Supersymmetry and Geometry of Hyperbolic Monopoles

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Moustafa Gharamti*)

To the strongest and most kind human I know

my mother

Abstract

This thesis studies the geometry of hyperbolic monopoles using supersymmetry in four and six dimensions. On the one hand, we show that starting with a four dimensional supersymmetric Yang-Mills theory provides the necessary information to study the geometry of the complex moduli space of hyperbolic monopoles. On the other hand, we require to start with a six dimensional supersymmetric Yang-Mills theory to study the geometry of the real moduli space of hyperbolic monopoles. In chapter two, we construct an off-shell supersymmetric Yang-Mills-Higgs theory with complex fields on three-dimensional hyperbolic space starting from an on-shell supersymmetric Yang-Mills theory on four-dimensional Euclidean space. We, then, show that hyperbolic monopoles coincide precisely with the configurations that preserve one half of the supersymmetry. In chapter three, we explore the geometry of the moduli space of hyperbolic monopoles using the low energy linearization of the field equations. We find that the complexified tangent bundle to the hyperbolic moduli space has a 2-sphere worth of integrable structures that act complex linearly and behave like unit imaginary quaternions. Moreover, we show that these complex structures are parallel with respect to the Obata connection, which implies that the geometry of the complexified moduli space of hyperbolic monopoles is hypercomplex. We also show, as a requirement of analysing the geometry, that there is a one-to-one correspondence between the number of solutions of the linearized Bogomol'nyi equation on hyperbolic space and the number of solutions of the Dirac equation in the presence of hyperbolic monopole. In chapter four and five, we shift the focus to supersymmetric Yang-Mills theories in six dimensional Minkowskian spacetime. Via dimensional reduction we construct a supersymmetric Yang-Mills Higgs theory on $\mathbb{R}^3$ with real fields

which we then promote to H^3. Under certain supersymmetric constraints, we show that hyperbolic monopoles configurations of this theory preserve, again, one half of the supersymmetry. Then, through investigating the geometry of the moduli space we show that the moduli space is described by real coordinate functions (zero modes), and we construct two sets of 2-sphere of real complex structures that act linearly on the tangent bundle of the moduli space, but don't behave like unit quaternions. This result coincides with the result of Bielawski and Schwachhöfer, who called this new type of geometry pluricomplex geometry. Finally, we show that in the limiting case, when the radius of curvature H^3 is set to infinity, the geometry becomes hyperkähler which is the geometry of the moduli space of euclidian monopoles.

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Contents

Chapter 1

Introduction

1.1 Monopoles and the Higgs mechanism

In the mid sixties and early eighties of the previous century, two major ideas were suggested to answer some unsolved questions in Particle Physics and Cosmology. These theories play, also, significant roles in the monopoles study at the level of constructing them and solving their existence problem. The general acceptance of these two theories has led to enormous experimental researches for more than forty years, which have recently flourished into a grand discovery for one of them and some promising results for the other.

At one end of the spectrum we have the famous discovery of the Higgs boson on the 4th of July 2012 at the CERN's large hadron collider via a collaborative work of the ATLAS and CMS experiments [1, 2]. The discovery of the Higgs boson didn't just mark the finding of the last missing fundamental particle, but it, more importantly, verifies the correctness of our understanding for the mechanism responsible for mass generation, the Higgs mechanism. The Higgs mechanism has been used in locally gauge invariant theories to give masses to bosons and fermions. The importance of the role of this mechanism has guaranteed François Englert and Peter Higgs the Nobel prize in physics on the 8th of October 2013:

