UNIVERSITÀ DI ROMA
"LA SAPIENZA"

FACOLTÀ DI SCIENZE MATEMATICHE FISICHE E NATURALI

Ph.D. THESIS IN PHYSICS

Propagation kernel techniques for loop quantum gravity

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XVII ciclo del Dottorato di Ricerca in Fisica

A.A. 2001-2004

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Hark to the gentle gradient of the breeze:
It whispers of a more ergodic zone.

from "The Cyberiad" by Stanislaw Lem

Contents

Chapter 1

Introduction

The problem of describing the quantum regime of the gravitational field is still open (see for example [1, 2, 3] and references therein). There are tentative theories, and competing research directions. The two largest research programs are string theory and loop quantum gravity.

String theory (see for example [4, 5, 6, 7, 8] for reviews and introductory material) is by far the research direction which is presently most investigated. String theory presently exists at two levels. First, there is a well developed set of techniques that define the string perturbation expansion over a given metric background. Second, the understanding of the nonperturbative aspects of the theory has much increased in recent years [9] and in the string community there is a widespread belief, supported by numerous indications, in the existence of a full non-perturbative theory, capable of generating the perturbation expansion. There are attempts of constructing this non-perturbative theory, generically denoted M theory. One of these attempts is Matrix-theory [10, 11, 12]. The claim that string theory solves quantum gravity is based on two facts. First, the string perturbation expansion includes the graviton. More precisely, one of the string modes is a massless spin two particle with helicity ± 2. Such a particle necessarily couples to the energy-momentum tensor of the rest of the fields [13, 14] and gives general relativity to a first approximation. Second, the perturbation expansion is consistent if the background geometry over which the theory is defined satisfies a certain consistency condition; this condition turns out to be a high energy modification of the Einstein's equations. The hope is that such a consistency condition for the perturbation expansion will emerge as a full-dynamical equation from the yet-to-be-found nonperturbative theory.

The second most popular approach to quantum gravity is loop quantum gravity [2, 15, 16, 17, 18, 19, 20, 21, 22]. Loop quantum gravity is presently the best developed alternative to string theory. Like strings, it is not far from a complete and consistent theory and it yields a corpus of definite physical predictions, testable in principle, on quantum spacetime. Loop quantum gravity, however, attacks the problem from the opposite direction than string theory. It is a non-perturbative and background independent theory to start with. In other words, it is deeply

rooted into the conceptual revolution generated by general relativity. In fact, the main lesson of general relativity is that, unlike in any other interaction, space-time geometry is fully dynamical. This special feature of gravity precludes the possibility of representing fields on a fixed background geometry and severely constrains the applicability of standard techniques that are successful in the description of other interactions.

In fact, successes and problems of loop quantum gravity are complementary to successes and problems of strings. Loop quantum gravity is successful in providing a consistent mathematical and physical picture of non perturbative quantum spacetime; but the connection to the low energy dynamics is not yet completely clear. The work presented in this thesis is an attempt in the direction of connecting spinfoam models of loop quantum gravity with measurable quantities.

The general idea on which loop quantum gravity is based is the following. The core of quantum mechanics is not identified with the structure of conventional

quantum field theory, because conventional quantum field theory presupposes a background metric spacetime, and is therefore in conflict with general relativity. Rather, it is identified with the general structure common to all quantum systems. The core of general relativity is identified with the absence of a fixed observable background spacetime structure, namely with active diffeomorphism invariance. Loop quantum gravity is thus a quantum theory in the conventional sense: a Hilbert space and a set of quantum field operators, with the requirement that its classical limit be general relativity with its conventional matter couplings. But it is not a quantum field theory over a metric manifold. Rather, it is a quantum field theory on a differentiable manifold, respecting the manifold's invariances and where only coordinate independent quantities are physical.

Technically, loop quantum gravity is based on two inputs. The first is the formulation of classical general relativity based on the Ashtekar connection [23, 24], in which parallel transport, rather than the metric, plays the main role. The version of the connection now most popular is not the original complex one, but an evolution of the same one, in which the connection is real. The second is the choice of the holonomies of this connection, denoted loop variables, as basic variables for the quantum gravitational field [25, 26]. This second choice determines the peculiar kind of quantum theory being built. Physically, it corresponds to the assumption that excitations with support on a loop are normalizable states. This is the key technical assumption on which everything relies. It is important to notice that this assumption fails in conventional 4d Yang Mills theory, because loop-like excitations on a metric manifold are too singular: the field needs to be smeared in more dimensions Equivalently, the linear closure of the loop states is a far too big non-separable state space. What makes general relativity different from 4d Yang Mills theory, however, is nonperturbative diffeomorphism invariance. The gauge invariant states, in fact, are not localized at all. They are, pictorially speaking, smeared by the gauge diffeomorphism group all over the coordinates manifold. More precisely, factoring away the diffeomorphism group takes down from the state space of the

loop excitations, which is too big, to a separable physical state space of the right size [27, 28]. Thus, the consistency of the loop construction relies heavily on diffeomorphism invariance. In other words, the diff-invariant invariant loop states (more precisely, the diff-invariant spin network states) are not physical excitations of a field on spacetime. They are excitations of spacetime itself.

More specifically[1], the configuration variable is an $SU(2)$ connection A_a^i on a 3-manifold Σ representing space. The canonical momenta are given by the densitized triad E_i^a. The latter encode the (fully dynamical) Riemannian geometry of Σ and are the analog of the "electric fields" of Yang-Mills theory. In addition to diffeomorphisms there is the local $SU(2)$ gauge freedom that rotates the triad and transforms the connection in the usual way. According to Dirac, gauge freedoms result in constraints among the phase space variables which conversely are the generating functionals of infinitesimal gauge transformations. In terms of connection variables the constraints are: $\mathcal{G}_i = \mathcal{D}_a E_i^a = 0$, $\mathcal{C}_a = E_k^b F_{ba}^k = 0$, $\mathcal{S} = \epsilon^{ijk} E_i^a E_j^b F_{abk} + \cdots = 0$, where $\mathcal{D}_a$ is the covariant derivative and F_{ba} is the curvature of A_a^i. $\mathcal{G}_i$ is the familiar Gauss constraint - analogous to the Gauss law of electromagnetism - generating infinitesimal $SU(2)$ gauge transformations, $\mathcal{C}_a$ is the vector constraint generating space-diffeomorphism, and $\mathcal{S}$ is the scalar constraint generating "time" reparameterization (there is an additional term that has been omitted for simplicity). Loop quantum gravity is defined using Dirac quantization. One first represents the constraints as operators in an auxiliary Hilbert space $\mathcal{H}$ and then solves the constraint equations $\hat{\mathcal{G}}_i \Psi = 0$, $\hat{\mathcal{C}}_a \Psi = 0$, $\hat{\mathcal{S}} \Psi = 0$. The Hilbert space of solutions is the so-called physical Hilbert space $\mathcal{H}_{phys}$. In a generally covariant system quantum dynamics is fully governed by constraint equations. In the case of loop quantum gravity they represent quantum Einstein's equations. States in the auxiliary Hilbert space are represented by wave functionals of the connection $\Psi(A)$ which are square integrable with respect to a natural diffeomorphism invariant measure, the Ashtekar-Lewandowski measure [29]. This space can be decomposed into a direct sum of orthogonal subspaces $\mathcal{H} = \oplus_\gamma \mathcal{H}_\gamma$ labeled by a graph γ in Σ. The fundamental excitations are given by the holonomy $h_l(A) \in SU(2)$ along a path l in Σ. Elements of $\mathcal{H}_\gamma$ are given by functions $\Psi_{f,\gamma}(A) = f(h_{l_1}(A), \ldots, h_{l_n}(A))$, where h_l is the holonomy along the links $l \in \gamma$ and $f : SU(2)^n \to \mathbb{C}$ is (Haar measure) square integrable. They are called cylindrical functions and represent a dense set in $\mathcal{H}$ denoted Cyl. Gauge transformations generated by the Gauss constraint act non-trivially at the end-points of the holonomy, i.e., at nodes of graphs. The Gauss constraint $\mathcal{G}_i$ is solved by looking at $SU(2)$ gauge invariant functionals of the connection. The fundamental gauge invariant quantity is given by the holonomy around closed loops. An orthonormal basis of the kernel of the Gauss constraint is defined by the so called spin network states $\Psi_{\gamma,\{j_l\},\{i_n\}}(A)$ [30, 31, 32]. Spin-networks are defined by a graph γ in Σ, a collection of spins $\{j_l\}$ - unitary irreducible representations of $SU(2)$ - associated with links $l \in \gamma$ and a collection of $SU(2)$ intertwiners $\{i_n\}$ associated to nodes $n \in \gamma$.

[1] I will describe the basics of loop quantum gravity and spinfoam models in a more detailed way in section 2.3.

The spin-network gauge invariant wave functional $\Psi_{\gamma,\{j_l\},\{i_n\}}(A)$ is constructed by first associating an $SU(2)$ matrix in the j_l representation to the holonomies $h_l(A)$ corresponding to the link l, and then contracting the representation matrices at nodes with the corresponding intertwiners i_n.

A great deal of progress has been made within the theory. At the mathematical level, the main achievement is the rigorous definition of the Hilbert space of quantum geometry, the regularization of geometric operators and the rigorous definition of the quantum Hamiltonian constraint (defining the quantum dynamics). States of quantum geometry are given by spin network states. From the physical viewpoint the main prediction of loop quantum gravity is the discreteness of geometry at the Planck scale. This provides a clear-cut understanding of the problem of UV divergences in perturbative general relativity: at the Planck scale the classical notion of space and time simply ceases to exist; therefore, it is the assumption of a fixed smooth background geometry (typically space-time) in perturbation theory that becomes inconsistent at high energies. The theory successfully incorporates interactions between quantum geometry and quantum matter in a way that is completely free of divergences [33]. The quantum nature of space appears as a physical regulator for the other interactions. Dynamics is governed by the quantum Hamiltonian constraint. Even when this operator is rigorously defined [34] it is technically difficult to characterize its solution space. This is partly because the 3 + 1-decomposition of space-time (necessary in the canonical formulation) breaks the manifest 4-diffeomorphism invariance of the theory making awkward the analysis of dynamics. The situation is somewhat analogous to that in standard quantum field theory. In the Hamiltonian formulation of standard quantum field theory manifest Lorentz invariance is lost due to a particular choice of time slicing of Minkowski space-time. The formalism is certainly Lorentz invariant, but one has to work harder to show it explicitly. Manifest Lorentz invariance can be kept only in the Lagrangian (path-integral) quantization making the (formal) path integral a powerful device for analyzing relativistic dynamics. Consequently, there has been growing interest in trying to define dynamics in loop quantum gravity from a 4-dimensional covariant perspective. This has given rise to the so-called spin foam approach to quantum gravity [35, 36, 37, 38, 39].

The spinfoam techniques provide well defined expressions for a Misner-Hawking "sum over 4-geometries" [40, 41, 42], where finiteness results from the discreteness of space revealed by loop quantum gravity. The spinfoam formalism provides an amplitude for quantum states of gravity and matter on a 3d boundary [41, 42, 43]. But no formalism is yet available for deriving particles' scattering amplitudes from these boundary amplitudes.

In this thesis I indicate a direction to construct such formalism. The key ingredient for developing this formalism is the Minkowski vacuum state, namely the "no-particle" state, or the coherent semiclassical state associated to the classical Minkowski solution. The construction of this state is considered a major open problem in nonperturbative quantum gravity, and it is being studied using a variety of

different techniques [44]. Here, I propose a tentative explicit expression for computing the Minkowski vacuum from a spinfoam formalism. I begin by introducing a certain number of general tools, in the context of the quantum field theory of a free massive scalar field. Adopting Schrödinger's representation of quantum field theory, which I briefly review in Chapter 2, I extend to the case of fields the propagation kernel introduced by Feynman to describe the quantum mechanics of a single particle [45]. In this way I obtain a propagation kernel between field configurations $W[\varphi_1, t_1; \varphi_2, t_2]$. More precisely, $W[\varphi_1, t_1; \varphi_2, t_2]$ describes the evolution of the dynamical field ϕ in an infinite strip bounded by two hyperplanes at fixed times t_1 and t_2, with $\phi|_{t=t_1} = \varphi_1$ and $\phi|_{t=t_2} = \varphi_2$. $W[\varphi_1, t_1; \varphi_2, t_2]$ depends exclusively on the times t_i and the boundary conditions φ_i. I show how the single tool of the propagation kernel allows to reconstruct all the quantities that one may want to extract from a quantum field theory. For example, the propagation kernel allows to obtain the two-point functions, and through these, thanks to the LSZ reduction formula and Wick's theorem, any particle scattering amplitudes. Moreover, the propagation kernel also allows to reconstruct the vacuum state, of which I give two different definitions, distinguishing between the "Minkowski" vacuum state which minimizes the energy and the "nonperturbative" vacuum state which codes the dynamics.

At first sight it may seem that ordinary quantum field theory techniques like the ones I just mentioned can be of no help in a theory of quantum gravity. For example, if gravitation is to be a dynamical field a propagation kernel between hyperplanes at fixed times is not general enough. More generally, to understand quantum gravity, it is necessary to understand how to formulate quantum field theory in a background-independent manner. In the presence of a background, quantum field theory yields scattering amplitudes and cross sections for asymptotic particle states, and these are compared with data obtained in the laboratory. The conventional theoretical definition of these amplitudes involves infinitely extended spacetime regions and relies on symmetry properties of the background. In a background independent context this procedure becomes problematic. For instance, consider the 2-point function $W(x; y) = \langle 0| \phi(x) \phi(y) |0\rangle$. In quantum field theory over a background, the independent variables x and y can be related to the spacetime location of particle detectors. In a background independent context, general covariance implies immediately that $W(x; y)$ is constant for $x \neq y$, and therefore it is not clear how the formalism can control the localization of the detectors [46].

Therefore, the first step to make contact with quantum gravity will be to extend the formalism of the propagation kernel to the case of finite regions of spacetime. To this end an essential tool is the general boundary formalism, introduced by Robert Oeckl [47, 48, 49]. The general boundary formalism is an extension of the standard formalism of quantum mechanics with the aim of natural compatibility with general covariance. It should be applicable in particular to quantum general relativity.

The extension as compared to the standard formalism might be sketched as follows. In the standard formalism one associates a Hilbert space of states with each time-slice of a global foliation of space-time. An evolution takes place between

two such time-slices and is represented by a unitary operator. Associated with states in the two time-slices is a transition amplitude, whose modulus square determines the probability of finding the final state given that the initial one was prepared. More explicitly, Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of states are associated to the initial time t_1 and final time t_2. The evolution is described by an operator $U(t_1, t_2) : \mathcal{H}_1 \to \mathcal{H}_2$. The transition amplitude for an initial state $\psi_1 \in \mathcal{H}_1$ to evolve into a final state $\psi_2 \in \mathcal{H}_2^*$ is written as $\langle\psi_2| U(t_1, t_2) |\psi_1\rangle$. In terms of a space-time picture an evolution operator is associated to a region of space-time, namely the product of the time interval with all of space. The states (initial and final) are naturally associated with the boundary components of this region. Indeed, the first step of the general boundary formulation generalization consists of forgetting the a priori distinction between initial and final state. Instead one considers a state space which is the tensor product of the two state spaces associated with the time-slices. The new state space is naturally associated with the boundary as a whole. That such a formulation is consistent is a rather non-trivial fact. It crucially relies on a symmetry of quantum field theory coming out of the LSZ reduction that allows to exchange individual particles between the initial and final state without changing the amplitude. (Of course, a CPT transformation must be performed on the particle at the same time and phase space measures do change.)

Formalizing this, a generalized state space $\mathcal{H}_{[t_1,t_2]}$ is defined that is the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2^*$. A state ψ in $\mathcal{H}_{[t_1,t_2]}$ is a (linear combination of) tensor product(s) $\psi_1 \otimes \psi_2$ of states in $\mathcal{H}_1$ and $\mathcal{H}_2$. The transition amplitude is then a map $\mathcal{H}_{[t_1,t_2]} \to \mathbb{C}$ which is denoted by $\rho_{[t_1,t_2]}$. In terms of the conventional notation, $\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2) = \langle\psi_2| U(t_1, t_2) |\psi_1\rangle$. In terms of a measurement process, the initial and final state are both encoded in a state of the generalized state space. The evolution operator becomes a linear map from the generalized state space to the complex numbers, associating transition amplitudes to generalized states. The second step is to generalize from the special regions of space-time that are time intervals extended over all of space to more general regions. For this to be consistent it is necessary to introduce a composition property. This property requires that when gluing two regions of space-time together the evolution map associated with the composite must equal the composition of the evolution maps associated with the original pieces. This generalizes the composition of time evolutions in standard quantum mechanics. More precisely, one demands the properties of a topological quantum field theory [50]. However, one allows more general boundaries than equal-time-slices in euclidean or minkowskian space, and here is where this formulation goes beyond the use that is usually made of topological quantum field theory in physical contexts. In particular, boundaries might have time-like components and one may glue along such boundaries.

Given a space-time region (4-manifold) M with boundary Σ, $\mathcal{H}_\Sigma$ is the state space associated with the boundary. The evolution map (or amplitude) is $\rho_M : \mathcal{H}_\Sigma \to \mathbb{C}$. The situation of conventional quantum mechanics is recovered if M is the product of all of space $\mathbb{R}^3$ with a time interval $[t_1, t_2]$. Σ is then the union of two components

$\Sigma_1 \cup \Sigma_2$, each being all of space $\mathbb{R}^3$ times a point in time. By the axioms of topological quantum field theory this implies that $\mathcal{H}_\Sigma$ decomposes into a tensor product of vector spaces associated to the components $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_2} \otimes \mathcal{H}_{\Sigma_2}$. In this way the expression of $\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2)$ is recovered. But in general (especially if Σ is connected) there is no natural decomposition of $\mathcal{H}_\Sigma$ into a tensor product and thus no longer any natural distinction between preparation and observation in quantum mechanics. This has profound interpretational implications.

It might seem that one still supposes an a priori fixed space-time in contrast to the desire of seeing it emerge from the quantization. However, this is not really the case. What is presumed is only the topology, but not the geometry. The geometry and thus the dynamical degrees of freedom of gravity are really to be encoded in the state.

The main advantage of the general boundary formulation is that local measurement processes can be described using a local region of space-time only. Neither is recourse made to distant events in the universe nor is any knowledge of its global structure necessary. This frameworks allows me to define a propagation kernel which describes the evolution of the dynamical fields inside a finite region $\mathcal{R}$ of spacetime bounded by a closed surface Σ. In the case of the propagation of the field in an infinite strip the propagation kernel $W[\varphi_1, t_1; \varphi_2, t_2]$ depends on the times t_i and the boundary conditions φ_i; in the case of a finite region $\mathcal{R}$ the propagation kernel $W[\varphi, \Sigma]$ depends only on the surface $\Sigma = \partial\mathcal{R}$ and on the boundary condition $\varphi = \phi|_\Sigma$. Through $W[\varphi, \Sigma]$ it is possible to build a covariant formalism for quantum field theory entirely in terms of boundary data, in which no reference is made to infinitely extended spatial surfaces, infinite past or infinite future. I argue that all physical predictions on measurements performed in the region $\mathcal{R}$, including scattering amplitudes between particles detected in the laboratory, can be expressed in terms of $W[\varphi, \Sigma]$. The geometry of Σ codes the relative spacetime localization of the particle detectors. This picture is near to what actually happens in a laboratory experiment, where the initial and final state of a scattering event are confined into finite-size spacetime regions. The relation between particle states that can be defined in such a finite context and the usual particle states of quantum field theory, defined on an infinite spacelike region, is discussed in [51]. I derive the evolution equation for $W[\varphi, \Sigma]$, which turns out to be a generalization of the Tomonaga-Schwinger equation [52, 53]. This equation becomes a generalized Wheeler-DeWitt equation in the background independent context [15].

Next, I consider the application of this formalism to the gravitational context. At this point the choice of using Schrödinger's representation of quantum field theory and the field propagation kernel reveals its full utility. Indeed, in spinfoam models the framework is pretty much the same, as states are described through wave functionals and propagation amplitudes between spinnetworks, which in turn can be expressed as sums over spinfoams, are formally analogous to the propagation kernel between field configurations.

In the gravitational context, if $W[\varphi, \Sigma]$ is well defined, then background inde-

pendence implies that it is independent from local variations of the location of Σ. At first sight, this seems to give rise to the characteristic interpretative obscurity of background independent quantum field theory: the independence of $W[\varphi, \Sigma]$ from is equivalent to the independence of $W(x; y)$ from x and y mentioned above. But at a closer look, it is not so: in this context the boundary field φ includes the gravitational field, which is the metric, and therefore the argument of $W[\varphi, \Sigma] = W[\varphi]$ still describes the relative spacetime location of the detectors. This fact allows to express scattering amplitudes directly in terms of $W[\varphi]$ even in the background-independent context. I distinguish two distinct notions of vacuum. The first is the nonperturbative vacuum state $|0_\Sigma\rangle$ that the functional integral on the bulk defines on the (kinematical) Hilbert space associated to the boundary surface Σ. If the metric on is chosen to be to be spacelike, this is the Hartle-Hawking state [43]. In the context I am considering, instead, Σ is the boundary of a finite 4d region of spacetime, and $|0_\Sigma\rangle$ is a background-independent way of coding quantum dynamics. The second notion of vacuum is (the local approximation to) the Minkowski vacuum state $|0_M\rangle$. I will argue that this state is recovered for appropriate values of the boundary metric. One of the main results of this thesis is an equation connecting the two vacuum states, and an explicit formula for the Minkowski vacuum state $|0_M\rangle$, in terms of a spinfoam model.

Finally, I study the application of some of the propagation kernel techniques derived in this thesis to a toy model. The boundary picture I sketched up to now is pithy and appealing, but its implementation in the full 4d quantum gravity theory is difficult because of the technical complexity of the theory. It is useful to test and illustrate it in a simple context. I consider riemannian general relativity in three dimensions. Since the theory is topological, the integral defining $W[\varphi]$ is trivial. To further simplify the context, I triangulate spacetime, reducing the field variables to a finite number [54, 55, 56, 57]. Furthermore, I take a "minimalist" triangulation: a single tetrahedron with four equal edges. In this way the number of variables I deal with is reduced to a bare minimum. The result is an extremely simple system, which, nevertheless, is sufficient to realize the conceptual complexity of a background independent theory of spacetime geometry. I show that this simple system has in fact a background independent classical and quantum dynamics. The classical dynamics is governed by the relativistic Hamilton function [15], the quantum dynamics is governed by the relativistic propagator $W[\varphi]$. I compute both these functions explicitly. The classical dynamics, which is equivalent to the Einstein equations, fixes relations between quantities that can be measured on the boundary of the tetrahedron. The quantum dynamics gives probability amplitudes for ensembles of boundary measurements. I describe the two (equivalent) interpretations of the model, in the classical as well in the quantum theory. This work has been done in collaboration with Daniele Colosi, Winston Fairbairn, Leonardo Modesto, Karim Noui and Carlo Rovelli; my contribution is limited to the classical version of the model. Furthermore, I concretely illustrate the distinction between the nonperturbative vacuum state and the "Minkowski" vacuum that minimizes the

energy associated with the evolution in T, and I show that the technique I suggest in the previous Chapters of this thesis for computing the Minkowski vacuum state from the nonperturbative vacuum state works in this context.

This thesis is organized as follows. In Chapter 2 I briefly review the formalism which forms the basis of the work described in this thesis, that is, Feynman's path integral formulation of quantum mechanics, and Schrödinger's representation in quantum mechanics and quantum field theory, and sketch the basics of loop quantum gravity and spinfoam models. In Chapter 3 I introduce the propagation kernel defined on an infinite strip, showing how it allows to reconstruct the two-point function and through it scattering amplitudes, and the vacuum state, of which I give two different definitions. In Chapter 4 I then proceed to sketch the general boundary formulation of quantum mechanics and quantum gravity, as introduced by Robert Oeckl, and to develop the propagation kernel formalism in the case of a finite region of spacetime, describing a possible application to quantum gravity. In Chapter 5 I describe the derivation of the generalized Tomonaga-Schwinger equation, for the propagation kernel, and finally in Chapter 6 I outline an application of the tools derived in the preceding chapters to a toy model which describes of the dynamics of a tetrahedron. The new results are in Chapter 3, in the section 4.2 of Chapter 4 and in Chapters 5, 6, and they appear in the papers [58, 59, 60].

Chapter 2

Basic formalism

In the following I will adopt Schrödinger's representation for quantum field theory, and use the extension of the Feynman's path integral formulation in this representation. The reason for doing this has already been sketched in the introduction: the ultimate goal of this research is that of writing down scattering amplitudes in loop quantum gravity, and more precisely in its formulation through spinfoam models. The formalism that allows to make contact with this models more easily is Schrödinger's representation, since one of the fundamental tools of the spinfoam models is the propagation amplitude between spinnetworks [15], which is formally similar to the field-to-field propagator in Schrödinger's representation which I will define in Chapter 3. Therefore, to set up a convenient background for what follows, in this chapter I will briefly review Feynman's path integral formulation of quantum mechanics [45, 61] and Schrödinger's representation [62, 63, 64, 65, 66, 67]. In the final section of this Chapter I will briefly review the basics of loop quantum gravity and of spinfoam models.

2.1 Feynman's path integral formulation

The core of Feynman's path integral formulation of quantum mechanics consists in introducing a probability amplitude associated with every method whereby an event in nature can take place. This probability amplitude is proportional to the classical action associated to the specific method considered. It is possible to associate an amplitude, called *kernel*, with the overall event by adding together the amplitudes of each alternative method. For example, in the case of the propagation of a particle from point a to point b the kernel K is given by the sum of the amplitude associated to every possible path in space and time,

$$K(a,b) \propto \sum_{\substack{\text{all paths} \\ \text{from } a \text{ to } b}} \exp \frac{i}{h} S[x(t)] \quad , \tag{2.1}$$

where $S[x(t)]$ is the classical action, calculated over paths $x(t)$ such that

$$x(t_a) = x_a \tag{2.2}$$

$$x(t_b) = x_b \quad . \tag{2.3}$$

The absolute square of the overall amplitude is interpreted as the probability that the event will happen.

It is useful to make a brief comparison with the situation in classical mechanics. In the latter, the propagation of a particle from a to b is described by a unique path, the classical path $\bar{x}(t)$, determined by the principle of least action. The action calculated on the classical path is also called Hamilton function [15]. In quantum mechanics, not just the particular path of extreme action contributes; all paths contribute. In the classical approximation, even a small change in the path, small on the classical scale, will correspond to huge changes in the action, huge when compared to h; contributions to the action for generic paths will average out, except for the classical path, according to the principle of the stationary phase. Actually trajectories differing from the classical path can still contribute as long as their action is within h of its extremal value. The classical trajectory is indefinite to this slight extent, and this rule serves as a measure of the limitations of the precision of the classically defined trajectory.

(2.1) can be rewritten as a path integral, that is, a functional integral over all paths joining a and b:

$$K(a,b) = \int_a^b \exp \frac{i}{h} S[x(t)] \, \mathcal{D}[x(t)] \quad . \tag{2.4}$$

In quantum mechanics, the kernel K is solution of the Schrödinger equation, in both variables

$$i\hbar \frac{\partial}{\partial t_a} K(a,b) = H_a K(a,b) \tag{2.5}$$

$$i\hbar \frac{\partial}{\partial t_b} K(a,b) = H_b K(a,b) \quad ; \tag{2.6}$$

thus the knowledge of K relative to a system at a given time $\bar{t}$ implies its knowledge at all subsequent times, which translates in a complete knowledge of the evolution of the system. The knowledge of K allows a complete description of the system and its evolution in time. Given the wave function of the system at the time 0, the kernel K allows to calculate the wave function at a subsequent time t

$$\psi(x,t) = \int dy K(x,t;y,0) \, \psi(y,0) \quad . \tag{2.7}$$

This can be made clear by considering a representation of the kernel K in the base of eigenstates $|x\rangle$ of the position operator:

$$K(a,b) = \langle b | \, e^{-iH(t_b - t_a)} \, |a\rangle \quad , \tag{2.8}$$

where H is the hamiltonian of the system. Then it's easy to see why (2.7) holds:

$$\psi(x,t) = \langle x| e^{-iHt} |P\rangle = \int dy \langle x| e^{-iHt} |y\rangle \langle y|P\rangle = \int dy \langle x| e^{-iHt} |y\rangle \psi(y,0) \quad . \tag{2.9}$$

The representation (2.8) also allows to derive easily a representation of the kernel K in terms of eigenstates of the energy ϕ_n

$$\langle b| e^{-iH(t_b-t_a)} |a\rangle = K(a,b) = \sum_n \langle b|n\rangle \langle n| e^{-iH(t_b-t_a)} |a\rangle = \tag{2.10}$$

$$= \sum_n e^{-iE_n(t_b-t_a)} \langle b|n\rangle \langle n|a\rangle = \sum_n e^{-iE_n(t_b-t_a)} \phi_n(b) \phi_n^*(a) \quad , \tag{2.11}$$

where $|n\rangle$ are eigenkets of the energy.

2.2 Schrödinger's representation

In nonrelativistic quantum mechanics, the starting point is a hamiltonian operator which is canonically quantized by postulating commutation relations between position operators and their conjugate momenta. Using Schrödinger's representation amounts to choosing the basis where the position operator X is diagonal:

$$X|x\rangle = x|x\rangle \quad , \tag{2.12}$$

where $|x\rangle$ is the eigenstate with eigenvalue x. The coordinate representation of a state $|P\rangle$, that is, its projection on the eigenstates $|x\rangle$ is the corresponding wavefunction $\psi(x)$:

$$\psi(x) = \langle x|P\rangle \quad , \tag{2.13}$$

$\psi(x)$ is the probability density of finding the particle in the position x. The Schrödinger equation becomes a differential equation whose solutions, the eigenfunctions of the hamiltonian differential operator, represent possible states of the system.

This should not be confused with the set of formalisms, called Schrödinger's and Heisenberg's representations, which apply time evolution respectively to quantum states or to quantum operators.

I'll now transport this formalism in quantum field theory, obtaining a description of quantum field theory in terms of fields rather than particles. Obviously the two descriptions are equivalent; in particle physics the second is preferred because the object of study is the dynamics of particles. I'll consider the case of a free scalar field ϕ in the minkowskian, with lagrangian

$$\mathcal{L}[\phi] = \frac{1}{2}\partial_\mu\phi(x)\,\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) - g\phi^4(x) \quad . \tag{2.14}$$

The conjugate variables are $\phi(x)$ and $\dot{\phi}(x)$; there's a pair of such variables for every point of space (not of spacetime). $\phi(x)$ is a hermitian field, it satisfies the equal-time commutators:

$$\begin{aligned} \left[\phi(\vec{x},t),\phi(\vec{y},t)\right] &= 0 \\ \left[\dot{\phi}(\vec{x},t),\dot{\phi}(\vec{y},t)\right] &= 0 \\ \left[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)\right] &= i\delta^{(3)}(\vec{x}-\vec{y}) \end{aligned} \tag{2.15}$$

this means that field operators evaluated in different points can be simultaneously diagonalised. In the coordinate Schrödinger representation a basis for the Fock space is used where the operator $\phi(\vec{x})$, now time independent, is diagonal. Then in the space of states the following relation holds:

$$\phi(\vec{x})|\psi\rangle = \psi(\vec{x})|\psi\rangle \quad , \tag{2.16}$$

with the important difference that $\phi(\vec{x})$ is an operator while $\psi(\vec{x})$ is a function. $|\psi\rangle$ are the eigenstates of the field operator with eigenvalues $\psi(\vec{x})$. Coordinate representations of state vectors or elements of Fock space are given by the projection of a state $|P\rangle$ on the basis of eigenstates $|\psi\rangle$ of the field operator:

$$\langle\psi|P\rangle = \Psi[\psi] \quad ; \tag{2.17}$$

where $\Psi[\psi]$ is a wave functional which determines the possible field configurations. $\Psi[\psi]$ is a functional in ψ and represents the probability amplitude for a field measure on the state $|P\rangle$ to give the classical field $\psi(\vec{x})$. The situation is completely analogous to nonrelativistic quantum mechanics, where the scalar product $\langle x|P\rangle$ gives the probability amplitude for the particle in the generic state $|P\rangle$ to be found at position x.

