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Global monopoles in Horndeski theory

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Abstract In this article, global monopole solutions will be presented for the Horndeski theory by solving linearized field equations. We obtain two different classes of solutions depending on whether the Horndeski scalar field is massless or massive. Several properties of these solutions and the motion of test particles around them are discussed in detail, compared with the existing solutions in the literature. The mass due to an arbitrary potential of the scalar field changes the asymptotic behavior of the monopole compared to the massless case by applying a screening effect, which makes the monopole to asymptotically behave similarly to those in General Relativity. We also discussed the Galaxy rotation curves, and saw that only for the massless case and the massive case with very light scalar mass, the flattening of the galaxy rotation curves may be explained with global monopoles.

1 Introduction

Global monopoles are very peculiar, exotic objects with quite interesting properties. Their possible production in the early universe is brought forward in General Unified Theories [1,2] when a global $O(3)$ symmetry is spontaneously broken to $U(1)$ [3] during the symmetry breaking phase transitions. In General Relativity, these monopoles have distinct properties such as the spacetime around a global monopole has a solid deficit angle, and they possess no active gravitational mass [3], despite the tiny negative mass of the core [4]. They are believed to contribute to structure formation in the early universe [5], along with other topological defects, but observations revealed that they cannot be the primary mechanism and their contribution is rather limited, but not completely ruled out [6–8]. They also have interesting cosmological outcomes since they are very dense objects and should dominate the early universe after being formed. The inflation mechanism can resolve this monopole problem by the exponential expansion of the universe during this phase, which dilutes away the monopoles. For a review of topological defects and their astrophysical and cosmological implications, we refer to [9]. It is worth mentioning that there are numerous interesting physical applications of global monopoles [10–16]. Understanding the effects of global monopoles and other topological defects, and their interaction with the surrounding matter and other fields, is an important topic concerning their cosmological and gravitational implications.

Since the interest in modified gravity theories is not fading away due to the growing number of accumulated observational data and theoretical considerations [17–21], it may be worthwhile to study physical outcomes of well established concepts of standard general relativity in the modified gravity theories. Therefore, in this regard, we consider global monopoles and try to establish some of their physical outcomes into a particular modified theory. Therefore, in this article, we would like to investigate the gravitational field of a global monopole in the Horndeski theory [22]. It is the most general scalar–tensor theory leading to second order field equations in four dimensions. Hence, any scalar–tensor theory having this property, including General Relativity, Brans–Dicke theory (BD) [23], f(R) theories, etc, is a special case of Horndeski theory [24]. Thus, Horndeski theory is a framework that covers an important subclass of the modified gravity theories. Since the complexity of the field equations of this theory and also the fact that even in more simpler theories such as Brans–Dicke theory the corresponding global monopole solution is usually found in the linearized field equations for BD [25] and a scalar–tensor theory, [26], we also keep our discussion on the linearized level. A global monopole solution [27] is discussed for massless or massive dilaton gravity, where the scalar field is non-

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minimally coupled to matter-energy Lagrangian. Recently, a global monopole solution in the framework of extended Gauss-Bonnet theory with a positive ADM mass is presented in [28]. There are many other works [29–32] obtaining global monopole solutions in various modified gravity theories.

Despite staying in the linearized level, several phenomenological questions can be answered, such as the behavior of the spacetime outside the core of the monopole and the behavior of test particles around such monopoles. The generality of this theory will lead to a great phenomenology depending on the properties of the scalar field. It could involve a scalar potential that might change the local and global behavior of the underlying spacetime. In other words, the fact that the scalar field can be massless or massive in the Horndeski theory leads to different characteristics for the solutions obtained. Therefore, global monopole solutions and their physical properties reveal the effects of massive theory compared to massless one. The mass of the scalar field, originated from a scalar potential function, has the property that it generally leads to a massive theory [33–35], rendering the scalar field short range. Resulting massive theory will have dramatically different local and asymptotic behavior, such as showing a screening effect [36–38] where the scalar field is frozen beyond a specific range determined by the mass of the scalar field, compared to the massless theories where the scalar field has a long range as in original BD theory [23]. Therefore, the obtained solutions might be another good example of the realization of such a mechanism.

The article is organized as follows. In Sect. 2, we review the weak field equations and the energy momentum tensor of a global monopole in Horndeski theory. We present solutions of linearized field equations of Horndeski theory for a global monopole for both the massless and massive cases in Sect. 3. The next section will be devoted to obtaining several physical implications of these solutions, such as the angular deficit they possess, the force they apply on nonrelativistic test particles, and geodesic particle motion around such monopoles for massless and massive cases. Section 5 is devoted to galaxy rotation curves, which might contribute to both cases. We conclude this paper with a short discussion.

2 Field equations

The Lagrangian of the action $S = \int \mathcal{L}\sqrt{|g|}d^4x$ of Horndeski theory [22], also known as the generalized Galileon theory [39], can be written as [24]:

$$\begin{aligned}\mathcal{L} &= K(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R \\ &\quad + G_{4,X}\left[(\Box\phi)^2 - (\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)\right] \\ &\quad + G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ &\quad - \frac{G_{5,X}}{6}\Big[(\Box\phi)^3 - 3\Box\phi\,(\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi) \\ &\quad + 2(\nabla_\mu\nabla^\nu\phi)(\nabla_\nu\nabla^\lambda\phi)(\nabla_\lambda\nabla^\mu\phi)\Big], \end{aligned} \tag{1}$$

where $X = -\nabla^\mu\nabla_\mu\phi/2$ is a kinetic term and K, G_3, G_4, G_5, are arbitrary functions of the scalar field ϕ and the kinetic term X. The comma denote partial differentiation for the variable after the comma, such as $f_{,X} = \partial f/\partial X$. We can add an appropriate minimally coupled matter Lagrangian $\mathcal{L}_m$ to this Lagrangian to explore behavior of the matter-energy distribution in this theory. The field equations of this theory are rather complex and we will refer to the work [40] for those equations.