> *for the theoretical discovery of a mechanism that contributes to our understanding*
> *of the origin of mass of subatomic particles, and which recently was confirmed*

through the discovery of the predicted fundamental particle, by the ATLAS and

CMS experiments at CERN's Large Hadron Collider

For the magnetic monopoles the discovery of the Higgs boson is of crucial importance and very intimate mathematical connection. The original notion of a magnetic monopole, a stable particle carrying a magnetic charge, appeared as a natural suggestion for a symmetry between the electric and magnetic fields in the Maxwell's equations. Dirac in 1931 [3] was the first to give a convincing argument of the monopoles concept by constraining their existence to the quantization of the electric charge. However, the Dirac monopoles doesn't have regular magnetic potential. Any vector potential whose curl is equal to a field of Coulomb form must be singular along a line (Dirac string) running from the origin to spatial infinity. This physical anomaly in addition to the lack of experimental evidence led to a natural fade of the idea of magnetic monopoles. Moreover, Dirac's electric charge quantization argument was later replaced by the fact that the electromagnetic $U(1)$ gauge group in a unified gauge theory is compact, which implies that the electric charge operator obeys commutation relations with other operators of the theory and these relations require that the eigenvalues of the electric charge operator to be integer multiple of a fundamental unit.

The beautiful twist in the history of magnetic monopoles happened in 1974 when 't Hooft [4] and Polyakov [5] independently found finite energy solutions for the bosonic part of the Georgi-Glashow model. The 't Hooft-Polyakov solutions asymptotically behave like Dirac monopoles, however, unlike Dirac monopoles, 't Hooft-Polyakov monopoles enjoy regularity at every point in space. This regularity is due to the nonzero vacuum expectation value of the Higgs field which forms the cornerstone of the Higgs mechanism. When the Higgs mechanism was discovered [6, 7, 8, 9, 10], it was the first successful attempt that allows to incorporate mass in a gauge theory without breaking the renormalizability. The Georgi-Glashow model is an upgrade of the Yang-Mills theory [11] by coupling it to a Higgs field, and hence using the Higgs mechanism, via spontaneous breaking of the gauge symmetry, to generate mass for the particle contents of the theory which consist of two massive charged vector bosons, a massive neutral scalar, and a photon. Studying the particle spectrum of a gauge theory is com-

monly achieved by fixing the gauge and then expanding the theory around a vacuum configuration. This method only lays our hands on the perturbative spectrum, but it is always interesting to investigate the spectrum beyond the perturbation theory. This is equivalent to looking for static stable solutions of the field equations, other than the vacuum configuration, i.e. "soliton solutions". The energy finiteness implies that the fields must approach the vacuum configurations at spatial infinity (sphere). In the Georgi-Glashow model the vacuum configurations define a sphere of radius equal to the vacuum expectation value of the Higgs field. Hence each field configuration can be understood as a map from the spatial boundary S^2 to the vacuum manifold S^2_{vac} with a topological degree. Moreover, since the topological degree is an integer, it implies that configurations with different degree can't be transformed into each other via any continuos deformation. In addition, any field configuration satisfying the finite energy constraints must become abelian outside the core of the monopoles and hence a solution of a version of Maxwell's field equations [12]. If we consider the vacuum configuration to be the constant map, hence of degree zero, then the 't Hooft-Polyakov solutions satisfy all the required conditions. The last, but most important, imprint of the non-zero vacuum expectation value of the Higgs field on the monopole is via its nature. The nature of the monopole source can be best understood by evaluating the total flux of the magnetic field which gives an integer multiple of 2π, and hence invariant under any time evolution of the field configurations. This implies that the source of the field configurations, and in particular the 't Hooft-Polyakov monopole, is purely topological [13], and hence the everywhere smoothness of the non-abelian monopoles.

Although a pure Yang-Mills theory has no soliton-like solutions in three space dimensions [14], the addition of a Higgs field with a non zero vacuum expectation value leads to the generation of the monopoles in their modern form. This significant physical signature that the Higgs field and the Higgs mechanism has on monopoles makes the discovery of the Higgs particle a huge success for the monopole physics as well.