Just like in quantum mechanics the states $|x\rangle$ are normalised to a $\delta(x-x')$, here the eigenstates $|\psi\rangle$ are normalised to a functional delta:

$$\langle\psi'|\psi\rangle = \prod_x \delta(\psi(x)-\psi'(x)) \quad ; \tag{2.18}$$

that is, the scalar product is nonzero only if the two configurations coincide everywhere. This formula contains an infinite product, which will recur often when dealing with the functional formalism, potentially rendering equations ill-defined. However, infinities pose no real obstacle, since it has been proved that the Schrödinger representation is renormalizable, both in the case of static [68, 69] and time-dependent [70, 71] problems .

In this representation the field operator $\phi(\vec{x})$ acts in this way:

$$\psi(\vec{x})\Psi[\psi] = \langle\psi|\phi(\vec{x})|P\rangle = \psi(\vec{x})\langle\psi|P\rangle \quad ; \tag{2.19}$$

$\psi(\vec{x})\,\Psi[\psi]$ is a functional of ψ which depends also on x. This reproduces the situation of nonrelativistic quantum mechanics, where in the basis where x is diagonal one has

$$x\psi(x) = \langle x|\, x\,|P\rangle = x\,\langle x|P\rangle \quad . \tag{2.20}$$

The scalar product between wave functionals is also an obvious extension:

$$\langle\psi_2|\psi_1\rangle = \int \prod_x \delta\psi(x)\,\langle\psi_2|\psi\rangle\,\langle\psi|\psi_1\rangle = \int \prod_x \delta\psi(x)\,\Psi_2^*(\psi)\,\Psi_1(\psi) \quad . \tag{2.21}$$

In the Schrödinger representation of nonrelativistic quantum mechanics, one uses a differential representation of the commutators by replacing the conjugate momenta with derivatives:

$$\langle y|\,[\hat{x},\hat{p}]\,|\alpha\rangle = i\hbar\,\langle y|\alpha\rangle\,; \quad \langle y|\,\hat{p}\,|\alpha\rangle = -i\hbar\frac{\partial}{\partial y}\,\langle y|\alpha\rangle \tag{2.22}$$

$$\Rightarrow p \to -i\hbar\partial/\partial x \quad . \tag{2.23}$$

In quantum field theory, the equal-time commutators (2.15) are given a functional differential representation through similar steps:

$$\langle\psi|\,\dot{\phi}(\vec{x})\,|P\rangle = -i\hbar\frac{\delta}{\delta\psi(\vec{x})}\,\langle\psi|P\rangle = -i\hbar\frac{\delta}{\delta\psi(\vec{x})}\Psi[\psi] \tag{2.24}$$

$$\Rightarrow \dot{\phi}(\vec{x}) = -i\hbar\frac{\delta}{\delta\psi(\vec{x})} \quad , \tag{2.25}$$

where $\psi(x)$ is the function defined in (2.16). The differential representation of the field momentum turns the hamiltonian operator associated to the lagrangian (2.14) into a functional differential operator

$$H[\psi] = \int d^3x \left(-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\psi^2(\vec{x})} + \frac{1}{2}\left(\nabla\psi(\vec{x})\right)^2 + \frac{m^2}{2}\psi^2(\vec{x}) + g\psi^4(\vec{x})\right) \quad , \tag{2.26}$$

and the Schrödinger equation in a functional differential equation

$$i\frac{\partial}{\partial t}\Psi[\psi] = \int d^3x \left(-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\psi^2(\vec{x})} + \frac{1}{2}\left(\nabla\psi(\vec{x})\right)^2 + \frac{m^2}{2}\psi^2(\vec{x}) + g\psi^4(\vec{x})\right)\Psi[\psi] \tag{2.27}$$

whose solutions, the eigenfunctionals of the hamiltonian functional differential operator, represent possible states of the system. For time-independent hamiltonians it is possible to separate the variables

$$\Psi[\psi] = e^{-iEt}\Psi'[\psi] \quad , \tag{2.28}$$

obtaining a functional eigenvalue problem for the time-independent Schrödinger equation

$$\int d^3x \left(-\frac{\hbar^2}{2} \frac{\delta^2}{\delta\psi^2(\vec{x})} + \frac{1}{2} \left(\nabla\psi(\vec{x})\right)^2 + \frac{m^2}{2}\psi^2(\vec{x}) + g\psi^4(\vec{x}) \right) \Psi'[\psi] = E\Psi'[\psi] \quad , \tag{2.29}$$

where $\Psi'[\psi]$ is independent of time.

Thanks to the functional derivative representation of the operator $\dot{\phi}$ (2.25) it is possible to write down the creation and destruction operator in this representation, thus formalizing the particle interpretation of the theory:

$$a\left(\vec{k}\right) = \int d^3x e^{i\vec{k}\vec{x}} \left(\omega_k \psi(\vec{x}) + \frac{\delta}{\delta\psi(\vec{x})} \right) \tag{2.30}$$

$$a^\dagger\left(\vec{k}\right) = \int d^3x e^{-i\vec{k}\vec{x}} \left(\omega_k \psi(\vec{x}) - \frac{\delta}{\delta\psi(\vec{x})} \right) \quad . \tag{2.31}$$

The ground state $\Psi_0\left[\tilde{\psi}\right]$, written in term of the Fourier transform $\tilde{\psi}$ of the field ψ, can be easily found by imposing the condition $a\left(\vec{k}\right)\Psi_0\left[\tilde{\psi}\right] = 0$; the result is:

$$\Psi_0\left[\tilde{\psi}\right] = \prod_{\vec{k}} \left(\frac{\omega_k}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{1}{2} \int d^3k \frac{1}{(2\pi)^3} \omega_k \tilde{\psi}^2\left(\left|\vec{k}\right|\right) \right) \quad ; \tag{2.32}$$

where ω_k is the energy: $\omega_k = \sqrt{\vec{k}^2 + m^2}$. It is now evident that the ground state is the infinite product of ordinary harmonic oscillator ground state wave functions. The ground state energy eigenvalue, E_0, is given by

$$E_0 = \frac{1}{2} \int d^3k \omega_k \delta^3(0) \quad . \tag{2.33}$$

Excited states can be easily built from $\Psi_0\left[\tilde{\psi}\right]$:

$$\Psi_1\left[\tilde{\psi}\right] = \frac{a^\dagger\left(\vec{k}_1\right)}{\sqrt{2\omega_{k_1}(2\pi)^3}} \Psi_0\left[\tilde{\psi}\right] = \left(\frac{2\omega_{k_1}}{(2\pi)^3}\right)^{\frac{1}{2}} \tilde{\psi}\left(\vec{k}_1\right) \Psi_0\left[\tilde{\psi}\right] \quad . \tag{2.34}$$

The momentum operator, P^i, generates infinitesimal spatial displacements and in ordinary quantum mechanics it is represented by $-i\,\partial/\partial x_i$. The momentum operator acting on the field operator should have the same effect, namely

$$[P_i, \phi(\vec{x}, t)] = -i \frac{\partial}{\partial x^i} \phi(\vec{x}, t) \quad ; \tag{2.35}$$

it is possible to satisfy this requirement with the operator

$$P_i = -\int d^3x \phi(x)\, \partial_i \dot{\phi}(x) \quad , \tag{2.36}$$

through the equal-time commutators (2.15). In the Schrödinger representation P_i becomes the functional differential operator

$$P_i = i \int d^3x \psi(\vec{x}) \partial_i \frac{\delta}{\delta \psi(\vec{x})} \quad . \tag{2.37}$$

It is now possible to verify that $\Psi_1\left[\tilde{\psi}\right]$ is a momentum eigenstate:

$$P_i \Psi_1\left[\tilde{\psi}\right] = i \int d^3x \psi(\vec{x}) \frac{\partial}{\partial x^i} \left(\frac{2\omega_{k_1}}{(2\pi)^3}\right)^{\frac{1}{2}} \cdot \tag{2.38}$$

$$\cdot \int d^3y \exp\left(-i\vec{k}_1\vec{y}\right) \delta^{(3)}(x-y) \Psi_0\left[\tilde{\psi}\right] = \tag{2.39}$$

$$= (k_1)_i \Psi_1\left[\tilde{\psi}\right] \quad . \tag{2.40}$$

Since $\Psi_1\left[\tilde{\psi}\right]$ is an energy eigenstate with energy ω_{k_1} relative to the vacuum and a momentum eigenstate with momentum $\vec{k}_1$, this allows to identify $\Psi_1\left[\tilde{\psi}\right]$ as a state describing one particle with 4-momentum k_1 and mass m.

2.3 Basics of loop quantum gravity and spinfoam models

In this section I skecth the basics of loop quantum gravity and the spinfoam formulation. There is a vast literature, a good entry point are [2, 15, 16, 17, 18, 19, 20, 21, 22] and references therein.

2.3.1 Founding hypotheses

The main physical hypotheses on which loop quantum gravity relies are only general relativity and quantum mechanics. In other words, loop quantum gravity is a rather conservative quantization of general relativity, with its traditional matter couplings. Of course quantization is far from a univocal algorithm, particularly for a nonlinear field theory. Rather, it is a poorly understood inverse problem (find a quantum theory with the given classical limit). More or less subtle choices are made in constructing the quantum theory. I illustrate these choices below.

The main idea beyond loop quantum gravity is to take general relativity seriously. General relativity introduces the idea that the spacetime metric and the gravitational field are the same physical entity. Thus, a quantum theory of the gravitational field is a quantum theory of the spacetime metric as well. One could conventionally split the spacetime metric into two terms: one to be consider as a

background, which gives a metric structure to spacetime; the other to be treated as a quantum field. This, indeed, is the procedure on which old perturbative quantum gravity, perturbative strings, as well as current non-perturbative string theories (M-theory), are based. In following this path, one assumes, for instance, that the causal structure of spacetime is determined by the underlying background metric alone, and not by the full metric. Contrary to this, in loop quantum gravity one assumes that the identification between the gravitational field and the metric-causal structure of spacetime holds, and must be taken into account, in the quantum regime as well. Thus, no split of the metric is made, and there is no background metric on spacetime. Spacetime can still be described as a (differentiable) manifold (a space without metric structure), over which quantum fields are defined. A classical metric structure will then be defined by expectation values of the gravitational field operator. Thus, the problem of quantum gravity is the problem of understanding what is a quantum field theory on a manifold, as opposed to quantum field theory on a metric space. This is what renders loop quantum gravity different from ordinary quantum field theory. In all versions of ordinary quantum field theory, the metric of spacetime plays an essential role in the construction of the basic theoretical tools (creation and annihilation operators, canonical commutation relations, gaussian measures, propagators...); these tools cannot be used in quantum field over a manifold. Technically, the difficulty due to the absence of a background metric is circumvented in loop quantum gravity by defining the quantum theory as a representation of a Poisson algebra of classical observables which can be defined without using a background metric. The idea that the quantum algebra at the basis of quantum gravity is not the canonical commutation relation algebra, but the Poisson algebra of a different set of observables has long been advocated by Chris Isham [72], whose ideas have been very in in the birth of loop quantum gravity. The algebra on which loop gravity is based is the loop algebra [26]. The particular choice of this algebra is not harmless. Indeed, in choosing the loop algebra as the basis for the quantization, one is essentially assuming that Wilson loop operators are well defined in the Hilbert space of the theory. In other words, that certain states concentrated on one dimensional structures (loops and graphs) have finite norm. This is a subtle non trivial assumptions entering the theory. It is the key assumption that characterizes loop quantum gravity. If the approach turned out to be wrong, it will likely be because this assumption is wrong. The Hilbert space resulting from adopting this assumption is not a Fock space. Physically, the assumption corresponds to the idea that quantum states can be decomposed on a basis of Faraday lines excitations (as Minkowski quantum field theory states can be decomposed on a particle basis). Furthermore, this is an assumption that fails in conventional quantum field theory, because in that context well defined operators and finite norm states need to be smeared in at least three dimensions, and one-dimensional objects are too singular. The assumption does not fail, however, in two-dimensional Yang-Mills theory, which is invariant under area preserving diffeomorphisms, and where loop quantization techniques were successfully employed

[73]. What distinguishes gravity from Yang-Mills theories, however, and makes this assumption viable in gravity even if it fails for Yang-Mills theory is diffeomorphism invariance. The loop states are singular states that span a huge non-separable state space. (Non-perturbative) diffeomorphism invariance plays two roles. First, it wipes away the infinite redundancy. Second, it smears a loop state into a knot state, so that the physical states are not really concentrated in one dimension, but are, in a sense, smeared all over the entire manifold by the nonperturbative diffeomorphisms.

Conventional field theories are not invariant under a diffeomorphism acting on the dynamical fields. (Every field theory, suitably formulated, is trivially invariant under a diffeomorphism acting on everything.) General relativity, on the contrary is invariant under such transformations. More precisely every general relativistic theory has this property. Thus, diffeomorphism invariance is not a feature of just the gravitational field: it is a feature of physics, once the existence of relativistic gravity is taken into account. Thus, one can say that the gravitational field is not particularly special in this regard, but that diff-invariance is a property of the physical world that can be disregarded only in the approximation in which the dynamics of gravity is neglected. What is this property? What is the physical meaning of diffeomorphism invariance? Diffeomorphism invariance is the technical implementation of a physical idea, due to Einstein. The idea is a deep modification of the pre-general-relativistic notions of space and time. In pre-general-relativistic physics, it is assumed that physical objects can be localized in space and time with respect to a fixed non-dynamical background structure. Operationally, this background spacetime can be defined by means of physical reference-system objects, but these objects are considered as dynamically decoupled from the physical system that one studies. This conceptual structure fails in a relativistic gravitational regime. In general relativistic physics, the physical objects are localized in space and time only with respect to each other. Therefore if one displaces all dynamical objects in spacetime at once, one is not generating a different state, but an equivalent mathematical description of the same physical state. Hence, diffeomorphism invariance. Accordingly, a physical state in general relativity is not located somewhere [15, 74] (unless an appropriate gauge fixing is made). Loop quantum gravity is an attempt to implement this relational notion of spacetime localization in quantum field theory. In particular, the basic quantum field theoretical excitations cannot be localized somewhere as, say, photons are. Intuitively, one can understand from this discussion how knot theory plays a role in the theory. First, one defines quantum states that correspond to loop-like excitations of the gravitational field, but then, when factoring away diffeomorphism invariance, the location of the loop becomes irrelevant. The only remaining information contained in the loop is then its knotting (a knot is a loop up to its location). Thus, diffeomorphism invariant physical states are labeled by knots. A knot represents an elementary quantum excitation of space. It is not here or there, since it is the space with respect to which here and there can be defined. A knot state is an elementary quantum of space. In this manner, loop quantum gravity ties the new notion of space and time introduced by general relativity with

quantum mechanics. The existence of such elementary quanta of space is then made concrete by the quantization of the spectra of geometrical quantities.

2.3.2 Formalism of loop quantum gravity

The starting point of the construction of the quantum theory is classical general relativity, formulated in terms of the Sen-Ashtekar-Barbero connection [23, 75, 76]. Detailed introductions to the complex Ashtekar formalism can be found in [77, 78]. The real version of the theory is presently the most widely used. Classical general relativity can be formulated in phase space form as follows [77, 76]. Consider a three-dimensional manifold M (compact and without boundaries), a smooth real $SU(2)$ connection $A_a^i(x)$ and a vector density $\tilde{E}_i^a(x)$ (transforming in the vector representation of $SU(2)$) on M. The notation is as follows: $a, b, \ldots = 1, 2, 3$ for spatial indices and $i, j, \ldots = 1, 2, 3$ for internal indices. The internal indices can be viewed as labeling a basis in the Lie algebra of $SU(2)$ or the three axis of a local triad. Coordinates on M are indicated with x. The relation between these fields and conventional metric gravitational variables is as follows: $\tilde{E}_i^a(x)$ is the (densitized) inverse triad, related to the three-dimensional metric $g_{ab}(x)$ of constant-time surfaces by

$$g g^{ab} = \tilde{E}_i^a \tilde{E}_i^b \quad , \tag{2.41}$$

where g is the determinant of g^{ab}, and

$$A_a^i(x) = \Gamma_a^i(x) + \gamma k_a^i(x) \quad ; \tag{2.42}$$

$\Gamma_a^i(x)$ is the spin connection associated to the triad, defined by $\partial_{[a} e_{b]}^i = \Gamma_{[a}^i e_{b]j}$, where e_a^i is the triad. $k_a^i(x)$ is the extrinsic curvature of the constant time three surface. In (2.42), γ is a constant, denoted the Immirzi parameter, that can be chosen arbitrarily (it will enter the hamiltonian constraint) [79, 80, 81]. Different choices for γ yield different versions of the formalism, all equivalent in the classical domain. If γ is choosen to be equal to the imaginary unit, $\gamma = \sqrt{-1}$, then A is the standard Ashtekar connection, which can be shown to be the projection of the selfdual part of the four-dimensional spin connection on the constant time surface. With the choice $\gamma = 1$ one obtains the real Barbero connection. The hamiltonian constraint of lorentzian general relativity has a particularly simple form in the $\gamma = \sqrt{-1}$ formalism, while the hamiltonian constraint of euclidean general relativity has a simple form when expressed in terms of the $\gamma = 1$ real connection. Other choices of γ are viable as well. In particular, it has been argued that the quantum theories based on different choices of γ are genuinely physical inequivalent, because they yield geometrical quanta of different magnitude [82]. Apparently, there is a unique choice of γ yielding the correct 1/4 coefficient in the Bekenstein-Hawking formula [83, 84, 85, 86]. The spinorial version of the Ashtekar variables is given in terms of

the Pauli matrices σ_i, $i = 1, 2, 3$ or the $su(2)$ generators $\tau_i = -\frac{i}{2}\sigma_i$, by

$$\tilde{E}^a(x) = -i\tilde{E}^a_i(x)\,\sigma_i = 2\tilde{E}^a_i(x)\,\tau_i \tag{2.43}$$

$$A_a(x) = -\frac{i}{2}A^i_a(x)\,\sigma_i = A^i_a(x)\,\tau_i \quad . \tag{2.44}$$

Thus, $A_a(x)$ and $\tilde{E}^a(x)$ are 2×2 anti-hermitian complex matrices. The theory is invariant under local $SU(2)$ gauge transformations, three-dimensional diffeomorphisms of the manifold on which the fields are defined, as well as under (coordinate) time translations generated by the hamiltonian constraint. The full dynamical content of general relativity is captured by the three constraints that generate these gauge invariances [75, 77]. As already mentioned, the lorentzian hamiltonian constraint does not have a simple polynomial form if one uses the real connection (2.42). For a while, this fact was considered an obstacle for defining the quantum hamiltonian constraint; therefore the complex version of the connection was mostly used. However, Thiemann has succeeded in constructing a lorentzian quantum hamiltonian constraint [87, 88, 89] in spite of the non-polynomiality of the classical expression. This is the reason why the real connection is now widely used. This choice has the advantage of eliminating the old reality conditions problem, namely the problem of implementing non-trivial reality conditions in the quantum theory.

Loop algebra

Certain classical quantities play a very important role in the quantum theory. These are: the trace of the holonomy of the connection, which is labeled by loops on the three manifold; and the higher order loop variables, obtained by inserting the E field (in n distinct points, or "hands") into the holonomy trace. More precisely, given a loop α in M and the points $s_1, s_2, \ldots, s_n \in \alpha$ one defines

$$\mathcal{T}[\alpha] = -Tr[U_\alpha] \tag{2.45}$$

$$\mathcal{T}[\alpha](s) = -Tr\left[U_\alpha(s,s)\,\tilde{E}^a(s)\right] \tag{2.46}$$

and, in general

$$\mathcal{T}^{a_1\ldots a_N}[\alpha](s_1,\ldots,s_N) = -Tr\left[U_\alpha(s_1,s_N)\,\tilde{E}^{a_N}(s_N)\,U_\alpha(s_N,s_{N-1})\cdots\tilde{E}^{a_1}(s_1)\right] \tag{2.47}$$

where $U_\alpha(s_1,s_2) \sim \mathcal{P}\exp\left\{\int_{s_1}^{s_2} A_a(\alpha(s))\,ds\right\}$ is the parallel propagator of A_a along α, defined by

$$\frac{d}{ds}U_\alpha(1,s) = \frac{d\alpha_a(s)}{ds}A_\alpha(\alpha(s))\,U_\alpha(1,s) \quad , \tag{2.48}$$

see [28] for more details. These are the loop observables, introduced in Yang Mills theories in [90, 91], and in gravity in [25, 26]. The loop observables coordinatize the

phase space and have a closed Poisson algebra, denoted the loop algebra. This algebra has a remarkable geometrical flavor. For instance, the Poisson bracket between $\mathcal{T}[\alpha]$ and $\mathcal{T}^a[\beta](s)$ is non vanishing only if $\beta(s)$ lies over α; if it does, the result is proportional to the holonomy of the Wilson loops obtained by joining α and β at their intersection (by rerouting the 4 legs at the intersection). More precisely

$$\{\mathcal{T}[\alpha], \mathcal{T}^a[\beta](s)\} = \Delta^a[\alpha, \beta(s)]\left[\mathcal{T}[\alpha\#\beta] - \mathcal{T}\left[\alpha\#\beta^{-1}\right]\right] \quad . \tag{2.49}$$

Here

$$\Delta^a[\alpha, x] = \int ds \frac{d\alpha^a(s)}{ds}\delta^3(\alpha(s), x) \tag{2.50}$$

is a vector distribution with support on α and $\alpha\#\beta$ is the loop obtained starting at the intersection between α and β, and following first α and then β. β^{-1} is β with reversed orientation. A (non-$SU(2)$ gauge invariant) quantity that plays a role in certain aspects of the theory, particularly in the regularization of certain operators, is obtained by integrating the E field over a two dimensional surface S

$$E[S, f] = \int_S dS_a \tilde{E}^a_i f^i \quad , \tag{2.51}$$

where f is a function on the surface S, taking values in the Lie algebra of $SU(2)$.

Loop quantum gravity

The kinematics of a quantum theory is defined by an algebra of elementary operators (such as x and $i\hbar d/dx$, or creation and annihilation operators) on a Hilbert space $\mathcal{H}$. The physical interpretation of the theory is based on the connection between these operators and classical variables, and on the interpretation of $\mathcal{H}$ as the space of the quantum states. The dynamics is governed by a hamiltonian, or, as in general relativity, by a set of quantum constraints, constructed in terms of the elementary operators. To assure that the quantum Heisenberg equations have the correct classical limit, the algebra of the elementary operator has to be isomorphic to the Poisson algebra of the elementary observables. This yields the heuristic quantization rule: promote Poisson brackets to commutators. In other words, define the quantum theory as a linear representation of the Poisson algebra formed by the elementary observables. For the reasons illustrated in the preceding subsection, the algebra of elementary observables we choose for the quantization is the loop algebra, defined above. Thus, the kinematic of the quantum theory is defined by a unitary representation of the loop algebra. Here, this representation will be constructed following a simple path. One can start à la Schrödinger by expressing quantum states by means of the amplitude of the connection, namely by means of functionals $\Psi(A)$ of the (smooth) connection. These functionals form a linear space, which one promotes to a Hilbert space by defining a inner product. To define the inner product, one chooses a particular set of states, which are denoted cylindrical states and begin by defining the scalar product between these. Pick a graph Γ, say with n

links, denoted $\gamma_1, \ldots, \gamma_n$, immersed in the manifold M. For technical reasons, one requires the links to be analytic. Let $U_i(A) = U_{\gamma_i}$, $i = i, \ldots, n$ be the parallel transport operator of the connection A along γ_i. $U_i(A)$ is an element of $SU(2)$. Pick a function $f(g_1, \ldots, g_n)$ on $[SU(2)]^n$. The graph Γ and the function f determine a functional of the connection as follows

$$\psi_{\Gamma,f}(A) = f(U_1(A), \ldots, U_n(A)) \quad . \tag{2.52}$$

These states are called cylindrical states because they were introduced in [29, 92, 93] as cylindrical functions for the definition of a cylindrical measure. Notice that it is always possible to enlarge the graph, in the sense that if Γ is a subgraph of Γ' one can always write

$$\psi_{\Gamma,f}(A) = \psi_{\Gamma',f'}(A) \tag{2.53}$$

by simply choosing f' independent from the U_i's of the links which are in Γ' but not in Γ. Thus, given any two cylindrical functions, one can always view them as having the same graph (formed by the union of the two graphs). Given this observation, one defines the scalar product between any two cylindrical functions [94, 29, 92, 93] by

$$(\psi_{\Gamma,f}, \psi_{\Gamma,h}) = \int_{SU(2)^n} dg_1 \cdots dg_n \overline{f(g_1 \cdots g_n)} h(g_1 \cdots g_n) \quad , \tag{2.54}$$

where dg is the Haar measure on $SU(2)$. This scalar product extends by linearity to finite linear combinations of cylindrical functions. It is not difficult to show that (2.54) defines a well defined scalar product on the space of these linear combinations. Completing the space of these linear combinations in the Hilbert norm, one obtains a Hilbert space $\mathcal{H}$. This is the (unconstrained) quantum state space of loop gravity. $\mathcal{H}$ carries a natural unitary representation of the diffeomorphism group and of the group of the local $SU(2)$ transformations, obtained transforming the argument of the functionals. An important property of the scalar product (2.54) is that it is invariant under both these transformations. $\mathcal{H}$ is non-separable. At first sight, this may seem as a serious obstacle for its physical interpretation. But after factoring away diffeomorphism invariance it is possible to obtain a separable Hilbert space (see below). Also, standard spectral theory holds on $\mathcal{H}$, and it turns out that using spin networks (discussed below) one can express $\mathcal{H}$ as a direct sum over finite dimensional subspaces which have the structure of Hilbert spaces of spin systems; this makes practical calculations very manageable. Finally, in Dirac notation one can write

$$\Psi(A) = \langle A|\Psi\rangle \quad , \tag{2.55}$$

in the same manner in which one may write $\psi(x) = \langle x|\psi\rangle$ in ordinary quantum mechanics. As in that case, however, one should remember that $|A\rangle$ is not a normalizable state.

Loop states and spin network states

A subspace $\mathcal{H}_0$ of $\mathcal{H}$ is formed by states invariant under $SU(2)$ gauge transformations. The next step is defining an orthonormal basis in $\mathcal{H}_0$. This basis was introduced in [31] and developed in [32, 95]; it is denoted spin network basis. First, given a loop α in M, there is a normalized state $\psi_\alpha(A)$ in $\mathcal{H}$, which is obtained by taking $\Gamma = \alpha$ and $f(g) = -Tr(g)$. Namely

$$\psi_\alpha(A) = -TrU_\alpha(A) \quad . \tag{2.56}$$

By introducing a Dirac notation for the abstract states, this state is denoted as $|\alpha\rangle$. These states are called loop states. Using Dirac notation, one can write

$$\psi_\alpha(A) = \langle A|\alpha\rangle \quad . \tag{2.57}$$

It is easy to show that loop states are normalizable. Products of loop states are normalizable as well. Traditionally, α denotes also a multiloop, namely a collection of (possibly overlapping) loops $\{\alpha_1, \ldots, \alpha_n\}$; a multiloop state is

$$\psi_\alpha(A) = \psi_{\alpha_1}(a) \times \cdots \times \psi_{\alpha_n}(A). \tag{2.58}$$

(Multi-)loop states represented the main tool for loop quantum gravity before the discovery of the spin network basis. Linear combinations of multiloop states (over-)span $\mathcal{H}$, and therefore a generic state $\psi(A)$ is fully characterized by its projections on the multiloop states, namely by

$$\psi(\alpha) = (\psi_\alpha, \psi) \quad . \tag{2.59}$$

The old loop representation was based on representing quantum states in this manner, namely by means of the functionals $\psi(\alpha)$ over the loop space defined in (2.59). Equation (2.59) can be explicitly written as an integral transform. Next, consider a graph Γ. A coloring of Γ is given by the following. First, associate an irreducible representation of $SU(2)$ to each link of Γ. Equivalently, one may associate to each link γ_i a half integer number s_i, the spin of the irreducible representation, or, equivalently, an integer number p_i, the color $p_i = 2s_i$. Next, associate an invariant tensor v in the tensor product of the representations $s_1 \ldots s_n$, to each node of Γ in which links with spins $s_1 \ldots s_n$ meet. An invariant tensor is an object with n indices in the representations $s_1 \ldots s_n$ that transform covariantly. If $n = 3$, there is only one invariant tensor (up to a multiplicative factor), given by the Clebsh-Gordan coefficient. An invariant tensor is also called an intertwining tensor. All invariant tensors are given by the standard Clebsch-Gordan theory. More precisely, for fixed $s_1 \ldots s_n$, the invariant tensors form a finite dimensional linear space. Pick a basis v_j in this space, and associate one of these basis elements to the node. Notice that invariant tensors exist only if the tensor product of the representations $s_1 \ldots s_n$ contains the trivial representation. This yields a condition on the coloring of the links. For $n = 3$, this

is given by the well known Clebsh-Gordan condition: each color is not larger than the sum of the other two, and the sum of the three colors is even. A colored graph is indicated by $\{\Gamma, \vec{s}, \vec{v}\}$, or simply $S = \{\Gamma, \vec{s}, \vec{v}\}$, and called a spin network. Given a spin network S, it is possible to construct a state $\Psi_S(A)$ as follows. One takes the propagator of the connection along each link of the graph, in the representation associated to that link, and then, at each node, one contracts the matrices of the representation with the invariant tensor. The result is a state $\Psi_S(A)$, which is also written as

$$\psi_S(A) = \langle A|S\rangle \quad . \tag{2.60}$$

One can then show the following.

- The spin network states are normalizable. The normalization factor is computed in [28].
- They are $SU(2)$ gauge invariant.
- Each spin network state can be decomposed into a finite linear combination of products of loop states.
- The (normalized) spin network states form an orthonormal basis for the gauge $SU(2)$ invariant states in $\mathcal{H}$ (choosing the basis of invariant tensors appropriately).
- The scalar product between two spin network states can be easily computed graphically and algebraically. See [28] for details.

The spin network states provide a very convenient basis for the quantum theory. The spin network states defined above are $SU(2)$ gauge invariant. There exists also an extension of the spin network basis to the full Hilbert space (see for instance [96, 97], and references therein).

The representation

The quantum operators, corresponding to the $\mathcal{T}$-variables, can be defined as linear operators on $\mathcal{H}$. These form a representation of the loop variables Poisson algebra. The operator $\mathcal{T}[\alpha]$ acts diagonally

$$\mathcal{T}[\alpha]\Psi(A) = -TrU_\alpha(A)\Psi(A) \quad . \tag{2.61}$$

Indeed, products of loop states and spin network states are normalizable states. Higher order loop operators are expressed in terms of the elementary grasp operation. Consider first the operator $\mathcal{T}^a(s)[\alpha]$, with one hand in the point $\alpha(s)$. The operator annihilates all loop states that do not cross the point $\alpha(s)$. Acting on a loop state $|\beta\rangle$, it gives

$$\mathcal{T}^a(s)[\alpha]|\beta\rangle = l_0^2\Delta^a[\beta, \alpha(s)]\left[|\alpha\#\beta\rangle - \left|\alpha\#\beta^{-1}\right\rangle\right] \quad , \tag{2.62}$$

where the elementary length l_0 is such that

$$l_0^2 = \hbar G = \frac{16\pi\hbar G_{Newton}}{c^3} = 16\pi l_{Planck}^2 \tag{2.63}$$

and Δ^a is defined in (2.50), where also # is defined. This action extends by linearity, continuity and by the Leibniz rule to products and linear combinations of loop states, and to the full $\mathcal{H}$. In particular, it is not difficult to compute its action on a spin network state [28]. Higher order loop operators act similarly. It is simple to verify that these operators provide a representation of the classical Poisson loop algebra. All the operators in the theory are then constructed in terms of these basics loop operators, in the same way in which in conventional quantum field theory one constructs all operators, including the hamiltonian, in terms of creation and annihilation operators. The construction of the composite operators requires the development of regularization techniques that can be used in the absence of a background metric. These have been introduced in [98] and developed in [99, 28, 100, 101, 96].