2.1 Linearized field equations

In the paper we will consider linearized field equations and their solutions of Horndeski theory. To achieve that, we assume a weak field expansion of the metric tensor and the scalar field as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{2}$$

$$\phi = \phi_0 + \varphi, \tag{3}$$

with $\eta_{\mu\nu}$ is the Minkowski metric, $h_{\mu\nu}$ is the metric perturbation tensor with $|h_{\mu\nu}| \ll 1$, ϕ_0 a constant and φ is the scalar perturbation function with again $\varphi \ll 1$. An arbitrary function of the scalar field and the kinetic term X can be expanded as

$$\begin{aligned} f(\phi, X) &= f(\phi_0, X_0) + f_{,\phi}(\phi_0, X_0)(\phi - \phi_0) \\ &\quad + f_{,X}(\phi_0, X_0)(X - X_0) + \ldots \\ &= f(\phi_0, 0) + f_{,\phi}(\phi_0, 0)\varphi + O(\varphi^2) \end{aligned} \tag{4}$$

around $\phi = \phi_0$ and $X = X_0 = 0$. Hereafter we use the abbreviation $f(\phi_0, 0) = f(0)$ for clarity.

Reading up linear terms in [40] and adding matter-energy Lagrangian and corresponding energy–momentum tensor, and also setting $K(0) = K_{,\phi}(0) = 0$ to have an asymptotically flat background (characteristic deviations from asymptotic flatness due to monopole term in the energy–momentum tensor will be encoded into $h_{\mu\nu}$), since they can play the role of a cosmological constant [41], we have the following field equations in linear order in h and φ as [42,43]:

$$G_4(0)G^{(1)}_{\mu\nu} + G_{4,\phi}(0)\left(\eta_{\mu\nu}\Box_\eta\varphi - \varphi_{,\mu\nu}\right) = \kappa\, T^{(1)}_{\mu\nu}, \tag{5}$$

$$K_{\phi\phi}(0)\,\varphi + \left[K_{,X}(0) - 2G_{3,\phi}(0)\right]\Box_\eta\varphi + G_{4,\phi}(0)R^{(1)} = 0. \tag{6}$$

Using the Eq. (5) into (6) we bring the scalar field equation in the form as

$$(\Box_\eta - m^2)\varphi = \kappa'\, T^{(1)}, \tag{7}$$

with the mass of the scalar field, m, and coupling constant κ' are defined as

$$m^2 = -\frac{K_{,\phi\phi}(0)}{K_{,X}(0) - 2G_{3,\phi}(0) + 3G_{4,\phi}^2/G_4(0)}, \tag{8}$$

$$\kappa' = \frac{G_{4,\phi}(0)\,\kappa}{G_4(0)\left[K_{,X}(0) - 2G_{3,\phi}(0)\right] + 3G_{4,\phi}^2(0)}. \tag{9}$$

The Eq. (5) can also be put in a convenient form by expanding $G_{\mu\nu}^{(1)}$ in terms of $h_{\mu\nu}$ and then transforming the metric perturbation terms into a new perturbation tensor $\theta_{\mu\nu}$ defined as

$$\theta_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h - \eta_{\mu\nu}\frac{G_{4,\phi}(0)}{G_4(0)}\varphi. \tag{10}$$

We also employ a Lorenz gauge

$$\theta^{\mu}{}_{\nu;\mu} = 0, \tag{11}$$

then the Eq. (5) takes a much simple form

$$\Box_\eta \theta_{\mu\nu} = -\frac{2\kappa}{G_4(0)} T_{\mu\nu}^{(1)} = -2\kappa_1 T_{\mu\nu}^{(1)}, \tag{12}$$

where we have defined

$$\kappa_1 = \frac{\kappa}{G_4(0)} \tag{13}$$

for clarity. The Eqs. (7) and (12), together with the gauge condition (11), are the main equations we will employ to obtain the desired solutions. In the following subsection, we will discuss the linearized energy–momentum tensor that corresponds to global monopoles. The above equations agree with linearized Horndeski field equations employed in the works [42,43], mainly to the discussion of gravitational waves.

2.2 Global monopoles and their energy–momentum tensor

The Lagrangian giving rise to a global monopole is constructed by a scalar field triplet Φ_a, $a = 1, 2, 3$, given by

$$L_{\text{mon}}[\Phi^a] = \partial^\mu \Phi^a \partial_\mu \Phi^a - \frac{\lambda}{4}\left(\Phi^a\Phi^a - \eta^2\right)^2, \tag{14}$$

having a global $O(3)$ symmetry. By a spontaneous symmetry breaking mechanism, global monopoles are formed with the energy scale η. Since a global monopole has spherical symmetry, it might be convenient to consider the Minkowski metric in spherical coordinates with $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$. Hence, from now on, we use the Minkowski metric in spherical coordinates, where

$$\eta_{\mu\nu} = \text{diag}(-1, 1, r^2, r^2\sin^2\theta), \tag{15}$$

unless stated otherwise. The reason for using these coordinates is the fact that expressing the energy–momentum tensor of a monopole in Cartesian coordinates will give more cumbersome equations to deal with. Hence, from now on, the differential operators assume their flat spherical counterparts with partial derivatives replaced with covariant ones in the relevant equations [44]. Using the usual monopole ansatz [2]

$$\Phi^a = \frac{\eta f(r) x^a}{r}, \tag{16}$$

with $x^a x^a = r^2$, and considering the fact that $f(r) \simeq 1$ outside the core of the monopole [2], the nonzero components of the energy momentum tensor becomes

$$T_{00}^{\text{mon}} \approx -T_{rr}^{\text{mon}} \approx \frac{\eta^2}{r^2}. \tag{17}$$

We will consider this energy–momentum tensor (17) and its trace $T = -2\eta^2/r^2$ as a source into the Eqs. (7, 12) in obtaining weak field Horndeski solutions corresponding to global monopoles.