The popular description of the Higgs mechanism is a case of spontaneous local gauge symmetry breaking [15]. A local gauge symmetry is not a symmetry of nature

but of our description of the physics in nature, in other words a local gauge symmetry connects different mathematical descriptions of the same physical state. This means that we shouldn't expect any physical consequences from spontaneous gauge symmetry breaking. However, the Higgs mechanism's physical consequences are mathematically evident and experimentally measured. This conceptual problem has led many philosophers of physics to investigate the correctness of the mechanism and to question the existence of a grand unified theory inspired by the success of the quantum electrodynamics and based on analogy with its gauge symmetry [16, 17, 18, 19, 20, 21]. In addition, this conceptual discussion had also concerned some physicists like 't Hooft [22] and Witten [23]. Luckily this confusion about the Higgs mechanism can be gauged away by describing the mechanism in a gauge invariant way. This procedure can be done by first writing the action in terms of gauge invariant variables, hence rendering the theory independent of the involved gauge group. This will factor out the gauge symmetry and therefore will be no need to fix the gauge. Then we can proceed in the traditional way by studying small fluctuations around the ground state. This procedure was originally done by Higgs for abelian gauge theories [8] and by Kibble for the non abelian case [24]. Moreover, this gauge independent account for the Higgs mechanism is discussed in Rubakov [25], and more recently has been adopted in some reviews and papers [26, 27, 28, 29].

1.2   Monopoles and inflation

At the other end of the spectrum we have some promising recent results that support an inflationary scenario in the birth of the Universe, in addition to experimental confirmations of predictions made by the inflation theory. The inflation theory in its various models, the old [30], the new [31, 32], and the chaotic [33], has inserted in the very early history of the Universe an extra phase of exponential expansion, the inflationary phase, where in 10^{-36} seconds the distance between two points stretched by at least a factor of 10^{26}, which means that the size of an atomic nucleus became the size of the solar system [34]. This rapid exponential expansion solves the questions that the standard Big Bang Cosmology alone can't answer. Among these ques-

tions is the lack of experimental detection for the primordial monopoles that appear naturally in Grand Unified Theories [35]. The 't Hooft-Polyakov monopoles are solutions to the Georgi-Glashow model where the symmetry group $SU(2)$ of the model is spontaneously broken by the vacuum configuration to $U(1)$, which has a nontrivial fundamental group $\pi_1(U(1)) = \mathbb{Z}$. It turns out that topological solutions appear as natural property when the symmetry group breaks by the vacuum manifold into a subgroup that has a nontrivial fundamental group, this mechanism is known as the Kibble mechanism [24]. Grand Unified Theories are invariant under the action of simple gauge groups that must break down after phase transition to leave the $U(1)$ group of electromagnetism intact, hence formation of magnetic monopoles. If we assume that inflation took place at the energy scale of the Grand Unified Theory, then all the magnetic monopoles were produced during inflation, and therefore their density was diluted by the exponential expansion to an unobservable level.

Inflation has gone through many stages from being a very speculative idea to becoming part of the standard Cosmology. Inflation is not only a theory that was constructed to fit some preexisting facts, but it also made a bunch of predictions. After thirty years of constructing the inflation theory, all of its predictions have been confirmed except for the gravitational waves. The most important inflation predictions that have been very successful, thanks to the remarkable progress in the development of the microwave detectors starting with the *Cosmic Background Explorer* (COBE) [36], are:

Decrease in the curvature of the Universe: Just like the curvature of a balloon decreases as the balloon is inflated, the curvature of the Universe is decreasing since inflation ended, and the current *Cosmic Microwave Background* (CMB) measurements show that the curvature is at least four times the curvature of the observable Universe [37].

Non-invariant scale of the density perturbation: The inflationary phase has to end, so as time progresses the inflation rate slowly decreases. The quantum fluctuations generated during inflation are proportional to the inflation rate [38]. Hence, fluctuations generated earlier have bigger amplitude and were stretched more, whereas fluctu-

ations that were generated later have smaller amplitude and didn't stretch as much. Therefore, according to inflation we should expect to see density perturbations of larger amplitude on larger angular scale of the CMB angular power spectrum, and to see those of smaller amplitude on smaller angular scale. Strong evidence of a departure from scale invariance has indeed been found through analysis of the CMB angular power spectrum [39].

