Asymmetrically gauged coset theories and symmetry breaking D-branes

New boundary conditions in conformal field theory

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Zusammenfassung

Auf sehr kleinen Längenskalen erlaubt die Weltflächenbeschreibung über zweidimensionale konforme Feldtheorien eine störungstheoretische Definition der String-Theorie. Viele strukturelle Eigenschaften und phänomenologische Implikationen der letzteren können mit Hilfe von D(irichlet)-Branen untersucht werden, die in der zugrunde liegenden Weltflächentheorie durch konforme Randbedingungen beschrieben werden.

Etliche interessante Hintergründe für die String-Theorie erhält man über Gruppenmannigfaltigkeiten und Coset-Modelle. Neben wichtigen Beispielen wie $SL(2,\mathbb{R})$, $SU(2)$ und Gepner-Modellen, die für AdS- und Calabi-Yau-Kompaktifizierungen eine Rolle spielen, beinhalten sie außerdem weitere Beispiele wie den Nappi-Witten-Hintergrund oder den Raum T^{11}, die über eine asymmetrische Wirkung der Eichgruppe definiert sind und eine kosmologische Raumzeit mit Urknall- und Weltsturz-Singularitäten bzw. die Basis des Conifolds beschreiben.

Die vorliegende Arbeit bietet eine umfassende, auf den exakten Methoden der konformen Feldtheorie beruhende Analyse von asymmetrischen Coset-Modellen. Wegen der heterotischen Natur der zugrundeliegenden Symmetriealgebra erlauben diese Modelle nur Randbedingungen, die einen Teil der Symmetrie brechen. Nach einer allgemeinen Erläuterung der Grundidee für die Konstruktion von symmetriebrechenden Randbedingungen richtet sich das Hauptaugenmerk auf WZNW- und asymmetrische Coset-Modelle, die das Fundament nahezu aller bekannten konformen Feldtheorien bilden.

Mit Hilfe der erzielten Ergebnisse werden die Struktur sowie die Geometrie von D-Branen in den Gruppen $SL(2,\mathbb{R})$ und $SU(2)$, im Hintergrund $AdS_3 \times S^3$, in der kosmologischen Nappi-Witten-Raumzeit und in T^{pq}-Räumen untersucht. Die Techniken, die in dieser Arbeit entwickelt werden, erlauben jedoch ebenso die Behandlung von Rändern und Kontaktstellen in (1+1)- oder 2-dimensionalen kritischen Systemen, die in der Festkörpertheorie oder der statistischen Physik auftreten. Insbesondere können Defektlinien beschrieben werden, die weder totale Reflexion noch völlige Transmission aufweisen.

Schlagwörter:
String-Theorie, D-Branen, Konforme Feldtheorie, Ränder und Defekte

Abstract

At very small length scales, the world sheet approach in terms of two-dimensional conformal field theories provides a perturbative definition of string theory. Many structural properties and phenomenological implications of the latter can be explored using D(irichlet)-branes which may be identified with conformal boundary conditions in the underlying world sheet theory.

Several interesting backgrounds in string theory arise from group manifolds and coset theories. Apart from prominent examples such as $SL(2,\mathbb{R})$, $SU(2)$ and Gepner models which play a role in AdS and Calabi-Yau compactifications, they also include further instances like the Nappi-Witten background or the space T^{11} which are constructed using an asymmetric action of the gauge group and which describe a cosmological space-time with big-bang and big-crunch singularities and the base of the conifold, respectively.

The present thesis provides a comprehensive analysis of asymmetric cosets based on the exact methods of boundary conformal field theory. Due to the heterotic nature of the underlying symmetry algebra, the models only allow for conformal boundary conditions which break parts of the bulk symmetry. The universal ideas for the construction of symmetry breaking boundary conditions are indicated and applied in detail to WZNW and asymmetric coset theories which provide the basic building blocks of almost all known conformal field theories.

The general results are used to investigate the structure and shape of D-branes in the group manifolds $SL(2,\mathbb{R})$ and $SU(2)$, the background $AdS_3 \times S^3$, the cosmological Nappi-Witten space-time and T^{pq}-spaces. The techniques developed in this thesis also allow for a treatment of boundaries and junctions in (1+1)- or 2-dimensional critical systems in condensed matter theory and statistical physics. In particular, they enable us to describe defect lines which go beyond full reflection or transmission.

Keywords:
String theory, D-branes, Conformal field theory, Boundaries and defects

Gewidmet meinen Eltern in Dankbarkeit

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Chapter 1

Introduction

The two fundamental theories of space-time and matter, Einstein's General Theory of Relativity and the Standard Model of elementary particles, give a very accurate description of almost all accessible physical phenomena. Nevertheless there still exist severe conceptual and physical issues in this context which are far from being solved. While the dynamics of matter can be formulated in terms of a full quantum (field) theory, the gravity part for instance resisted all attempts of finding a similar quantum description up to now.

Many solutions of classical gravity predict regions of space-time in which the curvature becomes singular. In these strongly curved regions, however, quantum fluctuations of the metric are certainly not negligible. In particular, one cannot expect to obtain an accurate description of the universe close to the big-bang or in the vicinity of black hole singularities without an appropriate quantum theory of gravity. One may even doubt whether the usual notion of space-time is a valid concept there.

String theory possesses several appealing features which make it a natural candidate for a quantum unified theory of all interactions, including gravity. First of all, a consistent formulation of string theory automatically requires a background whose metric satisfies Einstein's equations of general relativity in the limit of high string tension. Quantum fluctuations of the metric can then be described in terms of scattering of gravitons which naturally arise as massless excitations of the string.

Beside the graviton a string possesses an infinite number of additional excitations which can be interpreted as different kinds of particles. A finite number of them corresponds to massless degrees of freedom which are associated to gauge symmetries. According to a widely accepted assumption, the more massive excitations do not contribute to an effective low energy description of the string because the gap between two energy levels is of order of the

Planck scale. One may thus hope to recover the spectrum of the standard model and to arrive at the familiar description of matter in a limiting regime where the strings become small compared to all observable length scales. As a pleasant side-effect the extended geometry of strings smoothens out certain singularities which arise in Feynman diagrams of point-particles.

Despite all the promising features of string theory there are also a number of drawbacks one has to be aware of. First of all, one usually considers a supersymmetric version of the string because no stable vacuum of the purely bosonic string has been found until today. Yet up to now, the existence of supersymmetry in nature could not be established in collider experiments. Secondly, a consistent quantization of the superstring requires it to be embedded in a ten-dimensional target space. Since we are obviously living in four-dimensional space-time, one has to explain how to get rid of the remaining dimensions.

This problem can be solved in principle by compactifying the additional dimensions on some "internal" manifold which is too small to be resolvable at standard energies. Up to now, however, there seems to exist no universal guiding principle (except preservation of supersymmetry) which constrains the potential shape of the compactification manifold. One thus encounters a problem similar to the fine tuning problem in the standard model.

Until today, no one has been able to recover exactly the spectrum of the standard model within string theory. The most promising candidates for realistic string compactifications arise from brane worlds [1, 2]. In these scenarios the visible particle spectrum and their non-abelian gauge symmetries are associated to massless excitations of open strings whose end points are bound to move on so-called D(irichlet)-branes. These D-branes are non-perturbative dynamical objects which are embedded in the target space and which may carry charges with respect to the background fields. In the low-energy approximation of string theory they correspond to solitonic solutions of (super)gravity [3].

Apart from the natural appearance of non-abelian gauge theories, D-branes also play a fundamental role in the conceptual understanding of string theory. They constitute one of the most important tools to explore the web of dualities which exists between different types of string theories [4, 5]. Other applications include a microscopic derivation of the entropy of extremal black holes [6] and different kinds of dualities between string theory and ordinary gauge theory [7]. Recently, the open/closed string duality has found renewed interest in this context [8, 9].

The internal structure of space-time may lead to many new interesting phenomena since D-branes and strings – in contrast to point-particles – may wrap non-trivial cycles for example. Also, due to their extended structure strings know about the local curvature, i.e. about holonomy. Hence, it is not particularly surprising that the excitations and symmetries of string theory contain a lot of information about geometry.

String theory in a weakly curved geometrical background can be formulated in terms of non-linear σ-model perturbation theory. In this kind of models one deals with coordinate fields which describe the embedding of the two-dimensional string world-sheet into the target space. Consistency requires *conformal invariance* of the two-dimensional world-sheet theory.

As soon as the geometry varies perceptible over distances of a string length, stringy effects become important and classical geometry starts to loose its meaning. In such backgrounds it is convenient to change the point of view and to rather *define* (perturbative) string theory in terms of an *abstract* conformal field theory (CFT). In this picture, geometrical notions are not fundamental anymore but become derived concepts which are based on the underlying world-sheet theory.

This picture has been rather successful in the past. It is well known for instance that many geometric features such as holonomy and cohomology may be encoded in the structure of a conformal field theory [10, 11, 12]. Recent results also indicate a deep connection between the dynamics of space-time on the one hand and renormalization group (RG) flows between CFT's on the other. This relation seems to be well-established for the open string sector. One was indeed able to relate the tension of a D-brane to a boundary entropy in the associated CFT description [13]. While the first quantity decreases in physical decays, the second was conjectured to decrease under RG flows [14]. These ideas led to a successful study of Ramond-Ramond (RR) charges of D-branes using boundary renormalization group flows [15, 16]. Although a similar picture for the background geometry, i.e. a bulk CFT, is far from being settled there has also been some progress in this context [17]. Last but not least, the abstract approach is very suitable for a background independent formulation of (open) string theory [18, 19, 20, 21].

The number of known exactly solvable curved string backgrounds is very limited. String backgrounds which are very well understood from both the geometric and the abstract point of view are group manifolds [22]. The associated CFT's are given by Wess-Zumino-Novikov-Witten (WZNW) theories [23, 24] which possess an affine Kac-Moody algebra symmetry (see, e.g., [25]). The closed string sector can be reconstructed from the harmonic analysis on the group after a natural truncation of the resulting spectrum, showing again the close relation between CFT and (quantum) geometry.

The investigation of D-branes in group manifolds was initiated in [26, 27, 28, 29]. These authors constructed a whole class of branes – nowadays called maximally symmetric – which wrap quantized twisted conjugacy classes. The quantization which naturally arises from the algebraic approach was later shown to correspond to a mechanism of flux stabilization [30, 31]. Since then, there has been remarkable progress in the understanding of maximally symmetric branes on group manifolds. First, it was realized that these D-branes possess a fuzzy world-volume which may be described in terms of some non-commutative algebra [32]. This observation could be used in turn to model the dynamics of the branes in the large volume limit using a non-commutative gauge theory [33, 34]. Finally, the conserved charges associated with these dynamics could be compared successfully to the twisted K-theory of the group manifold [15, 16, 35, 36].

Unfortunately, the number of group manifolds which can directly enter a consistent superstring background is very limited. The dimension of group manifolds rapidly exceeds the crucial threshold of ten total or six compact space-time dimensions in string theory. One is essentially left with the groups $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SU}(2)$ and powers of $\mathrm{U}(1)$ which describe AdS_3-spaces, the sphere S^3 and tori T^n, respectively. These examples arise in the context of the AdS_3-type solutions of supergravity (see, e.g., [37, 38, 39]), in the near-horizon limit of NS5-branes [40] and in the simplest Calabi-Yau compactifications.

The dimension of the target space may be reduced by gauging a continuous symmetry of a WZNW model [41, 42, 43, 44, 45, 46]. The resulting theory describes strings on coset spaces, at least in the geometric regime. With the GKO construction [47] one also possesses a completely algebraic tool to introduce coset models deep in the stringy regime. A detailed understanding of coset models seems to be of fundamental importance since almost all known conformal field theories can be formulated in terms of gauged WZNW models or products and/or orbifolds of them.

Coset models appear indeed almost everywhere in string theory. The $N = 2$ superconformal minimal models are particularly interesting examples. They have been used by Gepner to model string theory at a special point in the moduli space of Calabi-Yau compactifications [10, 48]. A more general construction which also leads to the $N = 2$ superconformal algebra was proposed by Kazama and Suzuki [49]. Similar considerations resulted in models which describe string theory on manifolds with G_2 or $\mathrm{Spin}(7)$ special holonomy [50].

The common definition of a coset theory employs the adjoint action to gauge the subgroup. Geometrically, these cosets are plagued with all kinds of unpleasant features like singularities, boundaries and corners. These pathologies may be circumvented in principle by using an asymmetric, i.e. non-adjoint, action of the subgroup when performing the gauging.

The Lagrangian description of asymmetrically gauged coset theories has been around in the literature for quite a long time [51, 52]. Also, one knows about a number of interesting geometrical examples and applications of these models. The Nappi-Witten geometry for example describes a time-dependent cosmological background with big-bang and big-crunch singularities [53]. Another prominent example is provided by the T^{pq} spaces. For special values of the parameters p, q these non-Einstein spaces constitute a relative of the base of the conifold where RR-fluxes are replaced by NSNS-fluxes [54]. The conifold geometry plays an important role in recent investigations of open / closed string and matrix model / gauge theory dualities [8, 9].

The aim of the present thesis is to develop a systematic and comprehensive description of asymmetric cosets. The main focus is put on the algebraic formulation in terms of abstract conformal field theory. A detailed understanding of the latter is required in order to be able to explore the theory deep in the stringy regime. The closed string sector turns out to be described by heterotic CFT's which possess different left and right moving symmetries. This observation enforces a non-trivial and wide-reaching generalization of the concepts which are used to describe ordinary adjoint cosets [55].

The heterotic nature of asymmetric cosets drastically complicates the construction of D-branes in these backgrounds. The description in terms of boundary conformal field theory demands to impose certain boundary conditions relating the left and right moving degrees of freedom. In asymmetric cosets, left and right moving sectors possess different symmetries and, as a consequence, there exists no natural way to glue the currents. In fact, the unique way out is to consider symmetry breaking D-branes in which only a subset of the currents participate in the determination of the boundary condition [55].

From the geometric perspective the construction of D-branes in asymmetric cosets is not very hard, at least superficially. One simply has to find D-branes on the underlying group manifold which may be consistently projected down to the coset. It is, however, not difficult to find examples where this simple procedure turns out to be impossible for all the *maximally* symmetric branes on the group. In other cases one only obtains a small subset of the expected branes in this way.

The observations of the last two paragraphs reveal the necessity of a general theory of symmetry breaking boundary conditions in conformal field theory. After introducing the basic ideas, the whole program is worked out in detail for WZNW theories [56]. If the target space is given by the group manifold G, it is natural to preserve the action of a subgroup H on the boundary. By iterating this idea one arrives at embedding chains $\mathrm{H} \hookrightarrow \cdots \hookrightarrow \mathrm{H}_1 \hookrightarrow \mathrm{G}$ from which a whole hierarchy of branes can be obtained. The resulting D-branes are analyzed both from the algebraic and the Lagrangian point of view. They are shown to wrap products of twisted conjugacy classes, one for each of the groups which constitute the embedding chain [57]. Under weak assumptions this description even allows the construction of space-filling branes. Finally, the non-commutative world-volume algebra of the branes in the large volume limit is obtained from their spectrum of open strings.

Most of the insights which have been obtained for WZNW models carry over to asymmetric coset theories in a natural way [55]. To formulate the gluing conditions for these models one has to reduce the different symmetries of left and right moving degrees of freedom to a common subsymmetry. If one breaks the symmetry down to the Virasoro algebra, the boundary theory will be non-rational and thus almost impossible to solve.

Yet, in many examples one is able to find an intermediate symmetry with respect to which the theory remains rational. A whole class of such theories is provided by cosets of so-called generalized automorphism type. These cosets are based on embedding chains as they appeared in the construction of symmetry breaking D-branes on group manifolds. They are distinguished by the property that left and right action of the subgroup just differ by automorphisms in the intermediate groups. It is thus not surprising that the geometry of the associated D-branes is also inherited from the group case.

The general scheme is applied to the most important examples of asymmetric coset theories which arise in the context of string theory. As a warming up, non-factorizing and symmetry breaking D-branes are constructed in the group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ which underlies the background $\mathrm{AdS}_3 \times S^3$ [38, 39]. Some of these branes may be projected down to the cosmological Nappi-Witten space-time where one recovers D-branes which are capable of passing a big-bang big-crunch singularity and connecting two of the universes in this way. Using similar arguments, one is also able to identify branes which wrap a sphere S^3 inside the base of the conifold.

Finally, it should be emphasized that symmetry breaking boundary conditions in conformal field theories are also interesting beyond the area of string theory. There exists indeed a widely accepted conjecture that physical systems may be approximated by conformal field theories in the vicinity of second order phase transitions [58]. The origin of this proposal is basically the scale invariance which follows from the divergence of the correlation length of fluctuations. The excitations and the critical exponents of physical quantities may then be encoded in the spectrum of some conformal field theory.

The conformal invariance in turn gives strong constraints on the structure of the correlation functions, especially in two dimensions where the associated symmetry algebra is infinite dimensional.

Boundary conditions enter when one intends to describe critical systems with boundaries or impurities. These type of models may also be used to study diffusion, dissipation and percolation for instance [59, 60, 61, 62]. Among the vast number of models which could be described by methods of (boundary) CFT are the Ising model [63], the Kondo effect (see [64] and references therein) and quantum wires [65]. Last but not least, also a system of two CFT's which are separated by a defect line may be reformulated in terms of a boundary theory by folding along the defect.

Defect lines are particularly interesting if they allow the exchange of particles etc. between the two adjacent systems. This situation can, however, not be achieved within the framework of maximally symmetric boundary conditions. The latter correspond either to full reflection or (only if the systems are identical) to full transmission. Defect lines which are at the same time partially reflecting and partially transmissive can only be constructed if one uses symmetry breaking boundary conditions [56]. In the presence of several defects or boundaries one may hope to use the CFT description to determine the Casimir forces between them. Note that defect lines also arise naturally in the AdS/CFT-correspondence [66, 67, 68, 69].

This thesis is organized as follows. The second chapter gives a short account of string theory and critical phenomena. The focus is put on the description of D-branes and systems with boundaries or impurities, respectively. It is indicated how their physical properties can be reformulated in terms of abstract conformal field theory. Afterwards we focus on the universal nature of CFT whose structures and properties are reviewed in detail. In the third chapter the general idea of constructing symmetry breaking boundary conditions in CFT's is introduced and discussed at length for the special case of WZNW theories. It provides a comprehensive description of both the algebraic and the geometric point of view. The construction of asymmetric cosets takes place in chapter four. The presentation includes both the stringy as well as the geometric regime. The boundary theory is solved for asymmetric cosets of generalized automorphism type and the shape of the D-branes is derived. Finally, the fifth chapter concludes with the application of the general results to concrete examples of string backgrounds and defect systems. Additional material and some rather technical derivations of several results which are just cited in the main text have been postponed to the appendices.

In view of the various distinct applications the presentation of our results will emphasize the universal character of the constructions. The specific language of the concrete applications is only employed where appropriate.

* * *

The present doctoral thesis is based on the publications [70, 71, 56, 34, 57, 55]. Section 2.4.3 and appendix B also contain material on permutation branes that has not been published before in this explicit form.

Chapter 2

Conformal field theory

The following chapter provides background material on two-dimensional conformal field theory and its appearance in physics. It starts with a short presentation of the two main applications, the string theory approach to a quantum unified theory of all interactions and critical phenomena in statistical systems. Particular focus is laid on the description of D-branes, impurities and defect lines which necessitate the introduction of boundaries. Afterwards a more abstract point of view is taken up and the universal structures and principles of (boundary) conformal field theory are reviewed. We also use the opportunity to introduce Wess-Zumino-Novikov-Witten models, coset theories and orbifolds.

2.1 Physical applications

2.1.1 String theory and D-branes

In string theory, the elementary constituents of matter are not point-like particles but one-dimensionally extended strings. The latter admit all kinds of vibration modes which may propagate along them. With energies that are available at present colliders we are not able to resolve the inner structure of strings. The excitations rather appear to be particles in a good approximation and the stringy nature of matter only shows up in a modification of the high energy behavior and in connection with topological degrees of freedom such as winding modes.

String dynamics

The dynamics of a string can be described by a two-dimensional non-linear σ-model. In this approach, the world-sheet Σ of the string is embedded into

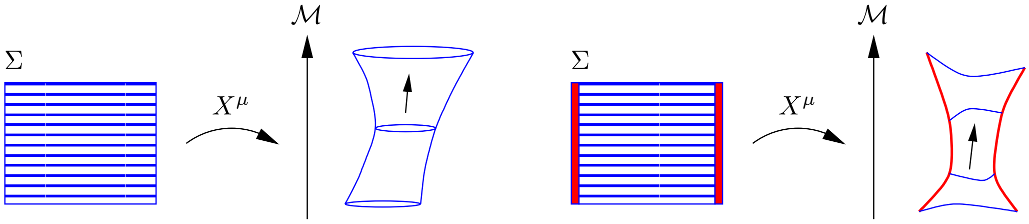

Figure 2.1: The non-linear σ-model for closed and open strings.

the target space $\mathcal{M}$ by coordinate fields $X^\mu : \Sigma \to \mathcal{M}$. The natural action functional of a string is the area it sweeps out in space-time. But as this so-called Nambu-Goto action contains square roots it is very hard to quantize. For this reason one usually considers an action functional which is equivalent to the previous one on the classical level but much easier to access. This so-called Polyakov action reads

$$\mathcal{S}(X,\gamma) \;=\; -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\Big[\sqrt{-\gamma}\,\Big(G_{\mu\nu}(X)\,\gamma^{ab}+B_{\mu\nu}(X)\,\epsilon^{ab}\Big)\partial_a X^\mu \partial_b X^\nu\Big] \quad .$$

The constant $1/\alpha'$ which normalizes the action is a measure for the string tension. We only wrote down the dependence on the world-sheet metric γ and on the coordinate functions X^μ. All fermionic degrees of freedom and the dilaton have been omitted. With these simplifications the action functional depends on two background fields, the metric $G_{\mu\nu}$ and a two-form field $B_{\mu\nu}$.

The Polyakov action is invariant under a number of symmetries which shall be preserved during the process of quantization. Apart from target space symmetries the most important are general diffeomorphism and Weyl invariance of the world-sheet. The latter corresponds to a local rescaling $\gamma^{ab} \mapsto \Omega(\sigma)\,\gamma^{ab}$ of the metric. Demanding this kind of conformal invariance in the quantum theory leads to strong restrictions for the background fields. In a scale invariant theory the β-functions for all the couplings have to vanish. Expressing the β-functions perturbatively as functions of the background fields $G_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$ thus leads to certain consistency conditions. To lowest order in α' one arrives at the equation

$$4\,R_{\mu\nu} - H_{\mu\rho\sigma}\,H_\nu{}^{\rho\sigma} \;=\; 0 \quad . \tag{2.1}$$

If the three-form field $H = \mathrm{d}B$ vanishes identically, we just recover Einstein's equations in the absence of matter. Higher orders then provide stringy corrections to classical general relativity.

Open strings enter the game if one allows for world-sheets Σ with boundaries. In the σ-model formulation one has to fix the boundary conditions of the coordinate fields X^μ such that energy-momentum is prevented from flowing across the end of the string. This condition is equivalent to the preservation of conformal invariance of the boundary theory. The boundary conditions restrict the regions which the end points of the open string can sweep out to certain subsets of the target space $\mathcal{M}$. The latter constitute the world-volumes of two D(irichlet)-branes.

An illustrative example: Flat Minkowski space

The σ-model above may easily be quantized in flat Minkowski space. In this background it just corresponds to an ensemble of massless free bosons. In addition one has to include ghosts in order to remove the negative norm states which appear due to the indefinite signature of the Minkowski metric. Eventually, one would also like to introduce free fermions (plus ghosts) if the final theory should obey supersymmetry. As it turns out, conformal invariance at the quantum level is only preserved in $D = 26$ or $D = 10$ space-time dimensions for the bosonic string or its supersymmetric analogue, respectively. These two conditions are equivalent to the vanishing of the total central charge c in the associated CFT.

The quantized string possesses an infinite tower of excitations corresponding to different oscillation modes which organize themselves in an equidistant energy spectrum. The difference between two energy levels scales with the string tension $1/\alpha'$ and is usually assumed to be of the order of the Planck mass. If this proposal is correct, we will even in the largest colliders only be able to observe the "massless" modes.

For the bosonic string, the lowest lying excitation has negative mass squared. The occurrence of this tachyon is usually interpreted as an instability of the whole theory. In contrast to this observation, the supersymmetric analogue is stable since the tachyon can be consistently projected out by means of the GSO-projection [72]. In this case the lowest excitations are found in the massless sector. The excitations with vanishing mass may be decomposed into a symmetric tensor, an antisymmetric tensor and a scalar (the trace of the tensor) which transform independently under Lorentz transformations. They may be interpreted as a graviton, a two-form field and a dilaton, respectively, and describe quantum fluctuations of the classical background.

The Hilbert space $\mathcal{H}$ of closed string excitations may be decomposed into representations of the underlying world-sheet symmetry algebra. In the present case we have an extended symmetry which enhances the conformal Virasoro algebra. Let us forget about fermions and ghosts and only consider the free boson part for a moment. The fourier modes α^μ_m and $\bar\alpha^\nu_n$ of the chiral derivative fields $\partial X^\mu(z) = \sum \alpha^\mu_m \, z^{-m-1}$ and $\bar\partial X^\nu(\bar z) = \sum \bar\alpha^\nu_n \, \bar z^{-n-1}$ generate several commuting copies of the Heisenberg (or $\widehat{\mathfrak{u}}(1)$) algebra. The conformal symmetry is built from the energy momentum tensor which is given by $\mathrm{T}(z) \sim \partial X^\mu \partial X_\mu$ and its antiholomorphic counterpart. The fourier modes L_n are normal ordered bilinears of the modes α^μ_m. The operator L_0 determines the mass of the string excitations.

The content of the Hilbert space $\mathcal{H}$ and the distribution of mass among the excitations may conveniently represented in terms of the closed string partition function

$$Z(q,\bar q) \;=\; \mathrm{tr}_{\mathcal{H}} \left[\, q^{L_0 - 1/24} \, \bar q^{\bar L_0 - 1/24} \right] \;\; .$$

It depends on one complex quantity τ, $\mathrm{Im}\,\tau > 0$, which enters through $q = e^{2\pi i \tau}$ and describes the modulus of a torus. The partition function can be further simplified if one decomposes $\mathcal{H}$ into irreducible representations $\mathcal{H}_k$ and introduces the characters $\chi_k(q) = \mathrm{tr}_{\mathcal{H}_k} q^{L_0 - 1/24}$ which describe the tower of excitations over a (chiral) highest weight state $|k\rangle$. The final result is then given by $Z(q,\bar q) \sim \int \mathrm{d}^D k \; \chi_k(q) \, \bar\chi_k(\bar q)$. It turns out to be modular invariant, i.e. invariant under the transformations $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$.

The label $k = (k^\mu)$ of the lowest weight state $|k,k\rangle$ corresponds to the (rescaled) eigenvalues of the zero-modes $\alpha^\mu_0 = \bar\alpha^\mu_0$. It can be interpreted as the space-time momentum of the excitations generated from the tachyon state $|k,k\rangle$. The latter is created out of the "vacuum" state $|0,0\rangle$ by means of the vertex operators

$$V_k(z,\bar z) \;=\, :e^{ikX(z,\bar z)}: \,=\; V_k(z) \otimes \bar V_k(\bar z) \;\; .$$

Since the state is lowest weight, it is annihilated by all modes $\alpha^\mu_m, \bar\alpha^\nu_n$ with $n > 0$. The higher excitations in the representation are generated by the negative modes and correspond to other types of particles.

In order to obtain physical predictions one has to calculate scattering amplitudes. All information about the scattering of string excitations is contained in the operator product expansion

$$V_k(z,\bar z) \, V_p(w,\bar w) \;=\; |z-w|^{\alpha' kp} \; V_{k+p}(w,\bar w) \;+\; \text{less singular} \;\; .$$

In the limit $\alpha' \to 0$ the conformal dimensions of the fields $V_k(z,\bar z)$ vanish and so does the dependence on the world-sheet coordinate in their OPE. As a consequence the OPE seems to reduce to the ordinary product of exponential functions. This observation indicates a close relation to classical geometry

[73, 74]. The exponential functions form indeed a complete orthonormal set of eigenfunctions of the Laplace operator on $\mathcal{M}$.

For the description of D-branes one has to find suitable boundary conditions for the fields X^μ. In the flat Minkowski background with vanishing B-field the latter can either be Neumann or Dirichlet. After appropriate choice of the coordinate system, the conditions read

$$\begin{aligned} \partial_n X^\mu &= 0 \qquad\qquad \text{for } \mu = 0, \cdots, 10-p \\ \partial_t X^\nu &= 0 \qquad\qquad \text{for } \nu = 10-p+1, \ldots, 10 \quad , \end{aligned}$$

where the symbols ∂_n and ∂_t denote derivatives normal and tangent to the boundary $\partial\Sigma$. Similar relations are obtained for the fermions. The interpretation of these conditions is quite simple. While the string end-points are free to move in the Neumann directions $\mu = 0, \ldots, 10-p$, they are bound to a specific point X_0^ν in the Dirichlet directions $\nu = 10-p+1, \cdots, 10$. All in all, the end-points of the string are confined to a p-dimensional submanifold of Minkowski space, a Dp-brane. In the low-energy approximation of string theory it corresponds to a solitonic solution of supergravity [3].

The boundary conditions for D-branes in flat Minkowski space in the presence of a constant non-vanishing B-field are modified. They interpolate smoothly between Dirichlet and Neumann. The effect of the B-field can be interpreted as introducing a non-commutative world-volume of the brane [75, 76]. Under these conditions the dynamics of D-branes in the decoupling limit may effectively be described by a non-commutative gauge theory [77]. Similar observations are expected for all backgrounds with non-vanishing B-field. Note, that the latter is automatically forced to be non-zero by the consistency condition (2.1) if the Ricci curvature of the target space does not vanish.

Non-trivial exact string backgrounds

In strongly curved backgrounds the σ-model perturbation theory in α' breaks down. In this case it is convenient to replace the geometrical model by an *abstract* conformal field theory. The latter has to satisfy a number of consistency conditions, which arise from physical grounds. Even though most of them have already been encountered in the example of Minkowksi space we will summarize them again in order to illuminate the general structure of string theory.

First of all, the central charge of the underlying Virasoro algebra has to vanish, which is equivalent to removing the conformal anomaly. Since the theory has to contain a ghost system of central charge $c_{\text{gh.}} = -15$, one is left

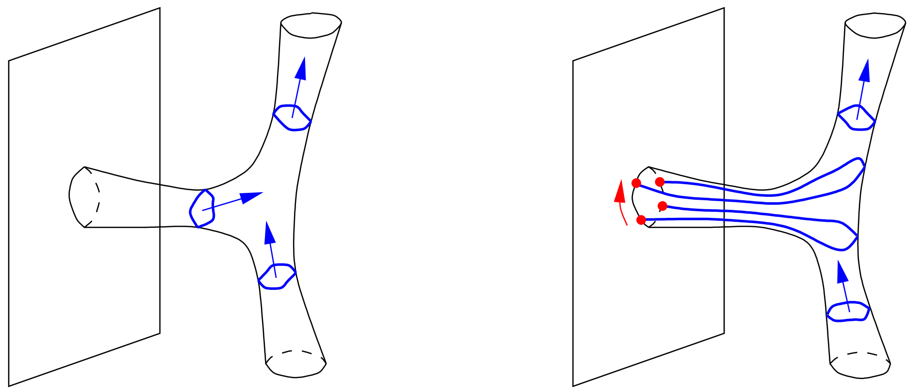

Figure 2.2: World-sheet duality.

with $c = 15$ for the matter part. If one further assumes the existence of four flat observable dimensions, the internal compactification space is described by a $c = 9$ abstract conformal field theory.

A second constraint arises from demanding target space supersymmetry in the uncompactified dimensions. This can be achieved by imposing $N = 2$ or $N = 1$ superconformal invariance of the world-sheet theory [10, 11]. In the geometric limit, the existence of supersymmetry is closely related to special holonomy properties of the target space or, equivalently, to the question of the existence of covariantly constant spinors.

All internal excitations of a closed string can be considered to be built out of elementary excitations which are completely chiral, i.e. which run only in one of the two directions along the string. They organize themselves with respect to some symmetry algebra which at least contains the Virasoro algebra generating conformal transformations. From fundamental physical principles such as locality, however, one obtains strong restrictions on the possibilities of combining left with right movers. As a special case one is led to the constraint of modular invariance which ensures the correct behavior under large world-sheet coordinate transformations. In exact string backgrounds which admit a geometric limit, the partition function and the content of vertex operators it describes are closely related to the algebra of functions on the target space.

Open strings which stretch between two D-branes possess excitations similar to the closed string. The end-points, however, constitute a natural defect where left and right movers are reflected into each other. For obvious reasons the spectrum of excitations of the open string depends crucially on the compatibility with this reflection. The latter may be formulated in terms of so-called gluing conditions of the chiral fields in a boundary conformal field theory (BCFT). The tension of a D-brane was argued to be given in terms of the boundary entropy of the associated boundary condition [78, 13].

It is a remarkable fact that any diagram which involves the emission or absorption of closed strings at a D-brane may be interpreted as an open string diagram (see figure 2.2). This so-called world-sheet duality facilitates the determination of the spectrum of open strings which can stretch between two D-branes because it can essentially be reduced to a calculation in the closed string sector. Details can be found in section 2.3 below.

2.1.2 Critical phenomena

Apart from string theory, the methods of conformal field theory possess immense applications in different areas such as condensed matter physics and statistical mechanics. More generally, these methods apply for models which show critical behavior such as percolation, dissipative quantum mechanics, diffusion limited aggregation and others.

Usually, physical systems with short range interactions have an intrinsic length scale such as the average distance between atoms or molecules. Correlations between physical quantities at different locations decay exponentially and vanish after a multiple of the correlation length. During a second order phase transition, however, the correlation length of the order parameter diverges. Hence, there does not exist a natural length scale anymore. In a very good approximation, the system can be treated as a scale invariant continuous field theory. Moreover, one even expects the system at the critical point to be described by a theory which is conformal [58]. It should be emphasized that (local) conformal invariance is much more restrictive than (global) scale invariance. Only in two dimensions it is already implied by the latter if one assumes the existence of a conserved energy momentum tensor. The continuum field theory which can be used to describe the critical point does not depend on the details of the interaction but only on the dimension, the symmetry and the range of the interactions. This follows from the hypothesis of universality.

Phase transitions show up in a characteristical singular behavior of physical quantities such as the specific heat, conductance, magnetization etc. near the critical point. In general, they follow power laws similar to the expression for the correlation length, which behaves like $\xi \sim (T - T_c)^{-\nu}$ in the vicinity of the critical temperature T_c. The quantity ν is called a critical exponent. The critical exponents of all the physical quantities can be extracted from the conformal dimensions of the fields in the CFT. Both the conformal dimensions and the possible form of the correlation functions are highly constrained by conformal symmetry [63, 79, 80].

The most important quantity of a statistical system is its partition function $Z = \sum_n e^{-\beta E_n}$. It contains information about all the excitations of the system and is the generating function for all thermodynamical quantities, in particular the free energy. As was argued by Cardy, the partition function – a natural object in every CFT – is strongly constrained by modular invariance [81].

In nature, the ideal of an infinitely extended homogeneous system will never be realized exactly. One will always have to deal with boundaries, impurities, defects and related phenomena. We will present some examples of the latter below and indicate how they may be described in a unified way within the framework of boundary conformal field theory.

Boundaries

Due to the limited extension of physical systems it is impossible to avoid finite size effects. While the physics far away from the boundary will not be affected by the choice of boundary conditions, the latter will for instance modify the form of correlation functions close to the boundary.

These ideas may be illustrated with the Ising model at the critical point. This model is described by an ensemble of quantum mechanical spins on a lattice which may either point up or down and which are coupled through nearest neighbor interactions. The Ising model allows for three boundary conditions which preserve the conformal invariance. Beside free boundary conditions one can also choose all spins on the boundary either to show up or down. If all spins at the boundary are fixed to show in one direction, their neighbors also will have an enhanced probability to show in the same direction.[1] Deep in the bulk, however, the boundary contribution will be shadowed by the effects of the other bulk spins.

Systems with finite geometry have the great advantage to be accessible to experiments and simulations on computers. Yet, they contain valuable information about the infinitely extended bulk theory. Let us for example consider an infinite strip of finite width L. It was argued that the free energy per unit length possesses the universal form [82, 83]

$$f \;=\; f_0\, L - \frac{\pi c}{6L} + O(L^{-2})$$

if one imposes periodic boundary conditions along the strip. Relations like this are known as finite-size scaling. They indicate that the central charge of

[1] Here, we assumed a ferromagnetic coupling.

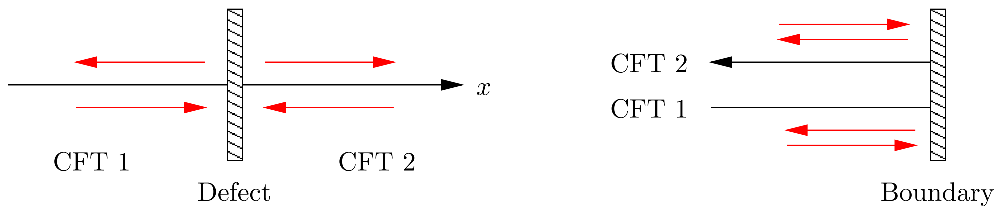

Figure 2.3: The folding trick relates a system on the real line with a defect to a tensor product theory on the half line.

a conformal field theory is a measure for the number of degrees of freedom of a system. This observation is also in agreement with Zamolodchikov's c-theorem according to which the central charge does not increase along renormalization group flows [84]. Let us recall that more and more degrees of freedom are integrated out as we follow a RG flow [85].

For more complicated boundary conditions, the factor in front of $c\,L^{-1}$ has to be modified slightly and one obtains two additional universal terms $\ln g_1 + \ln g_2$ which correspond to boundary entropies [14]. Similar relations also hold for one-dimensional quantum systems if one replaces the width L by the appropriate quantities.

Defect lines

Sometimes two different two-dimensional systems are adjacent along some defect line. It is not difficult to see that this kind of model may be mapped to a boundary problem by folding the system along the defect and gluing the world-sheets afterwards [65]. Although the two theories are now living on the same world-sheet, they do not interact except for the boundary. This implies that we have to deal with a tensor product of the two theories, whose factors are coupled exclusively at the boundary. The folding trick is illustrated in figure 2.3.

If a system possesses several defect lines and/or boundaries, the energy density in the different regions will differ in general. This means that the defect lines induce some sort of *Casimir force* which should in principle be accessible to measurements. Detailed computations of Casimir forces in the framework of BCFT can be found in [69].

A whole class of defects can be constructed from the individual boundary conditions of the two theories. This procedure would describe two systems which have two separated fully reflecting boundaries. As a consequence there is no interaction between the two systems at all in this case, neither in the bulk nor at the boundary. Another class of defects can be obtained if the defect separates two identical probes. One may easily construct fully transmissive defects in this case using the data of the two individual theories. The effect of the defect may then be interpreted as the generalization of a simple phase shift.

The situation becomes more complicated if one aims at describing defect lines which are neither fully reflecting nor fully transmissive. In this case, the symmetry which underlies both systems has to be broken to a common smaller symmetry before the usual scheme can be applied [56]. The construction of such defects will be possible with the methods of chapter 3. The example of two adjacent WZNW models will be considered in detail in section 5.5. For more examples of defect systems and further details we refer the reader to the literature [86, 87, 88].

Quantum impurities

Some extended systems whose degrees of freedom couple to a localized quantum mechanical impurity can be treated by methods of BCFT, at least in some regions in parameter space. The prototype of such systems is the Kondo model where we have an isolated magnetic impurity inside a three-dimensional conductor, interacting with the electrons in the conduction bands. Although the original problem is three-dimensional, it can be mapped to a one-dimensional system with boundary by assuming spherical invariance of the interactions, i.e. absence of spin-orbit coupling. At low temperature the quantum mechanical impurity degree of freedom is then modelled by a conformal boundary condition in a strongly interacting CFT (see [64] and references therein).

The Kondo model predicts an unexpected behavior of the conductivity. The electrons tend to screen the impurity and their mutual coupling turns out to become stronger at low temperature. As a consequence the conductivity decreases when the temperature is lowered and does not increase as would be expected from the usual picture that the conductivity is limited by the scattering of electrons with thermal phonons. In the BCFT description the process of screening corresponds to a renormalization group flow to a different boundary condition.

Other models with impurities which have been investigated using methods of boundary conformal field theory include the two-impurity Kondo model [89] and the Heisenberg spin chain [90, 91].

Quantum wires

A special subclass of impurity and defect systems is constituted by quantum wires. These are networks of one-dimensional conductors with junctions and boundaries. Each conductor is modelled by a one-dimensional gas of interacting electrons with spin and charge. In addition, the conductors may possess impurities which modify the interactions locally.

For the applications it is of fundamental importance to know the response of the system to a given input. In abstract language this corresponds to the determination of the transmission and reflection amplitudes at each junction or impurity. If an impurity is not completely localized but slightly extended, the phenomenon of resonant tunneling may occur. In this case one can even obtain full reflection or transmission depending on the exact design of the impurity.

At low temperature and in the limit of long wavelengths the quantum wires may be described by a conformal field theory [65]. For a single wire with an extended impurity the existence of four stable phases is known which correspond to either full or zero conductance of either charge or spin degrees of freedom. There also exist phases with partial transmission but they just separate two stable phases and have been shown to be unstable.

Percolation

The classical problem of percolation is concerned with clustering properties of objects which are randomly distributed over space. Percolation is known for a long time to be a critical phenomenon. Consider for instance a closed curve Γ which encircles a finite region $\mathcal{R}$. This region can be covered by square plaquettes of a certain area A. Let us associate a color to each of the plaquettes, white with probability p and black with probability $1-p$. In the limit $A \to 0$, the number of plaquettes diverges.

The plaquettes form clusters of equal color and one may ask for their average size in the limit $A \to \infty$. It turns out that below a certain critical threshold probability p_c all white clusters only contain a finite number of plaquettes. Above p_c in contrast there is a finite probability that a given point in the interior of $\mathcal{R}$ belongs to a white cluster of infinite size.

The cases $p \neq p_c$ are not very interesting since the white clusters form a subset in $\mathcal{R}$ of either zero or full measure. Much more interesting is the critical case $p = p_c$ which is exactly solvable in two dimensions. By assuming conformal invariance and using tools from BCFT in a cunning way, Cardy derived a formula for the probability that there exists a cluster which connects two given segments of the boundary curve. Similar methods led to a formula for the mean number of such crossing clusters (see [92] and references therein). The proof of conformal invariance and an independent confirmation of Cardy's formula was only recently achieved by Smirnov [93, 94].

Critical systems which are closely related to percolation and may be treated by similar methods are the Brownian motion, self avoiding walks and the stochastic Loewner evolution [95, 96, 62, 97].

Dissipative quantum mechanics

For studying macroscopic systems, one usually has to incorporate the influence of friction. The latter is the cumulative effect of degrees of freedom whose microscopic dynamics is not included in full detail in the description of the model. The simplest way to consider friction in a quantum mechanical model is to couple the particle to a bath of harmonic oscillators. Integrating out these degrees of freedom results in the standard friction term $-\eta\dot{x}$ proportional to the velocity [98].

The friction is an important ingredient for the localization of particles. Consider for instance a particle in one dimension subject to a periodic potential. Without the influence of the oscillators the particle tends to delocalize and smears over the full real axis. Yet, if one switches on the friction, there exists a critical threshold η_c above which the particle stays in the vicinity of its original well for all times, i.e. where delocalization turns into localization. This observation indicates that dissipation is a critical phenomenon. The critical behavior reveals itself in a logarithmic divergence of the mobility.

The connection between dissipative quantum mechanics at criticality and the BCFT approach to open string theory was worked out in [59]. The potential corresponds to a tachyon background and one can also discuss the effect of a space-time gauge field which couples to the end points of the open strings. The same scheme has also been applied successfully to the study of the dissipative Hofstadter model [60, 61].

2.2 Rational bulk conformal field theory

2.2.1 The bulk spectrum and modular invariance

The analysis of physical systems is usually greatly facilitated by the presence of symmetries. As was motivated in the last section, string theory and critical systems lead to field theories with conformal invariance. The associated symmetry algebra becomes infinite-dimensional in two dimensions and provides thus a powerful tool for the investigation of these models. It is also peculiar to CFT's that they may be formulated using a chiral decomposition into a left and a right moving sector which can be treated almost independently.

In this section we will review how the symmetry and additional consistency conditions like modular invariance constrain the spectrum of excitations in conformal field theories. The spectrum has to organize itself in terms of representations of the product $\mathcal{A}\otimes\bar{\mathcal{A}}$ of the two chiral symmetry algebras. We first describe the chiral part which is associated to the chiral algebra $\mathcal{A}$ and argue afterwards how left and right moving chiral sectors may be coupled to yield the full conformal field theory.

Chiral algebras and their representations

Let us consider a two-dimensional conformal field theory living on the complex plane $\Sigma = \mathbb{C}$. The most important field is given by the chiral energy momentum tensor $\mathrm{T}(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}$ (and its anti-holomorphic counterpart $\bar{\mathrm{T}}(\bar{z})$). Its modes L_n satisfy the relation

$$[L_m, L_n] \;=\; (m-n)\, L_{m+n} + c\, m(m^2-1)\,\delta_{m+n,0}/12 \quad ,$$

i.e. they generate the Virasoro algebra $\mathrm{Vir}(c)$ which is embedded into $\mathcal{A}$ and describes the conformal symmetry. The value of the central charge c depends on the model under consideration and is a measure for the number of degrees of freedom the system has. The sum of the zero modes $L_0+\bar{L}_0$ is proportional to the Hamiltonian and generates the dynamics of the system.

Occasionally, a CFT may contain additional chiral currents $W^i(z) = \sum_n z^{-n-h_i}\, W^i_n$ of (half-)integer conformal weight h_i. In this case the chiral algebra $\mathcal{A}$ is larger than the Virasoro algebra and we speak of an extended symmetry [23, 99]. The conformal dimensions are defined by the commutation relation

$$[L_m, W^i_n] \;=\; [(h_i-1)m-n]\, W^i_{m+n} \quad .$$

The most important examples of extended chiral algebras are provided by affine Kac-Moody and superconformal algebras. We will give a detailed account of models with the first type of symmetry in section 2.4 below.

The two chiral algebras $\mathcal{A}$ and $\bar{\mathcal{A}}$ can be identical but they do not have to. Most parts of this thesis are actually devoted to the study of asymmetric coset theories which allow for a different holomorphic and anti-holomorphic symmetry. As the description of these models, however, requires more elaborate techniques we will postpone their discussion to chapter 4 and assume the isomorphy $\mathcal{A}\cong\bar{\mathcal{A}}$ in this section. For technical reasons we will also restrict ourselves to so-called rational algebras $\mathcal{A}$ possessing a finite set $\mathrm{Rep}(\mathcal{A})$ of "physical" irreducible representations. By physical we mean that we deal with unitary representations in which the "energy" L_0 is bounded from below.

An appropriate representation $\mu \in \mathrm{Rep}(\mathcal{A})$ is characterized by a lowest weight vector $|\mu\rangle$ in a Hilbert space $\mathcal{H}_\mu$, i.e. $L_n|\mu\rangle = W^i_n|\mu\rangle = 0$ for $n > 0$. At the same time it labels an irreducible representation of the algebra generated by the zero-modes W^i_0. We will denote the conformal dimension of μ – the eigenvalue with respect to the "energy" operator L_0 – by the symbol h_μ. For physical reasons we demand the existence of a distinguished vacuum representation (0) with vanishing conformal dimension, $h_0 = 0$. Also, for every representation one can construct the conjugate (contragredient) representation μ^+ which may or may not be isomorphic to μ itself.

For $c \neq 0$ all representations $\mu \in \mathrm{Rep}(\mathcal{A})$ will necessarily be infinite-dimensional. It is thus useful to introduce the character

$$\chi_\mu(q) \;=\; \mathrm{tr}_{\mathcal{H}_\mu}\, q^{L_0 - c/24} \qquad \text{with } q = \exp(2\pi i\,\tau) \text{ and } \mathrm{Im}(\tau) > 0 \quad ,$$

which encodes the content of the irreducible representation. The characters constitute a unitary representation of the modular group. Under the action of the generators $\mathrm{T} : \tau \mapsto \tau + 1$ and $\mathrm{S} : \tau \mapsto -1/\tau$ they transform according to

$$\begin{aligned} \mathrm{T}\chi_\mu(\tau) &= \chi_\mu(\tau+1) = e^{2\pi i\,(h_\mu - c/24)}\,\chi_\mu(\tau) \\ \mathrm{S}\chi_\mu(\tau) &= \chi_\mu(-1/\tau) = \sum_{\nu \in \mathrm{Rep}(\mathcal{A})} S_{\mu\nu}\,\chi_\nu(\tau) \quad , \end{aligned}$$

where S is a unitary matrix. For later use it is convenient to summarize some properties of the modular S-matrix,

$$S_{\mu\nu} \;=\; S_{\nu\mu} \quad , \quad S_{\mu^+\nu} \;=\; \bar{S}_{\mu\nu} \quad , \quad \sum_{\lambda \in \mathrm{Rep}(\mathcal{A})} \bar{S}_{\mu\lambda}\, S_{\nu\lambda} \;=\; \delta^\mu_\nu \quad . \tag{2.2}$$

Explicit expressions for the modular S-matrix are known for many classes of conformal field theories. Some of them will be dicussed in section 2.4 below.

Bulk spectrum and modular invariance

For a complete specification of the bulk CFT we still need to couple holomorphic and anti-holomorphic degrees of freedom and to characterize its field content. The space of fields decomposes into irreducible representations of the product $\mathcal{A} \otimes \bar{\mathcal{A}}$ of the two chiral algebras,

$$\mathcal{H} \;=\; \bigoplus_{\mu,\bar{\mu} \in \mathrm{Rep}(\mathcal{A})} Z^{\mu\bar{\mu}}\; \mathcal{H}_\mu \otimes \bar{\mathcal{H}}_{\bar{\mu}} \tag{2.3}$$

with some numbers $Z^{\mu\bar{\mu}} \in \mathbb{N}_0$. We call the set of all pairs $(\mu, \bar{\mu})$ that contribute to $\mathcal{H}$ (including the multiplicities) the *spectrum* of the (bulk) theory and denote it by

$$\text{Spec} \; = \; \left\{ \, (\mu, \bar{\mu}|\eta) \, \middle| \, \eta = 1, \ldots, Z^{\mu\bar{\mu}} \, \right\} \; . \tag{2.4}$$

The spectrum of a conformal field theory, however, cannot be chosen arbitrarily but is subject to consistency conditions like modular invariance.

For the application in string theory it is indeed crucial to have an unambiguous description of our conformal field theory on arbitrary world-sheets.[2] The latter may be constructed by sewing certain elementary world-sheets. The consistency of this procedure boils down to two relations, crossing symmetry of the four-point function and modular invariance of the one-point function on the torus [100]. For practical purposes, however, one often just considers the consistency of the vacuum amplitude on the torus which already determines the allowed spectrum to a large extent [81]. The torus depends on one modulus τ with $\text{Im}(\tau) > 0$. The toroidal partition function of our theory then reads

$$Z(q, \bar{q}) \; = \; \text{tr}_{\mathcal{H}} \, q^{L_0 - c/24} \, \bar{q}^{\bar{L}_0 - c/24} \; = \sum_{\mu, \bar{\mu} \in \text{Rep}(\mathcal{A})} Z^{\mu\bar{\mu}} \; \chi_\mu(q) \, \bar{\chi}_{\bar{\mu}}(\bar{q})$$

where the argument $q = \exp(2\pi i \, \tau)$ is determined by the modulus of the torus. There exist, however, an infinite number of different representations for one and the same torus in terms of the parameter τ. Hence, one has to identify all values of τ which are related by a modular transformation, i.e. by a sequence of S- and T-transformations.

Imposing invariance of the toroidal partition function under modular transformations gives severe restrictions on the numbers $Z^{\mu\bar{\mu}}$. But nevertheless, there exist choices $Z^{\mu\bar{\mu}} = \delta^{\mu\bar{\mu}}$ and $Z^{\mu\bar{\mu}} = \delta^{\mu\bar{\mu}^+}$, the so-called diagonal and charge conjugate invariants, which are always allowed. Several techniques may be applied to construct further modular invariant starting from a diagonal or charge conjugate one. Non-trivial partition functions arise for example in orbifolds which are reviewed in section 2.4.3.

2.2.2 Fusion rules, simple currents and automorphisms

For the construction of further modular invariant partition functions it is necessary to understand the symmetries of the chiral algebras. These will also play a crucial role in the construction of boundary theories and in the

[2] In this section we only consider world-sheets without boundaries.

context of coset theories in the sections 2.3 and 2.4.2. We take this as an opportunity to summarize some well-known facts about fusion rules, simple currents and automorphisms.

Given two representations $\mu, \nu \in \mathrm{Rep}(\mathcal{A})$ of the chiral algebra $\mathcal{A}$ one may define the notion of a fusion product $\mu \star \nu$. This product is commutative and associative. It can be thought of as some generalization of the usual tensor product which however keeps fixed the value of the central charge[3] such that the product closes inside the set $\mathrm{Rep}(\mathcal{A})$.

According to Verlinde one may express the structure constants of the fusion product $\mu \star \nu = \sum_\lambda N_{\mu\nu}{}^\lambda \lambda$ in terms of the modular S-matrix [101]. The exact correspondence reads

$$N_{\mu\nu}{}^{\lambda} \;=\; \sum_{\sigma \in \mathrm{Rep}(\mathcal{A})} \frac{\bar{S}_{\lambda\sigma} S_{\mu\sigma} S_{\nu\sigma}}{S_{0\sigma}} \quad . \tag{2.5}$$

For later convenience let us summarize some important properties of fusion rules which may easily be proved by means of the Verlinde formula (2.5) and using the properties (2.2) of the modular S-matrix,

$$\begin{aligned} \text{Vacuum = Identity:} &\qquad N_{0\mu}{}^{\sigma} = \delta_\mu^\sigma \\ \text{Symmetry:} &\qquad N_{\mu\nu}{}^{\sigma} = N_{\nu\mu}{}^{\sigma} = N_{\mu\sigma^+}{}^{\nu^+} = N_{\mu^+\nu^+}{}^{\sigma^+} \\ \text{Associativity:} &\qquad \sum_{\sigma \in \mathrm{Rep}(\mathcal{A})} N_{\lambda\nu}{}^{\sigma} N_{\mu\sigma}{}^{\rho} = \sum_{\sigma \in \mathrm{Rep}(\mathcal{A})} N_{\mu\lambda}{}^{\sigma} N_{\sigma\nu}{}^{\rho} \quad . \end{aligned} \tag{2.6}$$

These fusion coefficients contain useful information about the symmetries of the algebra $\mathcal{A}$. We will describe two of them in the following, the simple current symmetries and the automorphisms.

Simple current symmetries

Simple currents $J \in \mathrm{Rep}(\mathcal{A})$ are characterized by the property that the fusion product $J \star \mu$ of J with any other sector $\mu \in \mathrm{Rep}(\mathcal{A})$ contains exactly one representation $J\mu \in \mathrm{Rep}(\mathcal{A})$. Since the vacuum representation is a simple current and the fusion product is commutative, the set of all simple currents forms an abelian group $\mathcal{Z}(\mathcal{A})$, the *center* of the chiral algebra. The inverse of a simple current J is given by its conjugate, $J^{-1} = J^+$.

Let us now summarize some well-known properties of simple currents. It turns out that simple current transformations relate different elements of the

[3]In an ordinary tensor product the central charges would add up.

modular S-matrix according to

$$S_{J\mu,\nu} \;=\; e^{2\pi i\, Q_J(\nu)}\, S_{\mu\nu} \quad . \tag{2.7}$$

The real number $Q_J(\nu)$ is defined modulo integers and it is called the *monodromy charge* of ν with respect to the simple current $J \in \mathcal{Z}(\mathcal{A})$. It is possible to show that monodromy charges are related to conformal weights by the formula

$$Q_J(\nu) \;=\; h_J + h_\nu - h_{J\nu} \quad \bmod 1 \quad .$$

The relation (2.7) has some wide reaching consequences. In particular, by iterated application we find

$$\begin{aligned} Q_{J_1 J_2}(\nu) \;&=\; Q_{J_1}(\nu) + Q_{J_2}(\nu) \\ \text{and} \quad Q_J(J'\nu) + Q_{J'}(\mu) \;&=\; Q_{J'}(J\mu) + Q_J(\nu) \quad . \end{aligned}$$

It follows especially that $Q_{J^n}(\nu) = n\, Q_J(\nu)$. For a simple current J of order N, i.e. an element $J \in \mathcal{Z}(\mathcal{A})$ satisfying $J^N = 0$ (the vacuum representation), the last relation means that the number $\exp\bigl(2\pi i\, Q_J(\nu)\bigr)$ is an N^{th} root of unity.

Simple currents provide symmetries of the fusion rules. Indeed, the commutativity and associativity properties of the fusion product imply

$$N_{\mu\, J\nu}^{\;J\sigma} \;=\; N_{\mu\nu}^{\;\sigma} \quad .$$

We will encounter simple currents almost everywhere in this thesis as they play a prominent role in the construction of orbifold and coset conformal field theories.

Automorphisms

The chiral algebra $\mathcal{A}$ may possess a number of automorphisms, i.e. maps $\mathcal{A} \to \mathcal{A}$ which respect the commutation relations and the adjoint operation and which preserve the Virasoro algebra as well. We will denote this set by $\mathrm{Aut}(\mathcal{A})$. Every automorphism $\Omega \in \mathrm{Aut}(\mathcal{A})$ induces a permutation $\omega : \mathrm{Rep}(\mathcal{A}) \to \mathrm{Rep}(\mathcal{A})$ on the set of representations. One can easily show that this map has to satisfy $\omega(0) = 0$ (the vacuum representation) and $\omega(\mu^+) = \omega(\mu)^+$.

The automorphisms manifest themselves as symmetries of the S-matrix and of the fusion rules. In contrast to the simple currents, one obtains now

$$S_{\omega(\mu)\omega(\nu)} \;=\; S_{\mu\nu} \qquad \text{and} \qquad N_{\omega(\mu)\omega(\nu)}^{\;\omega(\lambda)} \;=\; N_{\mu\nu}^{\;\lambda} \quad . \tag{2.8}$$

The last relation – which follows from Verlinde's formula (2.5) – shows that ω indeed constitutes an automorphism of the fusion algebra.

The action of an automorphism may consistently be restricted to the center $\mathcal{Z}(\mathcal{A})$. This statement is a simple consequence of the relation $\omega(J)\star\omega(\mu) = \omega(J\mu)$ which holds for all simple currents $J \in \mathcal{Z}(\mathcal{A})$. From the equations (2.7) and (2.8) one infers that the monodromy charges satisfy the relation

$$Q_{\omega(J)}\big(\omega(\mu)\big) \;=\; Q_J(\mu) \quad . \tag{2.9}$$

This equation will play a crucial role in our treatment of asymmetric coset theories.

2.3 Rational boundary conformal field theory

2.3.1 Boundary states and Cardy's condition

In the presence of world-sheet boundaries, the previous picture has to be slightly modified. While the two different chiral degrees of freedom remain to be non-interacting in the bulk, left movers are reflected into right movers and vice versa on the boundary. The details of this reflection depend on the boundary condition we impose. To describe D-branes in string theory or impurities in critical systems we have to preserve at least conformal invariance on the boundary. This condition is equivalent to the absence of energy momentum flow across the boundary. In this section we will review the standard construction of boundary conditions which are maximally symmetric, i.e. preserve the full chiral algebra $\mathcal{A}$ (see e.g. [102, 103, 104, 105]). The remaining parts of this thesis will then be concerned with boundary conditions which break part of this symmetry.

Let us review the construction of boundary conditions which preserve the full chiral algebra $\mathcal{A}$. We may specify boundary theories through associated boundary states $|B\rangle$ which live in the bulk Hilbert space. The choice of the boundary condition is then implemented by gluing conditions of the form

$$\big(\phi(z) - (-1)^{h_{\bar\phi}}\,\Omega\bar\phi(\bar z)\big)\,|B\rangle = 0 \qquad \text{for} \qquad z = \bar z \quad . \tag{2.10}$$

Here, the reflection of left into right movers $\phi(z) \to \Omega\bar\phi(\bar z)$ is described by a gluing automorphism $\Omega \in \mathrm{Aut}(\mathcal{A})$ which must leave the energy momentum tensor invariant in order to preserve conformal symmetry. As was reviewed in section 2.2.2, the automorphism Ω induces a permutation $\omega : \mathrm{Rep}(\mathcal{A}) \to \mathrm{Rep}(\mathcal{A})$ on the set of representations which leaves the vacuum representation

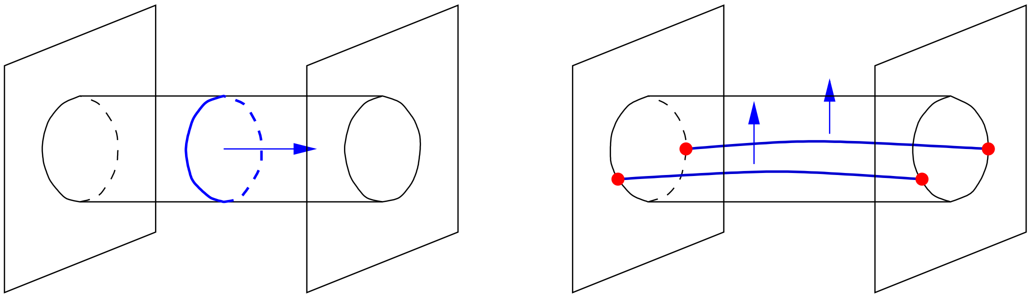

Figure 2.4: Emission and absorption of a closed string versus open string vacuum amplitude.

invariant and satisfies $\omega(\mu)^+ = \omega(\mu^+)$. It is then easy to see that for each element

$$(\mu,\eta) \in \mathrm{Spec}^\omega \;=\; \big\{\, (\mu,\eta) \;\big|\; (\mu,\bar\mu|\eta) \in \mathrm{Spec} \text{ and } \bar\mu = \omega(\mu^+) \big\} \tag{2.11}$$

in the ω-symmetric part of the spectrum (2.4) one can construct a so-called Ishibashi (or *generalized coherent*) state $|\mu,\eta\rangle\rangle$ [106]. These states are normalized by

$$\langle\langle \mu,\eta| q^{\frac{1}{2}(L_0+\bar L_0-c/12)} |\nu,\epsilon\rangle\rangle \;=\; \delta^\mu_\nu\, \delta^\eta_\epsilon\, \chi_\mu(q)$$

and they constitute a complete linear independent set of solutions to the linear equations (2.10). Although the Ishibashi states are often said to live in the bulk Hilbert space $\mathcal{H}$, one should bear in mind that they are not normalizable in the standard sense.