The exact solution of Einstein field equations corresponding to a global monopole with the energy–momentum tensor (17) was presented by Barriola and Vilenkin (BV) [3] as

$$ds^2 = -\left(1 - \kappa\,\eta^2 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \kappa\,\eta^2 - \frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{18}$$

where $\kappa = 8\pi G/c^4 = 8\pi$ (in the natural units $G = c = 1$) and M is either the core mass of the monopole itself or the mass of the black hole where the monopole is swallowed by. In this work, we study linearized solutions, the weak field form of (18) will be required in determining integration constants for those solutions. Hence, we consider the weak field line element of a global monopole as

$$ds^2 = -\left(1 - \kappa\,\eta^2 - \frac{2M}{r}\right)dt^2 + \left(1 + \kappa\,\eta^2 + \frac{2M}{r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{19}$$

Another form used in the literature is obtained by setting $M = 0$ and rescaling time and radial coordinates [3], which results

$$ds^2 = -dt^2 + dr^2 + (1 - \kappa\eta^2)\, r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{20}$$

describing a spacetime with a global angle deficit generated by the global monopole. The weak field global monopole solutions will serve us as a boundary condition to determine some arbitrary parameters obtained by solving the linearized field equations of Horndeski theory. Note that the solution (20) also describes a configuration described by Letelier [45], where an ensemble of radially distributed straight cosmic strings intersecting at a common point, which is also called the gravitational hedgehogs [46,47].

3 Global monopoles in linearized Horndeski theory

3.1 Massless case

Now let us derive the corresponding solution of a global monopole in the linearized Horndeski theory. Namely, we need first to solve the tensor part described by the Eq. (12) for the energy–momentum tensor (17). Already knowing particular forms (19, 20) of the global monopole solution [3] in linearized GR, we expect to have a diagonal $h_{\mu\nu}$ which may require a diagonal $\theta_{\mu\nu}$, then we employ the following ansatz

$$\theta_{\mu\nu} = \text{diag}\left(\theta_{00}, \theta_{rr}, \theta_{\theta\theta}\, r^2, \theta_{\theta\theta}\, r^2 \sin^2\theta\right). \tag{21}$$

The tensor field equations (12), for a global monopole source (17), with prime being derivative with respect to the radial coordinate, leading to the following equations

$$\theta''_{00} + \frac{2}{r}\theta'_{00} = -2\kappa_1 \frac{\eta^2}{r^2}, \tag{22}$$

$$\theta''_{rr} + \frac{2}{r}\theta'_{rr} - 4\frac{\theta_{rr}}{r^2} + 4\frac{\theta_{\theta\theta}}{r^2} = 2\kappa_1 \frac{\eta^2}{r^2}, \tag{23}$$

$$\theta''_{\theta\theta} + 2\frac{\theta'_{\theta\theta}}{r} - 2\frac{\theta_{\theta\theta}}{r^2} + 2\frac{\theta_{rr}}{r^2} = 0. \tag{24}$$

Note that the Eqs. (22–24) correspond to tensor Laplace equations in spherical coordinates and are more complicated than the usual scalar Laplace equations. Note also that the solutions of the Eqs. (22–24) should satisfy the Lorenz gauge condition (11). Indeed, we will use the equation obtained by using this gauge to eliminate one of the variables from the Eqs. (23, 24). Namely, the Lorenz gauge (11) gives

$$\theta'_{rr} + 2\frac{\theta_{rr}}{r} = 2\frac{\theta_{\theta\theta}}{r}. \tag{25}$$

Eliminating $\theta_{\theta\theta}$ using this equation in (23) gives

$$\theta''_{rr} + \frac{4}{r}\,\theta'_{rr} = 2\kappa_1 \frac{\eta^2}{r^2}. \tag{26}$$

Now, this equation and (22) involve a single variable and can be solved by elementary methods. Moreover, the remaining variable $\theta_{\theta\theta}$ can be obtained using the solution of via Eq. (25). The solutions we have found have the form

$$\theta_{00} = a_0 - \frac{a_1}{r} - 2\kappa_1\eta^2 \ln\frac{r}{r_0}, \tag{27}$$

$$\theta_{rr} = b_0 - \frac{b_1}{3r^3} + \frac{2}{3}\kappa_1\eta^2 \ln\frac{r}{r_0}, \tag{28}$$

$$\theta_{\theta\theta} = b_0 + \frac{b_1}{6r^3} + \frac{\kappa_1\,\eta^2}{3G_4(0)} + \frac{2}{3}\kappa_1\eta^2 \ln\frac{r}{r_0}. \tag{29}$$

Here a_0, a_1, b_0, b_1 and r_0 are arbitrary integration constants. Finally, the Eq. (24) should be satisfied, and indeed it is for the $\theta_{\theta\theta}$ term above, which provides a consistency check for the solutions obtained.

To present $h_{\mu\nu}$ from $\theta_{\mu\nu}$ using the inverse of (10), we need the solution of the scalar field equation (7) as well. To fulfill this, we first consider the massless case, $m = 0$, in this part, namely the case where the scalar field does not attain a mass. This requires $K_{\phi\phi} = 0$, since K involves the potential of the scalar field. First, we consider the case where the potential is massless or vanishing. Then, we will consider the massive case in the next subsection. Therefore, from (7) we have to solve

$$\varphi'' + \frac{2}{r}\varphi' = -2\kappa' \frac{\eta^2}{r^2}, \tag{30}$$

where κ' is defined in (9). The solution of this equation has the general form

$$\varphi(r) = \frac{2\kappa' M}{r} - 2\kappa'\eta^2 \ln\frac{r}{r_0}, \tag{31}$$

where a constant term is ignored since it can always be integrated into ϕ_0.

Using the above results, the metric perturbation tensor have the form

$$h_{00} = \frac{a_0}{2} - \frac{a_1}{2r} + \frac{3b_0}{2} + \frac{\kappa_1\eta^2}{3} + \tilde{\varphi}(r), \tag{32}$$

$$h_{rr} = \frac{a_0 - b_0}{2} - \frac{b_1}{3r^3} - \frac{a_1}{2r} - \frac{\kappa_1\eta^2}{3} - \frac{4}{3}\kappa_1\eta^2 \ln\frac{r}{r_0} - \tilde{\varphi}(r), \tag{33}$$

$$h_{\theta\theta} = \frac{a_0 - b_0}{2} + \frac{b_1}{6r^3} - \frac{a_1}{2r} - \frac{4}{3}\kappa_1\eta^2 \ln\frac{r}{r_0} - \tilde{\varphi}(r), \tag{34}$$

