscillator $\tilde{\alpha}^1_{-1}$ is along the first circle of the two-torus. The lightest such states that are invariant under the orbifold projection have $\mu_2=1$ and $\mu_1=\nu_1=\nu_2=0$ (*i.e.* one unit of momentum along the second circle of the two-torus) and their mass is given by $M_{\rm K}=1/\sqrt{U_2T_2}$.

Thus $M_{\rm G}/M_{\rm K}=|U|$ which is greater than one inside the fundamental domain $\mathcal{F}_0$ of $SL(2,\mathbb{Z})_U$. The other parts of the fundamental domain of $\Gamma(p)_U$ are obtained as $g\cdot\mathcal{F}_0$ for some $g\in SL(2,\mathbb{Z})$ and can be analyzed along the same lines, by transforming the shift vector accordingly.

Finally, one may wonder whether the vector multiplets moduli space of the putative non-perturbative $\mathcal{N}=2$ theory has an exact duality group, related to the perturbative groups $\mathcal{G}_p^{\rm HET}$ and $\mathcal{G}_p^{\rm IIA}$. On general grounds one expects that the heterotic vector multiplets moduli space gets corrected by NS5-branes instanton effects breaking the perturbative duality group (see however [55] as an exception to this rule). It has been shown for instance that T-dualities of heterotic strings on $K3\times T^2$ do not survive quantum effects as can be seen from the

8. Imposing those constraints on the modular form $\partial_T^3 f^{(1)}$ would actually be too stringent for most values of p.

Calabi–Yau type IIA dual, where the corresponding worldsheet instanton effects are known thanks to mirror symmetry [129]. In the present case, since there is no duality frame in which the vector multiplets moduli space is classical, there is no obvious way to adress this question.

Dualities acting on the $(SL(2,\mathbb{Z})/U(1))_U$ factor of the space (V.2) alone, given that there is no frame in which U is the axio-dilaton, may still be exact symmetries of the quantum theory if they appear on both sides of the duality. We have shown above that the IIA perturbative group contains a congruence subgroup $\Gamma^1(p)_U$, while the heterotic perturbative group contains a smaller congruence subgroup $\Gamma(p)_U$ of $SL(2,\mathbb{Z})$.

A duality $g \in \Gamma^1(p)_U \backslash \Gamma(p)_U$ is a symmetry on the type IIA side but does not belong to the factorized subgroup (V.14) on the heterotic side. If we consider the larger subgroup (V.2.2.c)) of the heterotic duality group, $g \in \Gamma^1(p)_U \backslash \Gamma(p)_U$ remains a symmetry of the theory if accompanied by a non-trivial transformation in $SL(2,\mathbb{Z})_T$.

From the type IIA side, this could be a problem as T is now the axio-dilaton. However, for any such $g = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ mod p, an appropriate transformation of T would be given by $g' = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, *i.e.* by an integral shift of the NS-NS axion $T \mapsto T + b$, $b \in \mathbb{Z}$. This transformation preserves the perturbative regime $\mathrm{Im}\,(T) \to \infty$ and this discrete Peccey-Quinn symmetry is expected to remain a symmetry of the quantum theory.

In conclusion, one may speculate that $\Gamma^1(p)_U$ acting on the vector moduli space is an exact duality of the $\mathcal{N} = 2$ quantum theory. Other exact dualities symmetries acting on the hypermultiplets moduli space, which does not receive g_s corrections, will be given in section V.3.

V.2.3 Heterotic case : singularities of the prepotential

Our goal in this subsection is to derive $\partial_T^3 f^{(1)}(T,U)$, which is a modular function of weight $(4,-2)$ in T and U, using its singularity structure and its behavior at the cusps, applying theorems of modular forms.

Understanding the location of the singularities is fairly easy from an effective field theory (EFT) point of view. To get an effective $\mathcal{N} = 2$ supergravity theory from the underlying string theory, one has to integrate all heavy fields ; there may be points in the vector moduli space where otherwise massive states become massive, resulting in a breakdown of the original effective field theory. As a result, the prepotential becomes singular at such a point leading in a pole of order one in $\partial_T^3 f^{(1)}$ [11].

From the conformal weights of the operators of the heterotic theory, one learns that the mass of a state satisfies :

$$\frac{m^2}{2} = \frac{|Q_L|^2}{2} + N_L + a_L = \frac{|Q_R|^2}{2} + N_R + a_R$$

with N_L (N_R) the excitation number and a_L (resp. a_R) the zero-point energies of the left-

(resp. right-) moving fields. a_L and a_R where explicitly computed in [8] and read :

$$a_L = \begin{cases} \frac{h}{p} - \frac{1}{2} \text{ if } h \leq \frac{p}{2} \\ \\ -\frac{h}{p} + \frac{1}{2} \text{ if } h \geq \frac{p}{2} \end{cases}$$

$$a_R = \frac{h^2}{p^2} - \frac{h}{p} - \left(\frac{\mathrm{gcd}(h,p)}{p}\right)^2 \prod_{\substack{q|p \\ q \text{ prime}}} (-q)$$

in the h-th twisted sector (the inequalities in the expression of a_L being valid for the representative of h in $\mathbb{Z}_p$ such that $0 < h < p$). In the untwisted sector, $a_L = -\frac{1}{2}$ and $a_R = -1$ as usual; a state with non-vanishing charge may therefore be massless in this sector if and only if

$$\begin{cases} |Q_L|^2 = 0 \\ \\ |Q_R|^2 = 2 \end{cases} .$$

While there is no such state in general, equation (V.7) implies that a state of charge $Q = (m_i, n_i) \in \mathbb{Z}^4$ may become massless if

$$T = \begin{pmatrix} -m_1 & m_2 \\ n_2 & n_1 \end{pmatrix} \cdot U \text{ and } \begin{pmatrix} -m_1 & m_2 \\ n_2 & n_1 \end{pmatrix} \in SL_2(\mathbb{Z})\,. \tag{V.15}$$

So far, the situation is the same as in the mother theory; the orbifold projection will furthermore select some allowed charges Q. At the end of the day, assuming as before that the basis is chosen such that the shift vector has non-vanishing components along the first cycle of the two-torus only, see eqn. (V.2.2.c)), the generically massive states which become massless at some points in the vector moduli space have charges satisfying :

$$\begin{pmatrix} -m_1 & m_2 \\ n_2 & n_1 \end{pmatrix} = \begin{pmatrix} a+\epsilon & b \\ c & a \end{pmatrix} \mod p\,,$$

with $\epsilon \in \{0, \pm 1\}$. The states satisfying the above equation with $\epsilon = 0$ belong to $\mathcal{N} = 2$ vector multiplets and correspond to non-abelian enhancements of the gauge symmetry; in contrast, the case $\epsilon = \pm 1$ corresponds to states belonging to charged hypermultiplets, then resulting in additional matter states without any enhancement of the gauge group. In either case, these states are responsible for the appearance of single poles in $\partial_T^3 f^{(1)}$ at the lines of the moduli space given in (V.15).

New singular lines absent in the mother theory could also occur if extra charged massless states come from the twisted sectors, which may happen only if the zero-point energy of the right-moving fields a_R is negative. It is worthwhile noticing that such a state would necessarily belong to a hypermultiplet, as only twisted oscillators of the T^4 have a small enough conformal dimension to fulfill the massless condition coming from the supersymmetric side of the CFT. As it turns out, even though the analysis of the situation goes along the same lines as in the untwisted sector case, it may not be performed keeping p (and h, the label of the twisted sector) generic.

As usual, finding which states may become massless or not for given values of the moduli T and U may also easily be done by computing the new supersymmetric index $\mathcal{I}$ of [124], whose worldsheet modular integral gives the one-loop Kähler potential (V.3). Defining $\mathcal{I}\left[\begin{smallmatrix} h \\ g \end{smallmatrix}\right]$ as the contribution from the h-th twisted sector with the insertion of the generator of the orbifold to the power g, it is easy to show that :

$$\mathcal{I}\begin{bmatrix} 0 \\ g \end{bmatrix} = -\frac{i}{\bar{\eta}^6(\tau)} \left[\prod_{d \left| \frac{p}{(g,p)} \right.} \bar{\eta}(d\tau)^{-\mu\left(\frac{p}{d \times (g,p)}\right)} \right]^{24/\varphi\left(\frac{p}{(g,p)}\right)} \left(\prod_{i=1}^{2} \bar{\vartheta}\begin{bmatrix} 1 \\ 1 + 2gs_i/p \end{bmatrix} \right) \Gamma\begin{bmatrix} 0 \\ g \end{bmatrix}$$

for $g \neq 0$ and $\mathcal{I}\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = 0$, as usual. [9] It is quite straightforward to obtain from there any $\mathcal{I}\left[\begin{smallmatrix} h \\ g \end{smallmatrix}\right]$ acting with elements of $SL_2(\mathbb{Z})$ on the above ; if p is prime, the contribution $\mathcal{I}_h$ from the h-th twisted sector to $\mathcal{I}$ then reads :

$$\mathcal{I}_0 = -\frac{ip^{12/(p-1)}}{p\bar{\eta}^6(\tau)} \left(\frac{\bar{\eta}(\tau)}{\bar{\eta}(p\tau)} \right)^{24/(p-1)} \sum_{g=1}^{p-1} \left(\prod_{i=1}^{2} \bar{\vartheta}\begin{bmatrix} 1 \\ 1 + 2gs_i/p \end{bmatrix} \right) \Gamma\begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\mathcal{I}_{h\neq 0} = \frac{ip^{12/(p-1)}}{p\bar{\eta}^6(\tau)} \sum_{g=0}^{p-1} \left(\frac{\bar{\eta}(\tau + h^{-1}g)}{\bar{\eta}\left(\frac{\tau + h^{-1}g}{p}\right)} \right)^{24/(p-1)} \left(\prod_{i=1}^{2} \bar{\vartheta}\begin{bmatrix} 1 + 2hs_i/p \\ 1 + 2gs_i/p \end{bmatrix} \right) \Gamma\begin{bmatrix} h \\ g \end{bmatrix}$$

with h^{-1} the inverse of h in $\mathbb{Z}_p^\times$.