Naively, one could think that all linear combinations

$$|b\rangle \;=\; \sum_{(\mu,\eta)\in \mathrm{Spec}^\omega} \frac{\psi_b{}^{(\mu,\eta)}}{\sqrt{S_{0\mu}}}\, |\mu,\eta\rangle\rangle \tag{2.12}$$

would lead to consistent boundary states. There exists, however, the important *Cardy constraint* which arises from world-sheet duality. The latter may easily be understood in the string theoretic picture where it corresponds to an exchange of open and closed string channel.

The string diagram in figure 2.4 may indeed be interpreted in a two-fold way. The left hand side shows the propagation of a closed string between two D-branes. The emission and absorption of the string are described by boundary states $|a\rangle$ and $|b\rangle$, respectively. On the right hand side one recognizes a vacuum diagram of an open string. Both diagrams are identical except for the direction of time. Nevertheless, they can be identified after a modular S-transformation. The evaluation of the diagram in the open string channel then gives

$$Z_{ab}(q) \;=\; \langle a|\,\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}|b\rangle \;=\; \sum_{(\mu,\eta)\in\mathrm{Spec}^\omega} \frac{\bar{\psi}_a{}^{(\mu,\eta)}\psi_b{}^{(\mu,\eta)}}{S_{0\mu}}\,\chi_\mu(\tilde{q})$$

$$= \sum_{\substack{(\mu,\eta)\in\mathrm{Spec}^\omega\\ \nu\in\mathrm{Rep}(\mathcal{A})}} \frac{\bar{\psi}_a{}^{(\mu,\eta)}\psi_b{}^{(\mu,\eta)}S_{\nu\mu}}{S_{0\mu}}\,\chi_\nu(q) \;\equiv\; \sum_{\nu\in\mathrm{Rep}(\mathcal{A})} \big(n_\nu\big)_b{}^a\,\chi_\nu(q)$$

where $\tilde{q}$ is obtained from q by modular transformation, i.e. $\tilde{q} = e^{-2\pi i/\tau}$. All characters $\chi_\nu(q)$ in the second line must have non-negative integer coefficients,

$$\big(n_\nu\big)_b{}^a \;=\; \sum_{(\mu,\eta)\in\mathrm{Spec}^\omega} \frac{\bar{\psi}_a{}^{(\mu,\eta)}\psi_b{}^{(\mu,\eta)}S_{\nu\mu}}{S_{0\mu}} \in \mathbb{N}_0 \quad , \tag{2.13}$$

since we want to interpret the whole expression as an open string partition function. The latter describes the excitations of open strings stretching between the two D-branes.

Consistent boundary states which correspond to the gluing automorphism Ω may be generated from a set $\mathcal{B}^\omega$ of elementary boundary states. As a criterion for elementarity we shall use the requirement $\big(n_0\big)_b{}^a = \delta_b^a$. It states that the identity field should only live between identical boundary conditions and that it should appear with multiplicity one.[4] Every consistent boundary state may be represented as a superposition of elementary boundary states with non-negative integer coefficients. In the language of string theory, non-elementary boundary states correspond to superpositions of D-branes and lead to Chan-Paton degrees of freedom. Let us emphasize that the boundary states do not only depend on the gluing automorphism Ω, but also on the bulk partition function under consideration.

One can show that the matrices n_ν form a non-negative integer valued matrix representation (NIM-rep) of the fusion ring of the CFT [105], i.e.

$$n_\lambda\, n_\mu \;=\; \sum_{\nu\in\mathrm{Rep}(\mathcal{A})} N_{\lambda\mu}^{\nu}\, n_\nu \qquad \text{and} \qquad n_{\lambda^+} \;=\; (n_\lambda)^T \quad , \tag{2.14}$$

where the fusion rules of $\mathcal{A}$ are denoted by $N_{\lambda\mu}^{\nu}$. Let us also remark that the classification of NIM-reps for a given fusion ring is not sufficient to construct

[4]This definition of elementarity is a working hypothesis. The exact definition would require to look at factorization properties of correlation functions.

consistent BCFT's. In fact, many NIM-reps are known to possess no physical interpretation [107].

There is a class of boundary conditions which was constructed by Cardy more than ten years ago [102]. In the original setup, these boundary conditions require Ω to be the identity and that we are working with the charge conjugate modular invariant, i.e. $Z^{\mu\bar{\mu}} = \delta^{\mu\bar{\mu}^+}$. Hence, Spec^{id} and $\text{Rep}(\mathcal{A})$ can be identified, so that it is easy to solve the Cardy condition by the boundary states

$$|\nu\rangle \;=\; \sum_{\lambda\in\text{Rep}(\mathcal{A})} \frac{S_{\nu\lambda}}{\sqrt{S_{0\lambda}}}\,|\lambda\rangle\!\rangle \quad , \tag{2.15}$$

where $\nu \in \mathcal{B}^{\text{id}} \cong \text{Rep}(\mathcal{A})$. Indeed, the Verlinde formula for fusion coefficients (2.5) immediately implies

$$Z_{\mu\nu}(q) \;=\; \sum_{\lambda\in\text{Rep}(\mathcal{A})} N_{\mu^+\nu}^{\lambda}\,\chi_\lambda(q) \quad .$$

The consistency of these boundary conditions is ensured by the properties (2.6) of the fusion coefficients. The first of these equations guarantees that the identity field propagates only between identical boundary conditions. In addition, these relations imply that the matrices $(N_\lambda)_\mu{}^\sigma = N_{\lambda\mu}^{\sigma}$ form a representation (2.14) – the adjoint representation – of the fusion algebra.

2.3.2 Symmetries of boundary data

Since the matrices n_ν constitute a representation of the fusion algebra, they share many properties with fusion rule coefficients. It is thus worth to have a closer look at the symmetry properties of annulus coefficients with respect to simple currents and automorphisms.

Let us consider annulus coefficients of the form n_J, where $J \in \mathcal{Z}(\mathcal{A})$ is a simple current. From the properties (2.14) and $J^+ = J^{-1}$ we conclude that the matrices n_J are orthogonal. Since the entries are all non-negative integers, this relation further implies that the matrices n_J describe a permutation. In this way we may define an action of the center $\mathcal{Z}(\mathcal{A})$ on the set of boundary labels $\mathcal{B}^\omega$ by $J \star a = \sum_b \big(n_J\big)_a{}^b\, b$. In Cardy's case this prescription reduces to the usual simple current action.

The structure constants ψ which have been introduced in the last section possess a very simple transformation behavior under the action of the center [108]. From eq. (2.13) we immediately deduce that

$$\psi_{Ja}{}^{(\mu,\eta)} \;=\; \sum\nolimits_b \big(n_J\big)_a{}^b\, \psi_b{}^{(\mu,\eta)} \;=\; e^{2\pi i\, Q_J(\mu)}\, \psi_a{}^{(\mu,\eta)} \quad . \tag{2.16}$$

Given this relation one can ask whether there is a similar action of $\mathcal{Z}(\mathcal{A})$ on the Ishibashi labels (μ, η), cf. eq. (2.11). Up to now, such action could not be established in full generality. Yet, all the examples indicate that there is indeed a well-defined action which induces a relation of the form [108]

$$\psi_a{}^{J(\mu,\eta)} \;=\; e^{2\pi i\,\tilde{Q}_J(a)}\,\psi_a{}^{(\mu,\eta)} \quad , \tag{2.17}$$

at least after choosing a suitable basis. These last two equations will be crucial for constructing boundary states in coset theories and orbifolds.

2.4 Model building

2.4.1 Wess-Zumino-Novikov-Witten theories

The Wess-Zumino-Novikov-Witten (WZNW) theories play a prominent role in conformal field theory. Almost all known CFT's can be obtained from them by a suitable construction. They are also an important ingredient in the asymmetrically gauged coset theories we are interested in.

The WZNW theories can be described as two-dimensional non-linear σ-models with a group manifold G as target space. This means that we have a theory of fields $g : \Sigma \to \mathrm{G}$ which map the two-dimensional world-sheet into the group. The WZNW action functional $\mathcal{S}_{\mathrm{WZNW}}(g|k)$ depends on a non-negative real parameter k which can be thought of measuring the size of the group manifold. The model possesses a $\mathrm{G}(z) \times \mathrm{G}(\bar{z})$ loop group symmetry, i.e. the action functional is invariant under the transformations $g' \mapsto g_L(z)\, g'\, g_R^{-1}(\bar{z})$ for arbitrary functions $g_L(z), g_R(\bar{z}) \in \mathrm{G}$ of the prescribed chirality. This symmetry is generated by two chiral currents $J(z)$ and $\bar{J}(\bar{z})$ which take values in the Lie algebra $\mathfrak{g}$ of the group G. We will postpone the discussion of the Lagrangian approach to chapter 3 and restrict ourselves to a description of the algebraic properties at this place. As before, it suffices to consider only the holomorphic sector. For simplicity we will assume G to be a simple simply-connected compact Lie group. Under these circumstances the level k is quantized to be a non-negative integer.

Chiral properties

The modes J^a_n of the chiral currents $J^a(z) = \sum_n z^{-n-1} J^a_n$ generate an affine Kac-Moody algebra $\hat{\mathfrak{g}}$ at level k. This means that they obey the commutation relations

$$[J^a_m, J^b_n] \;=\; i f^{ab}{}_c\, J^c_{m+n} + k\, m\, \kappa^{ab}\, \delta_{m+n,0} \quad .$$

The symbol κ^{ab} denotes the Killing form of $\mathfrak{g}$. Let us note that the affine Kac-Moody algebra $\hat{\mathfrak{g}}_k$ contains the Lie algebra $\mathfrak{g}$ as a subalgebra which is generated by the zero modes J_0^a. In fact, many features of $\hat{\mathfrak{g}}_k$ can be expressed by quantities of $\mathfrak{g}$. For this reason, we provide a summary of the most frequently used notions in the context of semi-simple Lie algebras in appendix A.

We still have to specify the energy momentum tensor of the theory. It turns out that the Virasoro field is defined by means of the Sugawara construction [109]

$$\mathrm{T}(z) \;=\; \frac{(J^a J_a)(z)}{2(k+g^\vee)} \quad ,$$

where $(\cdot)$ denotes conformal normal ordering. This coincides with the classical expression except for the shift by two times the dual coxeter number $g^\vee$ of $\mathfrak{g}$ which is due to renormalization. The central charge can be evaluated by standard methods and is given by $c = \frac{k \dim \mathrm{G}}{k+g^\vee}$. In the limit $k \to \infty$ this equals the value for an ensemble of $\dim \mathrm{G}$ free bosons as would be expected from a flat target space. The fields $J^a(z)$ possess unit conformal weight with respect to $\mathrm{T}(z)$. Their modes satisfy

$$[\, L_m, J_n^a \,] \;=\; -n\, J_{m+n}^a \quad ,$$

which implies that the representations of $\hat{\mathfrak{g}}_k$ decompose into representations of $\mathfrak{g}$ for fixed value of L_0. This statement is in particular valid for the ground states which possess lowest conformal weight.

The set Rep(G) of physical irreducible representations of the Kac-Moody algebra at level k can be described in a simple way. They are labeled by integral dominant weights μ of the Lie algebra $\mathfrak{g}$ subject to the cut-off $(\mu,\theta) \leq k$ with θ being the highest root of $\mathfrak{g}$. The representation has an integer grading with respect to L_0 starting with the eigenvalue $h_\mu = \frac{C_\mu}{2(k+g^\vee)}$ which is determined by the quadratic Casimir C_μ. The states of lowest energy L_0 form the irreducible representation μ of $\mathfrak{g}$.

The content of representations of affine Kac-Moody algebras is well understood and there exist explicit formulas for the characters. As their detailed form is not important for the remaining part of this thesis we will not bother to write them down. They can be found in [25, 110] for instance. The precise form of the associated modular S-matrix is given in eq. (A.4) in the appendix. This formula may be inserted into Verlinde's formula (2.5) to obtain an expression for the fusion rules in WZNW theories. There exists in fact an efficient algorithm for the calculation of fusion rules. A generalization thereof to twisted fusion rules can be found in appendix B. In the limit

$k \to \infty$ the fusion coefficients reduce to tensor product coefficients of the Lie algebra $\mathfrak{g}$.

The groups of automorphisms and simple currents in a WZNW theory based on the algebra $\hat{\mathfrak{g}}_k$ are easily described. The simple current group $\mathcal{Z}(\mathrm{G})$ is in one-to-one correspondence with the symmetries of the Dynkin diagram of $\hat{\mathfrak{g}}$ (i.e. the extended Dynkin diagram of $\mathfrak{g}$) which have not already been present in the Dynkin diagram of $\mathfrak{g}$. In all cases except $E_8^{(1)}$ at level two it also coincides with the center of the associated group G. It is almost as simple to describe the automorphism group Aut(G). First of all, the relevant automorphisms of $\hat{\mathfrak{g}}$ are inherited from those of $\mathfrak{g}$ in an obvious way. An arbitrary automorphism Ω of $\mathfrak{g}$ on the other hand may be decomposed into a diagram (or outer) automorphism Ω_D and an inner automorphism $\mathrm{Ad}_g(X) = gXg^{-1}$ for some element $g \in \mathrm{G}$: $\Omega = \Omega_D \circ \mathrm{Ad}_g$. The diagram automorphism corresponds to a symmetry of the Dynkin diagram of $\mathfrak{g}$ and acts by permuting the roots. While the inner automorphism Ad_g acts trivially on the Cartan subalgebra, the outer one induces a permutation of the Dynkin labels which describe the weights. Note that automorphisms do not only have to respect the commutation relations but also the adjoint operation, i.e. the scalar product. This gives additional constraints on the allowed diagram automorphisms for groups that are not simple, in particular for $\mathfrak{u}(1)$ and direct sums of simple Lie algebras.

The bulk theory

In the case of a simple simply-connected compact group the spectrum of the CFT is described by a charge conjugate modular invariant partition function. In fact, one could as well use any other partition function $Z = \sum_\mu \chi_\mu \, \bar{\chi}_{\omega(\mu)}$ of automorphism type. This just corresponds to a reinterpretation of the action of $\mathrm{G}(\bar{z})$ in the Lagrangian description using an automorphism Ω of G.

The form of the partition function may be understood from a semi-classical argument. The algebra of closed string vertex operators for the ground states should be identical to the algebra of functions on the group G in the limit $k \to \infty$. The latter, however, may be decomposed into representations of the left and right regular action of G, $\mathcal{F}(\mathrm{G}) = \oplus V_\mu \otimes V_{\mu^+}$. A finite value of the level k leads to a natural cut-off and suggests the use of the partition function proposed before.

The situation becomes slightly more complicated when we are dealing with non-simply-connected simple compact groups. In this case one should interprete the theory as an orbifold of the simply-connected covering group. The orbifold construction will be explained in section 2.4.3 below.

Maximally symmetric boundary conditions

Maximally symmetric boundary states in WZNW theories can be constructed for each gluing of chiral currents,

$$\bar{J}(\tau) \;=\; \big[\Omega(J)\big](\tau) \qquad \text{for} \quad \tau \in \partial\Sigma \quad , \tag{2.18}$$

with Ω being an arbitrary automorphism of the Lie algebra $\mathfrak{g}$. These boundary conditions admit a very nice geometric interpretation. The bulk symmetry $\mathrm{G}(z) \times \mathrm{G}(\bar{z}) : g' \mapsto g_L(z)\, g'\, g_R^{-1}(\bar{z})$ of the Lagrangian is broken to the diagonal subgroup $\mathrm{G}(\tau) : g' \mapsto g\, g'\, \Omega(g^{-1})$ with $g \equiv g(\tau)$ and $\tau \in \partial\Sigma$. On the level of Lie algebras this reduces to the gluing conditions (2.18).

In string theory, D-branes which are constructed from gluing conditions of the form (2.18) wrap twisted conjugacy classes

$$\mathcal{C}_f^{\mathrm{G}}(\Omega) \;=\; \big\{\, g\, f\, \Omega(g^{-1}) \,\big|\, g \in \mathrm{G} \,\big\} \tag{2.19}$$

in the target space [26, 27, 29]. They are parametrized by a group element f which may be chosen from the part of the invariant Cartan torus $\mathcal{T}_\Omega = \big\{\, t \in \mathcal{T} \,\big|\, \Omega(t) = t \,\big\}$ which is connected to the unit element. The element $f = e^{2\pi\alpha h/k}$ has to satisfy certain quantization constraints. The reason for the quantization can be understood both from the Lagrangian and from the algebraic point of view. At this place, we will focus on the second point of view and postpone the discussion of the Lagrangian picture to chapter 3.

It was shown in [28] that twisted boundary conditions are labeled by twisted representations of the affine Kac-Moody algebra $\hat{\mathfrak{g}}_k$. The latter are in one-to-one correspondence with representations of the twisted affine Kac-Moody algebra $\hat{\mathfrak{g}}_k^{(N)}$ related to Ω. The number N is the order of the automorphism Ω. In the case of a trivial automorphism, the twisted affine Lie algebra is equal to the untwisted one and the boundary conditions are given by the set $\mathcal{B}^{\mathrm{id}} = \mathrm{Rep}(\mathrm{G})$. This is in complete agreement with Cardy's description.

There exist rather explicit formulas for the annulus coefficients $(n_\mu)_\alpha{}^\beta$ which are simply twisted fusion coefficients $N_{\mu\alpha}^{\beta}$ [28, 71]. The latter describe the fusion of an untwisted representation μ with a twisted representation α of $\hat{\mathfrak{g}}_k$ [111]. In the sequel we will hardly need any concrete expressions for the annulus coefficients. For those who are interested in more details we provide a comprehensive overview in appendix B. This includes an efficient algorithm for the determination of twisted fusion rules [71] as well as the evaluation of the large k limit and its connection to non-commutative geometry [70, 34].

2.4.2 Coset theories

One of the basic tools to build new conformal field theories is the so-called coset or GKO construction [112, 47]. In the Lagrangian formalism it arises from gauging a continuous subgroup H in the WZNW model for the group G [41, 42, 43, 44, 45, 46]. The resulting action functional is then invariant under all transformations $g \mapsto \epsilon(h)\, g\, \epsilon(h^{-1})$ for arbitrary functions $h \equiv h(z,\bar{z}) \in \mathrm{H}$ which depend on *both* variables z and $\bar{z}$. The symbol ϵ denotes the embedding of H in G.

Chiral properties

Let us review the GKO construction and the determination of coset chiral algebras [47]. It suffices to restrict ourselves to the holomorphic sector in what follows. The embedding $\mathrm{H} \hookrightarrow \mathrm{G}$ descends to an embedding of the corresponding Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{g}$. The latter then induces an embedding of affine Kac-Moody algebras $\hat{\mathfrak{h}}_{k'} \hookrightarrow \hat{\mathfrak{g}}_k$. The levels are related by the equation $k' = x_\epsilon\, k$, where x_ϵ is the index of the embedding ϵ. We will denote the chiral algebras which are generated by $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{h}}_{k'}$ by $\mathcal{A}(\mathrm{G})$ and $\mathcal{A}(\mathrm{H})$, respectively. The coset chiral algebra $\mathcal{A}(\mathrm{G/H})$ is the maximal subalgebra of $\mathcal{A}(\mathrm{G})$ which commutes with the image of $\mathcal{A}(\mathrm{H})$ in $\mathcal{A}(\mathrm{G})$. It possesses a Virasoro subalgebra $\mathrm{Vir}(c^{\mathrm{G/H}})$ which is generated by the Virasoro field $\mathrm{T}^{\mathrm{G/H}} = \mathrm{T}^{\mathrm{G}} - \mathrm{T}^{\mathrm{H}}$. The central charge of the coset chiral algebra is given by the difference $c^{\mathrm{G/H}} = c^{\mathrm{G}} - c^{\mathrm{H}}$.

Before we can define the full CFT which corresponds to the gauged WZNW model, we first have to understand the representation theory of the chiral algebra $\mathcal{A}(\mathrm{G/H})$. At least two approaches are available, one more geometrical and one more algebraical. The possible generalization to arbitrary CFT's in mind we will focus on the simple current approach [113, 114, 115, 116].

It is convenient to distinguish the sectors of the G and H theories by using different types of labels,

$$\hat{\mathfrak{g}}_k :\ \ \mu,\nu,\rho,\ldots \in \mathrm{Rep(G)} \qquad\qquad \hat{\mathfrak{h}}_{k'} :\ \ a,b,c,\ldots \in \mathrm{Rep(H)} \quad .$$

The decomposition of an irreducible representation μ of $\mathcal{A}(\mathrm{G})$ into representations of the subalgebra $\mathcal{A}(\mathrm{G/H}) \otimes \mathcal{A}(\mathrm{H})$ reads

$$\mathcal{H}^{\mathrm{G}}_{\mu} \;=\; \bigoplus_{a\in\mathrm{Rep(H)}} \mathcal{H}^{\mathrm{G/H}}_{(\mu,a)} \otimes \mathcal{H}^{\mathrm{H}}_{a} \quad . \tag{2.20}$$

The so-called branching spaces $\mathcal{H}^{\mathrm{G/H}}_{(\mu,a)}$ constitute a representation of $\mathcal{A}(\mathrm{G/H})$. We will assume for a moment that all of them are even irreducible. Note that some of these representations may be trivial, i.e. zero-dimensional. This phenomenon occurs if a representation $\mathcal{H}_a^{\rm H}$ is not contained in a representation $\mathcal{H}_\mu^{\rm G}$.

These branching selection rules may be understood on the level of the horizontal subalgebras $\mathfrak{h}$ and $\mathfrak{g}$. Up to isomorphism, the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$ of the corresponding Lie algebras can be defined by specifying a projection $\mathcal{P} : L^{(\mathfrak{g})} \to L^{(\mathfrak{h})}$ from the weight lattice of $\mathfrak{g}$ to the one of $\mathfrak{h}$. This projection is dual to the injection of Cartan subalgebras. Any allowed pair (μ, a) in the decomposition (2.20) has to satisfy the constraint $\mathcal{P}\mu - a \in \mathcal{P}\mathcal{Q}$ where $\mathcal{Q}$ denotes the root lattice of $\mathfrak{g}$. If this relation were not satisfied, there would be no chance to find a weight in the weight system of μ that is projected onto a. We denote the set of allowed coset labels by

$$\mathrm{All(G/H)} \;=\; \big\{\, (\mu, a) \,\big|\, \mathcal{P}\mu - a \in \mathcal{P}\mathcal{Q} \,\big\} \;\subset\; \mathrm{Rep(G)} \times \mathrm{Rep(H)} \quad .$$

In addition, certain pairs (μ, a) need to be identified because they give rise to exactly the same sector [117, 118]. Generically, this field identification corresponds to elements in the common center $\mathcal{Z}(\mathrm{G}) \cap \epsilon\big(\mathcal{Z}(\mathrm{H})\big)$ of the groups G and H.[5]

Before we continue, let us make the last statement more precise. For two elements $J \in \mathcal{Z}(\mathrm{G})$ and $J' \in \mathcal{Z}(\mathrm{H})$, we say that the pair (J, J') lies in the common center if the relation

$$Q_J(\mu) \;=\; Q_{J'}(\mathcal{P}\mu) \tag{2.21}$$

holds for all weights $\mu \in \mathrm{Rep(G)}$, ensuring that both elements act in the same way. The abelian group of all pairs (J, J') satisfying condition (2.21) shall be denoted by $\mathcal{G}_{\rm id}$. By construction, $\mathcal{G}_{\rm id}$ is a subgroup of the product $\mathcal{Z}(\mathrm{G}) \times \mathcal{Z}(\mathrm{H})$. Sometimes this group is also called identification group. Note that it depends explicitly on the embedding ϵ we have used.

The first application of the identification group $\mathcal{G}_{\rm id}$ is that it allows to reformulate the branching selection rule which has been formulated above in a completely algebraic way. It turns out that the allowed weights may be described by

$$\mathrm{All(G/H)} \;=\; \big\{\, (\mu, a) \,\big|\, Q_J(\mu) = Q_{J'}(a) \text{ for all } (J, J') \in \mathcal{G}_{\rm id} \,\big\} \quad . \tag{2.22}$$

Below we shall also need a projector which implements the previous branching selection rule. This is rather easy to introduce by the explicit formula

[5] In so-called Maverick cosets (see e.g. [119]) and cosets arising from conformal embeddings this statement is not true. Conformal embeddings, however, are restricted to $k = 1$ and all known Maverick cosets also are a low level phenomenon.

$$\mathrm{P}(\mu, a) \;=\; \frac{1}{|\mathcal{G}_{\rm id}|} \sum_{(J,J')\in\mathcal{G}_{\rm id}} e^{2\pi i\,[Q_J(\mu)-Q_{J'}(a)]} \quad . \tag{2.23}$$

The definition (2.22) of All(G/H) directly implies that $\mathrm{P}(\mu, a) = 1$ for all (μ, a) in the set All(G/H) and that it vanishes otherwise.

Moreover, we can now also address the issue of field identification. Let us note that for generic sectors $(\mu, a), (\nu, b) \in$ All(G/H), the modular S-matrix of the coset theory is given by

$$S^{\rm G/H}_{(\mu,a)(\nu,b)} \;=\; |\mathcal{G}_{\rm id}|\; S^{\rm G}_{\mu\nu}\; \bar{S}^{\rm H}_{ab} \quad , \tag{2.24}$$

which follows (up to the prefactor) from the modular transformation properties of eq. (2.20). If we act on the first weight by an element $(J, J') \in \mathcal{G}_{\rm id}$, we obtain

$$S^{\rm G/H}_{(J\mu,J'a)(\nu,b)} \;=\; e^{2\pi i\,[Q_J(\nu)-Q_{J'}(b)]}\; S^{\rm G/H}_{(\mu,a)(\nu,b)} \;=\; S^{\rm G/H}_{(\mu,a)(\nu,b)}$$

from equation (2.7). The phase factor vanishes because of the branching selection rule expressed in relation (2.22). This hints towards an identification of the sectors $(J\mu, J'a)$ and (μ, a). In fact, under certain simplifying assumptions one can show that inequivalent irreducible representations of the coset theory are labeled by orbits

$$[\mu, a] \;\in\; \mathrm{Rep(G/H)} \;=\; \mathrm{All(G/H)}/\mathcal{G}_{\rm id} \quad . \tag{2.25}$$

Complications arise when there exist fixed points, i.e. sectors with the property $(J\mu, J'a) = (\mu, a)$ for at least one pair $(J, J') \in \mathcal{G}_{\rm id}$. In this case, the representation spaces carry (reducible) representations of the relevant stabilizer subgroup of $\mathcal{G}_{\rm id}$. Determining the irreducible constituents and the associated modular data is known as fixed point resolution [120, 115, 121, 122]. We will circumvent these technical difficulties by assuming that our field identification has no fixed points. Under these circumstances, all orbits $\mathcal{G}_{\rm id} \cdot (\mu, a)$ have the same length $|\mathcal{G}_{\rm id}|$.

From the expression (2.24) and Verlinde's formula (2.5) we can easily deduce the following expression for the fusion coefficients of the coset model,

$$N_{[\mu,a],[\nu,b]}^{\;[\sigma,c]} \;=\; \sum_{(J,J')\in\mathcal{G}_{\rm id}} N_{\mu\nu}^{J\sigma}\; N_{ab}^{J'c} \quad . \tag{2.26}$$

This result already constrains the possible form of automorphisms and simple currents in coset theories to a large extent.

The set Aut(G/H) of coset automorphisms is the subset of all tupels $(\Omega_{\rm G}, \Omega_{\rm H})$ of automorphisms of the groups G and H such that $\epsilon = \Omega_{\rm G} \circ \epsilon \circ \Omega_{\rm H}^{-1}$. On the algebraic level this relation implies that the element $(\Omega_{\rm G}(\mu), \Omega_{\rm H}(a))$ is an allowed coset weight if and only if (μ, a) is. Geometrically, it allows to replace the embedding ϵ of H into G by the concatenation $\Omega_{\rm G} \circ \epsilon \circ \Omega_{\rm H}^{-1}$, but still to describe the same coset. This procedure corresponds to a coset CFT with a modular invariant of automorphism type. The simple currents in coset theories are also easily described. They are simply given by the set $\mathcal{Z}({\rm G/H}) = \big(\mathcal{Z}({\rm G}) \times \mathcal{Z}({\rm H}) \cap {\rm All}({\rm G/H})\big)/\mathcal{G}_{\rm id}$.

Bulk and boundary theory

The natural modular invariant partition function of a coset theory is associated to the charge conjugate Hilbert space

$$\mathcal{H}^{\rm G/H} \;=\; \bigoplus_{[\mu,a]\in {\rm Rep}({\rm G/H})} \mathcal{H}^{\rm G/H}_{(\mu,a)} \otimes \bar{\mathcal{H}}^{\rm G/H}_{(\mu,a)^+} \quad .$$

Yet, this is not the only choice as there also exist orbifold models or partition functions of automorphism type. The latter just correspond to a reinterpretation $\epsilon = \Omega_{\rm G} \circ \epsilon \circ \Omega_{\rm H}^{-1}$ of the action of the subgroup which is divided out.

The boundary theory of the coset model for a given gluing automorphism $(\Omega_{\rm G}, \Omega_{\rm H})$ may be constructed once the boundary theories of its constituents G and H are known [123, 108, 124]. For simplicity, we shall assume that the symmetric part of the coset spectrum can be determined from the set

$${\rm All}^\omega({\rm G/H}) \;=\; {\rm Spec}^{\omega_{\rm G}}({\rm G}) \times {\rm Spec}^{\omega_{\rm H}}({\rm H}) \,\cap\, {\rm All}({\rm G/H}) \quad .$$

Unfortunately, the action of the field identification group $\mathcal{G}_{\rm id}$ does not need to close inside this set. Let us therefore define the group $\mathcal{G}^\omega_{\rm id}$ which is the largest subgroup of $\mathcal{G}_{\rm id}$ whose action closes on the set ${\rm All}^\omega({\rm G/H})$. We are now able to define the set of Ishibashi states

$${\rm Spec}^\omega({\rm G/H}) \;=\; {\rm All}^\omega({\rm G/H}) \,/\, \mathcal{G}^\omega_{\rm id} \quad .$$

All orbits have the same length since $\mathcal{G}_{\rm id}$ was assumed to act free of fixed points.

Let $\psi^{\rm G}$ and $\psi^{\rm H}$ denote the structure constants which enter in the construction of boundary states in the theories G and H. The boundary states in the coset are then given by

$$|(\rho,r)\rangle \;=\; \sqrt{\frac{|\mathcal{G}^\omega_{\rm id}|\,|\mathcal{G}^{\mathcal{B}}_{\rm id}|}{|\mathcal{G}_{\rm id}|}} \sum_{(\mu,a)\in{\rm Spec}^\omega({\rm G/H})} \frac{(\psi^{\rm G})_\rho{}^\mu}{\sqrt{S^{\rm G}_{0\mu}}} \, \frac{(\bar\psi^{\rm H})_r{}^a}{\sqrt{\bar S^{\rm H}_{0a}}} \; |(\mu,a)\rangle\rangle \quad ,$$

where the group $\mathcal{G}_{\rm id}^{\mathcal{B}}$ is defined below. This expression is independent of the chosen representatives for the Ishibashi states provided that the boundary labels satisfy the selection rule

$$\tilde{Q}_J(\rho) \;=\; \tilde{Q}_{J'}(r) \qquad \text{for } (J,J') \in \mathcal{G}_{\rm id}^{\omega} \qquad \text{(see eq. (2.17))} \quad .$$

The set of allowed boundary labels will be denoted by $\mathrm{All}^{\mathcal{B}}(\mathrm{G/H})$. Let $\mathcal{G}_{\rm id}^{\mathcal{B}}$ be the quotient of the field identification group $\mathcal{G}_{\rm id}$ by dividing out its stabilizer on $\mathrm{All}^{\mathcal{B}}(\mathrm{G/H})$. As can be checked explicitly, the expressions for the boundary labels $|(\rho,r)\rangle$ and $|(J\rho,J'r)\rangle$ with $(J,J')\in\mathcal{G}_{\rm id}^{\mathcal{B}}$ coincide. Hence, these labels have to be identified. Possible fixed points would have to be resolved but we simply assume absence of the latter as usual. The set of boundary states in our coset theory is then given by

$$\mathcal{B}^{\omega} \;=\; \mathrm{All}^{\mathcal{B}}(\mathrm{G/H})\,\big/\,\mathcal{G}_{\rm id}^{\mathcal{B}} \quad .$$

Using world-sheet duality one may finally determine the spectrum of boundary excitations between two boundary conditions. The result reads

$$Z^{\mathrm{G/H}}_{[\rho',r'][\rho,r]} \;=\; \sum_{[\nu,b]\in\mathrm{Rep(G/H)}}\;\sum_{(J,J')\in\mathcal{G}_{\rm id}^{\mathcal{B}}} (n_\nu)_{J\rho}{}^{\rho'}\,(n_b)_{J'r}{}^{r'}\,\chi^{\mathrm{G/H}}_{(\nu,b)}(q)$$

and can be shown to satisfy all consistency requirements.

The maximally symmetric D-branes in the coset G/H which we just constructed possess a nice geometrical interpretation on the "covering" space G. They wrap the subset of G/H which is obtained by projecting the product of twisted conjugacy classes [125, 126, 127]

$$\mathcal{C}^{\mathrm{G}}_{f_1}(\Omega_{\mathrm{G}})\cdot\Omega_{\mathrm{G}}\circ\epsilon\big(\mathcal{C}^{\mathrm{H}}_{f_2}(\Omega_{\mathrm{H}}^{-1})\big) \;=\; \big\{g\,f_1\,\Omega_{\mathrm{G}}\big(g^{-1}\epsilon(h\,f_2\,\Omega_{\mathrm{H}}^{-1}(h^{-1}))\big)\,\big|\,g\in\mathrm{G},h\in\mathrm{H}\big\}$$

down to the coset. The compatibility with this projection follows from the defining property $\epsilon=\Omega_{\mathrm{G}}\circ\epsilon\circ\Omega_{\mathrm{H}}^{-1}$ of coset automorphisms.

2.4.3 Orbifolds and products

In conformal field theory there exist several tools to construct new models out of a given one. An example was already provided by the coset construction in the last section. In this section we will give a detailed account of the orbifold construction which can be used to describe the quotient of an arbitrary CFT by a discrete subgroup of its center. For completeness we also discuss product CFT's. Special focus is laid on permutation branes.

Orbifold theories

Assume that we have given a conformal field theory G based on the chiral algebra $\mathcal{A}(\mathrm{G})$ and a charge conjugate partition function. If the theory possesses a non-trivial center, one can define an orbifold G/Γ of the original theory with respect to a simple current subgroup $\Gamma \subset \mathcal{Z}(\mathcal{A})$. The consistency of this construction requires all elements of Γ to have integer conformal weight. The chiral algebra $\mathcal{A}(\mathrm{G}/\Gamma)$ which we use to describe the orbifold is still given by $\mathcal{A}(\mathrm{G})$ but there is a projection onto invariant representations and the modular invariant becomes non-charge-conjugate. The central charge is not affected by the orbifolding procedure.

Before we are able to construct the modular invariant partition function, we need some further preparations. First of all, only sectors $\mu \in \mathrm{Rep}(\mathcal{A})$ with vanishing monodromy charge $Q_\Gamma(\mu) = 0$ with respect to the whole group Γ contribute to the theory. They constitute the set of allowed orbifold representations $\mathrm{All}(\mathrm{G}/\Gamma)$. The selection rule may be implemented by means of the projector

$$\mathrm{P}(\mu) \;=\; \frac{1}{|\Gamma|} \sum_{J\in\Gamma} e^{2\pi i\, Q_J(\mu)} \quad .$$

The relevant set of representations $\mathrm{Rep}(\mathrm{G}/\Gamma)$ then consists of orbits $[\mu]$ of elements $\mu \in \mathrm{All}(\mathrm{G}/\Gamma)$ with respect to Γ. Denote by $\mathcal{S}_{[\mu]}$ the stabilizer of a representative μ of $[\mu]$ under the action of Γ. The appearance of non-trivial stabilizers indicates the existence of orbifold fixed points which in principle have to be resolved.

The modular invariant partition function which corresponds to the orbifold is given by the expression

$$Z^{\mathrm{orb}} \;=\; \sum_{[\mu]\in\mathrm{Rep}(\mathrm{G}/\Gamma)} \left|\mathcal{S}_{[\mu]}\right| \left(\sum_{\nu\in[\mu]} \chi_\nu(q)\right)\left(\sum_{\sigma\in[\mu]} \bar{\chi}_{\sigma^+}(\bar{q})\right) \quad .$$

If one of the stabilizers $\mathcal{S}_{[\mu]}$ is non-trivial, the associated representations are not irreducible. In this case the fixed points have to be resolved [115, 121]. Throughout this thesis we will assume absence of fixed points for simplicity, i.e. $\left|\mathcal{S}_{[\mu]}\right| = 1$.

Boundary states for orbifolds may easily be constructed using the data of its covering theory. To start with, the automorphisms of an orbifold theory are those of the covering theory which map the group Γ to itself. This condition guarantees that the action $\omega([\mu]) = [\omega(\mu)]$ on the representation labels is well-defined. As was described in section 2.3.2, the boundary states $|a\rangle$ of the covering theory admit an action of the simple current group Γ.

The basic idea is then to form invariant combinations of these boundary states which may thus be projected down to the orbifold. More precisely, one obtains

$$|[a]\rangle^{\mathrm{orb}} \;=\; \frac{1}{\sqrt{|\Gamma|}}\sum_{J\in\Gamma}|Ja\rangle \;=\; \frac{1}{\sqrt{|\Gamma|}}\sum_{J\in\Gamma}\sum_{\mu}\frac{\psi_{Ja}{}^{\mu}}{\sqrt{S_{0\mu}}}|\mu\rangle\!\rangle \;=\; \sum_{\mu}\frac{\psi_{[a]}{}^{\mu}}{\sqrt{S_{0\mu}}}|\mu\rangle\!\rangle \quad .$$

This picture also suggests that the orbifold boundary labels are given by orbits of boundary labels in the covering theory. Eventually, there exist fixed points which have to be resolved, but this topic is beyond the scope of this introduction. The same intuitive picture of superposing branes in the covering space may be used to visualize the geometry of branes in the orbifold.

From eq. (2.16) one may deduce the relation $\psi_{[a]}{}^{\mu} = \sum_J \psi_{Ja}{}^{\mu}/\sqrt{|\Gamma|} = \sqrt{|\Gamma|}\,\mathrm{P}(\mu)\,\psi_a{}^{\mu}$ which implies that only Ishibashi states appearing in the orbifold partition function are taken into account. The spectrum of open strings between orbifold D-branes can then easily be evaluated to read

$$Z^{\mathrm{orb}}_{[a][b]} \;=\; \sum_{\mu\in\mathrm{Rep(G)}}\sum_{J\in\Gamma}\big(n_\mu\big)_{Jb}{}^{a}\,\chi_\mu \;=\; \sum_{\mu\in\mathrm{Rep(G)}}\big(n_\mu\big)_{[b]}{}^{[a]}\,\chi_\mu \quad .$$

This partition function satisfies all consistency requirements and can be shown to be invariant under the orbifold action.

Let us finally mention that there exist more general orbifolds than considered in this section. They will, however, not be needed in this thesis and therefore we decided not to include them in our presentation.

Product theories

Let G_i denote two conformal field theories based on the chiral algebras $\mathcal{A}(\mathrm{G}_i)$. Then it is very easy to define a chiral algebra for the product $\mathrm{G}_1\times\mathrm{G}_2$ by $\mathcal{A}(\mathrm{G}_1\times\mathrm{G}_2) = \mathcal{A}(\mathrm{G}_1)\otimes\mathcal{A}(\mathrm{G}_2)$. The resulting theory has central charge c_1+c_2 with respect to the Virasoro field $\mathrm{T}(z)=\mathrm{T}_1(z)+\mathrm{T}_2(z)$. The physical irreducible representations are given by the set $\mathrm{Rep}(\mathrm{G}_1\times\mathrm{G}_2)=\mathrm{Rep}(\mathrm{G}_1)\times\mathrm{Rep}(\mathrm{G}_2)$ and both – the representation spaces itself and the characters – factorize in a similar way. As a consequence the modular S-matrix is given by $S^{\mathrm{G}_1\times\mathrm{G}_2}_{(\mu_1,\nu_1)(\mu_2,\nu_2)} = S^{\mathrm{G}_1}_{\mu_1\mu_2}S^{\mathrm{G}_2}_{\nu_1\nu_2}$ from which factorization of the fusion rules follows as well. The modular invariant partition function is defined by $Z^{\mathrm{G}_1\times\mathrm{G}_2} = Z^{\mathrm{G}_1}Z^{\mathrm{G}_2}$. In terms of non-linear σ-models the product theory belongs to the sum of the two action functionals.

The boundary conditions in product theories are easily described. If the theories are different, the group of automorphisms factorizes, $\mathrm{Aut}(\mathrm{G}_1\times$

$G_2) = \mathrm{Aut}(G_1) \otimes \mathrm{Aut}(G_2)$. The same holds true for the maximally symmetric boundary states in this case. In the product of two *identical* CFT's $G = G_1 = G_2$ a new phenomenon appears[6]. The group of automorphisms possesses an additional generator Ω_{exc} which exchanges fields from G_1 and G_2.

Consider an arbitrary automorphism $\Omega = \Omega_{\mathrm{exc}} \circ (\Omega_1, \Omega_2)$. As usual, this map induces a permutation $\omega = \omega_{\mathrm{exc}} \circ (\omega_1, \omega_2)$ on the irreducible representations. If we restrict ourselves to two identical charge conjugate partition functions for simplity, the symmetric part of $\mathrm{Rep}(G \times G)$ is given by $\mathrm{Spec}^\omega = \big\{\big((\mu, \omega_1(\mu)\big)\,\big|\,\mu = \omega_2 \circ \omega_1(\mu)\big\}$. This set is isomorphic to the symmetric part of the spectrum of a *single* G-theory with gluing automorphism $\Omega_G = \Omega_2 \circ \Omega_1$. Assume that we have a solution ψ^G for the boundary theory related to this automorphism. Then the boundary states for the product theory are given by

$$|\rho\rangle \;=\; \sum_{\mu=\omega_2\circ\omega_1(\mu)} \frac{\left(\psi^G\right)_\rho{}^\mu}{S^G_{0\mu}}\;|(\mu,\omega_1(\mu))\rangle\!\rangle \quad .$$

These permutation branes (without the additional automorphisms Ω_1, Ω_2) have been constructed in [128]. Their spectrum reads

$$Z^{G\times G}_{\rho_1\rho_2}(q) \;=\; \sum_{\sigma,\mu_1,\mu_2} \left(n^G_\sigma\right)_{\rho_2}{}^{\rho_1} N_{\mu_1\mu_2}{}^\sigma\; \chi^G_{\mu_1}\,\chi^G_{\omega_1(\mu_2)}(q) \quad . \tag{2.27}$$

Note the existence of the additional twist in the label of the second character.

In WZNW theories there exists a simple geometric interpretation for these branes. According to the general scheme they are localized along twisted conjugacy classes $\mathcal{C}_f^{G\times G}(\Omega)$. This set may also be described as the preimage of the set $\mathcal{C}^G_{f^2}(\Omega_2 \circ \Omega_1)$ under the twisted "multiplication" $\mu(g_1, g_2) = g_1 \cdot \Omega_2(g_2)$ [129]. The considerations of this subsection can easily be extended to products of any (finite) number of conformal field theories.

[6]We emphasize that two WZNW theories based on the same group are different if their level does not agree.

Chapter 3

Symmetry breaking boundary conditions in RCFT

In the last chapter we discussed maximally symmetric boundary conditions in rational conformal field theories. Unfortunately, these are neither capable of reproducing all the charges of D-branes in string theory nor defect lines in statistical physics which go beyond full reflection or transmission. The present chapter provides a systematic analysis of more general boundary conditions which break parts of the symmetry. After a short sketch of the general ideas we turn our attention to WZNW models, where both the algebraic and the geometric constructions are discussed in great detail. The results and ideas of this chapter will also serve as the foundation for the description of boundary conditions in asymmetric cosets in the following chapter.

3.1 Symmetry breaking and conformal embeddings

In the description of maximally symmetric boundary conditions in section 2.3 we assumed that *every* field in the left chiral algebra $\mathcal{A}$ is glued with another one in the right chiral algebra $\bar{\mathcal{A}}$ by the use of some automorphism Ω. If more general boundary conditions shall be described, one obviously has to relax this condition.

The ultimate goal would be to characterize boundary conditions which *solely* preserve the common Virasoro subalgebra $\mathrm{Vir}(c)$ of the two chiral algebras $\mathcal{A}$ and $\bar{\mathcal{A}}$. This amounts to demand that only the Virasoro fields are glued on the boundary, $\mathrm{T}(\tau) = \bar{\mathrm{T}}(\tau)$ for $\tau \in \partial\Sigma$, without any additional restrictions on the other fields. In this case, however, one immediately runs into technical difficulties as the original theory will in general not be rational with respect to the left and right moving Virasoro algebra. This means that the spectrum becomes infinite and all the powerful tools which are familiar from rational BCFT such as Cardy's condition etc. break down.

To remain within the realm of rational conformal field theory one has to choose a suitable intermediate symmetry algebra $\mathcal{A}_{\rm red}$ to be preserved. The chiral algebra $\mathcal{A}_{\rm red}$ on the one hand has to contain the Virasoro subalgebra $\mathrm{Vir}(c) \hookrightarrow \mathcal{A}$ of the original theory. But at the same time it has to be sufficiently large such that the original CFT remains rational if we reinterpret it with respect to the smaller algebra. We will speak of a *rational conformal embedding* $\mathcal{A}_{\rm red} \hookrightarrow \mathcal{A}$ if these two conditions are satisfied.

Our aim is the classification of boundary conditions which preserve the chiral algebra $\mathcal{A}_{\rm red}$. According to the general procedure, we select an automorphism $\Omega \in \mathrm{Aut}(\mathcal{A}_{\rm red})$ and impose the gluing conditions (2.10) for fields in $\mathcal{A}_{\rm red}$. Since no conditions are imposed on the other fields of $\mathcal{A}$ which commute with $\mathcal{A}_{\rm red}$, the original symmetry of the bulk theory will in general be broken at the boundary. But as the algebra $\mathcal{A}_{\rm red}$ is *conformally* embedded into $\mathcal{A}$, their Virasoro fields coincide and at least the conformal symmetry of the larger chiral algebra will be preserved as well.

In order to construct the associated boundary states we have to find the Ishibashi states first. According to the general scheme one obtains an Ishibashi state for every Ω-symmetric combination of *irreducible* representations of $\mathcal{A}_{\rm red}$ in the bulk state space. The latter has been originally given in terms of representations of the chiral algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$. We thus have to know the branching rules which describe the decomposition[1]

$$\mathcal{H}_\mu \;=\; \bigoplus_{\alpha \in \mathrm{Rep}(\mathcal{A}_{\rm red})} b_\mu{}^\alpha \, \mathcal{H}_\alpha$$

of representations $\mathrm{Rep}(\mathcal{A})$ into representations $\mathrm{Rep}(\mathcal{A}_{\rm red})$. The numbers $b_\mu{}^\alpha \geq 0$ are called branching coefficients.[2] If this procedure is applied to the full bulk Hilbert space (2.3), we are led to

$$\mathcal{H} \;=\; \bigoplus_{\mu,\bar\mu,\alpha,\bar\alpha} Z^{\mu\bar\mu}\, b_\mu{}^\alpha\, b_{\bar\mu}{}^{\bar\alpha}\; \mathcal{H}_\alpha \otimes \bar{\mathcal{H}}_{\bar\alpha} \;=\; \bigoplus_{\alpha,\bar\alpha} Z^{\alpha\bar\alpha}\; \mathcal{H}_\alpha \otimes \bar{\mathcal{H}}_{\bar\alpha} \quad . \tag{3.1}$$

For the boundary theory to be rational the last sum should contain only a finite number of terms. As a consequence there will then exist just a finite number of Ishibashi and boundary states, respectively.

[1]We include the possibility of a decomposition into an infinite number or even a continuum of sectors, but assume the numbers $b_\mu{}^\alpha$ to be finite.

[2]They do not have to be confused with branching functions. Since we are dealing with conformal embeddings, the branching coefficients do not depend on q.

Up to now, no general procedure is known which allows to construct boundary states for the symmetry reduced state space (3.1). This is not very surprising as in most of the cases even the branching coefficients ${b_\mu}^\alpha$ are not known explicitly. In the remaining parts of this chapter we will therefore focus on WZNW theories where very concrete expressions can be found. The following chapter will then be devoted to the study of asymmetrically gauged WZNW models. The latter even *necessitate* the introduction of symmetry breaking boundary conditions as we will see.

Let us conclude this section with a short remark. One should expect to recover boundary conditions in the described framework which do not only preserve the algebra $\mathcal{A}_{\text{red}}$ but in fact some larger algebra or even the full chiral algebra $\mathcal{A}$. In this way a classification of all rational conformal embeddings of subalgebras into $\mathcal{A}$ will lead to a whole hierarchy of symmetry breaking boundary conditions.

3.2 Symmetry reduced gluing conditions for WZNW models

Our aim is now to apply the ideas of the preceding section to the case of WZNW theories. We will start with a discussion of rational conformal embeddings which arise naturally in these models and illuminate their geometric origin. Afterwards we will construct the associated boundary states and calculate the spectra of boundary excitations.

3.2.1 Which symmetry to preserve?

Before we dive into technical details let us pause for a moment to recall that maximally symmetric boundary conditions in WZNW models possess a geometric interpretation as twisted conjugacy classes $\mathcal{C}_f(\Omega) = \{\, g' \, f \, \Omega(g'^{-1}) \,\}$ [26, 27, 29]. As such they admit an obvious action $g' \mapsto gg'$ of the group G which induces translations along the conjugacy class. It is a natural assumption that symmetry breaking boundary conditions are associated to submanifolds $\mathcal{D}$ of G which only admit the action of some subgroup $\mathrm{H} \hookrightarrow \mathrm{G}$. On the level of chiral algebras this *naively* would correspond to preserving only an affine Kac-Moody subalgebra $\mathcal{A}(\mathrm{H}) \hookrightarrow \mathcal{A}(\mathrm{G})$.

This proposal, however, faces a severe problem, at least on the algebraic side. First of all, the embedding $\mathcal{A}(\mathrm{H}) \hookrightarrow \mathcal{A}(\mathrm{G})$ is not conformal in general. Yet, as was pointed out in the preceding section we need a conformal embedding to ensure that the energy momentum tensor is preserved. Embeddings of affine Kac-Moody algebras which are conformal can only occur if the level of the numerator G equals $k = 1$. This means that they constitute an almost irrelevant class of models.

A natural way out of this dilemma is to preserve not only the subalgebra $\mathcal{A}(\mathrm{H})$ but also the coset chiral algebra $\mathcal{A}(\mathrm{G/H})$. The embedding

$$\mathcal{A}(\mathrm{G/H}) \otimes \mathcal{A}(\mathrm{H}) \;\hookrightarrow\; \mathcal{A}(\mathrm{G}) \tag{3.2}$$

is indeed conformal by definition of the coset construction. Furthermore, the theory remains rational with respect to the smaller algebra. The embedding thus provides an explicit example of what we called before a rational conformal embedding.

It is natural to iterate and generalize the previous idea by considering an arbitrary embedding chain of subgroups

$$\mathrm{H} \;=\; \mathrm{U}_N \;\hookrightarrow\; \mathrm{U}_{N-1} \;\hookrightarrow\; \cdots \;\hookrightarrow\; \mathrm{U}_1 \;\hookrightarrow\; \mathrm{U}_0 \;=\; \mathrm{G}\;\;. \tag{3.3}$$

In this case, the chiral subalgebra which should be preserved is given by

$$\mathcal{A}(\mathrm{U}_0/\mathrm{U}_1) \otimes \mathcal{A}(\mathrm{U}_1/\mathrm{U}_2) \otimes \cdots \otimes \mathcal{A}(\mathrm{U}_{N-1}/\mathrm{U}_N) \otimes \mathcal{A}(\mathrm{U}_N) \;\hookrightarrow\; \mathcal{A}(\mathrm{G})\;\;. \tag{3.4}$$

Yet, before we are able to comment on the construction of symmetry breaking boundary states related to this decomposition, we first have to get acquainted with its geometric origin in more detail and to classify possible gluing conditions.

3.2.2 Gluing conditions and their geometric origin

In order to understand the more general situation it is very instructive to recall the parametrization and the geometric interpretation of maximally symmetric gluing conditions. On the level of the group the latter correspond to the breaking of the loop group symmetry $g' \mapsto g_{\mathrm{L}}(z)\, g'\, \Omega\big(g_{\mathrm{R}}^{-1}(\bar{z})\big)$ in the bulk[3] to the action $g' \mapsto g\, g'\, \Omega(g^{-1})$ of the diagonal subgroup $\mathrm{G}(\tau)$ on the boundary where $g \equiv g(\tau)$. In this prescription the only free parameter which could be introduced to relate left and right action was the automorphism Ω.

The situation, however, changes drastically in the present situation. If the diagonal subgroup $\mathrm{G}(\tau)$ of the bulk symmetry is further broken to a subgroup $\mathrm{H}(\tau)$, there are several distinct possibilities to define its action on elements of the target space G. The geometric intuition suggests that in

[3]The explicit inclusion of the automorphism Ω corresponds to an interpretation of the theory in terms of a partition function of automorphism type.

principle every action $g' \mapsto \epsilon_{\rm L}(h)\, g'\, \epsilon_{\rm R}(h^{-1})$ could be used where $\epsilon_{\rm L}$ and $\epsilon_{\rm R}$ denote two different embeddings[4] of H in G. But there are at least two ways to see that this simple picture does not hold true in general and that one therefore has to impose some additional consistency conditions.

Let us first comment on the geometric interpretation of symmetry reduction for the bulk theory. The usual affine currents $J^{\mathfrak{g}}(z)$ and $\bar{J}^{\mathfrak{g}}(\bar{z})$ are defined using the action $g' \mapsto g_{\rm L}(z)\, g'\, \Omega\big(g_{\rm R}^{-1}(\bar{z})\big)$ of the loop group $\mathrm{G}(z)\times\mathrm{G}(\bar{z})$. They are elements of the Lie algebra $\mathfrak{g}$. In the present situation, however, it is more natural to interpret the whole theory with respect to the smaller loop group symmetry $\mathrm{H}(z)\times\mathrm{H}(\bar{z})$ which is implemented by $g' \mapsto \epsilon_{\rm L}\big(h_{\rm L}(z)\big)\, g'\, \epsilon_{\rm R}\big(h_{\rm R}^{-1}(\bar{z})\big)$. This symmetry leads to two conserved currents $J^{\mathfrak{h}}(z)$ and $\bar{J}^{\mathfrak{h}}(\bar{z})$ which take values in the Lie algebra $\mathfrak{h}$. The gluing conditions we have been interested in before, may easily be rephrased to read $h_{\rm L}(\tau) = h_{\rm R}(\tau)$ or, equivalently, $J^{\mathfrak{h}}(\tau) = \bar{J}^{\mathfrak{h}}(\tau)$ for $\tau \in \partial\Sigma$. The last equation in particular implies a consistency condition for the choice of embeddings $\epsilon_{\rm L/R}$: they have to be chosen in such a way that holomorphic and anti-holomorphic currents both generate Kac-Moody algebras with *identical* levels.

Another constraint arises from the following observation: the decomposition of chiral algebras (3.2) may be different for the holomorphic and antiholomorphic part, respectively, since they might correspond to different GKO constructions. It is clear that under these circumstances no reasonable gluing conditions can be defined in which holomorphic and antiholomorphic coset fields are mapped one to one into each other. We thus have to find a sensible way to restrict the choices of embeddings $\epsilon_{\rm L/R}$. The same remarks apply to the iterated case (3.3) where one has to deal with embedding chains $\epsilon_{\rm L/R} = \epsilon^{(1)}_{\rm L/R} \circ \cdots \circ \epsilon^{(N)}_{\rm L/R}$ for left and right hand side, respectively.

A proposal for symmetry breaking boundary conditions

An elegant and natural way out is provided by demanding left and right embeddings to be related by automorphisms. In our context this means by definition that both of them may be defined in terms of one *single* set of embedding maps $\epsilon_i : \mathrm{U}_i \rightarrow \mathrm{U}_{i-1}$ and a collection of automorphisms Ω_i, one for each group U_i. More precisely, they should be expressible by

$$\boxed{\begin{aligned} &\epsilon_{\rm L} = \epsilon_1 \circ \epsilon_2 \circ \cdots \circ \epsilon_N \qquad \text{and} \\ &\epsilon_{\rm R} = \Omega_0 \circ \epsilon_1 \circ \Omega_1 \circ \epsilon_2 \circ \cdots \circ \epsilon_{N-1} \circ \Omega_{N-1} \circ \epsilon_N \circ \Omega_N \quad . \end{aligned}} \tag{3.5}$$

[4] We will use the phrase embedding to denote a group homomorphism which descends to an injective map on the level of the Lie algebras. This condition ensures that the embedding map preserves the dimension, but allows for non-trivial wrapping numbers.

	Maximal symmetry	Reduced symmetry
Bulk symmetries:	$\mathrm{G}(z) \times \mathrm{G}(\bar{z})$	$\mathrm{H}(z) \times \mathrm{H}(\bar{z})$
	$g' \mapsto g_{\mathrm{L}}(z)\, g'$	$g' \mapsto \epsilon_{\mathrm{L}}\big(h_{\mathrm{L}}(z)\big)\, g'$
	$g' \mapsto g'\, \Omega\big(g_{\mathrm{R}}^{-1}(\bar{z})\big)$	$g' \mapsto g'\, \epsilon_{\mathrm{R}}\big(h_{\mathrm{R}}^{-1}(\bar{z})\big)$
Chiral algebra:	$\mathcal{A}(\mathrm{G})$	$\otimes \mathcal{A}(\mathrm{U}_l/\mathrm{U}_{l+1}) \otimes \mathcal{A}(\mathrm{U}_N)$
Gluing condition:	$g_{\mathrm{R}}(\tau) = g_{\mathrm{L}}(\tau)$	$h_{\mathrm{L}}(\tau) = h_{\mathrm{R}}(\tau)$
Associated geometry:	$\mathcal{C}_f(\Omega)$	“ $\prod_{l=0}^{N} \mathcal{C}_{f_l}^{\mathrm{U}_l}(\Omega_l)$ ”

Table 3.1: The procedure of symmetry breaking based on eqs. (3.3) and (3.5).

This condition does not only ensure the equality of levels of the two chiral currents $J^{\mathfrak{h}}(z)$ and $\bar{J}^{\mathfrak{h}}(\bar{z})$. Furthermore, it also guarantees that the decomposition of the chiral algebra $\mathcal{A}(\mathrm{G})$ leads to isomorphic subalgebras of the form (3.4) for both holomorphic and antiholomorphic degrees of freedom.

The remaining part of this chapter will be devoted to the construction of boundary states associated to the decomposition proposed in the last paragraph. We will also show that they possess a geometric interpretation as a product of twisted conjugacy classes

$$\boxed{\mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\} \;=\; \mathcal{C}_{f_0}^{\mathrm{U}_0}(\Omega_0) \cdot \epsilon_{\Omega}^{\mathrm{U}_1\mathrm{G}}\Big(\mathcal{C}_{f_1}^{\mathrm{U}_1}(\Omega_1)\Big) \cdot \ldots \cdot \epsilon_{\Omega}^{\mathrm{U}_N\mathrm{G}}\Big(\mathcal{C}_{f_N}^{\mathrm{U}_N}(\Omega_N)\Big)} \tag{3.6}$$

which is embedded into the group manifold G. To make this expression well-defined we finally need to specify the embeddings

$$\begin{aligned} \epsilon_{\Omega}^{\mathrm{U}_l\mathrm{G}} &= \Omega_0 \circ \epsilon_1 \circ \Omega_1 \circ \epsilon_2 \circ \cdots \circ \epsilon_{l-1} \circ \Omega_{l-1} \circ \epsilon_l \\ \epsilon^{\mathrm{HU}_l} &= \epsilon_{l-1} \circ \cdots \circ \epsilon_{N-1} \circ \epsilon_N \quad . \end{aligned} \tag{3.7}$$

The maps ϵ^{HU_l} can be used to define an action of the subgroup H on the individual conjugacy classes. The whole procedure of symmetry breaking is summarized in table 3.1 in a compact way.

Implementing the symmetries

The defining property of the embedding chain (3.3) guarantees that every group U_l admits an action of the group $\mathrm{H} = \mathrm{U}_0$. Let us assume that this action is mediated by the embedding map ϵ^{HU_l} which has been defined in eq. (3.7). The twisted conjugacy classes $\mathcal{C}_{f_l}^{\mathrm{U}_l}(\Omega_l)$ which enter our proposal for the D-brane geometry posses an obvious action of the group U_l and thus also an action of H which is induced by ϵ^{HU_l}.

In order to describe the symmetry properties in detail it is useful to recall that the exact definition of the twisted conjugacy classes is given by

$$\mathcal{C}_{f_l}^{\mathrm{U}_l}(\Omega_l) \;=\; \left\{\; s_l\, f_l\, \Omega_l(s_l^{-1}) \;\middle|\; s_l \in \mathrm{U}_l \;\right\} \;\; .$$

Let us denote by c_l the elements of the image of these twisted conjugacy classes in G, i.e. $c_l \in \epsilon_\Omega^{\mathrm{U}_l\mathrm{G}}\big(\mathcal{C}_{f_l}^{\mathrm{U}_l}(\Omega_l)\big)$. Using this notation, the desired action of H on the twisted conjugacy classes can be formulated as follows,

$$s_l \;\mapsto\; \epsilon^{\mathrm{HU}_l}(h)\, s_l \;\Rightarrow\; c_l \;\mapsto\; \epsilon_\Omega^{\mathrm{U}_l\mathrm{G}} \circ \epsilon^{\mathrm{HU}_l}(h)\, c_l\, \epsilon_\Omega^{\mathrm{U}_l\mathrm{G}} \circ \Omega_l \circ \epsilon^{\mathrm{HU}_l}(h^{-1}) \;\; . \qquad (3.8)$$

Due to the recursion relations $\epsilon_\Omega^{\mathrm{U}_l\mathrm{G}} \circ \Omega_l \circ \epsilon_{l+1} = \epsilon_\Omega^{\mathrm{U}_{l+1}\mathrm{G}}$ also the product of twisted conjugacy classes (3.6) admits a well-defined action of H. An arbitrary element $x \in \mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\}$ transforms as

$$x \;\mapsto \epsilon_{\mathrm{L}}(h)\, x\, \epsilon_{\mathrm{R}}(h^{-1}) \in \mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\} \qquad (3.9)$$

and thus reproduces just the action of H which shall be preserved by our D-brane. As conjectured above, the subset $\mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\}$ of G thus indeed provides a natural candidate for a geometric description of the symmetry breaking D-branes which arise from the algebraic description via the decomposition (3.4).

