Advanced Studies in Theoretical Physics
Vol. 13, 2019, no. 3, 103 - 126
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/astp.2019.912

Geometric Engineering of Qubits from String Theory

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Abstract

Combining colored toric geometry and Lie symmetries, we engineer qubit systems in the context of the D-brane physics in type II superstrings. Concretely, we establish a correspondence between such quantum systems and a class of K3 isolated singularities using operation techniques of graph theory. We first analyze 1 andf 2-qubits in some details and show that they are associated with the geometric engineering of six dimensional gauge theories obtained from type IIA superstring in the presence of D2-branes probing one and two $su(2)$ singularities of the K3 surface, respectively. Using a possible factorization of vector fields, we reveal that the corresponding gauge symmetry breaking provides states of such qubit systems. After that, we discuss the corresponding entanglement. Applying graph theory operations to colored toric polyvalent geometry, we then investigate multi-qubits in terms of D2-branes probing several $su(2)$ isolated singularities of the K3 surface. The gauge field factorization generates abelian toric manifolds interpreted as Cartan sub-symmetries.

Subject Classifications: 32C35, 58A14, 81T30, 81P45, 83C57

Keyswords: String theory; Quantum information theory; Graph theory; Toric geometry; Mirror symmetry, Calabi-Yau singularities; Lie symmetries

1 Introduction

Recently, it has been remarked that quantum information theory (QIT) has received many interests mainly in relation with black hole physics and entanglement entropy [1, 2, 3]. In particular, qubit systems have been extensively investigated using different approaches like superstring and graph theories [4-16]. More precisely, a nice correspondence between the stringy black holes and such systems have been established by considering the compactification scenario. In this way, the supersymmetric STU black hole embedded in the II superstrings has been related to 3-qubits by exploiting hyperdeterminant calcultions [8, 9, 10]. This black hole/qubit correspondence has been enriched by many generalizations including superqubits using supermanifold computations[13]. Among others, works based on toric manifolds has been developed leading to a classification of qubit systems in terms of black hole charges in type II superstrings in the presence of D-brane objects[16]. Moreover, some studies have been proposed by using the Andinkra graph theory being exploited in the study of the supersymmetric representation theory [13, 14, 15, 16, 17]. In particular, these graphs have been elaborated to classify a class of qubits linked to extremal black branes in type II superstring compactified on complex manifolds [13]. Motivated by such results, colored toric graphs of a product of $\mathbb{CP}^1$ projective spaces have been also used to deal with concepts of QIT associated with logic gates [15]. Alternatively, a geometric method based on simple singularities corresponding to Arnold's classification has been explored to investigate the entanglement nature of pure qudit systems[18, 19, 20]. In this way, a special interest has been devoted to the case of D_4 ($so(8)$) singularity to approach 4-qubits.

The aim of this work is to contribute to these activities by using geometric engineering method developped in [21] to engineer qubit systems in the context of D2-branes in type II superstrings. Concretely, we give a correspondence between such quantum systems and a class of Lie symmetries via graph theory. The 1 and 2-qubit are analyzed in some details and are found to be associated with the geometric engineering of six dimensional theories obtained from type IIA superstring in the presence of D2-branes probing one and two isolated su(2) singularities of the K3 surface, respectively. Then, we discuss on the corresponding entanglement. Implementing graph operations of toric polyvalent geometry explored in string theory, we approach multi-qubits in terms of several $su(2)$ isolated singularities. Using colored toric geometry, we link the n-qubits to non zero roots of n copies of su(2) Lie symmetry carrying information on D2-brane charges. Precisely, the corresponding Weyl group can be identified with the $\mathbb{Z}_2^n$ symmetry associated with the existence of 2^n states in type IIA superstring obtained from D2-branes wrapping on n isolated $\mathbb{CP}^1$'s represented by colored hypercube graphs. In the corresponding six dimensional gauge symmetry breaking, n copies of local SU(2) can split $\mathrm{SU}(2)^n_{local} \to \mathrm{U}(1)^n_{local} \times \mathrm{SU}(2)^n_{global}$ producing $\mathbb{T}^n$ manifolds. This could be interpreted as the Cartan sub-symmetries of the studied Lie structures.

This paper is organized as follows. In section 2, we give a concise presentation on Lie symmetries. Section 3 concerns the geometric engineering of lower dimensional qubits in terms of D2-branes in type IIA superstring compactifications on the K3 surface with $su(2)$ isolated singularities. In section 4, we establish a correspondence between multi-qubits and the K3 singularities using colored graph theory of polyvalent type IIA geometry and its operations. Section 5 is devoted to discussions and including remarks.

2 Lie symmetries

For later use, we start by giving a concise review on Lie symmetry backgrounds, used in many physical area including high energy and condensed matter physics. Roughly, a Lie symmetry L is a vector space together with an antisymmetric bilinear bracket $[,] : L \times L \to L$ satisfying, among others, the famous Jacobi identity $([a,[b,c]] + [c,[a,b]] + [b,[c,a]] = 0)$. It is realized that any semi-simple Lie symmetry can be viewed as a direct sum of simple Lie symmetries [22, 23, 24]. It is recalled that the Cartan subalgebra H is generated by the all semi-simple elements, being the maximal toric abelian Lie sub-algebra playing a remarkable rôle in the study of the classification of Lie symmetries. Moreover, it is observed that L may then be written as the direct sum of H and the subspace E_Δ

$$L = H \oplus E_\Delta \tag{2.1}$$

where E_Δ is given by

$$E_\Delta = \oplus_\alpha L_\alpha. \tag{2.2}$$

It is recalled that L_α is defined by

$$L_\alpha = \{x \in L | [h, x] = \alpha(x)x\} \tag{2.3}$$

for x inside L and α ranges over all elements of the dual of H. In Lie symmetries, these vectors α are known by roots associated with vector particle states used in the standard model physics. In particular, the corresponding basic concepts of the root systems will be exploited later to approach QIT. It is noted that a root system Δ of a Lie symmetry is defined as a subset of an euclidean space E satisfying the following constraints

1. Δ is finite and spans E, 0 is not an element of it,
2. if α is a root of Δ, then $k\alpha$ is also but only for $k = \pm\alpha$,
3. for any root α in Δ, the later is invariant under reflection σ_α, where $\sigma_\alpha(\beta) = \beta - <\beta, \alpha> \alpha$,
4. if α and β are two roots of Δ, then the quantity $<\beta, \alpha>$ is in $\mathbb{Z}$.