Charged massless states give rise to divergences in $\partial_T^3 f^{(1)}$ through the contribution of unphysical tachyons coming from the non-supersymmetric side of the worldsheet CFT ; therefore, the knowledge of $\mathcal{I}_h$ allows to look for such tachyons in its expansion around $\tau \to i\infty$. This way, one may check for instance that no charged states coming from the twisted sector(s) become massless at any point of the T^2 moduli space for $p = 2$. Of course, the new supersymmetric index also gives information about the residues of $\partial_T^3 f^{(1)}$, even though those may also be determined by purely effective field theory considerations. In general, one finds [11]

$$\operatorname*{Res}_{U \to \gamma \cdot T} \partial_T^3 f^{(1)} = \frac{\beta_\gamma}{16\pi^2} \frac{\det^2(\gamma)}{J^4(\gamma, T)} \tag{V.17}$$

where β_γ is the beta function coefficient associated to the gauge group under which the corresponding charged massless fields are charged and where $J(\gamma, T) := cT + d$ for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

V.2.4 Vector multiplets moduli space : quantum corrections

In the following, we show that the above is sufficient to determine a closed form for $\partial_T^3 f^{(1)}$ in terms of modular functions of $\Gamma(p)$ for any value of p, at least in principle. We then proceed to the explicit computation of the corrections for the models with $p = 2$.

9. In the above derivation, equation (C.25) has been used in order to derive an expression more suited for numerical computations but equivalent to the more traditional form involving more ϑ-functions.

Type IIA viewpoint

The above analysis has been mainly focused on the heterotic side of the theory because the derivation of $\partial_T^3 f^{(1)}$ is more involved in this case. Indeed, in the perturbative type IIA regime, unlike in heterotic, there is no non-Abelian enhancement of the gauge symmetry in the (S,U) moduli space, hence no associated logarithmic singularities of the gauge couplings. We have checked as well that, at least for $p=2$, there are no charged hypermultiplets from the twisted sectors of the asymmetric orbifold of the Gepner model that could become massless.

As argued in [126, 11], using simple effective field theory considerations, the second derivatives of $f^{(1)}_{\text{IIA}}(S,U)$, *i.e.* the gauge couplings, should grow at most linearly in S and U in the decompactification limit hence one does not expect a pole of $f^{\text{IIA}}(S,U)$ at the cusps $S=i\infty$ and $U=i\infty$. Moreover, as we have argued in section V.2.2.b), the theory obtained when S or U tend to any other cusp may be equivalently described in terms of a theory isomorphic to the original ones, so that the same arguments show that $h^{\text{IIA}}(S,U)$ is holomorphic at all cusps.

In conclusion, using the fact that the space of negative weights modular forms is empty for any congruence sugroup, the type IIA perturbative corrections should vanish :

$$f^{(1)}_{\text{IIA}}(S,U) = 0\,.$$

Therefore, on the type IIA side, all the corrections to the prepotential in (V.2) are of non-perturbative nature.

Heterotic viewpoint

In the heterotic picture, the one-loop correction to the prepotential $f^{(1)}_{\text{HET}}$ cannot vanish as it would be inconsistent with the analysis of its poles and of their residues.

The idea here is to express $\partial_T^3 f^{(1)}_{\text{HET}}$ in terms of modular functions of $\Gamma(p)$, thereby making duality (V.14) manifest. It is a standard fact from modular form theory that a modular function f_k of weight p with respect to $\Gamma(p)$ satisfies [10] [30]

$$\sum_{x\in X(2)} \text{ord}_x(f_k) = \frac{k}{2} \qquad \text{if } p=2$$

$$\sum_{x\in X(p)} \text{ord}_x(f_k) = \frac{k}{24}\,|SL_2(\mathbb{Z}):\Gamma(p)| \qquad \text{if } p>2$$

where $X(p)$ is the compactification of the quotient of the upper-half plane $\mathcal{H}$ by $\Gamma(p)$ (that is the space $\mathcal{H}/\Gamma(p)$ to which one add the corresponding cusps) and where $\text{ord}_x(f_k)$ is the order of f_k at x in the complex analysis sense, that is the least power in the Laurent expansion of f_k around x with non-vanishing coefficient.

Let us then consider our function $\partial_T^3 f^{(1)}_{\text{HET}}(T,U)$ as a function of U with fixed parameter T for a moment. As we have seen, the only poles of $\partial_T^3 f^{(1)}_{\text{HET}}$ must be simple and located along singular lines of the form $U=\gamma\cdot T$ in the (T,U) moduli space ; we will denote in the following

10. The $p=2$ case must be treated separately because $\Gamma(2)$ is the only principal congruence subgroup containing $-\mathbb{I}$, leading to a difference from a factor of 2 between the two equations.

the set of all such γ's as Γ_{sing}. This means that the orders of $\partial_T^3 f^{(1)}_{\text{HET}}$ – seen as a function of U only – must satisfy :

$$\begin{cases} \text{ord}_U \partial_T^3 f^{(1)}_{\text{HET}} = -1 \ \forall U \in \Gamma_{\text{sing}} \cdot T \\ \\ \text{ord}_U \partial_T^3 f^{(1)}_{\text{HET}} \geq 0 \ \forall U \notin \Gamma_{\text{sing}} \cdot T \end{cases} .$$

Recalling that $\partial_T^3 f^{(1)}_{\text{HET}}$ is a modular function of weight $(4,-2)$ under $\Gamma(p)_T \times \Gamma(p)_U$, equation (V.18) shows that there is no much freedom left by these requirements ; to be more specific, there will be exactly 5 free parameters if $p = 2$ and

$$|\Gamma_{\text{sing}}| - \frac{1}{12} |SL_2(\mathbb{Z}) : \Gamma(p)|$$

free parameters if $p > 2$ left unfixed by just imposing the location of the poles of $\partial_T^3 f^{(1)}_{\text{HET}}$. Mathematically, these free parameters correspond to the location of the zeroes of $\partial_T^3 f^{(1)}_{\text{HET}}$ and are as such harder to fix through physical requirements only in general.

The additional data about the residues of $\partial_T^3 f^{(1)}_{\text{HET}}$ give even more information and allows to fix at least some of the above parameters. Focusing on the divisor of $\partial_T^3 f^{(1)}_{\text{HET}}$ seen as a function of U fixes it up to an overall multiplicative constant with respect to U, that is up to an overall multiplicative function of T only ; each residue computation lead to a constraint on the free parameters from above. Fixing the overall multiplicative function of T leaves then $|\Gamma_{\text{sing}}| - 1$ constraints in total, which are non-necessarily independent.

At the end of the day, we have more constraints than free parameters and are as such entitled to hope that these would be enough to completely determine $\partial_T^3 f^{(1)}_{\text{HET}}$ for any p. If it happened not to be the case for some orders p though, fixing the remaining free parameters would have to be done in a different but not necessarily cumbersome way ; indeed, computing only the first few terms in the expansion of $\partial_T^3 f^{(1)}_{\text{HET}}$ would be enough to fix it completely. At worst, our analysis would then allow to express an infinite series expansion in terms of somewhat easier to handle modular forms. We now turn to illustrating the above strategy in the $p = 2$ case.

V.2.5 $p = 2$ case : explicit derivation

In the following subsection, we will illustrate the strategy outlined above by computing explicitely the one-loop corrections to the prepotential for non-geometric models based on orbifolds of order $p = 2$, from their modular properties. [11]

The analysis of section V.2.3 show that there are 6 singular lines in the (T, U)-moduli space mod $\Gamma(2)_T \times \Gamma(2)_U$; more precisely, along the $T = U, U+1, -1/U$ and $U/(U+1)$ lines, one additional vector multiplet becomes massless, resulting in a $U(1)^2 \to SU(2) \times U(1)$ gauge symmetry enhancement. Along the $T = -1/(U+1)$ and $(U-1)/U$ lines, two charged hypermultiplets become massless, leading to additional matter content without gauge symmetry being enhanced.

11. This particular class of compactifications has previously been considered in the litterature [49], albeit formulated in a different way. Indeed, in the type IIA picture, such orbifold of Gepner model may be understood as acting as $(-1)^{F_L}$ (together with a momentum shift along the T^2). However, as far as the authors know, no derivation of the corrections to the prepotential had been given for this model before.

Accidentally, it turns out that a more compact way of parametrising the singular lines exist for $p=2$; indeed, one may notice from the above that there is actually a singular line at $T=\gamma\cdot U$ for any $\gamma\in SL_2(\mathbb{Z})/\Gamma(2)$ – which is not generalisable for arbitrary p.

Using the notations introduced in appendix E, $\partial_T^3 f^{(1)}_{\text{HET}}$ may therefore be written as :

$$\partial_3^T f^{(1)}_{\text{HET}} = f(T)\times\frac{g_{10}(T,U)}{\prod_{\gamma\in SL_2(\mathbb{Z})/\Gamma(2)}\left[V_\infty(U)(\lambda(U)-\lambda(\gamma\cdot T))\right]}$$

where the modular forms $V_\star(T)$ have a single zero at the corresponding cusp, see eqn. (E), and where $g_{10}(T,U)$ is a modular form of weight 10 with respect to $\Gamma(p)_U$; as such, it may be expanded on a basis of $\Gamma(p)_U$ modular forms in the variable U as :

$$g_{10}(T,U)=\sum_{n=0}^{5}a_n(T)X_1^{5-n}(U)X_2^n(U)\,.$$

Computing the residue of $\partial_T^3 f^{(1)}_{\text{HET}}$ when $U\to\gamma\cdot T$ is made especially easy using equation (E.28). One obtains :

$$\underset{U\to\gamma\cdot T}{\text{Res}}\,\partial_T^3 f^{(1)}_{\text{HET}} = f(T)\frac{g_{10}(T,\gamma\cdot T)}{\Delta(T)\partial j(T)}J^{-14}(\gamma,T)\overset{!}{=}-\frac{\epsilon_\gamma}{4\pi^2}\frac{\det^2(\gamma)}{J^4(\gamma,T)}$$

where equation (V.17) has been used to obtain the right-hand side. Here, ϵ_γ is 1 (resp. -1) for singular lines corresponding to vector multiplets (resp. hypermultiplets) ; these signs reflect the respective values of the beta-function coefficient β_γ, which is -4 for a SU(2) without charged hypermultiplet (resp. +4 for a U(1) with two hypermultiplets of charge 1 in absolute value) in the four-dimensional EFT.