Discussion

Our proposal (3.6) for the geometry of symmetry breaking D-branes reduces to the usual description of maximally symmetric D-branes after taking $\Omega_l = \mathrm{id}$ and $f_l = e$ for $l > 0$. This observation is in accordance with the algebraic results which will allow us as well to identify Cardy's states among the symmetry breaking boundary states.

These statements may be generalized and used to illustrate the hierarchical structure of the symmetry breaking D-branes which are described by the product of twisted conjugacy classes (3.6). Let us consider a fixed embedding chain (3.3). By choosing an automorphism Ω_l to act trivially and the corresponding element f_l to be given by the group unit we can achieve that the conjugacy class $\mathcal{C}_{f_l}^{\mathrm{U}_l}\big(\Omega_l\big)$ may be omitted from the expression (3.6) for the geometry of the D-brane. Hence, we could have equally well omitted the group U_l from the embedding chain (3.3) in order to describe the same D-brane. To obtain a classification of D-branes in the group manifold G which preserve an arbitrary continuous subgroup it is thus enough to find all *inequivalent* chains of *maximal* embeddings. Let us recall that an embedding is called maximal if there is no group which can be placed in between.

We conclude with a few remarks. The first concerns the dimension of the D-branes which correspond to the product of twisted conjugacy classes (3.6). For a naive evaluation of the dimension one would simply add the dimensions of the twisted conjugacy classes present in eq. (3.6). It is obvious that this procedure would rapidly exceed the dimension of the group itself if one takes embedding chains (3.3) with a large number of subgroups. Up to now we lack a general dimension formula for this kind of D-branes. Let us only emphasize at this point the remarkable fact that they tend to be more and more space-filling the more we break the symmetry.

We even believe that branes which cover the whole target space can be constructed for every WZNW model, at least for suitable choices of the level. A natural candidate for such a space-filling brane is obtained by taking the product of a non-degenerate ordinary conjugacy class of G and a distinguished twisted conjugacy class of its maximal torus $\mathcal{T} = \mathrm{U}(1)^{\mathrm{rank}\,\mathrm{G}}$. The first is isomorphic to $\mathrm{G}/\mathcal{T}$ [29], i.e. has dimension $\dim \mathrm{G} - \mathrm{rank}\,\mathrm{G}$, while the second is given by $\mathcal{T}$ itself. Space-filling branes have been argued to play an important role in phenomenological model building [130].

3.2.3 Symmetry breaking boundary states

We are now going to substantiate our proposal using the abstract tools of conformal field theory. To keep the algebraic discussion as transparent as possible we will restrict ourselves to a detailed presentation of the simplest case in large parts of this section, i.e. to an embedding chain $\mathrm{H} \hookrightarrow \mathrm{G}$ without additional subgroups U_i. We will work out a new set of boundary conditions extending the usual Cardy type conditions. Explicit expressions for the boundary states and the associated open string spectra are provided for different gluing conditions. Afterwards we state the expected results for more general embedding chains with additional intermediate subgroups.

Decomposition of the bulk theory

We will now describe the decomposition of the bulk Hilbert space with respect to the reduced symmetry algebra (3.2). For simplicity we will restrict ourselves to a simply-connected group manifold. As was explained in section 2.4.1, the latter is described by a WZNW theory with charge conjugate partition function. Non-simply-connected groups can be treated by means of the orbifold construction which has been reviewed in section 2.4.3.

Our considerations will be based on an embedding $\mathrm{H} \hookrightarrow \mathrm{G}$ which is defined in terms of homomorphisms $\epsilon_{\mathrm{L}} = \epsilon$ and $\epsilon_{\mathrm{R}} = \Omega_{\mathrm{G}} \circ \epsilon \circ \Omega_{\mathrm{H}}$. For simplicity we also restrict ourselves to choices of automorphisms such that $\epsilon_{\mathrm{L}} = \epsilon_{\mathrm{R}}$. This condition enables us to stay within the realm of ordinary adjoint coset constructions since the pair $(\Omega_{\mathrm{G}}, \Omega_{\mathrm{H}}^{-1})$ is a coset automorphism. More general choices are possible, but they require additional techniques and therefore we postpone their discussion to chapter 4 which deals with asymmetric cosets.

Under the restriction to $\mathcal{A}(\mathrm{G/H}) \otimes \mathcal{A}(\mathrm{H})$, the irreducible representations of $\mathcal{A}(\mathrm{G})$ can easily be reduced to

$$\mathcal{H}_{\mu}^{\mathrm{G}} \;=\; \bigoplus_{(\mu,a)\in \mathrm{All(G/H)}} \mathcal{H}_{(\mu,a)}^{\mathrm{G/H}} \otimes \mathcal{H}_{a}^{\mathrm{H}} \quad .$$

Note that the sum is restricted to those values of a for which the branching selection rule (2.22) is satisfied. Essentially the same calculation applies to the antiholomorphic sector. Yet, in this case one has to be aware of the additional automorphisms which enter the embedding ϵ_{R} in eq. (3.5) compared to the expression for ϵ_{L}. They do not change the interpretation of the coset but lead to a twist of the representation label. One thus obtains

$$\bar{\mathcal{H}}_{\mu^+}^{\mathrm{G}} \;=\; \bigoplus_{(\mu,\bar{a})\in \mathrm{All(G/H)}} \bar{\mathcal{H}}_{(\omega_{\mathrm{G}}(\mu),\omega_{\mathrm{H}}^{-1}(\bar{a}))^+}^{\mathrm{G/H}} \otimes \bar{\mathcal{H}}_{\bar{a}^+}^{\mathrm{H}} \quad ,$$

where we also employed that for our choice of automorphisms the label $(\omega_{\mathrm{G}}(\mu), \omega_{\mathrm{H}}^{-1}(\bar{a}))$ is allowed if and only if $(\mu, \bar{a})$ is allowed. By combining the last two formulas one can finally determine the decomposition of the full charge conjugate state space,

$$\mathcal{H}^{\mathrm{G}} \;=\; \bigoplus_{(\mu,a),(\mu,\bar{a})\in \mathrm{All(G/H)}} \mathcal{H}_{(\mu,a)}^{\mathrm{G/H}} \otimes \mathcal{H}_{a}^{\mathrm{H}} \otimes \bar{\mathcal{H}}_{(\omega_{\mathrm{G}}(\mu),\omega_{\mathrm{H}}^{-1}(\bar{a}))^+}^{\mathrm{G/H}} \otimes \bar{\mathcal{H}}_{\bar{a}^+}^{\mathrm{H}} \quad . \tag{3.10}$$

Let us stress that the resulting theory is not charge conjugate with respect to the smaller chiral algebra. In particular, the boundary states preserving the algebra $\mathcal{A}(\mathrm{G/H}) \otimes \mathcal{A}(\mathrm{H})$ cannot be constructed by Cardy's solution.

In our setting for symmetry breaking boundary conditions we are free to impose independent gluing conditions in the individual parts $\mathcal{A}(\mathrm{G/H})$ and $\mathcal{A}(\mathrm{H})$ of the reduced chiral algebra. But since we took already into account the automorphisms Ω_{G} and Ω_{H} when decomposing the Hilbert spaces we are now allowed to impose trivial gluing conditions,

$$\bigl(\phi(z) - (-1)^{h_{\bar{\phi}}}\, \bar{\phi}(\bar{z})\bigr)\, |B\rangle \;=\; \bigl(\psi(z) - (-1)^{h_{\bar{\psi}}}\, \bar{\psi}(\bar{z})\bigr)\, |B\rangle \;=\; 0 \quad ,$$

for arbitrary fields $\phi \in \mathcal{A}(\mathrm{G/H})$ and $\psi \in \mathcal{A}(\mathrm{H})$. Note that these conditions ensure the Virasoro field $\mathrm{T}^{\mathrm{G}} = \mathrm{T}^{\mathrm{G/H}} + \mathrm{T}^{\mathrm{H}}$ of the theory to be preserved along the boundary, i.e. conformal symmetry is conserved. One might think of other choices of gluing conditions, but they may all be rephrased in terms of the present trivial one by a redefinition of the automorphisms Ω_{G} and Ω_{H}. Note in particular that it suffices to specify two automorphisms in contrast to what one might have guessed at first instant. This feature is completely obvious from the geometric point of view. On the algebraic level it relies on the particular form of the partition function (3.10) where an additional automorphism could be removed by relabeling the index $\bar{a}$.

Naively, one might think that boundary states satisfying the previous gluing conditions can be factorized into boundary states of the two chiral algebras $\mathcal{A}(\mathrm{G/H})$ and $\mathcal{A}(\mathrm{H})$. However, this is not true because the partition function does not factorize. In the next subsection we shall discuss the special case in which both automorphisms Ω_{G} and Ω_{H} are trivial to get acquainted with the necessary techniques. Afterwards, we address more general possibilities of automorphisms.

Untwisted boundary states

Let us start with boundary conditions for which both automorphisms Ω_{G} and Ω_{H} reduce to the identity. This immediately implies $\Omega_{\mathrm{G/H}} \otimes \Omega_{\mathrm{H}} = \mathrm{id} \otimes \mathrm{id}$ which in turn reduces to the identity map $\omega = \mathrm{id} \times \mathrm{id}$ on the set $\mathrm{Rep}(\mathrm{G/H}) \times \mathrm{Rep}(\mathrm{H})$ of sectors. The constituents of the Hilbert space (3.10) which are left-right-symmetric are thus given by

$$\mathcal{H}^{\mathrm{G/H}}_{(\mu,a)} \otimes \mathcal{H}^{\mathrm{H}}_{a} \otimes \bar{\mathcal{H}}^{\mathrm{G/H}}_{(\mu,a)^+} \otimes \bar{\mathcal{H}}^{\mathrm{H}}_{a^+} \quad .$$

There are no elements of the form $(J, 0)$ in the field identification group $\mathcal{G}_{\mathrm{id}}$ of the coset G/H. Hence, Ishibashi states are labeled unambigously by pairs $(\mu, a) \in \mathrm{Spec}^{\mathrm{id}\times\mathrm{id}} = \mathrm{All}(\mathrm{G/H})$, i.e. μ, a run over all representations such that the branching selection rule (2.22) is satisfied. Let us point out that in these labels no field identification is made.

We choose the standard normalization of Ishibashi states such that

$$\langle\langle(\mu, a)|\, \tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}\, |(\nu, b)\rangle\rangle \;=\; \delta^{\mu}_{\nu}\, \delta^{a}_{b}\; \chi^{\mathrm{G/H}}_{(\mu,a)}(\tilde{q})\, \chi^{\mathrm{H}}_{a}(\tilde{q}) \quad .$$

As we shall see, the elementary boundary states are labeled by elements $[\rho, r]$ from the set $\mathcal{B}^{\mathrm{id}\times\mathrm{id}} = \big(\mathrm{Rep}(\mathrm{G}) \times \mathrm{Rep}(\mathrm{H})\big)/\mathcal{G}_{\mathrm{id}}$. Their expansion in terms of Ishibashi states reads

$$|[\rho, r]\rangle \;=\; \sum_{(\mu,a)\in\mathrm{All}(\mathrm{G/H})} B_{[\rho,r]}^{\;(\mu,a)}\, |(\mu, a)\rangle\rangle \tag{3.11}$$

with coefficients $B_{[\rho,r]}^{\,(\mu,a)}$ being determined by the modular S-matrix of the G and the H theory through the simple formula

$$B_{[\rho,r]}^{\,(\mu,a)} \;=\; \frac{S^{\rm G}_{\rho\mu}}{\sqrt{S^{\rm G}_{0\mu}}}\,\frac{\bar S^{\rm H}_{ra}}{\bar S^{\rm H}_{0a}} \quad . \tag{3.12}$$

The proof of this claim proceeds in several steps. Let us first note that the labels (ρ, r) and $(J\rho, J'r)$ lead to the same boundary state for $(J, J') \in \mathcal{G}_{\rm id}$. This is a simple consequence of eq. (2.7) and the definition (2.22) of All(G/H). We will now show that the proposed boundary states possess a consistent open string spectrum. Finally, it remains to demonstrate that the identity field propagates in between two boundary conditions if and only if the latter are identical.

Let us begin by computing the open string spectrum in between two boundary conditions $[\rho_1, r_1]$ and $[\rho_2, r_2]$,[5]

$$\begin{aligned} Z \;&=\; Z_{[\rho_1,r_1][\rho_2,r_2]}(q) \;=\; \langle[\rho_1,r_1]|\,\tilde q^{\frac12(L_0+\bar L_0-c/12)}\,|[\rho_2,r_2]\rangle \\ &=\; \sum_{(\mu,a),[\nu,b],c} \Big[\,\bar B_{[\rho_1,r_1]}^{\,(\mu,a)} B_{[\rho_2,r_2]}^{\,(\mu,a)}\, S^{\rm G/H}_{(\mu,a),(\nu,b)} S^{\rm H}_{ac}\,\Big]\,\chi^{\rm G/H}_{(\nu,b)}(q)\,\chi^{\rm H}_c(q) \\ &=\; |\mathcal{G}_{\rm id}| \sum_{(\mu,a),[\nu,b],c} \left[\,\frac{\bar S^{\rm G}_{\rho_1\mu} S^{\rm G}_{\rho_2\mu} S^{\rm G}_{\nu\mu}}{S^{\rm G}_{0\mu}}\;\frac{S^{\rm H}_{r_1a}\bar S^{\rm H}_{r_2a}\bar S^{\rm H}_{ba}S^{\rm H}_{ca}}{S^{\rm H}_{0a}\bar S^{\rm H}_{0a}}\,\right]\chi^{\rm G/H}_{(\nu,b)}(q)\,\chi^{\rm H}_c(q) \quad . \end{aligned}$$

In the second step we inserted our expression (3.12) for the coefficients of the boundary states and formula (2.24) for the S-matrix of the coset model. Note that the coefficients of the individual characters on the right hand side are not expected to be integers since we still sum over labels which are related by the action of the identification group. Now we use that the quantum dimensions S_{ra}/S_{0a} form a representation of the fusion algebra,

$$\frac{S^{\rm H}_{r_1a}}{S^{\rm H}_{0a}}\,\frac{\bar S^{\rm H}_{r_2a}}{\bar S^{\rm H}_{0a}} \;=\; \sum_{d\in{\rm Rep(H)}} N_{r_1r_2^+}^{\;\;d}\,\frac{S^{\rm H}_{da}}{S^{\rm H}_{0a}} \quad , \tag{3.13}$$

and obtain

$$Z \;=\; |\mathcal{G}_{\rm id}| \sum_{(\mu,a),[\nu,b],c,d} N_{r_1r_2^+}^{\;\;d} \left[\,\frac{\bar S^{\rm G}_{\rho_1\mu} S^{\rm G}_{\rho_2\mu} S^{\rm G}_{\nu\mu}}{S^{\rm G}_{0\mu}}\;\frac{S^{\rm H}_{da}\bar S^{\rm H}_{ba}S^{\rm H}_{ca}}{S^{\rm H}_{0a}}\,\right]\chi^{\rm G/H}_{(\nu,b)}(q)\,\chi^{\rm H}_c(q) \quad .$$

[5]To save space we omit the ranges of the summation indices. The summation rules are: $(\mu, a) \in$ All(G/H), $[\mu, a] \in$ Rep(G/H) and all other (single) indices run over Rep(G) or Rep(H), respectively.

If the sum over the pairs (μ, a) was not restricted by the branching selection rule (2.22), the quotients of S-matrices could be evaluated by means of the Verlinde formula (2.5). But as it stands, this step cannot be performed so easily. However, we can implement the constraint my means of the projector $\mathrm{P}(\mu, a)$ which has been defined in (2.23). This procedure yields

$$\begin{aligned} Z &= |\mathcal{G}_{\mathrm{id}}| \sum_{\mu,a,[\nu,b],c,d} \mathrm{P}(\mu,a)\, N_{r_1 r_2^+}{}^{d} \left[\frac{\bar{S}^{\mathrm{G}}_{\rho_1\mu} S^{\mathrm{G}}_{\rho_2\mu} S^{\mathrm{G}}_{\nu\mu}}{S^{\mathrm{G}}_{0\mu}} \, \frac{S^{\mathrm{H}}_{da} \bar{S}^{\mathrm{H}}_{ba} S^{\mathrm{H}}_{ca}}{S^{\mathrm{H}}_{0a}} \right] \chi^{\mathrm{G/H}}_{(\nu,b)}(q)\, \chi^{\mathrm{H}}_{c}(q) \\ &= \sum_{\substack{\mu,a,[\nu,b],c,d \\ (J,J')\in\mathcal{G}_{\mathrm{id}}}} \frac{e^{2\pi i\, Q_J(\mu)}}{e^{2\pi i\, Q_{J'}(a)}}\, N_{r_1 r_2^+}{}^{d} \left[\frac{\bar{S}^{\mathrm{G}}_{\rho_1\mu} S^{\mathrm{G}}_{\rho_2\mu} S^{\mathrm{G}}_{\nu\mu}}{S^{\mathrm{G}}_{0\mu}} \, \frac{S^{\mathrm{H}}_{da} \bar{S}^{\mathrm{H}}_{ba} S^{\mathrm{H}}_{ca}}{S^{\mathrm{H}}_{0a}} \right] \chi^{\mathrm{G/H}}_{(\nu,b)}(q)\, \chi^{\mathrm{H}}_{c}(q) \quad . \end{aligned}$$

We then use the fact that the exponentials may be pulled into the S-matrices with the help of eq. (2.7), which gives

$$Z = \sum_{\substack{\mu,a,[\nu,b],c,d \\ (J,J')\in\mathcal{G}_{\mathrm{id}}}} N_{r_1 r_2^+}{}^{d} \left[\frac{\bar{S}^{\mathrm{G}}_{\rho_1\mu} S^{\mathrm{G}}_{\rho_2\mu} S^{\mathrm{G}}_{J\nu\,\mu}}{S^{\mathrm{G}}_{0\mu}} \, \frac{S^{\mathrm{H}}_{da} \bar{S}^{\mathrm{H}}_{J'b\,a} S^{\mathrm{H}}_{ca}}{S^{\mathrm{H}}_{0a}} \right] \chi^{\mathrm{G/H}}_{(\nu,b)}(q)\, \chi^{\mathrm{H}}_{c}(q) \quad .$$

As the field identification demands $\chi^{\mathrm{G/H}}_{(\nu,b)} = \chi^{\mathrm{G/H}}_{(J\nu,J'b)}$, we may collect the summations over $(J, J') \in \mathcal{G}_{\mathrm{id}}$ and $[\nu, b] \in \mathrm{Rep(G/H)}$ to give a sum over $(\nu, b) \in \mathrm{All(G/H)}$. Then, applying in addition the Verlinde formula (2.5), we finally arrive at

$$Z_{[\rho_1,r_1][\rho_2,r_2]} = \sum_{\substack{(\nu,b)\in\mathrm{All(G/H)} \\ c,d\in\mathrm{Rep(H)}}} N_{\rho_1^+\rho_2}{}^{\nu}\, N_{r_1^+ r_2}{}^{d}\, N_{dc}{}^{b}\; \chi^{\mathrm{G/H}}_{(\nu,b)}(q)\, \chi^{\mathrm{H}}_{c}(q) \quad . \tag{3.14}$$

In the last step we also used some symmetries (2.6) of the fusion rule coefficients and charge conjugation invariance of the characters. Thereby we have shown that Z can be expanded into characters of the chiral algebra $\mathcal{A}(\mathrm{G/H}) \otimes \mathcal{A}(\mathrm{H})$ with manifestly non-negative integer coefficients.

Finally, we now wish to convince ourselves that the vacuum representation appears exactly once in the boundary partition function (3.14) with two identical elementary boundary conditions and that it does not contribute whenever the two boundary conditions are different. This is somewhat obscured by the possible field identification. By a calculation similar to the previous one it is possible to rewrite the partition function in the form

$$Z_{(\rho_1,r_1),(\rho_2,r_2)} = \sum_{(J,J')\in\mathcal{G}_{\mathrm{id}}} \delta^{\rho_1}_{J\rho_2} \delta^{r_1}_{J' r_2}\, \chi^{\mathrm{G/H}}_{(J0,J'0)}(q)\, \chi^{\mathrm{H}}_{0}(q) \;+\; \cdots \quad ,$$

where $\cdots$ stands for other contributions that do not contain the vacuum character of $\mathcal{A}(\mathrm{G/H})\otimes\mathcal{A}(\mathrm{H})$. Hence, the identity field only appears if (ρ_1, r_1) and (ρ_2, r_2) are identical up to the action of $\mathcal{G}_{\mathrm{id}}$, in agreement with our claim.

Let us conclude with some comments. First note that the partition function (3.14) cannot be obtained by decomposing the usual Cardy boundary partition function. In fact, if we decompose the latter into characters of $\mathcal{A}(\mathrm{G/H})\otimes\mathcal{A}(\mathrm{H})$, we obtain

$$Z_{\rho_1\rho_2} \;=\; \sum_{\nu\in\mathrm{Rep(G)}} N_{\rho_1^+\rho_2}^{\ \ \nu}\, \chi_\nu^{\mathrm{G}}(q) \;=\; \sum_{(\nu,b)\in\mathrm{All(G/H)}} N_{\rho_1^+\rho_2}^{\ \ \nu}\, \chi_{(\nu,b)}^{\mathrm{G/H}}(q)\, \chi_b^{\mathrm{H}}(q) \quad ,$$

where the labels b for the $\mathcal{A}(\mathrm{H})$-sector coincide with the second label of the coset theory. But this is not the case for most of the partition functions of our boundary states. In other words, our new boundary theories manifestly break some of the chiral symmetry $\mathcal{A}(\mathrm{G})$ of the bulk theory. Note, however, that the right hand side of the previous equation coincides with the partition function for the pair $[\rho_1, 0], [\rho_2, 0]$. In other words, for the states of the special form $|[\rho, 0]\rangle$, the maximal chiral symmetry is restored and these states can be identified with Cardy's boundary states.

Twisted boundary states

Our construction possesses a natural extension to cases in which we choose two arbitrary automorphism Ω_{G} and Ω_{H} which satisfy $\epsilon = \Omega_{\mathrm{G}}\circ\epsilon\circ\Omega_{\mathrm{H}}$. As argued before, the gluing in the factor $\mathcal{A}(\mathrm{H})$ can be chosen to be trivial. One then obtains a non-trivial gluing automorphism $\Omega_{\mathrm{G/H}} = (\Omega_{\mathrm{G}}, \Omega_{\mathrm{H}}^{-1})$ in the remaining factor $\mathcal{A}(\mathrm{G/H})$. We will assume that we know the solution of the auxiliary theories based on gluing automorphisms Ω_{G} and Ω_{H}^{-1} for the chiral algebras $\mathcal{A}(\mathrm{G})$ and $\mathcal{A}(\mathrm{H})$.

With this solution of the auxiliary problem in mind, we can now return to our main goal of finding new symmetry breaking boundary states for the G-theory. Once more, we have to determine which sectors in the decomposition (3.10) can contribute Ishibashi states. The condition is

$$\big((\mu,\bar a),\bar a\big) \;\sim\; \big((\omega_{\mathrm{G}}(\mu),\omega_{\mathrm{H}}^{-1}(a)),a\big) \quad .$$

We did not write "=" because the labels must be related only up to a field identification in the coset part. Since the Rep(H) part is not subject to any field identification, the previous relation immediately implies $\bar a = a$. Hence we are left to decide whether for given $(\mu, a)\in \mathrm{All(G/H)}$ we are able to find an element $(J, J')\in\mathcal{G}_{\mathrm{id}}$ of the field identification group such that

$$(J\mu, J'a) \;=\; \big(\omega_{\mathrm{G}}(\mu),\omega_{\mathrm{H}}^{-1}(a)\big) \quad .$$

Solutions to this equation are certainly obtained for labels (μ, a) which are themselves invariant under the given pair of automorphisms. We will actually assume in the following that these are the only ones. Additional solutions may eventually be found but they are always related to some kind of fixed point resolution [108, 124].

Let us denote the total gluing automorphism by $\Omega = \Omega_{\mathrm{G/H}} \times \mathrm{id}$. With the previous assumptions the set of Ishibashi states may be written as

$$\mathrm{Spec}^{\omega} \;=\; \bigl\{\bigl((\mu,a),a\bigr)\,\big|\,(\mu,a)\in \mathrm{All}(\mathrm{G/H}) \text{ and } \bigl(\omega_{\mathrm{G}}(\mu),\omega_{\mathrm{H}}^{-1}(a)\bigr)=(\mu,a)\bigr\} \quad .$$

As in the previous subsection no field identification is imposed. Using the coefficients ψ^{G} and ψ^{H} from the solution of the auxiliary G- and H-theory, we define boundary states by

$$|[\rho,r]\rangle \;=\; \sum_{((\mu,a),a)\in \mathrm{Spec}^{\omega}} \frac{\bigl(\psi^{\mathrm{G}}\bigr)_{\rho}^{\;\mu}}{\sqrt{S^{\mathrm{G}}_{0\mu}}}\,\frac{\bigl(\bar{\psi}^{\mathrm{H}}\bigr)_{r}^{\;a}}{\bar{S}^{\mathrm{H}}_{0a}}\;|((\mu,a),a)\rangle\!\rangle \quad . \tag{3.15}$$

This expression imitates the construction of the last subsection. The boundary states are given by orbits of the tupels (ρ, r) with respect to the identification group $\mathcal{G}_{\mathrm{id}}$. The absence of field identification in the Ishibashi labels implies the absence of selection rules in the boundary labels.

Along the line of our previous computations, one can work out the boundary partition function which is given by the formula

$$Z^{\omega\times\mathrm{id}}_{[\rho_1,r_1],[\rho_2,r_2]}(q) \;=\; \sum_{\substack{(\nu,b)\in\mathrm{All}(\mathrm{G/H})\\ c,d\in\mathrm{Rep}(\mathrm{H})}} \bigl(n^{\mathrm{G}}_{\nu}\bigr)_{\rho_2}^{\;\rho_1}\, N_{b^+c}^{\;\;d}\, \bigl(n^{\mathrm{H}}_{d}\bigr)_{r_1}^{\;r_2}\, \chi^{\mathrm{G/H}}_{(\nu,b)}(q)\,\chi^{\mathrm{H}}_{c}(q) \quad . \tag{3.16}$$

It contains the NIM-reps that come along with the solutions of the auxiliary G- and H-theories. It is easy to check that this expression satisfies all consistency requirements.

Extrapolation to the general case

It is not difficult to generalize the results which have been obtained in the last two subsections for the simplest case of an embedding $\mathrm{H} \hookrightarrow \mathrm{G}$ to the more general embedding chain (3.3) and to an arbitrary choice of embedding maps (3.5). An exact treatment would require an inadequate effort in comparison with the new insights which might be gained. Since the general expressions nevertheless turn out to be useful in the next section we decided to sketch the results while assuming absence of all technicalities such as field identification and branching selection rules.

The decomposition of the chiral algebras (3.4) is accompanied by the corresponding one

$$\mathcal{H}^{\mathrm{G}}_{\mu_0} \;=\; \bigoplus_{\{\mu_i\}} \mathcal{H}^{\mathrm{U}_0/\mathrm{U}_1}_{(\mu_0,\mu_1)} \otimes \cdots \otimes \mathcal{H}^{\mathrm{U}_{N-1}/\mathrm{U}_N}_{(\mu_{N-1},\mu_N)} \otimes \mathcal{H}^{\mathrm{U}_N}_{\mu_N}$$

of representation spaces $\mathcal{H}^{\mathrm{G}}_{\mu_0}$ of $\mathcal{A}(\mathrm{G})$. Like before the decomposition of the antiholomorphic part looks slightly different as it has to reflect the different choices of H-actions which are used to define the currents $J^{\mathfrak{h}}(z)$ and $\bar{J}^{\mathfrak{h}}(\bar{z})$. To be precise, we obtain

$$\bar{\mathcal{H}}^{\mathrm{G}}_{\mu_0} \;=\; \bigoplus_{\{\nu_i\}} \bar{\mathcal{H}}^{\mathrm{U}_0/\mathrm{U}_1}_{(\omega_0(\mu_0),\nu_1)} \otimes \cdots \otimes \bar{\mathcal{H}}^{\mathrm{U}_{N-1}/\mathrm{U}_N}_{(\omega_{N-1}(\nu_{N-1}),\nu_N)} \otimes \bar{\mathcal{H}}^{\mathrm{U}_N}_{\omega_N(\nu_N)} \quad .$$

The last two relations induce an analogous decomposition of the full charge conjugate partition function $\mathcal{H}^{\mathrm{G}}$.

Imposing trivial gluing conditions on all constituents of the reduced chiral algebra (3.4) enforces the conditions $\mu_0 = \omega_0(\mu_0)$ and $\mu_l = \nu_l = \omega_l(\nu_l)$ for the symmetric part of $\mathcal{H}^{\mathrm{G}}$ from which we can obtain Ishibashi states $|\mu_0,\cdots,\mu_N\rangle\!\rangle$. The associated boundary states

$$|\rho_0,\cdots,\rho_N\rangle \;=\; \sum_{\{\mu_i\}} \frac{\left(\psi^{\mathrm{U}_0}\right)_{\rho_0}^{\;\mu_0}}{\sqrt{S^{\mathrm{U}_0}_{0\mu_0}}} \, \frac{\left(\psi^{\mathrm{U}_1}\right)_{\rho_1}^{\;\mu_1}}{S^{\mathrm{U}_1}_{0\mu_1}} \cdots \frac{\left(\psi^{\mathrm{U}_N}\right)_{\rho_N}^{\;\mu_N}}{S^{\mathrm{U}_N}_{0\mu_N}} \,|\mu_0,\cdots,\mu_N\rangle\!\rangle \tag{3.17}$$

can be constructed using the solutions ψ^{U_l} for the auxiliary CFT's based on the chiral algebras $\mathcal{A}(\mathrm{U}_l)$. The calculation of bulk field propagation between two of these boundary states by world-sheet duality yields the Hilbert space

$$\mathcal{H}_{\rho'\rho} \;=\; \bigoplus_{\{\nu_i,\sigma_i,\lambda_i\}} \left(n^{\mathrm{U}_0}_{\nu_0}\right)_{\rho_0}^{\;\rho_0'} \left[\prod_{l=1}^{N} N_{\nu_l\sigma_l^+}^{\;\;\lambda_l} \left(n^{\mathrm{U}_l}_{\lambda_l}\right)_{\rho_l}^{\;\rho_l'}\right] \mathcal{H}^{\mathrm{U}_0/\mathrm{U}_1}_{(\nu_0,\sigma_1)} \otimes \cdots \otimes \mathcal{H}^{\mathrm{U}_N}_{\nu_N} \tag{3.18}$$

of boundary excitations. We used the abbreviation $\rho = (\rho_0,\cdots,\rho_N)$ to denote the boundary label. As before, the numbers N are fusion coefficients of the affine Lie algebras $(\hat{\mathfrak{u}}_l)_{k_l}$. The calculation proceeds in essentially the same way as those in the previous subsections.

Symmetry breaking from T-duality

An alternative way of constructing symmetry breaking boundary states was suggested by Maldacena, Moore and Seiberg [131, 35]. Their construction is based on a regular subgroup $\mathrm{H} \hookrightarrow \mathcal{T}$ of the Cartan torus of G. By definition, H is *abelian*. Under these circumstances one can proof the T-duality

$$\mathrm{G} \;=\; \big(\,\mathrm{G/H} \times \mathrm{H}\,\big)\,/\,\Gamma \quad , \tag{3.19}$$

where Γ is a suitable orbifold group. The right hand side should be understood as an orbifold based on the product theory $\mathrm{G/H} \times \mathrm{H}$ with *charge conjugate* modular invariant partition function. It is thus rather simple to construct boundary states for the right hand side of (3.19) along the lines of section 2.4.3. One can actually show that our approach contains the results of [131, 35] as a special case. More details can be found in appendix C.

3.3 Geometric interpretation

After these algebraic considerations we are prepared to justify our proposal for the geometry (3.6) which is associated to symmetry breaking boundary states. As this geometrical interpretation is mainly relevant for string theory, we will from now on also use the synonym D-brane to denote the conformal boundary conditions. Our symmetry breaking D-branes will turn out to admit a natural interpretation after a target space reinterpretation. Additional evidence comes from a comparison of open string spectra evaluated both from the algebraic and the geometric point of view.

3.3.1 Target space reinterpretation

The usual interpretation of a WZNW theory relies on the group G itself as target space. In the context of the decomposition (3.4) of the chiral algebra it is, however, more convenient to work with the space

$$\mathrm{G}^{\mathrm{new}} \;=\; \frac{\mathrm{U}_0 \times \mathrm{U}_1 \times \mathrm{U}_1 \times \cdots \times \mathrm{U}_N \times \mathrm{U}_N}{\mathrm{U}_1 \times \mathrm{U}_1 \times \cdots \times \mathrm{U}_N \times \mathrm{U}_N} \;=\; \frac{\mathrm{G} \times X}{X} \quad . \tag{3.20}$$

This statement is a generalization of a proposal which has been formulated in the context of coset theories [127]. The specific form of the auxiliary space X arises from the decomposition of chiral algebras (3.4). It is motivated by the close connection between coset CFT's $\mathrm{U}_l/\mathrm{U}_{l+1}$ and product CFT's $\mathrm{U}_l \times \mathrm{U}_{l+1}$ which itself is based on the similarity of modular properties. The extension of G by X introduces additional degrees of freedom which have to be removed by dividing through X again. The precise action of X on $\mathrm{G} \times X$ will be given in eq. (3.21) below.

The equivalence of the spaces G and G^{new} seems to be obvious at first sight. Nevertheless we have to be very careful as G carries additional structure which should be reflected in G^{new}. In particular, G admits an action of the group $G \times G$, i.e. the regular action from the left and from the right. The group $G \times G$ should be considered as the “constant” part of the WZNW symmetry $G(z) \times G(\bar{z})$. When we consider symmetry breaking boundary conditions which arise from the embedding chain (3.3), the action of $G \times G$ thus has to be broken to an action of the subgroup $H \times H$ where the embedding of the latter is given by the map (ϵ_L, ϵ_R). We will argue below that the *same* action of $H \times H$ can be found on G^{new} provided that one uses the correct action of X on $G \times X$ in the definition (3.20). To be precise, the elements of G^{new} should be given by tupels $(u_0, u_1', u_1, \cdots, u_{N-1}', u_{N-1}, u_N', u_N) \in G \times X$ subject to the identifications

$$\begin{aligned} (u_l', u_l) &\sim (u_l' \cdot t_l^{-1}, t_l \cdot u_l) \quad \text{and} \\ \big(u_{l-1}, u_l'\big) &\sim \big(u_{l-1} \cdot \Omega_{l-1} \circ \epsilon_l(s_l^{-1}), s_l \cdot u_l'\big) \quad \text{for } t_l, s_l \in U_l \quad . \end{aligned} \tag{3.21}$$

The remaining action of $H \times H$ on the target space G^{new} on the other hand should be defined by

$$(u_0, \cdots, u_N) \mapsto \big(\epsilon_L(h_1)\, u_0, \cdots, u_N\, \Omega_N(h_2^{-1})\big) \quad . \tag{3.22}$$

The identification (3.21) shows that the new target space G^{new} is a specific example of an asymmetric coset, for which a non-adjoint action of the subgroup is divided out. A general discussion of asymmetric coset spaces will follow in the next chapter.

In view of constructing the boundary WZNW functional for symmetry breaking boundary conditions in the following section it is useful to work out the natural representatives of elements in G^{new}. Using the embeddings $\epsilon_\Omega^{U_l G}$ which have been defined in eq. (3.7), the canonical representatives may be written

$$\big(u_0 \cdot \epsilon_\Omega^{U_1 G}(u_1' \cdot u_1) \cdot \ \ldots \ \cdot \epsilon_\Omega^{U_N G}(u_N' \cdot u_N),\, e, \ldots, e\big) \quad . \tag{3.23}$$

This relation shows that elements of G^{new} may be represented naturally as elements of G. Note however, that one and the same element of G is represented by a whole orbit of elements in $G \times X$. This shows explicitly the drastical increase of degrees of freedom which are associated to the decomposition (3.4) of the chiral algebra.

The reader may wonder why we had to choose such a complicated identification (3.21) to define the coset G^{new}. A partial answer is given by the remarkable relation between the representative (3.23) and the form of the product of twisted conjugacy classes (3.6). The deeper reason for this particular choice of identification, however, arises from demanding the equality of G and G^{new} including the given action of $H \times H$ on each of these groups. The latter is already suggested by the following observation: the action of $H \times H$ on G^{new} which was given in eq. (3.22) translates as desired into the action $x \mapsto \epsilon_L(h_L)\, x\, \epsilon_R(h_R)$ for an arbitrary representative $x \in G$ of the form (3.23). It can also be understood if one does not work on the level of manifolds, but descends to the algebras of functions $\mathcal{F}(G)$ and $\mathcal{F}(G^{new})$ which inherit the given action of $H \times H$ but allow for a *linear* representation. According to a theorem of Gel'fand and Naimark, also the topology of a manifold is completely contained in its algebra of functions. Showing the equality

$$\mathcal{F}(G) \;\cong\; \mathcal{F}(G^{new}) \;=\; \mathrm{Inv}_X\Big(\mathcal{F}\big(G \times X\big)\Big) \tag{3.24}$$

as $H \times H$ modules is thus enough to establish the equivalence of the target spaces G and G^{new} including the action of $H \times H$. The relation (3.24) may be proven by using the Peter-Weyl theorem which gives the decomposition of the algebra of functions on a group into irreducible representations under left and right regular action of the group G itself. Restricting the action to $H \times H$ and taking all twists into account we exactly recover equality (3.24).

The considerations of the last paragraph are the classical analogue of the decomposition (3.4) of chiral algebras. Let us recall that the interpretation of G as the target space of a WZNW theory is supported by the deep relation between the spectrum of closed strings and the algebra of functions on the group. According to the Peter-Weyl theorem the algebra of functions $\mathcal{F}(G)$ is recovered from the ground state structure of the charge conjugate partition function of the WZNW theory in the limit $k \to \infty$ when interpreted as a $G \times G$ module with respect to left and right regular action of G. As already mentioned above, the group $G \times G$ should be considered as the "constant" part of $G(z) \times G(\bar{z})$. After symmetry reduction, the decomposition (3.4) of the chiral algebras has to be accompanied by an analogous decomposition of the closed string Hilbert space. On the geometrical side this corresponds to the interpretation of the $G \times G$ module $\mathcal{F}(G)$ as an $H \times H$ module where the embedding is given by (ϵ_L, ϵ_R).

3.3.2 Justification of the proposed brane geometry

The target space reinterpretation (3.20) enables us to check the proposal (3.6) for the geometry of symmetry breaking D-branes. The coefficients entering the boundary states (3.17) are indeed identical with those of certain boundary states in the product space $G \times X$. The first factor for instance may be interpreted as an ordinary twisted brane in the factor $\mathrm{G} = \mathrm{U}_0$. A comparison with the results of section 2.4.3 in addition shows that each of the other factors corresponds to a twisted permutation brane in one of the factors $\mathrm{U}_l \times \mathrm{U}_l$ which is based on the automorphisms $\Omega'_l = \Omega^{(l)}_{\mathrm{exc}} \circ (\mathrm{id}, \Omega_l)$ involving an exchange of group factors. Altogether one concludes that the boundary state describes a D-brane in $\mathrm{G} \times X$ which wraps the (direct) product of twisted conjugacy classes

$$\mathcal{C}^{\mathrm{U}_0}_{f_0}\big(\Omega_0\big) \times \mathcal{C}^{\mathrm{U}_1\times\mathrm{U}_1}_{f'_1}(\Omega'_1) \times \ \ldots \ \times \mathcal{C}^{\mathrm{U}_N\times\mathrm{U}_N}_{f'_N}(\Omega'_N) \quad . \tag{3.25}$$

The labels $f'_l \in \mathrm{U}_l$ are determined by the labels ρ_l in the usual way and have thus to satisfy certain quantization conditions. Remembering the identifications (3.21) it is not difficult to check that the previous expression may be consistently projected down to the coset $\mathrm{G}^{\mathrm{new}} = \mathrm{G} \times X/X$. Or to express it in a different way, the action of X on an arbitrary element of the set (3.25) just leads to a translation *along* this submanifold.

The expression (3.25) considered as a subset of the manifold $\mathrm{G}^{\mathrm{new}} = \mathrm{G}\times X/X$ simplifies considerably if one works out the associated representative (3.23) which can be considered as an element of the group G itself. In this way we just recover the product of images of twisted conjugacy classes (3.6). The labels are related to the previous ones by $f_l = f'^2_l \in \mathrm{U}_l$. The quantization conditions may also be understood from the Lagrangian picture which will be discussed below.

One may ask whether arranging the twisted conjugacy classes in the definition (3.6) in different order would lead to new results. It is easy to show that this is not the case. Exchanging two conjugacy classes merely leads to a redefinition of embedding maps and automorphisms. Under these circumstances it is natural to work with one standard representative. In our case the latter is defined to be given by eq. (3.6).

The formalism which has been described in the previous sections provides a whole hierarchy of symmetry breaking D-branes. The classification of all these objects is greatly simplified by the following observation which is well known from maximally symmetric D-branes. Instead of allowing all choices of automorphisms we may restrict ourselves to the case of outer automorphisms. The appearance of an inner automorphism $\Omega_l(u_l) = b_l\, u_l\, b_l^{-1}$ in the product of twisted conjugacy classes (3.6) just corresponds to putting this automorphism to the identity in the expressions (3.5) and (3.6), to replace f_l by $f_l\, b_l$ and to multiply the resulting expression for eq. (3.6) with the element $\epsilon^{\mathrm{U}_l\mathrm{G}}_{\Omega}(b_l^{-1})$ from the right. Geometrically, this procedure induces an overall shift of the brane. The same idea also enables us to choose specific representatives for whole classes of outer automorphisms by separating their “inner part”.

There are several consistency checks which may be used to test our proposal for the geometry of symmetry breaking D-branes. First of all, we already know that the set (3.6) admits the correct action of the group H as given in table 3.1. We will now investigate the open string spectrum based on the geometric formulation and compare it with the result from the algebraic approach. As an independent check we will finally construct the boundary WZNW functional associated to these branes.

Comparison with the algebraic approach

In the large volume limit $k \to \infty$ one is able to extract geometric information out of the algebraic description of the D-brane. The ground state structure of the partition function for open strings which start and end on the same brane for instance should tend to the algebra of functions on the brane. Since the latter carries in our case an action of the subgroup H which is induced by its action on the D-brane world-volume (3.6), we are able to compare both approaches. In their common area of validity we will find full agreement between the algebraic and the geometric picture.

In the geometric picture only the ground state structure of the Hilbert space (3.18) can be recovered. Let us thus denote by $\mathcal{H}^{(0)}_{\rho\rho}$ the set of all ground states which are present in eq. (3.18) for $\rho' = \rho$. Our aim is to find an explicit expression for this space which solely contains geometric information in the limit $k \to \infty$. The ground states of affine representations transform in a representation of the underlying Lie algebra which is usually denoted by the same symbol. In contrast, it is more difficult to give a geometrical meaning to the coset representations. We can solely determine the number of ground states they contain. The latter is given by the branching coefficients which describe the embedding of the associated horizontal subalgebras. These considerations provide a dictionary of how to extract geometrical information out of eq. (3.18). We simply have to replace affine representations $\mathcal{H}^{\mathrm{U}_1}_{\nu_1}$ by representation spaces $V^{\mathrm{U}_1}_{\nu_1}$ of $\mathfrak{u}_1$ and branching spaces $\mathcal{H}^{\mathrm{U}_l/\mathrm{U}_{l+1}}_{(\nu_l,\sigma_{l+1})}$ by branching coefficients of the embedding $\mathfrak{u}_{l+1} \hookrightarrow \mathfrak{u}_l$. When represented as a module of H we end up with the following expression,

$$\mathcal{H}^{(0)}_{\rho\rho} \;=\; \bigoplus \left(n^{\mathrm{U}_0}_{\nu_0}\right)_{\rho_0}{}^{\rho_0} \left[\,\prod_{l=1}^{N} N_{\nu_l\sigma_l^+}{}^{\lambda_l} \left(n^{\mathrm{U}_l}_{\lambda_l}\right)_{\rho_l}{}^{\rho_l}\; b_{\nu_{l-1}}{}^{\sigma_l}\right] V^{\mathrm{U}_N}_{\nu_N} \quad , \tag{3.26}$$

for the space of ground states. The geometric limit of a WZNW theory is obtained by sending the level k of the affine Kac-Moody algebra $\hat{\mathfrak{g}}_k$ to infinity. This automatically forces the levels of the Kac-Moody subalgebras $(\hat{\mathfrak{u}}_l)_{k_l}$ to do the same. In this limit the fusion coefficients N entering eq. (3.18, 3.26) reduce to tensor product coefficients and the NIM-reps n^{U_l} have a natural geometrical interpretation as well. More details can be found an appendix B.2.

Let us now turn our attention to the evaluation of D-brane spectra in the geometric picture. As was shown in [32, 33] for ordinary conjugacy classes and then generalized in [34] to the twisted case, there is always a non-commutative geometry associated to these objects, see appendix B. In some sense this reflects the geometric limit of the non-commutative algebra of open string vertex operators. For the ground states this algebra indeed becomes coordinate independent in the large volume limit $k \to \infty$ as their conformal dimensions tend to zero. For an arbitrary D-brane wrapped around the twisted conjugacy class $\mathcal{C}^{\mathrm{U}_l}_{f_l}(\Omega_l)$ this non-commutative algebra admits an action of U_l under which it decomposes into modules $V^{\mathrm{U}_l}_{\nu_l}$ according to

$$\mathcal{A}\Big(\mathcal{C}^{\mathrm{U}_l}_{f_l}(\Omega_l)\Big) \;=\; \bigoplus_{\{\nu_l\}} \big(n_{\nu_l}\big)_{\rho_l}{}^{\rho_l}\; V^{\mathrm{U}_l}_{\nu_l} \quad . \tag{3.27}$$

The attentive reader might object that in our situation and for $l \neq 0$ not the previous conjugacy class is the relevant one, but the permutation branes which entered the geometry (3.25). Yet, it is not difficult to check that the same expression would be obtained for $\mathrm{Inv}_{\mathrm{U}_l}\, \mathcal{A}\big(\mathcal{C}^{\mathrm{U}_l\times\mathrm{U}_l}_{f'_l}(\Omega'_l)\big)$ where the invariance is defined with respect to the identification $(u'_l, u_l) \sim (u'_l\cdot t_l^{-1}, t_l\cdot u_l)$ for $t_l \in \mathrm{U}_l$ (see eq. (3.21)). For our further discussion it is not necessary to know the form of the matrices n^{U_l} in detail. It is only important to note that the numbers n^{U_l} entering eq. (3.26) coincide with those in eq. (3.27) in the limit $k \to \infty$. The proof can be found in appendix B.2.

The module structure of the algebra for a (direct) product of twisted conjugacy classes is given by the tensor product of the individual modules. We thus obtain the spectrum of open string ground states

$$\mathcal{A}\Big(\mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\}\Big) \;=\; \bigoplus_{\{\nu_l\}} \left[\,\prod_{l=0}^{N} \big(n_{\nu_l}\big)_{\rho_l}{}^{\rho_l}\right] V^{\mathrm{U}_0}_{\nu_0} \otimes \cdots \otimes V^{\mathrm{U}_N}_{\nu_N} \quad , \tag{3.28}$$

interpreted as a module of the group $\mathrm{U}_0 \times \cdots \times \mathrm{U}_N$. In our approach of section 3.2.2, the twisted conjugacy classes are not viewed as modules of the groups U_l, but as modules of the diagonally embedded $\mathrm{H} = \mathrm{U}_N$. Hence, we should fully decompose the module (3.28) with respect to H in order to read off the spectrum of ground states belonging to the D-brane described by $\mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\}$. In this way we get a number of additional branching and tensor product coefficients.

It is now straightforward to check the equality of the expressions on the right hand side of eqs. (3.26) and (3.28) in the limit $k \to \infty$, both considered as H modules. In other words we have just proven the relation $\mathcal{A}\big(\mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\}\big) \cong \mathcal{H}^{(0)}_{\rho\rho}$ in this limit which expresses the agreement of the open string spectra obtained both from an algebraic and a geometric point of view, respectively. Let us emphasize that in the present situation the geometric open string algebra $\mathcal{A}\big(\mathcal{D}\big\{U_l, \Omega_l, f_l\big\}\big)$ may not be identified with the algebra of functions on the D-brane world-volume $\mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\} \subset \mathrm{G}$. Again this may be interpreted as an effect in favour of working with the new target space $\mathrm{G}^{\mathrm{new}}$. When considered as an object in G, the points in the D-brane world-volume can be covered more than once. Obviously it is not possible to describe the new degrees of freedom which are associated to such multiple wrappings and superpositions of D-branes by the usual algebra of functions on the world-volume $\mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\} \subset \mathrm{G}$. Similar observations have been discussed in [131].

3.4 The boundary WZNW functional

An independent check of our proposal for the geometry of symmetry breaking D-branes may be achieved in the Lagrangian framework. We first review the general construction for WZNW models and specialize afterwards to the geometric situation we are interested in.

3.4.1 The general structure of the Lagrangian

Geometrically, a WZNW theory is described by a non-linear σ-model of fields $g : \Sigma \to \mathrm{G}$ which live on a two-dimensional world-sheet Σ and take values in the group manifold G. The latter will be assumed to be simple, but it is straightforward to generalize our results to reductive groups, i.e. to those which are a direct product of simple groups and U(1) factors. The action functional for this theory is given by [24, 27]

$$\mathcal{S}^{\mathrm{G}}_{\mathrm{WZNW}}\big(g|k;\mathcal{D}\big) \;=\; \mathcal{S}^{\mathrm{G}}_{\mathrm{kin}}(g|k) \;+\; \mathcal{S}^{\mathrm{G}}_{\mathrm{WZ}}(g|k) \;+\; \mathcal{S}^{\mathrm{G}}_{\mathcal{D}}(g|k) \tag{3.29}$$

and consists of three parts, the usual kinetic term, the so-called Wess-Zumino term and a boundary term. All of them contain a parameter $k > 0$ subject to certain consistency conditions (see below). The kinetic term is given by

$$\mathcal{S}^{\mathrm{G}}_{\mathrm{kin}}(g|k) \;=\; -\frac{k}{4\pi}\frac{2}{I_R}\int_\Sigma d^2z\,\mathrm{tr}_R\big\{\partial g g^{-1}\bar{\partial} g g^{-1}\big\} \quad .$$

The trace is evaluated in some non-trivial unitary representation R of the Lie algebra $\mathfrak{g}$ of G. This is indicated by the explicit appearance of the Dynkin index I_R of the representation. The symbol tr_R denotes the trace of $\dim R$-dimensional matrices. The combination $2/I_R\,\mathrm{tr}_R$ is a normalized trace which is independent of the representation R, see appendix A.

The Wess-Zumino term is defined in terms of its associated three-form ω^{WZ}. Its contribution to the boundary WZNW functional (3.29) is given by

$$\mathcal{S}^{\mathrm{G}}_{\mathrm{WZ}}(g|k) \;=\; -\frac{k}{4\pi}\frac{2}{I_R}\int_B \omega^{\mathrm{WZ}} \quad \text{with} \quad \omega^{\mathrm{WZ}}(g) \;=\; \frac{1}{3}\,\mathrm{tr}_R(g^{-1}dg)^3 \;\; . \tag{3.30}$$

This integral extends over a three-dimensional manifold B whose boundary is given by $\partial B = \Sigma \cup D$ where D is a disjoint union of (topological) discs filling the holes of Σ such that $\Sigma \cup D$ has no boundaries [27]. For notational simplicity we will assume that D consists of exactly one disc.

To complete the definition of the boundary WZNW functional (3.29) we finally have to define the boundary term. It is given by the integral

$$\mathcal{S}^{\mathrm{G}}_{\mathcal{D}}(g|k) \;=\; \frac{k}{4\pi}\frac{2}{I_R}\int_D \omega_{\mathcal{D}} \tag{3.31}$$

of a two-form $\omega_{\mathcal{D}}$ over the auxiliary disc D. We assume that the boundary of Σ and the whole disc D are mapped into the subset $\mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\}$ of G. The first condition justifies the use of the word D-brane when referring to this submanifold. The boundary two-form $\omega_{\mathcal{D}}$ entering the definition (3.31) depends crucially on the choice of the D-brane and will be specified below.

The physics which is described by the action functional (3.29) should not depend on the two auxiliary manifolds B and D. This leads to restrictions such as quantization conditions for the level [132] and the allowed transverse position of the D-branes [27, 125, 126]. Let us be a little bit more specific. For compact simply-connected simple Lie groups topological considerations regarding the Wess-Zumino term force k to be an integer. This may be different for non-simply-connected or non-compact groups. In the case of $\mathrm{G} = \mathrm{SO}(3)$ the level k has to be even for instance and for $\mathrm{G} = \mathrm{SL}(2,\mathbb{R})$ we obtain no additional constraints on the level. To show the invariance of the action functional (3.29) under infinitesimal deformations of the disc it suffices to prove the relation $\mathrm{d}\omega_{\mathcal{D}} = \omega^{\mathrm{WZ}}\big|_{\mathcal{D}}$. Taking global aspects of the embedding of the disc into account one recovers certain quantization conditions which render the path integral well-defined. For maximally symmetric branes, i.e. twisted conjugacy classes, these have been shown to coincide with the CFT description [27].

Let us assume for a moment the absence of a world-sheet boundary. In this case, only the first two terms of the action functional (3.29) survive and it is not difficult to show that it is invariant under all transformations $g \mapsto g_{\mathrm{L}}(z)\, g\, g_{\mathrm{R}}^{-1}(\bar{z})$. This is the famous loop group symmetry $\mathrm{G}(z) \times \mathrm{G}(\bar{z})$ of the WZNW theory that we already encountered before at different stages. This symmetry is generated by two Lie algebra valued currents $J(z) = -k\, \partial g g^{-1}$ and $\bar{J}(\bar{z}) = k\, g^{-1} \bar{\partial} g$ which are chiral by the equations of motion. As a consequence one obtains two commuting copies of the affine Kac-Moody algebra $\hat{\mathfrak{g}}_k$ at level k.

In the presence of a world-sheet boundary the previous arguments break down. For Σ being the upper half plane for instance, the two coordinates z and $\bar{z}$ cannot be considered as independent anymore on the real axis $z = \bar{z}$. Calculating the variation of $\mathcal{S}^{\mathrm{G}}_{\mathrm{kin}}(g|k) + \mathcal{S}^{\mathrm{G}}_{\mathrm{WZ}}(g|k)$ under the transformation $g \mapsto g_{\mathrm{L}}(\tau)\, g\, g_{\mathrm{R}}^{-1}(\tau)$ with $\tau \in \partial\Sigma$ thus leads to a non-vanishing result similar to an anomaly. The latter has to be compensated by the boundary term $\mathcal{S}^{\mathrm{G}}_{\mathcal{D}}(g|k)$ which also necessitates to relate g_{L} and g_{R} on the boundary. This procedure strongly reminds one of gauging a global symmetry and this analogy will be confirmed in the next chapter.

3.4.2 The boundary two-form and consistency

The two-form $\omega_{\mathcal{D}}$ which enters the boundary term (3.31) depends on the D-brane one intends to describe. For the D-brane $\mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\}$ which wraps the product of twisted conjugacy classes (3.6) it is given by the expression

$$\omega_{\mathcal{D}} \;=\; \sum_{l=0}^{N} \sum_{k=0}^{l} \omega_{\mathcal{D}}(c_k, \cdots, c_l) \quad . \tag{3.32}$$

The elements $c_l \in \mathrm{U}_l$ parametrize the individual twisted conjugacy classes, see our statements in the vicinity of eq. (3.8). The boundary two-forms entering the previous definition are specified by [57]

$$\begin{aligned} \omega_{\mathcal{D}}(c_l) \;&=\; \mathrm{tr}_R\big\{\epsilon^{\mathrm{U}_l\mathrm{G}}\big(s_l^{-1}\mathrm{d}s_l\, f_l\, \Omega_l(s_l^{-1}\mathrm{d}s_l)\, f_l^{-1}\big)\big\} \\ \omega_{\mathcal{D}}(c_k, \cdots, c_l) \;&=\; -\mathrm{tr}_R\big\{c_l^{-1} \cdots c_k^{-1}\mathrm{d}c_k c_{k+1} \cdots c_{l-1}\mathrm{d}c_l\big\} \quad . \end{aligned} \tag{3.33}$$

If $\mathcal{D}\big\{\mathrm{U}_l, \Omega_l, f_l\big\}$ is a single twisted conjugacy class, the expression (3.32) reduces to that found for maximally symmetric D-branes [27]. For a product of two twisted conjugacy classes we recover boundary terms which have been used to describe maximally symmetric D-branes in coset spaces [125, 126]. Our expressions also contain as a special case the results of [133] by taking the "embedding chain" $\mathrm{U}(1) \hookrightarrow \mathrm{G}$ and a particular choice of automorphisms.

We will now argue that the action functional (3.29) is quantum mechanically well-defined, i.e. single-valued under the path integral. First, infinitesimal deformations of the disc do not alter the value of the action functional as is shown in detail in appendix D.1. Different embeddings of the disc which are not continously connected on the other hand will in general lead to different values of the action. If one imposes certain quantization conditions on the allowed values of the elements f_l, the action can be shown to differ by multiples of 2π, thus rendering the path integral well-defined. This quantization just coincides with the integrality conditions for representation labels of affine Kac-Moody algebra and selection rules for coset sectors which arise in the exact CFT approach [27, 125, 126].

Note that the boundary two-form $\omega_{\mathcal{D}}$ does not only depend on the values of the field g on the boundary but also on the exact decomposition into a product $g = c_0 \cdots c_N$ of elements of the individual twisted conjugacy classes. This observation is very important since under certain circumstances the sets $\mathcal{D}\{\mathrm{U}_l, \Omega_l, f_l\}$ and $\mathcal{D}\{\mathrm{U}'_l, \Omega'_l, f'_l\}$ are identical although their parameters disagree. Yet, the algebraic analysis of the previous sections suggests that they should describe *different* D-branes with different spectrum and different mass density. This becomes particularly important for products of twisted conjugacy classes which cover the whole group G. The solution to this puzzle is connected to the target space reinterpretation which has been presented in section 3.3.1. In the larger "covering" space $\mathrm{G} \times X$ the shape of all our D-branes is indeed distinct.

It remains to be shown that the boundary WZNW functional (3.29) is invariant under the action (3.8, 3.9) of H on the boundary. Let us thus determine the variation of the boundary WZNW functional under an arbitrary infinitesimal action of $h(\tau) = 1 + i\,\omega(\tau) \in \mathrm{H}(\tau)$. A lengthy but straightforward calculation results in

$$\begin{aligned}\delta\omega_{\mathcal{D}} \;=\; -i \sum_{l=0}^{N} \mathrm{tr}\Big\{ \, & \mathrm{d}\omega_{\mathrm{L}}^{(0)} c_0 \cdots c_{l-1} \mathrm{d}c_l c_l^{-1} \cdots c_0^{-1} \; + \\ & + \mathrm{d}\omega_{\mathrm{R}}^{(N)} c_N^{-1} \cdots c_l^{-1} \mathrm{d}c_l c_{l+1} \cdots c_N \Big\} \;\; ,\end{aligned}$$

where the abbreviations $\omega_{\mathrm{L}}^{(0)} = \epsilon^{\mathrm{HU}_0}(\omega)$ and $\omega_{\mathrm{R}}^{(N)} = \epsilon_{\Omega}^{\mathrm{U}_N\mathrm{G}} \circ \Omega_N(\omega)$ have been introduced for convenience. The variation of the Wess-Zumino term may be determined from

$$\delta\omega^{\mathrm{WZ}} \;=\; -i\, \mathrm{d}\, \mathrm{tr}\Big\{ \mathrm{d}\omega_{\mathrm{L}}^{(0)} \mathrm{d}g g^{-1} + \mathrm{d}\omega_{\mathrm{R}}^{(N)} g^{-1} \mathrm{d}g \Big\} \;\; . \tag{3.34}$$

After integration it will give two contributions which arise from the boundary $\Sigma \cup D$ of B. The first one belonging to Σ is canceled by the variation of the kinetic term. If we restrict the discussion to the disc which is mapped to the set $\mathcal{D}\{\mathrm{U}_l, \Omega_l, f_l\}$, the variation (3.34) further simplifies to

$$\delta\omega^{\mathrm{WZ}}\big|_D \; = \; -i \sum_{l=0}^{N} \mathrm{d}\,\mathrm{tr}\Big\{ \mathrm{d}\omega_{\mathrm{L}}^{(0)} c_0 \cdots c_{l-1} \mathrm{d}c_l c_l^{-1} \cdots c_0^{-1} \; + \\ + \, \mathrm{d}\omega_{\mathrm{R}}^{(N)} c_N^{-1} \cdots c_l^{-1} \mathrm{d}c_l c_{l+1} \cdots c_N \Big\} \quad .$$

Obviously, the contributions from $\delta\omega^{\mathrm{WZ}}$ and $\delta\omega_{\mathcal{D}}$ cancel each other exactly. The details of the calculation can be found in appendix D.1. This completes the proof of the symmetry of the D-branes $\mathcal{D}\{\mathrm{U}_l, \Omega_l, f_l\}$ under the action of the subgroup H.

Chapter 4

Asymmetric cosets

An important class of conformal field theories is constituted by coset models. They do not only provide realizations of the minimal models for all kinds of conformal and superconformal algebras. They also describe string theory on coset spaces G/H and have been the core of our previous constructions which led to symmetry breaking boundary states in WZNW theories. The usual notion of a coset refers to gauging the adjoint action of a subgroup H in a WZNW model based on the group G. In the present chapter we will deliver a comprehensive study of asymmetric cosets which result from gauging more general actions of the subgroup H.

4.1 The bulk theory

In this first subsection we are going to describe the bulk geometry of asymmetric cosets. We will start with a detailed formulation of the general setup and of the conditions that conformal invariance imposes on the basic data. The origin of the latter can be explained with the help of the classical actions which we shall briefly recall in the second subsection. We then provide expressions for the bulk partition functions and establish their modular invariance. Finally, we present some examples showing the wide applicability of asymmetric cosets. In an appendix to this section we correct some earlier results of Guadagnini et al. [51, 134].

4.1.1 The geometry of asymmetric cosets

Two Lie groups G and H enter the construction of a coset G/H. Both of them are assumed to be reductive so that they split into a product of simple groups and U(1) factors. Let the number of these factors be n and r, respectively, i.e.

we take G and H to be of the form $\mathrm{G} = \mathrm{G}_1 \times \cdots \times \mathrm{G}_n$ and $\mathrm{H} = \mathrm{H}_1 \times \cdots \times \mathrm{H}_r$. Furthermore we assign to each factor G_i in the decomposition of G a level k_i. It is convenient to combine the set of all these levels into a vector $k = (k_1, \cdots, k_n)$.