where we have used the abbreviation

$$\tilde{\varphi}(r) = \frac{G_{4,\phi}(0)}{G_4(0)}\,\varphi(r). \tag{35}$$

There are four arbitrary integration constants a_0, a_1, b_0 and b_1 in the obtained solutions (32–34) which need to be determined using the previously known limiting solutions such as (19). In the following, we will use the conditions that this solution to reduce the weak field form of the BV global monopole solution (19) to fix the values of these constants. Since there is no $1/r^3$ term in the BV solution (19) in g_{rr} and $g_{\theta\theta}$ metric terms in GR, we should set $b_1 = 0$. Also comparing h_{00} with g_{00} term in BV solution (19) or similar solution in BD theory [25], we can deduce that we should have $a_1 = -4M/G_4(0)$ and $a_0 = -3b_0 + 4\kappa_1\eta^2/3$. Then the line element becomes

$$\begin{aligned} ds^2 = &-\left[1 - \kappa_1\eta^2 - \frac{2M}{G_4(0)\,r} - \tilde{\varphi}(r)\right]dt^2 \\ &+\left[1 - 2b_0 + \frac{2M}{G_4(0)\,r} + \frac{\kappa_1\eta^2}{3} - \frac{4}{3}\kappa_1\eta^2 \ln\frac{r}{r_0} - \tilde{\varphi}(r)\right]dr^2 \\ &+\left[1 + \frac{2M}{G_4(0)\,r} - 2b_0 + \frac{2\kappa_1\eta^2}{3}\right. \\ &\left.-\frac{4}{3}\kappa_1\eta^2 \ln\frac{r}{r_0} - \tilde{\varphi}(r)\right]^2 d\Omega^2, \end{aligned} \tag{36}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Note that since $\varphi(r)$ is already at the linear order in M and η^2, and since here we only consider linearized solutions, we can write the solution in a conformal form as

$$ds^2 = \left[1 - \tilde{\varphi}(r)\right]\Bigg[-\left(1 - \kappa_1\eta^2 - \frac{2M}{r}\right)dt^2 + \left(1 - 2b_0 + \frac{2M}{G_4(0)\,r} + \frac{\kappa_1\eta^2}{3} - \frac{4}{3}\kappa_1\eta^2 \ln\frac{r}{r_0}\right)dr^2 + \left(1 + \frac{2M}{G_4(0)\,r} - 2b_0 + \frac{2\kappa_1\eta^2}{3} - \frac{4}{3}\kappa_1\eta^2 \ln\frac{r}{r_0}\right)r^2 d\Omega^2\Bigg], \tag{37}$$

which will help to compare previous solutions in the literature, such as the BD solution [25]. This solution is still not in the conventional global monopole form, when the scalar field vanishes, such as the Schwarzschild type solution (19) presented in [3]. Hence, let us proceed to obtain a solution of the Schwarzschild form (19) of global monopole solution first, which means no monopole term in the angular part, with the following coordinate transformation

$$r = R\left(1 + \frac{2}{3}\kappa_1\eta^2 \ln\frac{r}{r_0}\right), \tag{38}$$

together with

$$b_0 = \frac{\kappa_1\eta^2}{3} \tag{39}$$

$$ds^2 = \left[1 - \tilde{\varphi}(R)\right]\Bigg[-\left(1 - \kappa_1\eta^2 - \frac{2M}{G_4(0)\,R}\right)dt^2 + \left(1 + \frac{2M}{G_4(0)\,R} + \kappa_1\eta^2\right)dR^2 + R^2\left(1 + \frac{2M}{G_4(0)\,R}\right)d\Omega^2\Bigg], \tag{40}$$

at the order of $\mathcal{O}(M)$ and $\mathcal{O}(\eta^2)$. We are still a coordinate transformation away from the BV solution in its usual form. Namely the transformation

$$R = -\frac{M}{G_4(0)} + \sqrt{\frac{M^2}{G_4(0)^2} + r^2} \tag{41}$$

brings the weak field solution to the desired form

$$ds^2 = \left[1 - \tilde{\varphi}(r)\right]\Bigg[-\left(1 - \kappa_1\eta^2 - \frac{2M}{G_4(0)\,r}\right)dt^2 + \left(1 + \kappa_1\eta^2 + \frac{2M}{G_4(0)\,r}\right)dr^2 + r^2 d\Omega^2\Bigg]. \tag{42}$$

When the scalar field vanishes, and $G_4(0) = 1$, the obtained solution is precisely the linearized form of the BV solution [3] given in Eq. (19), corresponding to a global monopole swallowed by a black hole in the order $\mathcal{O}(M)$ and $\mathcal{O}(\eta^2)$. We could also recover another well known form of the solution by applying the following transformation

$$r = \sqrt{1 - \kappa_1\eta^2}\,\tilde{r}, \quad t = \frac{\tilde{t}}{\sqrt{1 - \kappa_1\eta^2}}, \tag{43}$$

resulting

$$ds^2 = \left[1 - \tilde{\varphi}(\tilde{r})\right]\Bigg[-\left(1 - \frac{2M}{G_4(0)\,\tilde{r}}\right)d\tilde{t}^2 + \left(1 + \frac{2M}{G_4(0)\,\tilde{r}}\right)d\tilde{r}^2 + \tilde{r}^2(1 - \kappa_1\eta^2)d\Omega^2\Bigg], \tag{44}$$

in which, in the vanishing M limit, gives rise to a solution conformal to the renowned global monopole metric (20)

$$ds^2 = \left[1 + 2\kappa'\eta^2 \ln\frac{\tilde{r}}{r_0}\right]\left[-d\tilde{t}^2 + d\tilde{r}^2 + (1 - \kappa_1\eta^2)\,\tilde{r}^2 d\Omega^2\right], \tag{45}$$

describing a spacetime with solid angle deficit in the case of vanishing scalar field.

It is clear that this solution is conformal to the GR solution (20) and reduces to it, and also the BD [25] and scalar–tensor [26] solutions in the appropriate limits. Hence, we have found the linearized global monopole solution in the massless Horndeski theory. Note that if M given in Eq. (42) represents a compact object in which a global monopole is swallowed by, then the Newtonian limit requires $G_{4,\phi}\kappa' + 1 = G_4(0)$, which reduces to the well known BD result $\phi_0 = (2\omega + 4)/(2\omega + 3)$ [23] in which $G_4(\phi) = \phi,\ G_3(\phi) = G_5(\phi) = 0,\ K = 2\omega X/\phi$.