This data provides the necessary information about the behaviour of $g_{10}(T,U)$ under the action of $SL_2(\mathbb{Z})$ on its second variable. As moreover $\Gamma(p)$ is normal in $SL_2(\mathbb{Z})$ for any value of p, the ring of modular functions of the former is closed under the action of the latter. Therefore, $g_{10}(T,\gamma\cdot T)$ is a modular form of weight 10 with respect to $\Gamma(2)$ whose explicit form may be computed in terms of $\{a_n(T)\}$, leading to 5 independent equations allowing one to fix all the a_n's (up to an overall multiplicative function of T).

These equations may easily be solved and finally leads to the fully explicit result (see appendix E) :

$$\partial_T^3 f^{(1)}_{\text{HET}} = \frac{i}{995328\pi}\frac{P_{10,10}(T,U)}{V_0(T)V_1(T)V_\infty(T)\Delta(U)\left[j(T)-j(U)\right]} \qquad \text{(V.19)}$$

with $P_{10,10}(T,U)$ a modular form of weight $(10,10)$ with respect to $\Gamma(p)_T\times\Gamma(p)_U$ that is given by eqn. (E.29) in appendix E.

Since $T\leftrightarrow U$ exchange is part of the heterotic perturbative duality group, we get immediately the expression of $\partial_U^3 f^{(1)}_{\text{HET}}$ by exchanging the roles of T and U on the right-hand side of (V.19).

V.3 Hypermultiplets moduli space

In this section we describe the manifold $\mathcal{M}_{\text{H}}$ spanned by the massless scalars in the neutral hypermultiplets of the $\mathcal{N}=2$ low-energy four-dimensional theory, which is a quaternionic Kähler manifold [130, 131].

V.3.1 Hypermultiplet moduli space in the type IIA description

From the type IIA perspective, the situation is very different from the usual case of compactifications on Calabi–Yau three-folds. While in the latter case $\mathcal{M}_{\text{H}}$ contains the complex structure moduli of the CY_3, the scalars from the Ramond–Ramond forms and the axio-dilaton (hence receives g_s corrections), in the present case the axio-dilaton lies in a vector multiplet and there are no massless fields from the Ramond–Ramond sector [80]. Hence one does not expect any correction in the string coupling g_s, either perturbative or non-perturbative.

A preliminary analysis of $\mathcal{M}_{\text{H}}$ in type IIA was done in [7] and we will summarize now the main results. The models can be viewed as orbifolds of a product of a $K3$ Gepner model $\mathcal{G}$ and a T^2 by a $\mathbb{Z}_p$ cyclic group acting as a non-geometric automorphism of the Gepner model $\mathcal{G}$ and as a shift along the T^2. Massless hypermultiplets are obtained from the moduli of the type IIA compactification on the $K3$ surface (around the Gepner point $\mathcal{G}$) invariant under the action of the orbifold. Since the orbifold is freely-acting, we cannot get any new hypermultiplet from the twisted sectors of the orbifold.

These considerations indicate that $\mathcal{M}_{\text{H}} \subset \mathcal{M}_\sigma$, *i.e.* that this moduli space is a subset of the moduli space (II.9) of type IIA compactifications on $K3$. An incomplete description of this moduli space was obtained in [80]. Consider a $K3$ surface X described by a hypersurface of the form

$$z_1^p + f(z_2, z_3, z_4) = 0$$

in a weighted projective space, where f is a quasi-homogeneous polynomial of the appropriate degree. One considers the automorphism $\sigma_p : z_1 \mapsto e^{2i\pi/p} z_1$ and denote by $S(\sigma_p)$ the sublattice of the $K3$ lattice $\Gamma_{3,19}$ invariant under the action of σ_p on the second cohomology. If the K3 surface X is *polarized* by the lattice $S(\sigma_p)$ one has an unambiguous notion of complex structure on X. Then, following recent mathematical results [132], one can determine the moduli space $\mathcal{M}^p_{\text{CS}}$ of complex structures on X compatible with the action of σ_p.

Consider in the same way $X^\vee$, the Greene–Plesser/Berglund–Hübsch mirror of the surface (V.3.1), which is a quotient of a hypersurface of the form

$$\tilde{z}_1^p + f^\vee(\tilde{z}_2, \tilde{z}_3, \tilde{z}_4) = 0\,,$$

in terms of the transpose polynomial $f^\vee$ [68], and also admits an order p automorphism acting as $\tilde{\sigma}_p : \tilde{z}_1 \mapsto e^{2i\pi/p} \tilde{z}_1$. One can determine in the same way the moduli space $\widetilde{\mathcal{M}}^p_{\text{CS}}$ of complex structures on the $S(\tilde{\sigma}_p)$-polarized surface $X^\vee$ that are compatible with the action of $\tilde{\sigma}_p$.

As shown in [80], the *mirrored automorphism* $\widehat{\sigma}_p$ can be viewed as the diagonal action of the automorphisms σ_p and of $\tilde{\sigma}_p$ on a conformal field theory with target space X. Then $\mathcal{M}^p_{\text{CS}} \times \widetilde{\mathcal{M}}^p_{\text{CS}} \subset \mathcal{M}_\sigma$ is a sub-manifold of the moduli space of conformal field theories on $K3$ surfaces invariant under the action of $\hat{\sigma}_p$, that can be viewed as the moduli space of CFTs on $S(\sigma_p)$-polarized $K3$ surfaces invariant under the action of $\hat{\sigma}_p$.

Using the mathematical results known to us, it was not immediate to infer from $\mathcal{M}^p_{\text{CS}} \times \widetilde{\mathcal{M}}^p_{\text{CS}}$ the full hypermultiplets moduli space $\mathcal{M}_{\text{H}}$. As we will see below, on the heterotic side of the duality, one can get on the nose the exact form of $\mathcal{M}_{\text{H}}$ by rather standard arguments.

V.3.2 Heterotic perspective

As in type IIA, the heterotic dilaton sits in an $\mathcal{N} = 2$ vector multiplet, therefore the hypermultiplet moduli space $\mathcal{M}_{\text{H}}$ does not receive corrections either in the string coupling on the heterotic side, hence can be computed exactly at the perturbative level.

The heterotic description of the non-geometric models [8] is an order p orbifold acting as an order p $O(\Gamma_{4,20})$ isometry of a heterotic compactification on T^4 together with an order p shift along an extra two-torus. The free action of this orbifold prevents any (neutral) moduli to arise from the twisted sectors so that the moduli space of the theory should be directly inherited of that of the parent theory, that is the heterotic string on $T^4 \times T^2$, and the moduli lying in hypermultiplets come from the allowed deformations of the $\Gamma_{4,20}$ lattice associated with the T^4 compactification.

Considering the quotient by the automorphism $\hat{\sigma}_p$ only makes sense for lattices $\Gamma_{4,20}$ which admit $\hat{\sigma}_p$ as a symmetry, which means that the moduli space we are looking for may be interpreted as the space of deformations of such lattices. Phrased differently, the local form of $\mathcal{M}_H$ may be accessed by picking such a particular lattice and studying its deformations compatible with $\hat{\sigma}_p$-invariance.

At this stage, one may emphasise the peculiarity of the $p = 2$ model : first of all, one may notice that equation (III.6) implies that the matrix M_2 associated with the action of $\hat{\sigma}_2$ is simply $M_2 = -\mathbb{I}_{24}$; as any lattice admits this order two symmetry, all the deformations of $\Gamma_{4,20}$ are still allowed in the orbifolded theory, which is clearly not the case for any other admissible value of p. Therefore, in the $p = 2$ case, the hypermultiplets moduli space is directly given by the T^4 moduli space of the six-dimensional compactification :

$$\mathcal{M}_{\text{H}}^{p=2} \cong O(\Gamma_{4,20})\backslash O(4,20)/O(4) \times O(20)$$

Let us then consider the other cases where $p > 2$. The deformations of the Narain lattice $\Gamma_{4,20}$ in the parent theory correspond locally to choices of embedding of this lattice into $\mathbb{R}^{4,20}$, that is to fixing a space-like 4-dimensional plane $\Pi_L(\Gamma_{4,20})$ in the ambient space of $\Gamma_{4,20}$ (or equivalently to fixing a time-like 20-dimensional plane $\Pi_R(\Gamma_{4,20})$).

Given that, in the type IIA description, the automorphism σ_p acts by definition on the holomorphic two-form ω as $\omega \mapsto \zeta_p \omega$ and that $\int \omega \wedge \bar{\omega} > 0$, there exists a space-like eigenspace associated with the eigenvalue ζ_p of the automorphism. Further, on the heterotic side, $\mathcal{N} = 2$ space-time supersymmetry of the asymmetric orbifold [8] indicates that there must exist a basis of $\Pi_L(\Gamma_{4,20}) \otimes \mathbb{C}$ in which $M_p = \text{diag}(\zeta_p \mathbb{I}_2, \zeta_p^{-1}\mathbb{I}_2)$.