Along with the two groups G and H we need to specify an action of H on G. We take the latter to be of the form $g \mapsto \epsilon_{\mathrm{L}}(h)\, g\, \epsilon_{\mathrm{R}}(h^{-1})$ where $\epsilon_{\mathrm{L/R}} : \mathrm{H} \to \mathrm{G}$ denote two group homomorphisms which descend to embeddings of the corresponding Lie algebras. In the usual coset theories ϵ_{L} and ϵ_{R} are the same. An asymmetry in the coset construction arises when we drop this condition and allow for two different maps.

The coset space G/H consists of orbits under the action of H on G,

$$\mathrm{G/H} \;=\; \left\{\, g \in \mathrm{G} \,\middle|\, g \sim \epsilon_{\mathrm{L}}(h)\, g\, \epsilon_{\mathrm{R}}(h^{-1}) \text{ for } h \in \mathrm{H} \,\right\} \quad .$$

To be precise, we should display the dependence on the choice of $\epsilon_{\mathrm{L/R}}$. But since we consider these maps to be fixed once and for all, we decided to suppress them from our symbol G/H for the coset space.

Let us stress, however, that the geometry is very sensitive to the choice of $\epsilon_{\mathrm{L/R}}$. The ordinary adjoint cosets always have fixed points, e.g. the unit element. These lead to all kinds of singularities in the coset geometry, including boundaries and corners. A lot of freedom in model building can be obtained by considering asymmetric cosets. Some of the new theories even possess smooth background geometries like the five-dimensional base $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1)$ of the conifold. Other models have isolated singularities such as the big-bang singularity in the four-dimensional Nappi-Witten geometry $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)/\mathbb{R} \times \mathbb{R}$. The general insights which will be gained in the present chapter will prove useful for the detailed study of the last two models in chapter 5.

The basic data we have introduced so far, i.e. the two groups G, H, the vector k of levels and the maps ϵ_{L}, ϵ_{R}, will enter the construction of two-dimensional non-linear σ-models with target space G/H. To ensure conformal invariance, however, these data have to obey one important constraint which we can formulate using the notion of an embedding index $x_\epsilon \in \mathrm{Mat}(n \times r)$ for the homomorphism $\epsilon : \mathrm{H} \to \mathrm{G}$. To define x_ϵ we split ϵ into a matrix of homomorphisms $\epsilon^{si} : \mathrm{H}_s \hookrightarrow \mathrm{G}_i$ where $s = 1, \ldots, r$, and $i = 1, \ldots, n$, run through the factors of H and G, respectively. The embedding index $x_\epsilon = x = (x^{si})$ is a matrix with elements of the form

$$x^{si} \;=\; \frac{I_s}{I_i}\, \frac{\mathrm{tr}_i\big\{R_i \circ \epsilon^{si}(X)\, R_i \circ \epsilon^{si}(Y)\big\}}{\mathrm{tr}_s\big\{R_s(X)\, R_s(Y)\big\}} \qquad \text{for} \qquad X, Y \in \mathfrak{h}_s \backslash \{0\} \quad . \tag{4.1}$$

The traces are evaluated using two arbitrary non-trivial representations R_i and R_s of the Lie subalgebras $\mathfrak{g}_i$ and $\mathfrak{h}_s$, respectively, whose Dynkin indices I_i, I_s appear as a prefactor. Observe that the number that is computed by the expression on the right hand side does not depend on the choice of the elements X, Y and representations R_i, R_s. Let us also note that the map ϵ^{si} is allowed to map H_s onto the unit element in G_i for some choices of i and s. In this case, the corresponding matrix element x^{si} vanishes.

Let us now consider the embedding indices x_L and x_R for the two homomorphisms ϵ_L and ϵ_R. A conformal theory with target space G/H exists for our choice of levels k, provided that the latter obey the following constraint[1]

$$\boxed{x_\mathrm{L}\, k \;=\; x_\mathrm{R}\, k \quad .} \tag{4.2}$$

In other words, the vector of levels must lie in the kernel of $x_\mathrm{L} - x_\mathrm{R}$. For symmetric cosets this condition is trivially satisfied with any choice of k. Asymmetric cosets, however, constrain the admissible levels.

4.1.2 The classical action functional

Using the basic data we have introduced in the previous subsection, we can write down the classical action of the asymmetrically gauged WZNW model. As usual, this consists of several pieces. To begin with, there is the WZNW action for the numerator group G,

$$\mathcal{S}^{\mathrm{G}}_{\mathrm{WZNW}}\big(g|k\big) \;=\; \sum_{i=1}^{n} \mathcal{S}^{\mathrm{G}_i}_{\mathrm{WZNW}}(g_i|k_i) \tag{4.3}$$

with $g = g_1 \cdots g_n$. This action is a sum over the WZNW actions for the individual groups G_i without any interaction terms. These building blocks are explicitly given by

$$\mathcal{S}^{\mathrm{G}_i}_{\mathrm{WZNW}}(g_i|k_i) \;=\; -\frac{k_i}{4\pi}\frac{2}{I_i}\int_\Sigma d^2z\; \mathrm{tr}_i\big\{\partial g_i g_i^{-1}\,\bar{\partial} g_i g_i^{-1}\big\} + \mathcal{S}^{\mathrm{G}_i}_{\mathrm{WZ}}(g_i|k_i) \quad .$$

The fields g_i are evaluated in some (unspecified) non-trivial representation of the group G_i, whose Dynkin index I_i appears as a prefactor. The last contribution is defined as usual in terms of the Wess-Zumino three-forms ω_i^{WZ} given in eq. (3.30). Consistency of the associated quantum theories enforces quantization constraints on the levels k_i. For simply-connected simple constituents G_i the level k_i has to be an integer. For the U(1)-part and non-simply-connected groups the constraint is different.

[1]If there are two or more identical groups among the H_s or G_i, this equation has to hold up to a possible relabeling of these groups on one side.

The action functional (4.3) is invariant under "global" transformations of the form $g(z,\bar{z}) \mapsto g_{\rm L}(z)\, g(z,\bar{z})\, g_{\rm R}^{-1}(\bar{z})$ where $g_{\rm L}(z)$ and $g_{\rm R}(\bar{z})$ are arbitrary (anti-) holomorphic G-valued functions. Our subgroup H along with the two homomorphisms $\epsilon_{\rm L/R}$ can be used to gauge some part of this WZNW symmetry. To this end we consider the model

$$\mathcal{S}^{\rm G/H}\big(g, A, \bar{A}\big|\, k, \epsilon_{\rm L/R}\big) \;=\; \sum_{i=1}^{n} \mathcal{S}^{\rm G_i}_{\rm WZNW} + \sum_{i=1}^{n}\sum_{s=1}^{r} \mathcal{S}^{\rm G_i/H_s}_{\rm int} \quad . \tag{4.4}$$

For lack of space we left out the functional dependence on the fields and the parameters. The building blocks $\mathcal{S}^{\rm G_i/H_s}_{\rm int} \equiv \mathcal{S}^{\rm G_i/H_s}_{\rm int}\big(g_i, A_s, \bar{A}_s\big|\, k_i, \epsilon^{si}_{\rm L/R}\big)$ of the second term are given by [52]

$$\begin{aligned}\mathcal{S}^{\rm G_i/H_s}_{\rm int} \;=\; & \frac{k_i}{4\pi}\frac{2}{I_i}\int_\Sigma d^2z\ {\rm tr}_i\big\{\, 2\,\epsilon_{\rm L}(\bar{A}_s)\,\partial g_i g_i^{-1} - 2\,\epsilon_{\rm R}(A_s)\, g_i^{-1}\bar{\partial} g_i \\ & + 2\,\epsilon_{\rm L}(\bar{A}_s)\, g_i\, \epsilon_{\rm R}(A_s)\, g_i^{-1} - \epsilon_{\rm L}(\bar{A}_s)\,\epsilon_{\rm L}(A_s) - \epsilon_{\rm R}(\bar{A}_s)\,\epsilon_{\rm R}(A_s)\big\} \quad .\end{aligned} \tag{4.5}$$

In this formula we omitted the superscripts ^{si} on $\epsilon_{\rm L/R}$. The gauge fields $A_s, \bar{A}_s$ take values in the Lie algebra $\mathfrak{h}_s$. It is not difficult to check that the full action (4.4) is invariant under the following set of infinitesimal gauge transformations

$$\begin{aligned}\delta A_s \;&=\; i\,\partial\omega_s + i\,[\omega_s, A_s] \quad , \quad \delta\bar{A}_s \;=\; i\,\bar{\partial}\omega_s + i\,[\omega_s, \bar{A}_s] \quad , \\ \delta g_i \;&=\; i\,\epsilon_{\rm L}(\omega_s)\, g_i - i\, g_i\,\epsilon_{\rm R}(\omega_s) \qquad \text{for} \quad \omega_s = \omega_s(z,\bar{z}) \in \mathfrak{h}_s\end{aligned}$$

provided that the levels k_i obey the constraint (4.2). In fact, under gauge transformations the action behaves according to

$$\begin{aligned}\delta\mathcal{S}^{\rm G/H} \;=\; \sum_{i=1}^{n}\sum_{s=1}^{r} \frac{k_i}{4\pi}\frac{2}{I_i}\int_\Sigma d^2z\ {\rm tr}_i\big\{\, & \epsilon^{si}_{\rm L}(\bar{A}_s)\,\partial\epsilon^{si}_{\rm L}(\omega_s) - \epsilon^{si}_{\rm R}(\bar{A}_s)\,\partial\epsilon^{si}_{\rm R}(\omega_s) \\ & + \epsilon^{si}_{\rm R}(A_s)\,\bar{\partial}\epsilon^{si}_{\rm R}(\omega_s) - \bar{\partial}\epsilon^{si}_{\rm L}(\omega_s)\,\epsilon^{si}_{\rm L}(A_s)\big\}\end{aligned}$$

and so it vanishes whenever eq. (4.2) holds true. We have therefore shown that the data introduced above indeed label different two-dimensional conformal field theories.

4.1.3 Algebraic description of asymmetric cosets

Our aim now is to present a few elements of the exact solution. We shall begin with some remarks on the relevant chiral algebras and then address the construction of the modular invariant partition function for our asymmetric coset theories.

Chiral algebras and heteroticity

In the following let us denote the chiral algebra of the WZNW model for the group G and levels k_i by $\mathcal{A}(\mathrm{G})$. This algebra is generated by a sum of affine Lie algebras with levels k_i, one for each factor in the decomposition of the reductive group G. The two maps $\epsilon_{\mathrm{L/R}}$ give rise to two embeddings of the chiral algebra $\mathcal{A}(\mathrm{H})$ into $\mathcal{A}(\mathrm{G})$. Let us note that $\mathcal{A}(\mathrm{H})$ is generated by a sum of affine algebras, one for each factor in the product $\mathrm{H} = \mathrm{H}_1 \times \cdots \times \mathrm{H}_r$. The levels of these affine algebras form a vector $(k'_s)_{s=1,\ldots,r}$ whose entries are related to the levels of $\mathcal{A}(\mathrm{G})$ by $k' = x_{\mathrm{L/R}}\, k$ (matrix notation). Our assumption (4.2) means that $\epsilon_{\mathrm{L/R}}$ give rise to two (possibly *inequivalent*) embeddings of the *same* chiral algebra $\mathcal{A}(\mathrm{H})$ into $\mathcal{A}(\mathrm{G})$. Given these embeddings, we employ the usual GKO construction to obtain two coset algebras $\mathcal{A} = \mathcal{A}(\mathrm{G/H}, \epsilon_{\mathrm{L}})$ and $\bar{\mathcal{A}} = \mathcal{A}(\mathrm{G/H}, \epsilon_{\mathrm{R}})$ which form the left and right chiral algebras of the asymmetric coset model. Note that these two chiral algebras can be different if the two maps ϵ_{L} and ϵ_{R} are not identical. In this sense, asymmetric coset models of the kind that we consider in this note are heterotic conformal field theories.

A proposal for the partition function

The state space of any conformal field theory decomposes into representations of the chiral algebras. Our task now is to find a combination of these representations which does not only reflect the geometry of the target space G/H but is at the same time also consistent from a conformal field theory point of view. The second requirement implies that the vacuum must be unique and that the partition function is modular invariant.

The first condition, namely the relation of our exact solution to the space G/H, means that in the limit of large levels k the space of ground states has to reproduce the space of functions on G/H. Actually, we can turn this around for a moment and use the harmonic analysis on G/H to get some ideas about the structure of the state space. To this end, let us recall that the algebra $\mathcal{F}(\mathrm{G})$ of functions on G may be considered as a $\mathrm{G} \times \mathrm{G}$-module under left and right regular action. The Peter-Weyl theorem states that this module decomposes into irreducibles according to

$$\mathcal{F}(\mathrm{G}) \;=\; \bigoplus V_\mu \otimes V_{\mu^+} \quad ,$$

where μ^+ is the conjugate of μ. Since we want to divide G by the action of H, it is convenient to decompose the space of functions on G into representations of H. The space of functions on G/H is then obtained as the H-invariant part of $\mathcal{F}(\mathrm{G})$. We easily find

$$\begin{aligned}\mathcal{F}(\mathrm{G}) &\cong \bigoplus V_\mu \otimes V_{\mu^+} \\ &\cong \bigoplus (b_\mathrm{L})_\mu{}^a\,(b_\mathrm{R})_{\mu^+}{}^{c^+}\; V_a \otimes V_{c^+} \\ &\cong \bigoplus (b_\mathrm{L})_\mu{}^a\,(b_\mathrm{R})_{\mu^+}{}^{c^+}\; N_{ac^+}^{\;d}\; V_d \quad .\end{aligned} \tag{4.6}$$

The symbols $b_{\mathrm{L/R}}$ denote the branching coefficients of the inclusion $\epsilon_{\mathrm{L/R}} : \mathrm{H} \hookrightarrow \mathrm{G}$. The coefficients $N_{ac^+}^{\;d}$ for the decomposition of the tensor product of representations of H enter when we restrict the action of $\mathrm{H} \times \mathrm{H}$ to its diagonal subgroup $\mathrm{H} = \mathrm{H}_D$. Taking the invariant part of (4.6) corresponds to putting $d = 0$ or, equivalently, $a = c$ and hence we have shown that

$$\mathcal{F}(\mathrm{G/H}) \;=\; \mathrm{Inv}_{\mathrm{H}_D}\big(\mathcal{F}(\mathrm{G})\big) \;\cong\; \bigoplus\, (b_\mathrm{L})_\mu{}^a\,(b_\mathrm{R})_{\mu^+}{}^{a^+}\;\mathbb{C} \quad .$$

This is the space that we want to reproduce from the ground states of our exact solution when we send the levels to infinity. With a bit of experience in coset chiral algebras and their representation theory it is not too difficult to come up with a good proposal for the conformal field theory state space that meets this requirement.

The rough idea is to replace the branching coefficients $b_\mu{}^a$ by coset sectors $\mathcal{H}^{\mathrm{G/H}}_{(\mu,a)}$. But this rule is a bit too simple and has to be refined in several directions. First of all, the coset sectors are labeled by representations of the affine Lie algebras $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{h}}$ not by those of the groups G and H. This observation leads to a natural cut-off restricting the representations which one should take along. It will be crucial in the following that due to relation (4.2) this cut-off will be the same for the representations of the holomorphic and the antiholomorphic chiral algebra.

Further issues which always enter the representation theory of coset chiral algebras are selection rules and field identification. In this case, however, the sets of representations for the holomorphic and antiholomorphic part do not necessarily coincide and the corresponding rules may disagree. As a consequence, it is an absolutely non-trivial task to combine left and right representations in such a way that they lead to a modular invariant partition function. Fortunately, at least the sets of *labels* for the representations are identical for left and right movers due to condition (4.2).

In the following we label sectors of $\mathcal{A}(\mathrm{G})$ by $\mu, \nu, \ldots$, and we use the letters $a, b, \ldots$, for sectors of $\mathcal{A}(\mathrm{H})$. Let us denote by $(\mathrm{G/H})_{\mathrm{L/R}}$ the cosets which are based on the embeddings $\epsilon_{\mathrm{L/R}} : \mathrm{H} \hookrightarrow \mathrm{G}$ and by $\mathcal{G}_{\mathrm{id}}(\mathrm{L/R})$ the associated field identification groups. Based on these quantities we can introduce the functions $\mathrm{P}_{\mathrm{L/R}}(\mu, a)$ which project onto allowed coset labels All(L/R). More details of these constructions have been provided in section 2.4.2.

Having introduced all the relevant notions from the representation theory of coset chiral algebras we are finally able to spell out our proposal for the state space,

$$\boxed{\mathcal{H}^{\mathrm{G/H}} \;=\; \bigoplus_{[\mu,a]\in\mathrm{Rep(G/H)}} \mathcal{H}^{(\mathrm{G/H})_{\mathrm{L}}}_{(\mu,a)} \otimes \bar{\mathcal{H}}^{(\mathrm{G/H})_{\mathrm{R}}}_{(\mu,a)^+} \quad ,} \tag{4.7}$$

where the set Rep(G/H) is defined by

$$\begin{aligned} \mathrm{Rep(G/H)} \;&=\; \mathrm{All(G/H)}/\mathcal{G}_{\mathrm{id}} \quad \text{with} \\ \mathrm{All(G/H)} \;&=\; \mathrm{All(G/H)_L} \cap \mathrm{All(G/H)_R} \quad \text{and} \quad \mathcal{G}_{\mathrm{id}} \;=\; \mathcal{G}_{\mathrm{id}}(\mathrm{L}) \cap \mathcal{G}_{\mathrm{id}}(\mathrm{R}) \quad . \end{aligned} \tag{4.8}$$

Note that the field identification group $\mathcal{G}_{\mathrm{id}}$ admits a natural interpretation as the stabilizer of the action $g \mapsto \epsilon_{\mathrm{L}}(h)\, g\, \epsilon_{\mathrm{R}}(h^{-1})$, i.e.

$$\mathcal{G}_{\mathrm{id}} \;\cong\; \big\{(z, z') \,\big|\, z' \in \mathcal{Z}(\mathrm{H}),\; z = \epsilon_{\mathrm{L}}(z') = \epsilon_{\mathrm{R}}(z') \in \mathcal{Z}(\mathrm{G})\big\} \quad .$$

In writing our formula (4.7) we implicitly assumed that the action of the field identification group $\mathcal{G}_{\mathrm{id}}$ on All(G/H) possesses no fixed points or, equivalently, that all orbits $[\mu, a]$ have the same length. It should be stressed that fixed points for the action of $\mathcal{G}_{\mathrm{id}}(\mathrm{L/R})$ on $\mathrm{All(G/H)_{L/R}}$ are not ruled out by this assumption.

Although the expression (4.7) looks very simple and almost like a charge conjugate state space, it is actually quite non-trivial. Let us in particular emphasize once more that the representations which appear in the holomorphic and antiholomorphic parts are completely different and are by no means comparable even though they are *labeled* by the same set.

Modular invariance

As we mentioned before, our proposal (4.7) for the state space has to pass a number of tests before we can accept it as a candidate for the state space of our conformal field theory. From our discussion above it is not difficult to see that at large level, the space of ground states coincides with the space of functions on G/H. Moreover, taking the quotient of allowed weights with respect to $\mathcal{G}_{\mathrm{id}}$ in eq. (4.8) ensures that there is a unique vacuum in $\mathcal{H}^{\mathrm{G/H}}$. Hence, it only remains to demonstrate that our Ansatz also leads to a modular invariant partition function. To this end it is convenient to write the partition function in the form[2]

$$Z(q,\bar q) \;=\; \frac{1}{|\mathcal{G}_{\rm id}|}\sum_{\mu,a} \mathrm{P}_{\rm L}(\mu,a)\,\mathrm{P}_{\rm R}(\mu,a)\;\chi^{(\mathrm{G/H})_{\rm L}}_{(\mu,a)}(q)\,\bar\chi^{(\mathrm{G/H})_{\rm R}}_{(\mu,a)}(\bar q)\quad .$$

The factor $1/|\mathcal{G}_{\rm id}|$ in front of this expression removes a common factor from the whole expression in such a way that the vacuum character possesses a trivial prefactor. The summation in the previous expression runs over all labels μ and a and we enforce the restriction to the allowed coset labels by inserting the corresponding projectors.

It is now rather straightforward to compute the behaviour of this partition function under the modular S-transformation that replaces $q = \exp(2\pi i\,\tau)$ by $\tilde q = \exp(-2\pi i/\tau)$,

$$\mathrm{S}Z(q,\bar q) \;=\; \sum_{\mu,a,\nu,\lambda,b,c} \frac{\mathrm{P}_{\rm L}(\mu,a)\,\mathrm{P}_{\rm R}(\mu,a)}{|\mathcal{G}_{\rm id}|}\; S^{\rm G}_{\mu\nu}\bar S^{\rm H}_{ab}\bar S^{\rm G}_{\mu\lambda}S^{\rm H}_{ac}\;\chi^{(\mathrm{G/H})_{\rm L}}_{(\nu,b)}(q)\,\bar\chi^{(\mathrm{G/H})_{\rm R}}_{(\lambda,c)}(\bar q)\quad .$$

We would like to use the unitarity of the S-matrices to simplify this expression. But before we can do so, we have to eliminate the projectors. To this end we insert the explicit formulas (2.23) for the projectors in terms of monodromy charges and then pull the latter into the S-matrices by shifting their indices with the action of simple currents, see eq. (2.7). This gives

$$\mathrm{S}Z(q,\bar q) \;=\; \sum_{\substack{(J_1,J_1')\in\mathcal{G}_{\rm id}(\mathrm{L})\\ (J_2,J_2')\in\mathcal{G}_{\rm id}(\mathrm{R})}}\;\sum_{\mu,a,\nu,\lambda,b,c} \frac{S^{\rm G}_{\mu J_1\nu}\bar S^{\rm H}_{aJ_1'b}\bar S^{\rm G}_{\mu J_2\lambda}S^{\rm H}_{aJ_2'c}\;\chi^{(\mathrm{G/H})_{\rm L}}_{(\nu,b)}(q)\bar\chi^{(\mathrm{G/H})_{\rm R}}_{(\lambda,c)}(\bar q)}{|\mathcal{G}_{\rm id}|\cdot|\mathcal{G}_{\rm id}(\mathrm{L})|\cdot|\mathcal{G}_{\rm id}(\mathrm{R})|}\quad .$$

Now we are able to perform the sum over μ and a to obtain

$$\mathrm{S}Z(q,\bar q) \;=\; \sum_{\substack{(J_1,J_1')\in\mathcal{G}_{\rm id}(\mathrm{L})\\ (J_2,J_2')\in\mathcal{G}_{\rm id}(\mathrm{R})}}\;\sum_{\nu,\lambda,b,c} \frac{\delta^{J_2\lambda}_{J_1\nu}\,\delta^{J_1'b}_{J_2'c}\;\chi^{(\mathrm{G/H})_{\rm L}}_{(\nu,b)}(q)\,\bar\chi^{(\mathrm{G/H})_{\rm R}}_{(\lambda,c)}(\bar q)}{|\mathcal{G}_{\rm id}|\cdot|\mathcal{G}_{\rm id}(\mathrm{L})|\cdot|\mathcal{G}_{\rm id}(\mathrm{R})|}\quad .$$

At this stage we may resum the label. Then we see that part of the prefactor cancels and we are left with

$$\mathrm{S}Z(q,\bar q) \;=\; \frac{1}{|\mathcal{G}_{\rm id}|}\sum_{\nu,b} \chi^{(\mathrm{G/H})_{\rm L}}_{(\nu,b)}(q)\,\bar\chi^{(\mathrm{G/H})_{\rm R}}_{(\nu,b)}(\bar q)\quad .$$

[2]For notational reasons we omit the charge conjugation in the second character. This simplification does not affect the outcome of the calculation.

This is exactly the behavior modular invariance requires from our partition function. Note that the restriction to allowed coset labels is implicitly contained in the previous expression since coset characters vanish if the relevant branching selection rule is not satisfied. Let us finally add that our partition function is also invariant under modular T-transformations which send τ to $\tau+1$.

4.1.4 Special cases and examples

Our general construction includes a number of interesting special cases. We shall therefore use this subsection to discuss two different classes of asymmetric cosets. These will suffice to describe the most familiar examples, the Nappi-Witten background [53] and the T^{pq}-spaces [54].

Asymmetric cosets from automorphisms

In the simplest imaginable setup for asymmetric cosets the left and right embeddings are related by automorphisms. More precisely, we are thinking of situations in which the left homomorphism $\epsilon_{\rm L} = \epsilon$ is related to $\epsilon_{\rm R} = \Omega_{\rm G} \circ \epsilon \circ \Omega_{\rm H}^{-1}$ by composition with two automorphisms $\Omega_{\rm G}$ and $\Omega_{\rm H}$ of G and H, respectively. Let us notice that the concatenation of an embedding with an automorphism gives another embedding with the same embedding index. This observation guarantees the validity of the anomaly cancellation condition (4.2). In contrast to our treatment of cosets in earlier chapters we will now *not* assume that left and right embedding coincide. Otherwise we would simply end up with an ordinary adjoint coset with a partition function of automorphism type.

For the explicit construction of the state space (4.7) we have to know the centers $\mathcal{G}_{\rm id}({\rm L})$ and $\mathcal{G}_{\rm id}({\rm R})$ in detail. Note that every element $(J, J') \in \mathcal{G}_{\rm id}({\rm L})$ is mapped to an element $\big(\omega_{\rm G}(J), \omega_{\rm H}(J')\big) \in \mathcal{G}_{\rm id}({\rm R})$ by the action of the pair $(\omega_{\rm G}, \omega_{\rm H})$. The right center is thus the image of the left center, $\mathcal{G}_{\rm id}({\rm R}) = (\omega_{\rm G}, \omega_{\rm H})\big(\mathcal{G}_{\rm id}({\rm L})\big)$, and the common center is the intersection of these two sets. Similarly the allowed coset labels are related by ${\rm All}({\rm G/H})_{\rm R} = (\omega_{\rm G}, \omega_{\rm H})\big({\rm All}({\rm G/H})_{\rm L}\big)$. To prove this statement one employs the invariance property (2.9) of the monodromy charges with respect to automorphisms.

These observations enable us to find a rather explicit expression for the state space. In our example the general formula (4.7) can be simplified due to the fact that left and right moving chiral algebra are isomorphic. We will therefore express the state space in terms of quantities of the left chiral algebra only. All we need to do is to replace the coset representations $\mathcal{H}^{({\rm G/H})_{\rm R}}_{(\mu,a)}$

through $\mathcal{H}^{(G/H)_L}_{(\omega_G(\mu),\omega_H(a))}$. By construction, the latter is non-trivial if and only if the first one was. If we combine these facts we finally arrive at

$$\mathcal{H}^{G/H} \;=\; \bigoplus_{[\mu,a]\in\text{Rep}(G/H)} \mathcal{H}^{(G/H)_L}_{(\mu,a)} \otimes \bar{\mathcal{H}}^{(G/H)_L}_{(\omega_G(\mu),\omega_H(a))^+} \quad .$$

Let us emphasize once more that the coset sectors are *both* defined with respect to the *same* embedding $\epsilon = \epsilon_L$ in this expression. The asymmetry enters in the explicit appearance of the twists of labels and in an (implicit) reduction of labels over which we sum.

The most prominent example of asymmetric cosets of the type considered in this subsection is provided by the Nappi-Witten background [53]. It is obtained as a coset of the product group $G = SL(2,\mathbb{R}) \times SU(2)$ with respect to some abelian subgroup $H = \mathbb{R}\times\mathbb{R}$. In this case, the automorphism Ω_G is trivial while Ω_H exchanges the two factors of $\mathbb{R}$. The model will be discussed in detail in section 5.3.

Examples of GMM-type

Let us now consider a slightly more complicated family of examples in which the numerator group is a product $G_1 \times G_2$ of two groups G_1 and G_2 which possess a common subgroup H. Our aim is to describe the coset $G_1 \times G_2/H$ where the first homomorphism $\epsilon_L = e \times \epsilon'_2$ embeds H into the group G_2 and $\epsilon_R = \epsilon'_1 \times e$ sends elements of H into G_1. The Lagrangian description of such models was developed by Guadagnini, Martellini and Mintchev (GMM) more than fifteen years ago [51, 134]. In appendix D.2 we show how their results can be recovered from the more general expression (4.4). We also use the opportunity to correct some statements of GMM concerning the current algebra relations and the validity of the affine Sugawara/GKO construction for this type of coset models.

The Lagrangian treatment of appendix D.2 and algebraic intuition lets us expect manifestly heterotic coset models with a symmetry based on the product of chiral algebras

$$\mathcal{A}\big((G_1)_{k_1}\big) \otimes \mathcal{A}\big((G_2)_{k_2}/H_k\big) \otimes \bar{\mathcal{A}}\big((G_1)_{k_1}/H_k\big) \otimes \bar{\mathcal{A}}\big((G_2)_{k_2}\big) \quad .$$

One can easily see that the field identification group for the coset $G_1 \times G_2/H$ is given by

$$\mathcal{G}_{\text{id}} \;=\; \big\{(0,0,J') \,\big|\, (0,J') \in \mathcal{G}_{\text{id}}(G_1/H) \cap \mathcal{G}_{\text{id}}(G_2/H)\big\} \quad .$$

The allowed coset labels consist of triples (μ,α,a) such that (μ,a) and (α,a) are allowed for the G_1/H and G_2/H cosets, respectively. Coset representations are then obtained by dividing out the field identifications $\mathcal{G}_{\text{id}}$. The resulting state space simply reads

$$\mathcal{H} \;=\; \bigoplus_{[\mu,\alpha,a]\in \mathrm{Rep}(\mathrm{G}/\mathrm{H})} \mathcal{H}_{\mu}^{\mathrm{G}_1} \otimes \mathcal{H}_{(\alpha,a)}^{\mathrm{G}_2/\mathrm{H}} \otimes \bar{\mathcal{H}}_{(\mu,a)^+}^{\mathrm{G}_1/\mathrm{H}} \otimes \bar{\mathcal{H}}_{\alpha^+}^{\mathrm{G}_2} \quad . \tag{4.9}$$

It reflects the fact that in both the left and the right moving algebra one still finds a residual current symmetry.

For the physical applications we are particularly interested in a special choice of product group and subgroup, namely $\mathrm{G}_1 = \mathrm{G}_2 = \mathrm{SU}(2)$ and $\mathrm{H} = \mathrm{U}(1)$. Under these circumstances the GMM-model describes five-dimensional non-Einstein T^{pq}-spaces [54]. The special case $p = q = 1$ admits a direct interpretation as the base of the conifold (see, e.g., [135]). This example will be discussed in detail in section 5.4.

4.2 The boundary theory

In this section we will investigate the construction of boundary states in asymmetric coset theories and their geometric interpretation. The deep connection to the corresponding analysis in WZNW models will be illuminated. It turns out that *any* boundary functional for a WZNW theory which respects some arbitrary symmetry H on the boundary may be used to describe the boundary functional for the associated *gauged* WZNW theory in which the given action of H is divided out. The inverse statement holds also true. We comment on the algebraic considerations which are necessary to construct boundary states and provide a rough classification of asymmetric cosets. The heterotic nature of these models will force us to break part of the bulk symmetry which in some cases presumably leads to a non-rational boundary theory.

4.2.1 The Lagrangian approach

The action functional of a bulk gauged WZNW theory is invariant under all transformations $g' \mapsto \epsilon_{\mathrm{L}}(h)\, g'\, \epsilon_{\mathrm{R}}(h^{-1})$ for arbitrary functions $h \equiv h(z,\bar{z})$ which take values in the subgroup H. In contrast to the group case one is thus not necessarily faced with the problem of having to relate the purely holomorphic symmetry $g' \mapsto g_{\mathrm{L}}(z)\, g'$ with the purely antiholomorphic symmetry $g' \mapsto g'\, g_{\mathrm{R}}(\bar{z})$ on the boundary where z and $\bar{z}$ are not independent anymore. Naively, it should suffice to map the field g on the boundary into an arbitrary subset of G which is invariant under the action of H to be gauged away.

As one might have expected, this simple picture fails to succeed. If Σ has a boundary, one indeed has to modify the Wess-Zumino term of the bulk action (4.4) since there does not exist a manifold B such that its boundary is given by Σ. As in the WZNW case, one has to glue a set D of discs to the boundaries of Σ such that the disjoint union $\Sigma \cup D$ possesses no boundary. But if one then calculates the anomaly under the transformation $g' \mapsto \epsilon_\mathrm{L}(h)\, g'\, \epsilon_\mathrm{R}(h^{-1})$, one ends up with an integral over the discs D which remains from the variation of the Wess-Zumino term. Since this anomaly obviously is identical to the one which arises in the WZNW case, it can for instance be compensated by adding one of the boundary terms which have been worked out in section 3.4.

The results of the last two paragraphs may be summarized as follows:

> Every D-brane on G whose world-volume is invariant under the action $x \mapsto \epsilon_\mathrm{L}(h)\, x\, \epsilon_\mathrm{R}(h^{-1})$ of a subgroup H may be projected down to the corresponding coset G/H. Vice versa, every brane on an arbitrary coset G/H of the previous form can be interpreted as a brane on the group G itself. The only difference in the boundary action functionals consists in the *bulk* interaction term which has to be added for the coset theories in order to obtain the desired local gauge invariance.

These statements up to now are solely based on the classical action. However, we will soon see that the general picture is supported in the full quantum theory which results from the algebraic description.

Let us conclude with a remark. The previous results imply that all the branes on group manifolds which have been constructed in section 3.4 can be considered as a brane in a suitably chosen coset. Yet, they do not rule out the possibility to obtain further branes both for groups and cosets.

4.2.2 The algebraic description

The construction of boundary states relies on formulating gluing conditions for the chiral currents in a *common* conformal subalgebra of $\mathcal{A}$ and $\bar{\mathcal{A}}$. Despite of the possibility of asymmetric cosets being heterotic such an algebra always exists since both chiral algebras contain an identical copy of the Virasoro algebra. In most of the cases, however, the original theory will not be rational with respect to the Virasoro algebra. We thus require a systematic method to recover intermediate symmetries.

The canonical way to construct subalgebras of a coset algebra $\mathcal{A}(\mathrm{G/H})$ consists of considering some intermediate group $\mathrm{H} \hookrightarrow \mathrm{U} \hookrightarrow \mathrm{G}$. In this framework one is naturally led to the rational conformal embedding

$$\mathcal{A}(\mathrm{G/U}) \otimes \mathcal{A}(\mathrm{U/H}) \;\hookrightarrow\; \mathcal{A}(\mathrm{G/H}) \quad . \tag{4.10}$$

As in the group case this procedure may be iterated based on arbitrary embedding chains of the form (3.3). Nevertheless, it might turn out to be impossible to find a *common* subalgebra using this procedure if one starts with arbitrary holomorphic and antiholomorphic coset chiral algebras $\mathcal{A}(\mathrm{G}/\mathrm{H}, \epsilon_{\mathrm{L/R}})$ arising from different embeddings of the subgroup H.

To illustrate this statement assume a numerator of the product form $\mathrm{G} = \mathrm{SU}(3)_k \times \mathrm{SU}(3)_{3k}$ where the ratio of the group sizes is fixed as indicated. There exist two different ways to embed the subgroup $\mathrm{H} = \mathrm{SU}(2)_{4k}$ into G. To see how this works we have to recall that there are two ways of embedding SU(2) into SU(3), one with embedding index $x_\epsilon = 1$ and the other one with $x_\epsilon = 4$. Let us denote the associated embeddings by ϵ_1 and ϵ_4, respectively. The left action of H on G is then defined using the diagonal embedding $\epsilon_{\mathrm{L}} = (\epsilon_1, \epsilon_1)$ while for the right action one takes $\epsilon_{\mathrm{R}} = (\epsilon_4, e)$, i.e. the subgroup is solely embedded into the first group factor. Obviously, the consistency condition (4.2) is satisfied such that the coset is well-defined.

For the holomorphic chiral algebra there only exist intermediate groups of the form $\mathrm{SU}(2)_k \times \mathrm{SU}(2)_{3k}$ or the diagonal $\mathrm{SU}(3)_{4k}$. Yet, none of these is suitable for a decomposition of the antiholomorphic chiral algebra because the embedding of SU(2) in SU(3) is already maximal. Consequently, in this case it is impossible to identify a common subsymmetry of left and right movers using decompositions of the form (4.10). This fact suggests that for this model there exists *no* common *rational* subalgebra at all although we do not have a rigorous proof for this conjecture.

The reasoning of the last few paragraphs leads us to the following conclusion. For a rough classification of asymmetric coset theories it is convenient to distinguish two cases which require a qualitatively completely different treatment:

1. There exists *no* common conformal subalgebra with respect to which the theory remains rational. In this case the boundary theory will inevitably be non-rational despite the rationality of the bulk theory.

2. If there exists at least one common subalgebra $\mathcal{A}_{\mathrm{red}}$ with respect to which the theory stays rational, the same will hold for the boundary theory which preserves this algebra.

An example which presumably falls into the first class was already given in the previous paragraphs. Among the models which belong to the second class one finds in particular all the WZNW theories and ordinary adjoint coset models. In the next section we will introduce and discuss a rather large

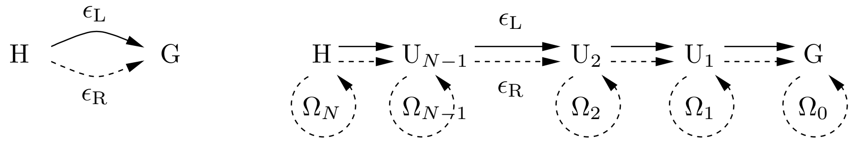

Figure 4.1: Asymmetric cosets of generalized automorphism type.

class of so-called cosets of generalized automorphism type which also pertain to this class.[3]

4.3 Cosets of generalized automorphism type

A large class of asymmetric cosets is introduced which covers all the examples of section 4.1.4 and whose boundary theory may be solved. Our construction relies on the identification of smaller chiral symmetries for which the boundary theory remains rational. Formulas for the boundary states and the partition functions of the boundary theories will be provided at the end of the section. Finally, we comment on the geometric interpretation of the boundary states.

4.3.1 Definition and examples

We will call an asymmetric coset G/H to be of generalized automorphism type whenever the following condition holds: there should exist a chain of subgroups U_i together with embedding maps $\epsilon_i : \mathrm{U}_i \to \mathrm{U}_{i-1}$ and a collection of automorphisms Ω_i such that

$$\boxed{\begin{aligned} \mathrm{H} &= \mathrm{U}_N \hookrightarrow \mathrm{U}_{N-1} \hookrightarrow \cdots \hookrightarrow \mathrm{U}_1 \hookrightarrow \mathrm{U}_0 = \mathrm{G} \;, \\ \epsilon_{\mathrm{L}} &= \epsilon_1 \circ \epsilon_2 \circ \cdots \circ \epsilon_N \qquad \text{and} \\ \epsilon_{\mathrm{R}} &= \Omega_0 \circ \epsilon_1 \circ \Omega_1 \circ \epsilon_2 \circ \cdots \circ \epsilon_{N-1} \circ \Omega_{N-1} \circ \epsilon_N \circ \Omega_N \;. \end{aligned}} \tag{4.11}$$

This definition is strongly inspired by the treatment of symmetry breaking boundary conditions in chapter 3. Note, however, that we did not demand the left and right action of H on G to coincide, i.e. we allow for $\epsilon_{\mathrm{L}} \neq \epsilon_{\mathrm{R}}$. An illustration of the conditions (4.11) can be found in figure 4.1.

Our general prescription includes a number of very familiar classes of CFT's. First of all, one can recover ordinary cosets with a partition function

[3] In the last few lines the statement of rationality is only valid if the numerator group is compact.

Figure 4.2: The action of the subgroup in GMM- and Nappi-Witten models.

of automorphism type whenever $\epsilon_{\rm L} = \epsilon_{\rm R}$. For H being the trivial group one ends up with the WZNW case. For the latter our new construction has already been applied successfully to determine symmetry breaking boundary conditions. This observation supports our claim that the boundary theory for asymmetric cosets of generalized automorphism type should be rather easy to access. Our assumptions on the existence of a chain of embeddings and its properties indeed guarantee that the resulting theories are rational with respect to the common conformal subalgebra

$$\mathcal{A}_{\rm red} \;=\; \mathcal{A}({\rm U}_0/{\rm U}_1) \otimes \mathcal{A}({\rm U}_1/{\rm U}_2) \otimes \cdots \otimes \mathcal{A}({\rm U}_{N-1}/{\rm U}_N) \;\hookrightarrow\; \mathcal{A}({\rm G}/{\rm H}) \quad (4.12)$$

of the holomorphic and antiholomorphic chiral algebra. But before we present the solution of the boundary theory, let us first show how the physically most important examples fit into the general scheme.

Examples

Remarkably, all explicit examples of asymmetric cosets which have been introduced in section 4.1.4 can easily be seen to be of generalized automorphism type. This statement particularly holds true for the Nappi-Witten background as well as for all GMM models.

In the GMM models one can take the product group $\mathrm{U}_1 = \mathrm{H} \times \mathrm{H}$ as an intermediate group which sits in between $\mathrm{U}_2 = \mathrm{H}$ and the numerator $\mathrm{U}_0 = \mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2$. Denote by $\epsilon_{\rm L} = (e, \epsilon_2')$ and $\epsilon_{\rm R} = (\epsilon_1', e)$ the embeddings in terms of which the coset has been defined originally. Let us then define the embedding $\epsilon_2 = (e, \mathrm{id})$ of H in $\mathrm{H} \times \mathrm{H}$, the embedding $\epsilon_1 = (\epsilon_1', \epsilon_2')$ of the latter in $\mathrm{G}_1 \times \mathrm{G}_2$ and the automorphism $\Omega_1(h_1, h_2) = (h_2, h_1)$. In terms of these new objects the original embeddings may be rephrased according to $\epsilon_{\rm L} = \epsilon_1 \circ \epsilon_2$ and $\epsilon_{\rm R} = \epsilon_1 \circ \Omega_1 \circ \epsilon_2$. This means that GMM models are of generalized automorphism type as proposed before. The common subalgebra of the holomorphic and antiholomorphic chiral algebra which corresponds to this interpretation is given by

$$\mathcal{A}_{\rm red} \;=\; \mathcal{A}\big(({\rm G}_1)_{k_1}/{\rm H}_k\big) \otimes \mathcal{A}\big(({\rm G}_2)_{k_2}/{\rm H}_k\big) \otimes \mathcal{A}({\rm H}_k) \quad . \qquad (4.13)$$

This appears to be the largest subalgebra which can be preserved on the boundary.

Essentially the same considerations apply to the Nappi-Witten background. In this case there is even no need to introduce an additional intermediate subgroup. Only an automorphism which interchanges the two factors of the subgroup $\mathbb{R} \times \mathbb{R}$ has to be specified. For both models we provided an illustration in figure 4.2.

4.3.2 Solution of the boundary theory

With the experience we have gained in the group case in chapter 3 it is now very simple to solve the boundary theory for asymmetric cosets of generalized automorphism type. Yet, one has to be aware of the fact that there may exist several different representations of the asymmetric coset under consideration in terms of embedding chains of the form (4.11). In general, they will (like in the group case) lead to different boundary theories. We will therefore assume that we have fixed a distinguished set of intermediate groups U_i, embeddings ϵ_i and automorphisms Ω_i.

The chiral algebra which should be preserved by the boundary conditions has already been identified in eq. (4.13). We are again free to choose trivial gluing conditions for all the currents in the coset chiral algebras $\mathcal{A}(\mathrm{U}_i/\mathrm{U}_{i+1})$ since alternative choices could be expressed in a redefinition of the automorphisms Ω_i. In order to determine the Ishibashi states, the symmetry reduction must be accompanied by a decomposition of the bulk Hilbert space (4.7). For the individual coset representations we find

$$\begin{aligned} \mathcal{H}^{\mathrm{G/H}}_{(\mu,a)} &= \bigoplus \mathcal{H}^{\mathrm{U}_0/\mathrm{U}_1}_{(\mu,\mu_1)} \otimes \cdots \otimes \mathcal{H}^{\mathrm{U}_{N-2}/\mathrm{U}_{N-1}}_{(\mu_{N-2},\mu_{N-1})} \otimes \mathcal{H}^{\mathrm{U}_{N-1}/\mathrm{U}_N}_{(\mu_{N-1},a)} \\ \bar{\mathcal{H}}^{\mathrm{G/H}}_{(\mu,a)^+} &= \bigoplus \bar{\mathcal{H}}^{\mathrm{U}_0/\mathrm{U}_1}_{(\omega_0(\mu),\nu_1)^+} \otimes \cdots \otimes \bar{\mathcal{H}}^{\mathrm{U}_{N-1}/\mathrm{U}_N}_{(\omega_{N-1}(\nu_{N-1}),\omega_N^{-1}(a))^+} \quad . \end{aligned}$$

We had to include the automorphisms Ω_i in one half of the coset representations because in the original formulation of the symmetry reduction left and right chiral algebra are just isomorphic, not identical. By inserting the automorphisms Ω_i explicitly we are able to formulate the theory in terms of one single chiral algebra $\mathcal{A}_{\mathrm{red}}$. These considerations are similar to those that have already been encountered in section 4.1.4.

The conditions for the Ishibashi states may easily be written down. Yet, due to the presence of field identification in the individual coset chiral algebras the system of relations

$$\begin{aligned} (\mu,\mu_1) &\sim (\omega_0(\mu),\nu_1) \\ (\mu_i,\mu_{i+1}) &\sim (\omega_i(\nu_i),\nu_{i+1}) \\ (\mu_{N-1},a) &\sim (\omega_{N-1}(\nu_{N-1}),\omega_N^{-1}(a)) \end{aligned} \tag{4.14}$$

is extremely difficult to solve. The further discussion will thus focus on two special cases, $N = 2$ with particular assumptions on the automorphisms on one hand and absence of field identification on the other.

Exact treatment of a special case

To provide a solution of the boundary problem, let us first restrict to embedding chains of depth $N = 2$. This does not only cover all the examples to be discussed later on, but it simplifies our notations as well. We will also set $\Omega_0 = \mathrm{id} = \Omega_2$ and write $\mathrm{U}_1 = \mathrm{U}$. The remaining automorphism will be abbreviated by $\Omega_1 = \Omega$.

With the previous assumptions being made the system of equations (4.14) can now be solved. From the G/U cosets we obtain the condition $\mu_1 \equiv \nu_1$ modulo field identifications of the form $(0, J) \in \mathcal{G}_{\mathrm{id}}(\mathrm{G/U})$. From the U/H cosets one gets $\mu_1 = \omega(\nu_1)$. This is due to the fact that elements of the group $\mathcal{G}_{\mathrm{id}}(\mathrm{U/H})$ cannot have the form $(J, 0)$. The first condition then translates into $\mu_1 = J\,\omega(\mu_1)$. We will assume that this condition can only be fulfilled for $\omega(\mu_1) = \mu_1$.[4] Generalized coherent states $|\mu, \alpha, a\rangle\rangle$ for this setup are thus labeled by triples μ, α, a such that

$$(\mu, \alpha) \in \mathrm{All}(\mathrm{G/U}) \quad , \qquad (\alpha, a) \in \mathrm{All}(\mathrm{U/H}) \quad , \qquad \omega(\alpha) = \alpha \quad .$$

In addition we have to identify these generalized coherent states according to the identification rule

$$|J\mu, \alpha, J'a\rangle\rangle \sim |\mu, \alpha, a\rangle\rangle \quad \text{for} \quad (J, J') \in \mathcal{G}_{\mathrm{id}} \quad .$$

Let ${\psi_z}^\alpha$ be the solution for the boundary theory with Ω-twisted gluing conditions in the auxiliary algebra $\mathcal{A}(\mathrm{U})$. Then we may define boundary states for the asymmetric coset by

$$|\rho, z, r\rangle = \sum \mathrm{P}(\mu, \alpha)\,\mathrm{P}(\alpha, a)\, \frac{S^{\mathrm{G}}_{\rho\mu}}{\sqrt{S^{\mathrm{G}}_{0\mu}}}\, \frac{{\psi_z}^\alpha}{S^{\mathrm{U}}_{0\alpha}}\, \frac{\bar{S}^{\mathrm{H}}_{ra}}{\sqrt{\bar{S}^{\mathrm{H}}_{0a}}}\, |\mu, \alpha, a\rangle\rangle \quad .$$

The boundary states have to satisfy the selection rule $Q_J(\rho) = Q_{J'}(r)$ since the formula should not depend on the specific representative of the Ishibashi states. Also, we may implement the identification of boundary states

$$|J\rho, z, J'r\rangle \sim |\rho, z, r\rangle \quad \text{for} \quad (J, J') \in \mathcal{G}_{\mathrm{id}} \quad .$$

[4]This condition can be non-trivial only if there exist elements in the center of H which are mapped to the unit element by both ϵ_{L} and ϵ_{R}.

Using world-sheet duality, it is not difficult to derive formulas for the boundary partition functions. As usual we start from the following expression involving the coefficients of boundary states

$$Z \;=\; \sum \mathrm{P}(\mu,\alpha)\,\mathrm{P}(\alpha,a)\,\frac{\bar{S}^{\mathrm{G}}_{\rho_1\mu}S^{\mathrm{G}}_{\rho_2\mu}}{S^{\mathrm{G}}_{0\mu}}\,\frac{\bar{\psi}_{z_1}^{\;\alpha}\,\psi_{z_2}^{\;\alpha}}{S^{\mathrm{U}}_{0\alpha}S^{\mathrm{U}}_{0\alpha}}\,\frac{S^{\mathrm{H}}_{r_1a}\bar{S}^{\mathrm{H}}_{r_2a}}{S^{\mathrm{H}}_{0a}}\,\chi^{\mathrm{G/U}}_{(\mu,\alpha)}\,\chi^{\mathrm{U/H}}_{(\alpha,a)}(\tilde{q})$$

and perform the modular S-transformation to obtain

$$Z \;=\; \sum \frac{\bar{S}^{\mathrm{G}}_{\rho_1\mu}S^{\mathrm{G}}_{\rho_2\mu}S^{\mathrm{G}}_{\nu\mu}}{S^{\mathrm{G}}_{0\mu}}\,\frac{\bar{\psi}_{z_1}^{\;\alpha}\,\psi_{z_2}^{\;\alpha}\bar{S}^{\mathrm{U}}_{\beta\alpha}S^{\mathrm{U}}_{\gamma\alpha}}{S^{\mathrm{U}}_{0\alpha}S^{\mathrm{U}}_{0\alpha}}\,\frac{S^{\mathrm{H}}_{r_1a}\bar{S}^{\mathrm{H}}_{r_2a}\bar{S}^{\mathrm{H}}_{ba}}{S^{\mathrm{H}}_{0a}}\;\times$$

$$\times\;\mathrm{P}(\mu,\alpha)\,\mathrm{P}(\alpha,a)\;\chi^{\mathrm{G/U}}_{(\nu,\beta)}\,\chi^{\mathrm{U/H}}_{(\gamma,b)}(q)\quad .$$

We now want to pass to an unrestricted sum over μ,α,a (α still has to be symmetric). This can be achieved if we express the projectors in terms of monodromy charges and pull the corresponding simple currents into the S-matrices, see eqs. (2.23) and (2.7). We are thus led to

$$Z \;=\; \sum \frac{\bar{S}^{\mathrm{G}}_{\rho_1\mu}S^{\mathrm{G}}_{\rho_2\mu}S^{\mathrm{G}}_{J_1\nu\mu}}{S^{\mathrm{G}}_{0\mu}}\,\frac{\bar{\psi}_{z_1}^{\;\alpha}\psi_{z_2}^{\;\alpha}\bar{S}^{\mathrm{U}}_{J_1'\beta\alpha}S^{\mathrm{U}}_{J_2\gamma\alpha}}{S^{\mathrm{U}}_{0\alpha}S^{\mathrm{U}}_{0\alpha}}\,\frac{S^{\mathrm{H}}_{r_1a}\bar{S}^{\mathrm{H}}_{r_2a}\bar{S}^{\mathrm{H}}_{J_2'ba}}{S^{\mathrm{H}}_{0a}}\,\frac{\chi^{\mathrm{G/U}}_{(\nu,\beta)}\chi^{\mathrm{U/H}}_{(\gamma,b)}(q)}{|\mathcal{G}^{\mathrm{G/U}}_{\mathrm{id}}|\;|\mathcal{G}^{\mathrm{U/H}}_{\mathrm{id}}|}\quad .$$

The expression may be evaluated directly by means of the Verlinde formula (2.5) and eq. (3.13), so that the final result is given by

$$\begin{aligned} Z \;&=\; \frac{1}{|\mathcal{G}^{\mathrm{G/U}}_{\mathrm{id}}|\;|\mathcal{G}^{\mathrm{U/H}}_{\mathrm{id}}|}\sum N_{\rho_2\,J_1\nu}^{\;\rho_1}\,N_{(J_1'\beta)^+\,J_2\gamma}^{\;\delta}\,\big(n_\delta\big)_{z_1}^{\;z_2}\,N_{r_1r_2^+}^{\;J_2'b}\;\chi^{\mathrm{G/U}}_{(\nu,\beta)}\,\chi^{\mathrm{U/H}}_{(\gamma,b)}(q)\\ &=\;\sum N_{\rho_2\nu}^{\;\rho_1}\,N_{\beta^+\gamma}^{\;\delta}\,\big(n_\delta\big)_{z_1}^{\;z_2}\,N_{r_1r_2^+}^{\;b}\;\chi^{\mathrm{G/U}}_{(\nu,\beta)}\,\chi^{\mathrm{U/H}}_{(\gamma,b)}(q)\quad . \end{aligned}$$

Before we propose a geometric interpretation which is consistent with this spectrum, we shall first discuss a more general situation.

Sketch of the general case

Let us now proceed to a second case which can be treated exactly. In the absence of field identification and, equivalently, branching selection rules the equations (4.14) are solved by the set of all symmetric labels $\mu_i = \omega_i(\mu_i)$ (we have set $\mu_0 = \mu$ and $\mu_N = a$). The different boundary conditions are then associated with the boundary states

$$|\rho_0,\cdots,\rho_N\rangle \;=\; \sum_{\{\mu_i=\omega_i(\mu_i)\}} \frac{\big(\psi^{\mathrm{U}_0}\big)_{\rho_0}^{\;\mu_0}}{\sqrt{S^{\mathrm{U}_0}_{0\mu_0}}}\,\frac{\big(\psi^{\mathrm{U}_1}\big)_{\rho_1}^{\;\mu_1}}{S^{\mathrm{U}_1}_{0\mu_1}}\cdots\frac{\big(\bar{\psi}^{\mathrm{U}_N}\big)_{\rho_N}^{\;\mu_N}}{\sqrt{\bar{S}^{\mathrm{U}_N}_{0\mu_N}}}\,|\mu_0,\cdots,\mu_N\rangle\!\rangle\quad .$$

The structure constants ψ^{U_l} are taken from maximally symmetric solutions for the auxiliary BCFT's based on the chiral algebras $\mathcal{A}(\mathrm{U}_l)$. The associated spectra of boundary excitations

$$\mathcal{H}^{\mathrm{G/H}}_{\rho'\rho} = \bigoplus_{\{\nu_i,\lambda_i\}} \left(n^{\mathrm{U}_0}_{\nu_0}\right)_{\rho_0}^{\ \rho'_0} \left[\prod_{l=1}^{N-1} N_{\nu_l\sigma_l^+}^{\ \ \lambda_l} \left(n^{\mathrm{U}_l}_{\lambda_l}\right)_{\rho_l}^{\ \rho'_l}\right] \left(n^{\mathrm{U}_N}_{\sigma_N}\right)_{\rho_N}^{\ \rho'_N} \mathcal{H}^{\mathrm{U}_0/\mathrm{U}_1}_{(\nu_0,\sigma_1)} \otimes \cdots \otimes \mathcal{H}^{U_{N-1}/\mathrm{U}_N}_{(\nu_{N-1},\sigma_N)}$$

are obtained following the standard arguments of world-sheet duality. For the details of the calculation we refer the reader to the last subsection. Let us emphasize the structural relationship of the previous formulae with the results that have been obtained in the WZNW case in chapter 3. In fact, this similarity is not accidental but just confirms the outcome of the Lagrangian description of boundary conditions in asymmetric cosets in section 4.2.1.

Geometric interpretation

In accordance with our general statements in section 4.2.1, the geometric interpretation of our boundary conditions carries directly over from the WZNW case. For an asymmetric coset of generalized automorphism type G/H, which is represented by eq. (4.11) one finds a whole set of branes which cover the product of twisted conjugacy classes (3.6) on the "covering" group G. Note that all of them may be projected down to the coset G/H. Of course, the validity of our proposal could also be explicitly checked once again by hand. The authors of [136] suggested a different brane geometry, but they did not provide any algebraic evidence for their prescription.

Chapter 5

Applications

In the last chapter of this thesis we present several applications of the rather general results which have been obtained before. In the cosmological Nappi-Witten background $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)/\mathbb{R}\times\mathbb{R}$ we will be able to identify branes which pass through the big-bang big-crunch singularities and connect different closed universes. Our analysis is based on the study of branes in the numerator groups, where especially symmetry breaking and – for the group manifold $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)$ or its cover $\mathrm{AdS}_3\times\mathrm{SU}(2)$ – non-factorizing branes will attract our attention. The second example is constituted by T^{pq}-spaces, cosets of the form $\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{U}(1)$, which are close relatives to the base $S^2\times S^3$ of the conifold. Finally, symmetry breaking defect lines which may be used to separate two WZNW theories based on the same Kac-Moody algebra, but at different levels, will be worked out explicitly.

5.1 D-branes in $\mathbb{R}$, $\mathbf{U}(1)$, $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{R})$

The first section of this chapter will be devoted to the study of branes in group manifolds of dimension less or equal to three. Since an immanent feature of our construction is the hierarchical structure which comes with the embedding chains of the form (3.3), we start with the two abelian groups of dimension one before we proceed to cases of increasing complexity. The results of the following subsections will have direct implications for our investigations of branes in the Nappi-Witten background and in the T^{pq}-spaces.

5.1.1 Branes in $\mathbb{R}$ and U(1)

Algebraic description

The target spaces $\mathcal{M} = \mathbb{R}$ and $\mathcal{M} = \mathrm{U}(1)$ are particularly simple from the string theory point of view. The associated conformal field theory is given by a free boson X which is compactified on a circle for $\mathcal{M} = \mathrm{U}(1)$. In both cases the spectrum generating algebra is described by two commuting copies of the Heisenberg algebra $\widehat{\mathfrak{u}}(1)$ which are generated by the modes of the fields ∂X and $\bar{\partial} X$. In the first case the representations which have to be employed form a continuum which can be thought of as labeling possible values of the momentum. In the second case the representations form an infinite discrete set which is labeled by a quantized momentum and a winding number. For special values of the compactification radius the symmetry is enhanced. At these points the representations organize themselves in a finite number of representations of the larger $\widehat{\mathfrak{u}}(1)_k$ algebra. The number $k \in \mathbb{N}$ is the analogue of the level in an affine Lie algebra.

The maximally symmetric conformal boundary conditions for the free boson are well-known. They can be either of Neumann or Dirichlet type,

$$\text{Neumann:} \quad \partial X = \bar{\partial} X \quad , \qquad\qquad \text{Dirichlet:} \quad \partial X = -\bar{\partial} X \quad .$$

They correspond to gluing automorphisms of the form $\Omega_\pm(J) = \pm J$. For the boson on the real line there exists exactly one boundary state which corresponds to the Neumann condition. In contrast it admits Dirichlet boundary states for every real number x_0. They can be interpreted as a space-filling brane and point-like branes sitting at $x = x_0$, respectively.

Let us now discuss the theory which is based on the rational algebra $\widehat{\mathfrak{u}}(1)_k$ in slightly more detail. The latter appears naturally in our construction of symmetry breaking boundary states for $\mathrm{SU}(2)_k$ and higher rank compact groups. The algebra possesses $2k$ integrable irreducible representations which we will denote by $a = -k+1, \cdots, k$. Their conformal weights are given by $h_a = a^2/4k$. The fusion product reads $a \star b = (a+b)$ which is understood modulo $2k$, i.e. all sectors consist of simple currents. The associated monodromy charges read $Q_a(b) = -ab/2k$. The automorphisms $\Omega_\pm$ act on the representations as $\omega_\pm(a) = \pm a$. Due to the periodicity of the representation labels there are now *two* solutions to the equation $\omega_-(\mu) = \mu$, namely $\mu = 0, k$. They yield two boundary states

$$|\pm\rangle \;=\; \sum_{\mu=0,k} \frac{{\psi_\pm}^\mu}{\sqrt{S_{0\mu}}}\, |\mu\rangle\!\rangle \;=\; (k/2)^{1/4} \left[\, |0\rangle\!\rangle \pm |k\rangle\!\rangle \right] \quad ,$$

which correspond to space-filling branes with a Wilson line. During the calculation we used the explicit expressions

$$S_{\mu\nu} \;=\; \frac{1}{\sqrt{2k}}\, e^{-i\pi\mu\nu/k} \qquad \text{and} \qquad {\psi_{\pm}}^{\mu} \;=\; \frac{1}{\sqrt{2}}\big(\,\delta_{\mu 0} \,\pm\, \delta_{\mu k}\big) \quad .$$

As usual, world-sheet duality determines the spectrum of open strings

$$Z_{\epsilon\eta} \;=\; \sum_{\nu} \, (n_\nu)_{\eta}{}^{\epsilon}\, \chi_\nu \;=\; \sum_{\nu} \frac{1}{2}\left[1+\epsilon\eta\,(-1)^{\nu}\right]\chi_\nu \tag{5.1}$$

stretching between two D-branes of type ϵ and η. From the expression for the matrices n_ν we can read off the action $a\star\eta = (-1)^a\,\eta$ of a simple current a on a boundary label η, see our discussion in section 2.3.2.

Geometric interpretation

Both spaces $\mathbb{R}$ and U(1) can be considered as abelian groups under addition and multiplication, respectively. It is thus not surprising that the previous features can also be formulated in purely geometrical terms, i.e. using the notion of twisted conjugacy classes.

The automorphisms $\Omega_\pm$ which have been used to formulate the gluing conditions above may be integrated to yield automorphisms of the associated groups. On the group level they simply induce an inversion,

$$\Omega^{\mathrm{U}}_{\pm}\big(e^{i\phi}\big) \;=\; e^{\pm i\phi} \qquad\qquad \text{and} \qquad\qquad \Omega^{\mathbb{R}}_{\pm}(\lambda) \;=\; \pm\lambda \quad . \tag{5.2}$$

The careful reader might object that for the real numbers the multiplication with any non-zero number gives an automorphism. Let us recall, however, that in our definition the automorphism also had to preserve the scalar product in the Lie algebra. This condition restricts the possibilities to the multiplication with ± 1.