Algebraic version (loop representation) and differential version (connection representation) of the formalism, and their equivalence

It is possible to build directly the quantum theory in the spin-network (or loop) basis, without ever mentioning functionals of the connections. This representation of the theory is denoted the loop representation. A section of the first paper on loop quantum gravity by Rovelli and Smolin [26] was devoted to a detailed study of transformation theory (in the sense of Dirac) on the state space of quantum gravity, and in particular on the relations between the loop states

$$\psi(\alpha) = \langle\alpha|\psi\rangle \tag{2.64}$$

and the states $\psi(A)$ giving the amplitude for a connection field configuration A, and defined by

$$\psi(A) = \langle A|\psi\rangle \quad . \tag{2.65}$$

Here $|A\rangle$ are eigenstates of the connection operator, or, more precisely (since the operator corresponding to the connection is ill defined in the theory) the generalized states that satisfy

$$\mathcal{T}[\alpha]\,|A\rangle = -Tr\left[\mathcal{P}e^{\int_\alpha A}\right]|A\rangle \quad . \tag{2.66}$$

However, at the time of [26] the lack of a scalar product made transformation theory quite involved. On the other hand, the introduction of the scalar product (2.54) gives a rigorous meaning to the loop transform. In fact, one can write, for every spin network S, and every state $\psi(A)$

$$\psi(S) = \langle S|\psi\rangle = (\psi_S, \psi) \quad . \tag{2.67}$$

This equation defines a unitary mapping between the two presentations of $\mathcal{H}$: the loop representation, in which one works in terms of the basis $|S\rangle$; and the connection representation, in which one uses wave functionals $\psi(A)$. The complete equivalence of these two approaches has been firmly established. In particular, the work of Roberto De Pietri [102] has proven the unitary equivalence of the two formalisms. For a more recent discussion see also [103].

Diffeomorphism invariance

The next step in the construction of the theory is to factor away diffeomorphism invariance. This is a key step for two reasons. First of all, $\mathcal{H}$ is a huge non-separable space. However, most of this redundancy is gauge, and disappears when one solves the diffeomorphism constraint, defining the diff-invariant Hilbert space $\mathcal{H}_{Diff}$. This is the reason for which the loop representation, as defined here, is of great value in diffeomorphism invariant theories only. The second reason is that $\mathcal{H}_{Diff}$ turns out to have a natural basis labeled by knots. More precisely by s-knots. An s-knot s is an equivalence class of spin networks S under diffeomorphisms. An s-knot is characterized by its abstract graph (defined only by the adjacency relations between links and nodes), by the coloring, and by its knotting and linking properties, as in knot-theory. Thus, the physical quantum states of the gravitational field turn out to be essentially classified by knot theory. There are various equivalent way of obtaining $\mathcal{H}_{Diff}$ from $\mathcal{H}$. One can use regularization techniques for defining the quantum operator corresponding to the classical diffeomorphism constraint in terms of elementary loop operators, and then find the kernel of such operator. Equivalently, one can factor $\mathcal{H}$ by the natural action of the diffeomorphism group that it carries. Namely

$$\mathcal{H}_{Diff} = \frac{\mathcal{H}}{Diff(M)} \quad . \tag{2.68}$$

There are several rigorous ways for defining the quotient of a Hilbert space by the unitary action of a group. See in particular the construction in [100], which follows the ideas of Marolf and Higuchi [104, 105, 106]. Apparently, there is the problem that a scalar product is not defined on the space of solutions of a constraint $\hat{C}$, defined on a Hilbert space $\mathcal{H}$. This, however, is a false problem. It is true that if zero is in the continuum spectrum of $\hat{C}$ then the corresponding eigenstates are generalized states and the $\mathcal{H}$ scalar product is not defined between them. But the generalized eigenspaces of $\hat{C}$, including the kernel, inherit nevertheless a scalar product from $\mathcal{H}$. This can be seen in a variety of equivalent ways. For instance, it can be seen from the following theorem. If $\hat{C}$ is self adjoint, then there exist a measure $\mu(\lambda)$ on its spectrum and a family of Hilbert spaces $\mathcal{H}(\lambda)$ such that

$$\mathcal{H} = \int d\mu(\lambda)\, \mathcal{H}(\lambda) \quad , \tag{2.69}$$

where the integral is a continuous sum of Hilbert spaces. $\mathcal{H}(\lambda)$ is the kernel of $\hat{C}$ equipped with a scalar product. There are two distinct ways of factoring away the

diffeomorphisms in the quantum theory, yielding two distinct version of the theory. The first way is to factor away smooth transformations of the manifold. In doing so, finite dimensional moduli spaces associated with high valence nodes appear [107], so that the resulting Hilbert space is still nonseparable. The physical relevance of these moduli parameters is unclear at this stage, since they do not seem to play any role in the quantum theory. Alternatively, one can consistently factor away continuous transformations of the manifold. This possibility has been explored by Zapata in [108, 27], and seems to lead to a consistent theory free of the residual non separability.

Dynamics

Finally, the definition of the theory is completed by giving the hamiltonian constraint. A number of approaches to the definition of a hamiltonian constraint have been attempted in the past, with various degrees of success. Recently, however, Thiemann [87, 88, 89] has succeeded in providing a regularization of the hamiltonian constraint that yields a well defined, finite operator. In the following I describe the final form of the constraint, following [109]. For the euclidean hamiltonian constraint, one has

$$\hat{H}\left|s\right\rangle = \sum_{i} \sum_{(IJ)} \sum_{\epsilon=\pm 1} \sum_{\epsilon'=\pm 1} A_{\epsilon\epsilon'}\left(p_1 \dots p_n\right) \hat{D}_{i;(IJ),\epsilon\epsilon'}\left|s\right\rangle \quad . \tag{2.70}$$

Here i labels the nodes of the s-knot s; (IJ) labels couples of (distinct) links emerging from i. $p_{1\dots}p_n$ are the colors of the links emerging from i. $\hat{D}_{i;(IJ),\epsilon\epsilon'}$ is the operator that acts on an s-knot by: (i) creating two additional nodes, one along each of the two links I and J; (ii) creating a novel link, colored 1, joining these two nodes, (iii) assigning the coloring $p_I + \epsilon$ and, respectively, $p_I + \epsilon$ to the links that join the new formed nodes with the node i. The coefficients $A_{\epsilon\epsilon'}\left(p_1 \dots p_n\right)$, which are finite, can be expressed explicitly (but in a rather laborious way) in terms of products of linear combinations of $6-j$ symbols of $SU\left(2\right)$, following the techniques developed in detail in [28]. Some of these coefficients have been explicitly computed [97]. The lorentzian hamiltonian constraint is given by a similar expression, but quadratic in the $\hat{D}$ operators. The operator defined above is obtained by introducing a regularized expression for the classical hamiltonian constraint, written in terms of elementary loop observables, turning these observables into the corresponding operators and taking the limit. The construction works thanks the fact, first noticed in [110], that certain operator limits $\hat{O}_\epsilon \to \hat{O}$ turn out to be finite on diff invariant states, thanks to the fact that for ϵ and ϵ' sufficiently small, $\hat{O}_\epsilon\left|\Psi\right\rangle$ and $\hat{O}_{\epsilon'}\left|\Psi\right\rangle$ are diffeomorphic equivalent. Thus, here diff invariance plays again a crucial role in the theory.

2.3.3 Spinfoam models

Just as a spin network is a graph with edges labeled by spins and vertices labeled by intertwining operators, a spin foam [35, 36, 37, 38, 39] is a 2-dimensional piecewise

linear cell complex - roughly, a finite collection of polygons attached to each other along their edges - with faces labeled by spins and edges labeled by intertwining operators. As with spin networks, one may think of spin foams either abstractly or embedded in spacetime. Either way, a generic slice of a spin foam "at fixed time" gives a spin network. Edges of this spin network come from faces of the spin foam, while vertices of the spin network come from edges of the spin foam. As we move the slice "forwards in time", the spin network changes topology only when the slice passes a vertex of the spin foam. In their joint work, Reisenberger and Rovelli [111] arrive at spin foams through the study of quantum gravity on a manifold of the form $\mathbb{R} \times S$ for some 3-manifold S representing space. They begin with the Hamiltonian constraint H with constant lapse function as an operator on the space of kinematical states. The actual form of this operator is currently a matter of controversy: Thiemann has proposed a formula [87], but it is far from universally accepted. Luckily, Reisenberger and Rovelli's argument depends only on some general assumptions as to the nature of the operator H. Under these assumptions, they are able to compute the transition amplitude $\langle \Psi, \exp(-itH)\,\Phi \rangle$ as a formal power series in t for any spin network states Ψ and Φ. The coefficient of t^n in this power series is a sum over certain equivalence classes of spin foams embedded in spacetime with Ψ as their initial slice, Φ as their final slice, and a total of n foam vertices. Each spin foam contributes an amplitude given by a product over its vertices of certain "spin foam vertex amplitudes".

Spin foams play a role much like that of Feynman diagrams. In standard quantum field theory transition amplitudes are computed as sums or integrals over graphs with edges labeled by irreducible unitary representations of the relevant symmetry group. Typically this group is the product of the Poincaré group and some internal symmetry group, so the edges are labeled by momenta, spins, and certain internal quantum numbers. To compute the transition amplitude from one basis state to another, one sums over graphs going from one set of points labeled by representations (and vectors lying in these representations) to some other such set. The contribution of any graph to the amplitude is given by a product of amplitudes associated to its vertices and edges. Each vertex amplitude depends only on the representations labeling the incident edges, while each edge amplitude, or propagator, depends only on the label of the edge itself. The propagators are usually computed using a free theory about which one is doing a perturbative expansion, while the vertices represent interactions. Similarly, one can consider spin foam models for an arbitrary symmetry group G, generalizing the case considered by Reisenberger and Rovelli, in which $G = SU(2)$. In this more general context a spin network is defined as a graph with oriented edges labeled by irreducible unitary representations of G and vertices labeled by intertwining operators from the tensor product of the representations labeling "incoming" incident edges to the tensor product of representations labeling "outgoing" edges. By analogy, a spin foam is defined to be a 2-dimensional piecewise linear cell complex with oriented faces labeled by irreducible unitary representations of G and oriented edges labeled by intertwining operators. In such a

model we compute the transition amplitude between two spin networks as a sum over spin foams going from the first spin network to the second. Each spin foam contributes to the amplitude an amount given by a product of amplitudes associated to its vertices, edges, and faces. In a model with crossing symmetry, the amplitude of any vertex depends only on the isotopy class of the spin network obtained by intersecting the spin foam with a small sphere centered at the vertex. The amplitude of any edge depends only on the intertwining operator labeling that edge and the representations labeling the incident faces. The amplitude of any face depends only on the representation labeling that face. By analogy with Feynman diagrams, the edge and face amplitudes can be thought of as propagators. Spin foam vertices can be thought of as interactions, and the vertex amplitudes characterize the nontrivial dynamics of the theory.

Chapter 3

Propagation kernel defined on an infinite strip

In this Chapter I will give a first definition of the propagation kernel between field configurations, and investigate some of its properties. The results here described are reported in the paper [58].

3.1 Definition

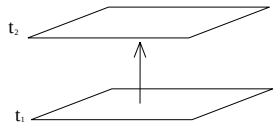

Figure 3.1: Infinite strip

The main tool used in this thesis is the propagation kernel between field configurations. In this Chapter I deal with the propagation kernel $W[\varphi_1, t_1; \varphi_2, t_2]$ which evolves the dynamical field on an infinite strip, that is, an infinite region of spacetime bounded by two hyperplanes at fixed time, see figure 3.1. More precisely, $W[\varphi_1, t_1; \varphi_2, t_2]$ propagates the field ϕ from the field configuration $\phi|_{t=t_1} = \varphi_1$ defined on the spatial hyperplane at time t_1 to the field configuration $\phi|_{t=t_2} = \varphi_2$ defined on the spatial hyperplane at time t_2. It is an extension of the propagation kernel $K(a, b)$ propagating a particle from position x_a at time t_a to position x_b at time t_b. I will now proceed to define it both on minkowskian and in euclidean space; I will denote the propagation kernel in the minkowskian and euclidian case respectively as W_M and W_E, leaving the notation W for situations where both cases are concerned. In the minkowskian case the propagation kernel can be defined by

generalizing either (2.4)

$$W_M\left[\varphi_1, t_1; \varphi_2, t_2\right] = \int_{\varphi_1, t_1; \varphi_2, t_2} \exp \frac{i}{h} S\left[\phi\right] \mathcal{D}\phi \quad , \tag{3.1}$$

or (2.8)

$$W_M\left[\varphi_1, t_1; \varphi_2, t_2\right] = \left\langle \varphi_2 \right| e^{-iH(t_2 - t_1)} \left| \varphi_1 \right\rangle \quad . \tag{3.2}$$

From this last definition an extension of (2.11) easily follows, inserting sums on eigenstates of the energy,

$$W_M\left[\varphi_1, t_1; \varphi_2, t_2\right] = \left\langle \varphi_2 \right| e^{-iH(t_2 - t_1)} \left| \varphi_1 \right\rangle = \tag{3.3}$$

$$= \sum_{mn} \left\langle \varphi_2 | \Psi_n \right\rangle \left\langle \Psi_n \right| e^{-iH(t_2 - t_1)} \left| \Psi_m \right\rangle \left\langle \Psi_m | \varphi_1 \right\rangle = \tag{3.4}$$

$$= \sum_{n} \Psi_n\left[\varphi_2\right] \Psi_n^*\left[\varphi_1\right] e^{-iE_n(t_2 - t_1)} \quad . \tag{3.5}$$

It is now easy to see why the kernel W is a field-to-field propagator; indeed:

$$\Psi\left[\varphi_2, t_2\right] = \left\langle \varphi_2 | \Psi \right\rangle = \int D\varphi_1 \left\langle \varphi_2 \right| e^{-iH(t_2 - t_1)} \left| \varphi_1 \right\rangle \left\langle \varphi_1 | \Psi \right\rangle = \tag{3.6}$$

$$= \int D\varphi_1 W\left[\varphi_2, t_2; \varphi_1, t_1\right] \Psi\left[\varphi_1\right] \quad . \tag{3.7}$$

The state space at time t_1, $\mathcal{H}_{t_1}$, is Fock space, where the hamiltonian H is defined. The corresponding definitions of the propagation kernel in the euclidean case are straightforward extensions of the above expressions to imaginary time.

3.2 Calculation

I will now calculate the explicit expression of the propagation kernel in a specific case. I will consider a real massive scalar field ϕ, both in minkowskian space and in euclidean space; for simplicity I will consider the free case, and set $t_1 = 0$, $t_2 = T$.

3.2.1 Minkowskian case

In this case, (3.1) reads

$$W_M\left[\varphi_1, 0; \varphi_2, T\right] = \int_{\varphi_1, 0; \varphi_2, T} D\phi \exp\left(\frac{i}{2} \int_0^T d^4x \left(\partial_\mu \phi\left(x\right) \partial^\mu \phi\left(x\right) - m^2 \phi^2\right)\right) \quad , \tag{3.8}$$

where I introduced the notation

$$\int_0^T d^4x = \int_0^T dx_0 \int_{-\infty}^{+\infty} d^3x \quad . \tag{3.9}$$

(3.8) is a gaussian integral; as such it can be solved by finding the extremal value of the exponent, which corresponds to the classical solution, and then solving the functional integral as the integrand calculated in the classical solution times the integral calculated on a fluctuation around the classical solution, with the appropriate boundary conditions.

The first step is to find the extremal value of the exponent. The variation of the action

$$\delta S = i \int_0^T d^4x \left(\partial_\mu \phi (x) \, \delta \partial^\mu \phi (x) - m^2 \phi \delta \phi \right) = \tag{3.10}$$

$$= i \, \partial_\mu \phi (x) \, \delta \phi (x) |_0^T - i \int_0^T d^4x \Box \phi (x) \, \delta \phi (x) - i \int_0^T d^4x m^2 \phi \delta \phi \quad , \tag{3.11}$$

after imposing the boundary conditions

$$\delta \phi (t = 0) = \delta \phi (t = T) = 0 \tag{3.12}$$

gives

$$\delta S = -i \int_0^T d^4x \left(\Box_x \phi (x) + m^2 \phi (x) \right) \delta \phi (x) \quad ; \tag{3.13}$$

the requirement $\delta S = 0$ gives the classical equation

$$\left(\Box_x + m^2 \right) \phi (x) = 0 \quad . \tag{3.14}$$

It is now necessary to solve this equation in the infinite strip bounded by the two hyperplanes $t = 0$ and $t = T$, with the boundary conditions

$$\begin{aligned} \phi (\vec{x}, 0) &= \varphi_1 (\vec{x}) \\ \phi (\vec{x}, T) &= \varphi_2 (\vec{x}) \quad . \end{aligned} \tag{3.15}$$

This is easily done by considering the Fourier transform $\tilde{\phi} (k)$ of the field $\phi (x)$; the resulting classical solution $\bar{\phi} (x)$ is

$$\bar{\phi} (x) = \int_{-\infty}^{+\infty} \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3y e^{-i\vec{k}(\vec{x}-\vec{y})} \frac{\varphi_2 (\vec{y}) \sin \omega_k t - \varphi_1 (\vec{y}) \sin \omega_k (t - T)}{\sin \omega_k T} \quad , \tag{3.16}$$

with

$$\omega_k = \sqrt{\vec{k}^2 + m^2} \quad . \tag{3.17}$$

Now the functional integral can be solved by substituting $\phi (x) = \bar{\phi} (x) + \eta (x)$, where $\eta (x)$ is a fluctuation:

$$W_M [\varphi_1, 0; \varphi_2, T] = \int_{\varphi_1, 0; \varphi_2, T} D\phi \exp \left(\frac{i}{2} \int_0^T d^4x \left(\partial_\mu \phi (x) \right)^2 - m^2 \phi^2 \right) = \tag{3.18}$$

$$= \int_{\varphi_1, 0; \varphi_2, T} D\eta \exp \left(\frac{i}{2} \int_0^T d^4x \left(\partial_\mu \bar{\phi} (x) + \partial_\mu \eta (x) \right)^2 - m^2 \left(\bar{\phi} (x) + \eta (x) \right)^2 \right) \quad ; \tag{3.19}$$

by using (3.14) and the fact that the boundary conditions for $\eta(x)$ are $\eta(\vec{x}, t=0) = \eta(\vec{x}, t=T) = 0$, this becomes

$$W_M[\varphi_1, 0; \varphi_2, T] = \tag{3.20}$$

$$= \int_{\varphi_1,0;\varphi_2,T} D\eta \exp\left(\frac{i}{2}\int_0^T d^4x \left(\left(\partial_\mu \bar{\phi}(x)\right)^2 - m^2\bar{\phi}^2(x) + \left(\partial_\mu \eta(x)\right)^2 - m^2\eta^2(x)\right)\right) \quad ; \tag{3.21}$$

which can in turn be simplified

$$W_M[\varphi_1, 0; \varphi_2, T] = \exp\left(\frac{i}{2}\int_0^T d^4x \left(\left(\partial_\mu \bar{\phi}(x)\right)^2 - m^2\bar{\phi}^2\right)\right) \cdot \tag{3.22}$$

$$\cdot \int_{0,x_1;0,x_2} D\eta \exp\left(\frac{i}{2}\int_0^T d^4x \left(\left(\partial_\mu \eta(x)\right)^2 - m^2\eta^2\right)\right) = \tag{3.23}$$

$$= \exp\left(\frac{i}{2}\int_{-\infty}^{+\infty} d^3x\ \bar{\phi}(x)\,\partial_{x^0}\bar{\phi}(x)\Big|_0^T\right)\left(\det\left(-\Box - m^2\right)\right)^{-\frac{1}{2}} \quad . \tag{3.24}$$

Using the explicit expression of $\bar{\phi}(x)$ (3.16) this is easily found to be

$$W_M[\varphi_1, 0; \varphi_2, T] = \exp\left(\frac{i}{2}\int_{-\infty}^{+\infty} d^3x\ \bar{\phi}(x)\,\partial_{x^0}\bar{\phi}(x)\Big|_{t_1}^{t_2}\right)\left(\det\left(-\Box - m^2\right)\right)^{-\frac{1}{2}} = \tag{3.25}$$

$$= \frac{1}{\sqrt{\det\left(-\Box - m^2\right)}} \exp\left(\frac{i}{2}\int_{-\infty}^{+\infty} d^3x \int_{-\infty}^{+\infty} d^3y \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}(\vec{y}-\vec{x})}\omega_k \cdot \right. \tag{3.26}$$

$$\left. \cdot \left(-\frac{\varphi_1(\vec{y})\,\varphi_2(\vec{x}) + \varphi_1(\vec{x})\,\varphi_2(\vec{y})}{\sin\omega_k T} + \cot\omega_k T\left(\varphi_2(\vec{y})\,\varphi_2(\vec{x}) + \varphi_1(\vec{y})\,\varphi_1(\vec{x})\right)\right)\right) \quad , \tag{3.27}$$

where again $\omega_k = \sqrt{\vec{k}^2 + m^2}$. The infinite factor $\left(\det\left(-\Box - m^2\right)\right)^{-\frac{1}{2}}$ will be dealt with later, in section 3.3, where I will find the normalization factor of the propagation kernel.

3.2.2 Euclidean case

I will now calculate explicitly the propagation kernel for a free massive scalar field in the euclidean space, using a slightly different method that will be used again in Chapter 5 in the calculation of the generalized Tomonaga-Schwinger equation.

The notation is in strict analogy with the preceding subsection dealing with the minkowskian case. The definition (3.1) of the propagation kernel this time reads

$$W_E[\varphi_1, 0; \varphi_2, T] = \int_{\varphi_1,0;\varphi_2,T} D\phi \exp\left(-\frac{1}{2}\int_0^T d^4x \left(\partial_\mu\phi(x)\,\partial^\mu\phi(x) + m^2\phi^2\right)\right) \quad ; \tag{3.28}$$

with identical steps one arrives at the classical equation

$$\left(\Box_x - m^2\right)\phi\left(x\right) = 0 \quad . \tag{3.29}$$

I will now solve this equation with a Green function technique. That is, I will look for the function $G\left(x,y\right)$ that solves

$$\left(\Box_x - m^2\right)G\left(x,y\right) = -\delta^{(4)}\left(x-y\right) \quad . \tag{3.30}$$

As a first step I will rewrite the classical solution $\bar{\phi}\left(x\right)$ to the equation (3.29) in terms of $G\left(x,y\right)$. (3.30) can be rewritten as

$$\int_0^T d^4x \left(G\left(x,y\right)\left(\Box_x - m^2\right)\bar{\phi}\left(x\right) - \bar{\phi}\left(x\right)\left(\Box_x - m^2\right)G\left(x,y\right)\right) = \tag{3.31}$$

$$= \int_0^T \delta^{(4)}\left(x-y\right)\bar{\phi}\left(x\right)d^4x \quad , \tag{3.32}$$

that is

$$\bar{\phi}\left(y\right) = \int_0^T d^4x \left(G\left(x,y\right)\Box_x\phi\left(x\right) - \phi\left(x\right)\Box_x G\left(x,y\right)\right) = \tag{3.33}$$

$$= \int_0^T dx^0 \int_{-\infty}^{+\infty} d^3x \left[\partial_{x^i}\left(G\left(x,y\right)\overleftrightarrow{\partial}_{x^i}\bar{\phi}\left(x\right)\right) + \partial_{x^0}\left(G\left(x,y\right)\overleftrightarrow{\partial}_{x^0}\bar{\phi}\left(x\right)\right)\right] \quad , \tag{3.34}$$

where I introduced the notation $G\overleftrightarrow{\partial}\phi = G\partial\phi - \phi\partial G$. Supposing that $G\left(x,y\right)$ and $\phi\left(x\right)$ go to zero fast enough at spatial infinity the first term of the sum is zero, and

$$\bar{\phi}\left(y\right) = \int_{-\infty}^{+\infty} d^3x \left(G\left(\vec{x},T,y\right)\partial_{x^0}\bar{\phi}\left(x\right)\big|_{x^0=T} - \partial_{x^0}G\left(x,y\right)\big|_{x^0=T}\varphi_2\left(\vec{x}\right) + \right. \tag{3.35}$$

$$\left. - G\left(\vec{x},0,y\right)\partial_{x^0}\bar{\phi}\left(x\right)\big|_{x^0=0} + \partial_{x^0}G\left(x,y\right)\big|_{x^0=0}\varphi_1\left(\vec{x}\right)\right) \quad . \tag{3.36}$$

To reproduce the boundary conditions (3.15) the Green function G must be zero in $x_0 = 0$, $x_0 = T$, $y_0 = 0$ and $y_0 = T$. Then

$$\bar{\phi}\left(y\right) = \int_{-\infty}^{+\infty} d^3x \left(\partial_{x^0}G\left(\vec{x},0,y\right)\varphi_1\left(\vec{x}\right) - \partial_{x^0}G\left(\vec{x},T,y\right)\varphi_2\left(\vec{x}\right)\right) \quad . \tag{3.37}$$

I now solve (3.30) with a "spatial" Fourier transform:

$$G\left(x,y\right) = \int \frac{d^3k}{\left(2\pi\right)^3}\tilde{G}\left(x_0,y_0\right)e^{-i\vec{k}\left(\vec{x}-\vec{y}\right)} \tag{3.38}$$

$$\delta^{(4)}\left(x-y\right) = \int \frac{d^3k}{\left(2\pi\right)^3}\delta\left(x_0-y_0\right)e^{-i\vec{k}\left(\vec{x}-\vec{y}\right)} \quad ; \tag{3.39}$$

the solution is easily found by imposing $\tilde{G}(0, y_0) = \tilde{G}(T, y_0) = 0$: it is

$$G(x,y) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}(\vec{x}-\vec{y})} \left(\frac{1}{2\omega_k} e^{-\omega_k |x_0 - y_0|} + \right. \tag{3.40}$$

$$\left. + \frac{e^{-\omega_k y_0} e^{-\omega_k T} - e^{-\omega_k |T - y_0|}}{2\omega_k \left(e^{\omega_k T} - e^{-\omega_k T}\right)} e^{\omega_k x_0} + \frac{e^{-\omega_k (T - y_0)} - e^{-\omega_k y_0} e^{\omega_k T}}{2\omega_k \left(e^{\omega_k T} - e^{-\omega_k T}\right)} e^{-\omega_k x_0} \right) \quad ; \tag{3.41}$$

now substituting in (3.37) it is straightforward to find the classical solution:

$$\phi(x) = \int_{-\infty}^{+\infty} d^3y \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}(\vec{x}-\vec{y})} \frac{\sinh \omega t \varphi_2(\vec{y}) - \sinh \omega (t - T) \varphi_1(\vec{y})}{\sinh \omega T} \quad . \tag{3.42}$$

The functional integral can be solved exactly like in the preceding minkowskian case, to obtain

$$W_E[\varphi_1, 0; \varphi_2, T] = \frac{1}{\sqrt{\det(-\Box + m^2)}} \cdot \tag{3.43}$$

$$\cdot \exp\left(-\frac{1}{2} \int_{-\infty}^{+\infty} d^3x \int_{-\infty}^{+\infty} d^3y \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}(\vec{y}-\vec{x})} \omega_k \cdot \right. \tag{3.44}$$

$$\left. \cdot \left(-\frac{\varphi_1(\vec{y}) \varphi_2(\vec{x}) + \varphi_1(\vec{x}) \varphi_2(\vec{y})}{\sinh \omega_k T} + \coth \omega_k T \left(\varphi_2(\vec{y}) \varphi_2(\vec{x}) + \varphi_1(\vec{y}) \varphi_1(\vec{x}) \right) \right) \right) \quad ; \tag{3.45}$$

again, the normalization will be fixed later, in section 3.3.

3.3 Normalisation factor

3.3.1 Minkowskian case

To find the correct normalisation factor for the propagation kernel, it is useful to recur to the Schrödinger equation, which the propagation kernel must solve

$$i \frac{\partial}{\partial T} W_M[\varphi_1, 0; \varphi_2, T] = \tag{3.46}$$

$$= \frac{1}{2} \int d^3x \left(-\frac{\delta^2}{\delta \varphi_2^2(\vec{x})} + |\nabla \varphi_2(\vec{x})|^2 + m^2 \varphi_2^2(\vec{x}) \right) W_M[\varphi_1, 0; \varphi_2, T] \quad , \tag{3.47}$$

with

$$|\nabla \phi|^2 = \partial_i \phi \partial^i \phi \quad . \tag{3.48}$$

By rewriting the propagation kernel as $W_M[\varphi_1, 0; \varphi_2, T] = M(T) \exp(iS(\varphi_1, \varphi_2))$, (3.46) reads as

$$i \exp(iS(\varphi_1, \varphi_2)) \frac{\partial}{\partial T} M(T) - W_M[\varphi_1, 0; \varphi_2, T] \frac{\partial}{\partial T} S(\varphi_1, \varphi_2) = \tag{3.49}$$

$$= \frac{1}{2} \int d^3 z W_M[\varphi_1, 0; \varphi_2, T] \cdot \tag{3.50}$$

$$\cdot \left(\left(-i \frac{\partial^2 S(\varphi_1, \varphi_2)}{\partial \varphi_2^2(\vec{z})} + \left(\frac{\partial S(\varphi_1, \varphi_2)}{\partial \varphi_2(\vec{z})} \right)^2 \right) + \left(|\nabla \varphi_2(\vec{z})|^2 + m^2 \varphi_2^2(\vec{z}) \right) \right) \quad ; \tag{3.51}$$

this can be greatly simplified by remembering that $S(\varphi_1, \varphi_2)$ is the Hamilton principal function, that is, the classical action calculated on the boundary conditions, and as such it must obey

$$0 = \frac{\partial S}{\partial T} + H\left(\varphi_2, \frac{\partial S}{\partial \varphi_2}\right) = \frac{\partial S}{\partial T} + \frac{1}{2} \int d^3 x \left(\left(\frac{\partial S}{\partial \varphi_2} \right)^2 + |\nabla \varphi_2|^2 + m^2 \varphi_2^2 \right) = 0 \tag{3.52}$$

(see [15]), to obtain

$$i \exp(iS(\varphi_1, \varphi_2)) \frac{\partial}{\partial T} M(T) = -\frac{i}{2} \int d^3 z W_M[\varphi_1, 0; \varphi_2, T] \frac{\partial^2 S(\varphi_1, \varphi_2)}{\partial \varphi_2^2(\vec{z})} \quad . \tag{3.53}$$

Now inserting the explicit expression of the propagation kernel one obtains

$$\frac{\partial}{\partial T} M(T) = -\frac{1}{2} M(T) V \int \frac{d^3 k}{(2\pi)^3} \sqrt{\vec{k}^2 + m^2} \cot\left(\sqrt{\vec{k}^2 + m^2} T \right) \quad , \tag{3.54}$$

where V is a volume. The normalization factor $M(T)$ can then be written as

$$M(T) = \prod_k \sqrt{\frac{m\omega_k}{2\pi\hbar}} \exp\left(-\frac{V}{2} \int \frac{d^3 k}{(2\pi)^3} \ln \sin \omega T \right) \quad , \tag{3.55}$$

where the infinite product before the exponential can be found by direct analogy with the single particle case.