Then, given the diagonal action of M_p onto $\Pi_L(\Gamma_{4,20}) \otimes \mathbb{C}$, the only freedom left amounts to choosing which directions correspond to left-movers in the eigenspace of $\hat{\sigma}_p$ corresponding to, say, ζ_p. Moreover, equation (III.6) states that the eigenspace of any eigenvalue of $\hat{\sigma}_p$ has dimension $24/\varphi(p)$, so that the moduli lying in hypermultiplets may be understood as arising from the freedom of choice of a space-like 2-dimensional complex plane into a $24/\varphi(p)$-dimensional complex space. Therefore, $\mathcal{M}_H$ may be locally understood as a Grassmannian space of complex spaces and has the local form :

$$\mathcal{T}_{\text{H}}^{p\neq 2} \cong SU\left(2, \tfrac{24}{\varphi(p)} - 2\right)/S\left[U(2) \times U\left(\tfrac{24}{\varphi(p)} - 2\right)\right].$$

The global form of $\mathcal{M}_{\text{H}}$ is then obtained by identifying the corresponding duality group ; this is done by noticing that this theory inherits its dualities from the mother toroidal theory. Indeed, as the orbifold procedure keeps only $\hat{\sigma}_p$-invariant states, any element g of the duality group must commute with the induced action $\hat{\sigma}_p^{\star}$ of the automorphism on the states of the theory. Then, any element g of the duality group of the original theory, that is $O(\Gamma_{4,20})$, satisfying such a condition must belong also to the duality group of the resulting theory. Furthermore, a little bit of thought also shows that no other duality element may be present here, as $O(\Gamma_{4,20})$ already includes all acceptable duality relations inside a given sector ; indeed, new elements would necessarily mix states from different sectors, which is not possible as they would have different conformal weights due to the free action of the orbifolds we are interested in. In summary, defining

$$\hat{O}_p := \left\{ \gamma \in O(\Gamma_{4,20}) \middle| \gamma \circ \hat{\sigma}_p^\star = \hat{\sigma}_p^\star \circ \gamma \right\},$$

the full moduli space spanned by scalars in hypermultiplets reads

$$\mathcal{M}_{\rm H}^{p\neq 2} \cong \hat{O}_p \backslash SU\left(2, \tfrac{24}{\varphi(p)} - 2\right) / S\left[U(2) \times U\left(\tfrac{24}{\varphi(p)} - 2\right)\right].$$

We have checked that the above analysis is also compatible with the BPS indices obtained in [8], from which one can infer in particular the difference $n_V - n_H$ between the number of massless vector and hypermultiplets. It may also be noted that similar types of hypermultiplets moduli spaces have already been considered in the literature, see *e.g.* [133], where it was noticed in particular that (V.3.2) was indeed a quaternionic Kähler manifold.

This hypermultiplets moduli space does not receive by construction corrections in the string coupling g_s. Crucially, one can argue that it does not receive α' corrections as well. The moduli space is derived in the heterotic description from an exact toroidal CFT on the worldsheet and, due to the freely-acting nature of the orbifold, there are no moduli from the twisted sectors. Besides this, as was shown in chapter III, the heterotic models at hand do not admit any non-abelian gauge bundle hence there are no small-instanton singularities anywhere in the moduli space.

V.4 Summary

In this chapter, we have derived the moduli space of $\mathcal{N} = 2$ four-dimensional compactifications on non-geometric backgrounds, using both their description in the type IIA duality frame as non-geometric Calabi-Yau backgrounds [7] and their description in the heterotic frame as asymmetric and freely-acting toroidal orbifolds [8].

We have first analyzed the vector multiplets moduli space, which receives corrections in the string coupling both in type IIA and in heterotic frames. While, as we have shown, there are only non-perturbative corrections to the prepotential in the type IIA variables, the heterotic prepotential receives both one-loop and non-perturbative corrections w.r.t. the heterotic dilaton.

Thanks to an analysis of the perturbative duality group acting on the heterotic vector multiplets moduli space – or at least of a subgroup of it – we have shown how to obtain an explicit expression of the third derivative of the one-loop prepotential, using that the latter is a modular form in the T and U variables (*i.e.* the moduli of the heterotic T^2) with respect to a $\Gamma(p) \times \Gamma(p)$ subgroup of the duality group. We have given explicitely the result for mirrored automorphisms of order $p = 2$ and explained how to generalize this result to mirrored automorphisms of arbitrary order $p > 2$. It would be interesting to obtain explicit results in those cases as well.

Finally we have studied the hypermultiplets moduli space, which is exact in the string coupling constant on both sides of the duality. While obtaining the hypermultiplets moduli space from a type IIA perspective is not trivial (see [7] for a discussion), the heterotic description of the models as asymmetric toroidal orbifold allowed us to get an exact description of these moduli spaces both in α' and in g_s.

For the models studied here the situation is in some way the opposite of what was found for dualities between type IIA on Calabi-Yau threefolds and heterotic on $K3 \times T^2$. In the latter case, there exists a duality frame in which the vector moduli space can be computed classically, while the hypermultiplets receives corrections in both frames (either in g_s or in α') that are not yet fully understood. In the present case, while the hypermultiplet moduli space is exact (in a rather mundane way) the vector multiplets moduli space receives g_s corrections in any duality frame.

Conclusion

In this thesis, we have been interested in giving better understanding and control over a recent class of non-geometric constructions developed in [7] which provided the first explicit example of mirror-fold. From the adiabatic argument, we have been able to derive non-perturbative dualities between these models and toroidal orbifolds of the heterotic string. From this dual perspective, it became clear that the original type II orbifold had to be modified to be non-perturbatively consistent ; since the corresponding modification has no effect on the type IIA perturbation theory, a dual picture was necessary to realise this. The story is quite similar to what happens in the FHSV model [55] in this respect. From the heterotic picture, modular covariance imposes the rotation of the $\Gamma_{4,20}$ lattice to be accompanied not only by a shift along the two-torus T^2 but also along its T-dual $\tilde{T}^2$; in this sense, the heterotic backgrounds may be seen as realisations of T-folds.

We have also analysed the moduli spaces of such models and while the whole story has not yet been fully understood, very non-trivial results have been derived. The moduli space of four-dimensional $\mathcal{N} = 2$ theories is known to split into two contributions $\mathcal{M}_{\text{V}}$ and $\mathcal{M}_{\text{H}}$ spanned by scalars living in vector and hypermultiplets respectively. As opposed, this time, to the FHSV model, the dilaton lies in a vector multiplet in both dual frames. This means in particular that non-perturbative corrections to $\mathcal{M}_{\text{V}}$ may not be understood directly from either perspective. On the other hand, this allowed us to determine the hypermultiplet moduli space exactly in g_S, from what we just said, and in α', since the derivation has been made directly from a solvable CFT description. This is very unusual in Calabi-Yau compactifications where one typically has a lot less control over $\mathcal{M}_{\text{H}}$ than over $\mathcal{M}_{\text{V}}$. Moreover, while it is not clear how to determine the full $\mathcal{M}_{\text{H}}$ from the type IIA perspective, the problem was tackled in the heterotic frame using standard arguments. The perturbative corrections to $\mathcal{M}_{\text{V}}$, which is universal at tree-level, were shown to vanish in the type II theory from complex analysis requirements ; in the heterotic picture we have been able to compute the perturbative corrections to $\mathcal{M}_{\text{V}}$ extending the modular form analysis of [10, 11].

A possible extension of the work summarised in this thesis would be to try and systematise the construction of dual pairs related to those described in chapter III. In particular, these kinds of models may easily be handled from a heterotic perspective ; instead of motivating the research of a heterotic dual by an interesting type IIA model, one could hope to derive new non-geometric background for the type IIA string from heterotic toroidal orbifolds. In the same direction, we noticed that our theories could not admit points of gauge enhancement from states charged under the $\Gamma_{4,20}$ lattice. From the type II picture, this is expected as all space-time supersymmetry comes from the left-movers, implying the absence of Ramond-Ramond ground states. From the heterotic perspective, it may be understood by the fact that the quotienting automorphism leaves no $\Gamma_{4,20}$ sublattice invariant. An intriguing question worth investigating would therefore be to understand whether similar orbifolds could preserve such a sublattice in the heterotic picture and, if they could, to understand the gauge symmetry enhancement in the dual type IIA theory. It would also be interesting to generalise this construction in order to obtain $\mathcal{N} = 1$ dual pairs instead. However, although non-geometric Calabi-Yau backgrounds are known for the type II string [134, 135, 136], it is far from being obvious whether the duality may still hold in that case ; indeed, as far as four-dimensional theories are concerned, compactification on a Calabi-Yau three-fold leaves no room for a shift along an extra circle allowing to apply the adiabatic argument. One may therefore expect the duality to take a different form from what we have seen here, if any.

Another interesting direction would come from the gauged supergravity aspect of this work. Indeed, the low-energy limit of the models described here correspond to gauged supergravities that is, from what we discussed in II.4.1, to non-trivial solutions to the embedding tensor constraints. In particular, it should be possible to relate the corresponding gauging to fluxes as we emphasised at the end of subsection II.4.2. Working out the embedding tensor components in details would therefore be an interesting extension of this work. Finally, in addition to their utility as duality consistency checks, the supersymmetry-protected indices computed in III.3 might theoretically be used to derive results about the macroscopic entropy of the corresponding new $\mathcal{N} = 2$ black hole solutions.

Appendices

A Calabi-Yau manifolds

A.1 Calabi-Yau manifolds

Because of the relation between space-time supersymmetry and holonomy of the underlying internal space, manifolds of restricted holonomy play a special role in string compactifications. In this section, we recall a few useful facts needed throughout this thesis.

In general, the holonomy group $\mathfrak{h}$ of orientable d-dimensional Riemannian manifolds is contained in $SO(d)$. Restricting $\mathfrak{h}$ to be contained in a subgroup of $SO(d)$ puts constraints on the allowed geometries ; in particular, for even-dimensional manifolds with $d := 2n$, manifolds with holonomy group contained in $U(n)$ and $SU(n)$ are respectively Kähler and Calabi-Yau manifolds [137]. These complex manifolds may be defined as follows.

Definition A.1 (Kähler manifold)**.** Let X be a complex manifold equipped with a Hermitian metric g and with an almost complex structure I. Define the Kähler form J by

$$J(u, v) := g(Iu, v)$$

for any two tangent vectors u and v. X is defined to be a Kähler manifold if its Kähler form J is closed.