It is has been shown that the root system Δ involves several information about the associated Lie symmetry structure. These information will be relevant in the present work to deal with qubit systems. In finite Lie symmetries, it has been shown that there is a nice relation given by

$$\dim L = \dim H + |\Delta| \tag{2.4}$$

where $|\Delta|$ indicates the number of the roots associated with L. The dimension of H, noted $\dim H$, is called also the rank which is identified with the number of the simple roots which could be interpreted as the basis of Δ. To make contact with qubits, we will consider a particular Lie symmetry constrained by

$$\dim L = n + 2n \tag{2.5}$$

where we have used

$$\dim H = n, \qquad |\Delta| = 2n. \tag{2.6}$$

We will see later that n will be identified with the order of the qubit. An alternative way to classify these symmetries is to use the Cartan matrices obtained from the scalar product between the simple roots. For such symmetries, we will consider the Cartan matrix $A = (A_{ij})$ of size n by n given by

$$A_{ij} = 2\delta_{ij}. \tag{2.7}$$

It is recalled that this matrix can be encoded in a nice graph called Dynkin graphs, or diagrams. In this graph, the diagonal elements correspond to vertices and the non diagonal elements describe the number of the edges between them. In fact, the number of the edges between a vertex i and a vertex j is given by $A_{ij}A_{ji}$.

In what follows, we combine these Lie symmetries and the geometric engineering method, used in the construction of six dimensional theories from string theory in the presence of D-branes, to investigate quantum information systems.

3 Geometric engineering of lower dimensional qubits

In this section, we engineer qubit systems from D-branes wrapped on $\mathbb{CP}^1$'s used in the deformation of $su(2)$ isolated singularities of the K3 surface.

3.1 Qubits from Type IIA supertring on the K3 surface

It is noted that the qubit has been extensively investigated from different physical and mathematical aspects[4, 5, 6, 7]. Using quantum mechanics notation (Dirac notation), 1-qubit is described by the following state

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \tag{3.1}$$

where c_i are complex numbers verifying the probability condition

$$|c_0|^2 + |c_1|^2 = 1. \tag{3.2}$$

The condition can be interpreted geometrically in terms of the so-called Bloch sphere. Similarly, the 2-qubits are represented by the following state

$$|\psi\rangle = c_{00}|00\rangle + c_{10}|10\rangle + c_{01}|01\rangle + c_{11}|11\rangle. \tag{3.3}$$

In this case, the probability relation becomes

$$|c_{00}|^2 + |c_{10}|^2 + |c_{01}|^2 + |c_{11}|^2 = 1 \tag{3.4}$$

describing a 3-dimensional complex projective space $\mathbb{CP}^3$ generalizing the Bloch sphere. This analysis can be extended to n-qubits associated with 2^n configuration states. Using the binary notation, the general state reads as

$$|\psi\rangle = \sum_{i_1\ldots i_n=0,1} c_{i_1\ldots i_n}|i_1\ldots i_n\rangle, \tag{3.5}$$

where the complex coefficients $c_{i_1\ldots i_n}$ verify the normalization condition

$$\sum_{i_1\ldots i_n=0,1} c_{i_1\ldots i_n}\bar{c}_{i_1\ldots i_n} = 1. \tag{3.6}$$

This equation generates the $\mathbb{CP}^{2^n-1}$ complex projective space. An inspection shows that the qubit systems can be engineered form D-branes placed at certain singularities of type IIA polyvalent geometry [21]. In what follows, we refer to it as n-valent geometry. More precisely, we will show that this geometry produces the states of n-qubit systems. In this way, a quantum state could interpreted in terms of D2-branes wrapping isolated $\mathbb{CP}^1$'s in type IIA superstring compactificatied on the K3 surface. We expect that the n-valent geometry should encode certain data on the associated D-brane physics offering a new take on the geometric realization of QIT using techniques based on a quantum geometry, Lie symmetries, and colored graph theory. To establish this link, we investigate first the the case of lower dimensional qubits. Then, we give a general statement for multi-qubits in the next section using graph theory operations including sum and cartesian products. The 1-qubit can be elaborated using the geometry associated with a local description of the K3 surface where the manifold develops a singularity corresponding to vanishing intersecting $\mathbb{CP}^1$'s. It is recalled that the K3 surface is a 2-dimensional Calabi-Yau manifold involving the Kähler structure permitting the existence of a global nonvanishing holomorphic 2-form assured by a SU(2) Holonomy group[25, 26]. This manifold can be constructed using different approaches, including the orbifold one. The latter can be done in terms of the $\mathbb{T}^4$ manifold modulo discrete isometries of the SU(2) holonomy group. In particular, it can be built by using the complex coordinates of such a torus

$$z_i = z_i + 1, \qquad z_i = z_i + \imath, \qquad i = 1, 2 \tag{3.7}$$

and imposing an extra $\mathbb{Z}_2$ symmetry acting on the complex variables z_i as follows

$$z_i \to -z_i. \tag{3.8}$$

This action involves 16 fixed points given by

$$(z_1^i, z_2^i) = (0,0), (0, \frac{1}{2}), (0, \frac{1}{2}\imath), (0, \frac{1}{2} + \frac{1}{2}\imath)), (\frac{1}{2}, 0) \ldots (\frac{1}{2} + \frac{1}{2}\imath, \frac{1}{2} + \frac{1}{2}\imath). \tag{3.9}$$

It has been shown that each fixed point corresponds to a vanishing 2-sphere identified with $\mathbb{CP}^1$. To see that, one considers a local version of the K3 surface. Indeed, the orbifold $\mathbb{T}^4/\mathbb{Z}_2$ can be viewed as a non compcat space given by $\mathbb{C}^2/\mathbb{Z}_2$, which is known by $su(2)$ (A_1) space described by the following equation

$$xy = z^2 \tag{3.10}$$

where x, y and z are $\mathbb{Z}_2$-invariants. In this way, they are related to the local coordinates z_1 and z_2 of $\mathbb{C}^2$ as follows

$$z = z_1 z_2, \quad x = z_1^2, \quad y = z_2^2, \tag{3.11}$$

The local geometry is singular at $x = y = z = 0$ which can be replaced by $\mathbb{S}^2$ which is isomorphic to $\mathbb{CP}^1$. The blowing up of 16 fixed points (singular points) change the Hodge diagram of $\mathbb{T}^4$ by adding the twisted sector. This diagram is given by