3.3.2 Euclidean case

In the euclidian case, the equation that the propagation kernel has to obey is the euclidianized version of the Schrödinger equation:

$$-\frac{\partial}{\partial T} W_E[\varphi_1, 0; \varphi_2, T] = \tag{3.56}$$

$$= \frac{1}{2} \int d^3 x \left(\frac{\delta^2}{\delta \varphi_2^2(\vec{x})} + |\nabla \varphi_2(\vec{x})|^2 + m^2 \varphi_2^2(\vec{x}) \right) W_E(\varphi_1, 0; \varphi_2, T) \quad . \tag{3.57}$$

By rewriting the propagation kernel as $W_E[\varphi_1, 0; \varphi_2, T] = E(T) \exp(-S(\varphi_1, \varphi_2))$ this becomes

$$-\exp(-S(\varphi_1,\varphi_2))\frac{\partial}{\partial T}E(T) + W_E[\varphi_1,0;\varphi_2,T]\frac{\partial}{\partial T}S(\varphi_1,\varphi_2) = \tag{3.58}$$

$$= \frac{1}{2}\int d^3z W_E[\varphi_1,0;\varphi_2,T]\cdot \tag{3.59}$$

$$\cdot\left(\left(-\frac{\partial^2 S(\varphi_1,\varphi_2)}{\partial\varphi_2^2(\vec{z})} + \left(\frac{\partial S(\varphi_1,\varphi_2)}{\partial\varphi_2(\vec{z})}\right)^2\right) + \left(|\nabla\varphi_2(\vec{z})|^2 + m^2\varphi_2^2(\vec{z})\right)\right) \quad , \tag{3.60}$$

which can be simplified by remembering that the Hamilton principal function $S(\varphi_1, \varphi_2)$ has to obey the euclidianized version of (3.52)

$$0 = -\frac{\partial S}{\partial T} + H\left(\varphi_2, \frac{\partial S}{\partial\varphi_2}\right) = -\frac{\partial S}{\partial T} + \frac{1}{2}\int d^3x\left(\left(\frac{\partial S}{\partial\varphi_2}\right)^2 + |\nabla\varphi_2|^2 + m^2\varphi_2^2\right) = 0 \quad . \tag{3.61}$$

From now on the calculation proceeds exactly like in the minkowskian case, to obtain

$$E(T) = \prod_k \sqrt{\frac{m\omega_k}{2\pi\hbar}} \exp\left(\frac{V}{2}\int \frac{d^3k}{(2\pi)^3}\ln\sinh\omega T\right) \quad . \tag{3.62}$$

3.4 Physical dimensions of the kernel

The normalization factor allows to find the physical dimensions of the propagation kernel. As one could expect, they are infinite. Indeed, considering for example the minkowskian case,

$$[W_M[\varphi_1,0;\varphi_2,T]] = [M(T)] = \left[\prod_k \sqrt{\frac{m\omega_k}{2\pi\hbar}}\right] = \prod\left[L^{-1}\right] \quad ; \tag{3.63}$$

the propagation kernel has the physical dimensions of an infinite product of factors L^{-1}.

3.5 Properties of the kernel

It is easy to check that the propagation kernel must obey the following properties.

1. Limit $T \to 0$:

$$\lim_{T\to 0} W[\varphi_1,0;\varphi_2,T] = \prod_x \delta(\varphi_1(\vec{x}) - \varphi_2(\vec{x})) \tag{3.64}$$

where the functional delta must be interpreted as specified in section 2.2.

2. Convolution property:

$$\int dz W\left[\varphi_1, 0; \varphi_2, T_2\right] W\left[\varphi_2, T_2; \phi_3, T_3\right] = W\left[\varphi_1, 0; \phi_3, T_3\right] \quad ; \tag{3.65}$$

this nonlinear equation fixes unambiguously the normalisation of the propagation kernel, which can however be calculated more easily with the method outlined in section 3.3.

3.6 Relation with the vacuum state

The vacuum state wave functional is (see chapter 2, where however it was written in terms of the Fourier transform $\tilde{\psi}$ of the function ψ):

$$\Psi_0\left[\psi\right] = \exp\left(-\frac{1}{2}\int d^3y d^3z \int \frac{d^3p}{\left(2\pi\right)^3} e^{i\vec{p}(\vec{y}-\vec{z})}\sqrt{p^2+m^2}\psi\left(\vec{z}\right)\psi\left(\vec{y}\right)\right) \quad . \tag{3.66}$$

This functional corresponds to the vacuum state defined as the state with the lowest energy. In the following I will call *Minkowski vacuum* the vacuum state defined in this way, to distinguish it from another vacuum state which I'm going to define shortly.

First of all, using the explicit expression of the propagation kernel it's easy to check that $\Psi_0\left[\psi\right]$ propagates into itself, as it should:

$$\Psi_0\left[\psi\right] = \int D\phi W\left[\phi, 0; \psi, T\right]\Psi_0\left[\phi\right] \quad . \tag{3.67}$$

More importantly, the Minkowski vacuum can be calculated using the propagation kernel alone. Indeed, from the definition (3.5) of the propagation kernel

$$\lim_{T\to\infty} W\left[\varphi_1, 0; \varphi_2, T\right] = \lim_{T\to\infty}\sum_n \Psi_n\left[\varphi_2\right]\Psi_n^*\left[\varphi_1\right]e^{-iE_nT} = \Psi_0\left[\varphi_2\right]\Psi_0^*\left[\varphi_1\right] \quad ; \tag{3.68}$$

this limit being valid in theory with a mass gap, that is, if E_0 and E_1 are separated. To obtain precisely the vacuum state it is necessary to set $\varphi_1 = 0$:

$$\lim_{T\to\infty} W\left[0, 0; \phi, T\right] = \Psi_0\left[\phi\right] \quad . \tag{3.69}$$

This result can be easily verified with the explicit expression of the propagation kernel and of the vacuum state functional. In the euclidian case the limit $T \to \infty$ is straightforward, in the minkowskian case it can be made rigorous with stationary phase arguments.

Considering now only the euclidian case, in which it is easier to render calculations rigorous, I'm going to give a new definition of vacuum, which I will call *nonperturbative vacuum*. First of all, consider the surface Σ_T consisting of the union

of the two spatial hyperplanes at times $t = 0$ and $t = T$. I define a *kinematical* Hilbert space $\mathcal{K}_{\Sigma_T}$, associated to the entire surface Σ_T, as the tensor product

$$\mathcal{K}_{\Sigma_T} = \mathcal{H}^*_{t=T} \otimes \mathcal{H}_{t=0} \quad , \tag{3.70}$$

where the notation $\mathcal{H}^*$ indicates the dual of the Hilbert space $\mathcal{H}$; of course $\mathcal{H}^*$ is canonically isomorphic to $\mathcal{H}$. $\mathcal{K}_{\Sigma_T}$ can be called a kinematical Hilbert space for reasons that will become clear in the course of what follows. I will denote a field on Σ_T by $\phi = (\varphi_1, \varphi_2)$. The field basis of the Fock space induces in $\mathcal{K}_{\Sigma_T}$ the basis

$$|\phi\rangle = |\varphi_1, \varphi_2\rangle = \langle\varphi_2|_{t=T} \otimes |\varphi_1\rangle_{t=0} \quad , \tag{3.71}$$

which in the language of wave functionals translates as

$$\Psi[\phi] = \Psi[\varphi_1, \varphi_2] = \langle\varphi_1, \varphi_2|\Psi\rangle \quad . \tag{3.72}$$

In the kinematical Hilbert space $\mathcal{K}_{\Sigma_T}$ the propagation kernel $W[\varphi_1, 0; \varphi_2, T]$ defines the preferred (bra) state

$$\langle 0_{\Sigma_T}|\phi\rangle = W[\varphi_1, 0; \varphi_2, T] \tag{3.73}$$

in this Hilbert space. I call the state $|0_{\Sigma_T}\rangle$ the *nonperturbative vacuum*, or *covariant vacuum*. This state expresses the dynamics from $t = 0$ to $t = T$. As a state in $\mathcal{K}_{\Sigma_T}$, which is the tensor product of two Hilbert spaces, it defines a linear mapping between the two spaces $\mathcal{H}_{t=0}$ and $\mathcal{H}_{t=T}$. This linear mapping is precisely the imaginary time evolution $\exp(-TH)$. Indeed, from the above equations and the euclidian version of the definition (3.2) one finds

$$\langle 0_{\Sigma_T}|\left(\langle\varphi_2| \otimes |\varphi_1\rangle\right) = \langle\varphi_2| e^{-TH} |\varphi_1\rangle \quad , \tag{3.74}$$

which can be rewritten as

$$\langle 0_{\Sigma_T}|\varphi_1\rangle = e^{-TH} |\varphi_1\rangle \quad . \tag{3.75}$$

It is important to point out that the bra/ket mismatch is apparent only, as the three states live in different Hilbert spaces. Taking the limit $T \to \infty$ of (3.74) gives the result of (3.68)

$$\lim_{T\to 0} \langle 0_{\Sigma_T}|\left(\langle\varphi_2| \otimes |\varphi_1\rangle\right) = \langle\varphi_2|0_M\rangle \langle 0_M|\varphi_1\rangle \quad , \tag{3.76}$$

where $|0_M\rangle$ denotes the state that I called Minkowski vacuum. I can therefore write the relation between the two notions of vacuum that I have defined as

$$\lim_{T\to 0} |0_{\Sigma_T}\rangle = |0_M\rangle \otimes \langle 0_M| \quad . \tag{3.77}$$

The tensor product of two quantum state spaces describes the ensemble of the measurements described by the two factors. Therefore $\mathcal{K}_{\Sigma_T}$ is the space of the

possible results of all measurements performed at $t = 0$ and at $t = T$ [15, 112, 113, 114, 115, 47, 48, 49]. Observations at two different times are correlated by the dynamics. Hence $\mathcal{K}_{\Sigma_T}$ is a kinematical state space in the sense that it describes more outcomes than the physically realizable ones. Dynamics is then a restriction on the possible outcome of observations [15, 112, 113, 114, 115, 47, 48, 49]. It expresses the fact that measurement outcomes are correlated. The state $\langle 0_{\Sigma_T}|$, seen as a linear functional on $\mathcal{K}_{\Sigma_T}$, assigns an amplitude to any outcome of observations. This amplitude gives the correlation between outcomes at $t = 0$ and outcomes at $t = T$. Therefore the theory can be represented as follows. The Hilbert space $\mathcal{K}_{\Sigma_T}$ describes all possible outcomes of measurements made on Σ_T. Dynamics is given by the single linear functional

$$\rho : \mathcal{K}_{\Sigma_T} \to C \tag{3.78}$$

$$|\Psi\rangle \mapsto \langle 0_{\Sigma_T}|\Psi\rangle \quad . \tag{3.79}$$

For a given collection of measurement outcomes described by a state $|\Psi\rangle$, the quantity $\langle 0_{\Sigma_T}|\Psi\rangle$ gives the correlation probability amplitude between these measurements.

3.7 Relation with the two-point function

I now wish to clarify the relation between the propagation kernel and the ordinary particle propagator, also called Feynman propagator. The latter is defined as [116] the two-point function

$$i\Delta_F(x_1 - x_2) = \langle 0|\, T\left(\phi(x_1)\,\phi(x_2)\right)|0\rangle \quad , \tag{3.80}$$

where the fields ϕ must be fundamental fields, that is, fields that create the particle whose propagation the propagator describes and not composite operators of any kind. In this case the two-point function $\langle 0|\, T\left(\phi(x_1)\,\phi(x_2)\right)|0\rangle$ is also the Green function for the equations of motion of the particle. It is defined as

$$\langle 0|\, T\left(\phi(x_1)\,\phi(x_2)\right)|0\rangle = \tag{3.81}$$

$$= \theta\left(x_1^0 - x_2^0\right)\langle 0|\,\phi(x_1)\,\phi(x_2)\,|0\rangle \pm \theta\left(x_2^0 - x_1^0\right)\langle 0|\,\phi(x_2)\,\phi(x_1)\,|0\rangle \quad , \tag{3.82}$$

where the plus sign holds for bosons and the minus for fermions. It is also possible to write an extension to a greater number of particles, for example the three-point function is

$$\langle 0|\, T\left(\phi(x_1)\,\phi(x_2)\,\phi(x_3)\right)|0\rangle = \tag{3.83}$$

$$= \theta\left(x_1^0 - x_2^0\right)\theta\left(x_2^0 - x_3^0\right)\langle 0|\,\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,|0\rangle + \text{permutations} \quad . \tag{3.84}$$

In the minkowskian case, the two-point function $\langle 0|\, T\left(\phi\left(x_1\right)\phi\left(x_2\right)\right)|0\rangle$ can be expressed via the propagation kernel in this way

$$\langle 0|\, T\left(\phi\left(x_1\right)\phi\left(x_2\right)\right)|0\rangle = \langle 0|\,\phi\left(\vec{x}_2\right)\exp\left(-iH\left(t_2-t_1\right)\right)\phi\left(\vec{x}_1\right)|0\rangle = \tag{3.85}$$

$$= \lim_{T\to\infty}\int D\varphi_1 D\varphi_2 W_M\left[\varphi_2,t_2;0,T\right]\varphi_2\left(\vec{x}\right)\cdot \tag{3.86}$$

$$\cdot\, W_M\left[\varphi_1,t_1;\varphi_2,t_2\right]\varphi_1\left(\vec{x}\right)W_M\left[0,-T;\varphi_1,t_1\right] \quad , \tag{3.87}$$

where I assumed $x_1^0 > x_2^0$ (otherwise it is enough to rename them). Since the equation involves a limit $T\to\infty$ it will be easier to perform the calculation in the euclidian case, where it is easy to find

$$\langle\phi\left(x_1\right)\phi\left(x_2\right)\rangle = \tag{3.88}$$

$$= \lim_{T\to\infty}\int D\varphi_1 D\varphi_2 W_E\left[\varphi_2,t_2;0,T\right]\varphi_2\left(\vec{x}_2\right)\cdot \tag{3.89}$$

$$\cdot\, W_E\left[\varphi_1,t_1;\varphi_2,t_2\right]\varphi_1\left(\vec{x}_1\right)W_E\left[0,-T;\varphi_1,t_1\right] = \tag{3.90}$$

$$= \int\frac{d^3q}{2\left(2\pi\right)^3}\frac{e^{-i\vec{q}\left(\vec{x}_2-\vec{x}_1\right)}}{\sqrt{\vec{q}^2+m^2}}\exp\left(-\sqrt{\vec{q}^2+m^2}\left(t_2-t_1\right)\right) \quad , \tag{3.91}$$

where the last line is the euclidian particle propagator for a free scalar field (see for example [116]). This shows that with the mere use of the propagation kernel it is possible to write down the two-point function, that is, the particle propagator for the given theory. However, this result allows to do much more than that. Indeed, the quantities that allow to make contact with experiment, that is, the scattering amplitudes like $\langle q_1\ldots q_n \text{ out } |p_1\ldots p_m \text{ in}\rangle$, can be rewritten in terms of n-point functions via the Lehmann-Symanzik-Zimmerman reduction formulas [116, 117]. For example, in the case of four scalar particles,

$$\langle q_1q_2 \text{ out } |p_1p_2 \text{ in}\rangle = \left(\frac{i}{\sqrt{Z_0}}\right)^4\int d^4x_1d^4x_2d^4x_3d^4x_4\cdot \tag{3.92}$$

$$\cdot\, f_{p_1}\left(x_1\right)f_{p_2}\left(x_2\right)f^*_{q_1}\left(x_3\right)f^*_{q_2}\left(x_4\right)\cdot \tag{3.93}$$

$$\cdot\left(\Box_{x_1}+m^2\right)\left(\Box_{x_2}+m^2\right)\left(\Box_{x_3}+m^2\right)\left(\Box_{x_4}+m^2\right)\cdot \tag{3.94}$$

$$\cdot\,\langle 0|\, T\left(O\left(x_1\right)O\left(x_2\right)O\left(x_3\right)O\left(x_4\right)\right)|0\rangle \quad , \tag{3.95}$$

where Z_0 is the residue in the pole of the Fourier transform of the propagator. $f\left(x\right)$ is a solution of the equation of motion, which in this case is Klein-Gordon's equation, so that $f\left(x\right)$ can be for example a plane wave; and $f^*\left(x\right)$ is its complex conjugate. The operator O can be any operator that has nonzero expectation value between the vacuum and the particle states $|p_1\rangle$, $|p_2\rangle$, $|q_1\rangle$, $|q_2\rangle$.

In turn, n-point functions can be rewritten in terms of two-point functions, that is, propagators, thanks to Wick's theorem. This leads to the crucial result that the single tool of the propagation kernel allows to reconstruct scattering amplitudes via the propagators, that is, it allows to reconstruct any quantity of physical interest that can be derived from quantum field theory. Therefore the propagation kernel formulation, whose usage I advocate in order to make contact with the case of loop quantum gravity, allows to completely reformulate quantum field theory.

3.8 Description of the interaction

I have shown in the previous section how with the use of the propagation kernel it is possible to reconstruct particle propagators, in the first place, and from those any scattering amplitude occurring in ordinary quantum field theory. In the final section of this chapter I wish to clarify the relationship between the descriptions of the interaction given by the propagation kernel formulation and the usual Feynman propagator formulation. Usually scattering amplitudes are written through Feynman propagators in momentum space; here I will consider the Feynman propagators written in spacetime which, though more awkward, allow to see more clearly where and when the interaction happens.

The first obvious difference between the two formulations is that the propagation kernel formulation deals with fields, and describes the field evolution from a given configuration to another, whereas the Feynman propagator deals with particles. But there is another difference which at first look might appear deeper: the propagation kernel describes the evolution only between the times corresponding to the initial and the final field configuration, while the Feynman propagator, even though it deals with incoming and outgoing particles that are located in specified positions in space and at given moments of time, it allows the interaction to happen everywhere, at any moment, even at times previous or later than all the times corresponding to incoming and outgoing particles respectively. If the interaction concerns two incoming particles at positions x_1 and x_2 at time t_1 and two outgoing particles at positions x_3 and x_4 at time t_2, the description of the interaction given by the propagation kernel can be pictorially sketched as in figure 3.2, while the description of the interaction given by the Feynman propagator can be represented as in figure 3.3. This can be understood for example by looking at the explicit expression of the scattering amplitude for the scattering of four particles in a scalar theory. The four-point function is given by

$$W\left(x_1, x_2, x_3, x_4\right) = \int D\phi\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\phi\left(x_4\right)e^{\frac{iS}{\hbar}} \quad , \tag{3.96}$$

where x_1, x_2 are the locations of the incoming particles, and x_3, x_4 those of the outgoing ones; for simplicity I will suppose that there is an ordering $x_1^0 = x_2^0 = t_i < x_3^0 = x_4^0 = t_f$. The action S can be separated in a free action S_0 and an interaction

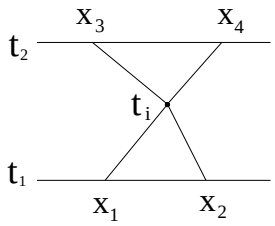

Figure 3.2: Interaction described by the propagation kernel

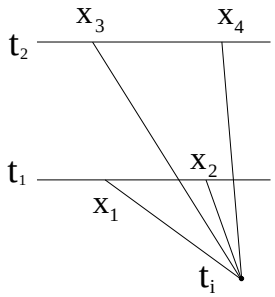

Figure 3.3: Interaction described by the Feynman propagator

term

$$S = S_0 + \lambda \int d^4x \phi^4 (x) \quad . \tag{3.97}$$

Discarding disconnected components and to first order in λ, the four-point function can be rewritten as

$$W(x_1, x_2, x_3, x_4) = \int_{-\infty}^{+\infty} dx_0 \int_{-\infty}^{+\infty} d^3x W(x_1, x) W(x_2, x) W(x_3, x) W(x_4, x) \quad , \tag{3.98}$$

where $W(x_i, x)$ are the propagators that connect the x_i and the point x where the interaction happens. From the limits of the integration on $\vec{x}$ and x_0 it can be seen that indeed the interaction can happen anywhere and at any moment. However, it is easy to verify that by imposing a restriction on the integrand the interaction can be constrained to happen at times later than x_2^0 and earlier than x_3^0. Indeed, by requiring that the integrand in (3.98) contains only positive frequencies, one finds that (3.98) can be rewritten as

$$W(x_1, x_2, x_3, x_4) = \int_{t_i}^{t_f} dx_0 \int_{-\infty}^{+\infty} d^3x W(x_1, x) W(x_2, x) W(x_3, x) W(x_4, x) \quad . \tag{3.99}$$

This is an integration over the spacetime bounded by two infinite spatial hyperplanes at $t = t_i$ and $t = t_f$, that is, over a region analogous to the one the propagation kernel deals with. To sum up, if one writes down scattering amplitudes via the Feynman

propagators, and then impose that the integrals only contain positive frequencies, that is, that the propagation of every particle be only forward in time, one obtains an interaction constrained to happen later than the beginning and earlier than the end of the experiment, that is, the very description of the interaction the propagation kernel gives.

Chapter 4

Extension to the case of a finite region of spacetime

The formalism developed in the past Chapter 3 referred to the evolution of fields in an infinite region of spacetime bounded by two infinite spatial hyperplanes at fixed time. To be able to apply the tools developed in the past Chapter to a quantum theory of general relativity, however, it is necessary to consider the evolution of fields in regions of spacetime bounded by more general surfaces. In this Chapter I'm going to consider the case of closed surfaces. The theoretical framework for studying the evolution of fields in closed surfaces is provided by the so called "general boundary formulation" of quantum field theory, introduced by Robert Oeckl [47, 48, 49]. Therefore, as a first step I review the basics of the general boundary formulation for quantum field theory and quantum gravity, as introduced by Robert Oeckl. Then I extend the propagation kernel formalism in the case of a finite spacetime region.

4.1 General boundary formulation

One obstacle in the quest for a quantum theory of gravity appears to be the fact that the foundations of quantum mechanics are inherently non-covariant. On the other hand, quantum field theory can be formulated in a covariant way. The price for this is a globality (manifest in the path integral expression for n-point functions) that fixes space-time to be minkowskian. Well known difficulties result from this already for the extension of quantum field theory to curved space-times. The general boundary formulation is an approach to formulating quantum theories that is at the same time local and inherently compatible with special or general covariance. The main idea is firstly, to associate state spaces with boundaries of general regions of space-time. Secondly, amplitudes are determined by a complex function for each region and associated state space. Crucially, (and contrary to standard quantum mechanics) connected boundaries of compact regions are the main focus of attention. In this sense the formulation is holographic, i.e. the information about the interior of a region is encoded through the states on the boundary. These structures are

required to be coherent in the sense of topological quantum field theory [50]. This does not mean that the underlying structure is necessarily topological. For quantum mechanics and quantum field theory the relevant background structure is the metric. Only for quantum gravity the theory would be topological (more precisely differentiable) in the usual sense. This does not imply a lack of local degrees of freedom. Since the association of states with possibly time-like hypersurfaces is a quite radical step for quantum mechanics it is crucial to understand their physical meaning. This is particularly true for particle states in quantum field theory. Thus it makes sense even with the goal of quantum gravity in mind first to reformulate non-relativistic quantum mechanics and quantum field theory in the general boundary sense. It turns out that Feynman's path integral approach is particularly suitable to achieve this goal, and the general boundary formulation is a rather straightforward generalization of it. It is designed so as to produce general boundary theories of the type described above. It is holographic not only in the sense mentioned above but also in the sense that the underlying classical configuration space on the boundary should be chosen such that it uniquely encodes a solution of the equations of motions in the interior.

The word "holography" calls to mind the holography principle introduced by 't Hooft, and more generally, the works of Bousso, Susskind, Maldacena (there is a vast literature, some entry points can be [118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129]), so it is in order to clarify the difference between the way it is used in the context of the general boundary formulation and by the authors I just mentioned[1]. While I am no expert of their work, there are a few key points which can clarify this difference.

First it is important to point out that in both cases the word "holography" is called into play because the theory attempts to describe the physics in a bounded region (called bulk) through the information encoded in the boundary, that is, a region of smaller dimensions. In the classical works on the holographic principle [118, 119, 120, 121, 123, 124, 125, 126], the bulk is a three dimensional region of space, so that the boundary is a two-dimensional surface; Bousso [122] extended the principle to arbitrary space dimensions. While in the general boundary formulation, the bulk is a region of four-dimensional spacetime, and consequently the boundary is a three-dimensional region of spacetime. Generally the scope of the works on the holographic principle is to make use of it to produce entropy bounds, which at the moment are not considered in the general boundary formulation. The holographic principle is also present in Maldacena's AdS/CFT conjecture [127, 128, 129], which states that superstring theory in a curved ten-dimensional space, which is a five-dimensional anti-de Sitter space times a five-dimensional sphere, is equivalent to a conformal field theory in four dimensions with $N = 4$ supersymmetry. It is evident that this is a very specific setting, different from the one adopted in the general boundary formulation.

[1] I would like to thank Robert Oeckl for clarifications on this point.

In the following sections I will follow the analysis presented in [47, 48, 49] of a quantum mechanical measurement, explaining how such an analysis suggests the introduction of a general boundary formulation, and describe the general boundary formulation for quantum field theory and quantum gravity.

4.1.1 A quantum mechanical measurement procedure

Conventionally, the problem of quantum mechanical measurement is treated as something that can be considered in the context of a classical (and even non-special-relativistic) space-time. This is what the formalism of quantum mechanics is based on. It is then assumed that a quantization of space and time is merely a second step which can be performed within the formalism thus set up. But the application of principles of quantum mechanics itself to space-time suggests a modification (or rather generalization) of the standard formalism. Consider a very schematic quantum mechanics measurement procedure: a box which contains some quantum mechanical system which after a time Δt may or may not have changed its state[2]. The experimenter closes the box (that is, isolates the system from its environment) and after a time Δt checks the state of the system. Quantum mechanics allows to predict the probability $p(t)$ with which the system will change its state, but not the outcome of any single experiment, that is, it forbids to assume a definite classical evolution to take place inside the closed box. Generally, the measurement process involves a quantum domain and a classical domain. The system on which the measurement is performed (here the interior of the box) is part of the quantum domain while the observer is part of the classical domain (here the surroundings). In the quantum domain no definite classical evolution takes place and it does not make sense there to ascribe classical states to the system. The conventional mathematical description of the experiment is as follows. Associated with the system is a Hilbert space $\mathcal{H}$ of states. At the time $t_1$the observer prepares the system in a state $\psi \in \mathcal{H}$. Then the observer isolates the system (closes the lid) and lets the system evolve for a time Δt, which is described by a unitary operator $U(\Delta t)$ acting on $\mathcal{H}$. At time t_2 the observation is performed. That is, the observer checks whether the system is in a state $\eta \in \mathcal{H}^*$. The probability p that the system changed its state is the modulus square of the corresponding transition amplitude: $p(\Delta t) = |\langle\eta| U(\Delta t) |\psi\rangle|$.

So fat it has been implicitly assumed that space and time provide a fixed classical background structure. But taking seriously the principle that nothing classical is known about the interior of the box implies extending it to space and time also. Outside the box on the other hand time remains a classical entity as part of the classical domain of observations. To examine the implications of this it is necessary to pay more attention to how the notion of time enters into my measurement process. After closing the lid the observer continuously watches an external clock to check that time Δt has elapsed. But how can the system know about the time Δt elapsed on the external clock when no definite evolution of time inside the closed box can be

[2] An example of this is Schrödinger's cat's thought experiment.

assumed? The sensible answer seems to be that the system does stay in contact with its environment while the lid is closed. More precisely, it stays in contact with the space-time as classically experienced by the observer. The information about the space-time structure surrounding the box is a boundary condition to the experiment. It encodes in particular the elapsed time Δt on the clock. This boundary condition must be regarded an integral part of the quantum mechanical measurement process. Putting the clock inside the box would of course not help to avoid this conclusion, since it would not be possible for the observer to know that a time Δt has elapsed.

On the conceptual level the above argument can also be seen as stating something about the observer. This is what in [47, 48, 49] is called the "principle of the integrity of the observer". This means that the whole measurement process (including preparation and observation proper) pertains to one connected classical domain in which the observer describes reality. In the above thought experiment this connectedness is manifest in the clock and in the observer watching the clock while the box is closed. This means that it does not make sense for the observer to consider preparation and observation as disconnected interactions between classical and quantum domains. To the contrary, to relate the two it is essential that the observer has a classical existence in between (with a classical time duration Δt).

The consequences of the thought experiment can be formalized as follows. To simplify it is easier to assume that the classical and quantum domains can be strictly identified with the corresponding regions of space-time. In the present experiment the quantum domain is thus the world 4-volume of the box in the time interval $[t_1, t_2]$. The classical domain is everything outside. The ambient classical space-time is a boundary condition to the experiment in the following sense. According to quantum mechanics the interaction between observer and system (preparation and observation) should take place at the interface between the classical and quantum domains. Thus, the relevant spatio-temporal information should reside in the metric space-time field on the three- dimensional boundary that separates the two. This three-dimensional connected boundary B consists of three parts: the space-like boundary S_1 consisting of the inside of the box at the time t_1 of preparation, the time-like boundary T of the spatial boundary of the box while waiting for the time Δt to elapse and again the space-like boundary S_2 of the inside of the box at the time t_2 of observation. To what extent a metric on a connected surface such as B determines or over-determines a solution of the Einstein equations inside is a difficult initial value problem. Due to the similarity with the thick sandwich problem [130] one can assume that the intrinsic metric is sufficient. However, the exact validity of such an assumption is probably not a crucial ingredient for a quantum theory of general relativity.

4.1.2 General boundary formulation: definition

To calculate a transition amplitude, then, three pieces of information are needed. The initial state ψ on S_1, the final state η on S_2 and the intrinsic metric g on

the whole of $B = S_1 \cup T \cup S_2$. This suggests a mathematical description as follows. Associated with B is a state space $\mathcal{H}_B$ and (ψ, η, g) can be thought of as determining an element in this space[3]. The amplitude for such a state is given by a map $\rho : \mathcal{H}_B \to \mathbb{C}$. The associated probability (density) ρ is as usual the modulus square of the amplitude, that is

$$p = |\rho(\psi, \eta, g)|^2 \quad . \tag{4.1}$$

To recover the conventional mathematical description of the experiment the state space can be split into a tensor product corresponding to the boundary components $\mathcal{H}_B = \mathcal{H}_{S_1} \otimes \mathcal{H}_T \otimes \mathcal{H}_{S_2}$. Correspondingly, the metric living on the different components is labeled by g_1, g_T, g_2. Then one can recover $\mathcal{H} = (\mathcal{H}_{S_1}|g_1)$, that is, the state space $\mathcal{H}$ is the space of states in $\mathcal{H}_{S_1}$ which are partly fixed to g_1 (namely in their metric information), correspondingly $\mathcal{H}^* = (\mathcal{H}_{S_2}|g_2)$. $\mathcal{H}$ and $\mathcal{H}^*$ will be called reduced state spaces. The amplitude is then equal to

$$\rho(\psi, \eta, g) = \langle \eta_{g_2}| U(g_T) |\psi_{g_1}\rangle \quad . \tag{4.2}$$

The indices on the states indicate that they live in the reduced state spaces with fixed metric and the argument of U that it depends (apart from the state spaces $\mathcal{H}$ and $\mathcal{H}^*$) on the metric g_T. In particular g_T contains the information about the time duration Δt.

I will now give the definition of general boundary formulation. It consists of a suitably adapted notion of topological quantum field theory. Let M be a region of space-time, that is, a four-dimensional 2 manifold, and S its boundary hypersurface. For the moment I do not specify what background structure this entails[4]. The following properties hold.

T1 Associated with each such boundary S is a vector space $\mathcal{H}_S$ of states

T2 If S decomposes into disconnected components $S = S_1 \cup \cdots \cup S_n$ then the state space decomposes into a tensor product $\mathcal{H}_S = \mathcal{H}_{S_1} \otimes \cdots \otimes \mathcal{H}_{S_n}$.

T3 Besides considering boundaries as "in" one can also regard them as "out". For a given boundary S, changing its orientation, that is the side on which it bounds a region M, corresponds to replacing $\mathcal{H}_S$ with the dual space $\mathcal{H}_S^*$.

T4 Associated with M is a complex function $\rho_M : \mathcal{H}_S \to \mathbb{C}$ which associates an amplitude to a state. One may also dualize boundaries. This means that one may convert $\rho_M : \mathcal{H}_{S_1} \otimes \cdots \otimes \mathcal{H}_{S_n} \to \mathbb{C}$ to a function $\rho_M : \mathcal{H}_{S_1} \otimes \cdots \otimes \mathcal{H}_{S_k} \to \mathcal{H}^*_{S_{k+1}} \otimes \cdots \otimes \mathcal{H}^*_{S_n}$, replacing spaces with dual spaces. Mathematically both

[3] In a truly quantum description of space and time g should be imagined as a quantum state peaked at a classical metric rather than a classical metric itself.

[4] For quantum gravity the background structure coming with space-time regions and their boundaries is just a differentiable structure. One obtains essentially a proper topological quantum field theory. For quantum mechanics and quantum field theory the background structure is a fixed metric (usually that inherited from minkowskian space).

versions of M are equivalent, giving one determines the other (hence the same notation).