Calabi-Yau manifolds are a special kind of Kähler manifolds which may be defined in several equivalent ways. We have chosen here to present a definition involving the holonomy group as it may lead to the most straightforward connection to space-time supersymmetry considerations as we discussed in II.1.

Definition A.2 (Calabi-Yau manifold)**.** A Calabi-Yau n-fold X is a compact Kähler manifold of complex dimension n satisfying one of the following equivalent conditions [138] :
- The holonomy of X is contained in $SU(n)$.
- X admits a globally defined nowhere vanishing holomorphic n-form Ω.

In particular, one may show using Yau's theorem that a Calabi-Yau manifold always admits a Ricci-flat metric [138] ; consequently, a flux-free background consisting of a Calabi-Yau metric with constant dilaton provides a solution to the string equations of motion. Another slightly different definition commonly found in the literature is to require to holonomy group to be identified to $SU(n)$.

Truncations of dimensionally reduced models to the massless sector typically involves harmonic forms of the internal space. The Hodge decomposition gives a natural isomorphism between the vector space of harmonic p-forms and the p-th cohomology group which motivates the study of the latter. From general theorems, one may derive relations between the Dolbeault cohomology groups $H^{p,q}(X)$ of a manifold X. In particular, the Hodge numbers $h^{(p,q)} := \dim\left(H^{p,q}(X)\right)$ of a complex manifold X must satisfy

$$b_{p,q} = b_{n-p,n-q} \tag{A.20a}$$

$$b_{p,q} = b_{q,p} \tag{A.20b}$$

where (A.20a) and (A.20b) are a consequence of Serre duality [139] and of the fact that $H^{p,q}(X)$ and $H^{q,p}(X)$ are related by complex conjugation. In addition, for Calabi-Yau manifolds the $(n, 0)$-form naturally induces an isomorphim $H^{p,0}(X) \cong H^{n-p,0}(X)$ by associating to any $(p, 0)$-form $\alpha_{(p,0)}$ the unique $(n-p, 0)$-form $\beta_{(n-p,0)}$ satisfying

Cohomology group	Basis	Indices
$H^{1,1}(X)$	ω_a	$a = 1, ..., h^{(1,1)}$
$H^{2,2}(X)$	$\tilde{\omega}^a$	$a = 1, ..., h^{(1,1)}$
$H^{2,1}(X)$	χ_k	$k = 1, ..., h^{(2,1)}$
$H^3(X)$	$\left(\alpha_K, \beta^K\right)$	$K = 0, ..., h^{(2,1)}$

TABLE A.1 – Basis of harmonic forms for a Calabi-Yau three-fold X with holonomy $SU(3)$.

$$\int \alpha_{p,0} \wedge \beta_{n-p,0} \wedge \overline{\Omega} = 1.$$

Consequently, Hodge numbers associated to Calabi-Yau manifolds satisfy the additional constraint

$$h^{(p,0)} = h^{(n-p,0)}. \tag{A.21}$$

We now restrict temporarily to the case where X is a Calabi-Yau three-fold with $SU(3)$ holonomy. In this case, one may show that $h^{(1,0)} = 0$ [138]. Equations (A.20) and (A.21) then imply that the Hodge diamond of X reads

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The remaining unconstrained topological invariants are therefore the Hodge numbers $h^{(1,1)}$ and $h^{(2,1)}$.

We fix a basis $\left\{\omega_a, a = 1, ..., h^{(1,1)}\right\}$ of complex forms ω_a for the cohomology group $H^{1,1}(X)$; this also naturally suggests to define a basis $\left\{\tilde{\omega}^a, a = 1, ..., h^{(1,1)}\right\}$ for elements of $H^{2,2}(X)$ as

$$\int \omega_a \wedge \tilde{\omega}^b = \delta_a^b.$$

There are two convenient ways to expand harmonic 3-forms which we use here, depending on the context. The first one is to define a complex basis $\left\{\chi_k, k = 1, ..., h^{(2,1)}\right\}$ for $H^{(2,1)}(X)$; the second one is to define a basis of real forms $\left\{\left(\alpha_K, \beta^K\right), K = 0, ..., h^{(2,1)}\right\}$ for the whole $H^3(X)$ satisfying

$$\int \alpha_K \wedge \beta^L = \delta_K^L.$$

We summarise our notations in table A.1.

A.2 K3 surfaces

A Calabi-Yau manifold of dimension 2 may fall into two categories [12] : when equipped with a Ricci-flat metric, its holonomy group may either be trivial or be equal to $SU(2)$. The corresponding manifold is either a 2-dimensional complex torus in the former case or a K3 surface in the latter. In the following, we will focus on the latter which will be of the utmost importance in building theories in section III.1. Using equations (A.20a) and (A.20b), the Hodge diamond of a K3 surface is

$$\begin{array}{ccccc} & & 1 & & \\ & h^{(1,0)} & & h^{(1,0)} & \\ 1 & & h^{(1,1)} & & 1 \\ & h^{(1,0)} & & h^{(1,0)} & \\ & & 1 & & \end{array} .$$

Computing $h^{(1,0)}$ and $h^{(1,1)}$ may be done quite straightforwardly by noticing a very important mathematical fact about K3 surfaces : unlike 3-dimensional Calabi-Yau manifolds with $SU(3)$ holonomy which may have very different topologies, any two K3 surfaces are diffeomorphic to each other [34]. This means in particular that one could study any suitable K3 surface X and deduce all the topological invariants of any other K3 surface X' at once. This is done, *e.g.*, in [34] and lead to the result

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} .$$

The middle line of the above Hodge diamond plays an important role in studying K3 surfaces ; consequently, we shall take some time to talk about some of its features. First of all, one may show that the cup product defines a non degenerate symmetric bilinear form on $H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z})$ for any K3 surface X and that $H^2(X,\mathbb{Z})$ is torsion-free, giving $H^2(X,\mathbb{Z})$ a lattice structure [45] ; we will refer to this lattice in the following as the intersection lattice, the scalar product between two elements u and v of $H^2(X,\mathbb{Z})$ being equivalently given by the intersection number $D_u.D_v$ of the associated divisors D_u and D_v. Moreover, one may show that this lattice must have signature $(3,19)$ and that it is even and unimodular [37] ; by theorem I.3.1, it is therefore isomorphic to

$$\Gamma_{3,19} := E_8^{\oplus 2} \oplus U^{\oplus 3}$$

with $E_8(-1)$ and U defined in the aforementioned theorem. $\Gamma_{3,19}$ is usually referred to as the K3 lattice in the litterature. This lattice structure of the second cohomology group of K3 surfaces is the reason why lattice theory plays such an important role in the study of Calabi-Yau 2-folds.

In the following, we shall define additional mathematical objects which will play a crucial role in understanding the models to be defined in section III.1. The first of these is the Picard group of a variety.

12. Some authors define Calabi-Yau manifolds as compact Kähler manifold with holonomy $SU(n)$, unlike what we defined in A.2 where the honolomy condition is weaker. According to this modified definition, K3 surfaces would be the only kind of Calabi-Yau manifolds of dimension 2.

Definition A.3 (Picard group)**.** The Picard group $\mathrm{Pic}(X)$ of a variety X is the abelian group of isomorphism classes of line bundles on X. The addition law is the following : let L and L' be line bundles over X. Then, we define

$$[L] + [L'] := [L \otimes L']$$

with $[L]$, $[L']$ the corresponding elements of $\mathrm{Pic}(X)$.

Equivalently, one may think about the Picard group in the following sense : to any element g of $\mathrm{Pic}(X)$ corresponds a line bundle L such that $c_1(L) = g$ [34]. Compatibility with the above definition A.3 is ensured by the fact that $c_1(L \otimes L') = c_1(L) + c_1(L')$ for any two line bundles L and L' (see *e.g.* [140]). For K3 surfaces, the Picard group may be inferred from the cohomology groups and in particular from the intersection lattice ; the connection between these two objects is given by the Lefschetz theorem on $(1, 1)$-classes.

Theorem A.1 (Lefschetz theorem on (1,1)-classes [37])**.** *Let X be a compact Kähler manifold of dimension 2. Then, the Picard group of X satisfies*[13]

$$\mathrm{Pic}(X) \cong H^{1,1}(X) \cap H^2(X, \mathbb{Z}).$$

The above theorem shows in particular that the Picard group is defined with respect to a given complex structure and that the rank $\rho(X)$ of the Picard group of a K3 surface X - also known as the Picard number of X - may not take arbitrary large values as $\dim H^{1,1}(X) = 20$. In fact, one may show that if X is a projective K3 surface, then $\rho(X) \geq 1$ and moreover the Picard group has signature $(1, \rho(X) - 1)$ [34].

The second lattice we need to introduce for our analysis is the transcendental lattice $T(X)$ of a K3 surface X.

Definition A.4 (Transcendental lattice)**.** Let X be a K3 surface. A sub-Hodge structure $L \in H^2(X, \mathbb{Z})$ is said to be *primitive* if the quotient group $H^2(X, \mathbb{Z})/L$ is torsion-free. The transcendental lattice $T(X)$ of X is defined as the minimal primitive sub-Hodge structure of X whose complexification $T_{\mathbb{C}}(X)$ satisfies

$$T^{2,0}_{\mathbb{C}}(X) = H^{2,0}(X).$$

It is fairly easy to see that definition A.4 implies that the transcendental lattice is the orthogonal complement of the Picard lattice in $H^2(X, \mathbb{Z})$ for any K3 surface X. Indeed, it implies that any integral class of X orthogonal to $T(X)$ must in particular be orthogonal to $H^{2,0}(X)$ and therefore be an element of $\mathrm{Pic}(X)$ by theorem A.1, leading to $T(X)^{\perp} \subset \mathrm{Pic}(X)$. Moreover, theorem A.1 also implies that $H^{2,0}(X)$ is orthogonal to $\mathrm{Pic}(X)$; by minimality of $T(X)$, this in turn shows that $T(X) \subset \mathrm{Pic}(X)^{\perp}$. Recalling that $L \subset L^{\perp\perp}$ for any lattice L, the two above inclusions prove that $\mathrm{Pic}(X) = T(X)^{\perp}$. Since the intersection form is non-degenerate as we have seen, this is equivalent to saying that $T(X) = \mathrm{Pic}(X)^{\perp}$.