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

where $h^{p,q}$ denotes the number of the holomorphic and the anti-holomorphic forms of degree (p, q). It turns that the local version of the K3 surface can involve more complicated singularities associated with ADE Lie symmetries. For instance, near the $su(3)$ singular point, the geometry can be viewed as an asymptotically locally Euclidean (ALE) complex space which is algebraically given by the blowing down of two intersecting $\mathbb{CP}^1$ spaces according to the $su(3)$ Dynkin graph. In this context, this configuration is considered as a 1-valent geometry since each $\mathbb{CP}^1$ intersects only another one which can be represented by two vertices as the $su(3)$ Dynkin graph in type IIB mirror geometry. Cutting such a 1-valent vertex, we get one vertex corresponding to the $su(2)$ Dynkin graph associated with the $su(2)$ singularity of the K3 surface discussed before. The singularity can be removed by blowing up the singular point by a $\mathbb{CP}^1$ complex curve. It has been revealed that this type IIA geometry generates a six dimensional SU(2) gauge model associated with $W^{0,\pm}$ gauge field bosons. The W^0 gauge vector is identified with a vector field A_1 obtained from the RR three-form A_3 according to the following decomposition

$$A_3 \to A_1 \wedge w \tag{3.12}$$

where w is 2-form on $\mathbb{CP}^1$ of the K3 surface compactification. However, the $W^\pm$ vector multiplet arises from the D2-brane wrapped on $\mathbb{CP}^1$ used to remove $su(2)$ singularity of the K3 surface. In this compactification, a D2-brane wrapping around such a complex curve gives two states depending on the two possible orientations for the wrapping procedure[21]. In Lie algebras, this procedure can be supported by the root system decomposition of $su(2)$ Lie symmetry. It is recalled that $su(2)$ is a 3-dimensional Lie symmetry defined by following commutation relations

$$[h, e] = e, \qquad [h, f] = -f, \qquad [e, f] = 2h, \tag{3.13}$$

where $\{e, h, f\}$ is called the Cartan basis. Its Cartan factorization reads as

$$su(2) = H \oplus L_{+\alpha} \oplus L_{-\alpha} \tag{3.14}$$

where H is the Cartan subalgebra generated by h [22, 23, 24]. In terms of the Cartan basis, it can be rewritten as

$$su(2) = \mathbb{C}h \oplus \mathbb{C}e \oplus \mathbb{C}f. \tag{3.15}$$

Adopting a binary digit notation, according to which $+$ corresponds to 0 and $-$ corresponds to 1, we will show that the two states of 1-qubit, which will be denoted $|\pm\rangle$ is linked to the two-dimensional vector space associated with the root system

$$\Delta(su(2)) = \{\alpha, -\alpha\}. \tag{3.16}$$

This space is given by the following quotient space

$$E_\Delta = \frac{su(2)}{H} = L_{+\alpha} \oplus L_{-\alpha}. \tag{3.17}$$

Inspired by D-brane physics, we can show that the correspondence

$$L_{+\alpha} \;=\; \mathbb{C}|+\rangle, \qquad L_{-\alpha} = \mathbb{C}|-\rangle$$

gives rise to a mapping between the qubit states and the states of the charged D2-brane appearing in the K3 surface compactification with the $su(2)$ singularity used in the geometric engineering of six dimensional gauge theories. Like in four dimensions, the SU(2) gauge group could be associated with $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ of the electroweak theory mediated by the weak gauge bosons, the photon with the isospin and the hypercharge as corresponding charges. In this way, SU(2) vector field $W_\mu^{0,\pm}(x)$, can split into a U(1) vector field $\lambda_\mu(x)$ and generators σ^ℓ of global SU(2) in the fundamental representation as

$$W_\mu^\ell(x) = e^{-i\lambda_\mu(x)}\sigma^\ell, \qquad \ell = 0, \pm \tag{3.18}$$

where the generators σ^ℓ are the Pauli matrices which satisfy the $su(2)$ commutation relations

$$\left[\sigma^k, \sigma^\ell\right] = 2i\epsilon_{k\ell m}\sigma^m \tag{3.19}$$

where $\epsilon_{k\ell m}$ is an antisymmetric tensor. The corresponding gauge symmetry breaking of the local SU(2) gauge group is

$$\mathrm{SU}(2)_{local} \to \mathrm{U}(1)_{local} \times \mathrm{SU}(2)_{global}. \tag{3.20}$$

Therefore, the above expression (3.18) shows an hidden quantum feature (spin $\frac{1}{2}$) of a classical non-abelian gauge field. In this way, the SU(2) gauge theory can be linked to 1-qubit system. Combining these data, the 1-qubit system can be obtained from the particle states of SU(2) interaction mediated by $W^{\pm,0}$. A possible connection could be done by proposing the following linear combination form between the D2-brane states

$$\begin{aligned} |+\rangle &= a|W^+\rangle + b|W^0\rangle \\ |-\rangle &= -b|W^-\rangle + a|W^0\rangle \end{aligned} \tag{3.21}$$

where a and b are real parameters satisfying

$$a^2 + b^2 = 1. \tag{3.22}$$

It is noted that this condition can be interpreted as the geometric realization of the U(1) gauge symmetry in terms of one-dimensional circle S^1 identified with the h generator of $su(2)$ Lie symmetry. Using this geometric constraint, the normalization conditions are now satisfied

$$\langle +|+\rangle = \langle -|-\rangle = a^2 + b^2 = 1 \tag{3.23}$$

$$\langle +|-\rangle = \langle -|+\rangle = -ab + ab = 0. \tag{3.24}$$

In the context of superstring compactifications, the probability of measuring the qubit in certain states could be determined in terms of winding numbers on $\mathbb{CP}^1$. This will be investigated in other occasions.