T5 A crucial property is the composition rule. Let M_1 and M_2 be two regions of space-time that share a common boundary S. Let M_1 also have a boundary S_1 and M_2 a boundary S_2. Consider $\rho_{M_1} : \mathcal{H}_{S_1} \to \mathcal{H}_S$ and $\rho_{M_2} : \mathcal{H}_{S_2} \to \mathcal{H}_S$. (The state spaces are chosen with respect to suitable orientations of the boundaries.) Then gluing gives $M = M_1 \cup M_2$ with boundaries S_1 and S_2. The composition rule demands the equality $\rho_M = \rho_{M_2} \circ \rho_{M_1}$.

The crucial point is that one allows more general boundaries than equal-time-slices in euclidean or minkowskian space, and here is where this formulation goes beyond the use that is usually made of topological quantum field theory in physical contexts. In particular, boundaries might have time-like components and one may glue along such boundaries. Rather than a composition of time-evolutions this would be a composition in space. Physically, this might for example correspond to the formation of a composite system out of separate systems. Since this is really a quantum composition in the same sense as the composition of time-evolutions is, the consistency of this operation is ensured.

An implication of allowing rather arbitrary boundaries is that the distinction between "in" and "out" states becomes arbitrary, in turn blurring the distinction between preparation and observation proper in the measurement. Even the "in" and "out" notions of topological quantum field theory become inadequate. More precisely, they become purely technical notions that can essentially be turned around at will (by dualization, see above). The physical notions of "in" (as preparation) or "out" (as observation) become necessarily disentangled from this technical one.

In fact, there is no need for this separation into physical "in" and "out". Indeed, this is manifest in quantum field theory in a remarkable feature of the LSZ reduction [117]. Consider the time ordered correlation function (in momentum space) of n fields, $\langle 0| T(\phi(p_1) \cdots \phi(p_n)) |0\rangle$. Its modulus square is a probability density in several ways. Given incoming particles with momenta $p_1, \ldots, p_k$ it is essentially the probability density for observing outgoing particles with momenta $p_{k+1}, \ldots, p_n$. How many particles are regarded as incoming (that is, which value is taken for k) is arbitrary. For each choice the right answer is given by the very same quantity. How the state is split up into prepared part and observed part is arbitrary. Note though that exchanging "in" and "out" states requires to exchange positive with negative energy. But this fits the associated orientation reversal in the time direction of the topological quantum field theory description. In the same sense the function ρ is to be regarded as giving an amplitude for a state. Whether a part of this state is to be considered as prepared or as observed does not alter the associated probability density. It is rather to be viewed as an ingredient of the experimental circumstances.

4.1.3 Application to quantum field theory and quantum gravity

Considering first a classical field theory will help clarify the meaning of the properties T1-T5 above. Consider the theory of a single scalar field with action $S[\phi]$. Now let K_S be the space of field configurations on a hypersurface S bounding a region M. The guiding principle is here, that the amount of boundary data encoded in K_S should be such that it essentially uniquely determines a classical solution inside M in a generic situation (e.g. M a 4-ball). The space of states $\mathcal{H}_S$ associated with S is the space of complex valued functions $C(K_S)$ on K_S. This amounts to adopting a state functional picture. The amplitude ρ_M for a state $\psi \in \mathcal{H}_S$ is given by

$$\rho_M(\psi) = \int_{K_S} \mathcal{D}\phi_0 \psi(\phi_0) \int_{\phi|_S=\phi_0} \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]} \quad . \tag{4.3}$$

The first integral is over field configurations ϕ_0 on S. The second integral is over all (not necessary classical) field configurations ϕ inside M that match the boundary data ϕ_0 on S. Note that adopting a state functional picture gives a prescription for T1 and ensures T2, since for $S = S_1 \cup S_2$ a disjoint union, $K_S = K_{S_1} \times K_{S_2}$ and hence $C(K_S) = C(K_{S_1}) \otimes C(K_{S_2})$. T4 is determined by (4.3). The dualization of boundaries corresponds simply to leaving the evaluation with a state on those boundaries open. Let M have boundaries S_1 and S_2 and consider states $\psi_1 \in \mathcal{H}_{S_1}$ and $\psi_2 \in \mathcal{H}_{S_2}$. Then $\rho_M(\psi_1)$ is an element of $\mathcal{H}^*_{S_2}$, that is a linear map $\mathcal{H}_{S_2} \to \mathbb{C}$ by mapping ψ_2 to

$$\int_{K_{S_1} \times K_{S_2}} \mathcal{D}\phi_1 \mathcal{D}\phi_2 \psi_1(\phi_1) \psi_2(\phi_2) \int_{\substack{\phi|_{S_1}=\phi_1 \\ \phi|_{S_2}=\phi_2}} \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]} \quad . \tag{4.4}$$

This also explains T3. The composition property T5 is also rather natural. Consider an integral over all field configurations in two regions with fields fixed on a common boundary and integrate also over the boundary values. Then this is the same as doing the unrestricted integral over field configurations in the union of the regions. This heuristic quantization procedure based on the path integral thus leads to general boundary type quantum theories. The topological quantum field theory-like axioms T1-T5 are automatically satisfied. It can be called holographic quantization since the information about the interior of a region is encoded through the states on the boundary. The following step would be to divide out symmetries either from the configuration space K_S or from the functions $C(K_s)$ on it to arrive at the physical state space $\mathcal{H}_S$. In quantum gravity an important step would be to divide out diffeomorphism symmetry.

I now move to quantum field theory. Space-time is minkowskian and coordinates will be denoted by $x = (\vec{x}, t)$. I start by considering regions R determined by time intervals $[t_1, t_2]$. In this case, the formalism proceeds as I described in Chapter 3; through the use of the propagation kernel it is possible to reconstruct all the information one may wish to extract from the field theory. Having once established

the formalism for the special time-slice boundaries the generalization to arbitrary boundaries is straightforward. States on time-slices can be pulled-back to any kind of boundary using the composition rule T5, as ensured by the form of (4.3). The resulting description is not only local but also natural in terms of typical experimental setups. Consider for example a scattering experiment in high energy physics. A typical detector has roughly the form of a sphere with the scattering happening inside (for example, a collision of incoming beams). The entries for particles and the individual detection devices are arranged on the surface. At some time t_1 the beam is switched on and at t_2 it is switched off. The space-time region M relevant for the experiment is the region inside the sphere times the time interval $[t_1, t_2]$. The particle inflow and detection happens on the boundary S of M . What seems unusual is that the parts of S that are really interesting and carry the particle states are its timelike components. On the spacelike components at t_1 and t_2 there are no particles (one can imagine the switching to happen smoothly). Concerning the interaction term in the lagrangian it is now natural to turn it on only inside M. Indeed, the particles detected on the boundary S should (as usual) be thought of as free. For calculational reasons it will usually be still advantageous to use particle states that are asymptotic (indeed, the difference has negligible effects on the resulting amplitudes, as I will show in the following section 4.2.1). However, there no longer seems to be a fundamental reason to do this. This becomes rather important for the construction of a non-perturbative theory of quantum gravity. There, asymptotic states in terms of a minkowskian space are not expected to be a useful fundamental concept. The advantage of a local description is thus crucial.

Both, the general boundary formulation of quantum theories as well as the holographic quantization prescription are mainly designed for a quantum theory of gravity. Then, the background structure for the topological quantum field theory-type axioms is just that of differentiable manifolds and their boundaries. Going down to three dimensions it is well known that pure quantum gravity is a topological quantum field theory [131], that is, it satisfies T1- T5. What is more, this topological quantum field theory is obtainable by following the quantization prescription sketched at the beginning of this section. Using connection variables the configuration space K_S associated with a boundary S of a region M is basically the space of flat spin connections on S. The path integral (4.3) is rigorously defined through a discretization of M as a spin foam model. The role of diffeomorphism invariance is the following here: if one thinks of the spin connection as specified by a connection 1-form $A_\mu(x)$ then K_S is the space of equivalence classes of such 1-forms under general gauge transformations. These general gauge transformations are now both the $SU(2)$ gauge transformations and diffeomorphisms. Also four-dimensional quantum gravity can be approached through the general boundary formulation. Indeed, the path integral approach to quantum gravity is well established [40, 41]. The crucial new ingredient is the admission of arbitrary (in particular connected) boundaries and their interpretation. A promising context for a non-perturbative realization of this appear to be spin foam models [37] in connection with a renormalization

procedure.

In the previous section 4.1.1 it has been shown how careful analysis of a quantum mechanical measurement procedure naturally leads to a description of quantum mechanics in terms of closed boundaries. Such a description can be called holographic in the sense mentioned above, that all relevant information is coded in the boundary of the spacetime region of interest. Significantly, the principle of the integrity of the observer implies the connectedness of the boundary at the interface between classical and quantum domain. One might object that ordinary quantum mechanics can very well deal with disconnected boundaries. But this is due rather to simplifications (especially due to the fixed space-time background) than to fundamental reasons. A typical system of interest has a finite extent. Outside this extent nothing relevant happens that requires really a quantum mechanical treatment. This argument also remains valid if the system is infinitely extended. The crucial point is that the observer remains excluded so that there is a boundary between him and the system. Pushing this further one might even consider the observer's world line as surrounded by a boundary outside of which the quantum mechanics happens. The connectedness of the boundary is rather significant for the interpretation of theories of quantum gravity and quantum cosmology. I will now compare this to the more traditional point of view that is often adopted in approaches to quantum gravity (for example in the Wheeler-DeWitt approach [132, 133], in euclidean quantum gravity [41] and also in loop quantum gravity [15, 19]). Consider two spacelike boundaries (say Cauchy surfaces) S_1 and S_2 which are closed and extend to infinity in the universe. One then considers transition amplitudes between quantum states of the metric on S_1 and on S_2. The question how a time duration Δt (along some path) between an event on S_1 and an event on S_2 is to be encoded is then answered as follows. Given a solution of Einstein's equations consider the two non-intersecting spatial hypersurfaces S_1 and S_2. Then, generically, it is conjectured that this solution can be reconstructed (up to diffeomorphism) given the intrinsic metrics on S_1 and S_2. This is the thick sandwich problem [133]. This implies that such intrinsic metrics contain the information about the time difference in the above sense. At least for quasi- classical states in a suitable sense it should be appropriate to talk about time durations t (with some uncertainty) between initial and final state. Nevertheless, this approach has the disadvantage that it cannot be directly related to a measurement process of the type considered above. What has been called the principle of the integrity of the observer is violated. To remedy this one would presumably have to fix some spatial region (where the observer lives) and its world-line to be classical. But this would essentially amount to introducing extra boundary components that connect S_1 and S_2, thus introducing a connected boundary through the back door. In the proposed approach the relevant boundary S is connected from the outset. There is no need to refer to temporal distances between boundaries. Temporal (or spatial) distances related to a measurement process can be evaluated on paths on the boundary using the intrinsic metric only.

Apart from an analysis of the measurement process there are other reasons to

look for a topological quantum field theory type description of quantum gravity using compact connected boundaries. An important reason is locality. Compact connected boundaries allow for an adaption of the mathematical description to the size of the system considered. There is no need a priori to include things (even empty space) at infinity. Of course, going to infinity might not change the mathematical description much or might even simplify it (like the usage of asymptotic states in quantum field theory). However, while this is certainly true in quantum mechanics and quantum field theory on minkowskian space, it is very unlikely to be true in a non-perturbative theory of quantum gravity. Also, the program of euclidean quantum gravity uses compact manifolds and connected boundaries. However, there the motivation is a mathematical one rather than a physical one. Indeed the interpretation of corresponding quantities in that program markedly differs from the interpretation that emerges in the present context [43].

There is a widespread belief that classical general relativity is not a topological theory and hence its quantization cannot be a topological quantum field theory. However, this could be due to a misuse of the word topological. In the first instance it refers to the classical theory having no local degrees of freedom, in the second to the fact that the background structures of the quantum theory are topological manifolds and their cobordisms (although differentiable would be more appropriate here). The second does not imply the first. Indeed, consider the classical limit. Then states on boundaries which are peaked at a classical metric determine up to diffeomorphisms essentially a unique classical solution of general relativity inside. There is no reason to think that a quantum theory cannot incorporate this, for example as the dominant contribution to a path integral. Indeed, this is for example a vital ingredient of the euclidean quantum gravity program. One should not be misled by the fact that many interesting topological quantum field theories that have been constructed can be viewed as quantizations of topological theories (see for example [134]). This seems rather related to the fact that the vector spaces associated with boundaries there are finite dimensional (which would not be expected for quantum gravity).

4.2 Propagation kernel defined on a finite region of spacetime

I now extend the definition of the propagation kernel to the case of a finite region of spacetime [58]. That is, I consider the case where the region $\mathcal{R}$ where the fields propagate, instead of being the infinite strip between two spatial hyperplanes at $t = 0$ and $t = T$, is an arbitrary finite region of spacetime. Let Σ be the boundary of $\mathcal{R}$, that is a closed, connected 3d surface with the topology (but in general not the geometry) of a 3-sphere. Let φ be a scalar field on Σ and consider the functional

$$W\left[\varphi, \Sigma\right] = \int_{\phi|_\Sigma = \varphi} D\phi e^{-S[\phi]} \quad . \tag{4.5}$$

The integral (4.5) is over all 4d fields on $\mathcal{R}$ that take the value φ on Σ. In the following I will specialize to the euclidean case. In the free theory the integral is a well defined Gaussian integral and can be evaluated. The classical equations of motion with boundary value φ on Σ form an elliptic system and in general have a solution $\phi_{cl}[\varphi]$, which can be obtained by integration from the Green function for the shape $\mathcal{R}$. A change of variable in the integral reduces it to a trivial Gaussian integration times $e^{-S^E_{\mathcal{R}}[\phi]}$. Here $e^{-S^E_{\mathcal{R}}[\phi]}$ is the field theoretical Hamilton function: the action of the bulk field determined by the boundary condition φ. This function satisfies a local Hamilton-Jacobi functional equation and solves the classical field theoretical dynamics [15, 135].

4.2.1 Relation with the case of two infinite spatial hyperplanes

Choose now Σ to be a cylinder Σ_{RT}, with radius R and height T, with the two bases on the surfaces $t = 0$ and $t = T$. Given two compact support functions φ_1 and φ_2, defined on $t = 0$ and $t = T$ respectively, it is always possible to choose R large enough for the two compact supports to be included in the bases of the cylinder. Then it can be expected that

$$W_E[\varphi_1, 0; \varphi_2, T] = \lim_{R\to\infty} W[\varphi_1, \varphi_2, \Sigma_{RT}] \quad , \tag{4.6}$$

because the euclidean Green function decays rapidly and the effect of having the side of the cylinder at finite distance goes rapidly to zero as R increases. In section 3.7 I have shown how scattering amplitudes can be computed from $W_E[\varphi_1, 0; \varphi_2, T]$. In turn, equation (4.6) indicates how $W_E[\varphi_1, 0; \varphi_2, T]$ can be obtained from $W_E[\varphi, \Sigma]$, where Σ is the boundary of a finite region. Therefore the knowledge of $W_E[\varphi, \Sigma]$ allows to compute physical scattering amplitudes. I expect that this should remain true in the perturbative expansion of an interacting field theory as well, where $\mathcal{R}$ includes the interaction region.

$W_E[\varphi, \Sigma]$ can be directly defined in the minkowskian regime as well. For a cylindrical box in minkowskian space, let $\varphi = (\varphi_{out}, \varphi_{in}, \varphi_{side})$ be the components of the field on the spacelike bases and timelike side. Consider the field theory defined in the box, with time dependent boundary conditions φ_{side}, and let $U[\varphi_{side}]$ be the evolution operator from $t = 0$ to $t = T$ generated by the (time dependent) hamiltonian of the theory. Then

$$W_M[\varphi, \Sigma] = \langle \varphi_{out}| U[\varphi_{side}] |\varphi_{in}\rangle \quad . \tag{4.7}$$

When φ_{side} is constant in time, this can be obtained by analytic continuation from the euclidean functional.

At first sight, the limits $T, R \to \infty$ seem to indicate that arbitrarily large surfaces Σ are needed to compute vacuum and scattering amplitudes. Notice however that the convergence of $W_E[\varphi_1, 0; \varphi_2, T]$ to the vacuum projector is dictated by (3.5): it is

exponential in the mass gap E_1, or the Compton frequency of the particle. Thus T at laboratory scales is largely sufficient to guarantee arbitrarily accurate convergence. In the euclidean, rotational symmetry suggests the same to hold for the $R \to \infty$ limit. Thus the limits can be replaced by fixing R and T at laboratory scales. Problems could arise for the analytical continuation, which might not commute with the limits, but these problems do not affect the determination of the vacuum state, where no analytical continuation is required.

The fact that it is possible to define the vacuum state, or particle states, locally seems to contradict the fact that the notions of vacuum and particle states are global. Some comments on this delicate point are therefore in order. The conventional notions of vacuum and particle states are global, but particle detectors are finitely extended. In fact, it is possible to distinguish two distinct notions of particle [51]. Fock particle states are "global", while states detected by a localized detector (eigenstates of local operators describing detection) are "local" particles states. Local particle states are very close to (in norm), but distinct from, the corresponding "global" particle states. On a background, it is possible to conveniently approximate the local particle state detected by the detectors, using global particle states, which are far easier to deal with. The global nature of the conventional definition of vacuum and particles is therefore an approximation adopted for convenience, it is not dictated by physical properties of particles detected in the laboratory.

By replacing the limits $R \to \infty$ and $T \to \infty$ with finite macroscopic R and T I miss the exact global vacuum or n-particle state, but I can nevertheless describe local experiments. The restriction of quantum field theory to a finite region of spacetime must describe completely experiments confined to this region and states detected by finitely extended particle detectors.

4.3 Application to quantum gravity

In quantum gravity, making the formulation described above concrete is a complex task. The problem that I consider here is only how to interpret a functional integral for quantum gravity defining a functional of the boundary states, assuming this is somehow given. Concrete definitions of $W[\varphi, \Sigma]$ are rather well developed in the context of the spinfoam formalism. Lorentzian and Riemannian version of the formalism have been studied, and some finiteness results have been proven to all orders in a perturbative expansion [136, 137]. Background independence implies immediately that the gravitational functional $W[\varphi, \Sigma]$ defined by an appropriate version of (4.5) is independent from any local variation of Σ. Fixing the topology of Σ, it is possible to write

$$W[\varphi, \Sigma] = W[\varphi] \quad . \tag{4.8}$$

At first sight, this seems the sort of independence from position and time, that renders background-independent quantum field theory difficult to interpret. For instance, consider the 2-point function $W(x; y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$. In quantum

field theory over a background, the independent variables x and y can be related to the spacetime location of particle detectors. In a background independent context, general covariance implies immediately that $W(x;y)$ is constant for $x \neq y$, and therefore it is not clear how the formalism can control the localization of the detectors (see for instance [46].) The independence of $W[\varphi]$ from Σ is analogous to the independence of $W(x;y)$ from x and y just mentioned. However, the property of Σ that codes the relative spacetime location of the detectors is the metric of Σ. In the gravitational case, the metric of Σ is not coded in the location of on a manifold: it is coded in the boundary value of the gravitational field on Σ. Therefore the relative location of the detectors, lost with Σ because of general covariance, comes back with φ, as this includes the boundary value of the gravitational field. Therefore, if one is given a functional integral for gravity, one can interpret it exactly as I did for the scalar field. The boundary value of the gravitational field plays the double role previously played by φ and Σ. In fact, this is precisely the core of the conceptual novelty of general relativity: there is no a priori distinction between localization measurements and measurements of dynamical variables. Once this observation is acquired, it is possible to claim that $W[\varphi]$ allows to calculate particle scattering amplitudes (see [138] for details).

$W[\varphi]$ determines a preferred state $|0_\Sigma\rangle$, defined by $\langle 0_\Sigma|\varphi\rangle = W[\varphi]$ in the kinematical state space $\mathcal{K}$ associated with the boundary. This is the covariant vacuum, and codes the dynamics. It satisfies a dynamical equation analogous to the equation

$$n^\mu(\vec{\tau})\frac{\delta}{\delta x^\mu(\vec{\tau})}W[\varphi,\Sigma] = H(\vec{\tau})\,W[\varphi,\Sigma] \quad , \tag{4.9}$$

which in the Chapter 5 I will prove for the propagation kernel $W[\varphi,\Sigma]$, but with the difference that in the case of $W[\varphi]$ the operator $H(\vec{\tau})$ is the hamiltonian constraint density operator. But since $W[\varphi]$ is independent from Σ by general covariance, the left hand side of (4.9) vanishes, leaving

$$H(\vec{\tau})\,W[\varphi,\Sigma] = 0 \tag{4.10}$$

which is the (lorentzian) Wheeler-DeWitt equation [41, 42].

4.3.1 Minkowski vacuum in quantum gravity

In section 3.6 I defined the quantum state $|0_M\rangle$ that describes the Minkowski vacuum. Such a state is not singled out by the dynamics alone in quantum gravity. Rather, it is singled out as the lowest eigenstate of an energy H_T which is the variable canonically conjugate to a nonlocal function T of the gravitational field defined as the proper time along a given worldline.

This situation has an analogy in the simple quantum system formed by a single a relativistic particle. In the Hilbert space of such a system there is no preferred vacuum state. But it is possible to choose a preferred Lorentz frame, and therefore

a preferred Lorentz time x^0. The conjugate variable to x^0 is the momentum p_0, and there is a (generalized) state of minimum p_0.

To find the Minkowski vacuum state, I can repeat the very same procedure used above. The only difference is that the bulk functional integral is not over the bulk matter fields, but also over the bulk metric. This difference has no bearing on the above formulas, which regard the boundary metric, which, in the two cases, is an independent variable.

As a first example, a boundary metric can be defined as follows. Consider a three-sphere formed by two "polar" "in" and "out" regions and one "equatorial" "side" region. Let the matter+gravity field on the three-sphere be split as

$$\varphi = (\varphi_{out}, \varphi_{in}, \varphi_{side}) \quad . \tag{4.11}$$

Fix the equatorial field φ_{side} to take the special value φ_{RT} defined as follows. Consider a cylindrical surface Σ_{RT} of radius R and height T in R^4, as defined previously. Let Σ_{in} (and Σ_{out}) be a (3d) disk located within the lower (and upper) basis of Σ_{RT}, and let Σ_{side} be the part of Σ_{RT} outside those disks, so that

$$\Sigma_{RT} = \Sigma_{in} \cup \Sigma_{out} \cup \Sigma_{side} \quad . \tag{4.12}$$

Let g_{RT} be the metric of Σ_{side} and let $\varphi_{RT} = (g_{RT}, 0)$ be the boundary field on Σ_{side} determined by the metric being g_{RT} and all other fields being zero. Given arbitrary values φ_{out} and φ_{in} of all the fields, included the metric, in the two disks, consider $W[(\varphi_{out}, \varphi_{in}, \varphi_{RT})]$. In writing the boundary field as composed by three parts as $\varphi = (\varphi_{out}, \varphi_{in}, \varphi_{side})$ I am in fact splitting the kinematical state space $\mathcal{K}$ as

$$\mathcal{K} = H_{out} \otimes H^*_{in} \otimes H_{side} \quad . \tag{4.13}$$

Fixing $\varphi_{side} = \varphi_{RT}$ amounts to contracting the covariant vacuum state $|0_\Sigma\rangle$ in $\mathcal{K}$ with the bra state $\langle\varphi_{RT}|$ in H_{side}. For large enough R and T, one can expect the resulting state in $H_{out} \otimes H^*_{in}$ to reduce to the Minkowski vacuum. That is

$$\lim_{R,T\to\infty} \langle\varphi_{RT}|0_\Sigma\rangle = |0_M\rangle \otimes \langle 0_M| \quad . \tag{4.14}$$

Therefore for a generic "in" configuration, and up to normalization,

$$\Psi_M[\varphi] = \lim_{R,T\to\infty} W[(\varphi_{out}, \varphi_{in}, \varphi_{RT})] \tag{4.15}$$

gives the vacuum functional for large R and T. In the rest of this chapter I shall use a simpler geometry for the boundary.

One may hope that the convergence in R and T is fast. These formulas allow to extract the Minkowski vacuum state from a euclidean gravitational functional integral. n-particle scattering states can then be obtained by generalizations of the space formalism, and, if it is well defined, by analytic continuation in the single variable T. This is precisely the case of time independent φ_{side}, where analytical continuation may be well defined.

4.3.2 Spinnetworks and spinfoams

The argument of W is not a classical field: it is an element of the eigenbasis of the field operator. In the gravitational case, (functions of) the gravitational field operator can be diagonalized, but eigenvalues are not continuous fields. In loop quantum gravity, eigenstates of the metric are spin network states $|s\rangle$. Therefore the quantum gravitational W must be a function of spin network states $W[s]$ on Σ, and not of continuous gravitational fields on Σ. In fact, this is precisely what a spin foam model provides.

A spinfoam sum where the degrees of freedom are not cut off by the choice of a fixed triangulation is defined by the Feynman expansion of the quantum field theory over a group, studied in [139, 140, 141, 142]. I will recall here the basic equations of the formulations, referring to [139, 140, 141, 142, 15] for motivations and details. Let $\phi(g_1, \dots, g_4)$ be a field on $[SO(4)]^4$, satisfying

$$\phi(g_1, g_2, g_3, g_4) = \phi(g_1 g, g_2 g, g_3 g, g_4 g) \tag{4.16}$$

for all $g \in SO(4)$. Consider the action

$$S[\phi] = \frac{1}{2}\int \phi^2 + \frac{\lambda}{5!}\int (P_H \phi)^5 \quad . \tag{4.17}$$

Here P_H is defined by

$$P_H \phi(g_1, g_2, g_3, g_4) = \tag{4.18}$$

$$= \int_{H^4} dh_1 \cdots dh_4 \phi(g_1 h_1, g_2 h_2, g_3 h_3, g_4 h_4) \quad , \tag{4.19}$$

where H is a fixed $SO(3)$ subgroup of $SO(4)$, and $\int \phi^5$ is a short hand notation for

$$\int \phi^5 = \int \prod_{i=1}^{10} dg_i \phi(g_1, g_2, g_3, g_4)\, \phi(g_4, g_5, g_6, g_7) \cdot \tag{4.20}$$

$$\cdot \phi(g_7, g_3, g_8, g_9)\, \phi(g_9, g_6, g_2, g_{10})\, \phi(g_{10}, g_8, g_5, g_1) \quad . \tag{4.21}$$

The Feynman expansion of this theory is a sum over spinfoams and can be interpreted as a well-defined version of the Misner-Hawking sum over geometries. Transition amplitudes between quantum states of space can be computed as expectation values of $SO(4)$ invariant operators in the group field theory. In particular, the boundary amplitude of a 4-valent spin network s can be computed as

$$W[s] = \int D\phi f_s[\phi]\, e^{-S[\phi]} \quad . \tag{4.22}$$

The spinfoam polynomial is defined as

$$f_s[\phi] = \prod_n \int dg_{n_1} \cdots dg_{n_4} R^{(j_{n_1})\beta_{n_1}}_{\alpha_{n_1}}(g_{n_1}) \cdots R^{(j_{n_4})\beta_{n_4}}_{\alpha_{n_4}}(g_{n_4})\, v^{i_n}_{\beta_{n_1}\cdots\beta_{n_4}} \prod_l \delta^{l_1 l_2} \quad , \tag{4.23}$$

where $n_1, \ldots, n_4$ indicate four links adjacent to the node n, and $n_i = l_1$ (or $n_i = l_2$) if the i-th link of the node n is the outgoing (or ingoing) link l.

I can now implement equation (4.15) in this theory. Instead of the cylindrical boundary consider above, I will choose a simpler geometry. Let the spin network s' be composed by two parts connected to each other, $s' = s \# s_T$. Let s be arbitrary and s_T is to be a weave state [101] for the three-metric g_T. Take a 3-sphere of radius T in R^4. Remove a spherical 3-ball of unit radius. g_T is the three-metric of the three-dimensional surface (with boundary) formed by the sphere with removed ball. I recall that a weave state for a metric g is an eigenstate of (functions of the smeared) metric operator, whose eigenvalues approximate (functions of the smeared) g at distances large compared to the Planck length.

The quantity

$$\Psi_M [s] = \langle s | 0_M \rangle = \lim_{T \to \infty} \int D\phi f_{s \# s_t} [\phi] \, e^{-S[\phi]} \tag{4.24}$$

is then a tentative ansatz for the quantum state describing the Minkowski vacuum in a ball of unit radius. This quantity can be computed explicitly [139, 140, 141, 142] and may be finite at all orders in λ [136, 137].

Chapter 5

Generalized Tomonaga-Schwinger equation

In this Chapter I derive the equation of evolution for the propagation kernel $W[\varphi, \Sigma]$ defined on a closed surface Σ with boundary conditions φ on Σ, defined in equation (4.5) which I copy here for convenience,

$$W[\varphi, \Sigma] = \int_{\phi|_\Sigma = \varphi} D\phi e^{-S[\phi]} \quad . \tag{5.1}$$

This evolution equation will be a generalization of an equation studied by Tomonaga and Schwinger [52, 53]. To derive the equation I will be using a Green function technique, similar to the one which I used in the derivation of the propagation kernel in the euclidean case (see Subsection 3.2.2).

There have been worries [143] that such an equation would not be viable, since it would give rise to a nonunitary evolution. However, I wish to point out that the study [143] was carried out in a slightly different setup. Firstly, the authors considered the evolution from a flat initial surface to a generic final surface, which is clearly a different case from the present closed surface Σ. Secondly, the authors considered a compactified model of space. And finally, should even their worries apply to this case as well, I think it possible that nonunitarity be circumvented, and that a time evolution could be conceived even in that case.

The results described in this Chapter are reported in the paper [59].

5.1 Introduction

It was argued in [58] that $W[\varphi, \Sigma]$ should satisfy a local functional equation governing the variation of $W[\varphi, \Sigma]$ under arbitrary local deformations of Σ, namely an equation of the form

$$\frac{\delta W[\varphi, \Sigma]}{\delta \Sigma(s)} = H\left(\varphi(s), \nabla\phi(s), \frac{\delta}{\delta\varphi(s)}\right) \; W[\varphi, \Sigma] \quad . \tag{5.2}$$

Here s is a coordinate on Σ, and the precise meaning of the other symbols will be defined below. When Σ is formed by two parallel planes, (5.1) becomes the field propagator, and its variation under a parallel displacement of one of the planes is governed by the functional Schrödinger equation. The possibility of extending this equation to variations of arbitrary *spacelike* 3d surfaces was explored by Tomonaga and Schwinger already in the late forties [52, 53]. These authors derived a well-known local generalization of the functional Schrödinger equation, called the Tomonaga–Schwinger equation. The functional equation (5.2) generalizes the Tomonaga–Schwinger equation, since it holds for arbitrary boundary surfaces. Indeed, if the general philosophy of the boundary approach is correct, the distinction between the initial and final fields on spacelike portions of Σ on the one hand, and the boundary values of the field on the timelike portions of Σ, should become of secondary relevance.

Equation (5.2) has been derived in [144] on the basis of a lattice regularization of the functional integral (5.1) defining $W[\varphi, \Sigma]$, and under certain hypotheses on the existence of the continuum limit. Here, working in the context of a free euclidean theory, I show that this equation can be derived from the functional integral definition of $W[\varphi, \Sigma]$ directly in the continuum, using a formula by Hadamard which expresses the variation of a Green function under variation of the boundary [145]. Although incomplete, I think that this is a relevant step towards establishing the general viability of an equation of the form (5.2).

Furthermore, as a preliminary step in the derivation of (5.2), I derive its classical limit, which is a generalized Hamilton–Jacobi functional equation of the form

$$\frac{\delta S[\varphi, \Sigma]}{\delta \Sigma(s)} = H\left(\varphi(s), \nabla\phi(s), \frac{\delta S[\varphi, \Sigma]}{\delta \varphi(s)}\right) \tag{5.3}$$

satisfied by the classical Hamilton function $S[\varphi, \Sigma]$. This equation is a generalization of equation (3.52) previously mentioned in Chapter 3. The Hamilton function $S[\varphi, \Sigma]$ is the value of the action computed on the solution of the equations of motion that takes the value φ on Σ (see [15].) The generalized Hamilton–Jacobi equation (5.3) is extensively discussed in [15].