Gaussian perturbation: In vacuum a free field ϕ, the inflaton field in the inflationary scenario, has a Gaussian probability distribution (wave function of harmonic oscillator in its ground state is Gaussian). Using Taylor expansion, we see that the energy density perturbation $\delta\rho = (dV/d\phi)\delta\phi + \frac{1}{2}(d^2V/d\phi^2)\delta\phi^2$ departs from Gaussianity by the second derivative of the potential energy $V(\phi)$. However, for inflation to work the second derivative of the potential energy should be very small (the slow-roll condition of the inflaton field [38]). Hence, according to inflation the CMB fluctuations should be very precisely, but not exactly, Gaussian. According to the most recent *Planck* data [40] the CMB density fluctuations departure from Gaussianity is smaller than 0.1%. In addition to these significant evidences for inflation, there are some others that have already been experimentally confirmed, for a full review one can check the most recent review from the *Planck* collaboration [40] .

However, a key prediction of the inflation theory, the primordial gravitational waves, remains unconfirmed. The 2.7K photons left from the Big Bang is uniform as we cross the cosmos with a small deviation measured to be $\delta_S = \frac{\Delta T}{T} \sim 10^{-5}$ [41]. This anisotropy in the cosmic microwave background can be understood in the context of quantum perturbations in the gravitational and scalar fields during inflation. The effect of the quantum fluctuations in the scalar field on the inhomogeneity of the CMB is physically different than the effect caused by the fluctuations in the gravitational field. On the one hand, the quantum fluctuations in the scalar field, δ_s, caused the perturbations in the energy density, have become denser due to gravity and later formed galaxies and other stellar objects that fill the cosmos (e.g. Andromeda galaxy and Milky Way are approaching each other and will collide to form a bigger object). On the other hand, quantum fluctuations in the gravitational field during inflation

"tensor fluctuation := δ_T" formed the primordial gravitational waves which caused vorticity in the polarization field of the CMB. But, a signature of primordial gravitational waves on the polarization of the CMB, called "B modes", has not been found yet. Several balloon and ground based experiments are relentlessly collecting data to measure the CMB polarization so that they can tease out the signature of gravitational waves. In March of last year, the BICEP2 collaboration announced the detection of B modes whose power spectrum had an angular momentum consistent with inflation [42]. Subsequent data from the *Planck* collaboration [43] and, most recently, a collaborative cross-correlation of BICEP2 and *Planck* data sets [44] have demonstrated that the signal originally reported by BICEP2 is consistent with having arisen entirely from dust emission in our own galaxy. However, the *Planck*/BICEP2 collaboration was able to put an upper limit for the tensor to scalar ratio $r = \frac{\delta_T^2}{\delta_S^2}$, which turns out to be consistent with that obtained indirectly by the *Planck* collaboration in 2013 based on the analysis of the CMB temperature fluctuations only. Detecting the "B modes" and hence measuring r, would allow cosmologists to infer the energy scale of the inflationary potential and determine the right model for the potential energy. Bamba, Nbjiri, and Odintsov [45] have, recently, constructed different scalar field models for inflation that can be consistent with different limits of r. The quest for the B-modes of the CMB is not over, BICEP2 and *Planck* are collecting data on different frequencies, and there are two proposals for new generations of balloon experiments the LiteBIRD which is a polarization-sensitive microwave experiment planned to be launched in 2020, and the COrE+ proposal (COrE for Cosmic Origins Explorer) for a CMB detector with 10 times more sensitivity than *Planck* and planned to be launched in late 2020.

After all the empirical successes, inflation is by far the best candidate for the mechanism that generated the primordial density fluctuations and the primordial gravitational waves. The 2014 Kavli Prize in Astrophysics was awarded to Alan Guth, Andrei Linde, and Alexei Starbinsky for their leading work on cosmic inflation. The consequences of the inflation evidences along with the ATLAS/CMS results on the monopoles study are staggering. The only argument that was against studying monopoles, absence of experimental evidence, is now weakened by the inflation empirical

successes, and hopefully will soon be washed away once the B modes are detected.