Having discussed 1-qubit case, we move now to the 2-qubit systems defined in a 4 dimensional Hilbert space. The 2-qubit model, which is interesting from entanglement applications, will correspond to 2-valent geometry appearing in the A_3 type IIA superstring. In the geometric engineering method, it contains a central $\mathbb{CP}^1$ which intersects two other ones according to the A_3 Dynkin graph in type IIB mirror description [21]. Removing the central vertex, we get a graph which can be identified with the Dynkin graph of the $su(2) \oplus su(2)$ Lie symmetry. In this way, the corresponding type IIA geometry involves two isolated $\mathbb{CP}^1$'s in the K3 surface compctifications. Two D2-branes, will be denoted by A and B respectively, wrapping around such a geometry will give four states depending on the two possible orientations on each $\mathbb{CP}^1$. In Lie algebra formwork, these configurations correspond to the root system

$$\Delta = \{\alpha_A, -\alpha_A, \alpha_B, -\alpha_B\} \tag{3.25}$$

of the $su(2) \oplus su(2)$ Lie symmetry. In this way, the root system decomposition providing a four dimensional vector space

$$E_\Delta = E_{\alpha_A} \oplus E_{\alpha_B} \tag{3.26}$$

where the factors E_α are given by

$$E_{\alpha_A} = L_{\pm\alpha_A}, \qquad E_{\alpha_B} = L_{\pm\alpha_B}. \tag{3.27}$$

In this case, the Weyl group will be identified with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry associated with the existence of four states in type IIA superstring using two D2-branes wrapping two isolated $\mathbb{CP}^1$'s of the K3 surface in the type IIA compactification. In the six dimensional gauge symmetry breaking, two copies of local SU(2) can split as

$$\mathrm{SU}(2)_{local} \times \mathrm{SU}(2)_{local} \to \mathrm{U}(1)^2_{local} \times \mathrm{SU}(2)^2_{global} \tag{3.28}$$

The associated tensorial Hilbert space read as

$$\mathbb{C}|\pm\rangle_A \otimes \mathbb{C}|\pm\rangle_B \tag{3.29}$$

with the following constraints required by the 1-qubit realizations

$$a_A^2 + b_A^2 = 1, \qquad a_B^2 + b_B^2 = 1. \tag{3.30}$$

It is noted that this condition produces a geometric realization of the $U(1)^2$ gauge symmetry in terms of a trivial fibration of two circles producing a torus $\mathbb{T}^2$.

3.2 Quantum entanglement from gauge vector bosons

The main point of this subsection is to discuss the corresponding entanglement in terms of su(2) gauge filed bosons. It is recalled that to approach this concept one should compute the concurrence quantity providing a tool to study the separability defined in terms the spin-flip operation. For pure states associated with 2-qubit states $|\psi>_{AB}$, the concurrence noted by $C(|\psi>_{AB})$ is given by

$$C(|\psi>_{AB}) = |<\psi|\tilde{\psi}>| \tag{3.31}$$

where $|\tilde{\psi}>= \sigma_y \otimes \sigma_y |\psi^*>_{AB}$ and where σ_y is the Pauli matrix along the y direction [27]. In this way, $|\psi^*>_{AB}$ indicates the complex conjugate of $|\psi>_{AB}$. For later use, the notation will be changed. In particular, one considers the standard basis $\{|-->, |-+>, |+->, |++>\}$ associated with the gauge vector boson charges of six dimensional gauge theories. In this case, the pure state reads as

$$|\psi>_{AB}= c_{++}|++> + c_{-+}|-+> + c_{+-}|+-> + c_{++}|++> \tag{3.32}$$

where c_{++}, c_{-+}, c_{+-} and c_{++} are complex coefficients satisfying the normalization condition

$$|c_{++}|^2 + |c_{+-}|^2 + |c_{-+}|^2 + |c_{--}|^2 = 1. \tag{3.33}$$

The calculation shows that the quantity $C(|\psi>)$ takes the following form

$$C(|\psi>) = 2|c_{++}c_{--} - c_{+-}c_{-+}|. \tag{3.34}$$

The state $|\psi>$ is separable if one has

$$c_{++}c_{--} = c_{+-}c_{-+}. \tag{3.35}$$

Other ways, it is entangled if one has

$$c_{--}c_{++} \neq c_{-+}c_{+-}. \tag{3.36}$$

We expect that these conditions can be satisfied by the charges of D2-branes wrapping non trivial cycles in the K3 surface. A possible connection can be provided by writing the state $|\psi>_{AB}$ in the basis $\{|W^+>, |W^->, |W^0>\}$ associated with the $su(2)$ brane charges in six dimensions. Indeed, one can consider the six dimensional vector field decomposition

$$\begin{array}{rcl} |++> &=& a_Aa_B|W^+W^+>_{AB} + a_Ab_B|W^+W^0>_{AB} + b_Aa_B|W^0W^+>_{AB} + b_Ab_B|W^0W^0>_{AB} \\ |+-> &=& -a_Ab_B|W^+W^->_{AB} + a_Aa_B|W^+W^0>_{AB} - b_Ab_B|W^0W^->_{AB} + b_Aa_B|W^+W^0>_{AB} \\ |-+> &=& -b_Aa_B|W^-W^+>_{AB} - b_Ab_B|W^-W^0>_{AB} + a_Aa_B|W^0W^+>_{AB} + a_Ab_B|W^0W^0>_{AB} \\ |--> &=& b_Ab_B|W^-W^->_{AB} - b_Aa_B|W^-W^0>_{AB} + a_Ab_B|W^0W^0>_{AB} - a_Ab_B|W^0W^->_{AB} \end{array}$$

Motivated with the usual computation, one may propose the following matrix associated with $su(2)$ D2-brane charges

$$M = \begin{pmatrix} c_{--}a_Aa_B + c_{-+}a_Ab_B + c_{+-}b_Aa_B + c_{++}b_Ab_B & -c_{+-}b_Ab_B - c_{--}a_Ab_B & c_{-+}a_Aa_B + c_{++}b_Aa_B \\ -c_{-+}b_Ab_B - c_{--}b_Aa_B & c_{--}b_Ab_B & c_{-+}b_Aa_B \\ c_{+-}a_Aa_B + c_{++}a_Ab_B & -c_{+-}a_Ab_B & c_{++}a_Aa_B \end{pmatrix} \qquad (3.37)$$

The corresponding invariant $detM$ can encodes information of the corresponding two qubits. In fact, the connection can be given

$$C(|\psi>) = \frac{detM}{2a_A^2a_Bb_Ab_Bc_{-+}}. \qquad (3.38)$$

It follows from this equation that the data of 2-qubits states can be extracted from two D2- brane charges associated with two copies of 3 vector states of the $su(2)$ Lie symmetry. We believe that this observation on the quantum entanglement information deserves deeper investigation.