Since, as shown in Fig. 1 the conformal factor in (45) increases indefinitely with increasing r, the solution cannot be applicable for asymptotically large values of r. In other words the linear approximation is not valid for large values of r and an exact treatment might be necessary. The solutions (42, 45) we have presented for massless Horndeski theory also valid for massless sub theories of it such as original BD theory, scalar–tensor theories without potential, etc.

3.2 Massive case

If we consider a massive scalar field, namely $m \neq 0$, since tensorial and scalar equations decoupled by defining auxiliary tensor $\theta_{\mu\nu}$ in (10), the only change in the field equations will be in the scalar field equation (7), which now becomes a massive scalar field equation

$$\varphi'' + \frac{2}{r}\varphi' - m^2\varphi = -2\kappa'\frac{\eta^2}{r^2}, \tag{46}$$

where its well behaving solution reads

$$\varphi = \frac{2\kappa' M\, e^{-mr}}{r} + \kappa'\eta^2\left[\frac{e^{-mr}\,\mathrm{Ei}(m\,r)}{m\,r} - \frac{e^{m\,r}\,\mathrm{Ei}(-m\,r)}{m\,r}\right]. \tag{47}$$

Here $\mathrm{Ei}(x)$ is the exponential integral function [48]. Note that the general forms of the solution obtained in the previous section are still valid for massive Horndeski theory, since the mass of the scalar field does not affect the tensor $\theta_{\mu\nu}$. Hence, the global monopole solution corresponding to linearized massive Horndeski theory can be given by either

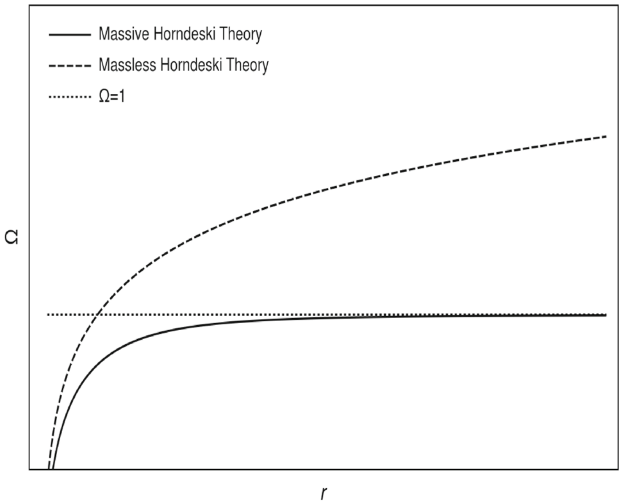


Fig. 1 Behavior of the conformal factors ($\Omega = 1 - \tilde{\varphi}$) of massless (45), massive (49) and general relativity ($\Omega = 1$) cases. It is clear that in the massive case, the conformal factor approaches to unity, i.e., the GR limit for large r values, whereas in the massless case, it increases by increasing r. Note that plots are not scaled. We choose consistent values such that these behaviors are clear to see on a single graph

(42), (44) or (45) where the scalar field φ should be given by (47). Inspecting the mass term of the compact object, M, reveals that now we have an effective gravitational coupling term $G(r) = \left(G_{4,\phi}(0)\kappa' e^{-mr} + 1\right)/G_4(0)$, which reduces to corresponding sub cases for example the massive BD case [34,49]. In the limit of vanishing monopole, $\eta = 0$, the solution (47) reduces to that of a point particle in massive Horndeski theory [42].

We can again ignore the mass of the monopole term, M, for the time being, and set $M = 0$ in φ. The metric perturbation tensor $h_{\mu\nu}$ have exactly the same form as in (32, 33, 34) with the scalar field term reads as

$$\tilde{\varphi} = \frac{G_{4,\phi}(0)}{G_4(0)}\varphi = \frac{G_{4,\phi}(0)}{G_4(0)}\kappa'\eta^2 \times \left[\frac{e^{-mr}\,\mathrm{Ei}(m\,r)}{m\,r} - \frac{e^{m\,r}\,\mathrm{Ei}(-m\,r)}{m\,r}\right]. \tag{48}$$

The general form of the metric (42) is also valid for massive Horndeski theory, with the only difference being that the scalar field now is given by (48). Namely we have

$$ds^2 = (1-\tilde{\varphi})\left[-dt^2 + dr^2 + \left(1-\kappa_1\eta^2\right)r^2 \times \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right], \tag{49}$$

where the term $\tilde{\varphi}$ now carries the effects of the mass of the scalar field, m, given in Eq. (48).

When we look at the behavior of the scalar field, we see that far from the monopole, it vanishes, and the monopole metric effectively becomes similar to GR monopoles (20), namely

$$ds^2_{r\to\infty} \approx -dt^2 + dr^2 + \left(1-\kappa_1\eta^2\right) \times r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{50}$$

Hence, the mass of the scalar field produces a screening mechanism that diminishes the effects of the scalar field in the far region. This shows that in the presence of a massive scalar field, the behavior of the global monopoles changes dramatically from those living in massless scalar fields in Horndeski theory. Asymptotic behavior of global monopoles in massive Horndeski theory resembles that of GR. This behavior is seen in Fig. 1. The monopole solutions we have derived for massive Horndeski theory also valid for sub theories of Horndeski, such as BD with a potential, scalar–tensor theories with a potential, $f(R)$ theory, etc, which were not discussed before, as far as we know.

We summarize some of the differences between massless and massive cases in the following section.

4 Some properties of the global monopoles in Horndeski theory

4.1 Deficit angle, Newtonian force on nonrelativistic test particles and light deflection

Let us now discuss some of the physical properties of the solutions obtained. The area of a sphere of radius r around a monopole in massless Horndeski theory, from (45), is given by

$$\mathcal{A} = 4\pi r^2\left(1 - \kappa_1\eta^2 - \tilde{\varphi}\right), \tag{51}$$

which resembles with the GR [9] and the BD [25] results in the appropriate limits. The area of a sphere of radius r and the deficit angle is now clearly a function of the scalar field in accordance with the corresponding BD solution instead of its constant value in the GR.