13. The notation $H^2(X, \mathbb{Z})$ is a little misleading here but lighter than a more rigorous one. To be precise, what we call $H^2(X, \mathbb{Z})$ here is actually its image under the natural embedding $H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{C})$.

A.3 K3 automorphisms

We conclude this appendix by discussing properties of the automorphism group of K3 surfaces, which are of great importance for building the models reviewed in III.1. To begin with, we first define what we call *Hodge isometries* in order to state a theorem which plays a crucial role in analysing and classifying K3 automorphisms, namely Torelli theorem.

Definition A.5 (Hodge isometry)**.** A Hodge isometry between two K3 surfaces X and Y is a map $\varphi : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ which preserves the cup product and maps $H^{2,0}(X)$ to $H^{2,0}(Y)$.

Theorem A.2 (Global Torelli theorem [37])**.** *Two K3 surfaces X and Y are isomorphic if and only if there exists a Hodge isometry $\varphi : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$. If φ maps a Kähler class on X to a Kähler class on Y, then the unique isomorphism $f : X \xrightarrow{\sim} Y$ identifies to the pullback ϕ^* of the above Hodge isometry.*

In particular, the second part of theorem A.2 states a very important mathematical fact : any isomorphism from a given K3 surface X to itself - that is any automorphism of X - may be equivalently described as an automorphism of the underlying lattice $H^2(X,\mathbb{Z})$. Phrased differently, lattice theory may be (and is) used in order to understand automorphisms of K3 surfaces.

That being said, there is an immediate consequence one can note about automorphism groups of finite order of K3 surfaces. Indeed, let X be a K3 surface, $G \subset \mathrm{Aut}(X)$ be a subgroup of its automorphisms and ω_X be a generator of the 1-dimensional Dolbeault cohomology group $H^{2,0}(X)$. As any element g of G preserves the decomposition of $H^2(X,\mathbb{C})$ into $(p, 2-p)$ forms and as $\dim H^{2,0}(X) = 1$, it follows in particular that the pullback g^* may only act on ω_X by multiplication by a non vanishing scalar $\alpha(g)$. One may construct this way a group homomorphism $\alpha : G \to \mathbb{C}^\times$. Therefore, if G has finite order, $\alpha(G)$ is a subgroup of $\mathbb{C}^\times$ of finite order and must then be cyclic. Two situations may then happen : either α maps g to $1 \in C^\times$ or to a non-trivial element of $C^\times$. In the first case, g therefore acts trivially on the holomorphic $(2,0)-$form and is said to be a *symplectic* automorphism ; in the latter, g acts non-trivially on ω_X and is said to be non-symplectic.

Classifying automorphisms of K3 surfaces depending on whether they preserve the holomorphic $(2,0)-$form has an immediate interest from a physical point of view. As we have seen before, a globally defined $(2,0)-$form may only exist on a manifold whose holonomy in contained in $SU(2)$. Let us then consider a string theory compactified on the quotient X/G. Depending on whether G preserves ω_X or not, ω_X will still be globally defined on X/G or not. If it is the case, quotienting by G still preserves half the maximal number of supercharges whereas if it is not, all supersymmetry is generically broken [141].

In constructing the models we will consider in the following, we will only deal with purely non-symplectic automorphisms of K3 surfaces, that is with automorphisms which act on the holomorphic $(2,0)-$form as multiplication by a primitive root of unity. As a matter of fact, any K3 surface admitting a non-symplectic automorphism is algebraic [45] and therefore projective [142]. This will allow us to restrict to such K3 surfaces in section III.1 and to study them as target spaces of sigma models.

B Buscher rules

This short section is devoted to give a set of equations known as the Buscher rules and originally derived in [143, 144]. The Buscher rules allow to relate background fields from two theories T-dual to each other. T-duality may be understood for NLSMs with curved background from a path integral approach. In particular, each vector field k with a nowhere vanishing norm and satisfying

$$\begin{aligned} \mathcal{L}_k G &= 0 \\ \iota_k H &= \mathrm{d}v \\ \mathcal{L}_k \phi &= 0 \end{aligned} \tag{B.22}$$

may be used to generate a T-duality transformation acting on the NLSM with background fields (G, B, ϕ). The first line of (B.22) shows that k is a Killing spinor of the metric G while the second one defines a one-form v up to an exact form. Working in a set of coordinates $\{x^i, \theta\}$ such that $k = \partial_\theta$, one shows that the background fields $\left(\hat{G}, \hat{B}, \hat{\phi}\right)$ of the corresponding dual theory may be expressed as [60]

$$\begin{aligned} \hat{G}_{ij} &= G_{ij} + \frac{v_i v_j - k_i k_j}{|k|^2}, \\ \hat{G}_{i\hat{\theta}} &= \frac{v_i}{|k|^2}, \\ \hat{G}_{\hat{\theta}\hat{\theta}} &= \frac{1}{|k|^2}, \\ \hat{B}_{ij} &= B_{ij} + \frac{k_i v_j - k_j v_i}{|k|^2}, \\ \hat{B}_{i\hat{\theta}} &= \frac{k_i}{|k|^2}, \\ \hat{\phi} &= \phi - \frac{1}{2}\log |k|^2. \end{aligned} \tag{B.23}$$

where $|k|^2 := k^i G_{ij} k^j$ and with $\hat{\theta}$ the coordinate dual to θ. These equations may also be seen in the simpler context where $\mathcal{L}_k B$ is actually vanishing instead of just being exact ; in this case, one may choose to set $v = -\iota_k B$, leading to the more familiar rules

$$\begin{aligned} \hat{G}_{ij} &= G_{ij} + \frac{B_{i\theta} B_{j\theta} - G_{i\theta} G_{j\theta}}{G_{\theta\theta}}, \\ \hat{G}_{i\lambda} &= \frac{B_{i\theta}}{G_{\theta\theta}}, \\ \hat{G}_{\lambda\lambda} &= \frac{1}{G_{\theta\theta}}, \\ \hat{B}_{ij} &= B_{ij} + \frac{G_{i\theta} B_{j\theta} - G_{j\theta} B_{i\theta}}{G_{\theta\theta}}, \\ \hat{B}_{i\lambda} &= \frac{G_{i\theta}}{G_{\theta\theta}}. \end{aligned}$$

C Partition Function Computations

ϑ functions

In this section, we give our conventions for ϑ functions and recall some of their modular properties that are useful in our computations. We define the Jacobi ϑ function with characteristic as

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau|v) := \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{\alpha}{2}\right)^2} e^{2i\pi\left(n+\frac{\alpha}{2}\right)\left(v+\frac{\beta}{2}\right)},$$

where $\alpha, \beta \in \mathbb{R}$ and where q is defined, as usual, by $q := \exp(2i\pi\tau)$. ϑ also admits the product representation [20]

$$\frac{\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau|v)}{\eta(\tau)} = e^{i\pi\alpha\left(v+\frac{\beta}{2}\right)} q^{\frac{\alpha^2}{8}-\frac{1}{24}} \prod_{n=1}^{\infty}\left(1+q^{n+\frac{\alpha-1}{2}}e^{2i\pi\left(v+\frac{\beta}{2}\right)}\right)\left(1+q^{n-\frac{\alpha+1}{2}}e^{-2i\pi\left(v+\frac{\beta}{2}\right)}\right),$$

where $\eta(\tau)$ is the Dedekind η function defined by

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

The well-known modular properties of the ϑ functions makes them functions are a powerful tool in constructing modular invariant quantities. Their behaviour under the generators of $SL(2,\mathbb{Z})$ are given by

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau+1|v) = e^{-\frac{i\pi}{4}\alpha(\alpha-2)}\vartheta\begin{bmatrix}\alpha\\ \alpha+\beta-1\end{bmatrix}(\tau|v)\,,$$

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(-\frac{1}{\tau}\middle|\frac{v}{\tau}\right) = e^{\frac{i\pi}{2}\alpha\beta+\frac{i\pi}{\tau}v^2}\vartheta\begin{bmatrix}-\beta\\ \alpha\end{bmatrix}(\tau|v)\,.$$

It is also easy to show that the arguments α and β satisfy the periodicity properties

$$\vartheta\begin{bmatrix}\alpha+2\\ \beta\end{bmatrix}(\tau|v) = \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau|v) \quad\text{and}\quad \vartheta\begin{bmatrix}\alpha\\ \beta+2\end{bmatrix}(\tau|v) = e^{i\pi\alpha}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau|v)\,.$$

In the following, we will drop the explicit τ dependence of the ϑ functions and write only $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](v)$ (or simply $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ if $v=0$). An especially useful identity when it comes to computing BPS indices for instance is the famous Jacobi abstruse identity which allows one to sum over spin structures and which reads [19]

$$\frac{1}{2}\sum_{\alpha,\beta=0}^{1}(-1)^{\alpha+\beta+\alpha\beta}\prod_{i=1}^{4}\vartheta\begin{bmatrix}\alpha+h_i\\ \beta+g_i\end{bmatrix}(v_i) = -\prod_{i=1}^{4}\vartheta\begin{bmatrix}1-h_i\\ 1-g_i\end{bmatrix}(v_i')$$

provided that $\sum_i h_i = \sum_i g_i = 0$; here,

$$v_1' = \frac{1}{2}(-v_1+v_2+v_3+v_4) \quad , \quad v_2' = \frac{1}{2}(v_1-v_2+v_3+v_4)$$
$$v_3' = \frac{1}{2}(v_1+v_2-v_3+v_4) \quad , \quad v_4' = \frac{1}{2}(v_1+v_2+v_3-v_4).$$

As recalled in equation (III.6), the eigenvalues of the $\Gamma_{4,20}$ automorphism $\hat{\sigma}_p$ used to generate the $\mathbb{Z}_p$ orbifold of section III.1.2 are all primitive p-th roots of unity, each with the same multiplicity. Consequently, when computing one-loop quantities, the product

$$\Theta_g^{(p)}(\tau) := \prod_{\substack{t=1\\(t,p)=1}}^{p-1} \vartheta\begin{bmatrix}1\\1-2gt/p\end{bmatrix}(\tau)$$

usually appears through the contribution of the untwisted sector. Here, (t,p) is a shorthand notation for $\gcd(t,p)$. It will then be helpful, at least for numerical computations, to notice that $\Theta_g^{(p)}$ may be rewritten as

$$\Theta_g^{(p)}(\tau) = e^{\frac{i\pi}{2}(1-g)\varphi(p)}\eta^{\varphi(p)}(\tau)\left[\Phi_{x_g}(1)\prod_{d|x_g}\eta(d\tau)^{2\mu\left(\frac{x_g}{d}\right)}\right]^{\frac{\varphi(p)}{\varphi(x_g)}} \tag{C.25}$$

with $x_g := \frac{p}{(g,p)}$, φ the Euler totient function as before, μ the Möbius function and $\Phi_n(x)$ the n-th cyclotomic polynomial.