4 Multi-qubits from graph theory of polyvalent type IIA geometry

A close inspection shows that multi-qublits needs new materials inspired by graph theory operations. It has been suggested that such systems could be approached using quiver methods dealing with several gauge factors. However, these quantum systems should be dealt with differently since they will be associated with graphs having non linear vertices. In particular, such quivers should involve polyvalent vertices being connected to more than two other ones as discussed in the previous section. In the geometric engineering method of four-dimensional gauge theories obtained from type II superstrings compactified on Calabi-Yau manifolds, the associated graphs have been explored to deal with singularities based on indefinite Lie algebras generalizing the finite and affine symmetries. A close inspection shows that the n-valent geometry can involve a nice graph representation by combining toric geometry and Lie symmetry structures. It is recalled that a graph G is defined by a pair of sets $G = (V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ indicates the edge set [28, 29, 30]. Two vertices are adjacent if they are connected by a edge. For each non oriented graph G, we associate a symmetric squared matrix called adjacency matrix $I(G) = (I_{ij})$, whose elements are either 0 or 1

$$I_{ij} = \begin{cases} 1, & (i,j) \in E(G), \\ 0, & (i,j) \notin E(G). \end{cases} \tag{4.1}$$

It has been observed that this nice matrix, which plays a primordial rôle to provide connections with many areas in mathematical and physics, encodes all the information on the graph in question. These data can be exploited to present a graphic representation of complicated physical systems including standard models of particle physics, or more generally non trivial quiver gauge models built from string theory compactified on singular Calabi-Yau manifolds. A special example of graphs which is relevant in the present work is n-valent geometry sharing similar properties with the star graph containing a cental vertex connected with n ones. In Lie symmetry, these graphs have been used to represent indefinite Lie algebras generalizing the finite and affine symmetries. For $n = 3$, one can recover the 3-valent geometry appearing in the finite $so(8)$ Lie algebra, known also by D_4. For $n = 4$, however, one can get the 4-valent vertex of the affine $\widehat{so(8)}$ Dynkin graph, known also by $\widehat{D_4}$. More generally, the graph which represents the n-valent geometry could be discussed in the context of toric manifolds explored in type II superstring compactifications [21]. It is recalled that a m-dimensional toric manifold $\mathbf{X^m}$ can be represented by a toric graph $G(\mathbf{X}^m)$ spanned by $m + n$ vertices v_i belonging to the m-dimensional lattice $\mathbb{Z}^m$ [31, 32, 33, 34, 35]. These vertices satisfy the following r relations

$$\sum_{i=1}^{m+n} q_i^\ell v_i = 0, \quad \ell = 1, \ldots, n, \tag{4.2}$$

where q_i^ℓ are integers called Mori vectors carrying many mathematical and physical data. In particular, this geometry can be built physically using a two-dimensional $\mathcal{N} = 2$ supersymmetric linear sigma model described in terms of $n+m$ chiral superfields Φ_i with charge q_i^ℓ, $i = 1, \ldots, n+m$; $\ell = 1, \ldots, n$ under $\mathrm{U}(1)^{\otimes n}$ gauge symmetry [36]. We then have

$$\Phi_i \qquad \rightarrow \qquad e^{i \sum_\ell \vartheta_\ell q_i^\ell} \Phi_i, \qquad i = 1, \ldots, n+m, \tag{4.3}$$

where the ϑ_ℓ's are the gauge group parameters. The D^ℓ-term equations are obtained by minimizing the Kahler potential of the $2D$ $\mathcal{N} = 2$ superfield action

$$\mathcal{S}_{\mathcal{N}=2} = \int d^2\sigma d^4\theta \mathcal{K} + \left(\int d^2\sigma d^2\theta \mathrm{W} + cc \right) \tag{4.4}$$

with respect to the gauge superfields V_ℓ. It is recalled that $\mathcal{K}$ is the usual gauge invariant Kahler term, while W is a chiral superpotential with superfield dependence as,

$$\begin{aligned} \mathcal{K} &= \mathcal{K}\left[\Phi_1, ..., \Phi^+_{n+m}; V_1, ..., V_\ell\right], \\ \mathrm{W} &= \mathrm{W}\left[\Phi_1, ..., \Phi_{n+m}\right], \end{aligned} \tag{4.5}$$

as well as coupling constant moduli which have not been specified. Using the explicit expression of $\mathcal{K}$ and putting back into

$$D^\ell = \frac{\partial \mathcal{K}}{\partial V_\ell}|_{\theta=0} = 0, \tag{4.6}$$

we get the following D^ℓ-term equations

$$\sum_{i=1}^{n+m} q_i^\ell |\phi_i|^2 = r^\ell, \qquad \ell = 1, \ldots, \ell, \tag{4.7}$$

where the r^ℓ's are FI coupling parameters and where the ϕ_i's are the leading scalar fields of the chiral superfield Φ_i. Dividing by $\mathrm{U}(1)^{\otimes n}$ gauge symmetry, one gets a m-dimensional toric variety, represented by a graph $G(X^n)$. It turns out that toric geometry can be extended by introducing a color data leading to a colored toric geometry. The toric data $\{v_i, q_i^\ell\}$ will be replaced by

$$\{v_i, q_i^\ell, c_\ell\}, \qquad \ell = 1, \ldots, n, \tag{4.8}$$

where ℓ indicates also the color of the edges carrying physical information of qubit states. In this way, one may propose the following mapping

$$\mathcal{N} = 2 \text{ linear sigma model} \leftrightarrow \text{colored graph} \tag{4.9}$$

which be useful in the discussion of the geometric engineering of qubits.

The simplest example in colored toric geometry, considered as the building block of higher dimensional varieties, is $\mathbb{CP}^1$. This geometry is obtained by taking $m = n = 1$. In this case, the Mori vector charge is $q = (1, 1)$ describing the 1-valent geometry. It involves one U(1) toric symmetry acting on the local coordinate x of $\mathbb{CP}^1$ as follows

$$x \to e^{i\theta} x. \tag{4.10}$$

This symmetry has two fixed points v_{-1} and v_1 placed on the real line. These two points (vertices), which describe the North and south poles respectively of $\mathbb{CP}^1$ satisfy the following constraint toric equation

$$v_{-1} + v_1 = 0. \tag{4.11}$$

In this way, $\mathbb{CP}^1$ is represented by a colored toric graph identified with an interval $[v_{-1}, v_1]$ with a circle on top. It vanishes at the vertices v_{-1} and v_1. In graph theory, this graph is known by K_2.