The essential role that magnetic monopoles play in Grand Unified Theories, in addition to the mathematical beauty they carry in their constructions or in the geometry of their moduli spaces, make them a very interesting topic and a fruitful domain of research.

1.3   BPS monopoles

The mathematical beauty of the geometry of the monopole moduli space geometry represents the heart of this thesis. The monopole addressed are the Bogomol'nyi-Prasad-Sommerfield, "BPS", monopoles with arbitrary charge, $2\pi N$, on hyperbolic space. For charge 2π, a spherically symmetric BPS monopole is a 't Hooft-Polyakov monopole with minimum energy. Prasad and Sommerfield [46] first studied static minimum energy solutions for the Georgi-Glashow model. They obtained a limiting example of the 't Hooft Polyakov monopoles when they investigated the spherically symmetric case with charge equal to 2π. Later, Bogomol'nyi [47] analyzed static solutions with minimum energy, and derived the field equation describing them, namely, the Bogomol'nyi equation.

1.3.1   BPS monopoles on $\mathbb{R}^3$

BPS monopoles—that is, the solutions of the Bogomol'nyi equation—have been under the microscope by mathematicians and physicists for a long time. This equation and its solutions can be studied on any oriented Riemannian 3-manifold, but they are particularly interesting in Euclidean and hyperbolic spaces. One inspiring observation about BPS monopoles in these spaces is that they can be viewed as instantons in four-dimensional Euclidean space left invariant under the action of a one-parameter subgroup of isometries: translations (resp. rotations) in the case of Euclidean (resp. hyperbolic) BPS monopoles. Another way of saying this is that the Bogomol'nyi equation results from the four-dimensional self-duality equation by demanding independence on one of the coordinates.

To begin with, consider the Bogomol'nyi equation in Euclidean space

$$\nabla_A \phi = - \star F_A, \tag{1.1}$$

where ϕ satisfies some suitable boundary conditions that make the L^2 norm of F_A finite and $\star$ is the Hodge operator of $\mathbb{R}^3$. For a detailed treatment of Euclidean monopoles, one can check [48, 49, 50]. The ingredients of the Bogomol'nyi equation can be cast into a geometrical framework, where A can be viewed as a connection on a principal G-bundle P over $\mathbb{R}^3$ and F_A as its curvature. The Higgs field ϕ is a section of the adjoint bundle $\mathrm{ad}P$ over $\mathbb{R}^3$; that is, the associated vector bundle to P corresponding to the adjoint representation of G on its Lie algebra, and ∇_A is the covariant derivative operator induced on $\mathrm{ad}P$. A pair (A, ϕ) satisfying equation (1.1) is what we call a Euclidean monopole. If we now interpret ϕ as being the x_4 component of the connection, then equation (1.1) becomes the self-duality Yang-Mills equation on $\mathbb{R}^4$

$$F_A = \star F_A, \tag{1.2}$$

where all the fields are independent of the x_4 coordinate, and the $\star$-operation is now with respect to the flat Euclidean metric on $\mathbb{R}^4$.

1.3.2 BPS monopoles on H^3

For the case of hyperbolic monopoles we simply replace the Euclidean base space $\mathbb{R}^3$ with hyperbolic space H^3. To construct hyperbolic monopoles from instantons, instead of considering translationally invariant solutions of equation (1.2) we will, however, look for rotationally invariant solutions [51]. To be specific consider the flat Euclidean metric in $\mathbb{R}^4$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2. \tag{1.3}$$

If we choose the rotations to be in the (x_1, x_2)-plane and we let r and θ be the polar coordinates in that plane, we have

$$
\begin{aligned}
ds^2 &= dr^2 + r^2 d\theta^2 + dx_3^2 + dx_4^2 \\
&= r^2 \left(d\theta^2 + \frac{dr^2 + dx_3^2 + dx_4^2}{r^2} \right) .
\end{aligned}
\tag{1.4}
$$

The rotations now act simply as shifts in the angular variable θ. This coordinate system is valid in the complement $\mathbb{R}^4 \setminus \mathbb{R}^2$ of the $x_1 = x_2 = 0$ plane. Inside the parenthesis we recognise the metric on $S^1 \times H^3$, which is therefore shown to be conformal to $\mathbb{R}^4 \setminus \mathbb{R}^2$.