5.2 Definitions

5.2.1 Surface and surface derivative

Consider a finite region $\mathcal{R}$ in the euclidean 4d space R^4. I use cartesian coordinates $x, y, z \ldots$ on R^4, where $x = (x^\mu)$, $\mu = 1, 2, 3, 4$. Let $\Sigma = \partial\mathcal{R}_\Sigma$ be a compact 3d surface that bounds a finite region $\mathcal{R}_\Sigma$. I denote $s, t, u \ldots$ coordinates on Σ, where $s = (s^a)$, $a = 1, 2, 3$. Then Σ is given by the embedding

$$\Sigma : s^a \longmapsto x^\mu(s) \quad . \tag{5.4}$$

The euclidean 4d metric, which raises and lowers the μ indices, induces the 3d metric

$$q_{ab}(s) = \frac{\partial x^\mu(s)}{\partial s^a} \frac{\partial x_\mu(s)}{\partial s^b} \tag{5.5}$$

on Σ, which can be used to raise and lower the three dimensional a indices. The surface gradient ∇ is defined as

$$\nabla^a = q^{ab} \frac{\partial}{\partial s^b} \quad . \tag{5.6}$$

The normal one-form to the surface is

$$\tilde{n}_\mu(s) = \epsilon_{\mu\nu\rho\sigma} \frac{\partial x^\nu(s)}{\partial s^1} \frac{\partial x^\rho(s)}{\partial s^2} \frac{\partial x^\sigma(s)}{\partial s^3} \quad . \tag{5.7}$$

I orient the coordinate system s so that $\tilde{n}_\mu$ is outward directed. Its norm is easily seen to be equal to the determinant of the induced metric

$$\tilde{n}_\mu \tilde{n}^\mu = \det q \quad . \tag{5.8}$$

In the following, I will use the normalized normal

$$n_\mu \equiv (\det q)^{-\frac{1}{2}} \ \tilde{n}_\mu \tag{5.9}$$

and the induced volume element on Σ

$$d\Sigma(s) \equiv (\det q(s))^{\frac{1}{2}} \ d^3 s \quad . \tag{5.10}$$

(I shall simply write $d\Sigma$ when the integration variable is clear from the context.) Notice that the combination $d\Sigma \ n_\mu$ does not depend on the metric, since

$$d\Sigma(s) \ n_\mu(s) = d^3 s \ \tilde{n}_\mu(s) \quad . \tag{5.11}$$

Given a functional $F[\Sigma]$ that depends on the surface, I define the functional derivative with respect to the surface as the normal projection of the functional derivative with respect to the embedding (5.4) that defines the surface

$$\frac{\delta}{\delta\Sigma(s)} \equiv n^\mu(s) \frac{\delta}{\delta x^\mu(s)} \quad . \tag{5.12}$$

Here the functional derivative on the right hand side is defined in terms of the volume element $d\Sigma$. That is [145], $\delta F\,[\Sigma]/\delta\Sigma(s)$ is defined as the distribution that satisfies

$$\int d\Sigma(s) \ N(s) \ \frac{\delta F\,[\Sigma]}{\delta\Sigma(s)} = \int d\Sigma \ N(s) \ n^\mu(s) \ \frac{\delta F\,[\Sigma]}{\delta x^\mu(s)} = \lim_{\varepsilon \to 0} \frac{F\,[\Sigma_{\varepsilon N}] - F\,[\Sigma]}{\varepsilon} \quad , \tag{5.13}$$

where $N(s)$ is the field describing the amount of displacement in the normal direction, and $\Sigma_{\varepsilon N}$ is the surface defined by the embedding

$$\Sigma_{\varepsilon N} : s^a \longmapsto x^\mu(s) + \varepsilon N(s) n^\mu(s) \quad . \tag{5.14}$$

Geometrically, this derivative gives the variation of the functional under an infinitesimal displacement of the surface in the normal direction.

The surface derivative takes a simple form for certain simple functionals. Consider a scalar field $f(x)$ on R^4. Let $F_f[\Sigma]$ be the integral of f in the region $\mathcal{R}_\Sigma$ bounded by Σ; that is

$$F_f[\Sigma] = \int_{\mathcal{R}} d^4x f(x) \quad . \tag{5.15}$$

Then easily

$$\frac{\delta F_f[\Sigma]}{\delta \Sigma(s)} = f(x(s)) \quad . \tag{5.16}$$

That is, the variation of the bulk integral under a normal variation of the surface is the integrand in the variation point. Similarly, consider a vector field $v^\mu(x)$ on R^4. Let $F_v[\Sigma]$ be the flux of $v^\mu(x)$ across Σ; that is

$$F_v[\Sigma] = \int_\Sigma d\Sigma(s)\ n_\mu(s) v^\mu(s) \quad . \tag{5.17}$$

Then Stokes theorem gives easily

$$\frac{\delta F_v[\Sigma]}{\delta \Sigma(s)} = \partial_\mu v^\mu(x(s)) \quad . \tag{5.18}$$

That is, the variation of the flux by a normal variation of the surface is the divergence of the vector field.

5.2.2 Field theory and Tomonaga-Schwinger equation

I consider a free euclidean scalar field $\phi(x)$, defined in $\mathcal{R}$, with assigned boundary conditions $\varphi(s)$ on Σ. That is

$$\phi(x(s)) = \varphi(s) \quad . \tag{5.19}$$

The dynamics of the field is governed by the lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 \quad , \tag{5.20}$$

and the equation of motion is

$$\left(-\Box_x + m^2\right)\phi\left(x\right) = 0,\ x \in V \tag{5.21}$$

with the boundary conditions (5.19) on Σ. Here $\Box_x = \partial/\partial x_\mu \; \partial/\partial x^\mu$. By choosing one of the coordinates x^μ, say x^4 as a "time" coordinate, it is possible to derive the hamiltonian density

$$H(\phi, \nabla\phi, \Pi) = \frac{1}{2}\Pi^2 - \frac{1}{2}\left(\nabla\phi\right)^2 - \frac{1}{2}m^2\phi^2 \quad ; \tag{5.22}$$

where Π is the canonical momentum associated to the field ϕ and ∇ is the gradient on the $x^4 = constant$ surface.

For this theory, the integral in (5.1) is a well defined gaussian functional integral. Therefore $W[\varphi, \Sigma]$ is well defined. I show in this paper that $W\left[\varphi, \Sigma\right]$ satisfies the generalized Tomonaga-Schwinger equation

$$\begin{aligned} \frac{\delta W[\varphi,\Sigma]}{\delta\Sigma(s)} &= H\left(\varphi(s), \nabla\varphi(s), -\frac{\delta}{\delta\varphi(s)}\right) \; W[\varphi,\Sigma] \\ &= \left(\frac{1}{2}\left(-\frac{\delta}{\delta\varphi(s)}\right)^2 - \frac{1}{2}\left(\nabla\varphi(s)\right)^2 - \frac{1}{2}m^2\varphi^2(s)\right) W[\varphi,\Sigma] \quad . \end{aligned} \tag{5.23}$$

5.3 Green functions

The tool I use to prove (5.23) is a Green-function technique, analogous to the strategy sketched in subsection 3.2.2. Given Σ, let the Green function $G_\Sigma(x, y)$ be the solution of the inhomogeneous equation

$$\left(-\Box_x + m^2\right) G_\Sigma\left(x, y\right) = \delta^{(4)}(x - y) \tag{5.24}$$

in the region $\mathcal{R}$, satisfying the boundary condition

$$G_\Sigma\left(x(s), y\right) = 0 \quad , \tag{5.25}$$

namely that vanishes when x is on Σ. I introduce the following useful notation:

$$\partial_n^s G_\Sigma(s, y) \equiv n^\mu(s) \; \frac{\partial G_\Sigma(x, y)}{\partial x^\mu}\bigg|_{x=x(s)} \tag{5.26}$$

and similarly for $\partial_n^s G_\Sigma(y, s)$, $\partial_n^s \partial_n^t G_\Sigma(s, t)$, and so on. The solution of (5.21), with the boundary condition (5.19), namely with the boundary value φ on Σ, is then given by

$$\phi_{\varphi,\Sigma}\left(y\right) = -\int_\Sigma d\Sigma(s) \; \varphi(s) \; \partial_n^s G_\Sigma(s, y) \quad . \tag{5.27}$$

To see that this is indeed the solution, one can write the equality

$$\phi\left(y\right) = \int d^4x \left(\phi\left(x\right)\left(-\Box_x + m^2\right) G_\Sigma\left(x, y\right) - G_\Sigma\left(x, y\right)\left(-\Box_x + m^2\right)\phi\left(x\right)\right) \quad , \tag{5.28}$$

which is satisfied because of (5.24) and (5.21). The right hand side can easily be written as a surface integral, which gives (5.27) using the boundary conditions (5.19) and (5.25).

The propagator $W[\varphi, \Sigma]$ can be written explicitly in terms of the Green function. In fact, it is possible to solve the gaussian integral (5.1) using the fact that the classical solution (5.27) is an extremal value of the exponent in (5.1). The integral is then a gaussian integral over the fluctuations. The action on the classical solution with given boundary conditions is called the Hamilton function [146]:

$$S[\varphi, \Sigma] \equiv S[\phi_{\varphi,\Sigma}] \quad = \quad \frac{1}{2}\int_{V_\Sigma} d^4x \left[\left(\partial_\mu \phi_{\varphi,\Sigma}\right)^2 + m^2 {\phi_{\varphi,\Sigma}}^2 \right] \quad . \tag{5.29}$$

Inserting (5.27), it is possible to write this as

$$S[\varphi, \Sigma] = -\frac{1}{2}\int_\Sigma d\Sigma(s) d\Sigma(t)\ \varphi(s)\ \varphi(t)\ \partial_n^s \partial_n^t G_\Sigma(s,t) \quad . \tag{5.30}$$

It is then easy to perform the gaussian integration in (5.1), which gives

$$\begin{aligned} W[\varphi, \Sigma] &= \frac{\exp\left(-S\left[\phi_{\varphi,\Sigma}\right]\right)}{\sqrt{\det\left(-\Box + m^2\right)}} = \\ &= \sqrt{\det G_\Sigma\left(x,y\right)}\ \exp\left(\frac{1}{2}\int_\Sigma d\Sigma(s) d\Sigma(t)\ \varphi(s)\ \varphi(t)\ \partial_n^s \partial_n^t G_\Sigma(s,t)\right) \quad . \end{aligned} \tag{5.31}$$

In the following, it is necessary to have an expression for the variations of Green function G_Σ with respect to a displacement of the surface Σ. This can be done using Hadamard's formula [145]

$$\frac{\delta G_\Sigma\left(x,y\right)}{\delta\Sigma(s)} = \partial_n^s G_\Sigma(s,x)\ \partial_n^s G_\Sigma(s,y) \quad . \tag{5.32}$$

I sketch a proof of this formula in the Appendix A.

5.4 Hamilton–Jacobi equation

In this section I prove that the Hamilton function (5.30) satisfies the generalized Hamilton–Jacobi equation (5.3). From (5.22), this reads

$$\frac{\delta S[\varphi,\Sigma]}{\delta\Sigma(s)} + \frac{1}{2}\left(\frac{\delta S[\varphi,\Sigma]}{\varphi(s)}\right)^2 - \frac{1}{2}m^2\varphi^2(s) - \frac{1}{2}\left(\nabla\varphi(s)\right)^2 = 0 \quad . \tag{5.33}$$

To calculate the first term of this expression, I use the form (5.29) of the Hamilton function

$$\frac{\delta S[\varphi,\Sigma]}{\delta\Sigma(s)} = \frac{\delta}{\delta\Sigma(s)}\ \ \frac{1}{2}\int_{\mathcal{R}_\Sigma} d^4x \left[\left(\partial_\mu \phi_{\varphi,\Sigma}\right)^2 + m^2 {\phi_{\varphi,\Sigma}}^2 \right] \quad . \tag{5.34}$$

I use (5.16) for the variation of the integration region, integrate by parts, use the equations of motion, to obtain

$$\frac{\delta S[\varphi,\Sigma]}{\delta\Sigma(s)} = \frac{1}{2}\left(\left(\partial_\mu\phi_{\varphi,\Sigma}(x(s))\right)^2 + m^2\phi_{\varphi,\Sigma}(x(s))^2\right) + \int_{\mathcal{R}_\Sigma} d^4x\ \partial_\mu\left(\partial_\mu\phi_{\varphi,\Sigma}\frac{\delta\phi_{\varphi,\Sigma}}{\delta\Sigma(s)}\right) \quad . \tag{5.35}$$

The second term can be written as a surface integral. Using (5.27), this becomes

$$\begin{aligned}\frac{\delta S[\varphi,\Sigma]}{\delta\Sigma(s)} =& \frac{1}{2}\left(\left(\partial_\mu\phi_{\varphi,\Sigma}(x(s))\right)^2 + m^2\phi_{\varphi,\Sigma}(x(s))^2\right) + \\ &- \int_\Sigma d\Sigma(t)d\Sigma(t)\ \varphi(t)\ \partial_n^t\partial_n^u G_\Sigma(t,u)\frac{\delta\phi_{\varphi,\Sigma}(x(u))}{\delta\Sigma(s)} \quad .\end{aligned} \tag{5.36}$$

To compute $\delta\phi_{\varphi,\Sigma}(x)/\delta\Sigma(s)$ observe that by defining the field $\varphi(x)$ in the neighborhood of the surface by $\varphi(x(s)) = \varphi(s)$ and $\partial_n\varphi(x(s)) = 0$, the right hand side of (5.27) is the flux of the vector field

$$v^\mu(x) = -\varphi(x)\ \frac{\partial}{\partial x^\mu}G_\Sigma(x,y) \quad , \tag{5.37}$$

Therefore it is possible to use (5.18) to obtain

$$\frac{\delta\phi_{\varphi,\Sigma}(x)}{\delta\Sigma(s)} = -\frac{\partial}{\partial y_\mu}\left(\varphi(y(s))\frac{\partial}{\partial y^\mu}G_\Sigma(y(s),x)\right) + \tag{5.38}$$

$$- \int_\Sigma d\Sigma(t)\ \varphi(t)\ n^\mu(t)\ \frac{\partial}{\partial y_\mu}\frac{\delta G_\Sigma(y(t),x)}{\delta\Sigma(s)} \quad . \tag{5.39}$$

In the first term, I separate the normal and tangential component of the sum over μ and recall that $\varphi(x)$ is constant in the normal direction, and in the second term I use the Hadamard formula (5.32). This gives

$$\begin{aligned}\frac{\delta\phi_{\varphi,\Sigma}(x)}{\delta\Sigma(s)} =& -\nabla_a\varphi(s)\nabla_a G_\Sigma(y(s),x) - \varphi(s)\Box_y G_\Sigma(y(s),x) + \\ &- \int_\Sigma d\Sigma(t)\ \varphi(t)\ n^\mu(t)\ \partial_n^s G_\Sigma(s,x)\ \partial_\mu^t\partial_n^s G_\Sigma(s,t) \quad .\end{aligned} \tag{5.40}$$

But the tangential derivative of the Green function on the boundary surface vanishes, since the Green function itself vanishes. So does its Dalambertian, since $(-\Box_z + m^2)G_\Sigma(z\in\Sigma, y) = 0$). Therefore the first two terms vanish, and using (5.27) again, the result is

$$\frac{\delta\phi_{\varphi,\Sigma}(x)}{\delta\Sigma(s)} = \partial_n^s G_\Sigma(s,x)\,\partial_n^s\phi_{\varphi,\Sigma}(s) \quad , \tag{5.41}$$

where I have used the notation (5.26) also for $\phi_{\varphi,\Sigma}$. After inserting this in (5.36), and using (A.2) the result is

$$\begin{aligned}\frac{\delta S[\varphi,\Sigma]}{\delta\Sigma(s)} =& \frac{1}{2}\left(\left(\partial_\mu\phi_{\varphi,\Sigma}(x(s))\right)^2 + m^2\phi_{\varphi,\Sigma}(x(s))^2\right) + \\ &+ \int_\Sigma d\Sigma(t)\ \varphi(t)\ \partial_n^s\phi(s)\ \partial_n^s\partial_n^t G_\Sigma(t,s) \quad .\end{aligned} \tag{5.42}$$

The derivative $\delta S[\varphi,\Sigma]/\delta\varphi\,(z)$ can be easily obtained from the definition (5.30). The Hamilton-Jacobi equation (5.33) finally reads

$$\begin{aligned}&\int_\Sigma d\Sigma(t)\ \partial_n^s\phi(z)\ \varphi(t)\ \partial_n^s\partial_n^t G_\Sigma(t,s) + \frac{1}{2}\left(\partial_n^s\phi(s)\right)^2 + \\ &+ \frac{1}{2}\int_\Sigma d\Sigma(t)d\Sigma(u)\ \varphi(t)\ \varphi(u)\ \partial_n^t\partial_n^s G_\Sigma(s,t)\ \partial_n^u\partial_n^s G_\Sigma(s,u) = 0 \quad .\end{aligned} \tag{5.43}$$

Calculating $\partial_n\phi(s)$ from (5.27) it is easy to show that this is an identity.

5.5 Generalized Tomonaga-Schwinger equation

It is possible to write the propagator (5.31) in the form

$$W[\varphi,\Sigma] = \sqrt{\det G_\Sigma}\ \exp\left(-S[\varphi,\Sigma]]\right) = \exp\left(\frac{1}{2}tr\ln G_\Sigma - S[\varphi,\Sigma]\right) \quad . \tag{5.44}$$

The variation with respect to the surface (dropping the arguments from the notation) is

$$\frac{\delta W}{\delta\Sigma(s)} = W\frac{\delta}{\delta\Sigma(s)}\left(\frac{1}{2}tr\ln G_\Sigma - S\right) \quad ; \tag{5.45}$$

so that the generalized Tomonaga-Schwinger equation (5.23) becomes

$$\frac{1}{2}\frac{\delta(tr\ln G_\Sigma)}{\delta\Sigma(s)} - \frac{\delta S}{\delta\Sigma(s)} = \tag{5.46}$$

$$= -\frac{1}{2}\frac{\delta^2 S}{\delta\varphi(s)\delta\varphi(s)} + \frac{1}{2}\frac{\delta S}{\delta\varphi(s)}\frac{\delta S}{\delta\varphi(s)} - \frac{1}{2}(\nabla\phi(s))^2 - \frac{1}{2}m^2\phi^2(s) \quad . \tag{5.47}$$

Using the Hamilton-Jacobi equation (5.33) derived in the last section, this equation reduces to

$$\frac{\delta(tr\ln G_\Sigma)}{\delta\Sigma(s)} = -\frac{\delta^2 S}{\delta\varphi(s)\delta\varphi(s)} \quad . \tag{5.48}$$

The left hand side of this equation can be calculated again using Hadamard's formula (5.32) and (5.24)

$$\frac{\delta tr\ln G_\Sigma}{\delta\Sigma(s)} = tr\left[\left(-\Box_x + m^2\right)\frac{\delta G_\Sigma\,(x,y)}{\delta\Sigma(s)}\right] = \partial_n^t\partial_n^u G_\Sigma\,(z_1,z_2)\big|_{t=u=s} \quad , \tag{5.49}$$

while the right hand side can be directly obtained from (5.30), giving

$$\frac{1}{2}\frac{\delta^2}{\delta\varphi(s)\delta\varphi(s)}\int_\Sigma d\Sigma(s)d\Sigma(t)\;\varphi(s)\;\varphi(t)\;\partial_n^s\partial_n^t G_\Sigma(s,t) = \partial_n^t\partial_n^u G_\Sigma\left(z_1,z_2\right)\big|_{t=u=s} \quad , \tag{5.50}$$

showing that the equation is satisfied.

Chapter 6

Application to a toy model: dynamics of a tetrahedron

In this chapter I apply some of the techniques and formulas derived in Chapter 3 to a simple toy model, in order to understand how physical information can be extracted from a background independent quantum system. For example, I will be using the strategy outlined in section 3.6 to obtain the vacuum state, of which again two different definitions are given. There's however a slight difference with the limit used to obtain the vacuum state in section 3.6; indeed, there the limit was taken in an euclidian setting but in imaginary time; here the limit will be taken in real time.

I use an extremely simple system that models a finite region of 3d euclidean quantum spacetime with a single equilateral tetrahedron. I will show that the physical information can be expressed as a boundary amplitude, and illustrate how the notions of evolution in a boundary proper-time and vacuum can be extracted from the background independent dynamics.

This work has been done in collaboration with Daniele Colosi, Winston Fairbairn, Leonardo Modesto, Karim Noui and Carlo Rovelli [60]. My contribution is limited to the classical model, that is, to sections 6.1, 6.2, 6.3.

6.1 Elementary geometry of an equilateral tetrahedron

Consider a tetrahedron immersed in three-dimensional euclidean space. Let a be the length of one of the edges (I will call it the top edge) and b the length of the opposite (bottom) edge, namely the edge disjoint from the top edge. Assume that the other four (side) edges have equal length c. See Figure 6.1. I call such a tetrahedron "equilateral". I call θ_a, θ_b, θ_c the (respectively bottom, top and side) dihedral angles

Figure 6.1: The equilateral tetrahedron

at the edges with length a, b, c. Elementary geometry gives

$$\begin{aligned} \sin\frac{\theta_a}{2} &= \frac{b}{\sqrt{4c^2-a^2}} \\ \sin\frac{\theta_b}{2} &= \frac{a}{\sqrt{4c^2-b^2}} \\ \cos\theta_c &= \frac{ab}{\sqrt{(4c^2-a^2)\,(4c^2-b^2)}} \quad , \end{aligned} \tag{6.1}$$

where the last equation can be easily obtained from the scalar product of the normals to two adjacent triangles, by working in the orthonormal basis determined by the top and bottom edges, and the tetrahedron axis. It follows from (6.1) that

$$\cos\theta_c = \sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2} \quad . \tag{6.2}$$

For later purpose, I consider also the case in which $c \gg a, b$. In this case, to the first relevant order

$$\begin{aligned} \theta_a &= \frac{b}{c} \\ \theta_b &= \frac{a}{c} \\ \theta_c &= \frac{\pi}{2} - \frac{ab}{4c^2} = \frac{\pi}{2} - \frac{\theta_a\theta_b}{4} \quad , \end{aligned} \tag{6.3}$$

I consider also the three external angles at the edges

$$\begin{aligned} k_a\,(a,b,c) &= \pi - \theta_a\,(a,b,c) \\ k_b\,(a,b,c) &= \pi - \theta_b\,(a,b,c) \\ k_c\,(a,b,c) &= \pi - \theta_c\,(a,b,c) \quad . \end{aligned} \tag{6.4}$$

It is interesting to notice that they express the discretized extrinsic curvature of the surface of the tetrahedron. This is why I have denoted them with the letter k, often used for the extrinsic curvature. Using (6.1) and (6.4), the relation between the edge lengths a, b, c and the external angles k_a, k_b, k_c can be written in the form

$$\begin{aligned} a &= \sqrt{4c^2 - b^2}\cos\frac{k_b}{2} \\ b &= \sqrt{4c^2 - a^2}\cos\frac{k_a}{2} \\ ab &= -\sqrt{(4c^2 - a^2)\,(4c^2 - b^2)}\cos k_c \quad ; \end{aligned} \tag{6.5}$$

while (6.2) reads

$$\cos k_c = -\cos\frac{k_a}{2}\cos\frac{k_b}{2} \quad . \tag{6.6}$$

6.2 Classical theory

6.2.1 Regge action

Consider the action of general relativity, in the case of a simply connected finite spacetime region $\mathcal{R}$. In the presence of a boundary $\Sigma = \partial\mathcal{R}$ it is necessary to add a boundary term to the Einstein-Hilbert action, in order to have well defined equations of motion. The full action reads

$$S_{GR}[g] = \int_{\mathcal{R}} d^n x\sqrt{\det g}R + \int_{\Sigma} d^{n-1}x\sqrt{\det q}k \quad , \tag{6.7}$$

where g is the metric field, R is the Ricci scalar, n is the number of spacetime dimensions, while q is the metric, and k the trace of the extrinsic curvature, induced by g on Σ. For a discussion on different choices of boundary terms in three-dimensional gravity, see [147]; here I am focusing on variations of the action at fixed value of the boundary metric. In general, the Hamilton function of a finite dimensional dynamical system is the value of the action of a solution of the equations of motion, viewed as a function of the initial and final coordinates; the general solution of the equations of motion can be obtained from the Hamilton function by simple derivations [146]. In field theory, the Hamilton function can be defined as the value of the action of a solution of the equations of motion, integrated on a finite region $\mathcal{R}$, viewed as a function of value of the field on the boundary Σ [15]. In general relativity, the Hamilton function $S[q]$ is the value of the action (6.7), computed on the solution g_q of the equations of motion determined by the boundary value q:

$$S[q] = S_{GR}[g_q] \quad . \tag{6.8}$$

If g_q is not unique, $S[q]$ is multivalued. It is important to note that $S[q]$ is independent from (local deformations of) Σ, because of diffeomorphism invariance.

Since the bulk action vanishes on a vacuum solution of the equations of motion, the Hamilton function of general relativity reads

$$S[q] = \int_{\Sigma} d^{n-1}x \sqrt{\det q} k[q] \quad , \tag{6.9}$$

where the extrinsic curvature $k[q]$ is a nonlocal function, determined by the Ricci-flat metric g_q bounded by q. In the following I consider only the three-dimensional riemannian case, where $n = 3$ and the signature of g is $[+++]$. In this case, it is necessary to add an overall minus sign in (6.7) and (6.9) (see for instance the Appendix C of [148]). Furthermore, I consider the discretization of the theory provided by a Regge triangulation [54]. Let i be the index labelling the links of the triangulation and call l_i the length of the link i. In three dimensions, the bulk Regge action is

$$S_{\text{Regge}}(l_i) = -\sum_i l_i \left(2\pi - \sum_t \theta_{i,t}(l)\right) \quad , \tag{6.10}$$

where $\theta_{i,t}(l)$ is the dihedral angle of the tetrahedron t at the link i, and the angle in the parenthesis is therefore the deficit angle at i. The boundary term is

$$S_{\text{boundary}}(l_i) = -\sum_{\text{boundary } i} l_i \left(\pi - \sum_t \theta_{i,t}(l)\right) \quad , \tag{6.11}$$

where the angle in the parenthesis is the angle formed by the boundary, which can be seen as a discretization of the extrinsic curvature. I choose the minimalist triangulation formed by a single tetrahedron, and, furthermore, consider only the case in which the tetrahedron is equilateral. Then there are no internal links, the Regge action is the same as the Regge Hamilton function, and is given by

$$S(a,b,c) = -ak_a(a,b,c) - bk_b(a,b,c) - 4ck_c(a,b,c) \quad . \tag{6.12}$$

The expression for the dihedral angles as functions of the edges length, for a flat interior geometry, is given in (6.1) and (6.4). Inserting these equations into (6.12) gives the Hamilton function

$$\begin{aligned} S(a,b,c) =& a\left(2\arcsin\frac{b}{\sqrt{4c^2-a^2}} - \pi\right) + b\left(2\arcsin\frac{a}{\sqrt{4c^2-b^2}} - \pi\right) + \\ &+ 4c\left(\arccos\frac{ab}{\sqrt{(4c^2-a^2)(4c^2-b^2)}} - \pi\right) \quad . \end{aligned} \tag{6.13}$$

6.2.2 The dynamical model and its physical meaning

The Hamilton function (6.13) defines a simple relativistic dynamical model. The model has three variables, a, b and c. These are partial observables in the sense

of [112]. That is, they include both the independent (time) and the dependent (dynamical) variables, all treated on equal footing. The general formalism and the interpretation of these general relativistic systems is discussed in detail in [15]. The equations of motion are obtained following the general algorithm of the relativistic Hamilton Jacobi theory [15]: it is enough to define the momenta

$$p_a(a,b,c) = \frac{\partial S(a,b,c)}{\partial a} \tag{6.14}$$

$$p_b(a,b,c) = \frac{\partial S(a,b,c)}{\partial b} \tag{6.15}$$

$$p_c(a,b,c) = \frac{\partial S(a,b,c)}{\partial c} \tag{6.16}$$

and equate them to constants

$$\begin{aligned} p_a(a,b,c) &= p_a \\ p_b(a,b,c) &= p_b \\ p_c(a,b,c) &= p_c \quad . \end{aligned} \tag{6.17}$$

These equations give the dynamics, namely the solution of the equations of motion. Explicitly, the calculation of the momenta is simplified by the observation that the action is a homogeneous function of degree one, hence

$$S(a,b,c) = a\frac{\partial S(a,b,c)}{\partial a} + b\frac{\partial S(a,b,c)}{\partial b} + c\frac{\partial S(a,b,c)}{\partial c} \quad ; \tag{6.18}$$

this allows to identify immediately

$$\begin{aligned} p_a(a,b,c) &= -k_a(a,b,c) \\ p_b(a,b,c) &= -k_b(a,b,c) \\ p_c(a,b,c) &= -k_c(a,b,c) \quad . \end{aligned} \tag{6.19}$$

Inserting the explicit form (6.1) of the angles, it is easy to obtain the evolution equations

$$\begin{aligned} a &= \sqrt{4c^2 - b^2}\cos\frac{p_b}{2} \\ b &= \sqrt{4c^2 - a^2}\cos\frac{p_a}{2} \\ ab &= -\sqrt{(4c^2 - a^2)(4c^2 - b^2)}\cos\frac{p_c}{4} \quad , \end{aligned} \tag{6.20}$$

which reproduce (6.5). This result deserves various comments.

- First, a technical comment. Notice that the variation of the action with respect to the lengths is completely determined by the variation of the first length factor in (6.11): the variation of the length in the argument of the angles has no effect on the action. The fact that this variation vanishes was already pointed out by Regge [54]. It is the discrete analog of the well-known fact that in deriving the Einstein equations from the Einstein-Hilbert action I can ignore the change of the Levi-Civita connection under a variation of the metric.

- It is interesting to notice that boundary lengths a, b, c determine the intrinsic geometry of the boundary surface. Their conjugate momenta p_a, p_b, p_c are determined by the dihedral angles and are given by the external angles at the links. That is, they measure the extrinsic curvature of the boundary surface. This is precisely as in the ADM hamiltonian framework [149], where the momentum variable conjugate to the metric is the extrinsic curvature. Equation (6.19) is the discrete analog of the ADM relation between momenta and extrinsic curvature.

- The evolution equations (6.20) are not independent, as is always the case in relativistic systems (for instance, out of the four equations of motion of a relativistic particle, only three are independent). It is possible to take the first two equations as the independent ones. They express relations between the lengths and dihedral angles of the tetrahedron.

- How are the evolution equations (6.20) related to the Einstein equations? They are essentially equivalent. In three dimensions, the vacuum Einstein equations $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor, imply that the Riemann tensor vanishes, namely that spacetime is flat. This implies that the tetrahedron is immersed in a flat 3d spacetime. But if spacetime is flat, the extrinsic curvature of the boundary at the edge is exactly equal to π minus the dihedral angle. Hence these equations express the flatness of spacetime, namely they have the same content as the Einstein equations $R_{\mu\nu} = 0$. In other words, I have derived the relation (6.1) between length and angles assuming a flat 3d space: viceversa, the fact that these relations are satisfied implies that, in the approximation captured by the triangulation, 3d space is flat, namely the Einstein equations hold.

- The physical interpretation of the model is as follows. I assume that it is possible to measure the three lengths a, b and c and the three external angles k_a, k_b and k_c (these are six partial observables in the sense of [112]). These are all local observations that can be made on the boundary surface. They refer to the intrinsic as well as the extrinsic geometry of the surface itself. The classical theory establishes relations between these measurable quantities. These relations are the physical content of the theory and are given by the equations (6.20). They are equivalent to the statement that spacetime is flat (to the given approximation).

- The fact that the equations of motion are not independent is reflected in a relation between the momenta. The relation is of course the one given by equation (6.6), that is

$$H\left(p_a, p_b, p_c\right) = \cos\frac{p_c}{4} + \cos\frac{p_a}{2}\cos\frac{p_b}{2} = 0 \quad . \tag{6.21}$$

  From this it is possible to read out directly the Hamilton-Jacobi equation satisfied by $S\left(a, b, c\right)$

$$\cos\frac{1}{4}\frac{\partial S}{\partial c} + \cos\frac{1}{2}\frac{\partial S}{\partial a}\cos\frac{1}{2}\frac{\partial S}{\partial b} = 0 \quad . \tag{6.22}$$

  The function $H\left(p_a, p_b, p_c\right)$ given in (6.21) is the relativistic hamiltonian [15], or hamiltonian constraint, of the system.