$$\mathbb{C}P^1 \rightarrow K_2 \ : \quad (4.12)$$

This toric representation can be easily extended to higher dimensional geometries using non trivial graphs. For projective space $\mathbb{CP}^n$, the graphs become simplex geometries. In this way, the $\mathbb{S}^1$ circle fibration, of $\mathbb{CP}^1$, is replaced by $\mathbb{T}^n$ fibration over an n-dimensional simplex considered as a regular polytope in toric geometry. Indeed, the $\mathbb{T}^n$ collapses to a $\mathbb{T}^{n-1}$ on each of the n faces of the simplex, and to a $\mathbb{T}^{n-2}$ on each of the $(n-2)$-dimensional intersections of these faces, etc.
It turns our that non trivial geometries, which we will be interested in here, can be also approached using such a method. Concretely, the n-valent can be discussed using toric geometry data. Indeed, the Mori vectors associated with 2-valent geometry take the following form

$$q_i^a = (-2, 1, 1, 0, ..., 0). \quad (4.13)$$

In local geometries of the K3 surface known by the deformed ALE spaces, the 2-valent vertices represent a linear chain of divisors with self intersection (-2) and intersect two adjacent divisors once with contribution $(+1)$. The 3-valent geometry, however, involves both 2-valent and 3-valent vertices which has been appeared in different occasions. In string theory for instance, this geometry has been used to incorporate fundamental matters in the geometric construction of a quiver theory based on several SU gauge groups [21]. In toric realization of the ALE spaces, the 3-valent geometry contains a central divisor with self intersection (-2) intersecting three other divisors once with contribution $(+1)$. In associated graph theory, the corresponding Mori vector reads as

$$q = (-2, 1, 1, 1, 0, \ldots, 0). \quad (4.14)$$

It has been suggested that the Calabi-Yau condition requires that this Mori vector should be modified and takes the following form

$$q = (-2, 1, 1, 1, -1, 0, \ldots, 0). \quad (4.15)$$

A close inspection shows that the 3-valent geometry has been explored in the study of the complex deformation of the T_{p_1,p_2,p_3} singularity defined as the intersection of three chains type A_{p_1-1}, A_{p_2-1} and A_{p_3-1} appearing in the blowing up of elliptic exceptional singularities $E_{6,7,8}$ of the K3 surface:

$$\begin{array}{l} E_6 \rightarrow T_{3,3,3} : x_1^3 + x_2^3 + x_3^3 + \lambda x_1 x_2 x_3 \\ E_7 \rightarrow T_{2,4,4} : x_1^2 + x_2^4 + x_3^4 + \lambda x_1 x_2 x_3 \\ E_8 \rightarrow T_{2,3,6} : x_1^3 + x_2^2 + x_3^6 + \lambda x_1 x_2 x_3 \end{array} \quad (4.16)$$

where λ is a complex parameter[21]. This T_{p_1,p_2,p_3} singularity could be extended to $T_{p_1,p_3,\ldots,p_n}$ n-valent singularities by considering a non trivial intersection of

n chains of 2-valent geometries formed by spheres in the Calabi-Yau manifolds associated with 1 and 2-qubits as we have seen in the previous section.
In what follows, we would like to approach physically n-valent geometry. In type IIA superstring, associated with the middle-degree cohomology, a n-valent geometry can be described by a central sphere $\mathbb{CP}^1_0$, with self intersection (-2) intersecting n other 1-dimensional projective spaces ($\mathbb{CP}^1_\ell$ $(\ell = 1, \ldots, n)$ with contribution $(+1)$. In this way, the intersection numbers read as

$$\begin{aligned} \mathbb{CP}^1_0.\mathbb{CP}^1_0 &= -2 \\ \mathbb{CP}^1_0.\mathbb{CP}^1_\ell &= 1 \qquad \ell = 1, \ldots, n. \end{aligned} \tag{4.17}$$

In relation with the adjacency matrix in graph theory and toric geometry, the intersection numbers of the n-valent geometry can be written in terms of Mori vectors and Cartan matrices as follows

$$\mathbb{CP}^1_\ell.\mathbb{CP}^1_{\ell'} = q_\ell^{\ell'} = -2\delta_{\ell\ell'} + I_{\ell\ell'}.$$

However, the Calabi-Yau condition requires that one should add an extra non compact cycle with contribution $2-n$. In string theory compactification on the local K3 surface, this cycle does not affect the corresponding physics including QIT. In this way, the Mori vector representing the central $\mathbb{CP}^1_0$ is given by

$$q_i^0 = (-2, \underbrace{1, \ldots, 1}_{n}, 2-n). \tag{4.18}$$

Deleting the central vertex, we obtain the so-called empty graph having n vertices leading to

$$I_{ij} = 0. \tag{4.19}$$

In type IIB mirror geometry, this corresponds to the Dynkin graph of a particular Lie symmetry defined by n pieces of $su(2)$ denoted by $\underbrace{su(2) \oplus \ldots \oplus su(2)}_{n}$.

In this way, the intersection numbers can be written in terms of the corresponding Cartan matrix

$$\mathbb{CP}^1_\ell.\mathbb{CP}^1_{\ell'} = -A_{\ell\ell'}. \tag{4.20}$$

The Lie symmetry contains $2n$ roots and it is represented by a graph involving n isolated vertices associated with $\mathbb{CP}^1$'s in the mirror type IIB geometry. However, the n $\mathbb{CP}^1$'s are represented by $2n$ vertices in type IIA geometry associated with the non-zero root system decomposition

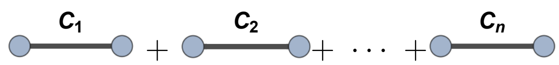
$$E_\Delta = E_{\alpha_{A_1}} \oplus \ldots \oplus E_{\alpha_{A_n}} \tag{4.21}$$

where the space $E_{\alpha_{A_\ell}}$ corresponds to the ℓ^{th} $su(2)$ symmetry defining the ℓ^{th} qubit. The vector space $E_{\alpha_{A_\ell}}$ which reads

$$E_{\alpha_{A_\ell}} = L_{+\alpha_\ell} \oplus L_{-\alpha_\ell}. \tag{4.22}$$

can be represented by a colored K_2 presented perviously.

For the vector space E_Δ, we need more methods to generate the corresponding graph using disjoint union operations of K_2's. To specific such a disjoint union, we write $K_2 + \ldots K_2$. It is given by the sum operation in graph theory associated with n colors.