Denote by H one of the groups U(1) or $\mathbb{R}$. Since H is abelian, the twisted conjugacy classes which correspond to the automorphisms $\Omega^{\mathrm{H}}_{\pm}$ consist either of a single point or they wrap the whole group. To be precise, one obtains

$$\mathcal{C}^{\mathrm{H}}_{f}(\Omega^{\mathrm{H}}_{\pm}) \;=\; \begin{cases} \{\,f\,\} & ,\ + \\ \ \mathrm{H} & ,\ - \ . \end{cases} \tag{5.3}$$

Following the algebraic discussion, the number f is quantized in the $\widehat{\mathfrak{u}}(1)_k$-theory where it can only take powers of the root of unity $e^{\pi i/k}$ or the values ± 1, respectively, depending on whether we consider Dirichlet or Neumann boundary conditions.

5.1.2 Branes in SU(2)

Algebraic details

The background SU(2) is described by a WZNW theory based on the affine Lie algebra $\widehat{\mathfrak{su}}(2)_k$. The value of the level $k \in \mathbb{N}$ is a measure for the size of the group manifold. The model possesses a geometric regime which can be reached by sending the level to infinity. For finite values of k, however, stringy effects are not negligible.

The affine Lie algebra $\widehat{\mathfrak{su}}(2)_k$ possesses $k+1$ integrable irreducible representations $\mu = 0, \cdots, k$ of conformal weight

$$h_\mu = \frac{\mu\,(\mu+2)}{4(k+2)} \quad .$$

Since the Dynkin diagram of $\mathfrak{su}(2)$ consists of just one point, it admits no non-trivial symmetries and hence there does not exist any non-trivial outer automorphism of SU(2). In contrast, the extended Dynkin diagram possesses a $\mathbb{Z}_2$-symmetry which corresponds to the simple current $J = k$ which acts as $J \star \mu = k - \mu$ and gives rise to the monodromy charges $Q_J(\mu) = \mu/2$. For even values of k this simple current symmetry can be used to define an orbifold theory which describes the space $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$.

Let us now describe the boundary theory of the $\widehat{\mathfrak{su}}(2)_k$ WZNW model. For the maximally symmetric branes the whole story is rather short. As there is no diagram automorphism at our disposal we just end up with $k+1$ Cardy branes. Additional – symmetry breaking – branes have been constructed in [131] using the T-duality

$$\mathrm{SU}(2) \;\cong\; \big(\mathrm{SU}(2)/\mathrm{U}(1) \times \mathrm{U}(1)\big)\big/\mathbb{Z}_k \quad .$$

In agreement with our investigations in appendix C, the same branes can be recovered from our approach by using an embedding chain $\mathrm{U}(1) \hookrightarrow \mathrm{SU}(2)$ and the decomposition (3.2) of chiral algebras.

The symmetry breaking branes on SU(2) fall into two classes, depending on the gluing condition in the U(1)-part. The selection of a trivial automorphism yields $k(k+1)$ so-called A-branes, which – according to our general description in section 3.2.3 – are parametrized by the set

$$\mathcal{B}^{\mathrm{id}\times\mathrm{id}} = \Big(\mathrm{Rep}\big(\mathrm{SU}(2)_k\big) \times \mathrm{Rep}\big(\mathrm{U}(1)_k\big)\Big)\Big/\mathcal{G}_{\mathrm{id}} \quad ,$$

and among which one recovers the $k+1$ Cardy branes. The non-trivial element (k,k) of the field identification $\mathcal{G}_{\mathrm{id}} \cong \mathbb{Z}_2$ of the coset SU(2)/U(1)

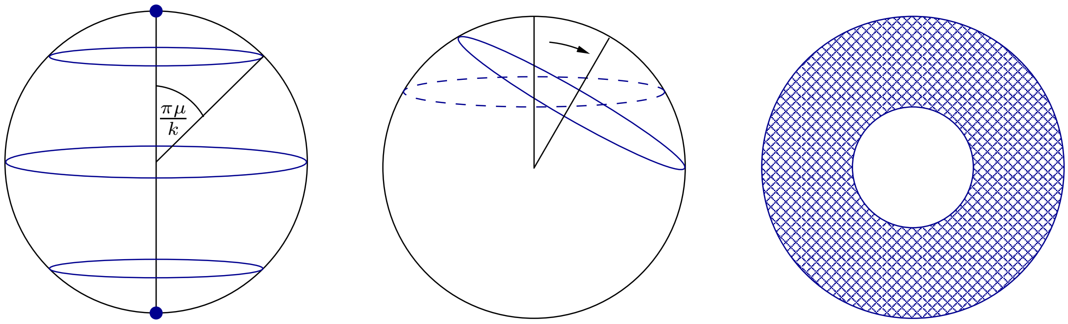

Figure 5.1: Maximally symmetric and symmetry breaking D-branes on SU(2). The latter arise from the rotation indicated in the central picture. They generically cover a three-dimensional subset of S^3.

acts by $(\mu, a) \mapsto (k-\mu, a+k)$. The B-branes which arise from choosing the inversion for gluing the U(1)-fields, are labeled by the set

$$\mathcal{B}^{\mathrm{id}\times\omega_-} \;=\; \big(\mathrm{Rep}(\mathrm{SU}(2)_k) \times \mathcal{B}^{\omega_-}(\mathrm{U}(1)_k)\big) \,/\, \mathcal{G}_{\mathrm{id}} \quad ,$$

where the boundary labels of U(1) are given by $\{\pm\}$ and the group $\mathcal{G}_{\mathrm{id}}$ induces the identification $(\mu, \pm) \sim \big(k-\mu, \pm(-1)^k\big)$. For odd k one obtains exactly $k+1$ B-branes. In the case of even k the action of $\mathcal{G}_{\mathrm{id}}$ possesses two fixed points $(k/2, \eta)$ which have to be resolved, and one ends up with $k+4$ B-branes. The associated spectra can be determined from eq. (3.16).

The geometric regime

The group manifold SU(2) may be realized as a subset of $\mathbb{C}^2$. To be more precise, the parametrization reads

$$\mathrm{SU}(2) \;=\; \Big\{ \left(\begin{smallmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{smallmatrix}\right) \;\Big|\; z_1, z_2 \in \mathbb{C} \text{ with } |z_1|^2 + |z_2|^2 = 1 \Big\} \quad . \tag{5.4}$$

Sometimes we will just write down the tupel (z_1, z_2) when we refer to elements of SU(2). The automorphisms of SU(2) are all inner automorphisms, i.e. they may be expressed using the adjoint action. It is, however, useful to introduce an extra notation for the automorphism

$$\gamma\left(\begin{smallmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{smallmatrix}\right) \;=\; \overline{\left(\begin{smallmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{smallmatrix}\right)} \;=\; \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \tag{5.5}$$

which corresponds to complex conjugation. It is inner because it can be represented as a conjugation with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SU}(2)$.

In order to construct symmetry breaking D-branes we have to classify all inequivalent chains of embeddings down to the smallest continuous subgroup

U(1). Actually, for the group SU(2) this is essentially the only subgroup which is at our disposal. There exist several inequivalent embeddings

$$\epsilon_n^{\mathrm{U,SU}}\big(e^{i\phi}\big) \;=\; \begin{pmatrix} \cos n\phi & \sin n\phi \\ -\sin n\phi & \cos n\phi \end{pmatrix} \qquad \text{with} \;\; n \in \mathbb{Z}\backslash\{0\} \;\; . \tag{5.6}$$

The integer n corresponds to a winding number. It determines the index of the embedding according to the formula $x_\epsilon = n^2$.

Let us now apply the results of the chapters 3 and 4 in order to identify the geometry of symmetry breaking D-branes on the group manifold SU(2). The necessary data, i.e. the embedding chain $\mathrm{U}(1) \hookrightarrow \mathrm{SU}(2)$ together with embedding maps $\epsilon_n^{\mathrm{U,SU}}$ as well as possible automorphisms has already been provided. According to eq. (3.6), the shape of the D-branes which may be constructed based on these informations is given by

$$\mathcal{C}_f^{\mathrm{SU}}(\Omega_{\mathrm{SU}}) \cdot \Omega_{\mathrm{SU}} \circ \epsilon_n^{\mathrm{U,SU}}\big(\mathcal{C}_{f'}^{\mathrm{U}}(\Omega_\pm)\big) \quad .$$

The automorphism Ω_{SU} may be chosen to act trivially without restrictions since its omission just induces an overall translation of the brane. Similarly, also the twisted conjugacy class of the U(1)-factor may be omitted whenever one uses the automorphism Ω_+. We are thus left with the investigation of the conjugacy classes of SU(2) and, eventually, the effects of its right multiplication with the subset $\epsilon_n^{\mathrm{U,SU}}\big(\mathrm{U}(1)\big) \subset \mathrm{SU}(2)$.

Let us start with a discussion of the conjugacy classes of SU(2) first [26]. If the element f takes values in the center, $f = \pm e$, the associated conjugacy class consists of just one point. For all other choices they are two-dimensional spheres S^2 which are embedded into $\mathrm{SU}(2) \cong S^3$. It is simple to find a parametrization for the conjugacy classes by taking the trace, since the latter is a class function, i.e. constant on adjoint orbits. For a WZNW theory at level k we have $k+1$ spheres S^2 which sit at the special values $\mathrm{Re}(z_1) = \cos\frac{\pi\mu}{k}$ with $\mu = 0, \cdots, k$. For $\mu = 0, k$ they degenerate to the zero-dimensional objects mentioned before. An illustration of these facts is given on the left hand side of figure 5.1.

To describe symmetry breaking D-branes we have to multiply the conjugacy classes of SU(2) by twisted conjugacy classes of U(1). Choosing a trivial automorphism simply amounts to a translation. When considering a non-trivial automorphism we have to take the union of all the shifted images. We will not write down the explicit expressions, but only refer to the illustration on the right hand side of figure 5.1. The symmetry breaking D-branes are either one- or three-dimensional. While the first ones are equatorial circles, the latter wrap a three-dimensional subset of the group, but generically leave some parts uncovered. Let us emphasize that we also find space-filling branes by considering the conjugacy classes of SU(2) with $\mu = k/2$ for even values of k. It is remarkable to note that a generic point of all these D-branes is covered twice. This observation is related to the fact that the space-filling branes can be further resolved into elementary branes. Hence, we found full agreement with the results of [131, 133] which have been obtained based on T-duality.

5.1.3 Branes in SL$(2,\mathbb{R})$

Algebraic aspects

The exact treatment of the background SL$(2,\mathbb{R})$ in terms of conformal field theory is a very delicate subject. Only recently, Maldacena and Ooguri proposed the use of a mixture of certain discrete and continuous representations for the description of the bulk spectrum and showed the consistency of their proposal [137]. Remarkably, the energy spectrum of the corresponding representations needs not to be bounded from below. An essential ingredient in the construction is the spectral flow operation which relates different representations and which can be interpreted as the action of elements of the loop group $\widehat{\mathrm{SL}}(2,\mathbb{R})$ which are not continuously connected to the identity.

In the present work we will have nothing new to say about the algebraic aspects of constructing D-branes in SL$(2,\mathbb{R})$ or its covering space AdS$_3$. We will merely use the geometric insights which have been gained in chapter 3 to obtain a picture of what should be expected in the exact CFT description. We actually anticipate that many features of our extrapolation will survive the step to the rigorous treatment.

Background geometry and basic definitions

The group SL$(2,\mathbb{R})$ may be described as a subset of four dimensional flat space. In this parametrization the connection to the matrix form is given by

$$\begin{pmatrix} X_0+X_3 & X_1+X_2 \\ X_1-X_2 & X_0-X_3 \end{pmatrix} \text{ with } X_i \in \mathbb{R} \text{ and } X_0^2-X_1^2+X_2^2-X_3^2 \;=\; 1 \quad . \tag{5.7}$$

It is convenient to introduce cylindrical coordinates r,θ and a *periodic* time τ. These take values in the domains $r \in [0,\infty[$ and $\theta,\tau \in [0,2\pi[$. The precise relation to the previous parametrization is given by

$$X_0+iX_2 \;=\; e^{i\tau}\cosh r \qquad \text{and} \qquad X_3+iX_1 \;=\; e^{i\theta}\sinh r \quad . \tag{5.8}$$

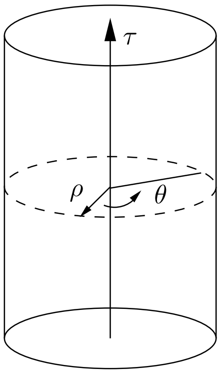

Figure 5.2: Parametrization of $SL(2,\mathbb{R})$.

In the cylindrical coordinates, the manifold $SL(2,\mathbb{R})$ may be depicted as in figure 5.2 with top and bottom of the cylinder identified. The covering space AdS_3 is obtained by resolving the periodicity of time, i.e. by extending its range to $\tau \in \mathbb{R}$. The region $r \to \infty$ describes the boundary of AdS_3.

In order to be able to characterize maximally symmetric branes, but also for the symmetry breaking ones, we need to determine the equivalence classes of outer automorphisms. In contrast to the SU(2)-case there indeed exists (up to concatenation with a conjugation) exactly one non-trivial outer automorphism of $SL(2,\mathbb{R})$ which reads

$$\Omega_0^{\mathrm{SL}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad . \tag{5.9}$$

Note that this automorphism allows a representation as conjugation with the matrix $M_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which is *not* an element of $SL(2,\mathbb{R})$.

For the construction of symmetry breaking D-branes we need to classify all subgroups of $SL(2,\mathbb{R})$ together with their embedding maps and automorphisms. It can easily be seen that there exist essentially two subgroups, a compact $SO(2) \cong U(1)$ and a non-compact $\mathbb{R}$. The relevant embedding maps of the former are given by

$$\epsilon_n^{\mathrm{U,SL}}\big(e^{i\phi}\big) = \begin{pmatrix} \cos n\phi & \sin n\phi \\ -\sin n\phi & \cos n\phi \end{pmatrix} \qquad \text{for} \quad n \in \mathbb{Z}\backslash\{0\} \quad . \tag{5.10}$$

They are unique up to conjugation. The parameter n gives the winding number and determines the value of the embedding index according to $x_\epsilon = n^2$. Let us recall that our definition of an embedding map just demanded it to be injective on the level of the Lie algebra, not on the level of the group. As a

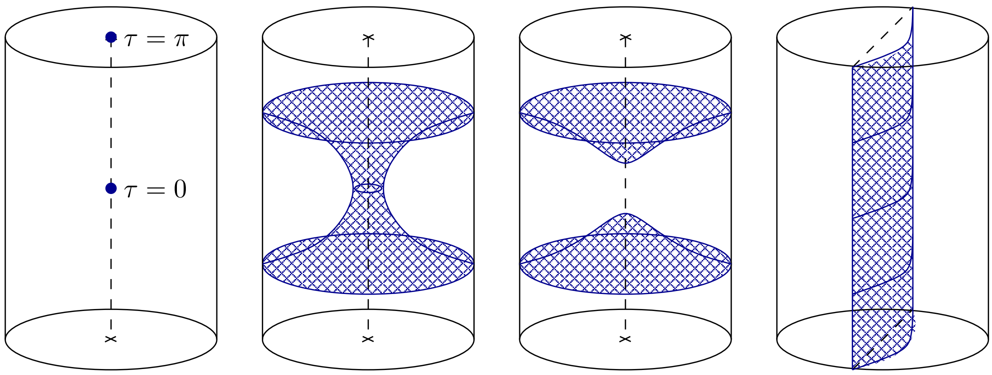

Figure 5.3: Representatives of maximally symmetric D-branes in $\mathrm{SL}(2,\mathbb{R})$. From left to right we have the following types: point-like, dS_2, H_2 and AdS_2 branes.

consequence we are also able to define two inequivalent classes of embedding maps for the subgroup $\mathbb{R}$,

$$\epsilon_\alpha^{\mathbb{R},\mathrm{SL}}(\lambda) \;=\; \begin{pmatrix} e^{\alpha\lambda} & 0 \\ 0 & e^{-\alpha\lambda} \end{pmatrix} \quad \text{and} \quad \tilde{\epsilon}_\beta^{\mathbb{R},\mathrm{SL}}(\lambda) \;=\; \begin{pmatrix} \cos\beta\lambda & \sin\beta\lambda \\ -\sin\beta\lambda & \cos\beta\lambda \end{pmatrix} \;, \tag{5.11}$$

with non-vanishing parameters α and β which determine the value of the embedding index. The automorphisms of the groups $\mathrm{U}(1)$ and $\mathbb{R}$ as well as the geometric interpretation of the associated twisted conjugacy classes have already been provided in section 5.1.1.

Brane geometry

The extrapolation of our exact results which have been obtained for compact groups lets us expect a large set of D-branes on $\mathrm{SL}(2,\mathbb{R})$. To be more precise, they should be located along the point-wise product

$$\mathcal{C}_f^{\mathrm{SL}}(\Omega_{\mathrm{SL}}) \cdot \Omega_{\mathrm{SL}} \circ \epsilon^{\mathrm{H,SL}}\big(\mathcal{C}_{f'}^{\mathrm{H}}(\Omega_\pm)\big) \tag{5.12}$$

of twisted conjugacy classes. The subgroup H may be chosen either to be $\mathrm{U}(1)$ or $\mathbb{R}$. Additional freedom comes with the selection of an embedding $\epsilon^{\mathrm{H,SL}}$ and our choice of the automorphisms.

Maximally symmetric D-branes belong to twisted conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$. They show up whenever the twisted conjugacy class of the abelian group H is chosen to be point-like, i.e. if the automorphism Ω_+ is used. Let us briefly review the classification of these maximally symmetric branes which traces back to the work of Stanciu, Bachas and Petropoulos [138, 139].

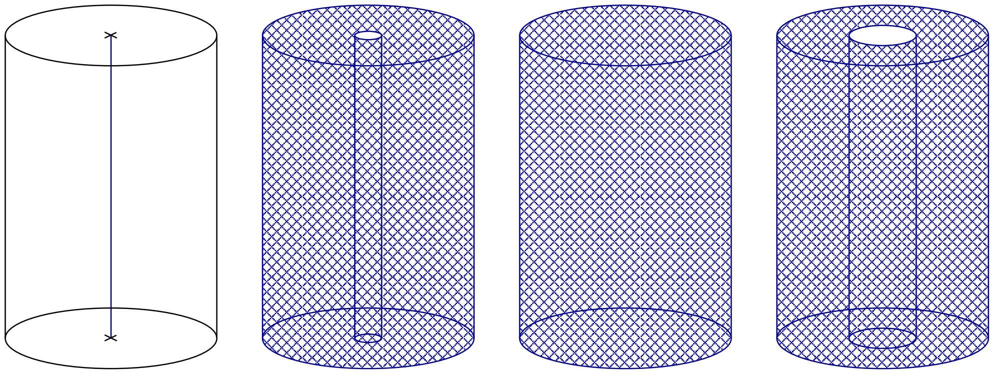

Figure 5.4: Certain classes of symmetry breaking D-branes on $SL(2,\mathbb{R})$. They are obtained from those in figure 5.3 by a simultaneous (θ,τ)-rotation.

Ordinary conjugacy classes which come with the automorphism $\Omega_{SL} = \mathrm{id}$ fall into three types. Let us recall that all elements of a conjugacy classes are mapped to the same number by taking the trace. Fixing X_0 in eq. (5.7) to some value $C \in \mathbb{R}$ while putting no constraints on the other coordinates therefore gives a first rough classification. The resulting submanifold may be disconnected showing that only demanding $X_0 = C$ does not lead to a complete solution of the classification problem. It is nevertheless useful to work with this description. The equation $X_0 = \cos\tau \cosh r = C$ admits very different types of solutions depending on whether $|C| < 1$ or $|C| > 1$. For $|C| > 1$ one recovers dS_2 branes while for $|C| < 1$ branes are obtained which are localized on hyperbolic planes H_2. In the limit $|C| \to 1$ they degenerate to two instantonic point-like D-branes at $\tau = 0, \pi$ which are associated to the center of $SL(2,\mathbb{R})$ and others sitting on the light cone. Representatives of this zoo of conjugacy classes are visualized in figure 5.3. It was argued in [139] that all these D-branes are unphysical. The H_2 and the point-like branes are instantonic objects while the dS_2 branes are spoiled by a supercritical electrical field.

Twisted conjugacy classes associated to the automorphism (5.9) are classified by the relation $\mathrm{tr}(M_0\, g) = 2X_2 = 2\,C$. According to eq. (5.8) this condition translates into $C = \sin\theta \sinh r$. In this situation there is no need to distinguish different cases. All these twisted conjugacy classes describe AdS_2 branes which are invariant under time translations and extend to the boundary of AdS_3 at $\theta = 0, \pi$ [139]. They are illustrated in the right-most picture of figure 5.3.

Let us now turn to the description of symmetry breaking D-branes. According to the expression (5.12), we may multiply the twisted conjugacy classes of $SL(2,\mathbb{R})$ by a twisted conjugacy class $\mathcal{C}^{H}_{f'}(\Omega_\pm)$ of $H = U(1)$ or $H = \mathbb{R}$. For Ω_+ the latter are point-like. This induces a translation of the original D-brane. The situation is more interesting for the automorphism Ω_-. In this case the twisted conjugacy class reduces to H itself and one has to consider the superposition of all shifted images.

This analysis is particularly simple for $H = U(1)$. In this case the multiplication of an element (r, θ, τ) of $SL(2,\mathbb{R})$ with an element $e^{i\lambda}$ of U(1) just induces the simultaneous rotation $(\theta, \tau) \mapsto (\theta \pm n\lambda, \tau \pm n\lambda)$ of angle and time coordinate. The sign and the wrapping number n are fixed by the choice of twist Ω^{SL} and embedding $\epsilon^{U,SL}$. As the twisted conjugacy class is given by the whole U(1), one may immediately evaluate the geometry of the resulting D-branes. By rotation of the dS_2 and the AdS_2 branes one obtains D-branes which fill all space outside a cylinder of radius $r_0 = \operatorname{arcosh}|C|$ or $r_0 = \operatorname{arsinh}|C|$, respectively. For degenerate cases they provide space-filling branes similar to those arising from the rotation of H_2 branes. If one rotates the 0-branes on the other hand, these sweep out the axis $r = 0$. To get some impression of these geometries we visualized all four of them in figure 5.4. Let us emphasize that the generic point in the world-volume of rotated dS_2 and AdS_2 branes is covered twice.

For $H = \mathbb{R}$ we have to distinguish two embeddings (5.11). The usage of $\tilde{\epsilon}^{\mathbb{R},SL}_\beta$ gives essentially the same result as for U(1). For $\epsilon^{\mathbb{R},SL}_\beta$, in contrast, the discussion becomes quite involved as the shift acts in a very intricate way – at least in our coordinates (r, θ, τ). To get an idea of what is going on, let us consider the case where the conjugacy class of $SL(2,\mathbb{R})$ reduces to a point. The D-brane is then parametrized by matrices of the form $\operatorname{diag}(\pm e^{\lambda}, \pm e^{-\lambda})$ with $\lambda \in \mathbb{R}$. It turns out that these are instantonic one-dimensional branes localized at times $\tau = 0, \pi$, respectively, and running all the way from $r = 0$ to $r = \infty$ in the directions $\theta = 0, \pi$. They do not seem to make sense physically and we will not discuss them in more detail. Notice that our results confirm the predictions for the geometry which arise from T-duality [131, 133].

5.2 D-branes in $AdS_3 \times S^3$

In this section we will apply our general framework to advance the classification of D-branes in the target space $G = SL(2,\mathbb{R}) \times SU(2)$. Breaking the symmetry down to a suitable subgroup allows us to construct *non-factorizing* D-branes whose world-volume cannot be written as a direct product of the form $\mathcal{D}_{SL} \times \mathcal{D}_{SU}$. Since the covering space of $SL(2,\mathbb{R})$ is AdS_3 and the group SU(2) is diffeomorphic to S^3 our results have immediate implications for the string backgrounds $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ [38, 39].

Following the general scheme for constructing symmetry breaking D-branes we first have to classify all inequivalent chains of maximal embeddings which reside in the product group $SL(2,\mathbb{R})\times SU(2)$. In addition one needs to classify every automorphism of all the subgroups which emerge during that procedure. *Common* subgroups of $SL(2,\mathbb{R})$ and $SU(2)$ can then be used to construct non-factorizing symmetry breaking D-branes.

5.2.1 Embedding chains and automorphisms

The continuous subgroups of the product $SL(2,\mathbb{R}) \times SU(2)$ are easily classified. There are essentially two choices of embedding chains which may be used for our construction. The first one is given by the maximal embedding

$$H_1 \times H_2 \hookrightarrow SL(2,\mathbb{R}) \times SU(2) \quad , \tag{5.13}$$

where H_1 and H_2 equal one of the groups $U(1)$ or $\mathbb{R}$. Without loss of generality we assume that H_1 is embedded into $SL(2,\mathbb{R})$ and H_2 is embedded into $SU(2)$. In this case the embedding map is given by $\epsilon^{H_1,SL} \times \epsilon^{H_2,SU}$. Also, we should demand $H_1 \neq H_2$ since otherwise the embedding chain could be enlarged by the common subgroup which sits diagonally in the product $H_1 \times H_2$.

The last statement already suggests the second possible choice for a chain of maximal embeddings. It is specified by

$$H_k \hookrightarrow H_{k'_1} \times H_{k'_2} \hookrightarrow SL(2,\mathbb{R})_{k_1} \times SU(2)_{k_2} \quad . \tag{5.14}$$

Again, the symbol H denotes either of the groups $U(1)$ or $\mathbb{R}$. The numbers k_i, k'_i, k are a measure for the size of the groups or, equivalently, the normalization of the scalar product which is used in the associated Lie algebra. The embedding map can be decomposed as $\left[\epsilon^{H,SL} \times \epsilon^{H,SU}\right] \circ \epsilon^{H,H\times H}$. Without any restriction we can assume the first embedding in relation (5.14) to have the diagonal form $\epsilon^{H,H\times H}(h) = (h,h)$. The remaining freedom is then solely contained in the embedding maps $\epsilon^{H,SL} \times \epsilon^{H,SU}$ from $H \times H$ to $SL(2,\mathbb{R}) \times SU(2)$. An embedding where the subgroup H is just mapped into one of the factors $H \times H$ can effectively be described by omitting the subgroup H and working with a chain of depth $N = 1$.

Finally, we have to discuss the automorphisms of all the groups which entered the expressions (5.13) and (5.14). For the product groups $SL(2,\mathbb{R}) \times SU(2)$ and $H_1 \times H_2$ the automorphisms factorize, i.e. they may be represented as a direct product of automorphisms of the individual constituents. The situation, however, is different for the embedding chain (5.14) and the product group $H_{k'_1} \times H_{k'_2}$. If both constituents have the same size, i.e. if $k'_1 = k'_2$, there exists an additional automorphism $\Omega_{\rm exc}$ which exchanges the two groups. On the algebraic side the condition imposed is equivalent to the statement that one deals with two *identical* copies of the chiral algebra $\mathcal{A}(\mathrm{H})$. Note that the numbers k_i' are already completely fixed given the numbers k_i and the embedding $\epsilon^{\mathrm{H,SL}} \times \epsilon^{\mathrm{H,SU}}$. In our applications based on the embedding chain (5.14) we will work with automorphisms of the form $\Omega_{\mathrm{H}\times\mathrm{H}} = \Omega \circ \big[\Omega_1^{\mathrm{H}} \times \Omega_2^{\mathrm{H}}\big]$, where $\Omega \in \{\mathrm{id}, \Omega_{\rm exc}\}$ denotes a possible exchange of the two group factors and $\Omega_1^{\mathrm{H}}, \Omega_2^{\mathrm{H}}$ are two arbitrary automorphisms of H.

We are now almost prepared to address the question of D-brane geometry in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$. All we still need is a better understanding of certain twisted conjugacy classes. For $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ and the subgroup $\mathrm{H}_1 \times \mathrm{H}_2$ which appears in the chain (5.13) the twisted conjugacy classes simply factorize in those of the individual consituents, cf. the previous section. It is slightly more complicated to find an expression for the twisted conjugacy classes $\mathcal{C}^{\mathrm{H}\times\mathrm{H}}\big(\Omega_{\mathrm{H}\times\mathrm{H}}\big)$. Considering first the case with $\Omega = \mathrm{id}$ the conjugacy classes factorize like before and we obtain

$$\mathcal{C}^{\mathrm{H}\times\mathrm{H}}_{(f_1,f_2)}\big(\Omega_{\mathrm{H}\times\mathrm{H}}\big)\Big|_{\Omega=\mathrm{id}} = \begin{cases} \{f_1\}\times\{f_2\} & , \ \Omega_1^{\mathrm{H}} = \mathrm{id}, \ \Omega_2^{\mathrm{H}} = \mathrm{id} \\ \{f_1\}\times \mathrm{H} & , \ \Omega_1^{\mathrm{H}} = \mathrm{id}, \ \Omega_2^{\mathrm{H}} \neq \mathrm{id} \\ \mathrm{H}\times\{f_2\} & , \ \Omega_1^{\mathrm{H}} \neq \mathrm{id}, \ \Omega_2^{\mathrm{H}} = \mathrm{id} \\ \mathrm{H}\times\mathrm{H} & , \ \Omega_1^{\mathrm{H}} \neq \mathrm{id}, \ \Omega_2^{\mathrm{H}} \neq \mathrm{id} \ . \end{cases} \tag{5.15}$$

In this case the element (f_1, f_2) has to satisfy $\Omega_i^{\mathrm{H}}(f_i) = f_i$. With non-trivial twist, i.e. with $\Omega \neq \mathrm{id}$, we have the restrictions $f_1 = \Omega_2^{\mathrm{H}}(f_2)$ and $f_2 = \Omega_1^{\mathrm{H}}(f_1)$. It thus suffices to work with one label $f = f_1$ which satisfies $\Omega_2^{\mathrm{H}} \circ \Omega_1^{\mathrm{H}}(f) = f$. A straightforward analysis yields

$$\mathcal{C}^{\mathrm{H}\times\mathrm{H}}_{f}\big(\Omega_{\mathrm{H}\times\mathrm{H}}\big)\Big|_{\Omega_{\rm exc}} = \begin{cases} \mathrm{H}\times\mathrm{H} & , \ \Omega_1^{\mathrm{H}} \neq \Omega_2^{\mathrm{H}-1} \\ \big\{\big(sf, \Omega_1^{\mathrm{H}}(fs^{-1})\big)\,\big|\, s\in\mathrm{H}\big\} & , \ \Omega_1^{\mathrm{H}} = \Omega_2^{\mathrm{H}-1} \ . \end{cases} \tag{5.16}$$

This concludes our presentation of the necessary tools for the determination of symmetry breaking D-branes in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$.

When constructing D-branes in the product geometry $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$, it is convenient to distinguish three cases which either lead to factorizing branes or two different mechanisms for obtaining non-factorizing branes. The discussion of this classification will be the subject of the following three subsections.

5.2.2 Factorizing branes

Factorizing D-branes whose world-volume may be written as a direct product of the form $\mathcal{D}_{\mathrm{SL}} \times \mathcal{D}_{\mathrm{SU}}$ arise naturally from the embedding chain (5.13). They may also be obtained from the chain (5.14) if one restricts oneself to the automorphism $\Omega = \mathrm{id}$ and if one in addition "excludes" the last subgroup H by hand. De facto, the latter can be achieved by choosing the trivial set $\{e\}$ for the conjugacy class which is associated to this group.

Let us discuss the geometry of these D-branes now. According to the general expression (3.6) they are localized along the product

$$\Big[\mathcal{C}^{\mathrm{SL}}_{f_1}\big(\Omega_{\mathrm{SL}}\big)\cdot\Omega_{\mathrm{SL}}\circ\epsilon^{\mathrm{H}_1,\mathrm{SL}}\Big(\mathcal{C}^{\mathrm{H}_1}_{f_2}\big(\Omega_{\mathrm{H}_1}\big)\Big)\Big]\;\times\;\Big[\mathcal{C}^{\mathrm{SU}}_{f_3}\big(\mathrm{id}\big)\cdot\epsilon^{\mathrm{H}_2,\mathrm{SU}}\Big(\mathcal{C}^{\mathrm{H}_2}_{f_4}\big(\Omega_{\mathrm{H}_2}\big)\Big)\Big]\quad .$$

This factorized geometry is completely under control using the dictionary which has been provided in the previous sections. The dimensions of these D-branes range from 0 to 6, the shape from point-like to space-filling. We will not bother to discuss these D-branes any further, but focus our attention on the description of non-factorizing D-branes.

5.2.3 Non-factorizing branes from diagonal embeddings

The first type of non-factorizing D-branes is obtained by using the embedding chain (5.14) and choosing an automorphism $\Omega_{\mathrm{H}\times\mathrm{H}} = \Omega\circ\big[\Omega^{\mathrm{H}}_1\times\Omega^{\mathrm{H}}_2\big]$ which does not involve an exchange of the two factors. In other words we demand $\Omega = \mathrm{id}$. The geometry associated to this kind of symmetry breaking D-brane is described by the product

$$\Big[\mathcal{C}^{\mathrm{SL}}_{f_1}\big(\Omega_{\mathrm{SL}}\big)\times\mathcal{C}^{\mathrm{SU}}_{f_2}\big(\mathrm{id}\big)\Big]\;\cdot\;\epsilon_1\Big(\mathcal{C}^{\mathrm{H}\times\mathrm{H}}_{(f_3,f_4)}\big(\Omega^{\mathrm{H}\times\mathrm{H}}\big)\Big)\;\cdot\;\epsilon_2\Big(\mathcal{C}^{\mathrm{H}}_{f_5}\big(\Omega_{\mathrm{H}}\big)\Big)\quad ,\qquad(5.17)$$

where the embeddings have been abbreviated by $\epsilon_1 = \big(\Omega_{\mathrm{SL}}\circ\epsilon^{\mathrm{H},\mathrm{SL}}\big)\times\epsilon^{\mathrm{H},\mathrm{SU}}$ and $\epsilon_2 = \big[\Omega_{\mathrm{SL}}\circ\epsilon^{\mathrm{H},\mathrm{SL}}\times\epsilon^{\mathrm{H},\mathrm{SU}}\big]\circ\big(\Omega^{\mathrm{H}}_1\times\Omega^{\mathrm{H}}_2\big)\circ\epsilon^{\mathrm{H},\mathrm{H}\times\mathrm{H}}$. Let us start our discussion with a given product of conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$. As can be seen from eq. (5.15) the effect of the multiplication with $\mathcal{C}^{\mathrm{H}\times\mathrm{H}}_{(f_3,f_4)}\big(\Omega_{\mathrm{H}\times\mathrm{H}}\big)$ is a combination of a translation and a factorized smearing as described in the sections 5.1.2 and 5.1.3. The effect of the multiplication with $\mathcal{C}^{\mathrm{H}}_{f_5}\big(\Omega_{\mathrm{H}}\big)$ in contrast is more interesting as it can provide the reason for non-factorizability. If this conjugacy class is 0-dimensional, it translates the whole D-brane by a constant amount leaving factorizability unaffected. On the other hand it may reduce to H itself. Under these circumstances one obtains a continuous superposition of shifted D-branes. Due to the diagonal embedding of H into $\mathrm{H}\times\mathrm{H}$, the shift acts on both factors $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ *simultaneously*. This feature is responsible for non-factorizability.

Since the discussion of the geometry of these D-branes becomes rather involved in the general case, we prefer to illustrate our considerations in two simple examples. Assume first that $\mathrm{H} = \mathbb{R}$, $\epsilon^{\mathrm{H},\mathrm{SL}} = \tilde{\epsilon}^{\mathbb{R},\mathrm{SL}}_{\alpha}$, $\epsilon^{\mathrm{H},\mathrm{SU}} = \epsilon^{\mathbb{R},\mathrm{SU}}_{\beta}$ and that we set $f_1 = f_2 = f_3 = f_4 = e$, $\Omega_{\rm SL} = {\rm id}$ and $\Omega_1^{\rm H} = \Omega_2^{\rm H} = {\rm id}$. This implies that the twisted conjugacy classes of $SL(2,\mathbb{R})$, $SU(2)$ and $H \times H$ entering (5.17) reduce to unit elements. In order to obtain a non-trivial result we choose $\Omega_{\rm H} \neq {\rm id}$ such that the remaining twisted conjugacy class in eq. (5.17) is given by H. After its embedding into $SL(2,\mathbb{R}) \times SU(2)$ one recovers the curve

$$\left(\begin{pmatrix} \cos\alpha\lambda & \sin\alpha\lambda \\ -\sin\alpha\lambda & \cos\alpha\lambda \end{pmatrix} , \begin{pmatrix} \cos\beta\lambda & \sin\beta\lambda \\ -\sin\beta\lambda & \cos\beta\lambda \end{pmatrix} \right) \qquad \text{with} \quad \lambda \in \mathbb{R} \quad .$$

The numbers α, β specifiy the embedding of $\mathbb{R}$ into the individual group factors. A closer comparison of this expression with eqs. (5.7, 5.8) shows that $\alpha\lambda$ may be identified with time τ. This configuration thus describes a number of D-particles each having a circular trajectory in the group factor SU(2) while sitting on the axis $r = 0$ of $SL(2,\mathbb{R})$. The number of D-particles is determined by the relative values of α and β. If the ratio is irrational one obtains an infinite number of particles which form a dense set in SU(2) at each instance of time. The appearance of multiple D-particles is due to the periodicity of time in $SL(2,\mathbb{R})$. This artifact disappears on the covering space AdS_3 which has a non-compact time-coordinate. Obviously, it is straightforward to generalize the previous idea to two-spheres which are rotating in the SU(2) factor in the evolution of time.

In our second example we choose $\Omega^{\rm SL} = \Omega_0^{\rm SL}$, but still fix $H = \mathbb{R}$, $\epsilon^{\rm H,SL} = \tilde{\epsilon}_\alpha^{\mathbb{R},\rm SL}$, $f_3 = f_4 = e$, $\Omega_1^{\rm H} = \Omega_2^{\rm H} = {\rm id}$ and $\Omega^{\rm H} \neq {\rm id}$. The set (5.17) is obtained from the product $AdS_2 \times S^2$ by performing simultaneous shifts in both factors. If we focus only on the AdS_2-part for a moment, we already know the resulting geometry from section 5.1.3. It is given by all points (r, θ, τ) which satisfy $r \geq r_0 = {\rm arsinh}|C|$ for some constant C. These points are generically not located on the twisted conjugacy class of $SL(2,\mathbb{R})$ we have started from. We thus have to decompose them into an element (r, θ', τ') and a shift λ such that $(\theta, \tau) = (\theta' + \lambda, \tau' + \lambda)$ and $\sin\theta' \sinh r = C$, i.e. such that (r, θ', τ') is an element of the twisted conjugacy class. With every solution θ'_1 we have another one $\theta'_2 = \pi - \theta'_1$. In the exceptional case $r = r_0$ we have only one solution $\theta'_1 = \theta'_2 = \pi/2$. For $r = r_0 = 0$ the angle can be chosen arbitrary. For simplicity we shall assume $r_0 > 0$ in what follows.

The two shifts $\lambda_{1/2}$ associated to the angles $\theta'_{1/2}$ have to come from the embedding $\tilde{\epsilon}_\alpha^{\mathbb{R},\rm SL}(\xi)$ of $\xi \in \mathbb{R}$ in $SL(2,\mathbb{R})$. As $\alpha\xi_{1/2}$ is only defined modulo 2π there are several choices $\xi_{1/2}^{(l)} = (\lambda_{1/2} + 2\pi\alpha l)/\alpha$ of elements in $\mathbb{R}$ which may be used to recover these shifts. These elements have to be used to implement the shift on the SU(2)-part. Using the embedding $\epsilon_\beta^{\mathbb{R},\rm SU}$ as before, these shifts are determined by the angles $\beta(\lambda_{1/2} + 2\pi\alpha l)/\alpha$. For $\alpha = \beta = 1$ we arrive at the following picture. The D-brane in $SL(2,\mathbb{R}) \times SU(2)$ is parametrized by points (r, θ, τ) in $SL(2,\mathbb{R})$ with $r \geq r_0$. Over each of these points one has two spheres S^2 which are generated out of the conjugacy class of SU(2) by the action of the shifts $\lambda_{1/2}(r)$. In the limiting regimes $r \to r_0$ and $r \to \infty$ the two-spheres move closer and closer until they finally coincide. For more general choices of α and β the number of two-spheres over each point in $SL(2,\mathbb{R})$ may be larger.

5.2.4 Non-factorizing branes from group interchanging twists

The second possibility to obtain non-factorizing branes is again associated to the embedding chain (5.14), but now it includes a twist $\Omega = \Omega_{\text{exc}}$ of the subgroup $\mathrm{H} \times \mathrm{H}$ which interchanges the two group factors. Let us recall that the existence of this automorphisms enforces some constraints on the relative size – the levels – of $SL(2,\mathbb{R})$ and SU(2) and the embeddings one uses.

After these remarks we can proceed as in the previous section. The geometry which belongs to our present choice of embedding chain may be read off from the product

$$\Big[\mathcal{C}^{\text{SL}}_{f_1}(\Omega_{\text{SL}}) \times \mathcal{C}^{\text{SU}}_{f_2}(\text{id})\Big] \;\cdot\; \epsilon_1\Big(\mathcal{C}^{\text{H}\times\text{H}}_{f_3}(\Omega_{\text{H}\times\text{H}})\Big) \;\cdot\; \epsilon_2\Big(\mathcal{C}^{\text{H}}_{f_4}(\Omega_{\text{H}})\Big) \quad , \tag{5.18}$$

where we used the abbreviations $\epsilon_1 = \big(\Omega_{\text{SL}} \circ \epsilon^{\text{H,SL}}\big) \times \epsilon^{\text{H,SU}}$ and $\epsilon_2 = \big[\Omega_{\text{SL}} \circ \epsilon^{\text{H,SL}} \times \epsilon^{\text{H,SU}}\big] \circ \Omega_{\text{H}\times\text{H}} \circ \epsilon^{\text{H,H}\times\text{H}}$. The discussion of the conjugacy class $\mathcal{C}^{\text{H}}_{f_4}(\Omega_{\text{H}})$ gives no new insights compared to the previous section. Despite of this fact there is still a significant qualitative difference, as we are now allowed to work with the expressions (5.16) for the conjugacy classes $\mathcal{C}^{\text{H}\times\text{H}}_{f_3}(\Omega_{\text{H}\times\text{H}})$. While the first possibility in (5.16) implies the usual factorized smearing, the second induces a superposition of simultaneous shifts in the two group factors similar to those which have been revealed for $\mathcal{C}^{\text{H}}_{f_4}(\Omega_{\text{H}})$. It is an interesting question to see whether the joint action of two independent simultaneous shifts will lead to new features.

To illustrate these considerations we choose a setup where $\mathrm{H} = \mathbb{R}$, $\epsilon^{\text{H,SL}} = \tilde{\epsilon}^{\mathbb{R},\text{SL}}_{\alpha}$, $\epsilon^{\text{H,SU}} = \epsilon^{\mathbb{R},\text{SU}}_{\beta}$, $f_1 = f_2 = e$, $\Omega_{\text{SL}} = \text{id}$ and $\Omega^{\text{H}}_1 = \Omega^{\text{H}}_2 = \Omega_\eta$ for $\eta = \pm 1$ as well as $\Omega_{\text{H}} = \Omega_-$. The product of the embedding of the last two twisted conjugacy classes (5.18) is parametrized by two real numbers λ, λ' and reads

$$\left(\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \,,\, \begin{pmatrix} \cos\psi' & \sin\psi' \\ -\sin\psi' & \cos\psi' \end{pmatrix} \right) \quad ,$$

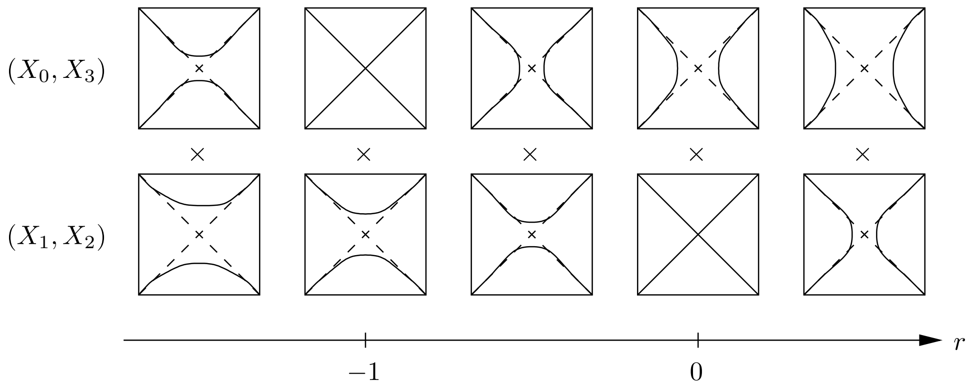

Figure 5.5: The group manifold $\mathrm{SL}(2,\mathbb{R})$.

with $\psi = \alpha(\eta\lambda + \lambda' + f_3)$ and $\psi' = \beta\eta(\lambda - \lambda' + f_3)$. We recognize that the joint action of simultaneous shifts for $\eta = 1$ leads to a factorized structure again as both ψ and ψ' are independent. For $\eta = -1$ the translations are "collinear" and one thus just recovers a shifted version of the non-factorizing D-brane which is already familiar from the previous subsection.

5.3 The big-bang big-crunch space-time

Recently, there has been renewed interest [140] in the Nappi-Witten background [53] which describes a closed universe between a big-bang and a big-crunch singularity. It was shown that the dynamics couples the closed universe to regions in space-time which formerly were believed to be unphysical. The full geometry is given by the coset $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)/\mathbb{R} \times \mathbb{R}$ where the groups in the denominator act asymmetrically on both factors in the numerator. Here we shall apply our general framework to the discussion of brane geometries in these asymmetric cosets. We believe that the construction of the corresponding boundary states in these non-compact backgrounds is possible using results from [141, 142].

5.3.1 The bulk geometry

Let us review the geometry of the target space first. For our purposes it is convenient to follow eq. (5.7) and to parametrize the group manifold $\mathrm{SL}(2,\mathbb{R})$ in terms of four real numbers $X_0, \cdots, X_3$ satisfying $X_0^2 - X_1^2 + X_2^2 - X_3^2 = 1$. This set can be depicted as a product of hyperbolas $X_1^2 - X_2^2 = r$ and $X_0^2 - X_3^2 = 1 + r$ in the (X_1, X_2)-plane and the (X_0, X_3)-plane, respectively. These hyperbolas are fibered over the real coordinate r and they degenerate

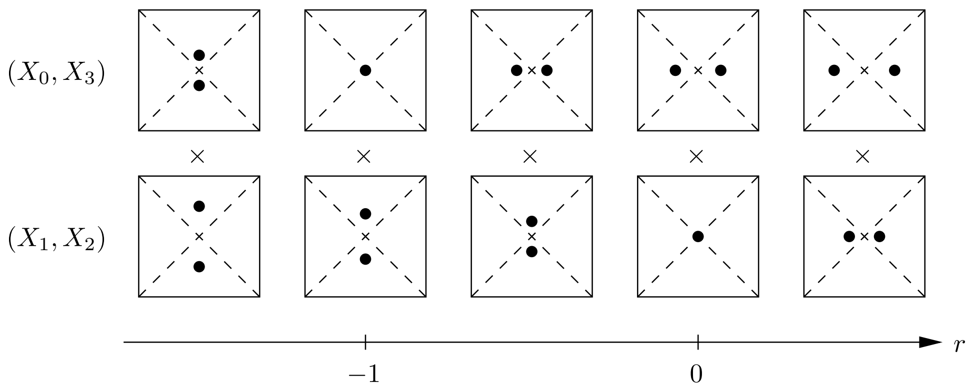

Figure 5.6: The group manifold $SL(2, \mathbb{R})$ after gauging.

in one of the two planes for $r = -1, 0$. We thus have to distinguish the regions $r > 0$, $0 > r > -1$ and $-1 > r$. The resulting geometry is pictured in figure 5.5 as a fibre over $r \in \mathbb{R}$. The parametrization of SU(2) in terms of complex coordinates z_1, z_2 has already been introduced in section 5.1.2.

In the next step we have to specify the action of the subgroup $\mathbb{R} \times \mathbb{R}$ on $SL(2, \mathbb{R}) \times SU(2)$. To make contact with the general setting of chapter 4 let us introduce the notation $G = G_1 \times G_2 = SL(2, \mathbb{R}) \times SU(2)$ and $H = \mathbb{R} \times \mathbb{R}$. The coset we want to consider is defined by using the identification $g \sim \epsilon_L(h)\, g\, \epsilon_R(h^{-1})$ where the left and right homomorphisms of the subgroup H are specified by [53]

$$\begin{aligned} \epsilon_L(\rho, \tau) &= \begin{pmatrix} e^{\rho} & 0 \\ 0 & e^{-\rho} \end{pmatrix} \times \begin{pmatrix} e^{i\tau} & 0 \\ 0 & e^{-i\tau} \end{pmatrix} \\ \epsilon_R(\rho, \tau) &= \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix} \times \begin{pmatrix} e^{-i\rho} & 0 \\ 0 & e^{i\rho} \end{pmatrix} \quad . \end{aligned}$$

Using these expressions it is not difficult to see that the action of H leaves the quantities $X_0^2 - X_3^2$, $X_1^2 - X_2^2$, $|z_1|$ and $|z_2|$ invariant. In fact, these transformations correspond to boosts on the hyperbolas and rotations in the complex planes. Deviating from the analysis in [140] we will perform the gauge fixing completely in the $SL(2, \mathbb{R})$ part of the target space. As can easily be seen, the gauge transformations allow to gauge the $SL(2, \mathbb{R})$ hyperbolas down to two disconnected points. This procedure completely removes the gauge freedom except for singular points at $r = -1, 0$. These locations correspond to the big-bang and big-crunch singularities and we will not be concerned too much with details of the geometry at these special points. The findings of these considerations are illustrated in the figures 5.6 and 5.7.

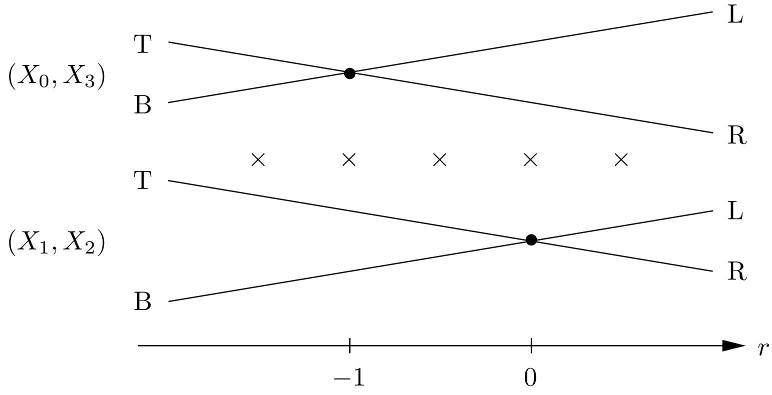

Figure 5.7: An alternative representation of the group manifold $SL(2, \mathbb{R})$ after gauging.

It is now only a short step to recover the results of [140]. Let us introduce the symbols L, R, T, B which are shorthand for left, right, top and bottom and specify the location of points relative to the origin of the coordinate system in the planes which are shown in figure 5.6. The regions of $SL(2, \mathbb{R})$ which appear in the fibre over $r \in \mathbb{R}$ can be described by pairs of symbols L, R, T, B, one for each of the planes. A short look at figure 5.6 reveals that only twelve different combinations are allowed. Working out the connectivity properties of these different regions we arrive at figure 5.8 which has also been obtained in [140]. In order to simplify the comparison with [140] we have adopted their notation in the last picture. The translation can be performed by means of table 5.1 (see also figure 5.9). From figure 5.8 we observe that there are four closed compact universes I–IV which are connected at the big-bang and big-crunch singularities. At each instant of time they have the topology of a three-sphere S^3 if one takes the SU(2) factor into account. The periodicity of time may be resolved by turning to the infinite cover AdS_3 of $SL(2, \mathbb{R})$. In addition to the closed universes there are eight whiskers which are also connected to the singularities. Over each point in the whisker one has a S^3.

5.3.2 The D-branes

Let us now begin to place branes into this geometry [143, 55]. Since the Nappi-Witten background falls into the class of asymmetric cosets of generalized automorphism type, there is a well-defined prescription for the geometry of many branes. According to the results of section 4.3 we first have to find embedding chains which interpolate between $\mathbb{R} \times \mathbb{R}$ and $SL(2, \mathbb{R}) \times SU(2)$.

Unfortunately, there is no way to construct a non-trivial embedding chain, so one is left with the chain of depth $N = 1$ where $\mathrm{U}_0 = \mathrm{G} = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ and $\mathrm{U}_1 = \mathrm{H} = \mathbb{R} \times \mathbb{R}$.

Our next task consists in finding a homomorphism $\epsilon : \mathrm{H} \hookrightarrow \mathrm{G}$ and all choices of automorphisms Ω_0 and Ω_1 such that the conditions (4.11) are satisfied. We are allowed to set $\epsilon = \epsilon_\mathrm{L}$. If we define an automorphism Ω of $\mathbb{R} \times \mathbb{R}$ by $\Omega(\rho, \tau) = (-\tau, -\rho)$, then the right action can be expressed as $\epsilon_\mathrm{R} = \epsilon \circ \Omega$ and the condition (4.11) simplifies to

$$\Omega_0 \circ \epsilon \circ \Omega_1 \;=\; \epsilon \circ \Omega \quad . \tag{5.19}$$

D-branes in the Nappi-Witten background should be localized along the following product of twisted conjugacy classes,

$$\Big[\mathcal{C}^{\mathrm{SL}}_{f_1}(\omega_0) \times \mathcal{C}^{\mathrm{SU}}_{f_2}(\omega_0')\Big] \cdot (\omega_0 \times \omega_0') \circ \epsilon\big(\mathcal{C}^{\mathbb{R}\times\mathbb{R}}_f(\Omega_1)\big) \quad , \tag{5.20}$$

before projecting to the coset. Here, we split $\Omega_0 = \omega_0 \times \omega_0'$ into the product of automorphisms for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$, respectively. There are several choices of automorphisms $\Omega_1, \omega_0, \omega_0'$ which satisfy our condition (5.19) and we will discuss all of them in the following.

Let us start with the discussion of the twisted conjugacy class $\mathcal{C}^{\mathbb{R}\times\mathbb{R}}_f(\Omega_1)$. The most general automorphism of the additive group $\mathbb{R} \times \mathbb{R}$ is implemented by a non-singular 2×2-matrix. In our situation, however, not all choices are allowed. The only choices which have the chance to be consistent with condition (5.19) are $\Omega_1(\rho, \tau) = (\eta\tau, \xi\rho)$ where $\eta, \xi = \pm 1$. The resulting geometry is given by

$$\mathcal{C}^{\mathbb{R}\times\mathbb{R}}_f(\Omega_1) \;=\; \begin{cases} \qquad\qquad \mathbb{R} \times \mathbb{R} & , \text{ for } \xi = -\eta \\ \big\{(f_1 + \lambda, f_2 - \eta\lambda) \,\big|\, \lambda \in \mathbb{R}\big\} & , \text{ for } \xi = \eta \quad . \end{cases} \tag{5.21}$$

Embedding these sets into $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ by means of the map $(\omega_1 \times \omega_1') \circ \epsilon$ leads to the same result in both cases after gauge fixing.

When investigating the geometry of the D-branes (5.20) in the big-bang big-crunch target space it is convenient to focus on the $\mathrm{SL}(2,\mathbb{R})$-part as all interesting features arise from this factor. We thus only have to distinguish two different cases, corresponding to the two types of twisted conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$. As most of the group $\mathrm{SL}(2,\mathbb{R})$ – except for two isolated points for each value of r, respectively – will be gauged away, it even suffices to address the following two questions:

1. Which ranges of r are covered by the twisted conjugacy classes?

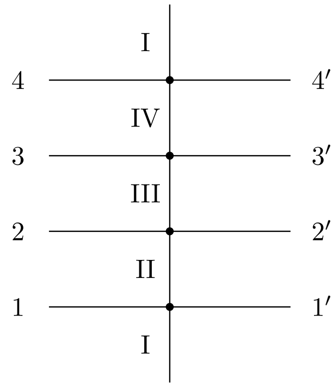

Figure 5.8: The big-bang big-crunch scenario.

2. Does the conjugacy class extend along one or even both branches of the hyperbolas, i.e. does the D-brane cover one or two points for fixed value of r after gauging?

The twisted conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ are easily described, see our discussion in section 5.1.3. In the untwisted case it is useful to distinguish two types. There are two point-like conjugacy classes which correspond to the center of $\mathrm{SL}(2,\mathbb{R})$ while all others are two-dimensional. The point-like branes are specified by $X_0 = \pm 1$ and $X_1 = X_2 = X_3 = 0$, i.e. they are localized at $r = 0$. After gauging they sit at the singularities between the closed universes I–II and III–IV, respectively. The two-dimensional conjugacy classes are of the form $X_0 = C = \text{const.}$ with arbitrary values of the remaining coordinates. According to the constraint (5.7) we obtain $r = C^2 - 1 - X_3^2 \leq C^2 - 1$. This means that the conjugacy class after gauging covers at least all four whiskers $2, 2', 4, 4'$. For $C \neq 0$ the conjugacy class grows into two of the four closed universes starting from the singularity which joins them. Depending on the sign of C these are the regions I–II (for $C > 0$) and III–IV (for $C < 0$). If $|C|$ reaches the value 1 (from below), the conjugacy class stretches completely through both of the closed universes. Increasing $|C|$ further, the conjugacy classes start to reach into two of the remaining whiskers – $1, 1'$ for $C > 1$ and $3, 3'$ for $C < -1$. Note, that the multiplication with the twisted conjugacy class of $\mathbb{R} \times \mathbb{R}$ has no influence on the possible values of r as it simply corresponds to some boost on the hyperbolas which will be gauge fixed to a point in any case.

The twisted conjugacy classes arise from the automorphism which reverses the sign of X_2 and X_3. The corresponding twisted conjugacy classes are given by $X_1 = C = \text{const.}$ According to the constraint (5.7) we obtain $r = C^2 - X_2^2 \leq C^2$. The discussion is similar as in the untwisted case. For

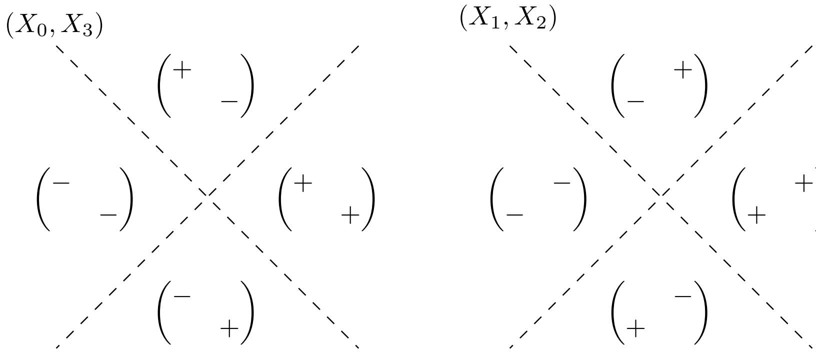

Figure 5.9: Different regions of $SL(2, \mathbb{R})$ and where they appear in our picture. The matrix elements indicate the sign of $X_0 \pm X_3$ and $X_1 \pm X_2$, respectively.

(R,B)	(R,T)	(L,T)	(L,B)	(R,R)	(R,L)	(B,T)	(T,T)	(L,L)	(L,R)	(T,B)	(B,B)
I	II	III	IV	1	$1'$	2	$2'$	3	$3'$	4	$4'$

Table 5.1: Translation table for the twelve different regions.

all values of C the twisted conjugacy classes pass through all four closed universes I–IV and the four whiskers $2, 2', 4, 4'$. For $C \neq 0$ the conjugacy classes also cover part of the whiskers $1, 3'$ ($C > 0$) or $1', 3$ ($C < 0$). The results of the last two paragraphs are illustrated in figure 5.10.

So far we have only considered the $SL(2, \mathbb{R})$-part of the target space. To obtain the complete picture we have also to take the SU(2)-part into account as well as the product with the twisted conjugacy class $\mathcal{C}_f^{\mathbb{R}\times\mathbb{R}}(\Omega_1)$. We already argued that the latter has no effect on the $SL(2, \mathbb{R})$-part as it does not affect the value of r and may thus be gauged away. This statement also implies that the resulting D-branes factorize (in the same sense as the gauge fixing factorized). If we try to solve condition (5.19) with $\omega_0 = \text{id}$, i.e. if we want to take the ordinary conjugacy classes in the $SL(2, \mathbb{R})$ part, we have to use an automorphism Ω_1 of $\mathbb{R} \times \mathbb{R}$ with $\eta = 1$. Depending on the choice of ξ we are still able to obtain both expressions for twisted conjugacy classes that appear in eq. (5.21). The same statement holds true for $\eta = -1$, i.e. for the case of a twisted conjugacy class in the $SL(2, \mathbb{R})$ part.

It is now very simple to describe the geometry of the D-branes in the SU(2)-part. We simply have to multiply the (shifted) conjugacy class of SU(2) with elements of the form $\text{diag}(e^{i\lambda}, e^{-i\lambda})$ for all values of λ. As was argued in section 5.1.2, this procedure corresponds to a smearing of the original conjugacy class.

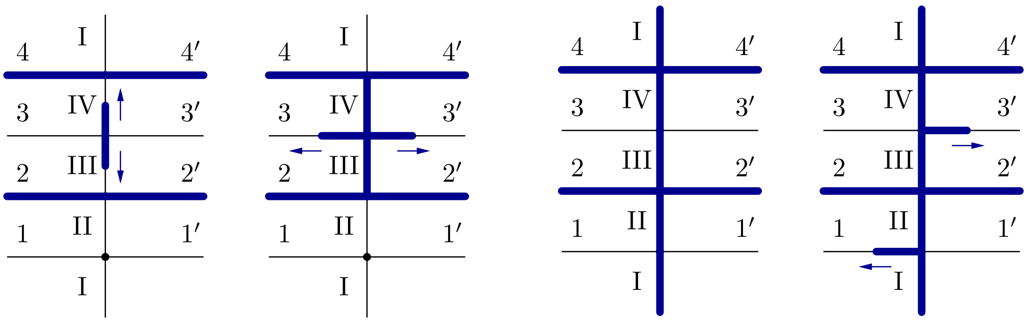

Figure 5.10: D-branes in the big-bang big-crunch scenario. The branes on the left hand side have been constructed with an ordinary conjugacy class of $SL(2,\mathbb{R})$ while for the right ones a twisted conjugacy class was employed.

Let us conclude with a short summary of our results. For convenience we illustrated all essential information about the target space and about its D-branes in the figure 5.10. While the D-branes cover the high-lightened regions in the $SL(2,\mathbb{R})$-part, we also have a three-sphere over each of these points which is partly covered by the D-brane. The geometry of the latter is either given by a circle around some equator or by a smeared two-sphere which covers a three-dimensional subset of S^3.