Restrictions on the shift vector

Let us first show that it is always possible to find a representative of the shift vector in $\frac{1}{p}\Gamma_{2,2}/\Gamma_{2,2}$ such that

$$p^2\alpha^i\beta_i = \Psi_p\,,$$

so equation (III.9) holds strictly, not just modulo p. First, one may note that Ψ_p and p must be coprime, as follows from $\gcd(s,p) = 1$ - the latter being imposed by equation (III.6). Assuming that one starts with a shift vector δ satisfying equation (III.9), this means now that $\gcd(\alpha^1,\alpha^2,p) = 1$ as well since equation (III.9) would not admit a solution for β_1 and β_2 otherwise. In such a case, it is always possible to define $\widetilde{\alpha}^1 := \alpha^1$ and $\widetilde{\alpha}^2 := \alpha^2 + tp$ for some integer t so that $\gcd(\widetilde{\alpha}^1,\widetilde{\alpha}^2) = 1$; indeed, the existence of a solution to

$$\begin{cases} t = 1 \bmod q & \forall \text{ prime } q \,|\, \gcd(\alpha^1,\alpha^2) \\ t = 0 \bmod q' & \forall \text{ prime } q' \mid \alpha^1 \text{ and } q' \nmid \gcd(\alpha^1,\alpha^2) \end{cases}$$

is guaranteed by the Chinese remainder theorem [14], and one may show that such an integer t would lead to $\gcd(\widetilde{\alpha}^1,\widetilde{\alpha}^2) = 1$ as required. Bézout's identity [15] then finally ensures us that there exist integers $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ with $\widetilde{\beta}_i = \beta_i \bmod p$ such that $\widetilde{\alpha}^i\widetilde{\beta}_i = \Psi_p$, so that we can indeed choose a representative of any given vector shift δ satisfying (III.9) strictly.

14. Which proves more generally the existence of a solution to

$$x = x_i \bmod p_i \quad i = 1, \ldots, n$$

for any set or pairwise coprime integers $\{p_i, i = 1, \ldots, n\}$.

15. Which states that

$$\alpha x + \beta y = \gamma$$

admits a solution for (α,β) if and only if $\gcd(x,y)\,|\,\gamma$; in particular, it therefore ensures the existence of solutions to the above equation for any integer γ in the case where x and y are coprime integers.

We now give more details about how one gets to equation (III.8). First, it may be shown (see *e.g.* [145]) that

$$\sum_{\substack{a=1\\ \gcd(a,p)=1}}^{p} a^k = \sum_{d|p} \mu\left(\frac{p}{d}\right)\left(\frac{p}{d}\right)^k \sum_{a=1}^{d} a^k$$

for any integers k and p, μ being here the Möbius function (that is the inverse of the constant function 1 under Dirichlet involution). This allows one to show in particular that

$$\sum_{\substack{a=1\\ \gcd(a,p)=1}}^{p} a = \frac{1}{2} p\varphi(p)$$

$$\sum_{\substack{a=1\\ \gcd(a,p)=1}}^{p} a^2 = \varphi(p)\left[\frac{1}{3}p^2 + \frac{1}{6}\prod_{\substack{q|p\\ q\,\text{prime}}}(-q)\right]$$

for all $p > 1$, where the product in the last equation runs over prime factors of p. The repartition of the eigenvalues of γ given in (III.6) then leads to the simplification (III.8) as claimed in section III.2.4.

D Modular forms

In this appendix, we briefly review important features of modular forms which play a key role in the discussion of chapter V. In string theory, the discrete group $SL_2(\mathbb{Z})$ of integer-valued matrices with unit determinant and subgroups thereof are commonly encountered. In particular, the complex structure of a two-torus is only defined up to $SL_2(\mathbb{Z})$ transformations which implies that any one-loop correlation function - that is computed on the torus - must be $SL_2(\mathbb{Z})$ invariant with respect to the modular parameter (denoted τ in this thesis). We also recall from section I.3.1 that compactifications involving a two-torus are expected to have

$$O(\Gamma_{2,2}) = (SL_2(\mathbb{Z})_T \times SL_2(\mathbb{Z})_U) \rtimes \mathbb{Z}_2$$

as a subgroup of the perturbative duality group. Orbifolds thereof typically break $SL_2(\mathbb{Z})$ to a subgroup $\Gamma(\mathbb{Z})$ as illustrated in V.2.2. Perturbative corrections to correlation functions are expected to be well-behaved under such duality transformations. This motivates our analysis of modular forms, to be defined in D.1. A complete introduction to the subject as well as proofs for every claim in this appendix may be found in [30].

An important subgroup of $SL_2(\mathbb{Z})$ is the *principal congruence subgroup of level N* $\Gamma(N)$, defined as

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

In particular, $\Gamma(1)$ identifies to the whole $SL_2(\mathbb{Z})$. A subgroup $\Gamma(\mathbb{Z})$ of $SL_2(\mathbb{Z})$ is called a *congruence subgroup* if there exists an integer N such that $\Gamma(N) \subset \Gamma(\mathbb{Z})$. Principal congruence subgroups trivially satisfy this condition ; another example appearing in this thesis is the *Hecke congruence subgroup of level N* $\Gamma_0(N)$ defined as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c = 0 \mod N \right\}.$$

In the following, we will be interested in functions of the upper-half complex plane $\mathcal{H} := \{\tau \in \mathbb{C}, \operatorname{Im}\tau > 0\}$ and more precisely of its "compactification" (in the mathematical sense) $\hat{\mathcal{H}} := \mathcal{H} \cup \{\infty\}$.. From what we have said above, it might seem interesting to find functions $f(\tau)$ such that $f(\gamma \cdot \tau) = f(\tau)$ for any γ in a given congruence subgroup $\Gamma(\mathbb{Z})$, where γ acts on τ as a fractional linear transformation

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}, \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(\mathbb{Z})$$

as in the string theoretical context. However, one may show that there are infinitely many linearly independent such functions. It turns out to be more fruitful to turn our invariance requirement into a covariance one and to add holomorphicity constraints ; introducing the *factor of automorphy* $j(\gamma, \tau)$ as

$$j(\gamma, \tau) = c\tau + d, \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(\mathbb{Z}),$$

we then define a *modular form of weight* k *with respect to* $\Gamma(\mathbb{Z})$ as follows.

Definition D.1 (Modular form with respect to a congruence subgroup)**.** Let $\Gamma(\mathbb{Z})$ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let k be an integer. A function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight k with respect to $\Gamma(\mathbb{Z})$ if

— f is holomorphic,

— f satisfies the covariance condition

$$f(\gamma \cdot \tau) = j^k(\gamma, \tau) f(\tau)$$

for all $\gamma \in \Gamma(\mathbb{Z})$,

— The function $f[\gamma]_k$ defined by

$$(f[\gamma]_k)(\tau) := j^{-k}(\gamma, \tau) f(\gamma \cdot \tau)$$

is holomorphic at ∞ for all $\gamma \in \Gamma(\mathbb{Z})$.

Modular functions may then be seen as functions of Riemann surfaces given by the quotient $\mathcal{F}_{\Gamma(\mathbb{Z})} := \hat{\mathcal{H}}/\Gamma(\mathbb{Z})$; $\mathcal{F}_{\Gamma(\mathbb{Z})}$ is known as the *fundamental domain* of $\Gamma(\mathbb{Z})$.

A modular function with respect to a congruence subgroup of level N then satisfies in particular $f(\tau + N) = f(\tau)$, as

$$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N).$$

Consequently, there exists a minimal integer $h \in \mathbb{N}$ such that $f(\tau+h) = f(\tau)$; f may therefore be expanded as [16]

16. The series (D.27) run over positive integers only as a consequence of the holomorphicity of f at ∞.

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^{n/h}. \tag{D.27}$$

Moreover, it is easy to see from definition D.1 that the space $\mathcal{M}_k$ of modular forms of weight k form a vector space over $\mathbb{C}$; by contrast, a very non-trivial statement is that this vector space has finite dimension [30]. This means that any modular form of weight k may be expressed in terms of a finite number of basis functions for $\mathcal{M}_k$. In other words, one only need to know a finite number of terms in the expansion (D.27) to fully characterise f.

Generically, a point in the upper-half plane $\mathcal{H}$ is only preserved by $\gamma \in \Gamma(\mathbb{Z})$ if $\gamma = \pm\mathbb{1}$. However, depending on the congruence subgroup of interest, some special points may be preserved by non-trivial transformations ; such points, if any, are known as *elliptic points.* One may show that elliptic points may be either related to i or to $\rho := \exp\left(\frac{2i\pi}{3}\right)$ by a $SL_2(\mathbb{Z})$ transformation ; the subgroup preserving them has order 4 in the former case and 6 in the latter. In a more physical context, elliptic points are typically related to points in the moduli space where states become massless. As an example, this explains - from a complex analysis point of view - the location of the points of enhanced symmetry in the vector multiplet moduli space of the reduction of the heterotic string on K3 $\times\, T^2$ analysed in [11].