(4.23)

In graph theory, the vector space E_Δ can be represented by the nK_2 which is the graph consisting of n pairwise disjoint copies of K_2. In this way, the root graph can be identified with a geometry involving $2n$ vertices. Geometrically, this could represent the homological class of $H^2(K3)$ given by

$$[\mathcal{C}] = \underbrace{\mathbb{CP}^1 \oplus \ldots \oplus \mathbb{CP}^1}_{n} \tag{4.24}$$

exhibiting $\mathrm{U}(1)^n$ toric actions associated with colors. To elaborate the sigma model describing the vector space E_Δ, we consider a type IIA geometry obtained by n U(1) gauge fields V_1,..., and V_n and $2n$ chiral superfields ϕ_i^ℓ with gauge charges with charges $q_i^\ell = \left(q_1^\ell, q_2^\ell, .., q_{2n}^\ell\right)$ as follows,

$$\begin{aligned} q_i^1 &= (1,1,0,0,0,...,0,0,0) \\ q_i^2 &= (0,0,1,1,0,...,0,0,0) \\ q_i^3 &= (0,0,0,0,1,1,0...,0,0,0) \\ ... &= ... \\ q_i^n &= (0,0,0,0,0,...,0,1,1)\,. \end{aligned} \tag{4.25}$$

More precisely, the associated D-terms reads

$$\begin{aligned} \left|\phi^1\right|^2 + \left|\phi_2\right|^2 &= r_1 \\ ... &= ... \\ \left|\phi_{n-1}\right|^2 + \left|\phi_n\right|^2 &= r_n. \end{aligned} \tag{4.26}$$

This sigma model suffers from at least two problems. The first one is the dimensionality problem. This can be solved using techniques of sigma models developed in [37]. The second one is associated with the state number of n-qubits. An examination shows that this problem comes from the fact that the only integral solutions of the equation

$$2n = 2^n \tag{4.27}$$

are

$$n = 1, \qquad n = 2. \tag{4.28}$$

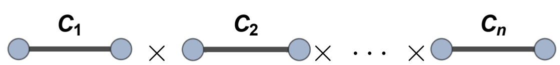
To solve this problem, an extra operation motivated by the graph operations are needed. To get the right states, one needs the cartesian product of colored K_2's. Indeed, consider type IIA string theory on the K3 surface. In this compactification, there are 2^n ways for n 2D-branes to wrap n distinguishable blowing up $\mathbb{CP}^1$ represented by $2n$ vertices. It is remarked that the $su(2)$ geometric spaces are distinguishable giving rise 2^n possible inequivalent states obtained from the cartesian product of colored K_2's

$$K_2 \times K_2 \ldots \times K_2 \tag{4.29}$$

describing the tensor vector space

$$\mathcal{H}_n = \mathbb{C}|\pm\rangle_{A_1} \otimes \ldots \otimes \mathbb{C}|\pm\rangle_{A_n}. \tag{4.30}$$

In graph theory, this space can be represented by



(4.31)

The graph shares a strong resemblance with the hypercube graph Q_n formed by 2^n vertices connected with $2^{n-1}.n$ edges where each vertex involves n colored edges incident to it [9]. In this way, one writes

$$K_2 \times K_2 \ldots \times K_2 = Q_n \tag{4.32}$$

In this type IIA geometry, the graph Q_n can represent wrapped D2-brane states

$$|D2/\mathbb{CP}^1_1, D2/\mathbb{CP}^1_2, \ldots D2/\mathbb{CP}^1_n\rangle \tag{4.33}$$

A close inspection show that these states can be associated with the trivial fibration of n $\mathbb{CP}^1$'s

$$\underbrace{\mathbb{CP}^1 \otimes \ldots \otimes \mathbb{CP}^1}_{n} \tag{4.34}$$

and their blow ups. In this way, the manifolds $\bigotimes_{i=1}^n \mathbb{CP}^1_i$ involves a $\mathrm{U}(1)^n$ toric actions exhibiting 2^n fixed points v_i. It can be represented by by 2^n vertices v_i which belongs to the $\mathbb{Z}^n$ lattice verifying n toric equations. To build the sigma model describing the Hilbert space $\mathcal{H}_n$, the type IIA geometry is obtained by $2^n - n$ U(1) gauge fields V_ℓ and 2^n chiral superfields $\Phi_{i_1 \ldots i_n}$ with gauge charges with charges $q^\ell_{i_1 \ldots i_n}$. In this way, one may also propose the following mapping

$$QIT \leftrightarrow N = 2 \text{ linear sigma model} \tag{4.35}$$

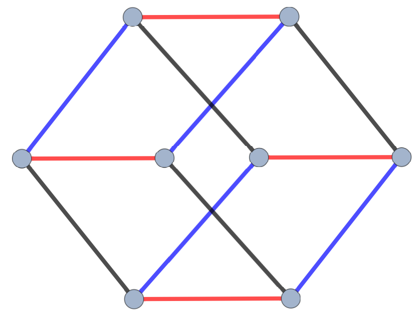
E_Δ	H_3
	

Table 1: Graphs associated with 3-qubits

To see this, let us consider the blow up of $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ type IIA geometry associated with 3-valent geometry. In colored sigma model, this manifold is described by the D-term equations of two-dimensional $\mathcal{N}=2$ supersymmetric linear sigma model described in terms of 2^3 chiral superfields $\Phi_{i_1i_2i_3}$ with charge $q^\ell_{i_1i_2i_3}$, $\ell = 1,\ldots,5$ under $\mathrm{U}(1)^{\otimes 5}$ gauge symmetry

$$\sum_{i_1i_2i_3=0,1} q^\ell_{i_1i_2i_3}|\phi_{i_1i_2i_3}|^2 = r_\ell, \qquad \ell = 1,\ldots,5, \tag{4.36}$$

where the charges are given by

$$\begin{aligned} q^1_{i_1i_2i_3} &= (1,0,0,1,0,0,0,0) \\ q^2_{i_1i_2i_3} &= (0,1,0,0,1,0,0,0) \\ q^3_{i_1i_2i_3} &= (0,0,1,0,0,1,0,0) \\ q^4_{i_1i_2i_3} &= (1,-1,0,0,0,0,1,0) \\ q^5_{i_1i_2i_3} &= (0,0,1,1,0,0,0,1). \end{aligned} \tag{4.37}$$

It has been shown that the corresponding vertices $v_{i_1i_2i_3}$ are given by

$$\begin{aligned} v_{000} = (1,0,0),\ v_{100} = (0,1,0),\ v_{010} = (0,0,1),\ v_{001} = (-1,0,0) \\ v_{110} = (0,-1,0),\ v_{101} = (0,0,-1),\ v_{110} = (-1,1,0),\ v_{111} = (0,-1,-1), \end{aligned} \tag{4.38}$$

being connected with three colors describing 3-qubits. In this way, the associated space are presented in table 1.

The D2-brane configurations of 3-qubits can be associated with 3-valent vertex with three colored legs extending the 1-valent and 2-valent corresponds to 1 and 2 qubits respectively. They are illustrated in table 2. It is observed that the n-qubits are associated with polyvalent D2-brane charges living on n colored legs.