5.4 T^{pq} spaces and the base of the conifold

The spaces T^{pq} that we are about to analyze next are simple generalizations of the space T^{11}. The latter is a close relative of the base of the conifold in which the RR-fluxes are replaced by a NSNS background field [54]. Our general theory provides a large class of boundary theories for this background, including branes that wrap one of the three-spheres in T^{11}. Related objects play an important role in the conifold geometry (see for example [135]).

5.4.1 The bulk geometry

The T^{pq} spaces are defined to be quotients $SU(2)_{k_1} \times SU(2)_{k_2}/U(1)_k$ where the $U(1)$ subgroup acts by twisted conjugation, i.e. according to $(g_1, g_2) \mapsto \big(g_1\, \epsilon_1'(h^{-1}), \epsilon_2'(h)\, g_2\big)$. The embeddings $\epsilon_1' = \epsilon_p^{U,SU}$ and $\epsilon_2' = \epsilon_q^{U,SU}$ are defined by

$$\epsilon_p^{U,SU}\big(e^{i\phi}\big) \;=\; \begin{pmatrix} e^{ip\phi} & 0 \\ 0 & e^{-ip\phi} \end{pmatrix} \;\cong\; \big(e^{ip\phi}, 0\big) \quad .$$

The action of the subgroup U(1) on the numerator SU(2) × SU(2) can be obtained from the choice of embedding maps

$$\epsilon_{\rm L} \;=\; e \times \epsilon_2' \quad , \quad \epsilon_{\rm R} \;=\; \epsilon_1' \times e \quad .$$

If we parametrize the first factor SU(2) by (z_1, z_2) as in section 5.1.2 and similarly use (z_1', z_2') for the second factor, we realize that the action of $h = \exp(i\tau) \in \mathrm{U}(1)$ can be stated more explicitly by the formula

$$(z_1, z_2, z_1', z_2') \;\mapsto\; (e^{-ip\tau} z_1, e^{ip\tau} z_2, e^{iq\tau} z_1', e^{iq\tau} z_2') \quad . \tag{5.22}$$

The corresponding gauged WZNW functional is free of anomalies provided that $k = k_1\, p^2 = k_2\, q^2$ (see condition (4.2)). Note that the resulting coset still has a SU(2) × SU(2) symmetry which is realized by $(g_1', g_2') \mapsto \bigl(g_1\, g_1', g_2'\, g_2\bigr)$.

The geometry of the coset may be deduced by using the action (5.22) to put z_1 to a real number. This prescription works fine except for $z_1 = 0$ where we have to gauge z_2 instead. The resulting geometry is based on a product of a two-sphere times a three-sphere. A detailed analysis yields

$$T^{pq} \;=\; \mathrm{SU}(2) \times \mathrm{SU}(2) \,/\, \mathrm{U}(1) \;=\; \bigl(S^2 \times S^3\bigr) \,/\, \mathbb{Z}_p \quad ,$$

since due to the non-trivial embeddings only part of the U(1) has to be used for the gauge fixing and an orbifold action remains.

5.4.2 The D-branes

Let us now have a look for the D-branes in this geometry [143, 55]. The T^{pq}-spaces fall into the GMM-class of asymmetric coset theories which has been introduced in section 4.1.4. The latter, however, just form a subset of asymmetric cosets of generalized automorphism type, see our discussion in section 4.3. Therefore there is a good chance to reveal the geometry of a large number of D-branes for T^{pq}-spaces.

According to the general procedure we are instructed to select a chain of subgroups. In the present situation, the only possibility turns out to be a chain of depth $N = 2$ consisting of $\mathrm{U}_2 = \mathrm{U}(1)_k$, $\mathrm{U}_1 = \mathrm{U}(1)_k \times \mathrm{U}(1)_k$ and $\mathrm{U}_0 = \mathrm{SU}(2)_{k_1} \times \mathrm{SU}(2)_{k_2}$. We also pick embedding maps $\epsilon_2 : \mathrm{U}_2 \to \mathrm{U}_1$ defined by $\epsilon_2(h) = e \times h$ and $\epsilon_1 : \mathrm{U}_1 \to \mathrm{U}_0$ given through $\epsilon_1 = \epsilon_p \times \epsilon_q$. All what remains to be done is to choose a set of three automorphisms $\Omega_0, \Omega_1, \Omega_2$ such that the conditions (4.11) are satisfied. The localization of the D-branes would then be given by the product

$$\mathcal{C}^{\rm SU\times SU}(\Omega_0) \cdot \Omega_0 \circ \epsilon_1\bigl(\mathcal{C}^{\rm U\times U}(\Omega_1)\bigr) \cdot \Omega_0 \circ \epsilon_1 \circ \Omega_1 \circ \epsilon_2\bigl(\mathcal{C}^{\rm U}(\Omega_2)\bigr) \quad ,$$

if considered on the "covering" space SU(2) × SU(2). Yet, although these branes possess the correct symmetry, there is some problem with this proposal in the present case since a naive application of this prescription would lead to cosets of the form U/U in the algebraic description.

The way out is suggested by the expression (4.13) for the maximal chiral algebra in GMM models which can be preserved on the boundary. It lets us suggest a natural target space reinterpretation of the form

$$\frac{\mathrm{G}_1\times\mathrm{G}_2}{\mathrm{H}} \;=\; \frac{\mathrm{G}_1\times\mathrm{G}_2\times\mathrm{H}\times\mathrm{H}\times\mathrm{H}}{\mathrm{H}\times\mathrm{H}\times\mathrm{H}\times\mathrm{H}} \quad . \tag{5.23}$$

It is a tricky issue to incorporate the correct action of the subgroup H^4. Let us propose the identifications

$$\begin{aligned}\big((g_1,g_2),(h_1,h_2,h_3)\big) \;&\sim\; \Big(\big(g_1\,\Omega_{\mathrm{G}_1}\circ\epsilon_1'(k^{-1}),g_2\big),\big(h_1,k\,h_2,h_3\big)\Big)\\ &\sim\; \Big(\big(g_1,g_2\,\Omega_{\mathrm{G}_2}\circ\epsilon_2'(k^{-1})\big),\big(h_1,h_2,k\,h_3\big)\Big)\\ &\sim\; \Big(\big(g_1,g_2\big),\big(k\,h_1,h_2\,\Omega_1^{-1}(k^{-1}),h_3\big)\Big)\\ &\sim\; \Big(\big(g_1,\epsilon_2'(k)\,g_2\big),\big(h_1\,k^{-1},h_2,h_3\big)\Big) \quad .\end{aligned}$$

In order to recover the original coset we have to constrain the automorphisms. The analogue of condition (4.11) in the present situation reads

$$\Omega_{\mathrm{G}_1}\circ\epsilon_1'\circ\Omega_1^{-1} \;=\; \epsilon_1' \quad . \tag{5.24}$$

Note that this equation does not refer to the automorphism Ω_{G_2} in any way. Finally, we define the remaining action of H

$$\big((g_1,g_2),(h_1,h_2,h_3)\big) \;\mapsto\; \big((\epsilon_1'(k)\,g_1,g_2),(h_1,h_2,h_3\,\Omega_2(k^{-1}))\big) \quad ,$$

which shall be preserved by the D-brane. One can easily see that the choice of automorphisms Ω_{G_1}, Ω_{G_2}, Ω_1 and Ω_2 exhausts all freedom of performing twists.

The simplest way to check our last assertion is to descend to the level of the algebra of functions, or equivalently, to the partition function of the associated CFT. Since our main aim is to obtain a rough idea of what is going on, we neglect field identification[1] and write

$$\mathcal{H} \;=\; \bigoplus \mathcal{H}^{\mathrm{G}_1/\mathrm{H}}_{(\mu,a)}\otimes\mathcal{H}^{\mathrm{G}_2/\mathrm{H}}_{(\alpha,b)}\otimes\mathcal{H}^{\mathrm{H}}_{a}\otimes\bar{\mathcal{H}}^{\mathrm{G}_1/\mathrm{H}}_{(\omega_{\mathrm{G}_1}(\mu),\omega_1(b))^+}\otimes\bar{\mathcal{H}}^{\mathrm{G}_2/\mathrm{H}}_{(\omega_{\mathrm{G}_2}(\alpha),c)^+}\otimes\bar{\mathcal{H}}^{\mathrm{H}}_{\omega_2(c)^+} \quad .$$

[1]This assumption is valid if p and q are relatively prime or in the infinite volume limit.

The only place where an additional twist could enter is in the a-variable. It would relate the variables a with c when we impose gluing conditions, but actually it can be absorbed in a redefinition of Ω_2. The gluing conditions also tell us that – for the symmetric part of the Hilbert space – b and c are actually completely determined by the choice of a. As a consequence there is only one H-parameter left for the Ishibashi states and hence also for the boundary states.

For the D-brane geometry, the previous remarks imply the occurrence of a *cyclically* twisted conjugacy class in the group $\mathrm{H}\times\mathrm{H}\times\mathrm{H}$ which enters the numerator of expression (5.23). To be more precise, we expect

$$\mathcal{C}_f^{\mathrm{H}\times\mathrm{H}\times\mathrm{H}}(\text{cycl.}) \;=\; \big\{\big(k_1 f k_2^{-1}, k_2 f \Omega_1^{-1}(k_3^{-1}), k_3\Omega_2(f k_1^{-1})\big)\,\big|\, k_1,k_2,k_3\in\mathrm{H}\,\big\}$$

to be the relevant object. For the constituents G_1 and G_2 we obtain ordinary twisted conjugacy classes $\mathcal{C}_{f_1}^{\mathrm{G}_1}(\Omega_{\mathrm{G}_1})$ and $\mathcal{C}_{f_2}^{\mathrm{G}_2}(\Omega_{\mathrm{G}_2})$ as usual.

In order to be able to identify the geometry of the brane we have to project the set

$$\mathcal{C}_{f_1}^{\mathrm{G}_1}(\Omega_{\mathrm{G}_1})\times\mathcal{C}_{f_2}^{\mathrm{G}_2}(\Omega_{\mathrm{G}_2})\times\mathcal{C}_f^{\mathrm{H}\times\mathrm{H}\times\mathrm{H}}(\text{cycl.}) \;\subset\; \mathrm{G}_1\times\mathrm{G}_2\times\mathrm{H}\times\mathrm{H}\times\mathrm{H}$$

down to the coset, i.e. to fix the gauge. For our purposes, the most suitable representative for elements in the coset (5.23) turns out to be

$$\Big(\big(g_1, \epsilon_2'(\Omega_1(h_2)h_1)\, g_2\, \Omega_{\mathrm{G}_2}\circ\epsilon_2'(h_3)\big), (e,e,e)\Big) \quad .$$

Since we are not dealing with single elements g_1, g_2 but with whole conjugacy classes, we can use translation invariance along $\mathcal{C}^{\mathrm{G}_2}(\Omega_{\mathrm{G}_2})$ to rewrite the previous expression as

$$\big(\mathcal{C}_{f_1}^{\mathrm{G}_1}(\Omega_{\mathrm{G}_1}), \mathcal{C}_{f_2}^{\mathrm{G}_2}(\Omega_{\mathrm{G}_2})\,\Omega_{\mathrm{G}_2}\circ\epsilon_2'(\Omega_1(h_2)h_1h_3)\big) \;\subset\; \mathrm{G}_1\times\mathrm{G}_2 \quad .$$

This form is the most convenient for the evaluation of the brane geometry since the gauge fixing, i.e. putting the complex coordinate z_1 parametrizing the G_1-part to a positive real number, may be performed straightaway.

The geometry and the dimensions of the resulting branes depend crucially on the exact form of the expression

$$\Omega_1(h_2)h_1h_3 \qquad \text{for} \qquad (h_1,h_2,h_3) \;\in\; \mathcal{C}_f^{\mathrm{U}\times\mathrm{U}\times\mathrm{U}}(\text{cycl.}) \quad .$$

Depending on the choice of automorphisms Ω_1 and Ω_2 one obtains either a point-like or a one-dimensional object which we will abbreviate by $\mathcal{C}^{\mathrm{U}}(\Omega)$ since it coincides geometrically with a single twisted conjugacy class of H. If

$\Omega_1 \neq \mathrm{id}$ or $\Omega_2 \neq \mathrm{id}$, the same holds true for Ω. Appropriate choices of the automorphisms Ω_{G_1}, Ω_1 and Ω_2 enable us to obtain both types for $\mathcal{C}^{\mathrm{H}}(\Omega)$. If we employ for example the inversion $\Omega_1 = \Omega_-^{\mathrm{U}}$ in combination with the inner automorphism $\Omega_{\mathrm{G}_1} = \gamma$ which has been defined in eq. (5.5), the set $\mathcal{C}^{\mathrm{U}}(\Omega)$ will be one-dimensional. In contrast, it will be point-like if all automorphisms are chosen to be trivial. One may easily check that the constraint (5.24) is satisfied in both cases.

Having all these details in mind, the brane world-volume can be written

$$\begin{aligned}\mathcal{C}^{\mathrm{SU}}_{f_1} \times \mathcal{C}^{\mathrm{SU}}_{f_2} \times \mathcal{C}^{\mathrm{U}\times\mathrm{U}\times\mathrm{U}}_{f}&(\mathrm{cycl.}) \\ &= \left\{ \left(r e^{\pm i\theta(r)}, \sqrt{1-r^2}\, e^{i\phi_2}, r' e^{\pm i\theta'(r')+iq\phi}, \sqrt{1-r'^2}\, e^{i\phi_2'-iq\phi} \right) \right\}\end{aligned}$$

if considered on the covering space SU(2)×SU(2). The signs $\pm$ can assume all four possible choices and the parameters r and r' take values in the intervals $c \leq r \leq 1$ and $c' \leq r' \leq 1$, where the non-negative constants c, c' depend on the labels of the conjugacy classes in the SU(2)-parts and parametrize their trace. The functions θ and θ' are implicitly defined by the relations $r\cos\theta(r) = c$, $r'\cos\theta'(r') = c'$. The angles ϕ_1, ϕ_2 are not constrained while the value of ϕ is restricted by the condition $e^{i\phi} \in \mathcal{C}^{\mathrm{U}}(\Omega)$.

The gauge may be fixed by using the transformation (5.22) with $p\,\tau = \pm\theta(r)$. After a redefinition of ϕ_1 and ϕ_2 we end up with the following subset

$$\begin{aligned}\mathcal{C}^{\mathrm{SU}} \times \mathcal{C}^{\mathrm{SU}} \times \mathcal{C}^{\mathrm{U}\times\mathrm{U}\times\mathrm{U}}_{f}&(\mathrm{cycl.}) \\ &= \left\{ \left(r, \sqrt{1-r^2}\, e^{i\phi_2}, r' e^{\pm i\theta'(r')\pm iq/p\,\theta(r)+iq\phi}, \sqrt{1-r'^2}\, e^{i\phi_2'} \right) \right\} \quad .\end{aligned}$$

This rather explicit expression immediately allows us to read off the dimensions of branes in the T^{pq}-spaces. The generic brane is thus four- or five-dimensional, depending on whether $\mathcal{C}^{\mathrm{U}}(\Omega)$ is point-like or one-dimensional. For each of the special choices $c = 1$ or $c' = 1$ one looses two dimensions, since the radial variables will be constrained to $r = 1$ or $r' = 1$ in this case. As a conclusion we found branes of all dimensionality inside of the T^{pq}-spaces. When the levels are even, it is in particular possible to reveal three-dimensional branes which fill one of the three-spheres of T^{pq}. Related objects play an important role for string theory on the conifold.

5.5 Defect lines in WZNW theories

Our final goal is to apply the general formalism to the construction of non-trivial defect lines between different conformal field theories. According to our remarks in section 2.1.2 this can be achieved by considering non-factorizing boundary states in products of two or more conformal field theories. While our previous examples for branes in such backgrounds centered around the geometric interpretation, we shall now emphasize the algebraic aspects. After a short reminder of the general setup and additional motivation we will illustrate the constructive power of our formulas by considering defects between WZNW models.

5.5.1 Boundary states in tensor products

In the last few sections we already encountered a number of examples in the context of string theory where product conformal field theories played a crucial role. Another important application of the latter arises by considering a one-dimensional quantum system with a defect (see e.g. [86, 144, 145, 146, 147] and [148, 149] for higher dimensional analogues), or, more generally, two different systems on the half-lines $x < 0$ and $x > 0$ which are in contact at the origin. The defect or contact at $x = 0$ could be totally reflecting, or more interestingly, it could be partially (or fully) transmissive. To fit such system into our general discussion, we apply the usual folding trick (see figure 2.3). After such a folding, the defect or contact is located at the boundary of a new system on the half-line. In the bulk, the new theory is simply a product of the two models that were initially placed to both sides of the contact at $x = 0$.

Factorizing boundary states for the new product theory on the half line correspond to totally reflecting defects or contacts. With our new boundary states we can go further and couple the two systems in a non-trivial way. Since we always start with conformal field theories $\mathrm{G}_1 = \mathrm{G}_<$ and $\mathrm{G}_2 = \mathrm{G}_>$ on either side of $x = 0$, it is natural to look for contacts that preserve conformal invariance. This requires to preserve the sum of the two Virasoro algebras of the individual theories. After folding the system, the preserved Virasoro algebra is diagonally embedded into the product theory $\mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2$. Of course, one can often embed a larger chiral algebra H and then look for defects that preserve this extended symmetry. This is exactly the setup to which our general ideas of chapter 3 apply.

The construction of non-factorizing boundary states in product theories may also shed new light on the K-theory conjecture [150, 151]. In the case of $S^3 \times S^3$, for example, stable factorizable branes carry only zero-brane charge. But K-theory predicts the existence of additional branes with non-vanishing three-brane charge which cannot be built up from branes on the factors S^3. Hence, there is a strong demand for additional boundary states.

5.5.2 Defect lines with jumping central charge

Our goal is now to construct examples of defect lines that join two conformal field theories with different central charge. Such situations are known to appear on the boundary of an AdS-space whenever there is a brane in the bulk which extends all the way to the boundary [66, 67, 68, 69]. To be specific, we will choose two WZNW models based on the same semi-simple Lie group H but at different levels k_i and hence with different central charges $c_i = k_i \dim \mathrm{H}/(k_i + g^\vee)$. The boundary states we shall discuss may also be interpreted as D-branes in the product geometry $\mathrm{H}_1 \times \mathrm{H}_2$ in which the two factors may have different volume.

In this setup, the "G-theory" is provided by the charge conjugate modular invariant partition function for $\mathcal{A}(\mathrm{G}) = \mathcal{A}(\mathrm{H}_1) \otimes \mathcal{A}(\mathrm{H}_2) = \mathcal{A}(\hat{\mathfrak{h}}_{k_1} \oplus \hat{\mathfrak{h}}_{k_2})$. Now we are instructed to choose some chiral subalgebra $\mathcal{A}(\mathrm{P})$. There are many different choices, but for simplicity we shall use the affine algebra $\hat{\mathfrak{h}}_{k_1+k_2}$ which is embedded diagonally into $\hat{\mathfrak{h}}_{k_1} \oplus \hat{\mathfrak{h}}_{k_2}$. In other words, $\mathrm{P} \cong \mathrm{H}_\mathrm{D}$ and $\mathcal{A}(\mathrm{H}_\mathrm{D}) = \mathcal{A}(\hat{\mathfrak{h}}_{k_1+k_2})$.

We start by introducing some pieces of notation. As we have to deal with three *different* affine algebras $\hat{\mathfrak{h}}$, it is convenient to use different labels for the sectors of each of these algebras,

$$\mu, \nu, \cdots \;\in\; \mathrm{Rep}(\mathrm{H}_1) \quad , \quad \alpha, \beta, \cdots \;\in\; \mathrm{Rep}(\mathrm{H}_2) \quad , \quad a, b, \cdots \;\in\; \mathrm{Rep}(\mathrm{H}_\mathrm{D}) \quad .$$

In the case under consideration, the projection is given by $\mathcal{P}(\mu, \alpha) = \mu + \alpha$ and hence the branching selection rule (2.22) reduces to $\mu + \alpha - a \in \mathcal{Q}$ where $\mathcal{Q}$ denotes the root lattice of $\mathfrak{h}$. Consequently, the coset fields are labeled by triples

$$\big((\mu, \alpha), a\big) \;\in\; \mathrm{Rep}(\mathrm{H}_1 \times \mathrm{H}_2 \times \mathrm{H}_\mathrm{D}) \qquad \text{with} \qquad \mu + \alpha - a \in \mathcal{Q}$$

which give rise to a set $\mathrm{All}(\mathrm{H}_1 \times \mathrm{H}_2/\mathrm{H}_\mathrm{D})$ of allowed tripels. Next we have to describe the field identifications. Let $\mathcal{Z}(\mathrm{H}_i)$ be the centers of the H_i-theories which – under our assumptions – are all ismorphic. The common center is given by the diagonal subset

$$\big\{\big((J, J), J\big)\big\} \;=\; \mathcal{G}_\mathrm{id} \;=\; \mathcal{Z}_\mathrm{D} \;\subset\; \mathcal{Z}(\mathrm{H}_1) \times \mathcal{Z}(\mathrm{H}_2) \times \mathcal{Z}(\mathrm{H}_\mathrm{D})$$

and leads us to the identification rules

$$\big((J\mu, J\alpha), Ja\big) \;\sim\; \big((\mu, \alpha), a\big) \quad .$$

Note in particular that no additional field identifications occur even in the case where the levels coincide, $k_1 = k_2 = k$, and the two types of fields μ, α take values in the same set.

Affine Lie algebra	$A_n^{(1)}$	$B_n^{(1)}$	$C_n^{(1)}$	$D_n^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$
Fixed points for k in	$\bigcup_{1\neq s\mid(n+1)} s\mathbb{N}_0$	$\mathbb{N}_0$	$2\mathbb{N}_0$	$2\mathbb{N}_0$	$3\mathbb{N}_0$	$2\mathbb{N}_0$

Table 5.2: Existence of fixed points under simple current actions.

After this preparation we can address the issue of field identification fixed points and spell out conditions for their absence. For the moment, let us focus on one of the factors and denote it by H. Every outer automorphism $J \in \mathcal{Z}(\mathrm{H})$ is associated with a unique permutation π_J of affine fundamental weights. Denoting affine weights by square brackets, this action may be written as

$$J\,[\,\lambda_0\,,\,\cdots\,,\,\lambda_r\,] \;=\; [\,\lambda_{\pi_J(0)}\,,\,\ldots\,,\,\lambda_{\pi_J(r)}\,] \quad .$$

Thus the existence of a field identification fixed point is equivalent to finding an affine weight such that $\lambda_i = \lambda_{\pi_J(i)}$ for all $i = 0, \cdots, r$ and at least one non-trivial $J \in \mathcal{Z}(\mathrm{H})$. The condition for the existence of such weights have been studied and the results for all simple Lie algebras are summarized in table 5.2. Note that the exceptional groups E_8, F_4 and G_2 have trivial centers and thus no field identification or selection rules.

To illustrate the rules summarized in table 5.2, let us derive them for the special case of A_2. The group $\mathcal{Z}\big(A_2^{(1)}\big) \cong \mathbb{Z}_3$ is generated by the shift

$$J\,[\,\lambda_0, \lambda_1, \lambda_2\,] \;=\; [\,\lambda_2, \lambda_0, \lambda_1\,] \quad .$$

In terms of non-affine weights, this action reads $J(\lambda_1, \lambda_2) = (k - \lambda_1 - \lambda_2, \lambda_1)$. Hence, a fixed point would have to satisfy $\lambda_1 = \lambda_2$ and $\lambda_1 = k - \lambda_1 - \lambda_2$, i.e. it should be given by $(k/3, k/3)$. Obviously this is not an element of the weight lattice unless the level k is a multiple of three.

Except from the B-series, we can always find levels for which the action of the center $\mathcal{Z}(\mathrm{H})$ on the weights has no fixed points. In the context of our construction, we have three different sets of labels on which this groups acts at the same time and it is sufficient if at least one of the values k_1, k_2 or $k_1 + k_2$ avoids the values specified in table 5.2. In the following we shall assume that this condition is satisfied. Otherwise one would have to resolve the fixed points following reference [122], which leads to technical difficulties but no conceptually new insights.

The rest is now straightforward. Let us recall from section 2.4.3 that the modular S-matrix of the "numerator theory" factorizes according to

$$S^{\mathrm{H}_1\times\mathrm{H}_2}_{(\mu,\alpha)(\nu,\beta)} \;=\; S^{\mathrm{H}_1}_{\mu\nu}\; S^{\mathrm{H}_2}_{\alpha\beta} \qquad \text{and} \qquad N_{(\mu,\alpha)(\nu,\beta)}^{\;\;(\rho,\gamma)} \;=\; N_{\mu\nu}^{\;\rho}\, N_{\alpha\beta}^{\;\gamma} \quad .$$

Our boundary states are now labeled by $\mathcal{G}_{\mathrm{id}}$-orbits of triples $\big((\rho,\gamma),r\big)$. When we finally insert these expressions into the formula (3.14), we can read off the boundary partition function,

$$Z(q) \;=\; \sum_{((\nu,\beta),b),c,d} \Big[N_{\rho_1^+\rho_2}^{\;\;\nu}\, N_{\gamma_1^+\gamma_2}^{\;\;\beta}\, N_{r_1^+r_2}^{\;\;d}\, N_{dc}^{b} \Big]\, \chi^{\mathrm{H}_1\times\mathrm{H}_2/\mathrm{H}_\mathrm{D}}_{((\nu,\beta),b)}(q)\, \chi^{\mathrm{H}_\mathrm{D}}_c(q) \quad ,$$

which describes the spectrum of fields living between the boundary conditions $\big[(\rho_1,\gamma_1),r_1\big]$ and $\big[(\rho_2,\gamma_2),r_2\big]$. When reinterpreted in terms of defects, our formulas provide us with a large set of possible junctions between two conformal field theories. Note that these have different central charge if $k_1 \neq k_2$.

Chapter 6

Conclusions and outlook

In the present work we provided a comprehensive study of two-dimensional boundary conformal field theories which are obtained from asymmetrically gauged Wess-Zumino-Novikov-Witten theories. Even though conformal invariance imposes strong constraints, a lot of freedom in model building is gained. Our investigations shed light on both the abstract algebraic approach and the geometric interpretation. The detailed analysis of several interesting examples showed the wide applicability of our results.

One of the main characteristics of asymmetric cosets is their heterotic symmetry algebra. As a consequence, the construction of the bulk theory required to go beyond the standard coset construction which applies for adjoint cosets. For the same reason, there is no chance to define conformal boundary conditions which preserve the full symmetry. Instead, we have been forced to introduce boundary conditions which break part of the symmetry [56, 57, 55].

The key ingredient in most of our constructions was a distinguished class of asymmetric cosets of so-called "generalized automorphism type". This rather large set contains ordinary adjoint cosets and WZNW theories as special cases but, among others, also all models of GMM-type [51]. In contrast to generic asymmetric cosets with rational spectrum which might lead to a non-rational boundary theory, we have been able to construct rational boundary theories for those of generalized automorphism type [55].

Let us briefly summarize some details of our results. The analysis of symmetry breaking boundary conditions in arbitrary CFT's was initiated in chapter 3 based on the idea of rational conformal embeddings. The latter allowed to reduce the symmetry of the model in a way which is compatible with conformal invariance. In WZNW theories based on a group G this has been attained by choosing an arbitrary continuous subgroup H. On the algebraic level, this choice had to be accompanied by the decomposition of the original chiral algebra $\mathcal{A}(\mathrm{G})$ into the tensor product $\mathcal{A}(\mathrm{H}) \otimes \mathcal{A}(\mathrm{G/H})$ of the smaller WZNW-symmetry and its associated coset algebra.

In this framework the construction of boundary states which only preserve the reduced symmetry has been achieved for different kinds of gluing conditions. The only input we needed was the solution for maximally symmetric boundary conditions in the G- and the H-theory, respectively. By iterating the symmetry reduction, i.e. by introducing whole embedding chains (3.3) of intermediate groups, an entire hierarchy of symmetry breaking boundary states could be recovered.

We proposed the product (3.6) of twisted conjugacy classes, one for each group in the embedding chain, for the localization of these branes. This was justified by symmetry considerations and the comparison of the algebraic outcome for the open string spectrum with expectations from non-commutative geometry. The proposal was further substantiated by the construction of the corresponding boundary WZNW functional. In view of future applications the identification of branes which cover (at least almost) all the space might play an important role [130].

In contrast to the case of WZNW models our study of asymmetric coset theories G/H in section 4 had to start nearly from scratch, i.e. with the construction of the bulk theory. Since the action $g \mapsto \epsilon_{\mathrm{L}}(h)\, g\, \epsilon_{\mathrm{R}}(h^{-1})$ of the subgroup H which is divided out, does not need to be symmetric, the resulting CFT is heterotic in many cases, i.e. the left and right moving chiral algebras disagree. After some refinements of the usual coset construction we have nevertheless been able to propose a reasonable partition function (4.7) which possesses the correct geometrical limit. The condition of modular invariance turned out to be equivalent to the condition (4.2) of local gauge invariance of the classical action functional (4.4). Both restrict the choices of embeddings ϵ_{L} and ϵ_{R} which are compatible with conformal invariance.

The heterotic nature of the symmetry algebra was also a serious obstacle for constructing boundary theories which by definition rely on gluing chiral fields from the left and right moving sector living in the *same* symmetry algebra. Although one knows about a natural prescription for symmetry reduction in coset chiral algebras which is closely related to that in the WZNW case, it may happen that all potential reductions of this sort are incompatible for left and right moving chiral algebra. In the worst case one would even have to break the symmetry down to the pure conformal symmetry thus leaving the realm of rational CFT and making a solution almost impossible.

The special class of asymmetric cosets of generalized automorphism type circumvents these difficulties. Basically, we demanded that both actions $\epsilon_{\mathrm{L/R}}$ of the subgroup H can be formulated in terms of a *single* embedding chain and that they should only differ by automorphisms in the intermediate groups, see eq. (4.11). In spite of these severe restrictions, all asymmetric cosets which attracted physicists so far, like the Nappi-Witten background or the base of the conifold, belong to this class.

Our solution of the boundary problem for asymmetric cosets of generalized automorphism type resembles the construction of symmetry breaking boundary conditions in WZNW models. Most of the results which have been obtained for the latter indeed carry over to the more general situation whenever certain simple consistency conditions are satisfied. The validity of this statement is suggested by the algebraic, the geometric but also by the Lagrangian approach.

The general investigation of asymmetric coset theories has been accompanied by a detailed discussion of a number of illustrative examples in chapter 5. The descriptive power of our approach was first used to illuminate the structure of symmetry breaking D-branes in the group manifolds SU(2) and $\mathrm{SL}(2,\mathbb{R})$, where we found full agreement with results obtained before by using arguments from T-duality [131, 133]. The insights which were gained in these toy models have then been used to propose non-factorizing branes on the product space $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$.

By dividing out the subgroup $\mathbb{R}\times\mathbb{R}$ we arrived at the cosmological Nappi-Witten background. Among others, we found branes, which stretch through the singularities and connect different universes. Other branes just cease to exist after a certain instant of time. As another example we considered branes in T^{pq}-spaces whose representative T^{11} is closely related to the base of the conifold. Here, we could in particular identify a brane which wraps one of the three-spheres. Finally, our general prescription could also be applied successfully to construct non-trivial defect lines between two WZNW theories based on the same Lie algebra, but at different levels.

It would be interesting to extend our results in several directions. First of all, the stability of our branes remains to be analyzed. For the target spaces SU(2) and SU(2)/U(1) it is known that most of them are unstable [131]. The same can be expected for groups of higher rank and their cosets, respectively. A comprehensive answer, however, would require to understand the full dynamics of our branes. This could be achieved by generalizing recent results on the dynamics of maximally symmetric branes in group manifolds [33, 16, 34] and ordinary coset theories [127, 152, 124]. A key ingredient for a solution of this problem will presumably be the target space reinterpretation of section 3.3.1 in combination with the results of appendix B.

It is still unknown whether symmetry breaking branes can carry charge. This would not only stabilize some of them, but might also help to verify the proposal according to which charges are classified by an appropriate version of K-theory [150, 151]. For group manifolds for instance there exists a mismatch between the branes known to carry charge and the prediction of twisted K-theory. Up to now, the K-theory conjecture could only be confirmed for SU(2) and SU(3), but already for groups such as SU(2) × SU(2) one still lacks one of the two expected charges if the sizes of the two factors disagree. The knowledge about the brane dynamics would also shed new light on this issue [16, 36].

Apart from these very interesting problems there also exist a number of more fundamental issues which are related to our research and which we shall briefly summarize now.

- The *exact* construction of branes on non-compact spaces is still not well-developed despite recent progress (see [153, 142, 141] and references therein). We hope that our results which have been extrapolated from the rational case will finally experience a strict justification.
- The inclusion of D-branes in the AdS_3/CFT_2-correspondence predicts CFT's with defects [67, 68, 69, 149]. Since one expects defects with jumping central charge, these provide a natural playground for our theory of symmetry breaking boundary conditions.
- An extension of our results to σ-models based on supergroups/cosets could lead to branes in backgrounds with RR-fluxes [154, 155, 156].
- Asymmetric cosets might eventually give rise to new realizations of both world-sheet and target space supersymmetry.

With the present work we have provided the foundation for further studies of exactly solvable models in string theory and critical phenomena in statistical physics. Its impact for future developments cannot be fully estimated at this time, but we already pointed out some of the very promising applications.

Appendix A

Semi-simple Lie algebras

This appendix provides a brief survey of notions related to semi-simple Lie algebras which have been used throughout the main text without further explanation. In particular, we will focus on embeddings of those algebras since they play a fundamental role in the coset construction and hence also in the investigation of symmetry breaking boundary conditions, see chapters 3 and 4. An important result of this chapter is an explicit formula for branching coefficients which will enable us in appendix B to identify the non-commutative world-volume algebra associated to twisted D-branes in the large volume limit.

A.1 Definition and properties

A Lie algebra $(\mathfrak{L}, [\cdot,\cdot])$ is a vector space $\mathfrak{L}$ over a field $\mathbb{K}$ which is equipped with a bilinear multiplication $[\cdot,\cdot] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ such that the following requirements hold,

$$\begin{aligned} \text{antisymmetry:} \qquad & [X,Y] \;=\; -[Y,X] \\ \text{Jacobi identity:} \qquad [X,[Y,Z]] + [Y,[Z,X]] + & [Z,[X,Y]] \;=\; 0 \quad . \end{aligned}$$

Given a basis T^i of $\mathfrak{L}$, the Lie algebra is completely specified by the structure constants which enter the multiplication $[T^i, T^j] = i f^{ij}{}_k T^k$. Every associative algebra $\mathfrak{A}$ over $\mathbb{K}$ can be made into a Lie algebra by using the commutator $[X,Y] = XY - YX$ as the Lie bracket. In particular, this holds true for the algebra $\mathfrak{gl}(V)$ of endomorphisms of a vector space V. A representation of the Lie algebra $\mathfrak{L}$ on a vector space V is a homomorphism $R : \mathfrak{L} \to \mathfrak{gl}(V)$ which preserves the Lie bracket. An example is given by the adjoint representation $\text{ad} : \mathfrak{L} \to \mathfrak{gl}(\mathfrak{L})$, where an element $X \in \mathfrak{L}$ is mapped to the endomorphism $\text{ad}_X = [X, \cdot\,]$. To each representation one can associate a module using the prescription $X \cdot v = R(X)v$. Often we will not distinguish between representations and modules and simply denote both of them by the symbol V of the involved vector space.

From now on we will focus our attention to finite-dimensional Lie algebras over the complex numbers. One of the most important quantities in the structure theory of these Lie algebras is the *Killing form* which is defined by

$$\kappa(X,Y) \;=\; \mathrm{tr}\big\{\, \mathrm{ad}_X \cdot \mathrm{ad}_Y \,\big\} \quad .$$

Lie algebras for which the Killing form is non-degenerate are called *semi-simple*. They may be represented as a direct sum of *simple* Lie algebras. The latter are defined by the condition that they are neither abelian nor possess a proper ideal.

A complete classification of simple Lie algebras has been achieved by Cartan. There exists four infinite series A_n, B_n, C_n and D_n as well as five exceptional ones of type E_6, E_7, E_8 and F_4, G_2. They are all completely specified in terms of their *Cartan matrix* or, equivalently, by their *Dynkin diagram*. For our purposes, however, it is enough to know about the root space decomposition of $\mathfrak{g}$,

$$\mathfrak{g} \;=\; \mathfrak{h} \,\oplus\, \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \quad ,$$

into a maximal commuting subalgebra $\mathfrak{h}$ and one-dimensional root-spaces $\mathfrak{g}_\alpha = \big\{ X \in \mathfrak{g} \,\big|\, [h,x] = \alpha(h)X,\, h \in \mathfrak{h} \big\}$. The roots form a finite subset Φ of the dual space $\mathfrak{h}^*$. All roots may be written as an integer linear combination of so-called *simple roots* $\alpha_{(i)}$ $(i = 1, \cdots, r)$, where $r = \dim \mathfrak{h}$ denotes the *rank* of $\mathfrak{g}$. If all coefficients in the expansion $\alpha = \sum \lambda_i\, \alpha_{(i)}$ are positive then the root α is called positive, $\alpha > 0$, otherwise negative. To each root α one may associate a root generator E_α which can be interpreted as a raising or a lowering operator, depending on whether α is positive or negative. The most important features of the commutation relations can be summarized in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

The Killing form induces a scalar product $(\,\cdot,\cdot\,)$ on the space $\mathfrak{h}^*$. It can be used to define the *simple coroots* $\alpha^\vee_{(i)} = 2\alpha_{(i)}/\big(\alpha_{(i)}, \alpha_{(i)}\big) \in \mathfrak{h}^*$ and its integer span $L^\vee$, the *coroot lattice*. The latter is dual to the *weight lattice* L which is generated by the *fundamental weights* $\Lambda_{(i)} \in \mathfrak{h}^*$. Their sum, $\rho = \sum \Lambda_{(i)}$, is called *Weyl vector*. The coefficients in the expansion $\lambda = \sum \lambda_i\, \Lambda_{(i)}$ of a weight into fundamental weights λ are called Dynkin label. On the weight lattice one may define an action of the *Weyl group* W which is generated by the reflections

$$s_i(\mu) \;=\; \mu \,-\, \big(\mu, \alpha^\vee_{(i)}\big)\, \alpha_{(i)}$$

at the hyperplanes perpendicular to the simple roots $\alpha_{(i)}$. Every element $w \in W$ can be represented as a product of these reflections. Depending on whether the number of the latter is even or odd we associate to w the value $\epsilon(w) = \pm 1$. The homomorphism ϵ is called the sign-function of W.

Let us now consider a finite-dimensional representation V_μ of $\mathfrak{g}$. Since all the (hermitian) basis elements h_i of the Cartan subalgebra commute, the associated matrices $R_\mu(h_i)$ may be diagonalized simultaneously by choosing a suitable basis $|\lambda_s\rangle \in V_\mu$. These vectors may be represented by a tuple $(\lambda_s)_i$ – the weight – of their eigenvalues with respect to the generators h_i. The action of a root generator E_α on a vector of weight λ leads to a vector of weight $\lambda + \alpha$ (if it does not vanish). The content of any representation μ can be encoded in the weight system M_μ which keeps track of all weights including their multiplicity.

Let us now consider the special class of irreducible *integrable highest weight representations*, which are finite-dimensional and unitary. The set of these representations is equal to the set

$$P^+ \;=\; L \big/ W \;\cong\; \Big\{ \mu \in L \,\Big|\, \mu = \sum \mu_i \, \Lambda_{(i)} \text{ and } \mu_i \in \mathbb{N}_0 \Big\}$$

of dominant weights. Algebraically, the irreducible highest weight representation $\mu \in P^+$ is specified as follows. There exists a vector $|\mu\rangle$ such that

$$h_i \,|\mu\rangle \;=\; \mu_i \,|\mu\rangle \qquad \text{and} \qquad E_\alpha \,|\mu\rangle \;=\; 0 \quad \text{for} \quad \alpha > 0 \quad ,$$

i.e. the vector possesses the correct weight and it is annihilated by the positive root generators. All states in the representation are generated by repeated application of the negative roots $E_{-\alpha}$ on the vector $|\mu\rangle$ for $\alpha > 0$. The multiplicity of individual weights in the weight system M_μ can be calculated using the Freudenthal recursion formula. The adjoint representation corresponds to a particular weight θ which is also known as highest root.

For several quantities in the context of irreducible and integrable highest weight representations there exist explicit formulas. The dimension of a module $\mu \in P^+$ for example is given by $\dim \mu = \prod_{\alpha>0} (\mu + \rho, \alpha)/(\rho, \alpha)$. The Dynkin index of the representation on the other hand can be expressed as

$$I_\mu \;=\; \frac{1}{\operatorname{rank} \mathfrak{g}} \sum_{\nu \in M_\mu} (\nu, \nu) \quad .$$

It determines the constant of proportionality which relates the Killing form κ with the bilinear form

$$\kappa_\mu(X, Y) \;=\; \operatorname{tr}\big\{ \, R_\mu(X) \cdot R_\mu(Y) \, \big\} \;=\; I_\mu / I_{\text{ad}} \; \kappa(X, Y) \quad .$$

These forms are all non-degenerate (for $\mu \neq 0$) and can serve as a metric. In order to obtain a metric which is independent of the representation μ it is convenient to employ the matrix

$$\kappa^{ij} \;=\; 1/I_\mu \; \kappa_\mu(T^i, T^j) \quad , \tag{A.1}$$

in a given basis $\{T^i\}$. The inverse κ_{ij} may be used to lower indices and allows us to define the quadratic Casimir $Q = \sum_{i,j} \kappa_{ij}\, T^i T^j \in \mathcal{U}(\mathfrak{g})$ which commutes with all elements of the Lie algebra $\mathfrak{g}$. According to Schur's Lemma it acts as a scalar if evaluated on *irreducible* representations. Its precise value can be determined from the equation

$$C_\mu \;=\; (\mu, \mu + 2\rho) \quad .$$

Let us recall that the quadratic Casimir enters the expression for the conformal dimension of primary fields in WZNW models. The latter also involves the dual Coxeter number $g^\vee$ which can be introduced by the equation $g^\vee = C_{\text{ad}}/2 = (\theta, \rho) + 1$. For additional details and useful tables of data related to simple Lie algebras including all Dynkin diagrams we refer the reader to [110, 157, 158].

A.2 Embeddings of semi-simple Lie algebras

A subspace of a Lie algebra $\mathfrak{g}$ which is itself closed under multiplication is called a *subalgebra*. More formally, one speaks of an *embedding* $\mathfrak{p} \hookrightarrow \mathfrak{g}$ of Lie algebras if there exists an *injective* homomorphism $\epsilon : \mathfrak{p} \to \mathfrak{g}$. Under these circumstances the image of ϵ is isomorphic to $\mathfrak{p}$ itself and constitutes a subalgebra of $\mathfrak{g}$. Often we will also directly refer to $\mathfrak{p}$ as a subalgebra. To each embedding $\epsilon : \mathfrak{p} \to \mathfrak{g}$ one can associate an *embedding index*. It is defined as the constant of proportionality entering the equation $I_R\, x_\epsilon = I_{R\circ\epsilon}$ which relates the Dynkin index of a representation R of $\mathfrak{g}$ with that of the induced one $R \circ \epsilon$ of $\mathfrak{p}$.

Given a finite-dimensional module of the Lie algebra $\mathfrak{g}$, it is an important and natural question to ask how this module decomposes under restriction of the action to the subalgebra $\mathfrak{p}$. This decomposition is described by non-negative integer numbers, the so-called branching coefficients. The aim of this section is to provide new tools for determining branching coefficients in the case where both $\mathfrak{p}$ and $\mathfrak{g}$ are finite-dimensional semi-simple Lie algebras. Several techniques have been developed to deal with this question. Among them are the use of generating functions, Schur functions and a generalization of Kostant's multiplicity formula as well as different kinds of algorithms. For details we refer the reader to [159, 160, 158, 110, 161] and references therein.

Here, we develop a new approach which uses the fact that a semi-simple Lie algebra $\mathfrak{g}$ is naturally embedded in its affine extension $\hat{\mathfrak{g}}$. This makes available the powerful techniques of affine Kac-Moody algebras (see e.g. [25]) and conformal field theories related to such algebras (see [110] for instance). To give an example, we remind the reader that Verlinde's formula [101] for fusion coefficients in $\hat{\mathfrak{g}}_k$ WZNW theories gives a generalization of the concept of tensor product coefficients of $\mathfrak{g}$. We will show that analogous relations hold for branching coefficients if we extend either $\mathfrak{g}$ or its subalgebra $\mathfrak{p}$ to the corresponding affine Kac-Moody algebra. In particular, in the first case there exists a relation to the theory of conformal boundary conditions and to the theory of fusion rings in WZNW models, see appendix B.

A.2.1 Verlinde-like formula for branching coefficients

We want to describe an embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}$ of one finite-dimensional semi-simple Lie algebra into another. For notational simplicity let us assume that $\mathfrak{p}$ actually is a simple Lie algebra but this does not restrict the validity of our results. Denote the weight lattices of $\mathfrak{p}$ and $\mathfrak{g}$ by $\bar{L}$ and L, respectively. Here and in what follows we will always use the convention that $\mu, \nu, \dots \in L$ and $a, b, \dots \in \bar{L}$. The finite-dimensional irreducible representations of the Lie algebras $\mathfrak{p}$ and $\mathfrak{g}$ are in one-to-one correspondence to the weights with non-negative integral Dynkin labels. These sets of integrable highest weights of $\mathfrak{p}$ are denoted by $\bar{P}^+ \cong \bar{L}/W_{\mathfrak{p}}$ with Weyl group $W_{\mathfrak{p}}$ and similarly for $\mathfrak{g}$. Let M_a and M_μ be the weight systems of the representations $a \in \bar{P}^+$ and $\mu \in P^+$ including the multiplicities. The embedding can be characterized by a projection $\mathcal{P} : \langle L \rangle \to \langle \bar{L} \rangle$ where $\langle \cdot \rangle$ means the span of the corresponding lattice over $\mathbb{C}$. Under this projection, the weight system M_μ of the representation $\mu \in P^+$ of $\mathfrak{g}$ decomposes into weight systems of representations of $\mathfrak{p}$ according to

$$\mathcal{P} M_\mu \;=\; \bigoplus_{a \in \bar{P}^+} b_\mu{}^a M_a \quad . \tag{A.2}$$

The numbers $b_\mu{}^a \in \mathbb{N}_0$ are called branching coefficients. Our aim is to find an explicit and general formula for the coefficients $b_\mu{}^a$ with $\mu \in P^+$ and $a \in \bar{P}^+$. To achieve this, we consider the untwisted affine extension $\hat{\mathfrak{p}}_k$ of $\mathfrak{p}$. The level k has to be chosen large enough and depends on the value of μ. This statement will be made precise below. The integrable highest weights of $\hat{\mathfrak{p}}_k$ are given by the set $\bar{P}^+_k = \bar{L}/(W_{\mathfrak{p}} \ltimes k\bar{L}^\vee)$ where we used the decomposition of the affine Weyl group into a semi-direct product of finite Weyl group and translations by k times the coroot lattice $\bar{L}^\vee$. If we introduce the notation $k(c) = (\theta, c)_{\mathfrak{p}}$ where θ is the highest root of $\mathfrak{p}$ we may write $\bar{P}^+_k = \{a \in \bar{P}^+ | k(a) \leq k\}$. The bracket $(\cdot,\cdot)_{\mathfrak{p}}$ denotes the scalar product on the weight space $\langle\bar{L}\rangle$ which is induced by the Killing form. It is given in terms of the quadratic form matrix $F_{\mathfrak{p}}$ if the weights are written using Dynkin labels, i.e. $(a,b)_{\mathfrak{p}} = a^T F_{\mathfrak{p}} b$. In the following we will always identify in a natural way an integrable highest weight representation $\hat{c} \in P_k^+$ of $\hat{\mathfrak{p}}_k$ with a highest weight $c \in \bar{P}^+$ of $\mathfrak{p} \hookrightarrow \hat{\mathfrak{p}}_k$.

Before we continue, let us briefly introduce further objects that will be needed as we proceed. The character of a highest weight representation $\mu \in P^+$ of $\mathfrak{g}$ is defined as

$$\chi_\mu(\cdot) \;=\; \sum_{\nu\in M_\mu} e^{(\nu,\cdot)_{\mathfrak{g}}} \tag{A.3}$$

and analogously for $\mathfrak{p}$. The second ingredient of our formula is the modular S-matrix of $\hat{\mathfrak{p}}_k$ which, for $a,b \in \bar{P}_k^+$, is given by the Kac-Peterson formula [25]

$$S_{ab} \;=\; i^{|\Delta_+|}|\bar{L}/\bar{L}^\vee|^{-1/2}(k+g^\vee)^{-r/2}\sum_{w\in W}\epsilon(w)\exp\Big\{-\frac{2\pi i}{k+g^\vee}\Big(w(a+\rho),b+\rho\Big)_{\mathfrak{p}}\Big\}\,. \tag{A.4}$$

This formula involves the rank of the Lie algebra r, the number of positive roots $|\Delta_+|$, the Weyl vector ρ, the dual Coxeter number $g^\vee = (\theta,\rho)_{\mathfrak{p}}+1$ and a sum over the Weyl group W including its sign function ϵ. We omit the index $\mathfrak{p}$ because we will not encounter the corresponding objects for the Lie algebra $\mathfrak{g}$. Due to Weyl's character formula we may write

$$\chi_a(\xi_b) \;=\; \frac{S_{ba}}{S_{b0}} \quad\text{where}\quad \xi_b \;=\; -\frac{2\pi i}{k+g^\vee}(b+\rho) \quad\text{and}\quad a,b\in\bar{P}_k^+ \;\;. \tag{A.5}$$

We are now prepared to state the first important result.

Theorem 1. *Consider an embedding $\mathfrak{p}\hookrightarrow\mathfrak{g}$ of two finite-dimensional semi-simple Lie algebras. Let $\mathcal{P}:\langle L\rangle\to\langle\bar{L}\rangle$ be the projection matrix characterizing the embedding and $a\in\bar{P}^+,\mu\in P^+$ be two arbitrary but integrable highest weights. Define a map $\mathcal{P}^* \;=\; F_{\mathfrak{g}}^{-1}\mathcal{P}^T F_{\mathfrak{p}}:\langle\bar{L}\rangle\to\langle L\rangle$ and let k be a number such that $k\geq\max\{k(c)|{b_\mu}^c\neq 0\}$. Then we have*

$${b_\mu}^a \;=\; \sum_{d\in\bar{P}_k^+}\sum_{\nu\in M_\mu}\bar{S}_{da}S_{d0}\,e^{-\frac{2\pi i}{k+g^\vee}(\mathcal{P}\nu,d+\rho)_{\mathfrak{p}}} \;=\; \sum_{d\in\bar{P}_k^+}\bar{S}_{da}S_{d0}\,\chi_\mu(\mathcal{P}^*\xi_d) \;\;. \tag{A.6}$$

Proof. For notational simplicity we assume $\mathfrak{p}$ to be simple. Let us first note that $\max\{k(c)|{b_\mu}^c\neq 0\}$ exists as all weight systems involved are finite. We then start by writing down the identity

$$\sum_{c\in\bar{P}_k^+}{b_\mu}^c\,\frac{S_{dc}}{S_{d0}} \;=\; \sum_{c\in\bar{P}^+}{b_\mu}^c\,\chi_c(\xi_d) \;=\; \chi_\mu(\mathcal{P}^*\xi_d) \;\;. \tag{A.7}$$

If we multiply both sides of (A.7) with $\bar{S}_{da}S_{d0}$ and sum over all $d \in \bar{P}_k^+$ we obtain the desired result due to the unitarity $\sum_d \bar{S}_{da}S_{dc} = \delta_c^a$ of the S-matrix. Thus we only have to motivate (A.7). The left equality simply results from (A.5) and the condition on the level k, but the right equality is more interesting. Let M_μ be the weight system of the representation μ including all multiplicities. We insert the definition (A.3) of the characters into (A.7). After this substitution, the sum on the right hand side of (A.7) is over M_μ and involves scalar products $(\nu, \cdot)_{\mathfrak{g}}$. In contrast to this, the sum in the middle is over the projected weights $\mathcal{P}M_\mu$ and therefore involves scalar products of the form $(\mathcal{P}\nu, \cdot)_{\mathfrak{p}}$. The sum in both cases runs essentially over the same set M_μ. Therefore the equality in (A.7) holds if we can identify the scalar products according to $(\mathcal{P}\nu, \cdot)_{\mathfrak{p}} = (\nu, \mathcal{P}^* \cdot)_{\mathfrak{g}}$. Writing this relation in terms of quadratic form matrices, we see that $\mathcal{P}^*$ was constructed exactly in a way that this identity holds. □

Notice the following remarkable observation. If we could rewrite $\mathcal{P}^*\xi_d$ as ξ'_ν for some integrable highest weight ν of $\hat{\mathfrak{g}}_{k'}$ at a certain level k', we could apply eq. (A.5) and eq. (A.6) would reduce to a Verlinde-like formula for branching coefficients. In general, this does not seem to be possible because $F_{\mathfrak{g}}^{-1}$ might cause negative entries in $\mathcal{P}^*$. We will see however in appendix B that in some specific cases we are able to recover a Verlinde-like formula using a different approach.

Let us briefly comment on the modifications if $\mathfrak{p}$ is finite-dimensional and semi-simple but not simple. Under these circumstances we have a decomposition $\mathfrak{p} \cong \oplus_{s=1}^n \mathfrak{p}_s$ of $\mathfrak{p}$ into simple Lie algebras $\mathfrak{p}_s$. In the affine extension, each simple factor obtains its own level: $\hat{\mathfrak{p}}_k \cong \oplus_{s=1}^n (\hat{\mathfrak{p}}_s)_{k_s}$ with $k = (k_1, \ldots, k_n)$. All relevant structures like the weight lattice, the Weyl group, the quadratic form matrix and the modular S-matrix "factorize" in some sense, i.e. they are given by a direct sum, a product, a block diagonal matrix or they factorize in the original sense of the word. Obviously, the proof of theorem 1 still remains valid if one takes these notational difficulties into account. In particular, the condition $k \geq \max\{k(c) | {b_\mu}^c \neq 0\}$ actually means $k_s \geq \max\{k_s(c) | {b_\mu}^c \neq 0\}$ in this case.

A.2.2 An algorithm for branching coefficients

We will now use formula (A.6) to give a simple derivation of a well-known algorithm [161] for the calculation of branching coefficients which is the basis of many computer algebra programs[1]. The algorithm exhibits some similarity with the Racah-Speiser algorithm for the calculation of tensor product

[1] I am grateful to M. van Leeuwen for providing this information.

multiplicities (see also [25, 162, 163, 164, 165, 166, 71] for its extension to fusion rules).

Theorem 2. *Consider an embedding* $\mathfrak{p} \hookrightarrow \mathfrak{g}$ *of finite-dimensional semi-simple Lie algebras. Let* $\mu \in P^+$ *be a highest weight of* $\mathfrak{g}$ *and* $\mathcal{P} : \langle L \rangle \to \langle \bar{L} \rangle$ *be the projection matrix characterizing the embedding. The decomposition* $\mathcal{P}M_\mu = \oplus_a {b_\mu}^a M_a$ *can be obtained by the following algorithm*[2]*.*

1. *Calculate the weight system of the representation* μ *including the multiplicities. This gives some set* $M_\mu \subset L$*.*
2. *Project this set to* $\bar{L}$ *and add the Weyl vector of the subalgebra* $\mathfrak{p}$*. Now we are dealing with the set* $Z_\mu = \mathcal{P}M_\mu + \rho \subset \bar{L}$ *including the multiplicities.*
3. *For each weight of* Z_μ *use a Weyl reflection to map it into the fundamental Weyl chamber where all Dynkin labels are non-negative. An algorithm in terms of elementary Weyl reflections can be found in [158] for example.*
4. *Drop all weights lying on the boundary of the fundamental Weyl chamber and subtract the Weyl vector* ρ *of the subalgebra* $\mathfrak{p}$ *from the remaining ones.*
5. *Add up all these contributions including the signs of the relevant Weyl reflections and the multiplicities. The coefficient obtained for each weight* $a \in \bar{P}^+$ *is just the number* ${b_\mu}^a$*.*

Proof. Again we assume $\mathfrak{p}$ to be simple without loss of generality. Essentially, the idea is to evaluate equation (A.6) for $k \to \infty$. We insert the definitions (A.3),(A.4) for the characters and the S-matrix. Denoting the prefactor by $\mathcal{N} = |\bar{L}/\bar{L}^\vee|^{-1}(k+g^\vee)^{-r}$ we obtain

$$ {b_\mu}^a \;=\; \mathcal{N} \sum_{d\in\bar{P}_k^+} \sum_{w_1,w_2\in W} \sum_{\nu\in M_\mu} \epsilon(w_1)\,\epsilon(w_2)\, e^{-\frac{2\pi i}{k+g^\vee}(\mathcal{P}\nu+w_1\rho-w_2(a+\rho)\,,\,d+\rho)_{\mathfrak{p}}} \tag{A.8} $$

where we already made use of the defining relation $(\nu,\mathcal{P}^*\xi_d)_{\mathfrak{g}} = (\mathcal{P}\nu,\xi_d)_{\mathfrak{p}}$ for $\mathcal{P}^*$. The next step consists in evaluating the sum over d. We define a function $f(d)$ by ${b_\mu}^a = \sum_{d\in\bar{P}_k^+} f(d+\rho)$. The function $f(c)$ as read of from eq. (A.8) has two important properties. First, it satisfies $f(wc) = f(c)$ for all $w \in W$.

[2] The author would like to thank I. Runkel and Ch. Schweigert for the collaboration on [71] which uses similar techniques in a different context.

Indeed, the Weyl reflection may be absorbed into a redefinition[3] of w_1, w_2 and ν. To derive the second property let us define the set $\bar{P}^{++}_{k+g^\vee} = \bar{P}^{+}_{k} + \rho$. It turns out that $\bar{P}^{++}_{k+g^\vee}$ exactly contains the elements of $\bar{P}^{+}_{k+g^\vee}$ which do not lie at the boundary of the corresponding affine Weyl chamber. This boundary is given by the set of all weights which are invariant under at least one elementary Weyl reflection including the shifted reflection at the k-dependent hyperplane described by $(\theta, \cdot)_{\mathfrak{p}} = k + g^\vee$. One may show that $f(c) = 0$ if c is invariant under an affine fundamental Weyl reflection. To see this, note that the function $g_x(c) = S_{x,c-\rho}$ which enters $f(d)$ satisfies $g_x(\hat{w}c) = \epsilon(\hat{w}) g_x(c)$ with respect to any affine Weyl transformation $\hat{w} \in W \ltimes (k + g^\vee)L^\vee$. These considerations lead to the simple relation

$$
\begin{aligned}
b_\mu{}^a &= \frac{1}{|W|} \sum_{d \in \bar{P}^+_k} \sum_{w \in W} f\big(w(d+\rho)\big) \\
&= \frac{1}{|W|} \sum_{c \in \bar{P}^+_{k+g^\vee}} \sum_{w \in W} f(wc) \;=\; \frac{1}{|W|} \sum_{c \in L/(k+g^\vee)L^\vee} f(c) \quad . \qquad \text{(A.9)}
\end{aligned}
$$

We are now in a situation where we are able to perform the sum over $c \in L/(k + g^\vee)L^\vee$. The sum over the exponentials in eq. (A.8) exactly gives a non-vanishing result if $\mathcal{P}\nu + w_1\rho - w_2(a + \rho) \in (k + g^\vee)L^\vee$. In this case it obviously compensates the normalization factor $\mathcal{N}$. In the limit $k \to \infty$ the previous condition reduces to a Kronecker symbol and we are left with the k-independent expression

$$
b_\mu{}^a \;=\; \frac{1}{|W|} \sum_{w_1 \in W} \sum_{w_2 \in W} \sum_{\nu \in M_\mu} \epsilon(w_1)\, \epsilon(w_2)\, \delta_{w_2(a+\rho), \mathcal{P}\nu + w_1\rho} \quad . \qquad \text{(A.10)}
$$

Next shift w_2 to the other side of the Kronecker symbol ($w_2^{-1} = w_2$) and resum $w_1 \mapsto w_2 w_1$ as well as $\mathcal{P}\nu \mapsto w_2 w_1 \mathcal{P}\nu$. The expression under the sum then obviously does not depend on w_2 anymore. By summing over w_2, we compensate the factor $1/|W|$. The final result is

$$
b_\mu{}^a \;=\; \sum_{\nu \in M_\mu} \sum_{w \in W} \epsilon(w)\, \delta_{a, w(\mathcal{P}\nu+\rho)-\rho} \quad . \qquad \text{(A.11)}
$$

For each weight $\mathcal{P}\nu + \rho$ lying at the boundary of a Weyl chamber there always exists an elementary Weyl reflection which leaves it fixed. These weights may be omitted because they would contribute twice with different sign. Inserting our result into equation (A.2) proves the theorem. □

[3]Note that the weight system which belongs to an arbitrary representation is invariant under Weyl transformations. In particular this holds for the set $\mathcal{P}M_\mu$.

Using theorem 1 and formula (A.6) one may explicitly check some well-known properties of branching coefficients. Thus one obtains

Corollary 1. *Let* $\mathfrak{h} \hookrightarrow \mathfrak{p} \hookrightarrow \mathfrak{g}$ *be an embedding chain of finite-dimensional semi-simple Lie algebras and denote the integrable highest weights by* $\alpha, \beta, \ldots$ *and* $a, b, \ldots$ *and* $\mu, \nu, \ldots$ *respectively. The branching coefficients have the following properties.*

1. *The trivial representation* $0 \in P^+$ *decomposes according to* ${b_0}^a = \delta_0^a$.

2. *Denoting the conjugate representation by* $(\cdot)^+$*, the relation* ${b_{\mu^+}}^{a^+} = {b_\mu}^a$ *holds.*

3. *The branching coefficients of the different embeddings in the chain* $\mathfrak{h} \hookrightarrow \mathfrak{p} \hookrightarrow \mathfrak{g}$ *are related by* ${b_\mu}^\alpha = \sum_a {b_\mu}^a {b_a}^\alpha$.