Let us discuss the area formula for massive Horndeski theory. Since the area of a sphere surrounding the monopole is given as in (51), the only change is the scalar function $\varphi(r)$. In this case, it is given by (48). Due to this, if the sphere radius is small, the expression for the areas deviates from both the GR and massless Horndeski monopoles. However, if the radius of the sphere increases, for large r, the area formula, as well as solid deficit angle, becomes similar to the GR one with only difference is the coupling constant κ_1.

The solution of massless Horndeski theory can be put into Galilean coordinates by the transformation

$$T = (1-\eta'^2)t, \tag{52}$$

$$R = r\left(1 - \frac{\kappa_1\eta^2}{2} + \frac{\kappa_1\eta^2}{2}\ln\frac{r}{r_0}\right), \tag{53}$$

giving

$$ds^2 = \left[1 - \tilde{\varphi}(R)\right]\left[-\left(1+\eta'^2\right)dT^2 + \left(1 - \eta'^2 \ln\frac{R}{r_0}\right) \times \left(dX^2 + dY^2 + dZ^2\right)\right] \tag{54}$$

$$= -\left[1 + \kappa_1\eta^2 + \tilde{\varphi}(R)\right]dT^2 + \left[1 - \kappa_1\eta^2 \ln\frac{R}{r_0} - \tilde{\varphi}(R)\right]\left(dX^2 + dY^2 + dZ^2\right), \tag{55}$$

with $R^2 = X^2 + Y^2 + Z^2$. One advantage of these Galilean coordinates is to obtain the motion of a non-relativistic test particle using the equation

$$\ddot{X}^i = -\frac{1}{2}\frac{\partial h_{00}}{\partial X^i} = \frac{1}{2}\frac{\partial \tilde{\varphi}(R)}{\partial X^i} = -2\kappa'\eta^2 \frac{G_{4,\phi}(0)}{G_4(0)}\frac{X^i}{R}. \tag{56}$$

These results indicate that, unlike the GR case in which the global monopole exerts no force on the test particles, nonrelativistic test particles around the global monopole in massless Horndeski theory experience an attractive force from it. In the BD limit, this expression reduces to the one given in [25].

Another change due to a massive scalar field is the force acting on non-relativistic test particles around the monopole in massive Horndeski theory. In the near region of the monopole, the particles feel a similar force. However, when the distance between the monopole and the test particles increases, the force reduces dramatically and vanishes practically at large distances. Hence, unlike the massless case, apart from the near region, the force acting on test particles becomes zero as in the GR case for massive Horndeski theory.

Note that since the Horndeski monopole solution (45) is conformally related to the corresponding GR solution [3], the behavior of light rays has the same behavior as in GR, with only differences being related to the gravitational coupling constant. Therefore, if one considers the case where a light source S and an observer O are perfectly aligned with the global monopole, the image becomes a ring with an angular diameter

$$\delta\Omega = \kappa_1\eta^2 l/(d+l), \tag{57}$$

where d is the distance between the monopole and the observer, and l is from the monopole to the source. Determining $\delta\Omega$ can lead to the determination of the coupling constants in κ_1. Note that the null geodesics and hence the angular diameter formula (57) do not change for both classes of monopoles since the effect of the massive scalar field manifests itself as a conformal factor, which does not affect the null geodesics.

4.2 Geodesic motion

4.2.1 Radial geodesics and effective potential

Let us investigate particle and light motion around a global monopole in Horndeski theory. Using the line element (45, 49) for *a bare monopole* without mass, let us consider the Lagrangian

$$2\mathcal{L} = -A\,\dot{t}^2 + B\,\dot{r}^2 + C\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) = \epsilon, \tag{58}$$

where λ is an affine parameter, the overdot represent derivative with respect to λ, $\epsilon = -1$ for massive particles and $\epsilon = 0$ for photons, $A = B = 1 - \tilde{\varphi}$ is the conformal factor in (45) for massless case or in (49) massive case and $C = A(1-\kappa_1\eta^2)r^2$. Since the problem is spherically symmetric, the motion takes place in a plane, and we can align coordinates such that the motion takes place in the equatorial plane with $\theta = \pi/2$. Then, the first integrals of the motion become

$$\dot{t} = -E/A, \tag{59}$$

$$\dot{\phi} = L/C, \tag{60}$$

where the integration constant E is the specific energy and L is the specific angular momentum of the particle. In this part, we consider a bare monopole by ignoring its mass. From the Lagrangian itself, we find

$$\dot{r}^2 = \frac{1}{A^2}\left[E^2 - \frac{L^2}{(1-\kappa_1\eta^2)r^2} + \epsilon A\right]. \tag{61}$$

Defining a new affine parameter as $d\gamma = A^{-1}d\lambda$ yields,

$$\left(\frac{dr}{d\gamma}\right)^2 = E^2 - V_{eff}^2,$$

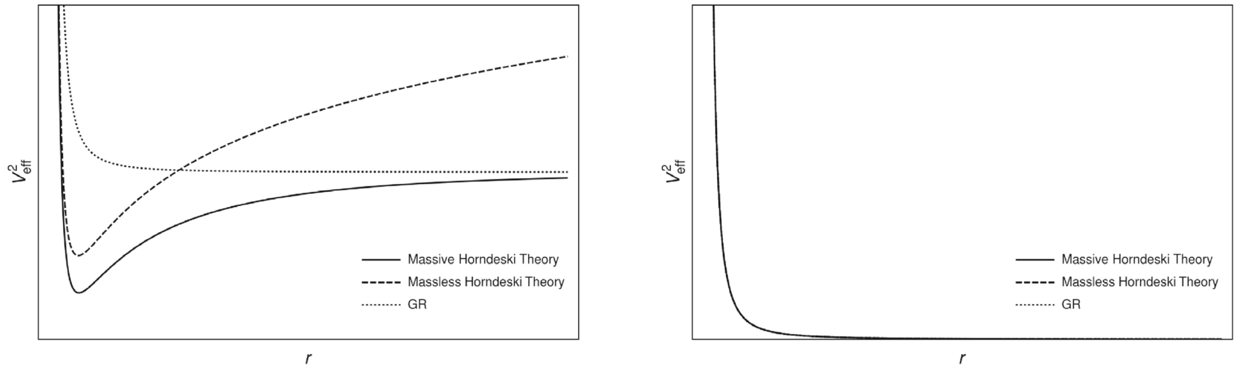
$$V_{eff}^2 = \frac{L^2}{(1-\kappa_1\eta^2)r^2} - \epsilon(1-\tilde{\varphi}), \approx (1+\kappa_1\eta^2)\frac{L^2}{r^2} - (1-\tilde{\varphi}), \tag{62}$$

where V_{eff} is the effective potential and at the last step we have used linear approximation we have used throughout in this paper.