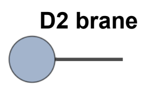
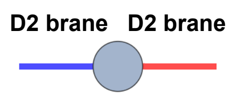
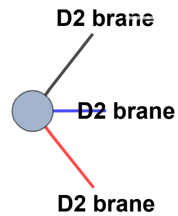
	$1-qubit$	$2-qubits$	$3-qubits$
D2-brane states			
geometry	1-valent	2-valent	3-valent

Table 2: Graphs of D2-brane charges associated with n-valent

Besides the root decomposition, proposed link can be supported by the corresponding Wyel group in type IIA geometry given by

$$\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{n}. \tag{4.39}$$

This symmetry can be associated with the hypercube graph involving n colors. In the corresponding gauge symmetry breaking associated with D2-branes, n copies of local SU(2) can split

$$\mathrm{SU}(2)^n_{local} \to \mathrm{U}(1)^n_{local} \times \mathrm{SU}(2)^n_{global} \tag{4.40}$$

producing a $\mathbb{T}^n$ manifold given by

$$a^2_{A_1} + b^2_{A_1} = 1, \quad \ldots \quad a^2_{A_n} + b^2_{A_n} = 1. \tag{4.41}$$

A close inspection reveals that there is a possible correspondence between $N = 2$ sigma model, quantum information physics, and graph theory. It is given in the table 3.

$N = 2$ Sigma modem	Quantum Information	Graph theory
Target space	Hilbert Space	Colored Graphs
Chiral fields	qubit states	vertices
Number of chiral fields	Dimension of the associated Hilbert space	numbers of vertices
Number of U(1) gauge symmetry	Number of qubits	Number of colors

Table 3: Correspondence between s Lie symmetries, graph theory, and quantum information physics.

This link may offer a novel way to study quantum geometry associated with quantum mechanical theory of strings and D-branes wrapping non trivial cycles in the Calabi-Yau manifolds.

5 Discussions and open questions

In this work, we have engineered multi-qubits from type II superstrings moving on the singular K3 surface. Concretely, we have combined n-valent geometry, appearing in the Calabi-Yau manifolds used in the geometric engineering method, Lie symmetries, and graph theory, to approach such quantum systems. In particular, we have investigated the 1 and 2- qubits in terms of linear chain geometry in type IIA superstring. The geometries are nothing but the A_2 and A_3 ALE spaces associated with the $su(3)$ and $su(4)$ singularities respectively. For $su(3)$ case, deleting one vertex we have obtained just one $\mathbb{CP}^1$ on which a D2-brane can wrap to give two states depending on the two possible orientations of the $\mathbb{CP}^1$. This mechanism reproduces the states of a 1-qubit system using the Cartan decomposition of su(2) Lie algebra. This can generate a two-dimensional vector space associated with the massive vector states of the geometric engineering of SU(2) gauge theory in six dimensions. In practice, this gauge symmetry can split into a U(1) vector field $\lambda_\mu(x)$ and generators of global SU(2) in the fundamental representation associated with 1-qubit system. A similar analysis has been presented for two isolated $\mathbb{CP}^1$'s producing 2-qubit systems. Then, a discussion on the corresponding entanglement has been given in terms of six-dimensional vector fiels obtained from D2-brane charges. Moreover, we have given a geometric engineering picture of n-qubits in terms of polyvalent geometry using graph operations and the Cartan decomposition of $\underbrace{su(2) \oplus \ldots \oplus su(2)}_{n}$ Lie symmetry. In this way, we have linked the n-qubits to non zero roots of such a Lie algebra associated with D2-brane charges using graph operations of colored toric geometry. Precisely, the Weyl group is identified with the $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ symmetry associated with the existence of 2^nstates in type IIA superstring using D2-branes wrapping two isolated $\mathbb{CP}^1$'s. These geometries have been elaborated using a colored toric graphs dual to $N = 2$ linear sigma model.

This work comes up with many open questions and remarks. The first remark that one should do concerns associated with the $n = 4$ matching with the results of the ADE correspondence in the context of QIF where 4-qubits are linked to the D_4 singularity [18, 19, 20]. In the present work, the 4-qubit systems correspond to the $\widehat{so(8)}$ affine Lie algebra. An inspection shows that the singularity can arise in the study of IIB mirror geometry of a toric Calabi-Yau manifold where the Mori vectors, up some details, are given in terms of the $\widehat{so(8)}$ Cartan matrix. Using the mirror symmetry calculation applied to $\widehat{so(8)}$ Dynkin graph[21], one can obtain the following algebraic geometry

$$P(x_1, x_2, x_3, x_4, w) = x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_1x_2x_3x_4 + w(x_1^2 + x_2^2 + x_3^2 + x_4^2) + w^2. \tag{5.1}$$

This is a quasihomogenous hypersurface in the weighted projective space $\mathbb{WCP}^4_{1,1,1,1,2}$ which may be considered as the deformation of a 4-valent singularity

$$T_{4,4,4,4} = x_1^4 + x_2^4 + x_3^4 + x_4^4 + \lambda x_1 x_2 x_3 x_4. \tag{5.2}$$

It is recalled that this is a quartic in $\mathbb{CP}^3$ identified with a K3 surface. In string theory compactification, this geometry can produce a singular Calabi-Yau 4-fold with a K3-fibration developing a 4-valent singularity. Type IIA superstring on such a four-fold singularity provides a two dimensional field model. Using the work of [21], the 4-valent singularity could be removed by a bouquet of four-spheres arranged as the $\widehat{so(8)}$ Dynkin graph. In this configuration, the 4-qubit states could be associated with wrapped D4-branes placed near such a 4-valent singularity.

The intersecting problem is the discussion of entanglement associated with higher dimensional n-valent singularities in type II geometry. This will be addressed elsewhere.

Acknowledgements. AB would like to thank N. Askour for shearing with him his office in Department of Mathematics, Faculty of Science and Technology, University Sultan Moulay Slimane (USMS), Beni-Mellal. AB woull like also to thank all members of the Departamento de Fśica Téorica, Universidad de Zaragoza for very kind hospitality and scientific supports during the realization of a part of this work, He also acknowledges the warm hospitality and support of Asorey, Diaz, Montanez, Naz and Segui families durante his travel in Spain and he thanks also Hajja Fatima (his mother) for patience and supports. The authors (AB and LM) thank S. Melliani head of the laboratory of mathematics and scientific computing at USMS for scientific helps and warm hospitality. LM thanks also I. El Ouargi and H. Jaffali for discussions. AS is supported by FPA2012-35453.

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Received: January 15, 2019; Published: February 25, 2019