4. *In the decomposition of a tensor product* $V_\mu \otimes V_\nu$ *both reductions are equivalent, i.e. the branching coefficients satisfy*

$$\sum_\lambda N_{\mu\nu}^\lambda \, {b_\lambda}^a \; = \; \sum_{c,d} {b_\mu}^c \, {b_\nu}^d \, N_{cd}^a \quad .$$

Proof. The first relation holds because $\chi_0(\cdot) = 1$. For the second relation one needs that the charge conjugation matrix satisfies $C = C^T = C^{-1}$ as well as $F \circ C = C \circ F$ and $C_\mathfrak{p} \circ \mathcal{P} = \mathcal{P} \circ C_\mathfrak{g}$. The third relation is due to the fact that $\mathcal{P}^*(\mathfrak{h} \hookrightarrow \mathfrak{p} \hookrightarrow \mathfrak{g}) = \mathcal{P}^*(\mathfrak{p} \hookrightarrow \mathfrak{g}) \circ \mathcal{P}^*(\mathfrak{h} \hookrightarrow \mathfrak{p})$. The last property can be checked using the Verlinde formula for N_{cd}^a (this is valid if we choose k large enough, see corollary 2), the unitarity of the S-matrix and the property $\chi_\mu \, \chi_\nu = \sum_\lambda N_{\mu\nu}^\lambda \, \chi_\lambda$ of characters. □

The diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$ is distinguished since its branching coefficients are deduced from the tensor product of representations in $\mathfrak{g}$. In this case theorem 1 implies

Corollary 2. *Let* $\mathfrak{g}$ *be a finite-dimensional semi-simple Lie algebra and* V_μ, V_ν *be two fixed integrable highest weight modules. There exists some* $k_0 \in \mathbb{N}$ *such that the coefficients in the decomposition* $V_\mu \otimes V_\nu = \oplus_\lambda N_{\mu\nu}^\lambda V_\lambda$ *may be expressed by the Verlinde formula*

$$N_{\mu\nu}^\lambda \;=\; \sum_{\rho \in P_k^+} \frac{\bar{S}_{\rho\lambda} \, S_{\rho\mu} \, S_{\rho\nu}}{S_{\rho 0}}$$

for all integers $k > k_0$.

Proof. This is a simple consequence of theorem 1 and the fact that the branching coefficients for the diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$ with projection $\mathcal{P}(\rho_1, \rho_2) = \rho_1 + \rho_2$ are given by the tensor product multiplicities of $\mathfrak{g}$. Using the definition one obtains $\mathcal{P}^*(\rho) = (\rho, \rho)$. The character of $\mathfrak{g} \oplus \mathfrak{g}$ in (A.6) decomposes into a product of two characters of $\mathfrak{g}$ with argument ξ_ρ. Applying equation (A.5) gives the desired result. □

From theorem 1 one may deduce integral formulae for branching coefficients. We will not provide a proof that this is always possible but only give the idea and a simple example for illustration. First we observe that the S-matrices and the character in (A.6) both have a dependence $\sim (d+\rho)/(k+g^\vee)$ on the summation index d. In addition, the two S-matrices give a total prefactor of the form $(k + g^\vee)^{-r}$ where r is the rank of the subalgebra, i.e. the number of independent components of d. Therefore it is likely that in many (if not all) cases we may rewrite the sum as an integral in the limit $k \to \infty$ and in this way recover an integral representation of branching coefficients.

We show how this works in a very simple example and rederive some integral formula for the (of course well-known) tensor product multiplicities of representations of A_1, i.e. the branching rules of the diagonal embedding $A_1 \hookrightarrow A_1 \oplus A_1$. The characters of A_1 read $\chi_a(x) = \sinh \frac{x}{2}(a+1)/\sinh \frac{x}{2}$ and the S-matrix is given by

$$S_{ab} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2}(a+1)(b+1) \quad .$$

Using the factorization of the $A_1 \oplus A_1$-character, equation (A.6) implies for all k greater than some k_0

$$\begin{aligned} N_{a_1 a_2}^{\;a} &= b_{(a_1,a_2)}{}^a \\ &= \frac{2}{k+2} \sum_{b=0}^{k} \frac{\sin \frac{\pi(a+1)(b+1)}{k+2} \sin \frac{\pi(a_1+1)(b+1)}{k+2} \sin \frac{\pi(a_2+1)(b+1)}{k+2}}{\sin \frac{\pi(b+1)}{k+2}} \\ &= 2 \int_0^1 dx \, \frac{\sin \pi(a+1)x \, \sin \pi(a_1+1)x \, \sin \pi(a_2+1)x}{\sin \pi x} \quad . \end{aligned}$$

For the last equality we consider the sum to be a Riemann sum with an equidistant partition of the interval $[1/(k+2), (k+1)/(k+2)]$ into intervals of length $\Delta x = 1/(k+2)$. Due to continuity we may extend the interval to $[0, 1]$. As the integral exists, it is given by the previous series in the limit $k \to \infty$. While such integral representations for general branching coefficients seem to be new, similar statements for tensor products can for example be found in [110, p. 534].

We now want to show how our construction is related to the theory of NIM-reps, see eq. (2.14) for a definition of the latter. Let $\mathfrak{p}$ be a subalgebra of $\mathfrak{g}$. Denote the tensor product multiplicities of $\mathfrak{p}$ by N_{ab}^{c} and the branching coefficients by ${b_\mu}^a$. One can easily show that the matrices ${\left(n_\mu\right)_b}^a = \sum_c {b_\mu}^c N_{cb}^{a}$ constitute a NIM-rep of the fusion ring of the WZNW model associated with $\hat{\mathfrak{g}}_k$ at level $k \to \infty$. In this limit the fusion rules $N_{\mu\nu}^{\lambda} = \lim_{k\to\infty} N_{\mu\nu}^{(k)\lambda}$ reduce to the tensor product multiplicities of $\mathfrak{g}$. The proof of the NIM-rep properties relies on the fact that the two possibilities of decomposing a module $V_\mu \otimes V_\nu$ of $\mathfrak{g}$ into modules of $\mathfrak{p}$ are equivalent (compare corollary 1) and on the associativity of tensor products. It is easy to generalize the theorems 1 and 2 and to obtain

$$\begin{aligned}
{\left(n_\mu\right)_b}^a &= \sum_{c\in\bar{P}+} {b_\mu}^c N_{cb}^{a} = \sum_{d\in\bar{P}_k^+} \bar{S}_{da} S_{db}\, \chi_\mu(\mathcal{P}^*\xi_d) \\
&= \sum_{\nu\in M_\mu} \sum_{w\in W} \epsilon(w)\, \delta_{a,w(\mathcal{P}\nu+b+\rho)-\rho} \quad .
\end{aligned} \tag{A.12}$$

While the equality in the first line holds for sufficiently large values of the level k, there is no reference to the level in the second line.

As will be seen in appendix B, some of the NIM-reps (A.12) admit a continuation to finite values of the level k where they possess an interpretation in conformal field theory. It is not known up to now whether this statement may be generalized. Yet, we believe that a possible continuation of general NIM-reps of the type (A.12) to finite values of k might be of importance for a representation theoretic understanding of embeddings of quantum groups at roots of unity as it provides a natural analogue to the transition from tensor product to fusion coefficients (cf. [165, 166]). This last point has to be clarified in future work. Note that there has been some progress recently in understanding subgroups of quantum groups [167, 168, 169, 170, 171].

Appendix B

Maximally symmetric D-branes on group manifolds

An essential ingredient of almost all the constructions in the present thesis are maximally symmetric D-branes on group manifolds. While the logical development of the subject only required a minimum knowledge about these objects we find it nevertheless useful to provide some additional information.

We split the presentation in two parts. In the first part we consider the case of finite level and work out an explicit expression for the spectrum of open strings which is suitable for numeric evaluation on a computer [71]. In the second part we investigate the theory in the large volume limit $k \to \infty$, where geometrical notions become available. We argue that the D-branes possess a new kind of non-commutative world-volume and give an alternative interpretation of the D-brane labels in this limit [70, 34].

B.1 The stringy regime

The aim of this section is to provide an exact analytic expression for the spectrum of open strings which stretch between two maximally symmetric D-branes on a group manifold G at finite level k. More precisely, we will work in the following setting. Let $\hat{\mathfrak{g}}$ be an untwisted affine Lie algebra and Ω an automorphism of order N of its horizontal subalgebra $\mathfrak{g}$. In the WZNW theory based on $\hat{\mathfrak{g}}$ at level k with modular invariant given by charge conjugation, we consider boundary conditions for which left movers and right movers are related by the automorphism Ω at the boundary. The spectrum of open strings living between two such boundary conditions α, β is encoded in the boundary partition function

$$Z_{\beta\alpha}(q) \; = \; \sum_{\mu} \; N_{\mu\alpha}^{\;\beta} \, \chi_\mu(q) \quad . \tag{B.1}$$

where the sum over μ runs over integrable highest weight representations of $\hat{\mathfrak{g}}$ at level k and $\chi_\mu(q)$ are the corresponding characters. The set of boundary conditions is given by twisted representations of $\hat{\mathfrak{g}}$ at level k and the annulus coefficients $N_{\mu\alpha}^{\;\beta}$ are the corresponding twisted fusion rules [28] (they have been introduced in [111] in a different context).

In order to describe the set of twisted representations we need to introduce some notation. We denote the weight lattice of the horizontal subalgebra $\mathfrak{g}$ by L. The lattice $L^\vee$ dual to L is the coroot lattice of $\mathfrak{g}$; a basis are the simple coroots $\alpha^\vee_{(i)}$. The lattices L and $L^\vee$ inherit an action of the automorphism Ω, which can be decomposed into an outer automorphism Ω_0 and an inner one Ω_i, $\Omega = \Omega_i \circ \Omega_0$. While the inner automorphism Ω_i can be chosen to be the adjoint action of an element of a Cartan subalgebra and therefore induces a trivial action on L and $L^\vee$, the outer part Ω_0 can be chosen to be a diagram automorphism of the Dynkin diagram of $\mathfrak{g}$. It acts on the lattices L and $L^\vee$ by the permutations $\omega_0(\Lambda_{(i)}) = \Lambda_{(\omega_0 i)}$ and $\omega_0\big(\alpha^\vee_{(i)}\big) = \alpha^\vee_{(\omega_0 i)}$ of fundamental weights or simple coroots, respectively. Without loss of generality we can therefore assume Ω to be a diagram automorphism. The length of the orbit $\big\{\Lambda_{(i)}, \omega(\Lambda_{(i)}), \omega^2(\Lambda_{(i)}), \ldots\big\}$ will be denoted by n_i. We also define the lattice of symmetric weights $L_\omega = \big\{\mu \in L \,\big|\, \omega(\mu) = \mu\big\}$ which inherits the scalar product from L.

An important ingredient in our algorithm is the subgroup [121, 172]

$$W_\omega = \big\{\, w \in W \,\big|\, w \circ \omega = \omega \circ w \,\big\}$$

of the Weyl group that commutes with the action of ω. It is a Coxeter group with the following generators $\tilde{s}_i$: for orbits of length 1, take $\tilde{s}_i = s_i$. If $i \neq \omega i$, take the product $\tilde{s}_i = s_i \, s_{\omega i} \cdots s_{\omega^{n_i - 1} i}$. This prescription needs to be modified, if the element $A_{i,\omega i}$ of the Cartan matrix is non-vanishing, which in our situation only happens for the outer automorphism of A_{2n} and the orbit consisting of the two nodes in the middle of the Dynkin diagram. In this case, take $\tilde{s}_i = s_i \, s_{\omega i} \, s_i = s_{\omega i} \, s_i \, s_{\omega i}$.

We also need the orthogonal projection $\mathcal{P}_\omega$ of weight space onto its symmetric subspace, defined by $\mathcal{P}_\omega = \frac{1}{N}\big(1 + \omega + \cdots + \omega^{N-1}\big)$, N being the order of ω. For the implementation on a computer, one uses directly the action of $\tilde{s}_i$ on symmetric weights:

$$\tilde{s}_i(\lambda) \; = \; \lambda - \frac{2(\lambda, \mathcal{P}_\omega \, \alpha_{(i)})}{(\mathcal{P}_\omega \, \alpha_{(i)}, \mathcal{P}_\omega \, \alpha_{(i)})} \; \mathcal{P}_\omega \, \alpha_{(i)} \quad . \tag{B.2}$$

While the symmetric weight lattice L_ω is not invariant under the full Weyl group, it admits an action of W_ω.

We may also define a symmetric coroot lattice $(L^\vee)_\omega = \{\beta \in L^\vee \,|\, \omega(\beta) = \beta\}$. Note that L_ω and $(L^\vee)_\omega$ are not dual to each other. Instead one finds that the lattice $\big((L^\vee)_\omega\big)^\vee$ dual to $(L^\vee)_\omega$ involves fractional symmetric weights. The projection $\mathcal{P}_\omega$ restricts to a surjective map from L to $\big((L^\vee)_\omega\big)^\vee$.

We summarize the expressions for the different lattices by comparing to the situation for inner automorphisms where just two lattices appear:

- Weight lattice: $L \;=\; \{\sum_i \lambda_i\, \Lambda_{(i)} \,|\, \lambda_i \in \mathbb{Z}\}$.
- Coroot lattice: $L^\vee \;=\; \{\sum_i \beta_i\, \alpha^\vee_{(i)} \,|\, \beta_i \in \mathbb{Z}\} \subset L$.

In addition there are four lattices related to the automorphism Ω.

- Symmetric weight lattice:
  $L_\omega = \{\sum_i \lambda_i\, \Lambda_{(i)} \,|\, \lambda_i \in \mathbb{Z},\, \lambda_{\omega i} = \lambda_i\} \subset L$.
- Symmetric coroot lattice:
  $(L^\vee)_\omega = \{\sum_i \beta_i\, \alpha^\vee_{(i)} \,|\, \beta_i \in \mathbb{Z},\, \beta_{\omega i} = \beta_i\} \subset L^\vee$.
- Fractional symmetric weight lattice:
  $\big((L^\vee)_\omega\big)^\vee = \{\sum_i \lambda_i\, \Lambda_{(i)} \,|\, n_i\, \lambda_i \in \mathbb{Z},\, \lambda_{\omega i} = \lambda_i\} \supset L_\omega$.
- Fractional symmetric coroot lattice:
  $(L_\omega)^\vee = \{\sum_i \beta_i\, \alpha^\vee_{(i)} \,|\, n_i\, \beta_i \in \mathbb{Z},\, \beta_{\omega i} = \beta_i\} \supset (L^\vee)_\omega$.

Recall that the n_i are the orbit lengths of fundamental weights.

The integrable highest weight modules of $\hat{\mathfrak{g}}$ at level k are in one-to-one correspondence with elements in $P_k^+ = L/(W \ltimes kL^\vee)$. The expression $W \ltimes kL^\vee$ is just the decomposition of the affine Weyl group into a semi-direct product of the finite Weyl group and the translations with respect to the scaled coroot lattice. Alternatively, the affine Weyl group is generated by finite Weyl reflections and one additional element, a shifted Weyl reflection. The latter is a combination of an elementary reflection at the highest root θ of $\mathfrak{g}$ and a translation. This amounts to an orthogonal reflection with respect to the hyperplane $(\theta, \cdot) = k$. An analogous construction can be performed with respect to the lattices L_ω and $\big((L^\vee)_\omega\big)^\vee$. This defines the sets $S_k^+ = L_\omega/\big(W_\omega \ltimes k(L^\vee)_\omega\big)$ and $B_k^+ = \big((L^\vee)_\omega\big)^\vee/\big(W_\omega \ltimes k(L_\omega)^\vee\big)$. While $W_\omega \ltimes k(L^\vee)_\omega$ is generated by W_ω and the shifted Weyl reflection at $(\theta, \cdot) = k$, the corresponding shifted Weyl reflection for $W_\omega \ltimes k(L_\omega)^\vee$ is at the hyperplane

$\mathfrak{g}$	A_{2n}	A_3	A_{2n+1}	D_4 (triality)	D_n	E_6
θ_ω	$2\big(\Lambda_{(1)}+\Lambda_{(2n)}\big)$	$2\Lambda_{(2)}$	$\Lambda_{(2)}+\Lambda_{(2n)}$	$\Lambda_{(1)}+\Lambda_{(3)}+\Lambda_{(4)}$	$2\Lambda_{(1)}$	$\Lambda_{(1)}+\Lambda_{(5)}$

Table B.1: The vector θ_ω in the labeling conventions of reference [25, p. 53].

$(\theta_\omega, \cdot) = k$. The vector θ_ω in weight space is defined in table B.1. For each of the three subsets there is a natural choice of a fundamental domain.

- Integrable highest weights

  $P_k^+ \;=\; \big\{\lambda = \sum_i \lambda_i\,\Lambda_{(i)} \,\big|\, \lambda_i \in \mathbb{N}_0 \text{ and } (\theta, \lambda) \leq k\big\}\;\;.$

- Symmetric integrable highest weights

  $S_k^+ \;=\; \big\{\lambda = \sum_i \lambda_i\,\Lambda_{(i)} \,\big|\, \lambda_i \in \mathbb{N}_0, (\theta, \lambda) \leq k \text{ and } \lambda_{\omega i} = \lambda_i\big\}\;\;.$

- Boundary labels correspond to twisted highest weight representations [28] or, equivalently, to irreducible integrable highest weight representations of the corresponding twisted Lie algebra. They are labelled by

  $B_k^+ \;=\; \big\{\beta = \sum_i \beta_i\,\Lambda_{(i)} \,\big|\, n_i\,\beta_i \in \mathbb{N}_0, (\theta_\omega, \beta) \leq k \text{ and } \beta_i = \beta_{\omega i}\big\}\;\;.$

There is a distinguished vector $\rho_\omega = \sum_i n_i^{-1}\Lambda_{(i)}$ in the lattice $\big((L^\vee)_\omega\big)^\vee$ which is a fractional analogue of the Weyl vector $\rho = \sum_i \Lambda_{(i)}$. We denote by P_k^{++}, S_k^{++} and B_k^{++} the subsets obtained from P_k^+, S_k^+ or B_k^+ after dropping elements which belong to the boundary of the respective Weyl chamber, i.e. are left invariant by at least one non-trivial element of $W \ltimes kL^\vee$, $W_\omega \ltimes k(L^\vee)_\omega$ or $W_\omega \ltimes k(L_\omega)^\vee$, respectively. It is not difficult to see that there exist identifications of the form $P_k^+ + \rho = P_{k+g^\vee}^{++}$, $S_k^+ + \rho = S_{k+g^\vee}^{++}$ and $B_k^+ + \rho_\omega = B_{k+g^\vee}^{++}$ where $g^\vee$ is the dual Coxeter number of $\mathfrak{g}$. These are a simple consequence of the fact that $(\theta, \rho) = (\theta_\omega, \rho_\omega) = g^\vee - 1$.

We are now prepared to state our result for the determination of twisted fusion rules. It is a generalization of the Racah-Speiser algorithm for tensor product multiplicities (see e.g. [158]) and the Kac-Walton formula [25, 162] (see also [165, 164]) for ordinary fusion rules.

Theorem 3. *The decomposition of the fusion product*

$$\mu \star \alpha \;=\; \sum_{\beta \in B_k^+} N_{\mu\alpha}^{\;\beta}\,\beta$$

of an untwisted representation $\mu \in P_k^+$ of $\hat{\mathfrak{g}}$ and a twisted representation $\alpha \in B_k^+$ into twisted representations can be obtained by the following algorithm:

1. *Compute the weight system M_μ, including multiplicities, of the finite dimensional irreducible highest-weight representation μ of the finite dimensional Lie algebra $\mathfrak{g}$.*

2. *Use $\mathcal{P}_\omega$ to project the set M_μ to the lattice of fractional symmetric weights.*

3. *Add the weight α and the twisted Weyl vector ρ_ω to the resulting weights.*

4. *Use the reflections (B.2) in W_ω and the shifted reflection at the plane $(\theta_\omega, \cdot) = k+g^\vee$ to map the set $\mathcal{P}_\omega M_\mu + \alpha + \rho_\omega$ to the fundamental domain $B^+_{k+g^\vee}$.*

5. *Discard weights on the boundary $B^+_{k+g^\vee} \backslash B^{++}_{k+g^\vee}$, i.e. those with at least one vanishing entry or scalar product with θ_ω equal to $k + g^\vee$. Supply each remaining contribution, counting multiplicities, with a sign depending on whether the number of reflections has been even or odd.*

6. *Subtract the twisted Weyl vector ρ_ω. Adding all contributions including the relevant multiplicities and signs gives the fusion product.*

We will split the proof into several steps. First, we summarize some earlier results which will be important in the sequel. It was shown in [28, (2.57)] that the twisted fusion coefficients for three weights $\mu \in P^+_k$ and $\alpha, \beta \in B^+_k$ are given by the formula

$$N_{\mu\alpha}^{\ \beta} \;=\; \sum_{\sigma \in S^+_k} \frac{\bar{S}^\omega_{\beta\sigma} S_{\mu\sigma} S^\omega_{\alpha\sigma}}{S_{0\sigma}} \quad . \tag{B.3}$$

where the matrix $S^\omega_{\alpha\sigma}$ is given by [28, (4.6)]

$$S^\omega_{\alpha\sigma} \;=\; (\text{phase}) \left| \, L_\omega \,/\, (k+g^\vee)(L^\vee)_\omega \right|^{-1/2} \sum_{w \in W_\omega} \epsilon_\omega(w) e^{-\frac{2\pi i}{k+g^\vee}(w(\alpha+\rho_\omega),\, \sigma+\rho)} \tag{B.4}$$

(see also [25, Theorem 13.9]). Note that it carries two different labels $\alpha \in B^+_k$ and $\sigma \in S^+_k$. The symbol ϵ_ω denotes the sign-function of W_ω. As the generators of W_ω may be products of several generators of W, in general the sign-function ϵ_ω of W_ω does not coincide with the restriction of the sign-function ϵ of W to the subgroup W_ω. Using Weyl's character formula, the quotient of S-matrices $S_{\mu\sigma}/S_{0\sigma}$ which appears in (B.3) may be expressed as

$$\frac{S_{\mu\sigma}}{S_{0\sigma}} \;=\; \chi_\mu\Big(-\frac{2\pi i}{k+g^\vee}(\sigma+\rho)\Big) \;=\; \sum_{\nu \in M_\mu} e^{-\frac{2\pi i}{k+g^\vee}(\nu,\, \sigma+\rho)} \tag{B.5}$$

where M_μ denotes the weight system of the finite-dimensional highest-weight module μ of $\mathfrak{g}$ including the multiplicities. If one inserts the expressions (B.4) and (B.5) into the definition (B.3), we may write

$$N_{\mu\alpha}^{\beta} \;=\; \sum_{\sigma\in S_k^+} f(\sigma+\rho) \;=\; \sum_{\sigma\in S_{k+g^\vee}^{++}} f(\sigma) \quad , \tag{B.6}$$

where we used the rule $S_k^+ + \rho = S_k^{++}$ and defined the function

$$\begin{aligned} f(\sigma) \;=\; & \left|L_\omega/(k+g^\vee)(L^\vee)_\omega\right|^{-1} \\ & \times \sum_{\nu\in M_\mu}\sum_{w_1,w_2\in W_\omega} \epsilon_\omega(w_1)\,\epsilon_\omega(w_2)\, e^{-\frac{2\pi i}{k+g^\vee}(\mathcal{P}_\omega\nu+w_1(\alpha+\rho_\omega)-w_2(\beta+\rho_\omega),\,\sigma)} \end{aligned} \tag{B.7}$$

which takes symmetric weights $\sigma\in L_\omega$ as arguments. Note that from the property $(\omega x,y)=(x,\omega^{-1}y)$ and the definition of $\mathcal{P}_\omega$ it follows $(\mathcal{P}_\omega\nu,\sigma)=(\nu,\sigma)$ for $\sigma\in L_\omega$.

Lemma 1. *The function f is invariant under the action of $W_\omega \ltimes (k+g^\vee)(L^\vee)_\omega$ and vanishes for elements on the boundary of the Weyl chambers, in particular on $S_{k+g^\vee}^{+}\backslash S_{k+g^\vee}^{++}$.*

Proof. The property $f(w\sigma)=f(\sigma)$ for $w\in W_\omega$ is proved by using $(wx,y)=(x,w^{-1}y)$, invariance of the weight system M_μ under Weyl transformations and redefinition of ν,w_1,w_2. Due to $\epsilon_\omega(w)^2=1$ possible signs cancel. As $\mathcal{P}_\omega\nu+w_1(\alpha+\rho_\omega)-w_2(\beta+\rho_\omega)\in\big((L^\vee)_\omega\big)^\vee$, the property $f\big(\sigma+(k+g^\vee)\beta\big)=f(\sigma)$ for $\beta\in(L^\vee)_\omega$ is obvious. To prove the second statement let us define the auxiliary function $g(\sigma)=S^\omega_{\alpha,\sigma-\rho}$ which enters each summand of the function $f(\sigma)$ as a factor. Similar as for $f(\sigma)$ one shows that $g\big(w\sigma+(k+g^\vee)\beta\big)=\epsilon_\omega(w)g(\sigma)$ for all $\beta\in(L^\vee)_\omega$ and $w\in W_\omega$. Let σ be an element of the boundary of the fundamental Weyl chamber, i.e. $\sigma\in S_{k+g^\vee}^{+}\backslash S_{k+g^\vee}^{++}$. Then it is either invariant under an elementary reflection or a combined action of a translation and an elementary reflection $w\in W_\omega$. The equation $g(\sigma)=\epsilon_\omega(w)g(\sigma)$ now implies that $g(\sigma)=0$ and thus $f(\sigma)=0$ for $\sigma\in S_{k+g^\vee}^{+}\backslash S_{k+g^\vee}^{++}$. □

Corollary 3. *Equation (B.6) can be rewritten as*

$$N_{\mu\alpha}^{\beta} \;=\; \frac{1}{|W_\omega|}\sum_{w\in W_\omega}\sum_{\sigma\in S_{k+g^\vee}^{+}} f(w\sigma) \;=\; \frac{1}{|W_\omega|}\sum_{\sigma\in L_\omega/(k+g^\vee)(L^\vee)_\omega} f(\sigma) \quad . \tag{B.8}$$

Lemma 2. *Let Γ be a lattice and $\Gamma_s\subset\Gamma$ be a sublattice of the same rank as Γ. Let $\Gamma^\vee$ and $\Gamma_s^\vee$ be the dual lattices to Γ,Γ_s with respect to an inner product $(\cdot,\cdot)$. For any $h\in\mathbb{N}$ and $x\in\Gamma_s^\vee$ we have*

$$\sum_{y\in\Gamma/h\Gamma_s} e^{2\pi i\,(x,y)/h} \;=\; \big|\Gamma/h\Gamma_s\big|\cdot\delta_{x\in h\Gamma^\vee} \quad .$$

Proof. We will use the fact that the characters χ of irreducible representations of a finite group $\mathcal{G}$ are orthogonal in the sense that $\sum_{g\in\mathcal{G}} \chi(g)\overline{\chi'(g)} = |\mathcal{G}|\cdot\delta_{\chi,\chi'}$. The quotient $\Gamma/h\Gamma_s$ is a finite abelian group. For $x \in \Gamma_s^\vee$ the function $\chi_x : \Gamma/h\Gamma_s \to \mathbb{C}$, $\chi_x(y) = e^{2\pi i\,(x,y)/h}$ is the character of an irreducible representation of $\Gamma/h\Gamma_s$ and the character χ_0 of the trivial representation is identical to one. The orthogonality relation reads, for $x \in \Gamma_s^\vee$,

$$\sum_{y\in\Gamma/h\Gamma_s} e^{2\pi i\,(x,y)/h} \;=\; \sum_{y\in\Gamma/h\Gamma_s} \chi_x(y)\,\overline{\chi_0(y)} \;=\; \big|\Gamma/h\Gamma_s\big|\cdot\delta_{\chi_x,\chi_0} \quad .$$

But $\chi_x \equiv \chi_0$ is equivalent to $x \in h\Gamma^\vee$. □

Proof of Theorem 3. We insert expression (B.7) for $f(\sigma)$ into (B.8) and apply Lemma 2 with $\Gamma = L_\omega$, $\Gamma_s = (L^\vee)_\omega$ and $h = k + g^\vee$. This results in

$$N_{\mu\alpha}^{\;\beta} \;=\; \frac{1}{|W_\omega|}\sum_{\nu\in M_\mu}\sum_{w_1,w_2\in W_\omega} \epsilon_\omega(w_1w_2)\,\delta_{\mathcal{P}_\omega\nu+w_1(\alpha+\rho_\omega)-w_2(\beta+\rho_\omega)\in(k+g^\vee)(L_\omega)^\vee} \quad .$$

Using the invariance of all quantities under W_ω we are lead to the final result

$$N_{\mu\alpha}^{\;\beta} \;=\; \sum_{\nu\in M_\mu}\sum_{w\in W_\omega} \epsilon_\omega(w)\,\delta_{w(\mathcal{P}_\omega\nu+\alpha+\rho_\omega)-(\beta+\rho_\omega)\,\in\,(k+g^\vee)(L_\omega)^\vee} \quad . \qquad \text{(B.9)}$$

The interpretation of the last formula then amounts to the algorithm of the theorem. Step 5 follows since $\beta + \rho_\omega$ is always in $B^{++}_{k+g^\vee}$. □

Note that for inner automorphisms the sets P_k^+, S_k^+ and B_k^+ all coincide and we recover the Kac-Walton formula for ordinary fusion rules. Formula (B.9) directly shows that the twisted fusion rules are integer numbers but does not show that they are non-negative. (However, non-negativity follows from the general theory [104].) We have also implemented the algorithm on a computer and have verified that no negative integers appear for the first few levels in the cases listed in table B.1.

B.2 Large volume limit and non-commutative geometry

In the second part of this appendix we will work out the limit $k \to \infty$ of the open string spectrum (B.9). In this limit, we are able to reveal a somewhat natural identification of the set of twisted representations with representations of the subgroup G^ω of G which is invariant under the automorphism Ω. The twisted fusion coefficients (B.9) in turn may be expressed in terms of branching rules of the embedding $G^\omega \hookrightarrow G$ and tensor product coefficients of G^ω [70, 34]. Our findings lead to a proposal for the non-commutative world-volume algebra of the ground states of open string vertex operators [34].

B.2.1 Evaluation of the open string spectrum at $k \to \infty$

The careful reader will already have noticed the remarkable structural similarity between the expressions (A.12) and (B.9). In this section we will make this correspondence more precise. Our reasoning will be as follows. We start with the assumption that we can distuingish a certain subalgebra $\mathfrak{h}^\omega$ of $\mathfrak{g}$ such that the equality

$$N_{\mu\alpha}^{\beta} \;=\; \sum\nolimits_c b_\mu{}^c\, N_{ca}^{\,b} \qquad \text{(B.10)}$$

holds in the limit $k \to \infty$ after an appropriate identification $\alpha = \Psi(a)$ and $\beta = \Psi(b)$ of the labels. To be precise, one has to find a bijection Ψ between integrable representations of $\mathfrak{h}^\omega$ and twisted representations of $\hat{\mathfrak{g}}$. This idea in mind, we employ several structural arguments to impose severe constraints on the allowed choices of the subalgebra $\mathfrak{h}^\omega$ which lead to a unique answer.

It turns out that in almost all cases $\mathfrak{h}^\omega$ is given by the invariant subalgebra $\mathfrak{g}^\omega$ [34]. The only case where our procedure predicts a different result is $\mathfrak{g} = A_{2n}$, where we are lead to the orbit Lie algebra $\mathfrak{h}^\omega = C_n$ and not to the invariant subalgebra $\mathfrak{g}^\omega = B_n$. Yet, in this specific case there exists a second identification which results from taking an inequivalent inductive limit and which leads to the invariant subalgebra [70].

Note that results closely related to those presented in this section have also been found independently in [173, 174] for finite k using different methods. In all cases for which such a finite k extension exists, i.e. for the A_n-series in [173] as well as the D_n-series and the A_{2n}-series in [174], our results may also be recovered from the existing literature by taking k to infinity. Let us emphasize, however, that in the cases of A_{2n-1}, D_4 (triality) and E_6 our treatment seems to indicate stronger statements, i.e. larger subgroups, for finite k than those proposed in [174].

The generic correspondence

In this first subsection we propose an identification of boundary labels with representations of a distinguished subalgebra $\mathfrak{h}^\omega$ of $\mathfrak{g}$ such that equation (B.10) is satisfied using the data of the embedding $\mathfrak{h}^\omega \hookrightarrow \mathfrak{g}$ after taking the limit $k \to \infty$. Neither is obvious that this will work a priori nor is clear which subalgebra one should take. Starting from certain assumptions we will

$\mathfrak{g}$	order	$\mathfrak{g}^\omega$	orbit Lie algebra $\breve{\mathfrak{g}}$	relevant subalgebra $\mathfrak{h}^\omega$
A_2	2	$A_1\,(x_\epsilon = 4)$	A_1	$A_1\,(x_\epsilon = 1)$
A_{2n-1}	2	C_n	B_n	C_n
A_{2n}	2	B_n	C_n	$C_n \hookrightarrow A_{2n-1}$
D_4	3	G_2	G_2	$G_2 \hookrightarrow B_3$
D_n	2	B_{n-1}	C_{n-1}	B_{n-1}
E_6	2	F_4	F_4	F_4

Table B.2: Some data related to outer automorphisms of simple Lie algebras.

first derive a set of consistency relations. Afterwards we will show that for each $\mathfrak{g}$ there is indeed a unique solution $\mathfrak{h}^\omega$ to these consistency equations and that in most cases it is given by the invariant subalgebra $\mathfrak{g}^\omega$.

It is now important to specify which structure our identification of boundary labels and representations is supposed to preserve. Let $\langle\mathcal{B}^\omega\rangle$ be the integer linear span of the set of boundary conditions $\mathcal{B}^\omega$, i.e. the full lattice generated by elements of $\mathcal{B}^\omega$. Both the lattice $\langle\mathcal{B}^\omega\rangle$ and the weight lattice $L^{(\mathfrak{h}^\omega)}$ permit an action of Weyl groups. In the first case this group is given by the symmetric part $W_\omega = \{w \in W_\mathfrak{g}\,|\,\omega \circ w = w \circ \omega\}$ of the Weyl group of $\mathfrak{g}$ and in the other case it is naturally given by $W_{\mathfrak{h}^\omega}$. Furthermore, in both cases we have a projection $\mathcal{P} : L^{(\mathfrak{g})} \to L^{(\mathfrak{h}^\omega)}$ and $\mathcal{P}_\omega : L^{(\mathfrak{g})} \to \langle\mathcal{B}^\omega\rangle$, respectively. The first defines the embedding $\mathfrak{h}^\omega \hookrightarrow \mathfrak{g}$ and the latter is given by the projection $\mathcal{P}_\omega = \frac{1}{N}(1 + \omega + \cdots + \omega^{N-1})$ onto the symmetric part of the weights, N being the order of ω. As we will see, we have to find an isomorphism $\Psi : L^{(\mathfrak{h}^\omega)} \to \langle\mathcal{B}^\omega\rangle$ between the fractional lattice generated by the boundary conditions and the weight lattice of the subalgebra which preserves all of these structures. In particular it should be accompanied with an isomorphism $\Psi : W_{\mathfrak{h}^\omega} \to W_\omega$ of the corresponding Weyl groups. To summarize, we have to find a subalgebra $\mathfrak{h}^\omega$ and a functor-like map Ψ such that the diagrams (B.11) commute. It turns out that the answer for both $\mathfrak{h}^\omega$ and Ψ is unique.

$$
\begin{array}{ccc}
L^{(\mathfrak{g})} & \xrightarrow{\mathcal{P}} & L^{(\mathfrak{h}^\omega)} \\
\Big\downarrow = & & \Big\downarrow \Psi \\
L^{(\mathfrak{g})} & \xrightarrow{\mathcal{P}_\omega} & \langle\mathcal{B}^\omega\rangle
\end{array}
\qquad\qquad
\begin{array}{ccc}
L^{(\mathfrak{h}^\omega)} & \xrightarrow{w \in W_{\mathfrak{h}^\omega}} & L^{(\mathfrak{h}^\omega)} \\
\Big\downarrow \Psi & & \Big\downarrow \Psi \\
\langle\mathcal{B}^\omega\rangle & \xrightarrow{\Psi(w) \in W_\omega} & \langle\mathcal{B}^\omega\rangle
\end{array}
\qquad (B.11)
$$

In the following we restrict ourselves to a non-trivial diagram automorphism $\omega \neq \mathrm{id}$ since our statements become trivial otherwise.

Remember that we only consider the limit $k \to \infty$. For $\alpha = \Psi(a)$, $\beta = \Psi(b)$ we want to proof the equality (B.10) where the last expression contains some branching coefficients for $\mathfrak{h}^\omega \hookrightarrow \mathfrak{g}$ and the tensor product coefficients of the subalgebra. The equations (A.12) and (B.9) may be employed to rewrite the left and right hand side in the limit $k \to \infty$ according to

$$\begin{aligned} N_{\mu\beta}^{\ \alpha} &= \sum_{\nu \in M_\mu, w \in W_\omega} \epsilon_\omega(w)\, \delta_{\alpha, w(\mathcal{P}_\omega \nu + \beta + \rho_\omega) - \rho_\omega} \\ \sum_c b_\mu{}^c\, N_{cb}^{\ a} &= \sum_{\nu \in M_\mu, w \in W_{\mathfrak{h}^\omega}} \epsilon(w)\, \delta_{a, w(\mathcal{P}\nu + b + \rho) - \rho} \quad . \end{aligned} \tag{B.12}$$

The abbreviation $\nu \in M_\mu$ means that ν runs over all weights in the weight system of μ. Both expressions are obviously equal to each other if the bijection Ψ is structure preserving, i.e. if

$$\begin{aligned} \Psi(\rho) &= \rho_\omega \\ \Psi \circ \mathcal{P} &= \mathcal{P}_\omega \\ \Psi(wa) &= \Psi(w)\, \Psi(a) \quad . \end{aligned} \tag{B.13}$$

The last condition already constrains the possible subalgebras to a large extent. Indeed, the Weyl group W_ω can be described as the Weyl group of the so-called orbit Lie algebra of $\mathfrak{g}$ with respect to the automorphism Ω (see [121]). In some special cases this orbit Lie algebra coincides with the invariant subalgebra $\mathfrak{g}^\omega$ while it does not for the whole A_n and D_n series. A survey of these relations can be found in table B.2 which in part has been taken from [28]. Note however that the Weyl groups of B_n and C_n are isomorphic to each other (see e.g. [157, p. 74]) which can most easily be seen by treating them as abstract Coxeter groups. Thus by imposing the last constraint we only have to decide which of possibly two subalgebras – the orbit Lie algebra or the invariant subalgebra – and which specific kind of embedding we should take. This choice is uniquely determined by the other two conditions. In the cases of A_{2n-1} and D_n the orbit Lie algebra not even is a subalgebra so that this possibility is ruled out immediately.

We will show in the most simple example of the Lie algebra A_2 how this procedure works and then state only results for all the other cases. Let us consider $\mathfrak{g} = A_2$ with outer automorphism $\omega(a_1, a_2) = (a_2, a_1)$. The relevant subalgebra is given by $\mathfrak{h}^\omega = A_1$ and the projection to fractional symmetric weights – which describe the boundary conditions of the theory – reads $\mathcal{P}_\omega(a_1, a_2) = \frac{1}{2}(a_1 + a_2, a_1 + a_2)$. There are two inequivalent embeddings $A_1 \hookrightarrow A_2$ given by projections $\mathcal{P}_{x_\epsilon}(a_1, a_2) = \sqrt{x_\epsilon}\,(a_1 + a_2)$ with embedding index $x_\epsilon = 1$ and $x_\epsilon = 4$, respectively [110, p. 534]. Imposing the first condition we see that

$$\Psi \circ \mathcal{P}_{x_\epsilon}(a_1, a_2) = \sqrt{x_\epsilon}\ \Psi(a_1 + a_2) \quad .$$

This only equals $\mathcal{P}_\omega(a_1, a_2)$ for

$$\Psi(a) \;=\; \frac{1}{2\sqrt{x_\epsilon}}\,(a,a) \quad .$$

The condition $\Psi(\rho) = \rho_\omega = \frac{1}{2}(1,1)$ forces us to use the projection with $x_\epsilon = 1$. One can also check explicitly that the Weyl groups correspond to each other. This is the first example where the relevant subalgebra is not given by the invariant subalgebra (which has embedding index $x_\epsilon = 4$) but by the orbit Lie algebra. The same statement holds for the whole A_{2n} series as we will see.

One can treat the whole ADE series using a case by case study. Let us emphasize that we use the labeling conventions for weights which can be found in [110, p. 540]. The projections have been determined using [159, p. 57-61] and the programs LiE [175] and SimpLie [176]. Note that LiE uses a different labeling convention for the weights. For a useful table of branching rules see also [177].

1. The case of A_{2n-1} is straightforward. The relevant subalgebra is given by the invariant subalgebra $C_n \hookrightarrow A_{2n-1}$. This is a maximal embedding and the identification reads

$$\mathcal{P}_\omega \;=\; \frac{1}{2}\begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & \iddots & \\ & & 1 & & 1 & & \\ & & & 2 & & & \\ & & 1 & & 1 & & \\ & \iddots & & & & \ddots & \\ 1 & & & & & & 1 \end{pmatrix} \;=\; \frac{1}{2}\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 2 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 & & & & 1 \\ & \ddots & & \iddots & \\ & & 1 & & \end{pmatrix} \;=\; \Psi\circ\mathcal{P} \quad .$$

2. The case A_{2n} is exceptional. Here the relevant subalgebra is given by the orbit Lie algebra which can be described by the sequence of maximal embeddings $C_n \hookrightarrow A_{2n-1} \hookrightarrow A_{2n}$ (for $n = 1$ we have $A_1 \hookrightarrow A_2$). The identification reads

$$\mathcal{P}_\omega \;=\; \frac{1}{2}\begin{pmatrix} 1 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 1 & 1 & & \\ & & 1 & 1 & & \\ & \iddots & & & \ddots & \\ 1 & & & & & 1 \end{pmatrix} \;=\; \frac{1}{2}\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}\begin{pmatrix} 1 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 1 & 1 & & \end{pmatrix} \;=\; \Psi\circ\mathcal{P} \quad .$$

There is a second identification related to the subalgebra $B_n \hookrightarrow A_{2n}$ which will be discussed in the next subsection and which is the relevant one for the main part of the paper.

3. The order 3 diagram automorphism of D_4 leads to the sequence of maximal embeddings $G_2 \hookrightarrow B_3 \hookrightarrow D_4$ and to the identification

$$\mathcal{P}_\omega \;=\; \frac{1}{3}\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \;=\; \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \;=\; \Psi\circ\mathcal{P} \quad .$$

4. For the order 2 automorphism of the D-series one obtains the maximal embedding $B_{n-1} \hookrightarrow D_n$ and

$$\mathcal{P}_\omega \;=\; \frac{1}{2}\begin{pmatrix} 2 & & & \\ & \ddots & & \\ & & 2 & \\ & & & \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \end{pmatrix} \;=\; \frac{1}{2}\begin{pmatrix} 2 & & & \\ & \ddots & & \\ & & 2 & \\ & & & \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \end{pmatrix}\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \begin{smallmatrix} 1 & 1 \end{smallmatrix} \end{pmatrix} \;=\; \Psi \circ \mathcal{P} \quad .$$

5. Also the last case E_6 behaves regular and yields the maximal embedding $F_4 \hookrightarrow E_6$ with

$$\mathcal{P}_\omega = \frac{1}{2}\begin{pmatrix} 1&0&0&0&1&0 \\ 0&1&0&1&0&0 \\ 0&0&2&0&0&0 \\ 0&1&0&1&0&0 \\ 1&0&0&0&1&0 \\ 0&0&0&0&0&2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&2&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 2&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&1&0&1&0&0 \\ 1&0&0&0&1&0 \end{pmatrix} = \Psi \circ \mathcal{P}.$$

These considerations show that in all cases but A_{2n} we may identify the boundary labels with representations of the invariant subgroup G^ω. Let us emphasize that it is possible to identify the set of Lie algebra representations $P^+_{\mathfrak{g}^\omega}$ with the set of group representations $P^+_{\mathrm{G}^\omega}$ in these cases as the corresponding groups G^ω are all simply-connected. The outcome of this section is summarized in table B.2. A detailed discussion of the identification one should use in the exceptional case of A_{2n} is postponed to the next subsection.

The special case A_{2n}

In the last section it was shown that under certain natural and well-founded assumptions the relevant subalgebra $\mathfrak{h}^\omega$ which describes the boundary labels is given by the orbit Lie algebra C_n in the case of A_{2n} and not by the invariant subalgebra B_n. This contradicts our geometrical intuition as we are expecting the invariant subalgebra (or even better: the invariant subgroup) to be the relevant structure.

There exists, however, a different inductive limit, i.e. an identification which involves the level k explicitly, which leads to the invariant subalgebra. Indeed, in writing (B.12) we implicitly already took a very special limit. That performing the limit $k \to \infty$ may have non-trivial effects can be seen from simple current symmetries in fusion rules. In this case, all simple current symmetries get lost and it becomes important on which “branch” of the simple current orbits one sits while taking the limit. There is also another point on which we have not been careful enough in the last subsection. The geometric picture suggests that we should work with representations of the invariant sub*group*, not necessarily with those of the invariant sub*algebra*. These two sets may differ as can easily be seen from the familiar example of SO(3) which only allows SU(2) representations of integer spin. Note that the automorphism Ω in the case of SU(3) is just given by charge conjugation and that SO(3) is exactly the invariant subgroup. Similar remarks hold for the whole series $SO(2N+1) \hookrightarrow SU(2N+1)$, i.e. the whole A_{2n} series. All this should be reflected in the new identification in a some way.

Let us now propose the new identification [70] and see whether it fits our requirements. The construction only works for even values of the level k. This fact may be reminiscent of the D-series modular invariants of SU(2) describing a SO(3) WZNW model. Therefore we will assume k to be even in what follows. This restriction will not be relevant in the limit $k \to \infty$.

The labels for the twisted boundary conditions in the WZNW model based on A_{2n} are given by half-integer symmetric weights α of A_{2n}. To be more specific, the Dynkin labels have to satisfy the relations $2\alpha_i \in \mathbb{N}_0$, $\alpha_i = \alpha_{2n+1-i}$ and $\sum_{i=0}^{n} \alpha_i \leq k/4$. These labels may be interpreted as labels of the invariant subalgebra B_n of A_{2n}. The map from weights of B_n to the boundary labels is given by

$$\Psi'\big(a_1, \cdots, a_n\big) \;=\; \frac{1}{4}\Big(2a_{n-1}\,,\,\cdots\,,\,2a_1\,,\,k-2\sum_{i=1}^{n-1} a_i - a_n\,,\cdots\Big) \quad . \qquad \text{(B.14)}$$

Note that this map involves the level k explicitly and that it is only well-defined for weights whose last Dynkin label a_n is even. This last condition has two interpretations. From the group theoretical point of view it restricts to representations of the Lie algebra B_n which may be integrated to single-valued representations of the group $SO(2n+1)$. From the Lie algebra point of view it corresponds to the branching selection rule of the embedding $B_n \hookrightarrow A_{2n}$ for which the map

$$\mathcal{P}\big(i_1, \cdots, i_{2n}\big) \;=\; \big(i_1 + i_{2n}, i_2 + i_{2n-1}, \cdots, 2(i_n + i_{n+1})\big)$$

is the relevant projection.

Up to now, we have just seen that the structure of the identification map (B.14) is consistent with our expectations, but of course the ultimate check consists in the verification of equality (B.10). For small values of rank and level we evaluated both sides using an implementation of the algorithms which have been stated in the theorems 2 and 3 on a computer and found full agreement. Until now we still lack a rigorous proof for arbitrary values of the number n. Only for the simplest example $n = 1$, i.e. for the WZNW model based on the Lie algebra A_2, more solid arguments are known which support our claim. We use the opportunity to summarize them briefly.

The Lie algebra $\mathfrak{g} = A_2$ has exactly one automorphism Ω related to a non-trivial Dynkin diagram symmetry, where it acts as a permutation of nodes. On the level of weights it thus acts as a permutation of Dynkin labels $\omega(a_1, a_2) = (a_2, a_1)$. The boundary labels are given by half-integer symmetric weights $\alpha, \beta = (0,0), (1/2, 1/2), \cdots, (\lfloor k/2 \rfloor/2, \lfloor k/2 \rfloor/2)$. Here, the symbol $\lfloor x \rfloor$ denotes the largest integer number smaller or equal to x. The relevant annulus coefficients [28]

$$N^{(k)\,\alpha}_{(\mu_1,\mu_2)\,\beta} \;=\; \sum_{\sigma=0}^{\lfloor k/2 \rfloor} \frac{\bar{S}^{\omega}_{\sigma\alpha} S^{\omega}_{\sigma\beta} S_{(\sigma,\sigma),(\mu_1,\mu_2)}}{S_{(\sigma,\sigma),(0,0)}} \tag{B.15}$$

may be calculated using the explicit formula

$$S^{\omega}_{\sigma\alpha} \;=\; \frac{2}{\sqrt{k+3}} \,\sin \frac{2\pi}{k+3}(\sigma+1)(2\alpha+1) \quad , \tag{B.16}$$

where we identified the tupel α with one of its (identical) entries. Note the remarkable similarity of this expression with the S-matrix of $A_1^{(1)}$.

In the last subsection we already encountered a bijection Ψ which related the annulus coefficient (B.15) to the embedding $A_1 \hookrightarrow A_2$ with embedding index $x_\epsilon = 1$. Yet, as we will show now, the map Ψ' which has been defined in eq. (B.14) yields the embedding with projection $\mathcal{P}(\mu_1, \mu_2) = 2(\mu_1 + \mu_2)$ and embedding index $x'_\epsilon = 4$. For any even weight a of A_1 the definition (B.14) reduces to $\Psi'(a) = (k/4, k/4) - (a/4, a/4)$. We use our new identification map Ψ' to rewrite (B.16) according to

$$S^{\omega\prime}_{\sigma a} \;=\; S^{\omega}_{\sigma\Psi'(a)} \;=\; \frac{2(-1)^\sigma}{\sqrt{k+3}} \,\sin \frac{\pi}{k+3}(\sigma+1)(a+1) \quad .$$

Apart from a factor $\sqrt{2}(-1)^\sigma$ this is just the S-matrix $S^{A_1}_{\sigma a}$ of $A_1^{(1)}$ at level $k+1$. Using (A.5) we are now able to write equation (B.15) as

$$N^{(k)\,\Psi'(a)}_{(\mu_1,\mu_2)\,\Psi'(b)} \;=\; 2\sum_{\sigma=0}^{k/2} \bar{S}^{A_1}_{\sigma a} S^{A_1}_{\sigma b}\, \chi^{A_2}_{(\mu_1,\mu_2)}\Big(-\frac{2\pi i}{k+3}(\sigma+1, \sigma+1)\Big) \quad .$$

Remembering the definitions of $\mathcal{P}'^*$ in theorem 1 and of ξ_σ in equation (A.5), the argument of the character can be identified to be $\mathcal{P}'^*\xi_\sigma$. By setting the index b to zero, theorem 1 implies

$$b_{(\mu_1,\mu_2)}{}^{a} \;=\; \lim_{k\to\infty} \sum_{\sigma=0}^{k+1} \bar{S}^{A_1}_{\sigma a}\, S^{A_1}_{\sigma 0}\, \chi^{A_2}_{(\mu_1,\mu_2)}\big(\mathcal{P}'^*\xi_\sigma\big) \;=\; \lim_{k\to\infty} N^{(k)\,\Psi'(a)}_{(\mu_1,\mu_2)\,\Psi'(0)} \quad .$$

This equality holds because we are allowed to use the prefactor 2 to extend the range of σ from $0, \cdots, k/2$ to $0, \cdots, k+1$. We have thus proved the equality (B.10) for a special choice of parameters.

B.2.2 Non-commutative D-brane world-volumes

According to the procedure suggested in [76, 32] (see also [74, 178] for similar proposals in case of closed strings), the world-volume geometry of branes can be read off from the correlators of boundary operators in the decoupling regime $k \to \infty$ [1]. Note that the conformal dimensions $h_\mu = C_\mu/2(k+g^\vee)$ of the boundary fields vanish in this limit so that the operator product expansion

$$\psi_\mu^{(ab)}(x)\,\psi_\nu^{(bc)}(y) \;\sim\; \sum\nolimits_\rho (x-y)^{h_\rho-h_\mu-h_\nu}\, C_{\mu\nu}^{(abc)\,\rho}\,\psi_\rho^{(ac)}(y) \quad \text{for} \quad x<y$$

of primary open string vertex operators becomes independent of the world-sheet coordinates. In particular, all conformal families in the boundary theory contribute to the massless sector and thus to the gauge theory which governs the low energy dynamics of the D-branes we are studying. The program we have just sketched has been carried out successfully for the untwisted branes on compact group manifolds. In case of A_1, this leads to the well-known fuzzy spheres [179, 32]. We will now describe the generalization to arbitrary Ω-twisted D-branes on compact simply-connected simple group manifolds G.

There will be two guiding principles which lead to the identification of the correct world-volume algebra. The first (and most important) is symmetry. As can be inferred from the expression for the open string spectrum (B.1) the primary boundary fields $\psi_\mu^{(ab)}$ indeed transform in an irreducible representation μ of the group G. The *same* G-module structure should be reproduced by our world-volume algebra. Also the multiplicative structure should be respected which only allows to multiply fields $\psi^{(ab)}$ and $\psi^{(b'c)}$ if the adjacent indices coincide, $b=b'$.

In the previous section we explained that all possible Ω-twisted D-branes can be labeled by the set $P^+_{\mathrm{G}^\omega}$ of representations of the invariant subgroup $\mathrm{G}^\omega = \{g \in \mathrm{G}\,|\,\Omega(g)=g\}$. This observation is already a major step towards a geometrical interpretation. Let us stress that we use representations of the group G^ω and not of its Lie algebra. The two sets of representations agree only if G^ω is simply-connected. This is the case for all Lie algebras but the A_{2n} series where $\mathrm{G}^\omega = \mathrm{SO}(2n+1)$ and $P^+_{\mathrm{G}^\omega} = \big\{\mu \in P^+_{\mathrm{G}^\omega}\big|\mu_n \text{ even}\big\}$. An overview over the relevant groups and Lie algebras is given in table B.3.

After these comments we can now formulate our proposal for the world-volume algebra. To this end, let V_a, V_b be two representation spaces for

[1] When taking the limit $k\to\infty$, we keep the open string data stable. In particular, the low energy spectrum of open string modes does not change. A limit where the closed string data is kept stable instead has been considered in [29].

$\mathfrak{g}$	order	$\mathfrak{g}^\omega$	x_ϵ	G	G^ω
A_2	2	A_1	4	SU(3)	SO(3)
A_{2n-1}	2	C_n	1	SU($2n$)	$\mathrm{Sp}(2n,\mathbb{C})\cap\mathrm{SU}(2n)$
A_{2n}	2	B_n	2	SU($2n+1$)	SO($2n+1$)
D_4	3	G_2	1	$\mathrm{Spin}(8)=\widetilde{SO}(8)$	$\widetilde{G}_2$
D_n	2	B_{n-1}	1	$\mathrm{Spin}(2n)=\widetilde{\mathrm{SO}}(2n)$	$\mathrm{Spin}(2n-1)=\widetilde{\mathrm{SO}}(2n-1)$
E_6	2	F_4	1	$\widetilde{E}_6$	$\widetilde{F}_4$

Table B.3: Simple Lie algebras, groups and more data related to outer automorphisms.

irreducible representations $a, b \in P^+_{\mathrm{G}^\omega}$ of G^ω. As we will argue below, the relevant algebraic structure governing strings stretching between two D-branes of type a and b, respectively, is given by

$$\mathcal{A}^{(a,b)} \;\cong\; \mathrm{Inv}_{\mathrm{G}^\omega}\big(\mathcal{F}^{(a,b)}\big) \quad \text{where} \quad \mathcal{F}^{(a,b)} \;:=\; \mathcal{F}(\mathrm{G})\otimes\mathrm{Hom}(V_a,V_b) \quad . \tag{B.17}$$

Here $\mathcal{F}(\mathrm{G})$ denotes the algebra of functions on the group G and $\mathrm{Hom}(V_a, V_b)$ is the vector space of linear transformations from V_a to V_b. The auxiliary space $\mathcal{F}^{(a,b)}$ can be regarded as a vector space of matrix valued functions on the group G. It carries an action of the product group $\mathrm{G}\times\mathrm{G}^\omega$ defined by

$$\big[(g,h)\cdot \mathrm{A}\big](g') \;=\; R_b(h)\,\mathrm{A}(g^{-1}g'h)\,R_a(h)^{-1} \quad , \tag{B.18}$$

where $R_a(h) \in \mathrm{GL}(V_a)$ and $R_b(h) \in \mathrm{GL}(V_b)$ represent h. In our construction of $\mathcal{A}^{(a,b)}$, we restrict to matrix valued functions $\mathrm{Inv}_{\mathrm{G}^\omega}\big(\mathcal{F}^{(a,b)}\big)$ which are invariant under the action of $\{\mathrm{id}\}\times\mathrm{G}^\omega \subset \mathrm{G}\times\mathrm{G}^\omega$. Let us note that this leaves us with an action of G on the space $\mathcal{A}^{(a,b)}$ of G^ω-invariants. Below we will show that the G-module structure exactly coincides with the one of the ground states in the open string partition function (B.1).

We can realize the G-module $\mathcal{A}^{(a,b)}$ explicitly in terms of G^ω-equivariant functions on the group G,

$$\mathcal{A}^{(a,b)} \;\cong\; \big\{\,\mathrm{A}\in\mathcal{F}^{(a,b)}\,\big|\,\mathrm{A}(gh) = R_b(h)^{-1}\mathrm{A}(g)R_a(h) \text{ for } h\in\mathrm{G}^\omega\,\big\} \quad . \tag{B.19}$$

When the two involved representations are trivial, i.e. $a = b = 0$, elements of $\mathcal{A}^{(a,b)}$ are simply invariant under right translations with respect to $\mathrm{G}^\omega \subset \mathrm{G}$.

There exists more structure on the spaces $\mathcal{A}^{(a,b)}$ if $a = b$. In fact, $\mathcal{A}^{(a)} = \mathcal{A}^{(a,a)}$ inherits an associative product from the pointwise multiplication of elements in $\mathcal{F}^{(a,a)}$. This turns the subspace $\mathcal{A}^{(a)}$ of G^ω-invariants into an associative matrix algebra. Note that it also makes sense to multiply elements from $\mathcal{A}^{(a,b)}$ with those from $\mathcal{A}^{(b,c)}$. The result will then be an element of $\mathcal{A}^{(a,c)}$ just as would be expected from the CFT description.

The constructions we have outlined so far may easily be generalized to arbitrary superpositions of D-branes. To this end we replace the irreducible representations V_a, V_b in (B.17) by reducible ones. Let V_Q be such a reducible representation, i.e. $V_Q \cong \oplus Q^a V_a$. It represents a superposition of $\sum Q^a$ D-branes in which Q^a branes of type $a \in P^+_{\mathrm{G}^\omega}$ are placed on top of each other. Strings ending on such a brane configuration Q give rise to an algebra $\mathcal{A}^{(Q)} = \mathcal{A}^{(Q,Q)}$ analogous to (B.17). For a stack of N identical branes of type $a \in P^+_{\mathrm{G}^\omega}$, the constructions specialize and produce the typical Chan-Paton factors,

$$\mathcal{A}^{N(a)} \;\cong\; \mathrm{Inv}_{\mathrm{G}^\omega}\big(\mathcal{F}^{(a,a)}\big) \otimes \mathrm{Mat}(N) \quad . \tag{B.20}$$

Obviously, the left translation of the group G turns this into a G-module with trivial action of G on $\mathrm{Mat}(N)$.

This concludes the formulation of our proposal for the algebra of "functions on twisted D-branes". We are now going to show that its predictions agree with the exact CFT results described in the last section. In particular, we shall confront the formula (B.19) with the CFT-spectrum of boundary fields (B.1). Before we carry out the details, let us note that all sewing constraints [180] are automatically satisfied by our construction if we manage to show that the spectra match. In fact, associativity is manifest in our proposal and it is the only content of the sewing constraints when we send the level k to infinity.

Hence, it remains to discuss the spectrum of open strings. The CFT description provides an expression eq. (B.10) for the NIM-rep which describes the spectrum of strings stretching between D-branes of type $a, b \in P^+_{\mathrm{G}^\omega}$ in the limit $k \to \infty$. We claim that in this limit, the G-module of ground states is isomorphic to the G-module $\mathcal{A}^{(a,b)}$. We will prove this by decomposing $\mathcal{A}^{(a,b)}$ into irreducibles. To do so, let us note that there is a canonical isomorphism $\mathrm{Hom}(V,W) \cong V^* \otimes W$. Furthermore, we may apply the Peter-Weyl theorem to decompose the algebra $\mathcal{F}(\mathrm{G})$ with respect to the regular action of $\mathrm{G} \times \mathrm{G}$ into

$$\mathcal{F}(\mathrm{G}) \;\cong\; \bigoplus\nolimits_\mu U^*_\mu \otimes U_\mu$$

where μ runs over all irreducible representations of G and the two factors of $\mathrm{G} \times \mathrm{G}$ act on the two vector spaces U^*_μ, U_μ, respectively. To make contact with our definition of $\mathcal{A}^{(a,b)}$, we have to restrict the right regular G action to the subgroup G^ω, which leaves us with the $\mathrm{G} \times \mathrm{G}^\omega$-module

$$\mathcal{F}(\mathrm{G}) \;\cong\; \bigoplus\nolimits_{\mu,c} b_\mu{}^c\, U^*_\mu \otimes V_c \quad .$$

The numbers ${b_\mu}^c \in \mathbb{N}_0$ are the branching coefficients which count the multiplicity of the G^ω-module V_c in U_μ. Combining these remarks we arrive at

$$\mathcal{F}^{(a,b)} \;\cong\; \bigoplus_{\mu,c} {b_\mu}^c\, U_\mu^* \otimes V_c \otimes V_a^* \otimes V_b \quad .$$

It remains now to find the invariants under the G^ω-action. Note that G^ω acts on the last three tensor factors. The number of invariants in the triple tensor product of irreducible representations is simply given by the fusion coefficients $N_{cb}^{\,a}$ of G^ω. Hence, as a G-module, we have shown that

$$\mathcal{A}^{(a,b)} \;\cong\; \bigoplus_{\mu,c} {b_\mu}^c\, N_{cb}^{\,a}\, U_\mu^* \tag{B.21}$$

This decomposition obviously agrees with the right hand side of the formula (B.10) for the CFT NIM-rep.

As a simple cross-check we consider the case of trivial automorphism $\Omega = \mathrm{id}$ where we can make contact to well-known results (see [32]). First we observe that the construction above simplifies considerably since $\mathrm{G}^\omega \cong \mathrm{G}$. This implies that all the latin labels can be replaced by greek letters. In particular, the boundary conditions are now labeled by representations of G itself. The corresponding G-module structure is now given by

$$\mathcal{A}^{(\mu,\nu)} \;\cong\; \bigoplus_{\rho} N_{\mu^+\nu}^{\;\;\rho}\, U_\rho \quad .$$

This is in complete agreement with the known CFT results which correspond to *Cardy's case* [102].

Another independent verification can be achieved by considering *permutation branes* in the product group $\mathrm{G} \times \mathrm{G}$ which are based on the automorphism $\Omega_{\mathrm{exc}}(g_1, g_2) = (g_2, g_1)$. In this case the invariant subgroup is G itself and the branching coefficients reduce to the tensor product coefficients, ${b_{(\mu,\nu)}}^\rho = N_{\mu\nu}^{\,\rho}$. Under these circumstances our general expression (B.21) for the non-commutative world-volume algebra reduces to

$$\mathcal{A}^{(\mu,\nu)} \;\cong\; \bigoplus_{\sigma,\rho,\lambda} N_{\rho\lambda}^{\,\sigma}\, N_{\sigma\nu}^{\,\mu}\, U_{(\rho,\lambda)}^* \quad .$$

This result is again in perfect agreement with that which can be obtained from the open string partition function (2.27) in the limit $k \to \infty$.