The turning points of the motion of a massive test particle is given by

$$E^2 = V_{eff}^2 = (1+\kappa_1\eta^2)\frac{L^2}{r^2} - \epsilon(1-\tilde{\varphi}). \tag{63}$$

The behavior of the conformal factor outside the global monopole core, $1-\tilde{\varphi}$ term, can be seen in the Fig. 1. We see that for both massive and massless Horndeski theory, the conformal factor is an increasing function of r. The difference is that in massless theory, massive test particles are always subject to a bounded motion around global monopoles. This cannot be said for massive theory since we only observe an



(a) Behavior of the Effective Potential (62) for massless (45), massive (49) cases for *massive* test particles. It is clear that for the global monopoles in the massive or massless Horndeski theory, the massive particles are bounded for small r. We used physical values such that all constants equal to unity to see the behavior of the graph. Hence, the graphs are not scaled.

(b) Behavior of the Effective Potential (62) for massless (45) and massive (49) cases for *massless* test particles. There are no bound states here as expected, and both massive and massless Horndeski theory reproduces the general relativity result. The scalar functions are used such that all constants equal to unity to see the behavior in the graph.

Fig. 2 Behavior of the effective potential for massive and massless Horndeski theory

upward trend until we hit the general relativity limit. These behaviors can be seen in the Fig. 2a. Hence, the behavior of effective potential clearly distinguishes massless or massive cases. The motion of photons in the spacetime of linearized Horndeski theory has similar characteristics. For both massive and massless theories, photons reach radial infinity, and there is no bounded motion. This behavior can be seen in the Fig. 2b.

4.2.2 Circular orbits

The form of the effective potential for both cases suggests that around a global monopole in Horndeski theory, circular orbits having $r = r_c$ with $\dot{r} = 0$ present for massive test particles. Indeed let us first calculate $E^2 = V^2_{eff}$ and $V'_{eff} = 0$ simultaneously to have the energy and angular momentum for a test particle to follow a circular geodesics for massless Horndeski case as:

$$E^2 = 1 + \frac{G_{4,\phi}(0)}{G_4(0)}\kappa'\eta^2\left[1 + 2\ln\frac{r_c}{r_0}\right], \tag{64}$$

$$L^2 = \frac{G_{4,\phi}(0)}{G_4(0)}\kappa'\,\eta^2 r_c^2. \tag{65}$$

This circular geodesics is stable since $V''_{eff} = 4G_{4,\phi}(0)\kappa\eta^2/(G_4(0)\,r_c^2) > 0$.

For massive Horndeski theory, the resulting expressions for circular timelike geodesics are more complicated. Hence, we refer to them implicitly as

$$E^2 = 1 + \frac{L^2(1+\kappa_1\eta^2)}{r_c^2} - \tilde{\varphi}(r_c), \tag{66}$$

$$L^2 = -\frac{1}{2}r_c^3\tilde{\varphi}'(r_c). \tag{67}$$

Similar to the massless case, the result has similar characteristics: The second derivative of the potential for r_c is positive. Hence, the circular geodesics around a bare monopole in massive Horndeski theory are also stable.

This results permit us to speculate that the form of the effective potential and the existence of stable circular orbits may imply that, unlike GR, bare global monopoles of Horndeski theory can contribute to the structure formation in the early universe, such as galaxy and black hole formation processes, among other known mechanisms [9], by accumulating material around them due to their attractive gravitational potential, which do not exist in GR global monopoles.

5 Galaxy rotation curves

It was shown in [50–54] that global monopoles in BD theory can play a role in the flattening of galaxy rotation curves. Here we investigate whether global monopoles of massive or massless Horndeski theory can also have the same property. To model this phenomenon, we consider a spherical static galaxy with mass M. We need to calculate the rotational velocity of stars around this galaxy with a global monopole at the center. For the Lagrangian (58) with first integrals of motion (59, 60) for a massive particle, we find the following radial equation

$$\dot{r}^2 + U(r) = 0, \tag{68}$$

with

$$U(r) = \frac{1}{B}\left(\frac{L^2}{C} - \frac{E^2}{A} + 1\right). \tag{69}$$

We want these stars to follow a circular, stable orbit around this galaxy. The stability condition for such orbits can be expressed as:

$$\dot{r} = U(r) = 0, \ \frac{dU}{dr} = 0, \ \frac{d^2U}{dr^2} > 0. \tag{70}$$

The first two conditions give the energy and the angular momentum of test particles following stable circular orbits as

$$E^2 = \frac{A^2C'}{AC' - A'C}, \tag{71}$$

$$L^2 = \frac{C^2A'}{AC' - CA'}. \tag{72}$$

The second derivative of the potential evaluated at the extrema reads

$$U'' = \frac{2A'C'}{ABC} + \frac{C'A'' - A'C''}{B(AC' - CA')}. \tag{73}$$

Note for $C(r) = r^2$ these expressions reduce to the ones found in [55].