Now a wonderful fact about the self-duality equation is its conformal invariance: the Hodge $\star$ is conformally invariant acting on middle-dimensional forms in an even-dimensional manifold. This allows us to drop the conformal factor r^2 from the metric without altering the equation. If we now impose the condition that the gauge potential A is S^1 invariant, i.e., rotationally symmetric in the (x_1, x_2)-plane, and if we define $A_\theta = \phi$, the self-duality equation becomes the Bogomol'nyi equation on H^3. The Bogomol'nyi equation on H^3 is also given by equation (1.1) but with the $\star$-operation of H^3. The first constructions of a monopole solution on hyperbolic space were given in [52, 53, 54].

A BPS monopole in hyperbolic space is labelled by a mass $m \in \mathbb{R}^+$ and a charge $k \in \mathbb{Z}^+$ given by

$$
\begin{aligned}
m &= \lim_{r \to \infty} |\phi(r)| \\
k &= \lim_{r \to \infty} \frac{1}{4\pi m} \int_{H^3} \mathrm{tr}(F_A \wedge \nabla_A \phi) ,
\end{aligned}
\tag{1.5}
$$

and it is known [55] that hyperbolic monopoles exist for all values of m and k. In contrast to the Euclidean monopoles, m cannot be rescaled to unity in the hyperbolic case, as the value of m affects the monopole solutions [56]. Alternatively, one can normalise the mass to unity, but only at the price of rescaling the hyperbolic metric to one of curvature $-1/m^2$. The rotationally invariant instanton on $\mathbb{R}^4 \setminus \mathbb{R}^2$ corresponding to a hyperbolic monopole of charge k and mass m will extend to a rotationally invariant instanton on all of $\mathbb{R}^4$ if (and only if) $m \in \mathbb{Z}$.

In [57] Manton interpreted low energy dynamics of monopoles as geodesic mo-

tion on the moduli space; that is, the space of solutions up to gauge equivalence, and this ushered in an era of much activity in the study of the geometry of the moduli space. For the case of Euclidean monopoles, Atiyah and Hitchin showed in [48] that the moduli space has a natural hyperkähler metric and they found the explicit form of the metric for the moduli space of charge 2. Moreover, the metric of the moduli space of well separated monopoles was found in [58], where the monopoles were treated as point particles carrying scalar, electric and magnetic charges.

The hyperbolic case is much less understood. In [51], where Atiyah introduced hyperbolic monopoles, he writes:

Moreover, by varying the curvature of hyperbolic space and letting it tend to zero, the Euclidean case appears as a natural limit of the hyperbolic case. While the details of this limiting procedure are a little delicate, and need much more careful examination than I shall give here, it seems reasonable to conjecture that the moduli of monopoles remains unaltered by passing to the limit.

Atiyah also showed [59] that the moduli space $\mathcal{M}_{k,m}$ of hyperbolic monopoles of charge k and mass m can be identified with the space of rational maps of the form

$$\frac{a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k}{z^k + b_1 z^{k-1} + \cdots + b_k} \quad \text{with } k \geqslant 1, \tag{1.6}$$

where the polynomials in the numerator and denominator are relatively prime. Since the $a_1, \ldots, a_k, b_1, \ldots, b_k$ are complex numbers, the moduli space has real dimension $4k$.

Most of the progress in the study of hyperbolic monopoles was focused on finding methods of constructing multimonopole solutions, either by building a hyperbolic version of the Nahm transform [56, 60, 61, 62] or by studying the spectral curves associated with hyperbolic monopoles [63, 64, 65]. Progress on the geometry of the moduli space was hindered by the early realisation [56] that the natural L^2 metric, which in the Euclidean case induces upon reduction a hyperkähler metric on the moduli space, does not converge in the case of hyperbolic monopoles, suggesting that the geometry of the moduli space is not in fact Riemannian. Nevertheless, Hitchin [66] constructed

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a family g_m of self-dual Einstein metrics on the moduli space of centered hyperbolic monopoles with mass $m \in \mathbb{Z}$, which in the flat limit $m \to \infty$ recovers the Atiyah–Hitchin metric. It is an interesting open question to relate Hitchin's construction to the physics of hyperbolic monopoles.