The world-volume algebra (B.20) for stacks of twisted branes is the central ingredient in the non-commutative gauge theory which governs their dynamics in the large volume limit [33, 34]. The same structures are known to appear in the effective description of D-branes in coset theories [127, 181]. We believe that the ideas presented in this section together with the target space interpretation (3.20) will also be crucial for understanding the dynamics of symmetry breaking D-branes. Let us recall that the construction of the latter could indeed essentially be reduced to an analysis of maximally symmetric branes in some intermediate groups, see section 3.2.3.

Appendix C

Symmetry breaking from T-duality

C.1 The basic idea

In [131, 35], Maldacena, Moore and Seiberg introduced a fascinating and very intuitive way of constructing symmetry breaking D-branes in $\mathrm{SU}(N)$ group manifolds. They made use of the T-duality

$$\mathrm{G} \;=\; \big(\mathrm{G}/\mathrm{H} \;\times\; \mathrm{H}\big)\,/\,\Gamma \;\;, \tag{C.1}$$

which is known to hold provided that H is some regular subgroup of the maximal torus $\mathrm{U}(1)^{N-1}$ of $\mathrm{G} = \mathrm{SU}(N)$ and Γ is a specific orbifold group. This duality is very attractive as it allows to use the results for maximally symmetric boundary states in $\mathrm{G}/\mathrm{H} \times \mathrm{H}$. By taking an Γ-invariant superposition of these states one may then construct boundary states which project down to the orbifold, i.e. to the group $\mathrm{SU}(N)$. Symmetry breaking is achieved by choosing a non-trivial gluing automorphism in the H-part.

The aim of this section is two-fold. We will first work out the conditions on the groups G, H and Γ which have to hold such that a T-duality of the form (C.1) is valid. If one restricts oneself to simple current orbifolds, one finds a concrete expression for Γ and strong constraints on the possible choices of the subgroup H. In a second step we show that the results for symmetry breaking boundary states which are obtained from T-duality in this special situation fit into our more general framework of chapter 3.

C.2 General considerations

To establish a T-duality of the form (C.1) we have to compare the state spaces for the CFT's on the left and on the right hand side. If they are identical, there is a good chance that the duality holds, otherwise this is certainly not the case. After a successful check of the spectrum one would in principle have to compare all the correlation functions, but we will simply assume the validity of the duality under these circumstances.

Let us start with a discussion of the CFT on the right hand side of (C.1). In the following we restrict ourselves to simple current orbifolds for which the group Γ is a subgroup of the center $\mathcal{Z}(\mathrm{G/H}) \times \mathcal{Z}(\mathrm{H})$. Note that all the examples in the existing literature fall into this class. A general survey of simple current orbifolds has been given in section 2.4.3 but it is useful to restate the main formulas for the present situation.

The sectors $\big([\mu,a],b\big)$ of $\mathcal{A}(\mathrm{G/H}) \otimes \mathcal{A}(\mathrm{H})$ fall into orbits $\big[[\mu,a],b\big]$ with respect to the action of Γ. With each of these orbits we associate two numbers, namely the monodromy charges $Q_J\big(\big[[\mu,a],b\big]\big)$, $J \in \Gamma$, and the order $|\mathcal{S}_{[[\mu,a],b]}|$ of the stabilizer subgroup. The orbifold bulk partition function is then given by

$$Z^{\mathrm{orb}}(q) \;=\; \sum_{Q_\Gamma([[\mu,a],b])=0} \Big|\mathcal{S}_{[[\mu,a],b]}\Big| \, \Bigg| \sum_{([\nu,c],d)\in[[\mu,a],b]} \chi^{\mathrm{G/H}}_{(\nu,c)}(q)\chi^{\mathrm{H}}_{d}(q) \Bigg|^2 \quad . \tag{C.2}$$

This expression has to be compared with the decomposed partition function (3.10) of the charge conjugate theory. It will turn out that the existence of an appropriate group Γ imposes strong constraints on the choice of H. Once these constraints have been formulated, we shall compare our new boundary states (3.11, 3.12) with those arising from the orbifold construction.

To formulate necessary conditions for the equivalence of the partition function (3.10) of the G-theory with one of the orbifold partition functions (C.2), we shall concentrate on terms that contain a factor $\chi_{[0,0]}\chi_0$ from the holomorphic sector,

$$Z^{\mathrm{G}} \;=\; \textstyle\sum_a \chi^{\mathrm{G/H}}_{[0,0]}\,\chi^{\mathrm{H}}_0\,\bar\chi^{\mathrm{G/H}}_{[0,a]}\,\bar\chi^{\mathrm{H}}_a \;+\;\cdots \quad ,$$

$$Z^{\mathrm{orb}} \;=\; \sum_{(J,J')\in\Gamma} \chi^{\mathrm{G/H}}_{[0,0]}\,\chi^{\mathrm{H}}_0\,\bar\chi^{\mathrm{G/H}}_{J[0,0]}\,\bar\chi^{\mathrm{H}}_{J'0} \;+\;\cdots \quad .$$

The summation over a in the first expression is restricted such that $(0,a) \in \mathrm{All(G/H)}$. We can now read off one important condition for the equivalence:

all the labels $a \in \text{Rep(H)}$ that appear in the summation must be simple currents of $\mathcal{A}$(H), i.e. elements of $\mathcal{Z}$(H). Under this condition we can set

$$\Gamma \;=\; \big\{ \big([0,a],a\big) \;\big|\; (0,a) \in \text{All(G/H)} \big\} \;\subset\; \mathcal{Z}(\text{G/H}) \times \mathcal{Z}(\text{H}) \quad .$$

By projection on the first or second factor, Γ can be identified with a subgroup of both $\mathcal{Z}$(H) and $\mathcal{Z}$(G/H). If the identification group $\mathcal{G}_{\text{id}}$ is trivial, it follows that all sectors of $\mathcal{A}$(H) must be simple currents, i.e. H must be abelian. In cases with non-trivial field identification, the orbifold construction with Γ can reproduce the partition function of the G-theory even if some of the sectors of H are not simple currents. We shall provide one example in section C.4 below.

A more detailed comparison of the bulk partition functions reveals a second necessary condition for the desired equivalence. Namely, one can see that the orbifold and the G-theory can only agree if Γ acts transitively on the sets $\text{All}_\mu(\text{G/H}) := \big\{(\mu,b) \in \text{All(G/H)}\big\}$. In particular, this implies that $\big|\text{All(G/H)}/\Gamma\big| = |\text{Rep(G)}|$.

C.3 Equivalence of boundary states

Let us assume a situation in which the duality (C.1) holds. Under these circumstances there exist two different pictures in which one can construct boundary states for one and the same CFT background. One can either apply the procedure which has been developed in section 3.2.3 or one could use the orbifold techniques of section 2.4.3. The aim of this section is to show their equivalence and to provide the exact correspondence between boundary states in both pictures. For simplicity we restrict ourselves to the case without any twist.

Boundary states of the orbifold theory can be obtained from the Cardy states $|[\mu,a],b\rangle = |[\mu,a]\rangle^{\text{G/H}} \otimes |b\rangle^{\text{H}}$ of the charge conjugate covering theory by averaging over the action of the orbifold group Γ. This leads to boundary states of the form

$$\big|\big[[\mu,a],b\big]\big\rangle \;=\; \frac{1}{\sqrt{|\Gamma|}} \sum_{(J,J')\in\Gamma} |J[\mu,a],J'b\rangle \tag{C.3}$$

where the labels $\big[[\mu,a],b\big]$ of boundary states now take values in the set $\big(\text{Rep(G/H)} \times \text{Rep(H)}\big)/\Gamma$. It is also easy to calculate the boundary partition function

$$Z^{\text{orb}}_{[[\mu,a],b],[[\nu,c],d]} \;=\; \sum_{(J,J')\in\Gamma} \; \sum_{[\sigma,e]\in\text{Rep(G/H)},f\in\text{Rep(H)}} N^{J[\sigma,e]}_{[\mu,a]^+[\nu,c]}\, N^{J'f}_{b^+d}\, \chi^{\text{G/H}}_{[\sigma,e]}\, \chi^{\text{H}}_f \;\; . \tag{C.4}$$

When the orbifold action has fixed points, some of these states may be resolved further, but we will not discuss this issue. The main point here was to outline how one can obtain boundary states in the background (C.2). They are labeled by elements of $\big(\mathrm{Rep}(\mathrm{G/H}) \times \mathrm{Rep}(\mathrm{H})\big)/\Gamma$.

We are now prepared to compare the brane spectra of the orbifold construction with the spectra obtained in section 3.2.3. In the following analysis we assume that H is abelian which is the case for all the examples considered in [131, 35]. The orbifold construction of the background works for a slightly larger class of cases, but in such cases the brane spectra can be different, at least before resolving possible fixed points of Γ (see below). Assuming that $\mathrm{Rep}(\mathrm{H}) = \mathcal{Z}(\mathrm{H})$, we want to verify first that both constructions provide the same number of boundary states. This amounts to saying that

$$\big(\mathrm{Rep}(\mathrm{G/H}) \times \mathrm{Rep}(\mathrm{H})\big) \,/\, \Gamma \;\cong\; \big(\mathrm{Rep}(\mathrm{G}) \times \mathrm{Rep}(\mathrm{H})\big) \,/\, \mathcal{G}_{\mathrm{id}} \quad . \tag{C.5}$$

By our assumption on H, the action of Γ has no fixed points. The same holds automatically true for the action of $\mathcal{G}_{\mathrm{id}}$. Therefore, our results of section 3.2.3 apply and it is easy to compute the order of the two sets in relation (C.5). For the set on the left hand side we find that

$$\left|\frac{\mathrm{Rep}(\mathrm{G/H}) \times \mathrm{Rep}(\mathrm{H})}{\Gamma}\right| \;=\; \frac{|\mathrm{All}(\mathrm{G/H})| \cdot |\mathrm{Rep}(\mathrm{H})|}{|\Gamma| \cdot |\mathcal{G}_{\mathrm{id}}|} \;=\; \frac{|\mathrm{Rep}(\mathrm{G})| \cdot |\mathrm{Rep}(\mathrm{H})|}{|\mathcal{G}_{\mathrm{id}}|} \quad .$$

This agrees with the number of new boundary states on the right hand side of eq. (C.5). If we drop the assumption $\mathrm{Rep}(\mathrm{H}) = \mathcal{Z}(\mathrm{H})$ the action of Γ can have fixed points so that the number of unresolved branes is smaller than the number of branes we obtained from our construction.

To compare the open string spectra of the two sets of branes we have to go a step further and choose an explicit isomorphism between the labels. Let us propose

$$\Xi : \;\; \big[[\mu, b], c\big] \;\mapsto\; (\mu, b - c) \quad .$$

Note that $b - c \in \mathrm{Rep}(\mathrm{H})$ makes sense for two elements $b, c \in \mathrm{Rep}(\mathrm{H})$ since we assume $\mathrm{Rep}(\mathrm{H}) = \mathcal{Z}(\mathrm{H})$ to be an abelian group. Furthermore, Ξ is well-defined because the action of Γ on the labels $\big([\mu, b], c\big) \in \mathrm{Rep}(\mathrm{G/H}) \times \mathrm{Rep}(\mathrm{H})$ adds the same a to b and c so that their difference $b - c$ is left invariant. In writing down the pair $(\mu, b - c)$ we have to pick a representative (μ, b) of the sector $[\mu, b]$. This is unique up to the action of the identification group $\mathcal{G}_{\mathrm{id}}$. But different representatives are mapped to the same $\mathcal{G}_{\mathrm{id}}$-orbit in $\mathrm{Rep}(\mathrm{G}) \times \mathrm{Rep}(\mathrm{H})$. Obviously, Ξ is surjective and hence, by our counting above, it is a bijection between the two sets of labels.

It is now straightforward to compare the boundary partition function resulting from our construction with those arising from the orbifold analysis. In the following we shall identify the elements $\big([0,a],a\big) \in \Gamma$ with $a \in \mathcal{Z}(\mathrm{H})$. We first calculate the boundary partition function from the orbifold point of view. Using the formula (2.26) we obtain

$$Z^{\mathrm{orb}}_{[[\mu,b_1],c_1],[[\nu,b_2],c_2]} \;=\; \sum_{\substack{a\in\Gamma,(\sigma,d)\in\mathrm{All}(\mathrm{G/H})\\ e\in\mathrm{Rep}(\mathrm{H})}} N_{\mu\nu}^{\;\sigma}\; N_{b_1^+\,b_2}^{\;a+d}\; N_{c_1^+\,c_2}^{\;a+e}\; \chi^{\mathrm{G/H}}_{(\sigma,d)}\,\chi^{\mathrm{H}}_{e} \quad .$$

In this particular example, the fusion coefficients for the H-part are well-known and parts of the sum may be carried out. A careful calculation leads to

$$Z^{\mathrm{orb}}_{[[\mu,b_1],c_1],[[\nu,b_2],c_2]} = \sum_{(\sigma,d)\in\mathrm{All}(\mathrm{G/H})} N_{\mu\nu}^{\;\sigma}\,\chi^{\mathrm{G/H}}_{(\sigma,d)}\,\chi^{\mathrm{H}}_{d+c_2-c_1+b_1-b_2} \quad .$$

Let us now consider the boundary partition function for the corresponding weights (μ,b_1-c_1) and (ν,b_2-c_2) in our approach. Again, a careful analysis yields

$$\begin{aligned} Z_{(\mu,b_1-c_1),(\nu,b_2-c_2)} \;&=\; \sum_{\substack{(\sigma,d)\in\mathrm{All}(\mathrm{G/H})\\ e,f\in\mathrm{Rep}(\mathrm{H})}} N_{\mu\nu}^{\;\sigma}\; N_{(b_1-c_1)^+\,(b_2-c_2)}^{\;f}\; N_{fe}^{\;d}\; \chi^{\mathrm{G/H}}_{(\sigma,d)}\,\chi^{\mathrm{H}}_{e} \\ &=\; \sum_{(\sigma,d)\in\mathrm{All}(\mathrm{G/H})} N_{\mu\nu}^{\;\sigma}\,\chi^{\mathrm{G/H}}_{(\sigma,d)}\,\chi^{\mathrm{H}}_{d+c_2-c_1+b_1-b_2} \quad . \end{aligned}$$

This agrees with the result of the orbifold construction and thus proves the equivalence of the two approaches in the case of an *abelian* subgroup H. Apart from being more general because it allows to use *non-abelian* subgroups, our prescription also provides new conceptual insights and a more natural geometric interpretation.

C.4 An instructive example

For our general comparison of brane spectra in the previous subsection we assumed that H is abelian, i.e. that all sectors of $\mathcal{A}(\mathrm{H})$ are simple currents. This assumption was sufficient for the equivalence of the bulk partition functions but not necessary when the identification group $\mathcal{G}_{\mathrm{id}}$ is non-trivial. In this subsection we shall present one example for the latter case.

Let us set $\mathcal{A}(\mathrm{G}) = \mathcal{A}\big(\mathrm{SU}(2)_{k_1}\times\mathrm{SU}(2)_{k_2}\big)$. This chiral algebra has several subalgebras $\mathcal{A}(\mathrm{H})$ that we could choose for our construction of boundary states. There are various abelian subalgebras that we could use such as

$\mathcal{A}(\mathrm{H}) = \mathcal{A}\big(\mathrm{U}(1)_{k_1}\big)$ or $\mathcal{A}(\mathrm{H}) = \mathcal{A}\big(\mathrm{U}(1)_{k_1} \otimes \mathrm{U}(1)_{k_2}\big)$ etc. To make things a bit more interesting we shall pick a non-abelian subalgebra, namely the chiral algebra that is generated by the diagonally embedded subalgebra $\widehat{\mathfrak{su}}(2)_{k_1+k_2}$. The corresponding projection of weights is given by $\mathcal{P}(\mu,\alpha) = \mu+\alpha$. Sectors of the coset theory are labeled by triples (μ,α,a) with $\mu \leq k_1, \alpha \leq k_2$, $a \leq k_1+k_2$ and the branching selection rule $\mu+\alpha-a = 0 \mod 2$. One can show that there is only one non-trivial field identification current $(k_1,\, k_2,\, k_1+k_2)$. It gives rise to the field identification

$$(\mu,\alpha,a) \;\sim\; (k_1-\mu,\, k_2-\alpha,\, k_1+k_2-a) \quad .$$

Since we want to avoid fixed points of the field identification we have to consider the situation where at least one of the levels is odd.

We now specialize to the case $k_1 = k_2 = 1$ for which the coset algebra is the chiral algebra of the Ising model. The relevant lists of sectors are,

$$\begin{aligned} \mathrm{Rep}(\mathrm{G}) &= \mathrm{Rep}\big(\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_1\big) \;=\; \big\{(0,0),(0,1),(1,0),(1,1)\big\} \\ \mathrm{Rep}(\mathrm{H}) &= \mathrm{Rep}\big(\mathrm{SU}(2)_2\big) \;=\; \big\{0,1,2\big\} \\ \mathrm{Rep}(\mathrm{G/H}) &= \big\{(0,0,0)\sim(1,1,2),\,(0,0,2)\sim(1,1,0),\,(0,1,1)\sim(1,0,1)\big\} \;. \end{aligned}$$

Next, we have to decompose the charge conjugated modular invariant partition function for $\mathcal{A}\big(\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_1\big)$ into characters of the reduced chiral algebra. In our case this reads,

$$\begin{aligned} Z &= \Big|\chi^{\mathrm{G}}_{(0,0)}\Big|^2 + \Big|\chi^{\mathrm{G}}_{(0,1)}\Big|^2 + \Big|\chi^{\mathrm{G}}_{(1,0)}\Big|^2 + \Big|\chi^{\mathrm{G}}_{(1,1)}\Big|^2 \\ &= \Big|\chi^{\mathrm{G/H}}_{(0,0,0)}\chi^{\mathrm{H}}_0 + \chi^{\mathrm{G/H}}_{(0,0,2)}\chi^{\mathrm{H}}_2\Big|^2 + 2\Big|\chi^{\mathrm{G/H}}_{(0,1,1)}\chi^{\mathrm{H}}_1\Big|^2 + \Big|\chi^{\mathrm{G/H}}_{(0,0,2)}\chi^{\mathrm{H}}_0 + \chi^{\mathrm{G/H}}_{(0,0,0)}\chi^{\mathrm{H}}_2\Big|^2 \end{aligned} \tag{C.6}$$

where we already took the field identification into account. Following the results of section 3.2.3 for trivial gluing conditions on the reduced chiral algebra, we find six boundary states with labels from the set

$$\begin{aligned} \mathcal{B} = \big\{&(0,0,0)\sim(1,1,2),\,(1,0,0)\sim(0,1,2),\,(0,1,0)\sim(1,0,2), \\ &(1,1,0)\sim(0,0,2),\,(0,0,1)\sim(1,1,1),\,(1,0,1)\sim(0,1,1)\big\} \;. \end{aligned}$$

The four boundary states which are in the $\mathcal{G}_{\mathrm{id}}$-orbit of the labels $(\mu,\alpha,0)$ with trivial last entry can be identified with the four Cardy states of the model. All of them preserve the full chiral algebra $\mathcal{A}(\mathrm{G})$. For the remaining two boundary states we find

$$
\begin{aligned}
Z_{(0,0,1),(0,0,1)} &= \chi^{\mathrm{G/H}}_{(0,0,0)}\chi^{\mathrm{H}}_0 + \chi^{\mathrm{G/H}}_{(0,0,2)}\chi^{\mathrm{H}}_2 + \chi^{\mathrm{G/H}}_{(0,0,0)}\chi^{\mathrm{H}}_2 + \chi^{\mathrm{G/H}}_{(0,0,2)}\chi^{\mathrm{H}}_0 = \chi^{\mathrm{G}}_{(0,0)} + \chi^{\mathrm{G}}_{(1,1)} \\
Z_{(0,0,1),(1,0,1)} &= 2\chi^{\mathrm{G/H}}_{(0,1,1)}\chi^{\mathrm{H}}_1 \qquad\qquad\qquad\qquad\qquad\qquad\quad = \chi^{\mathrm{G}}_{(1,0)} + \chi^{\mathrm{G}}_{(0,1)} \\
Z_{(1,0,1),(1,0,1)} &= \chi^{\mathrm{G/H}}_{(0,0,0)}\chi^{\mathrm{H}}_0 + \chi^{\mathrm{G/H}}_{(0,0,2)}\chi^{\mathrm{H}}_2 + \chi^{\mathrm{G/H}}_{(0,0,0)}\chi^{\mathrm{H}}_2 + \chi^{\mathrm{G/H}}_{(0,0,2)}\chi^{\mathrm{H}}_0 = \chi^{\mathrm{G}}_{(0,0)} + \chi^{\mathrm{G}}_{(1,1)} \,.
\end{aligned}
$$

In particular, these boundary conditions preserve the full chiral symmetry! This is rather accidental and it is related to the fact that $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_1$ possesses an outer automorphism which acts by exchanging the two summands. With our construction we just recovered the two boundary states which belong to the associated twisted gluing condition. Note, however, that the spectra of open strings which stretch in between the four Cardy and the two non-Cardy type branes do only preserve the reduced chiral symmetry.

Before we conclude this section let us observe that the partition function (C.6) actually is an orbifold partition function obtained with the orbifold group

$$\Gamma_0 \;=\; \big\{\big([0,0,0],0\big),\big([0,0,2],2\big)\big\} \;\cong\; \mathbb{Z}_2 \quad .$$

In fact, the partition function of our model is recovered from the general expression (C.2) with the help of $Q_{([0,0,2],2)}\big(\big[[\mu,\alpha,a],b\big]\big) = (b-a)/2$ and using that the weight $\big[[0,1,1],1\big]$ is invariant under Γ_0. On the other hand, $\mathrm{H} = \mathrm{SU}(2)_2$ is not abelian since $(1) \in \mathrm{Rep}\big(\mathrm{SU}(2)_2\big)$ is not a simple current.

The orbifold group Γ_0 acts on the set $\mathrm{Rep}(\mathrm{G/H}) \times \mathrm{Rep}(\mathrm{H})$. Under this action, the nine elements of the latter are grouped into four orbits of length 2 and one fixed point. Hence, before resolving of the fixed point one obtains five boundary states of the form (C.3). But the one brane $|([0,1,1],1)\rangle$ which is associated with the fixed point of Γ_0 can be resolved into a sum of two elementary branes. In this way we recover all the six branes with symmetry $\mathcal{A}(\mathrm{G/H}) \otimes \mathcal{A}(\mathrm{H})$ from the orbifold construction. Note that in our approach the issue of fixed point resolution did not arise.

Appendix D

More on the Lagrangian approach

D.1 The computational details for symmetry breaking branes

In this appendix we will provide the computational details which have been omitted in section 3.4. It is convenient to choose a slightly more general framework and to generalize the notion of twisted conjugacy classes. We will then be able to include recent proposals of [136] in our description.

In the general approach we start with a family U_l ($l = 0,\dots,N$) of continuous subgroups $\mathrm{H} \hookrightarrow \mathrm{U}_l \hookrightarrow \mathrm{G}$ which do not necessarily satisfy the embedding chain property (3.3). To each of these subgroups we associate three embeddings $\epsilon^{\mathrm{HU}_l} : \mathrm{H} \to \mathrm{U}_l$ and $\epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L/R}} : \mathrm{U}_l \to \mathrm{G}$. The indices L/R stand for left and right, respectively. We then define *generalized* twisted conjugacy classes of U_l in G by (cf. reference [136])

$$\mathcal{C}^{\mathrm{U}_l,\mathrm{G}}_{\tilde{f}_l}\big(\epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L/R}}\big) \;=\; \big\{\, c_l \;=\; \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L}}(s_l)\,\tilde{f}_l\,\epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{R}}(s_l^{-1}) \,\big|\, s_l \in \mathrm{U}_l \big\} \quad . \tag{D.1}$$

These generalized twisted conjugacy classes admit an action of H under which they transform as

$$s_l \;\mapsto\; \epsilon^{\mathrm{HU}_l}(h)\,s_l \quad\Rightarrow\quad c_l \;\mapsto\; \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L}} \circ \epsilon^{\mathrm{HU}_l}\big(h\big)\,c_l\,\epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{R}} \circ \epsilon^{\mathrm{HU}_l}\big(h^{-1}\big) \quad . \tag{D.2}$$

By putting $\epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L}} = \epsilon^{\mathrm{U}_l\mathrm{G}}$, $\epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{R}} = \epsilon^{\mathrm{U}_l\mathrm{G}}_{\Omega} \circ \Omega_l$ and $\tilde{f}_l = \epsilon^{\mathrm{U}_l\mathrm{G}}(f_l)$ we can recover ordinary twisted conjugacy classes in this more general framework, i.e. the setup of section 3.2.

One can easily verify that the set $\mathcal{D}\big\{\mathrm{U}_l, \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L/R}}, \tilde{f}_l\big\}$ which is generated by the following product of generalized twisted conjugacy classes,

$$\mathcal{D}\big\{\mathrm{U}_l, \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L/R}}, \tilde{f}_l\big\} \;=\; \mathcal{C}^{\mathrm{U}_0,\mathrm{G}}_{\tilde{f}_0}\big(\epsilon^{\mathrm{U}_0\mathrm{G}}_{\mathrm{L/R}}\big) \cdot \;\cdots\; \cdot \mathcal{C}^{\mathrm{U}_N,\mathrm{G}}_{\tilde{f}_N}\big(\epsilon^{\mathrm{U}_N\mathrm{G}}_{\mathrm{L/R}}\big) \;\subset\; \mathrm{G} \quad , \tag{D.3}$$

is invariant under the action of H provided that the embedding maps satisfy the relations

$$\epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}} \circ \epsilon^{\mathrm{HU}_l} \;=\; \epsilon_{\mathrm{L}}^{\mathrm{U}_{l+1}\mathrm{G}} \circ \epsilon^{\mathrm{HU}_{l+1}} \quad . \tag{D.4}$$

For later purposes we also have to require the embedding indices of the left and right embeddings $\epsilon_{\mathrm{L/R}}^{\mathrm{U}_l\mathrm{G}}$ being identical for fixed subgroup U_l. Both conditions are automatically satisfied if we restrict to the original setup of section 3.2. As a consequence of eq. (D.4) the elements $x \in \mathcal{D}\big\{\mathrm{U}_l, \epsilon_{\mathrm{L/R}}^{\mathrm{U}_l\mathrm{G}}, \tilde{f}_l\big\}$ transform according to

$$x \;\mapsto\; \epsilon_{\mathrm{L}}^{\mathrm{U}_n\mathrm{G}} \circ \epsilon^{\mathrm{HU}_n}\big(h\big)\, x\, \epsilon_{\mathrm{R}}^{\mathrm{U}_1\mathrm{G}} \circ \epsilon^{\mathrm{HU}_1}\big(h^{-1}\big) \quad . \tag{D.5}$$

To write down the boundary WZNW functional for D-brane candidates which are localized along $\mathcal{D}\big\{\mathrm{U}_l, \epsilon_{\mathrm{L/R}}^{\mathrm{U}_l\mathrm{G}}, \tilde{f}_l\big\}$ we have to generalize the definition (3.33) to

$$\omega_{\mathcal{D}}(c_l) \;=\; \mathrm{tr}_R\big\{\epsilon_{\mathrm{L}}^{\mathrm{U}_l\mathrm{G}}\big(s_l^{-1}\mathrm{d}s_l\big)\, \tilde{f}_l\, \epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}}\big(s_l^{-1}\mathrm{d}s_l\big)\, \tilde{f}_l^{-1}\big\} \quad .$$

The other two-forms $\omega_{\mathcal{D}}(c_k, \cdots, c_l)$ are not modified, but one has to be aware of the fact that the quantities c_l now parametrize elements of the generalized twisted conjugacy classes (D.1).

In this appendix we will present the computational details for the proof that a) the boundary WZNW functional (3.29) is invariant under the infinitesimal action (D.2, D.5) of $h = 1 + i\,\omega \in \mathrm{H}$ on the boundary and b) that it is well-defined with respect to infinitesimal deformations of the disc D. We will, however, not be concerned with global issues which give rise to a quantization of generalized twisted conjugacy classes and branching selection rules. These global topological properties may possibly lead to severe restrictions which prohibit certain subsets of the form (D.3). For example we would not know how to model maximally symmetric D-branes which are localized along the product of two conjugacy classes $\mathcal{C}_{f_1}^{\mathrm{G}} \cdot \mathcal{C}_{f_2}^{\mathrm{G}}$ in the algebraic description. This example suggests that there also might be problems with the conformal invariance of the boundary WZNW functional (3.29) on the quantum level even if the theory is classically well-defined. A justification of the more general setting in terms of an algebraic description is therefore profoundly desirable. These questions, however, have to be addressed in future work.

Let us start with the first item, i.e. the invariance of the action functional (3.29) under transformations of the form (D.2, D.5) on the boundary. Refering to the discussion in section 3.4 this amounts to a proof of the relation $\delta\omega^{\mathrm{WZ}}\big|_D = \mathrm{d}\delta\omega_{\mathcal{D}}$. The elements of the twisted conjugacy classes transform according to

$$\delta c_l \;=\; i\,\omega_{\mathrm{L}}^{(l)}\, c_l \;-\; i\, c_l\, \omega_{\mathrm{R}}^{(l)} \quad ,$$

where we introduced the short hand notations $\omega^{(l)}_{\mathrm{L/R}} = \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L/R}} \circ \epsilon^{\mathrm{HU}_l}(\omega)$. The condition (D.4) of mutual consistency of the embedding maps obviously translates into the relation $\omega^{(l)}_{\mathrm{R}} = \omega^{(l+1)}_{\mathrm{L}}$. Supplied with this information it is now very easy to calculate all variations

$$\delta(c_k c_{k+1} \cdots c_{l-1} c_l) \;=\; i\,\omega^{(k)}_{\mathrm{L}}\, c_k c_{k+1} \cdots c_{l-1} c_l \;-\; i\, c_k c_{k+1} \cdots c_{l-1} c_l\, \omega^{(l)}_{\mathrm{R}} \quad .$$

Similar relations hold for the inverse $\delta c_l^{-1} = i\,\omega^{(l)}_{\mathrm{R}} c_l^{-1} - i\, c_l^{-1} \omega^{(l)}_{\mathrm{L}}$ and for chains of the form $c_l^{-1} c_{l-1}^{-1} \cdots c_{k+1}^{-1} c_k^{-1}$. Finally, we also need to know the variation

$$\delta \mathrm{d}c_l \;=\; \mathrm{d}\big(i\,\omega^{(l)}_{\mathrm{L}}\, c_l \;-\; i\, c_l\, \omega^{(l)}_{\mathrm{R}}\big) \;=\; i\, \mathrm{d}\omega^{(l)}_{\mathrm{L}}\, c_l \;-\; i\, c_l\, \mathrm{d}\omega^{(l)}_{\mathrm{R}} + i\,\omega^{(l)}_{\mathrm{L}}\, \mathrm{d}c_l \;-\; i\, \mathrm{d}c_l\, \omega^{(l)}_{\mathrm{R}} \quad .$$

Due to these relations the transformation properties of $\omega_{\mathcal{D}}(c_k, \cdots, c_l)$ may easily be calculated. It turns out that all terms involving ω cancel each other. Only four terms involving $\mathrm{d}\omega$ survive. We summarize this result in

$$\begin{aligned}
\delta\omega_{\mathcal{D}}(c_k, \cdots, c_l) \;=\; & -i\, \mathrm{tr}\big\{ c_l^{-1} \cdots c_k^{-1}\, \mathrm{d}\omega^{(k)}_{\mathrm{L}}\, c_k \cdots c_{l-1} \mathrm{d}c_l \big\} \\
& + i\, \mathrm{tr}\big\{ c_l^{-1} \cdots c_{k+1}^{-1}\, \mathrm{d}\omega^{(k+1)}_{\mathrm{L}}\, c_{k+1} \cdots c_{l-1} \mathrm{d}c_l \big\} \\
& - i\, \mathrm{tr}\big\{ c_{l-1}^{-1} \cdots c_k^{-1} \mathrm{d}c_k c_{k+1} \cdots c_{l-1}\, \mathrm{d}\omega^{(l-1)}_{\mathrm{R}} \big\} \\
& + i\, \mathrm{tr}\big\{ c_l^{-1} \cdots c_k^{-1} \mathrm{d}c_k c_{k+1} \cdots c_l\, \mathrm{d}\omega^{(l)}_{\mathrm{R}} \big\} \quad .
\end{aligned}$$

When evaluating this expression special care has to be taken if $l = k+1$. In this case no factors $c_{k+1} \cdots c_{l-1}$ appear between the differentials in lines two and three.

Due to its different structure the variation of $\omega_{\mathcal{D}}(c_l)$ has to be treated separately. In this case we obtain

$$\delta\omega_{\mathcal{D}}(c_l) \;=\; -i\, \mathrm{tr}\big\{ \mathrm{d}\omega^{(l)}_{\mathrm{L}} \mathrm{d}c_l c_l^{-1} + \mathrm{d}\omega^{(l)}_{\mathrm{R}} c_l^{-1} \mathrm{d}c_l \big\} \quad .$$

During the calculation we made use of

$$\begin{aligned}
c_l^{-1}\mathrm{d}c_l \;&=\; \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{R}}\big(s_l\big)\, \tilde{f}_l^{-1}\, \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L}}\big(s_l^{-1}\mathrm{d}s_l\big)\, \tilde{f}_l\, \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{R}}\big(s_l^{-1}\big) - \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{R}}\big(\mathrm{d}s_l s_l^{-1}\big) \\
\mathrm{d}c_l c_l^{-1} \;&=\; \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L}}\big(\mathrm{d}s_l s_l^{-1}\big) - \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L}}\big(s_l\big)\, \tilde{f}_l\, \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{R}}\big(s_l^{-1}\mathrm{d}s_l\big)\, \tilde{f}_l^{-1}\, \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L}}\big(s_l^{-1}\big) \quad .
\end{aligned}$$

Indeed, these two relations imply

$$\begin{aligned}
& i\, \mathrm{tr}\big\{ \mathrm{d}\omega^{(l)}_{\mathrm{L}} \mathrm{d}c_l c_l^{-1} + \mathrm{d}\omega^{(l)}_{\mathrm{R}} c_l^{-1} \mathrm{d}c_l \big\} \\
& \qquad = \; -\delta\omega_{\mathcal{D}}(c_l) + \; i\, \mathrm{tr}\big\{ \mathrm{d}\omega^{(l)}_{\mathrm{L}} \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{L}}\big(\mathrm{d}s_l s_l^{-1}\big) - \mathrm{d}\omega^{(l)}_{\mathrm{R}} \epsilon^{\mathrm{U}_l\mathrm{G}}_{\mathrm{R}}\big(\mathrm{d}s_l s_l^{-1}\big) \big\} \quad .
\end{aligned}$$

If we rewrite the last term according to

$$\mathrm{tr}\big\{\epsilon_{\mathrm{L}}^{\mathrm{U}_l\mathrm{G}}\big(\epsilon^{\mathrm{HU}_l}(\mathrm{d}\omega)\mathrm{d}s_l s_l^{-1}\big) - \epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}}\big(\epsilon^{\mathrm{HU}_l}(\mathrm{d}\omega)\mathrm{d}s_l s_l^{-1}\big)\big\}$$

we see that it vanishes provided the two embeddings $\epsilon_{\mathrm{L/R}}^{\mathrm{U}_l\mathrm{G}}$ have the same embedding index.

Summing up all contributions and remembering that the variation of $\omega_{\mathcal{D}}(c_k, c_{k+1})$ shows some subtleties we obtain

$$\delta\omega_{\mathcal{D}} = -i\sum_{l=0}^{N}\mathrm{tr}\big\{\mathrm{d}\omega_{\mathrm{L}}^{(0)} c_0\cdots c_{l-1}\mathrm{d}c_l c_l^{-1}\cdots c_0^{-1} + \mathrm{d}\omega_{\mathrm{R}}^{(N)} c_N^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_N\big\}.$$

During the calculation we made use of several cancellations. Finally, we have to compare this expression with the variation of the Wess-Zumino term. A careful calculation gives

$$\delta\omega^{\mathrm{WZ}} \;=\; -i\,\mathrm{d}\,\mathrm{tr}\big\{\mathrm{d}\omega_{\mathrm{L}}^{(0)}\mathrm{d}g g^{-1} + \mathrm{d}\omega_{\mathrm{R}}^{(N)} g^{-1}\mathrm{d}g\big\} \quad .$$

This may easily be evaluated using the relations

$$\begin{aligned} g^{-1}\mathrm{d}g \;&=\; \sum_{l=0}^{N} c_N^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_N \\ \mathrm{d}g g^{-1} \;&=\; \sum_{l=0}^{N} c_0\cdots c_{l-1}\mathrm{d}c_l c_l^{-1}\cdots c_0^{-1} \quad . \end{aligned}$$

The variation then reads

$$\begin{aligned} \delta\omega^{\mathrm{WZ}} \;=\; -i\sum_{l=0}^{N}\mathrm{d}\,\mathrm{tr}\big\{&\mathrm{d}\omega_{\mathrm{L}}^{(0)} c_0\cdots c_{l-1}\mathrm{d}c_l c_l^{-1}\cdots c_0^{-1} + \\ &+\; \mathrm{d}\omega_{\mathrm{R}}^{(N)} c_N^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_N\big\} \quad . \end{aligned}$$

Obviously, the contributions from $\delta\omega^{\mathrm{WZ}}$ and $\delta\omega_{\mathcal{D}}$ cancel each other exactly. This proves that the product of generalized twisted conjugacy classes is indeed a valid candidate for the geometry of D-branes which preserve an action of the group H.

Now we are able to address item b), i.e. the invariance of the action functional (3.29) under infinitesimal deformations of the disc D. It is sufficient to proof the relation $\mathrm{d}\omega_{\mathcal{D}} = \omega^{\mathrm{WZ}}\big|_D$. The calculation turns out to be very involved if one tries to perform it directly. Therefore it is convenient to use an induction argument instead, i.e. we supply the boundary two-form (3.32) with an additional label N and write $\omega_{\mathcal{D}}(N)$. The number $N+1$ is just the number of generalized twisted conjugacy classes appearing in eq. (D.3). For $N=0$ we have $\omega_{\mathcal{D}}(0)=\omega_{\mathcal{D}}(c_0)$. Let us thus first determine

$$\begin{aligned}\mathrm{d}\omega_{\mathcal{D}}(c_l) &= -\,\mathrm{tr}\big\{\epsilon_{\mathrm{L}}^{\mathrm{U}_l\mathrm{G}}(s_l^{-1}\mathrm{d}s_l s_l^{-1}\mathrm{d}s_l)\,\tilde f_l\,\epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}}(s_l^{-1}\mathrm{d}s_l)\,\tilde f_l^{-1}\big\}\\ &\quad +\mathrm{tr}\big\{\epsilon_{\mathrm{L}}^{\mathrm{U}_l\mathrm{G}}(s_l^{-1}\mathrm{d}s_l)\,\tilde f_l\,\epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}}(s_l^{-1}\mathrm{d}s_l s_l^{-1}\mathrm{d}s_l)\,\tilde f_l^{-1}\big\}\quad .\end{aligned}$$

On the other hand we have

$$\begin{aligned}\omega^{\mathrm{WZ}}(c_l) &= \frac{1}{3}\,\mathrm{tr}\big\{\big[\epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}}(s_l)\,\tilde f_l^{-1}\,\epsilon_{\mathrm{L}}^{\mathrm{U}_l\mathrm{G}}(s_l^{-1}\mathrm{d}s_l)\,\tilde f_l\,\epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}}(s_l^{-1})-\epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}}(\mathrm{d}s_l s_l^{-1})\big]^3\big\}\\ &= \frac{1}{3}\,\mathrm{tr}\big\{\epsilon_{\mathrm{L}}^{\mathrm{U}_l\mathrm{G}}\big((s_l^{-1}\mathrm{d}s_l)^3\big)-\epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}}\big((\mathrm{d}s_l s_l^{-1})^3\big)\big\}\\ &\quad -\mathrm{tr}\big\{\tilde f_l^{-1}\,\epsilon_{\mathrm{L}}^{\mathrm{U}_l\mathrm{G}}\big(s_l^{-1}\mathrm{d}s_l s_l^{-1}\mathrm{d}s_l\big)\,\tilde f_l\,\epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}}\big(s_l^{-1}\mathrm{d}s_l\big)\big\}\\ &\quad +\mathrm{tr}\big\{\,\tilde f_l^{-1}\,\epsilon_{\mathrm{L}}^{\mathrm{U}_l\mathrm{G}}\big(s_l^{-1}\mathrm{d}s_l\big)\,\tilde f_l\,\epsilon_{\mathrm{R}}^{\mathrm{U}_l\mathrm{G}}\big(s_l^{-1}\mathrm{d}s_l s_l^{-1}\mathrm{d}s_l\big)\big\}\quad .\end{aligned}$$

The first two terms vanish as the two embeddings by assumption have the same embedding index. By specializing to $l=0$ we have proven $\mathrm{d}\omega_{\mathcal{D}}(N)=\omega^{\mathrm{WZ}}\big|_D$ for $N=0$.

Let us now turn to the case $N>0$. It is convenient to introduce the notation $g_N=c_0\cdots c_N=g_{N-1}\,c_N$. In addition we also need the recursion property $\omega_{\mathcal{D}}(N)=\omega_{\mathcal{D}}(N-1)+\sum_{l=0}^{N}\omega_{\mathcal{D}}(c_l,\cdots,c_N)$. Using the representation above we easily obtain $g_N^{-1}\mathrm{d}g_N=c_N^{-1}\mathrm{d}c_N+c_N^{-1}g_{N-1}^{-1}\mathrm{d}g_{N-1}c_N$ and thus we are able to calculate

$$\begin{aligned}\omega^{\mathrm{WZ}}(g_N) &= \omega^{\mathrm{WZ}}(g_{N-1})\;+\;\omega^{\mathrm{WZ}}(c_N)\\ &\quad +\;\mathrm{tr}\big\{(\mathrm{d}c_N c_N^{-1})^2 g_{N-1}^{-1}\mathrm{d}g_{N-1}+\;\mathrm{d}c_N c_N^{-1}(g_{N-1}^{-1}\mathrm{d}g_{N-1})^2\big\}\quad .\end{aligned}$$

By induction we have $\mathrm{d}\omega_{\mathcal{D}}(N-1)=\omega^{\mathrm{WZ}}(g_{N-1})$. We also proved already that $\mathrm{d}\omega(c_N)=\omega^{\mathrm{WZ}}(c_N)$. It thus remains to check whether

$$\sum_{l=0}^{N-1}\mathrm{d}\omega_{\mathcal{D}}(c_l,\cdots,c_N)\;=\;\mathrm{tr}\big\{(\mathrm{d}c_N c_N^{-1})^2 g_{N-1}^{-1}\mathrm{d}g_{N-1}+\mathrm{d}c_N c_N^{-1}(g_{N-1}^{-1}\mathrm{d}g_{N-1})^2\big\}\quad . \tag{D.6}$$

Indeed, for the left hand side we find

$$\begin{aligned}
\sum_{l=0}^{N-1} \mathrm{d}\omega_{\mathcal{D}}(c_l, \cdots, c_N) &= \sum_{l=0}^{N-1} \mathrm{tr}\big\{c_N^{-1}\mathrm{d}c_N c_N^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_{N-1}\mathrm{d}c_N\big\} \\
&+\sum_{l=0}^{N-1}\sum_{k=l}^{N-1} \mathrm{tr}\big\{c_N^{-1}\cdots c_k^{-1}\mathrm{d}c_k c_k^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_{N-1}\mathrm{d}c_N\big\} \\
&+\sum_{l=0}^{N-1}\sum_{k=l+1}^{N-1} \mathrm{tr}\big\{c_N^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_{k-1}\mathrm{d}c_k c_{k+1}\cdots c_{N-1}\mathrm{d}c_N\big\} \quad . \qquad \text{(D.7)}
\end{aligned}$$

To evaluate the right hand side of eq. (D.6) we use the explicit form of g_{N-1} as a product of c's and write

$$\begin{aligned}
g_{N-1}^{-1}\mathrm{d}g_{N-1} &= \sum_{l=0}^{N-1} c_{N-1}^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_{N-1} \\
\mathrm{d}g_{N-1}g_{N-1}^{-1} &= \sum_{l=0}^{N-1} c_0\cdots c_{l-1}\mathrm{d}c_l c_l^{-1}\cdots c_0^{-1} \quad .
\end{aligned}$$

Taking the square of the first expression we arrive at

$$\begin{aligned}
(g_{N-1}^{-1}\mathrm{d}g_{N-1})^2 &= \sum_{l=0}^{N-1}\sum_{k=0}^{N-1} c_{N-1}^{-1}\cdots c_l^{-1}\mathrm{d}c_l\cdots c_{N-1}c_{N-1}^{-1}\cdots \mathrm{d}c_k c_{k+1}\cdots c_{N-1} \\
&= \sum_{l=0}^{N-1}\sum_{k=l}^{N-1} c_{N-1}^{-1}\cdots c_k^{-1}\mathrm{d}c_k c_k^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_{N-1} \\
&\quad +\sum_{l=0}^{N-1}\sum_{k=l+1}^{N-1} c_{N-1}^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_{k-1}\mathrm{d}c_k c_{k+1}\cdots c_{N-1} \quad .
\end{aligned}$$

Plugging all this into the right hand side of eq. (D.6) we finally find that

$$\begin{aligned}
&\mathrm{tr}\big\{(\mathrm{d}c_N c_N^{-1})^2 g_{N-1}^{-1}\mathrm{d}g_{N-1} \,+\, \mathrm{d}c_N c_N^{-1}(g_{N-1}^{-1}\mathrm{d}g_{N-1})^2\big\} \\
= &\sum_{l=0}^{N-1} \mathrm{tr}\big\{\mathrm{d}c_N c_N^{-1}\mathrm{d}c_N c_N^{-1} c_{N-1}^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_{N-1}\big\} \\
&+\sum_{l=0}^{N-1}\sum_{k=l}^{N-1} \mathrm{tr}\big\{\mathrm{d}c_N c_N^{-1} c_{N-1}^{-1}\cdots c_k^{-1}\mathrm{d}c_k c_k^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_{N-1}\big\} \\
&+\sum_{l=0}^{N-1}\sum_{k=l+1}^{N-1} \mathrm{tr}\big\{\mathrm{d}c_N c_N^{-1} c_{N-1}^{-1}\cdots c_l^{-1}\mathrm{d}c_l c_{l+1}\cdots c_{k-1}\mathrm{d}c_k c_{k+1}\cdots c_{N-1}\big\} \quad .
\end{aligned}$$

This expression coincides with the expression (D.7) and so our induction argument is completed.

D.2 The GMM cosets revisited

In this appendix we would like to discuss further aspects of the GMM models which have been introduced in section 4.1.4. They first appeared in reference [51, 134]. These authors presented a Lagrangian formulation of these theories and considered the associated current algebra. In our opinion their discussion of the algebraic properties is not completely accurate. In particular, they argued that the energy momentum tensor is *not* obtained by the standard affine Sugawara [109] and coset constructions [47]. This statement seems to be incorrect. We take this as an opportunity to review the Lagrangian description and to correct some of their formulas.

The gauged WZNW functional (4.4) is quadratic in the gauge fields. It may thus be simplified – in principle – by integrating out the gauge fields. The resulting expressions will, however, remain quite formal in the general case (see however [41, 42, 43, 44, 182]). The reason for these difficulties is the third term in the interaction functional (4.5) which does not only contain the gauge fields A and $\bar{A}$ but also the group element g. For the Gaussian path integral to be performed one would need to diagonalize the quadratic form matrix which depends explicitly on g.

For our particular choice of embeddings the corresponding term vanishes and the path integral may easily be evaluated. The interaction functional (4.5) reduces to

$$\begin{aligned} S_{\text{int}}^{\text{G}_1\times\text{G}_2/\text{H}}&(g_1, g_2, A, \bar{A}|k, \epsilon_{\text{L/R}}) \\ &= \frac{k_1}{4\pi}\frac{2}{I_1}\int_\Sigma d^2z\ \text{tr}_1\big\{-2\epsilon_1(A)\, g_1^{-1}\bar{\partial} g_1 - \epsilon_1(\bar{A})\,\epsilon_1(A)\big\} \\ &\qquad + \frac{k_2}{4\pi}\frac{2}{I_2}\int_\Sigma d^2z\ \text{tr}_2\big\{2\epsilon_2(\bar{A})\,\partial g_2 g_2^{-1} - \epsilon_2(\bar{A})\,\epsilon_2(A)\big\}\quad . \end{aligned}$$

It is fairly simple to read off the quadratic form matrix from this expression and integrate out the gauge fields in full generality. We only have to be a bit careful about our notations.

We may decompose the $\mathfrak{h}$-valued gauge fields A and $\bar{A}$ according to $A = A_\alpha T^\alpha$ and $\bar{A} = \bar{A}_\alpha T^\alpha$. The *abstract* Lie algebra generators satisfy the commutation relations $[T^\alpha, T^\beta] = i f^{\alpha\beta}{}_\gamma T^\gamma$. Indices are raised and lowered using the Killing form $\kappa^{\alpha\beta}$ and its inverse, see eq. (A.1). We may choose generators $T^i \in \{\epsilon_1(T^\alpha), T^I\}$ of $\mathfrak{g}_1$ and generators $T^a \in \{\epsilon_2(T^\alpha), T^A\}$ of $\mathfrak{g}_2$. These satisfy $[T^i, T^j] = if^{ij}{}_k T^k$ and $[T^a, T^b] = if^{ab}{}_c T^c$. If all three indices take values in the subalgebra $\mathfrak{h}$, the structure constants by construction just reduce to the structure constants of $\mathfrak{h}$ in the given basis. This is only true as long as the index structure is as indicated because one would have to use different Killing forms to lower the indices. From (4.1) it follows that these forms are related by

$$\kappa^{\alpha\beta} \;=\; \kappa_1^{\alpha\beta}/x_1 \;=\; \kappa_2^{\alpha\beta}/x_2 \quad .$$

We see the embedding indices x_i entering this expression.

The last relations imply

$$\mathrm{tr}_1\big\{\epsilon_1(\bar{A})\,\epsilon_1(A)\big\} \;=\; I_1\,x_1\,\bar{A}_\alpha A^\alpha \quad , \quad \mathrm{tr}_2\big\{\epsilon_2(\bar{A})\,\epsilon_2(A)\big\} \;=\; I_2\,x_2\,\bar{A}_\alpha A^\alpha \quad .$$

The formula

$$\int d^n y\, e^{-\frac{1}{2}y^T \mathbb{A} y + b^T y} \;=\; \frac{(2\pi)^{n/2}}{\sqrt{\det \mathbb{A}}}\, e^{\frac{1}{2}\,b^T \mathbb{A}^{-1} b} \quad .$$

for the Gaussian path integral may thus be applied using the parameters

$$y \;=\; \begin{pmatrix} A^\alpha \\ \bar{A}^\beta \end{pmatrix} \qquad\qquad b \;=\; \begin{pmatrix} -\frac{k_1}{\pi I_1}\,\mathrm{tr}_1\big\{\epsilon_1(T^\alpha) g_1^{-1}\bar{\partial} g_1\big\} \\ \frac{k_2}{\pi I_2}\,\mathrm{tr}_2\big\{\epsilon_2(T^\beta)\partial g_2 g_2^{-1}\big\} \end{pmatrix}$$

and

$$\mathbb{A} \;=\; \begin{pmatrix} 0 & \frac{x_1 k_1}{\pi}\,\kappa_{\alpha\beta} \\ \frac{x_2 k_2}{\pi}\,\kappa_{\beta\alpha} & 0 \end{pmatrix} \quad .$$

The matrix $\mathbb{A}$ is symmetric as by assumption $k = x_1 k_1 = x_2 k_2$, see eq. (4.2), and may easily be inverted. After performing the Gaussian path integral the interaction term reads

$$\begin{aligned} &S_{\mathrm{int}}^{\mathrm{G}_1\times\mathrm{G}_2/\mathrm{H}}(g_1, g_2|\,k, \epsilon_{\mathrm{L/R}}) \\ &\quad = \; -\frac{\kappa_{\alpha\beta}\,k_1 k_2}{4\pi\,k}\,\frac{4}{I_1 I_2}\int_\Sigma d^2 z\;\mathrm{tr}_1\big\{\epsilon_1(T^\alpha)\,g_1^{-1}\bar{\partial} g_1\big\}\,\mathrm{tr}_2\big\{\epsilon_2(T^\beta)\,\partial g_2 g_2^{-1}\big\} \end{aligned} \tag{D.8}$$

This is exactly the expression that has been suggested in [51, 134].

The action functional (D.8) possesses a number of very interesting and useful symmetries. By construction it is invariant under the infinitesimal gauge transformations $(g_1, g_2) \mapsto \big(g_1(1 - i\,\epsilon_1(\omega)), (1 + i\,\epsilon_2(\omega))g_2\big)$ with $\omega = \omega(z,\bar{z}) \in \mathfrak{h}$. In addition, the model admits the symmetry $\mathrm{G}_1^{\mathrm{L}}(z) \times \mathrm{G}_2^{\mathrm{R}}(\bar{z})$, i.e.

it is invariant under $(g_1, g_2) \mapsto \big(g_1'(z)\, g_1, g_2\, g_2'^{-1}(\bar z)\big)$. The last symmetry is generated by the currents

$$\begin{aligned} J(z) &= -k_1\, \partial g_1 g_1^{-1} - \frac{k_2\, \kappa_{\alpha\beta}}{I_2\, x_1}\; g_1 \epsilon_1(T^\alpha) g_1^{-1}\; \mathrm{tr}_2\big\{\epsilon_2(T^\beta)\, \partial g_2 g_2^{-1}\big\} \\ \bar J(\bar z) &= k_2\, g_2^{-1} \bar\partial g_2 + \frac{k_1\, \kappa_{\alpha\beta}}{I_1\, x_2}\; g_2^{-1} \epsilon_2(T^\beta) g_2\; \mathrm{tr}_1\big\{\epsilon_1(T^\alpha)\, g_1^{-1} \bar\partial g_1\big\} \quad , \end{aligned}$$

which satisfy $\bar\partial J(z) = \partial \bar J(\bar z) = 0$ by the equations of motion. During the derivation we used the relation $x_1\, k_1 = x_2\, k_2$. Note that J takes values in the Lie algebra $\mathfrak{g}_1$, while $\bar J$ is from $\mathfrak{g}_2$. So the index structure is J^i, $\bar J^a$ which makes explicit the heterotic nature of our coset. Both currents are gauge invariant.

In the algebraic description of our asymmetric coset model we already took some properties for granted which would have been expected from a straightforward generalization of the GKO construction. We are now able to justify this procedure more rigorously by working out the energy momentum tensor and the commutation relations of the currents. Let us start with the latter. It is convenient to introduce the fields

$$\begin{aligned} J_1 &= -k_1\, \partial g_1 g_1^{-1} & \qquad \bar J_1 &= k_1\, g_1^{-1} \bar\partial g_1 \\ J_2 &= -k_2\, \partial g_2 g_2^{-1} & \qquad \bar J_2 &= k_2\, g_2^{-1} \bar\partial g_2 \end{aligned}$$

which correspond to the (former) G_1 and G_2 currents, respectively. In terms of these quantities one obtains

$$\begin{aligned} J(z) &= J_1 + \frac{\kappa_{\alpha\beta}}{I_2\, x_1}\; g_1 \epsilon_1(T^\alpha) g_1^{-1}\; \mathrm{tr}_2\big\{\epsilon_2(T^\beta) J_2\big\} \\ \bar J(\bar z) &= \bar J_2 + \frac{\kappa_{\alpha\beta}}{I_1\, x_2}\; g_2^{-1} \epsilon_2(T^\beta) g_2\; \mathrm{tr}_1\big\{\epsilon_1(T^\alpha) \bar J_1\big\} \quad , \end{aligned}$$

which shows how the gauge interaction term mixes the original currents.

The symmetry $\mathrm{G}_1^{\mathrm{L}}(z) \times \mathrm{G}_2^{\mathrm{R}}(\bar z)$ implies the Ward identities [110, (15.40)]

$$\begin{aligned} \delta_{\mathrm{L}}^{(1)} \langle X(w,\bar w)\rangle &= -\oint \frac{dz}{2\pi i}\; \omega_i \langle J^i(z) X(w,\bar w)\rangle \\ \delta_{\mathrm{R}}^{(2)} \langle X(w,\bar w)\rangle &= \oint \frac{d\bar z}{2\pi i}\; \omega_a \langle \bar J^a(\bar z) X(w,\bar w)\rangle \quad , \end{aligned}$$

which are related to the transformations $\delta_L^{(1)} g_1 = i\, \omega_i T^i\, g_1$ and $\delta_R^{(2)} g_2 =$

$-i\, g_2\, \omega_a T^a$. From the previous equations we may derive the non-trivial OPE's

$$\begin{aligned} J^i(z)\, J^j(w) &= \frac{i f^{ij}{}_k}{z-w}\; J^k(w) + \frac{k_1\, \kappa_1^{ij}}{(z-w)^2} \\ J^i(z)\, g_1(w,\bar{w}) &= -\frac{T^i\, g_1(w,\bar{w})}{z-w} \\ J^i(z)\, J_1^j(w,\bar{w}) &= \frac{i f^{ij}{}_k}{z-w}\; J_1^k(w,\bar{w}) + \frac{k_1\, \kappa_1^{ij}}{(z-w)^2} \quad . \end{aligned}$$

and

$$\begin{aligned} \bar{J}^a(\bar{z})\, \bar{J}^b(\bar{w}) &= \frac{i f^{ab}{}_c}{\bar{z}-\bar{w}}\; \bar{J}^c(\bar{w}) + \frac{k_2\, \kappa_2^{ab}}{(\bar{z}-\bar{w})^2} \\ \bar{J}^a(\bar{z})\, g_2(w,\bar{w}) &= \frac{g_2(w,\bar{w})\, T^a}{\bar{z}-\bar{w}} \\ \bar{J}^a(\bar{z})\, \bar{J}_2^b(w,\bar{w}) &= \frac{i f^{ab}{}_c}{\bar{z}-\bar{w}}\; \bar{J}^c(\bar{w}) + \frac{k_2\, \kappa_2^{ab}}{(\bar{z}-\bar{w})^2} \quad . \end{aligned}$$

All the remaining OPE's between the currents and the fields g_1 and g_2 vanish. Let us emphasize the asymmetry in the OPE's which already showed up in the algebraic construction.

Now, as the current symmetry is under control we can focus our attention on the conformal symmetry, i.e. on the energy momentum tensor. Due to the structure of the action functional for the asymmetric coset, the *classical* chiral energy momentum tensors are given by

$$\mathrm{T} \;=\; \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_{\mathrm{int}} \qquad \text{and} \qquad \bar{\mathrm{T}} \;=\; \bar{\mathrm{T}}_1 + \bar{\mathrm{T}}_2 + \bar{\mathrm{T}}_{\mathrm{int}} \quad .$$

The first two summands are the standard WZNW energy momentum tensors

$$\begin{aligned} \mathrm{T}_1 &= \frac{1}{2\,k_1\, I_1}\; \mathrm{tr}_1\big\{J_1 J_1\big\} & \bar{\mathrm{T}}_1 &= \frac{1}{2\,k_1\, I_1}\; \mathrm{tr}_1\big\{\bar{J}_1 \bar{J}_1\big\} \\ \mathrm{T}_2 &= \frac{1}{2\,k_2\, I_2}\; \mathrm{tr}_2\big\{J_2 J_2\big\} & \bar{\mathrm{T}}_2 &= \frac{1}{2\,k_2\, I_2}\; \mathrm{tr}_2\big\{\bar{J}_2 \bar{J}_2\big\} \quad . \end{aligned}$$

The extra summands are given by

$$\begin{aligned} \mathrm{T}_{\mathrm{int}} &= \frac{k_1\ \kappa_{\alpha\beta}}{x_2\, I_1\, I_2}\; \mathrm{tr}_1\big\{\epsilon_1(T^\alpha)\, g_1^{-1}\partial g_1\big\}\; \mathrm{tr}_2\big\{\epsilon_2(T^\beta)\, \partial g_2 g_2^{-1}\big\} \\ \bar{\mathrm{T}}_{\mathrm{int}} &= \frac{k_1\ \kappa_{\alpha\beta}}{x_2\, I_1\, I_2}\; \mathrm{tr}_1\big\{\epsilon_1(T^\alpha)\, g_1^{-1}\bar{\partial} g_1\big\}\; \mathrm{tr}_2\big\{\epsilon_2(T^\beta)\, \bar{\partial} g_2 g_2^{-1}\big\} \quad . \end{aligned}$$

It is very instructive to evaluate the expressions $\mathrm{tr}_1 JJ$ and $\mathrm{tr}_2 \bar{J}\bar{J}$. One is then naturally lead to

$$\begin{aligned} \mathrm{T} &= \frac{1}{2k_1}\, J_i J^i + \frac{1}{2k_2}\,(J_2)_a (J_2)^a - \frac{1}{2x_2 k_2}\,(J_2)_\alpha (J_2)^\alpha \;=\; \mathrm{T}^{\mathrm{G}_1}_{k_1} + \mathrm{T}^{\mathrm{G}_2}_{k_2} - \mathrm{T}^{\mathrm{H}}_{x_2 k_2} \\ \bar{\mathrm{T}} &= \frac{1}{2k_2}\, \bar{J}_a \bar{J}^a + \frac{1}{2k_1}\,(\bar{J}_1)_i (\bar{J}_1)^i - \frac{1}{2x_1 k_1}\,(\bar{J}_1)_\alpha (\bar{J}_1)^\alpha \;=\; \bar{\mathrm{T}}^{\mathrm{G}_1}_{k_1} + \bar{\mathrm{T}}^{\mathrm{G}_2}_{k_2} - \bar{\mathrm{T}}^{\mathrm{H}}_{x_1 k_1} \quad . \end{aligned}$$

The additional factors x_1 and x_2 arise due to the usage of the natural Killing form for $\mathfrak{h}$-quantities. After quantizing the theory the levels get shifted by the respective dual Coxeter numbers. Let us emphasize the following remarkable fact: Due to the condition $x_1\, k_1 \;=\; x_2\, k_2$ left and right moving Virasoro algebra possess the same central charge. This result also has been noted in [51, 134] but the algebraic reasons remained unclear. In particular in the last reference due to usage of inappropriate notation it was not realized that the energy momentum tensor is actually defined by a combination of the standard affine Sugawara and coset constructions.

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Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt zu haben und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Thomas Quella 10.3.2003