Another class of interesting points is given by the set of images of ∞ under $SL_2(\mathbb{Z})$ with $\Gamma(\mathbb{Z})$-related points identified ; these images belong to $(\mathbb{Q} \cup \{\infty\})/\Gamma(\mathbb{Z})$ and are known as the *cusps* of $\Gamma(\mathbb{Z})$. Understanding why these points are important from a physical point of view may be done by considering an example relevant to the analysis of chapter V. Let us then assume that we are looking at the moduli space of a theory reduced on a two-torus and that the congruence subgroup of interest is contained in $SL_2(Z)_T$, the part of $O(\Gamma_{2,2})$ acting non-trivially on the Kähler modulus T only. In this context, the cusps of $\Gamma(\mathbb{Z})$ correspond to limits of T leaving the complex structure U fixed. Let us consider first the cusp ∞, corresponding to the $T \to \infty$ limit ; as $\text{Im}(T)$ is the volume of the two-torus, this corresponds to a decompactification limit of the system [17]. As we argue in V.2.1, the other cusps may be also be understood in terms of decompactification limits as far as the models analysed in this thesis are concerned.

In the following section, we give explicit formulas for the modular functions of the principal congruence subgroup of level 2 $\Gamma(2)$ which are used in particular in section V.2.5 so as to give an example illustrating the strategy developed in chapter V.

E $\Gamma(2)$ modular forms

The ring of modular forms of $\Gamma(2)$ is known to be isomorphic to the ring of modular forms of $\Gamma_0(4)$ (the set of $SL_2(\mathbb{Z}$ elements with vanishing lower-left component modulo 4), see *e.g.* [121] for details.

A basis of the modular ring of $\Gamma(2)$ is given by the modular forms X_1 and X_2, defined as :

$$X_1(\tau) := E_2(\tau) - 2E_2(2\tau) \qquad X_2(\tau) := E_2\left(\frac{1}{2}\tau\right) - 4E_2(2\tau)\,,$$

17. More precisely, the, say, $T \to \infty$, U fixed limit correspond to the limit where the radii of both cycles of the two-torus tend to infinity.

with E_2 the Eisenstein series of weight 2 defined as :

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \,,$$

$\sigma_1(n)$ being the sum of the divisors of n, which is not a modular form.

Modular forms of weight $2k$ of $\Gamma(2)$ have exactly k zeroes ; it will therefore be useful to also define :

$$\begin{aligned} V_0(\tau) &:= \frac{6X_1(\tau) - X_2(\tau)}{48} \\ V_1(\tau) &:= -\frac{X_2(\tau)}{48} \\ V_\infty(\tau) &:= \frac{3X_1 - X_2(\tau)}{24} \end{aligned}$$

so that V_s vanishes as $\sqrt{q} + \mathcal{O}(q)$ around the cusp s, with $s = 0, 1, \infty$ and $q := \exp(2i\pi\tau)$.

The Hauptmodul for the congruence subgroup $\Gamma(2)$ is the well-known λ function, expressed in terms of the above as follows :

$$\lambda(\tau) := 16 \frac{V_1(\tau)}{V_\infty(\tau)} \,.$$

A useful relation, allowing one to compactly write the results of section V.2.5, is :

$$\prod_{\gamma \in SL_2(\mathbb{Z})/\Gamma(2)} \Big[\lambda(U) - \lambda(\gamma \cdot T) \Big] = \frac{\Delta(U)}{V_\infty^6(U)} \Big[j(U) - j(T) \Big] \tag{E.28}$$

with j the Klein j function, that is the $SL_2(\mathbb{Z})$ j-invariant, and $\Delta(U)$ the cusp form of $SL_2(\mathbb{Z})$.

This peculiar occurrence of modular functions of $SL_2(\mathbb{Z})$ instead of just $\Gamma(2)$ is proper to the $p = 2$ case. It is linked to the fact that the singular lines are located at $T = \gamma \cdot U$ for all γ in the coset $SL_2(\mathbb{Z})/\Gamma(2)$, so that in the end one has a singular line whenever $T = g \cdot U$ for any g in $SL_2(\mathbb{Z})$ (even though every such g would not lead to physically equivalent configurations).

The modular form $P_{10,10}$ appearing in equation (V.19) reads, in terms of the above modular forms,

$$
\begin{aligned}
P_{10,10}(T,U) = {} & -32X_1(T)^5X_2(U)^5 + 288X_1(T)^5X_1(U)X_2(U)^4 \\
& - 576X_1(T)^5X_1(U)^2X_2(U)^3 + 24X_1(T)^4X_2(T)X_2(U)^5 \\
& - 8X_1(T)^4X_2(T)X_1(U)X_2(U)^4 - 2304X_1(T)^4X_2(T)X_1(U)^2X_2(U)^3 \\
& + 12672X_1(T)^4X_2(T)X_1(U)^3X_2(U)^2 + 20736X_1(T)^4X_2(T)X_1(U)^5 \\
& - 25920X_1(T)^4X_2(T)X_1(U)^4X_2(U) - 4X_1(T)^3X_2(T)^2X_2(U)^5 \\
& - 192X_1(T)^3X_2(T)^2X_1(U)X_2(U)^4 + 2848X_1(T)^3X_2(T)^2X_1(U)^2X_2(U)^3 \\
& - 20736X_1(T)^3X_2(T)^2X_1(U)^5 - 13248X_1(T)^3X_2(T)^2X_1(U)^3X_2(U)^2 \\
& + 26496X_1(T)^3X_2(T)^2X_1(U)^4X_2(U) + 88X_1(T)^2X_2(T)^3X_1(U)X_2(U)^4 \\
& + 7488X_1(T)^2X_2(T)^3X_1(U)^5 - 1104X_1(T)^2X_2(T)^3X_1(U)^2X_2(U)^3 \\
& + 4960X_1(T)^2X_2(T)^3X_1(U)^3X_2(U)^2 - 9792X_1(T)^2X_2(T)^3X_1(U)^4X_2(U) \\
& - 1152X_1(T)X_2(T)^4X_1(U)^5 - 15X_1(T)X_2(T)^4X_1(U)X_2(U)^4 \\
& + 184X_1(T)X_2(T)^4X_1(U)^2X_2(U)^3 - 816X_1(T)X_2(T)^4X_1(U)^3X_2(U)^2 \\
& + 1576X_1(T)X_2(T)^4X_1(U)^4X_2(U) + 64X_2(T)^5X_1(U)^5 \\
& + X_2(T)^5X_1(U)X_2(U)^4 - 12X_2(T)^5X_1(U)^2X_2(U)^3 \\
& + 52X_2(T)^5X_1(U)^3X_2(U)^2 - 96X_2(T)^5X_1(U)^4X_2(U)
\end{aligned}
\tag{E.29}
$$

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Yoan GAUTIER **4 décembre 2020**

Sujet : Compactifications Calabi-Yau non-géométriques en théorie des cordes

Résumé : Les théories des cordes font partie des candidats à une description quantique de la gravité tout en offrant un cadre théorique unifié permettant de décrire les quatre interactions fondamentales connues à ce jour. Vers le milieu des années 1990 a été émis l'hypothèse que les cinq théories des supercordes cohérentes étaient reliées entre elles par des relations de dualité. Dans cette thèse, nous construisons les duaux hétérotiques de modèles non-géométriques obtenus à partir de la réduction de la corde de type IIA sur un orbifold libre de $K3 \times T^2$; en particulier, nous montrons comment des conditions de cohérence non-perturbatives doivent être prises en compte. Ces modèles préservent N=2 supersymétries en quatre dimensions. Nous calculons en particulier, dans l'approche hétérotique, des indices protégés par la supersymétrie construits à partir d'états privilégiés appelés états BPS et fournissant des renseignements sur les états solitoniques dans la théorie type IIA. Enfin, nous analysons l'espace des modules sous-jacent - c'est-à-dire la variété engendrée par les champs scalaires de masse nulle - dans les formulations hétérotiques et type IIA. En particulier, contrairement au cas standard de réduction (géométrique) sur une variété de Calabi-Yau où la sous-variété engendrée par les champs scalaires vivant dans des hypermultiplets est difficile à analyser, nous montrons qu'il est possible d'obtenir dans notre cas son expression exacte en théorie des perturbations.

Mots clés : Non-géométrique, dualité, non-perturbatif, modules, hétérotique, type IIA

Subject : Non-geometric Calabi-Yau compactifications in string theory

Abstract: String theories are candidates for a quantum formulation of gravity and grant a unified framework to describe all known interactions. In the mid-90s, it was conjectured that all superstring theories were related to one another through a web of dualities; these dualities play a key role in string theory as they allow to understand non-perturbative features which would otherwise remain out of reach. In this thesis, we derive dual heterotic models to a class of non-geometric constructions obtained from the reduction of the type IIA string on a free orbifold of $K3 \times T^2$; in particular, we show how non-perturbative consistency conditions invisible from either description alone must be taken into account. These models preserve N=2 supersymmetry in four dimensions; while such models only contain non-chiral multiplets and may therefore not, as such, be phenomenologically relevant, they are very important from a theoretical point of view as they are both simple enough to allow for a detailed analysis and unconstrained enough to exhibit interesting features. We compute in particular supersymmetry-protected indices, constructed from special states in the spectrum known as BPS states, in the heterotic frame which are expected to give informations about solitonic states in the type IIA perspective. Finally, we analyse the moduli space – that is the manifold spanned by massless scalar fields – from both heterotic and type IIA formulations; in particular, while the submanifold spanned by scalars living in hypermultiplets is hard to analyse for usual (geometric) Calabi-Yau compactifications of the type II string, we show that it may be exactly derived in perturbation theory in our case.

Keywords : Non-geometric, duality, non-perturbative, moduli, heterotic, type IIA