- Finally, in the limit in which $c \gg a, b$ the action is given simply by

$$S\left(a, b, c\right) = \frac{ab}{c} - \left(a + b + 2c\right)\pi \quad , \tag{6.23}$$

  and the evolution equations (6.20) become

$$a = c\left(p_b + \pi\right) \tag{6.24}$$

$$b = c\left(p_a + \pi\right) \tag{6.25}$$

$$ab = -c^2\left(p_c + 2\pi\right) \quad . \tag{6.26}$$

6.3 Time evolution

In the description given so far, no reference to evolution in a preferred time variable was considered. I now introduce it here. I consider the direction of the axis of the equilateral tetrahedron as a temporal direction. In particular, I interpret b as an initial variable and a as a final variable (b for before and a for after). The length c of the side links can then be regarded as a proper length measured in the temporal direction, namely as the physical time elapsed from the measurement of a to the measurement of b. Indeed, had the spacetime had signature $[++-]$, and assuming the tetrahedron axis had been oriented in a timelike direction, c would precisely be the physical time measured by a real clock on the boundary of the spatial region considered, the worldline of the clock running along one of the side edges. To emphasize this interpretation of the variable c, in this section I change its name, renaming c as T. The Hamilton function reads then $S\left(a, b, T\right)$ and can now be interpreted as the Hamilton function that determines the evolution in T of a variable a. The variable b is interpreted as measured at time $T = 0$ and the variable a at time T; therefore b can be viewed as an integration constant for the evolution

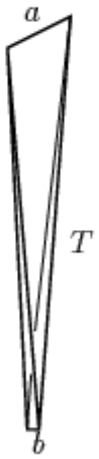

Figure 6.2: Large T, at constant b and θ_b, implies $\theta_a \to 0$ and $a \sim T$

of a in T. b is not necessarily the same variable as a, namely $T = 0$ does not imply $a = b$. For comparison, I recall that the Hamilton function of a free particle moving from a position b to a position a in a time T is

$$S_{\text{free particle}}(a, b, T) = \frac{m(a-b)^2}{2T} \tag{6.27}$$

which completely describes the free particle dynamics: equations (6.17) give in fact

$$p_a(a, b, T) = \frac{\partial S(a, b, T)}{\partial a} = m\frac{a-b}{T} = p_a \tag{6.28}$$

$$p_b(a, b, T) = \frac{\partial S(a, b, T)}{\partial b} = m\frac{b-a}{T} = p_b \tag{6.29}$$

$$p_c(a, b, T) = \frac{\partial S(a, b, T)}{\partial T} = -\frac{m(a-b)^2}{T^2} = p_T \quad , \tag{6.30}$$

which can be readily recognized as the evolution equation for coordinate and momentum

$$a(T) = a_0 + VT \tag{6.31}$$

$$p_a(T) = mV \quad , \tag{6.32}$$

where $a_0 = b$ and $V = -p_b/m$, and the relation between energy ($E \equiv -p_T$) and momentum

$$E = H(p_b) = \frac{p_b^2}{2m} \quad , \tag{6.33}$$

which defines the hamiltonian function $H(p_b)$. Returning to the system under study, the hamiltonian that evolves the system in the time T, which can be called *proper-time hamiltonian*, can obtained from the energy

$$E = -p_T = -\frac{\partial S(a,b,T)}{\partial T} = 4\pi - 4\arccos\frac{ab}{\sqrt{(4T^2-a^2)(4T^2-b^2)}} \tag{6.34}$$

by using the equations of motion to express the initial position b as a function of the position a and momentum p_a. This gives

$$H(a,p_a,T) = 4\pi - 4\arccos\left(\frac{a\cos\frac{p_a}{2}}{\sqrt{4T^2-(4T^2-a^2)\cos^2\frac{p_a}{2}}}\right) \quad . \tag{6.35}$$

Notice that the angle θ_c can vary between 0 and $\pi/2$, and therefore so does the arccos. Therefore the energy can vary between 2π and 4π. The fact that the domain of the energy is bounded has important consequences. For instance, one should expect time to become discrete in the quantum theory.

In this way, the relativistic background independent system can be reinterpreted as an evolution system, where the proper time on the boundary of the region of interest is taken as the independent time variable. The Hamilton equation generated by the hamiltonian for $a(T)$ and $p_a(T)$ are

$$\frac{da(T)}{dT} = \frac{\partial H}{\partial p_a} = \frac{4aT}{4T^2\sin^2\frac{p_a}{2} + a^2\cos^2\frac{p_a}{2}} \tag{6.36}$$

$$\frac{dp_a(T)}{dT} = -\frac{\partial H}{\partial a} = -\frac{4T\sin p_a}{4T^2\sin^2\frac{p_a}{2} + a^2\cos^2\frac{p_a}{2}} \quad . \tag{6.37}$$

The solution of these equations is

$$\begin{aligned} a(T) &= \sqrt{4T^2-b^2}\cos\frac{p_b}{2} \\ p_a(T) &= -2\arccos\frac{b}{\sqrt{4T^2\sin^2\frac{p_b}{2} + b^2\cos^2\frac{p_b}{2}}} \quad , \end{aligned} \tag{6.38}$$

where b and p_b are integration constants. These solutions are immediately recognized as the equations (6.5). Therefore the dynamics generated by the hamiltonian is the same as the general relativistic dynamics defined in a-temporal terms in the previous section.

It is interesting to consider the long time evolution of the system. In the large T limit the following behavior is found

$$\begin{aligned} a(T) &\to \text{const}\ T \\ p_a(T) &\to \frac{\text{const}}{T} - \pi \quad , \end{aligned} \tag{6.39}$$

which is precisely (6.3), identifying the two integration constants with the initial data θ_a and a. Therefore $p_a(T)$ tends to $-\pi$ as T increases. It is easy to understand this behavior geometrically. See Figure 6.2: at fixed values of the bottom length b and bottom angle $\theta_b = \pi + p_b$, as the side length T grows, the top angle $\theta_a = \pi + p_a \to 0$ and a grows proportionally to T.

The energy is not constant (there is no reason for the energy to be constant) and tends to

$$E(T) \to 2\pi \tag{6.40}$$

which is its minimal value. This result can also be obtained by considering the hamiltonian for large T. Starting from (6.23), it is possible to obtain

$$H = -\frac{\partial S(a,b,T)}{\partial T} = \frac{ab}{T^2} + 2\pi = \frac{a(\pi - p_a)}{T} + 2\pi \quad . \tag{6.41}$$

The equations of motion

$$\frac{da(T)}{dT} = \frac{\partial H}{\partial p_a} = \frac{a}{T} \tag{6.42}$$

$$\frac{dp_a(T)}{dT} = -\frac{\partial H}{\partial a} = -\frac{\pi + p_a}{T} \quad , \tag{6.43}$$

are solved by (6.39) and yield (6.40). Notice that the convergence of the "velocity" to the attraction point $p_a \to -\pi$ and the energy to its minimal value, resembles a dissipative system, such as a point particle under a constant force in a fluid.

6.3.1 Phase space and extremal configurations

Figure 6.3: The flat tetrahedron: the bottom and top edges touch.

Viewed as a dynamical system evolving in T, the system under study has a phase space coordinatized by $a \in [0, \infty[$ and $p_a \in [0, -\pi]$. The maximum value of the energy (6.35) on this phase space is $E_{\min} = 4\pi$, which is attained along the boundary p_a of the phase space Γ. These are states with vanishing external angle at the top edge. They are configuration in which the tetrahedron is flattened: its volume is zero, and the upper and bottom edges touch. The value of a is arbitrary. See Figure 6.3. Notice that these configurations evolve into one another. In fact, if $p_a = 0$, (6.38) gives

$$a(T) = \sqrt{4T^2 - b^2} \tag{6.44}$$

$$p_a(T) = 0 \quad . \tag{6.45}$$

Therefore these states grow in T remaining flattened and with the the energy remaining constant in T at the value $E = 4\pi$. In all the other states, the energy changes with time. As T grows a generic state evolves towards a state of the form

$$a = 2T \cos \frac{p_b}{2} \tag{6.46}$$

$$p_a = -\pi \quad , \tag{6.47}$$

with the energy converging to the value $E = 2\pi$. These states minimize the energy and form the boundary $p_a = -\pi$ of Γ. I call these states *Minkowski vacuum states*, since they minimize the energy. Notice that their definition depends on the choice of the time variables made. Therefore the 2d phase space has two notable subsets:

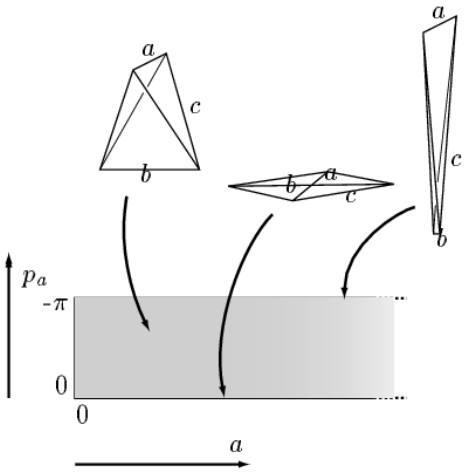

Figure 6.4: The phase space of the system with some typical configurations. The "Minkowski" states are the ones along the $p_a = -\pi$ boundary.

the line $p_a = 0$ forms an independent sector evolving into itself, given by the energy-maximizing states; while the line $p_a = -\pi$ is an attractor for the rest of the phase space, and is formed by the energy-minimizing states that I have called Minkowski states. See Figure 6.4. Notice that the variable T is bounded by $|T| > b/2$ from (6.38), therefore it is not possible to continue the solution for arbitrarily small T. It is natural to introduce the time variable

$$t = \sqrt{T^2 - \frac{b^2}{4}} \quad , \quad \text{for } T > \frac{b}{2} \tag{6.48}$$

which geometrically represents the height of the triangular face of the tetrahedron

with base b, and which arrives at zero. The evolution equations read then

$$a\left(T\right) = 2t\cos\frac{p_b}{2} \tag{6.49}$$

$$p_a\left(T\right) = -2\arccos\frac{b}{\sqrt{4t^2\sin^2\frac{p_b}{2} + b^2}} \quad . \tag{6.50}$$

Notice that the equations of motions can be extended also for negative t and negative a and p_a . It is natural to interpret this as an evolution in which the tetrahedron crosses the point $a = 0$, $p_a = 0$ in which it has zero volume, and grows on the other side, overturned as a glove. See Figure 6.5. By considering this extension, it is possible to take the phase space to be given by $a \in \mathbb{R}$ and $p_a \in [-\pi, \pi]$. In the following I will not consider this extension.

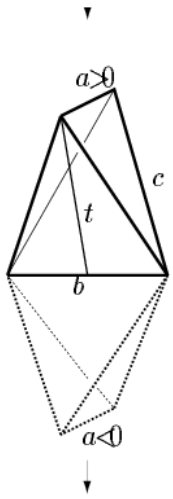

Figure 6.5: The extension to negative a and negative t.

6.4 Quantum theory

In this section I review the extension of the model described in the previous sections to the quantum case. This work is due to Daniele Colosi, Winston Fairbairn, Leonardo Modesto, Karim Noui and Carlo Rovelli [60].

6.4.1 Kinematics

The first step is to construct the boundary Hilbert space $\mathcal{K}$, on which the operators representing the boundary (partial) observables are defined. Consider the triad

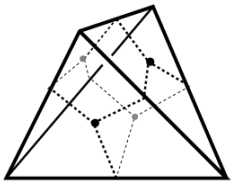

Figure 6.6: The tetrahedra T (continuous lines) and T^* (pointed lines).

formalism for three-dimensional euclidean general relativity. The variables are the triad $e^i_\mu(x)$, $\mu = 1,2,3$, $i = 1,2,3$ and its $SO(3)$ spin connection $A^i_a(x)$. The canonical boundary variables can be taken to be the $SO(3)$ connection $A^i_a(x)$, $a = 1,2$ and the inverse densitized triad $E^a_i(x)$ induced on the boundary surface.

Spacetime will be discretized in terms of a single tetrahedron T. Call f^p, $p = 1,2,3,4$ the faces of the tetrahedron, e^{pq} the oriented edge separating the face p from the face q (say oriented rightward in going from p to q). To define the discrete dynamical variables, consider the dual tetrahedron T^* defined by vertices v_p in the face f^p of T. The edges e_{pq} of T^* connect the vertex p to the vertex q; they are dual, and cut the corresponding edges e^{pq} of T. It is possible to discretize the boundary field $A^i_a(x)$ by replacing it with six group elements U_{pq} associated to the six edges e_{pq}, interpreted as the parallel transport matrix of the connection along e_{pq}. As usual in quantum gravity, it is possible to take $U_{pq} \in SU(2)$ (the classical theory is determined by the algebra, not the group) and write $U_{pq} = U^{-1}_{qp}$. Gauge transformations act on the vertices v_p; they are determined by four group elements V_p and the group elements U_{pq} transform as

$$U_{pq} \to V_p U_{pq} V_q^{-1} \quad . \tag{6.51}$$

The quantum theory can be defined starting from the Hilbert space $\mathcal{K}$ of the Haar-square-integrable functions $\psi(U_{pq})$ of the six dynamical variables U_{pq} that are gauge-invariant under the transformations (6.51), namely

$$\psi(U_{pq}) = \psi\left(V_p U_{pq} V_q^{-1}\right) \quad . \tag{6.52}$$

These gauge transformations depend on four group elements, therefore $\mathcal{K} = \mathcal{L}_2\left[(SU(2))^6 / (SU(2))^4\right]$ where the action of $(SU(2))^4$ on $(SU(2))^6$ is the one given in (6.51). The notation $\mathbf{U} = (U_{pq})$ represents the 6-tuplet of group elements, and thus write states as $\psi(\mathbf{U})$. Similarly, a 6-tuplet of spins is indicated as $\mathbf{j} = (j_{pq})$. As well known (see for instance [15]), a basis in $\mathcal{K}$ is given by the spin-network states

$$\begin{aligned} &\psi_{\mathbf{j}}(\mathbf{U}) = \langle \mathbf{U}|\mathbf{j}\rangle = v^{ijk} v^{olm} v^{prn} v^{qst} \cdot \\ &\cdot R^{j_{12}}_{io}(U_{12}) R^{j_{13}}_{jp}(U_{13}) R^{j_{14}}_{kq}(U_{14}) R^{j_{23}}_{lr}(U_{23}) R^{j_{24}}_{ms}(U_{24}) R^{j_{34}}_{nt}(U_{34}) \quad , \end{aligned} \tag{6.53}$$

where the R^j_{kl} are the matrix elements of the $SU(2)$ representation j and v^{ikl} are the normalized invariant tensors. The index structure of equation (6.53) is determined by the geometry of the tetrahedron. The function $\psi_{\mathbf{j}}(\mathbf{U})$ is the spin-network function for a spin network having T^* as graph. See [15] for details. 11 fi The left invariant vector field on each group can be identified as the operator associated to the triad field integrated along the edges of T. The integral of the $SU(2)$-norm of these gives the length of the edge; therefore the Casimir operators C_{pq} of the (pq)-th group

$$C_{pq}\,|\mathbf{j}\rangle = j_{pq}\,(j_{pq}+1)\,|\mathbf{j}\rangle \tag{6.54}$$

can then be naturally identified as the operator giving the length square of the edge e^{pq} [150]. The tensor structure of the algebra of the $SU(2)$ representations implements the triangular relations satisfied by the length. The spectrum of the length of the edges e_{pq} is therefore given by

$$l_{pq} = \sqrt{j_{pq}\,(j_{pq}+1)} \quad . \tag{6.55}$$

The fact that the lengths have discrete spectrum is an immediate consequence of their conjugate variables being angles, and thus vary on a compact domain. Following [15], it is possible to interpret the spectral properties of the partial observables as physical predictions of the quantum model.

6.4.2 Dynamics

The quantum dynamics is completely captured by the propagator, as I described in Chapter 3. In a general relativistic theory, the propagator is formally expressed as the function of the boundary variables given by (4.5). However, in general the propagator is not a function of classical boundary variables; the reason is that the boundary quantities may fail to have continuous spectrum. If they have discrete spectrum, the propagator depends on the quantum numbers that label the discrete eigenvectors of the boundary quantities, and not on the corresponding continuous classical variables [15]. In the present case, the propagator can be written in the basis (6.53), where it will be a function $W(j_{pq})$. To find this function, recall that the classical dynamics requires three-dimensional space to be flat. This means that any parallel transport along a three-dimensional closed path must be trivial. Consider the four "elementary" closed paths γ_p on T^*, where γ_4 is defined by the sequence of edges $e_{12}e_{23}e_{31}$, that circle the vertex opposite to the face f^4, and so on. The requirement is expressed by the parallel transport around each of these paths being trivial

$$U_{12}U_{23}U_{31} = \mathbf{1} \quad , \tag{6.56}$$

and similarly for the other three. Therefore it is possible to write the 3d flatness requirement (the Einstein equations) in the form

$$(U_{12}U_{23}U_{31} - \mathbf{1}) = 0 \quad , \tag{6.57}$$

where $p \neq q \neq r$. Then the main dynamical equation of the quantum theory can be written in the form

$$\left(U_{12}U_{23}U_{31} - \mathbf{1}\right)\psi_0\left(\mathbf{U}\right) = 0 \quad , \tag{6.58}$$

which can be interpreted as a Wheeler-DeWitt equation. Its general solution is

$$\psi_0\left(\mathbf{U}\right) = f\left(\mathbf{U}\right)\prod_{pqr}\delta\left(U_{pq}U_{qr}U_{rp}\right) \quad , \tag{6.59}$$

where the delta function is the one on the group (for the Haar measure) and $f\left(\mathbf{U}\right)$ is an arbitrary gauge invariant function. This equation defines the physical states ψ_0 that solve the dynamics of the theory. To express these states in the $\mathbf{j}$ basis, one simply projects them on the basis states (6.53)

$$\psi_0\left(\mathbf{j}\right) = \int d\mathbf{U}\bar{\psi}_{\mathbf{j}}\left(\mathbf{U}\right)f\left(\mathbf{U}\right)\prod_{pqr}\delta\left(U_{pq}U_{qr}U_{rp}\right) \quad . \tag{6.60}$$

It is easy to see that by gauge invariance, it is possible to gauge fix all U_{pq} to unity in the integral, giving

$$\psi_0\left(\mathbf{j}\right) = \int d\mathbf{U}\bar{\psi}_{\mathbf{j}}\left(\mathbf{U}\right)f\left(\mathbf{U}\right)\prod_{pq}\delta\left(U_{pq}\right) = \tag{6.61}$$

$$= c\psi_{\mathbf{j}}\left(1\right) = \tag{6.62}$$

$$= c\delta_{io}\delta_{jp}\delta_{kq}\delta_{lr}\delta_{ms}\delta_{nt}v^{ijk}v^{olm}v^{prn}v^{qst} \quad . \tag{6.63}$$

The constant $c = f\left(1\right)$ can be absorbed in the normalization. The last line is the definition of the Wigner 6-j symbol, usually written as

$$\psi_0\left(\mathbf{j}\right) = \begin{pmatrix} j_{12} & j_{13} & j_{14} \\ j_{34} & j_{24} & j_{23} \end{pmatrix} \equiv \langle\mathbf{j}|0\rangle \quad . \tag{6.64}$$

Thus, it is possible to conclude that there is a single state $|0\rangle$ (up to normalization) in $\mathcal{K}$ that solves the dynamics, and that this state is proportional to the Wigner 6-j symbol.

The physical amplitude of an arbitrary kinematical state $\psi \in \mathcal{H}$ is determined by its projection on the state that solve the dynamical equation, namely by its projection on the state $|0\rangle$

$$A\left(\psi\right) = \langle 0|\psi\rangle \quad . \tag{6.65}$$

The state $|0\rangle$ is called the "non-perturbative" vacuum state [15]. It expresses the dynamics of the theory. In other words, the physical amplitude for having the boundary configuration j_{pq} is the Wigner 6-j symbol. Namely the propagator of the theory is Wigner 6-j symbol

$$W\left(\mathbf{j}\right) \equiv \langle\mathbf{j}|0\rangle = \begin{pmatrix} j_{12} & j_{13} & j_{14} \\ j_{34} & j_{24} & j_{23} \end{pmatrix} \quad . \tag{6.66}$$

Now, this is precisely the result obtained by Ponzano and Regge on the basis of a physical ansatz on the discretization of the lengths, and a discretization of the Einstein-Hilbert action [151]. In this minimalist model, the functional integral (4.5) is trivial because there are no bulk degrees of freedom. Its result is therefore proportional to the exponential of the action. Ponzano and Regge found that the Wigner 6-j symbol (6.66) can in fact be viewed as a discretization of (the real part) of the exponential of the action. The result is also equivalent (up to a phase) to the specialization to a single tetrahedron of the boundary amplitude computed in [55] and in [147]. In the present case, the discretization of the length is not introduced as an ansatz, but it is a standard quantum-mechanical consequence of the conjugate variable being an angle.

6.4.3 Quantum equilateral tetrahedron

So far, an arbitrary quantum tetrahedron has been considered. Now the formalism will be specialized to the case of an equilateral tetrahedron. The simplest way to do so is to consider only the states where four of the six edge lengths are equal. More precisely, the equilateral tetrahedron is selected by setting

$$j_a \equiv j_{13} \tag{6.67}$$

$$j_b \equiv j_{24} \tag{6.68}$$

$$j_c \equiv j_{12} = j_{23} = j_{34} = j_{41} \quad , \tag{6.69}$$

and considering only the states

$$|j_a, j_b, j_c\rangle = |j_c, j_a, j_c, j_c, j_b, j_c\rangle \quad . \tag{6.70}$$

Accordingly, the states $\psi(U_{pq})$ can be restricted to the subset of $(SU(2))^6$ determined by $U_{12} = U_{23} = U_{34} = U_{41}$, and

$$U_a \equiv U_{13} \tag{6.71}$$

$$U_b \equiv U_{24} \tag{6.72}$$

$$U_c \equiv U_{12} = U_{23} = U_{34} = U_{41} \quad . \tag{6.73}$$

The gauge transformations that preserve the resulting subspace are the ones for which

$$V_1 = V_3 \equiv V_a \tag{6.74}$$

$$V_2 = V_4 \equiv V_b \quad , \tag{6.75}$$

under which the states $\psi(U_a, U_b, U_c)$ transform as

$$\psi(U_a, U_b, U_c) \to \left(V_a U_a V_a^{-1}, V_b U_b V_b^{-1}, V_a U_c V_b^{-1}\right) \quad . \tag{6.76}$$

Using these gauge transformations, it is possible to transform U_a, U_b, U_c to three rotations around three orthogonal axes, of three angles k_a, k_b, k_c. The interpretation of these angles is simple. Since spacetime is flat, it is possible to choose the gauge in which the internal space is directly identified with spacetime. Then the rotation along the edge e_{pq} of T^* can be identified as the physical rotation that one undergoes in crossing the edge e^{pq} of T. These are precisely the external angles that were denoted k_a, k_b, k_c in the previous section. For an $SU(2)$ matrix, $\mathrm{Tr}(U) = 2\cos(\phi/2)$, where ϕ is the rotation angle. Therefore one can consider the operator

$$T_a \equiv \mathrm{Tr}\,(U_a) = 2\cos\frac{k_a}{2} = 2\cos\frac{p_a}{2} \quad , \tag{6.77}$$

which is now gauge invariant. The action of this operator is easily obtained from $SU(2)$ representation theory:

$$T_a\,|j_a, j_b, j_c\rangle = \left|j_a + \frac{1}{2}, j_b, j_c\right\rangle + \left|j_a - \frac{1}{2}, j_b, j_c\right\rangle \quad , \tag{6.78}$$

and similarly for the other edges. In the next section it will be shown that the commutator between this operator and the length reproduces the classical Poisson brackets.

In summary, the boundary Hilbert state $\mathcal{K}$ is spanned by the states $|j_a, j_b, j_c\rangle$. The boundary observables a, b, c, p_a, p_b, p_c that measure the length of the edges of the tetrahedron and the external angles are represented by Casimir and trace operators, and the dynamics is given by the propagator

$$W(j_a, j_b j_c) = \begin{pmatrix} j_a & j_c & j_c \\ j_b & j_c & j_c \end{pmatrix} \quad , \tag{6.79}$$

which expresses the probability amplitude of measuring the lengths determined by j_a, j_b, j_c. This concludes the definition of the quantum theory. The predictions of the theory are given by the quantization of the lengths and by the relative probability amplitude (6.79).

6.5 Time evolution in the quantum theory

So far, the system under study has been viewed as a general relativistic system, in which predictions are expressed in terms of (probabilistic) relations between boundary partial observables, or probability amplitudes for boundary configurations. The system will now be reinterpreted as a system evolving in a time variable, as I did in the classical case. Thus, say, b and p_b are considered as initial variables, a and p_a as final variables, and j_c as a time parameter. It is necessary to identify the Hilbert space of the system at fixed time.

The final state is described by the operators C_a and T_a that act on the variable U_a. The boundary Hilbert state $\mathcal{K}$, spanned by the states $|j_a, j_b, j_c\rangle$ can be decomposed

as (a subspace, because of the Clebsch-Gordan relations of) the tensor product of three spaces $\mathcal{K}_a$, $\mathcal{K}_b$, $\mathcal{K}_c$ spanned by states $|j_a\rangle$, $|j_b\rangle$, $|j_c\rangle$ respectively. $\mathcal{K}_a$ can be interpreted as the state space at fixed time; it can be simply expressed as the space of the class functions $(\psi(U_a)$, that is, the functions satisfying

$$\psi(U_a) = \psi\left(V_a U_a V_a^{-1}\right) \quad . \tag{6.80}$$

The basis $|j_a\rangle$ is defined by the characters

$$\langle U_a|j_a\rangle = \chi_{j_a}(U_a) = \frac{\sin\left(\left(j_a + \frac{1}{2}\right)U_a\right)}{\sin\frac{U_a}{2}} \quad . \tag{6.81}$$

The Casimir and trace operators act as

$$C_a|j_a\rangle = j_a\left(j_a + \frac{1}{2}\right)|j_a\rangle \tag{6.82}$$

$$T_a|j_a\rangle = \left|j_a + \frac{1}{2}\right\rangle + \left|j_a - \frac{1}{2}\right\rangle \quad , \tag{6.83}$$

where the second relation is easily derived from the properties of the characters. It is convenient to define also the operator

$$S_a|j_a\rangle = i\left(\left|j_a + \frac{1}{2}\right\rangle + \left|j_a - \frac{1}{2}\right\rangle\right) \quad , \tag{6.84}$$

that satisfies $T_a^2 + S_a^2 = 4$, and is therefore a function of T_a

$$S_a = \sqrt{4 - T_a^2} \quad . \tag{6.85}$$

Since T_a has been identified with $2\cos(p_a/2)$, it follows that also S_a and $2\sin(p_a/2)$ can be identified. The classical Poisson brackets

$$\{a, p_a\} = 1 \tag{6.86}$$

give, for $T_a = 2\cos(p_a/2)$,

$$\{a, T_a\} = \sin\frac{p_a}{2} = \frac{1}{2}\sqrt{4 - T_a^2} \quad . \tag{6.87}$$

Consider the operator J_a defined by $C_a = J_a(J_a + 1)$ and acting as

$$J_a|j_a\rangle = j_a|j_a\rangle \quad ; \tag{6.88}$$

a straightforward calculation gives

$$[J_a, T_a] = \frac{i}{2}S_a = \frac{i}{2}\sqrt{4 - T_a^2} \quad . \tag{6.89}$$

Therefore the operators J_a and T_a define a linear representation of the classical Poisson algebra defined by the observables a and $2\cos(p_a/2)$. There are then two options. The first is to identify the classical quantity a with the operator J_a. The second is to identify a with the square root of the Casimir. Both choices give the correct classical limit, since they become the same in the limit of large quantum numbers. The first gives a quantum theory in which the length is quantized in half-integers j_a; the second gives a quantum theory in which the length is quantized as $\sqrt{j_a(j_a+1)}$. The second choice can be identified with the quantization defined in the previous section.

A discrete time evolution is determined by the propagator (6.79), seen as a propagator from the state $|j_b\rangle$ to the state $|j_a\rangle$ in a (discrete) time j_c. In the classical theory the long time evolution drives the system to the "Minkowski" configurations where $p_a = -\pi$. The next step is studying the quantum evolution for long times. For $j_c \to \infty$ the following relation holds [54]

$$W(j_a, j_b, j_c) = \begin{pmatrix} j_a & j_c & j_c \\ j_b & j_c & j_c \end{pmatrix} \to \frac{(-1)^{-(j_a+j_b+2j_c)}}{2j_c} \quad . \tag{6.90}$$

This can be written as

$$W(j_a, j_b, T) \underset{T\to\infty}{\to} \frac{(-1)^{-(j_a+j_b+2T)}}{2T} = \tag{6.91}$$

$$= \frac{e^{-2\pi(iT)}}{2T} e^{-ij_b\pi} e^{-ij_a\pi} = \frac{e^{-iE_0T}}{2T} \psi_0(j_a)\,\psi_0(j_b) \quad . \tag{6.92}$$

That is, for large T the evolution projects on the (generalized) state

$$\psi_0(j_a) \equiv \langle j_a | 0_M \rangle = e^{-ij_a\pi} \quad . \tag{6.93}$$

It is easy to see that this is the generalized eigenstate of p_a with eigenvalue $-\pi$ (since p_a itself is not an operator in the theory, this means, of course, a generalized eigenstate of $T_a = 2\cos(p_a/2)$ with eigenvalue $2\cos(-\pi/2) = 0$)

$$2\cos(p_a/2)\,\psi_0(j_a) = T_a e^{-ij_a\pi} = e^{-i(j_a+1/2)\pi} + e^{-i(j_a-1/2)\pi} = \tag{6.94}$$

$$= e^{-ij_a\pi}\left(e^{i\pi/2} + e^{-i2}\right) = 2\cos(-\pi/2)\,\psi_0(j_a) \quad . \tag{6.95}$$

Therefore it has been shown that the quantum dynamics converges to the classical dynamics on long times. It is appropriate to call $|0_M\rangle$ the "Minkowski" quantum state, since it minimizes the energy. It has been shown that the nonperturbative vacuum state $|0_M\rangle$ in $\mathcal{K}$ becomes a projector on $|0_M\rangle$ in the $T \to \infty$ limit. Therefore it is possible to write

$$\lim_{j_c\to 0} |0\rangle = |0_M\rangle\langle 0_M| \quad . \tag{6.96}$$

The bra/ket mismatch is only apparent: the l.h.s is a ket in $\mathcal{K}$, while the r.h.s. is an element of the tensor product between $\mathcal{K}_a$ and its dual, which can be identified with a subspace of $\mathcal{K}$ under

$$|j_a\rangle \langle j_b| \leftrightarrow |j_a, j_b, j_c\rangle \quad . \tag{6.97}$$

See [15] for details. Equation (6.96) is the expression I proposed in Chapter 3 for computing the Minkowski vacuum state for spinfoam transition amplitudes. So one finds that in the present case this equation is correct. Notice, however, that in this euclidean context the limit is taken for real times. Alternatively, it is possible to study the continuous time evolution determined by quantizing the classical hamiltonian (6.35). Notice that (6.35) can be easily written in terms of the operators that have been defined

$$H\left(a, p_a, T\right) = 4\pi - 4\arccos\left(\sqrt{C_a}\frac{1}{\sqrt{4T^2 - \left(4T^2 - C_a\right)\left(\frac{T_a}{2}\right)^2}}\frac{T_a}{2}\right) \quad . \tag{6.98}$$

Choosing this ordering (where the inverse and the arccos are defined by spectral decomposition) gives immediately that the eigenstate of the p_a with eigenvalues $-\pi$ is an eigenstates of the hamiltonian, with energy 2π, in accord with the corresponding classical result.

Chapter 7

Conclusions

In this thesis I put forward a proposal concerning a possible strategy to make contact between background independent quantum gravity and conventional calculations of particle scattering amplitudes. I studied thoroughly this strategy in the case of ordinary quantum field theory. In the following I will illustrate the logical scheme of this proposal.