Using the definition of four velocity $U^\mu = dx^\mu/d\tau = (\dot{t}, \dot{r}, \dot{\theta} = 0, \dot{\phi})$, and the proper time relation $d\tau^2 = -ds^2$ we obtain that

$$A(r)(U^0)^2(1 - v^2) = 1, \tag{74}$$

where U^0 is the time component of the four velocity and v is the spatial velocity of the particle given by

$$v^2 = \frac{1}{A}\left[B\left(\frac{dr}{dt}\right)^2 + C\left(\frac{d\phi}{dt}\right)^2\right] := (v^r)^2 + (v^\phi)^2, \tag{75}$$

with v^r and v^ϕ being the components of the velocity v observed in an orthonormal coordinate system. The angular component v^ϕ is given by

$$v^\phi = \sqrt{\frac{C}{A}}\,\Omega \tag{76}$$

where Ω is the angular velocity of the particle, which can be calculated using the first integrals of the motion as

$$\Omega = \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \frac{AL}{CE} = \frac{A'}{C'}, \tag{77}$$

in which we have used the circular geodesics conditions (71, 72) at the last step of the calculation. Using this result, the tangential velocity of a particle in a stable circular motion is found as

$$(v^\phi)^2 = \frac{C\,A'}{A\,C'}. \tag{78}$$

5.1 Massless Horndeski theory

Reading the metric functions $A(r)$, $B(r)$ and $C(r)$ from the line element (42) with the help of the scalar field (31) for the massless Horndeski theory, we find that

$$(v^\phi)^2 = \frac{M}{G_4(0)\,r} - \frac{r\,\tilde{\varphi}'}{2} \tag{79}$$

$$= \left[\frac{1 + \kappa' G_{4,\phi}(0)}{G_4(0)}\right]\frac{M}{r} + \frac{\kappa' G_{4,0}(0)}{G_4(0)}\eta^2. \tag{80}$$

We see that near the outside of the galactic center, we have the usual $1/r$ term due to the mass of the galaxy. Whereas far from it, the effect of the monopole is to contribute a constant term which may help to straighten the galaxy rotation curves. This result is in accordance with previous ones obtained for sub theories of massless Horndeski theory such as BD theory [52–54].

5.2 Massive Horndeski theory

For the massive Horndeski theory, similarly using the line element (42) with now the scalar field (31) we find that

$$(v^\phi)^2 = \left[\frac{1 + \kappa' G_{4,\phi}(0)}{G_4(0)}(1 + m\,r)e^{-mr}\right]\frac{M}{r} + \frac{G_{4,\phi}(0)^2}{2G_4(0)^2}\kappa'\eta^2 \times \left[\frac{(1 + mr)\,e^{-mr}\,\mathrm{Ei}(m\,r)}{m\,r} - \frac{(1 - mr)\,e^{m\,r}\,\mathrm{Ei}(-m\,r)}{m\,r}\right]. \tag{81}$$

Let us investigate this result for various cases. When we ignore the monopole term, then the expression of the rotational velocity can be put into the form:

$$(v^\phi)^2 = \frac{G_\infty M}{r}\left[1 + \alpha e^{-\frac{r}{r_0}}\left(1 + \frac{r}{r_0}\right)\right], \tag{82}$$

which is exactly the expression found in [56] by Sanders for a hypothetical Yukawa type phenomenological potential where r_0 is the length scale, and α is the coupling constant of this Yukawa type potential, and G_∞ is the gravitational coupling constant measured at the infinity. It was shown in [56] that for the range $-0.95 < \alpha < 0.92$, the general properties of extended galactic rotation curves are recovered. From (81) we have the identifications $r_0 = 1/m$, $\alpha = \kappa' G_{4,\phi}(0)/G_4(0)$ and $G_\infty = 1$. From these, we see that in order for the Yukawa potential originated from the mass of the scalar field to explain observed galaxy rotation curves is the coupling constants to satisfy $-0.95 < \kappa' G_{4,\phi}(0)/G_0(0) < -0.92$. For massive Brans–Dicke theory [34], this corresponds to $-2.04 < \omega < -2.02$ for BD parameter, ω, which is a very unfavorable and restricted value for this parameter. Hence, only for very restrictive and negative values of α, the flat rotational curves of galaxies can be attributed to the massive scalar field in Horndeski theory without a global monopole.

Now, let us discuss whether the existence of a global monopole changes this result. If the mass of the scalar field is "heavy", i.e., $m \to \infty$, then the second term in the expression (81) vanishes. Therefore, for a heavy massive scalar field, the monopole term cannot explain the flattening of the rotation curves. For intermediate values of m, the second term in (81) is a decreasing positive function of radial distance, hence it cannot explain the flat rotational curves. For a light scalar field, i.e., $m \to 0$, the second term in (81) reduces to (79), and it can contribute to the flattening of the rotational velocity of a star around a spherical galaxy, similar to the massless scalar field case. In summary, only if the mass of the scalar field is quite small, then we obtain the novel result that the global monopoles may possibly be responsible for the flattening of the rotational curves of the galaxies.

6 Conclusions

In this paper, we have presented two distinct solutions of linearized Horndeski theory corresponding to global monopoles obeying Barriola and Vilenkin's approximation for their energy–momentum tensor. The first case can be called the massless Horndeski theory, whereas the second case is the massive one, due to the existence of an arbitrary potential term in the action. The existence of the potential changes the asymptotical behavior of the spacetime as it possesses a screening mechanism to suppress the effects of the long range scalar field and effectively make it a short range one. Due to this mechanism, the asymptotic behavior of the global monopole becomes the same as the monopoles in General Relativity, as it is seen in Fig. 1.

We have also investigated several physical properties of the solution for both theories and see the effect of the mass and hence the screening effect in the asymptotical behavior of these solutions. For example, the geodesic equations clearly show the different behavior of these theories, as we have seen by investigating the radial motion of test particles. The effective potential for massive or null test particles clearly shows these as shown in Fig. 2a, b. Namely, the local properties of both cases have similar characteristics, whereas far from the monopole, the properties of the massive Horndeski theory, unlike the massless case, approach those of GR.

We have also investigated the effects of global monopoles on the galaxy rotation curves. We have seen that, similar to the Brans–Dicke theory, for a massless scalar field, global monopoles can explain the flattening of the galaxy rotation curves. For the case of the massive scalar field, however, only if the mass of the scalar field is "light", i.e., $m \to 0$, then the global monopoles can explain this phenomenon. For intermediate and heavy scalar field, however, the global monopoles cannot explain the flattening of the galaxy rotation curves. As far as we know, this and other novel results we have discussed throughout the paper, especially about monopoles on massive scalar–tensor theories, were not discussed before in the literature.

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