Field propagation kernel defined on an infinite strip The first step has been to write down a formulation of quantum field theory in terms of the propagation kernel $W[\phi_1, t_1; \phi_2, t_2]$ (see Chapter 3). In the context of a Schrödinger picture of quantum field theory, where the primary objects of interest are the fields, I considered a scalar massive field theory. I then calculated in this case the extension of the propagation kernel introduced by Feynman in the description of the quantum mechanics of a single particle [45, 61], that is, the propagation kernel between field configurations defined on infinite spatial hyperplanes at fixed time. I then performed some other calculations to show that through this single tool it is possible to reconstruct all physical information that can be extracted from a field theory. For example, the propagation kernel allows to calculate two-point functions and therefore, through Wick's theorem and the LSZ reduction formulas, any scattering amplitude. Furthermore, using the propagation kernel it is possible to obtain the vacuum state, of which I gave two different definitions. The first is the definition of the Minkowski vacuum state $|0_M\rangle$, the second is that of a nonperturbative vacuum state $|0_\Sigma\rangle$. The two are related by a limit on the time interval between the initial and final field configuration. The results described in this Chapter are illustrated in [58].

Extension to a finite region of spacetime Since the final scope of my study is to find a technique to a possible strategy with which to extract physical predictions, such as particle scattering amplitudes, from a quantum theory of spacetime, of course a propagation kernel defined on a special region of spacetime (an infinite strip) is not general enough. More generally, it is the picture

of initial state, evolution and final state, which was quite appropriate in the non-relativistic context of the founding days of quantum mechanics, which becomes inadequate when including space and time in the quantum mechanical realm itself. Therefore in Chapter 4 I proceeded to extend the formalism to generic regions of spacetime, and more specifically closed ones. This can be done by adopting a the framework of the so-called "general boundary formulation" of quantum field theory, introduced by Robert Oeckl [47, 48, 49], which consists in adopting a topological-quantum-field-theory-like description of quantum field theory allowing at the same time the evolution of the fields between generic boundary surfaces, for example closed surfaces. This framework entails a number of implications. Among them are a somewhat radical departure from the static Hilbert space picture of quantum mechanics. Related to that is a necessary duality between "in" and "out" states, or preparation and observation. In particular, states are physically meaningful even if associated with boundaries that have time-like components. In section 4.2 I then specialized the propagation kernel formalism to the case of a closed boundary surface enclosing a finite region $\mathcal{R}$ [58]. Quantum field theory can be formulated in terms of a state space $\mathcal{K}_\Sigma$ associated to the boundary Σ of $\mathcal{R}$. States in $\mathcal{K}_\Sigma$ represent measurement outcomes on Σ. I have defined the propagation kernel $W[\varphi, \Sigma]$ depending on Σ and the boundary values φ of the dynamical fields on Σ. $W[\varphi, \Sigma]$ can be used to compute the Minkowski vacuum state $|0_M\rangle$, taking the appropriate limit on Σ; it is enough to consider propagation in imaginary time for a laboratory scale. Through $W[\varphi, \Sigma]$ it is also possible to define the nonperturbative vacuum $\langle 0_\Sigma|$ in $\mathcal{K}_\Sigma$, which expresses the dynamics since it gives the amplitude for any complete set of measurements. Measurements of the boundary field φ are represented by the basis $|\varphi\rangle$. Particle detection determines particle states in $\mathcal{K}_\Sigma$, which can be obtained acting with the field operator on $|0_M\rangle$. This formulation can be used to modify present approaches to quantum gravity, with a view towards obtaining physically meaningful amplitudes. This applies notably to loop quantum gravity [19, 15] and spin foam models [35, 37]. In a background independent theory, n particle functions $W(x_1, \dots x_n)$ become meaningless, because they are independent from the coordinates; while $W[\varphi, \Sigma]$, which now must rather be written as $W[\varphi]$, maintains its physical meaning, in spite of the fact that it is independent from Σ. This is because in a gravitational theory the relative location of the detectors is coded in φ and not in Σ. Localization measurements are on the same footing as the dynamical variables measurements. Through the functional $W[\varphi]$ I also defined a state $|0_\Sigma\rangle$ that codes the dynamics of the theory by determining the correlation amplitudes between boundary measurements. The Minkowski vacuum state $|0_M\rangle$ can be computed from nonperturbative quantum gravity by choosing appropriate boundary values of the gravitational field; a tentative formula giving the Minkowski vacuum state in terms of a spinfoam model is given by equation (4.24). Relevant analytical continuation is in the proper

length of the boundary, not in the time coordinate.

Generalized Tomonaga-Schwinger equation The following step is studying which kind of evolution equation the propagation kernel $W[\varphi, \Sigma]$ obeys [59]. In Chapter 5 I have derived the generalized Tomonaga–Schwinger equation (5.2) from its functional integral definition (5.1), in the case of a free euclidean scalar field. To this end I used a Green function technique and in particular Hadamard's formula, which describes the change of Green functions under modifications of the surface over which they are defined. The result I derived undercuts some a priori arguments against the general possibility of an equation as the form (5.2), and renders the equation more plausible. The difficulty of using functional integral techniques is that integrals that converge in the euclidian oscillate in the lorentzian case. If convergence of the integrals can be controlled, I expect that it should be possible to prove the validity of the generalized Tomonaga–Schwinger in the lorentzian case using the same strategy as here.

Application to a toy model Finally, in Chapter 6 I considered the application of some of the techniques I developed in the previous Chapters to a toy model. The model is very simple, but it gives an idea of how quantum field theory can be defined and consistently interpreted in the absence of a background spacetime. This is a study I carried out in collaboration with Daniele Colosi, Winston Fairbairn, Leonardo Modesto, Karim Noui and Carlo Rovelli [60]; my contribution is limited to the study of the classical version of the model. The model consists of a single equilateral tetrahedron. Observables can be defined on a closed finite boundary. The classical dynamics can be expressed as a set of relations between these observables. The quantum theory can be defined in terms of a boundary Hilbert space $\mathcal{K}$, on which operators representing boundary observables are defined. The boundary observables are partial observables: they represent quantities whose measurements can be operationally defined in principle, but whose value cannot be predicted from the knowledge of the state, in general [112]. The spectral properties of these boundary operators are physical kinematical predictions of the theory. Dynamical predictions do not refer to values of partial observables, but rather to relations between these values. The quantum dynamics is captured by the nonperturbative vacuum state $|0\rangle$, or, equivalently, by the propagator (6.79). A temporal interpretation of the model is not necessary, but it is possible [152]. By interpreting the "side" length c as a time variable, the propagator $W(j_a, j_b, j_c)$ can be interpreted as the transition probability amplitude from the initial state $|j_b\rangle$ to the final state $|j_a\rangle$ in a discrete time j_c. The energy that drives this evolution has minimum value on a state (denoted as the "Minkowski" state) that can be obtained from the propagator: the propagator becomes proportional to a projector on this state in the large time limit.

Appendix A

Hadamard formula

I sketch here the demonstration of the Hadamard formula (5.32). Consider two surfaces Σ_1 and Σ_2, where Σ_2 coincides with Σ_1 except for an infinitesimal outward bulge around a point. Consider the difference $G_{\Sigma_2}(x,y) - G_{\Sigma_1}(x,y)$. This difference satisfies the equation of motion (5.21). If $x \in \Sigma_2$:

$$G_{\Sigma_2}(x,y) - G_{\Sigma_1}(x,y)\big|_{x\in\Sigma_2} = -G_{\Sigma_1}(x,y) \ ; \tag{A.1}$$

taking $y \in \Sigma$ in (5.27) shows that

$$-\partial_n^x G_\Sigma(x, y\in\Sigma) = \delta^{(3)}(x-y)\big|_{y\in\Sigma} \ , \tag{A.2}$$

so that

$$G_{\Sigma_2}(x,y) - G_{\Sigma_1}(x,y) = \int_{\Sigma_2} d\Sigma_2(s)\partial_n^z G_{\Sigma_2}(z,x)\, G_{\Sigma_1}(z,y) \ . \tag{A.3}$$

The region of Σ_2 where $\Sigma_2 \equiv \Sigma_1$ doesn't contribute to this integral because of (5.25). In the region where $\Sigma_2 \neq \Sigma_1$, $G_{\Sigma_1}(z,y)$ is infinitesimal, so it is possible to approximate

$$G_{\Sigma_1}(z,y) \simeq \partial_n^z G_{\Sigma_2}(z,y)\,\delta n \ , \tag{A.4}$$

where n is the normal to Σ_1, positive in the outgoing direction. As a consequence it is easy to find Hadamard's formula:

$$\frac{\delta G_\Sigma(x,y)}{\delta\Sigma(s)} = \partial_n^z G_\Sigma(z,x)\,\partial_n^z G_\Sigma(z,y) \ . \tag{A.5}$$

Bibliography

[1] L. Smolin, *How far are we from the quantum theory of gravity?*, hep-th/0303185

[2] C. Rovelli, *Strings, loops and others: A critical survey of the present approaches to quantum gravity*, gr-qc/9803024

[3] G. Amelino-Camelia, *The three perspectives on the quantum-gravity problem and their implications for the fate of Lorentz symmetry*, gr-qc/0309054

[4] E. Alvarez, P. Meessen, *String primer*, JHEP, **02**: (1999), 015, hep-th/9810240

[5] K. R. Dienes, *String Theory and the Path to Unification: A Review of Recent Developments*, Phys. Rept., **287**: (1997), 447–525, hep-th/9602045

[6] T. Mohaupt, *Introduction to string theory*, Lect. Notes Phys., **631**: (2003), 173–251, hep-th/0207249

[7] R. J. Szabo, *BUSSTEPP lectures on string theory: An introduction to string theory and D-brane dynamics*, hep-th/0207142

[8] J. Polchinski, *What is string theory?*, hep-th/9411028

[9] J. Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, Phys. Rev. Lett., **75**: (1995), 4724–4727, hep-th/9510017

[10] T. Banks, W. Fischler, S. H. Shenker, L. Susskind, *M theory as a matrix model: A conjecture*, Phys. Rev., **D55**: (1997), 5112–5128, hep-th/9610043

[11] N. Ishibashi, H. Kawai, Y. Kitazawa, A. Tsuchiya, *A large-N reduced model as superstring*, Nucl. Phys., **B498**: (1997), 467–491, hep-th/9612115

[12] D. Bigatti, L. Susskind, *Review of matrix theory*, hep-th/9712072

[13] S. Weinberg, *Photons and gravitons in S Matrix theory: derivation of charge conservation and equality of gravitational and inertial mass*, Phys. Rev., **135**: (1964), B1049–B1056

[14] S. Weinberg, *Photons and gravitons in perturbation theory: Derivation of Maxwell's and Einstein's equations*, Phys. Rev., **138**: (1965), B988–B1002

[15] C. Rovelli, *Quantum gravity*, Cambridge University Press (2004)

[16] C. Rovelli, P. Upadhya, *Loop quantum gravity and quanta of space: A primer*, gr-qc/9806079

[17] L. Smolin, *An invitation to loop quantum gravity*, hep-th/0408048

[18] C. Rovelli, *Notes for a brief history of quantum gravity*, gr-qc/0006061

[19] C. Rovelli, *Loop quantum gravity*, Living Rev. Rel., **1**: (1998), 1, gr-qc/9710008

[20] T. Thiemann, *Lectures on loop quantum gravity*, Lect. Notes Phys., **631**: (2003), 41–135, gr-qc/0210094

[21] T. Thiemann, *Introduction to modern canonical quantum general relativity*, gr-qc/0110034

[22] M. Gaul, C. Rovelli, *Loop quantum gravity and the meaning of diffeomorphism invariance*, Lect. Notes Phys., **541**: (2000), 277–324, gr-qc/9910079

[23] A. Ashtekar, *New variables for classical and quantum gravity*, Phys. Rev. Lett., **57**: (1986), 2244–2247

[24] A. Ashtekar, *New hamiltonian formulation of general relativity*, Phys. Rev., **D36**: (1987), 1587–1602

[25] C. Rovelli, L. Smolin, *Knot theory and quantum gravity*, Phys. Rev. Lett., **61**: (1988), 1155

[26] C. Rovelli, L. Smolin, *Loop space representation of quantum general relativity*, Nucl. Phys., **B331**: (1990), 80

[27] J. A. Zapata, *Combinatorial space from loop quantum gravity*, Gen. Rel. Grav., **30**: (1998), 1229–1245, gr-qc/9703038

[28] R. De Pietri, C. Rovelli, *Geometry Eigenvalues and Scalar Product from Recoupling Theory in Loop Quantum Gravity*, Phys. Rev., **D54**: (1996), 2664–2690, gr-qc/9602023

[29] A. Ashtekar, J. Lewandowski, *Projective techniques and functional integration for gauge theories*, J. Math. Phys., **36**: (1995), 2170–2191, gr-qc/9411046

[30] M. P. Reisenberger, *World sheet formulations of gauge theories and gravity*, gr-qc/9412035

[31] C. Rovelli, L. Smolin, *Spin networks and quantum gravity*, Phys. Rev., **D52**: (1995), 5743–5759, gr-qc/9505006

[32] J. C. Baez, *Spin network states in gauge theory*, Adv. Math., **117**: (1996), 253–272, gr-qc/9411007

[33] T. Thiemann, *QSD V: Quantum gravity as the natural regulator of matter quantum field theories*, Class. Quant. Grav., **15**: (1998), 1281–1314, gr-qc/9705019

[34] T. Thiemann, *Gauge field theory coherent states (GCS). I: General properties*, Class. Quant. Grav., **18**: (2001), 2025–2064, hep-th/0005233

[35] D. Oriti, *Spacetime geometry from algebra: Spin foam models for non-perturbative quantum gravity*, Rept. Prog. Phys., **64**: (2001), 1489–1544, gr-qc/0106091

[36] A. Perez, *Introduction to loop quantum gravity and spin foams*, gr-qc/0409061

[37] A. Perez, *Spin foam models for quantum gravity*, Class. Quant. Grav., **20**: (2003), R43, gr-qc/0301113

[38] J. C. Baez, *An introduction to spin foam models of BF theory and quantum gravity*, Lect. Notes Phys., **543**: (2000), 25–94, gr-qc/9905087

[39] J. C. Baez, *Spin foam models*, Class. Quant. Grav., **15**: (1998), 1827–1858, gr-qc/9709052

[40] C. W. Misner, *Feynman quantization of general relativity*, Rev. Mod. Phys., **29**: (1957), 497

[41] S. W. Hawking, *The path-integral approach to quantum gravity*, in: S. Hawking, W. Israel (editors), *General relativity: An Einstein centenary survey*, Cambridge University Press (1979), 746–789

[42] S. W. Hawking, *Quantum cosmology*, presented at Les Houches Summer School on Theoretical Physics, Les Houches, France, Jun 27 - Aug 4, 1983

[43] J. B. Hartle, S. W. Hawking, *Wave function of the universe*, Phys. Rev., **D28**: (1983), 2960–2975

[44] H. Sahlmann, T. Thiemann, O. Winkler, *Coherent states for canonical quantum general relativity and the infinite tensor product extension*, Nucl. Phys., **B606**: (2001), 401–440, gr-qc/0102038

[45] R. P. Feynman, A. R. Hibbs, *Quantum mechanics and path integrals*, McGraw-Hill (1965)

[46] N. C. Tsamis, R. P. Woodard, *Physical Green's function in quantum gravity*, Ann. Phys., **215**: (1992), 96–155

[47] R. Oeckl, *Schrödinger's cat and the clock: Lessons for quantum gravity*, Class. Quant. Grav., **20**: (2003), 5371–5380, gr-qc/0306007

[48] R. Oeckl, *A 'general boundary' formulation for quantum mechanics and quantum gravity*, Phys. Lett., **B575**: (2003), 318–324, hep-th/0306025

[49] R. Oeckl, *The general boundary approach to quantum gravity*, gr-qc/0312081

[50] M. Atiyah, *Topological quantum field theories*, inst. Hautes Etudes Sci. Publ. Math. (1989), n. 68, pp. 175-186

[51] D. Colosi, C. Rovelli, *Global particles, local particles*, gr-qc/0409054

[52] S. Tomonaga, *On a relativistically invariant formulation of the quantum theory of wave fields*, Prog. Theor. Phys., **1**: (1946), 27

[53] J. Schwinger, *Quantum electrodynamics I. A covariant formulation*, Phys. Rev., **74**: (1948), 1439

[54] T. Regge, *General relativity without coordinates*, Nuovo Cim., **19**: (1961), 558–571

[55] H. Ooguri, N. Sasakura, *Discrete and continuum approaches to three-dimensional quantum gravity*, Mod. Phys. Lett., **A6**: (1991), 3591–3600, hep-th/9108006

[56] J. Mäkelä, *Simplicial Wheeler-DeWitt equation in (2+1)-space-time dimensions*, Phys. Rev., **D48**: (1993), 1679–1686

[57] J. Mäkelä, *On the phase space coordinates and the Hamiltonian constraint of Regge calculus*, Phys. Rev., **D49**: (1994), 2882–2896

[58] F. Conrady, L. Doplicher, R. Oeckl, C. Rovelli, M. Testa, *Minkowski vacuum in background independent quantum gravity*, Phys. Rev., **D69**: (2004), 064019, gr-qc/0307118

[59] L. Doplicher, *Generalized Tomonaga-Schwinger equation from the Hadamard formula*, Phys. Rev., **D70**: (2004), 064037, gr-qc/0405006

[60] D. Colosi, L. Doplicher, W. Fairbairn, L. Modesto, C. Rovelli, K. Noui, *Background independence in a nutshell: the dynamics of a tetrahedron*, gr-qc/0408079

[61] G. W. Johnson, M. L. Lapidus, *The Feynman integral and Feynman's operational calculus*, Oxford University Press (2000)

[62] R. Jackiw, *Analysis on infinite dimensional manifolds: Schrödinger representation for quantized fields*, in: O. Eboli, M. Gomes, A. Santoro (editors), *Field theory and particle physics*, World Scientific (1990), presented at Seminar on Higher Mathematics, Montreal, Canada, Jun 1988

[63] B. Hatfield, *Quantum field theory of point particles and strings*, Addison-Wesley (1992)

[64] M. Testa, lectures

[65] G. C. Rossi, M. Testa, *The structure of Yang-Mills theories in the temporal gauge. 1. General formulation*, Nucl. Phys., **B163**: (1980), 109

[66] G. C. Rossi, M. Testa, *The structure of Yang-Mills theories in the temporal gauge. 2. Perturbation theory*, Nucl. Phys., **B176**: (1980), 477

[67] A. Corichi, J. Cortez, H. Quevedo, *On the relation between Fock and Schroedinger representations for a scalar field*, Ann. Phys., **313**: (2004), 446–478, `hep-th/0202070`

[68] K. Symanzik, Nucl. Phys., **B190 [FS3]**: (1983), 1

[69] M. Lüscher, *Schrödinger representation in quantum field theory*, Nucl. Phys., **B254**: (1985), 52–57

[70] F. Cooper, E. Mottola, *Initial value problems in quantum field theory in the large N approximation*, Phys. Rev., **D36**: (1987), 3114

[71] S.-Y. Pi, M. Samiullah, *Renormalizability of the time dependent variational equations in quantum field theory*, Phys. Rev., **D36**: (1987), 3128

[72] C. J. Isham, in: B. DeWitt, R. Stora (editors), *Relativity, groups and topology, Les Houches Session XL*, North Holland (1984)

[73] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, T. Thiemann, *SU(N) quantum Yang-Mills theory in two dimensions: A complete solution*, J. Math. Phys., **38**: (1997), 5453–5482, `hep-th/9605128`

[74] C. Rovelli, *What is observable in classical and quantum gravity?*, Class. Quant. Grav., **8**: (1991), 297–316

[75] A. Sen, *Gravity as a spin system*, Phys. Lett., **B119**: (1982), 89–91

[76] J. F. Barbero, *Real Ashtekar variables for Lorentzian signature space times*, Phys. Rev., **D51**: (1995), 5507–5510, `gr-qc/9410014`

[77] A. Ashtekar, *Lecture notes on non-perturbative canonical gravity*, World Scientific (2001)

[78] C. Rovelli, *Ashtekar formulation of general relativity and loop space non-perturbative quantum gravity: A Report*, Class. Quant. Grav., **8**: (1991), 1613–1676

[79] G. Immirzi, *Real and complex connections for canonical gravity*, Class. Quant. Grav., **14**: (1997), L177–L181, `gr-qc/9612030`

[80] G. Immirzi, *Quantum gravity and Regge calculus*, Nucl. Phys. Proc. Suppl., **57**: (1997), 65–72, `gr-qc/9701052`

[81] G. Immirzi, *Quantizing Regge calculus*, Class. Quant. Grav., **13**: (1996), 2385–2394, `gr-qc/9512040`

[82] C. Rovelli, T. Thiemann, *The Immirzi parameter in quantum general relativity*, Phys. Rev., **D57**: (1998), 1009–1014, `gr-qc/9705059`

[83] K. V. Krasnov, *On statistical mechanics of gravitational systems*, Gen. Rel. Grav., **30**: (1998), 53–68, `gr-qc/9605047`

[84] C. Rovelli, *Black hole entropy from loop quantum gravity*, Phys. Rev. Lett., **77**: (1996), 3288–3291, `gr-qc/9603063`

[85] C. Rovelli, *Loop quantum gravity and black hole physics*, Helv. Phys. Acta, **69**: (1996), 582–611, `gr-qc/9608032`

[86] A. Corichi, K. V. Krasnov, *Loop quantization of Maxwell theory and electric charge quantization*, `hep-th/9703177`

[87] T. Thiemann, *Anomaly-free formulation of non-perturbative, four- dimensional Lorentzian quantum gravity*, Phys. Lett., **B380**: (1996), 257–264, `gr-qc/9606088`

[88] T. Thiemann, *Quantum spin dynamics (QSD)*, Class. Quant. Grav., **15**: (1998), 839–873, `gr-qc/9606089`

[89] T. Thiemann, *Quantum spin dynamics (QSD) II*, Class. Quant. Grav., **15**: (1998), 875–905, `gr-qc/9606090`

[90] R. Gambini, A. Trias, *On the geometrical origin of gauge theories*, Phys. Rev., **D23**: (1981), 553

[91] R. Gambini, A. Trias, *Gauge dynamics in the C representation*, Nucl. Phys., **B278**: (1986), 436

[92] A. Ashtekar, J. Lewandowski, *Representation theory of analytic holonomy C* algebras*, gr-qc/9311010

[93] A. Ashtekar, J. Lewandowski, *Differential geometry on the space of connections via graphs and projective limits*, J. Geom. Phys., **17**: (1995), 191–230, hep-th/9412073

[94] J. Lewandowski, *Topological measure and graph differential geometry on the quotient space of connections*, Int. J. Mod. Phys., **D3**: (1994), 207–210, gr-qc/9406025

[95] J. C. Baez, *Spin networks in nonperturbative quantum gravity*, gr-qc/9504036

[96] A. Ashtekar, J. Lewandowski, *Quantum theory of geometry. I: Area operators*, Class. Quant. Grav., **14**: (1997), A55–A82, gr-qc/9602046

[97] R. Borissov, R. De Pietri, C. Rovelli, *Matrix elements of Thiemann's Hamiltonian constraint in loop quantum gravity*, Class. Quant. Grav., **14**: (1997), 2793–2823, gr-qc/9703090

[98] L. Smolin, *Recent developments in nonperturbative quantum gravity*, hep-th/9202022

[99] C. Rovelli, L. Smolin, *Discreteness of area and volume in quantum gravity*, Nucl. Phys., **B442**: (1995), 593–622, gr-qc/9411005

[100] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, T. Thiemann, *Quantization of diffeomorphism invariant theories of connections with local degrees of freedom*, J. Math. Phys., **36**: (1995), 6456–6493, gr-qc/9504018

[101] A. Ashtekar, C. Rovelli, L. Smolin, *Weaving a classical geometry with quantum threads*, Phys. Rev. Lett., **69**: (1992), 237–240, hep-th/9203079

[102] R. De Pietri, *On the relation between the connection and the loop representation of quantum gravity*, Class. Quant. Grav., **14**: (1997), 53–70, gr-qc/9605064

[103] J. Lewandowski, *Volume and quantizations*, Class. Quant. Grav., **14**: (1997), 71–76, gr-qc/9602035

[104] D. Marolf, *Quantum observables and recollapsing dynamics*, Class. Quant. Grav., **12**: (1995), 1199–1220, gr-qc/9404053

[105] D. Marolf, *The Spectral analysis inner product for quantum gravity*, gr-qc/9409036

[106] A. Higuchi, *Linearized gravity in de Sitter space-time as a representation of SO(4,1)*, Class. Quant. Grav., **8**: (1991), 2005–2021

[107] N. Grot, C. Rovelli, *Moduli-space structure of knots with intersections*, J. Math. Phys., **37**: (1996), 3014–3021, gr-qc/9604010

[108] J. A. Zapata, *A combinatorial approach to diffeomorphism invariant quantum gauge theories*, J. Math. Phys., **38**: (1997), 5663–5681, gr-qc/9703037

[109] C. Rovelli, *Outline of a generally covariant quantum field theory and a quantum theory of gravity*, J. Math. Phys., **36**: (1995), 6529–6547, gr-qc/9503067

[110] C. Rovelli, L. Smolin, *The Physical Hamiltonian in nonperturbative quantum gravity*, Phys. Rev. Lett., **72**: (1994), 446–449, gr-qc/9308002

[111] M. P. Reisenberger, C. Rovelli, *"Sum over surfaces" form of loop quantum gravity*, Phys. Rev., **D56**: (1997), 3490–3508, gr-qc/9612035

[112] C. Rovelli, *Partial observables*, Phys. Rev., **D65**: (2002), 124013, gr-qc/0110035

[113] C. Rovelli, *A note on the fundation of relativistic quantum mechanics. I: Relativistic observables and relativistic states*, to appear in the proceedings of the 15th SIGRAV Conference on General Relativity and Gravitational Physics, Rome, September 2002, gr-qc/0111037

[114] C. Rovelli, *A note on the fundation of relativistic quantum mechanics. II: Covariant hamiltonian general relativity*, to appear in the proceedings of the Plebanski fest, World Scientific, gr-qc/0202079

[115] C. Rovelli, *Covariant hamiltonian formalism for field theory: Hamilton-Jacobi equation on the space G*, to appear in the proceedings of the conference DICE 2002, Piombino, September 2002, gr-qc/0207043

[116] M. E. Peskin, D. V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley (1995)

[117] H. Lehmann, K. Symanzik, W. Zimmermann, *On the formulation of quantized field theories*, Nuovo Cimento, **1**: (1955), 205–225

[118] G. 't Hooft, *The holographic principle: Opening lecture*, hep-th/0003004

[119] G. 't Hooft, *Dimensional reduction in quantum gravity*, gr-qc/9310026

[120] R. Bousso, *The holographic principle*, Rev. Mod. Phys., **74**: (2002), 825–874, hep-th/0203101

[121] R. Bousso, *A Covariant Entropy Conjecture*, JHEP, **07**: (1999), 004, hep-th/9905177

[122] R. Bousso, *Holography in general space-times*, JHEP, **06**: (1999), 028, hep-th/9906022

[123] R. Bousso, *The holographic principle for general backgrounds*, Class. Quant. Grav., **17**: (2000), 997–1005, hep-th/9911002

[124] L. Susskind, *The world as a hologram*, J. Math. Phys., **36**: (1995), 6377–6396, hep-th/9409089

[125] L. Thorlacius, *Black holes and the holographic principle*, hep-th/0404098

[126] D. Bigatti, L. Susskind, *TASI lectures on the holographic principle*, hep-th/0002044

[127] H. Ooguri, *Gauge theory and string theory: An introduction to the AdS/CFT correspondence*, Nucl. Phys. Proc. Suppl., **83**: (2000), 77–81, hep-lat/9911027

[128] J. M. Maldacena, *TASI 2003 lectures on AdS/CFT*, hep-th/0309246

[129] K.-H. Rehren, *QFT lectures on AdS-CFT*, hep-th/0411086

[130] J. A. Wheeler, *Geometrodynamics and the issue of the final state*, in: C. DeWitt, B. DeWitt (editors), *Relativity, groups and topology*, Gordon and Breach (1964), 315–520

[131] H. Ooguri, *Partition functions and topology changing amplitudes in the 3-D lattice gravity of Ponzano and Regge*, Nucl. Phys., **B382**: (1992), 276–304, hep-th/9112072

[132] B. S. Dewitt, *Quantum theory of gravity. 1. The canonical theory*, Phys. Rev., **160**: (1967), 1113–1148

[133] J. A. Wheeler, *Superspace and the nature of quantum geometrodynamics*, in: C. M. DeWitt, J. A. Wheeler (editors), *Batelle Rencontres*, W. A. Benjamin (1968), 242–307

[134] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys., **121**: (1989), 351

[135] A. V. Stoyanovsky, *Generalized Schrödinger equation and constructions of quantum field theory*, math-ph/0301037

[136] A. Perez, *Finiteness of a spinfoam model for Euclidean quantum general relativity*, Nucl. Phys., **B599**: (2001), 427–434, gr-qc/0011058

[137] L. Crane, A. Perez, C. Rovelli, *A finiteness proof for the Lorentzian state sum spinfoam model for quantum general relativity*, gr-qc/0104057

[138] F. Conrady, *Free vacuum for loop quantum gravity*, gr-qc/0409036

[139] R. De Pietri, L. Freidel, K. Krasnov, C. Rovelli, *Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space*, Nucl. Phys., **B574**: (2000), 785–806, hep-th/9907154

[140] M. P. Reisenberger, C. Rovelli, *Spacetime as a Feynman diagram: The connection formulation*, Class. Quant. Grav., **18**: (2001), 121–140, gr-qc/0002095

[141] M. Reisenberger, C. Rovelli, *Spin foams as Feynman diagrams*, gr-qc/0002083

[142] A. Perez, C. Rovelli, *A spin foam model without bubble divergences*, Nucl. Phys., **B599**: (2001), 255–282, gr-qc/0006107

[143] C. G. Torre, M. Varadarajan, *Functional evolution of free quantum fields*, Class. Quant. Grav., **16**: (1999), 2651–2668, hep-th/9811222

[144] F. Conrady, C. Rovelli, *Generalized Schrödinger equation in Euclidean field theory*, hep-th/0310246

[145] V. Volterra, *Theory of functionals and of integral and integro-differential equations*, Dover (1959)

[146] W. R. Hamilton, *On the application to dynamics of a general mathematical method previously applied to optics*, British Association Report, 513–518

[147] M. O'Loughlin, *Boundary actions in Ponzano-Regge discretization, quantum groups and AdS(3)*, prepared for 9th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories (MG 9), Rome, Italy, 2-9 Jul 2000

[148] S. W. Hawking, C. J. Hunter, *The Gravitational Hamiltonian in the Presence of Non- Orthogonal Boundaries*, Class. Quant. Grav., **13**: (1996), 2735–2752, gr-qc/9603050

[149] R. Arnowitt, S. Deser, C. W. Misner, *The dynamics of general relativity*, in: L. Witten (editor), *Gravitation: an introduction to current research*, Wiley (1962), 227–265, gr-qc/0405109

[150] C. Rovelli, *The basis of the Ponzano-Regge-Turaev-Viro-Ooguri quantum gravity model in the loop representation basis*, Phys. Rev., **D48**: (1993), 2702–2707, hep-th/9304164

[151] G. Ponzano, T. Regge, *Semiclassical limit of Racah coefficients*, in: F. Bloch (editor), *Spectroscopic and group theoretical methods in physics*, North-Holland Publ. Co., Amsterdam (1968)

[152] C. Rovelli, *Time in quantum gravity: a hypothesis*, Phys. Rev., **D43**: (1991), 442456

Thanks to...

To my advisors, Carlo Rovelli and Massimo Testa, for their generous and constant help and encouragement

To the Physics Department in Marseille, for hospitality in March-May 2004

To my parents, for support, love and understanding

To the relativity lunch group in Marseille, for making me feel at home there

To Giovanni Amelino-Camelia, for always providing help and advice

To Robert Oeckl and Daniele Oriti, for last-minute advice

A kiss to Slava

A hug to all my friends in Rome, within and without the Physics Department