The situation has changed dramatically in recent times due to the seminal work of Bielawski and Schwachhöfer, based on earlier work of O. Nash [67]. Nash used a new twistorial construction of $\mathcal{M}_{k,m}$ to show that the complexification of the real geometry of the moduli space of hyperbolic monopoles is similar in some respects to the complexification of a hyperkähler geometry. Building on that work, Bielawski and Schwachhöfer [68] identified the real geometry of the moduli space of hyperbolic monopoles as "pluricomplex geometry", which is equivalent to saying that there is a $\mathbb{C}$-linear hypercomplex structure on the complexification $T_\mathbb{C}\mathcal{M}_{k,m}$ of the tangent bundle to the moduli space. Later in [69] Bielawski and Schwachhöfer studied the Euclidean limit of the pluricomplex moduli space of hyperbolic monopoles, and showed that in the limit one recovers an enhanced hyperkähler geometry, richer by an additional complex structure.

1.3.3 BPS monopoles geometries

Every textbook on monopoles, or solitons in general, dedicates a chapter on differential geometry and topology as a preliminary for the study of their moduli spaces. As mentioned in the previous section, the geometry of the moduli space of BPS monopoles can be very rich, depending on which background we study the Bogomol'nyi equation and, also, on the fields being complex or real, i.e. whether we are studying the real or the complex moduli space. In this section we review the geometries of the known moduli spaces of BPS monopoles. This is a brief description of each geometry, for a full review on the constructions of the objects of each geometry one can refer to chapter three of [50] and chapter three of [61], or [70, 71, 72] for very thorough references on the subject.

We will start by making some key definitions for these geometries. Suppose that a $2n$-dimensional manifold M admits a globally defined $(1, 1)$ tensor J with local ex-

pression $J_\mu{}^\nu dx^\mu \otimes \partial_\nu$, which enjoys the following properties:

$$J_\mu{}^\mu = 0, \tag{1.7}$$

$$J_\mu{}^\kappa J_\kappa{}^\nu = -\delta_\mu{}^\nu, \tag{1.8}$$

then the tensor is called an **almost complex structure** and M is called an **almost complex manifold**. Using the almost complex structure, we can define a mixed three-tensor, called the **Nijenhuis tensor** N, with components

$$N_{\mu\nu}{}^\rho = \frac{1}{6} J_\mu{}^\sigma \partial_{[\sigma} J_{\nu]}{}^\rho - (\mu \leftrightarrow \nu). \tag{1.9}$$

It can be proven that the Nijenhuis tensor vanishes identically if and only if the almost complex structure is a complex structure (also called **integrable complex structure**) (see e.g. [73]). The latter condition means that it is possible to find a holomorphic atlas on M, i.e. in every chart coordinates $\{z^m, \bar{z}^{\bar{m}}\}$ exist for which

$$J_m{}^n = i\delta_m{}^n, \qquad J_{\bar{m}}{}^{\bar{n}} = -i\delta_{\bar{m}}{}^{\bar{n}}, \qquad J_{\bar{m}}{}^n = J_m{}^{\bar{n}} = 0, \tag{1.10}$$

with $m, \bar{m} = 1, \ldots, n$. If an almost complex manifold is Riemannian and the metric satisfies

$$J_\mu{}^\rho J_\nu{}^\sigma g_{\rho\sigma} = g_{\mu\nu}, \tag{1.11}$$

the metric is called **almost hermitian**. This condition is equivalent to $J_{\mu\nu} = J_\mu{}^\rho g_{\nu\rho}$ being antisymmetric, and $J_{\mu\nu}$ is then called the fundamental two-form. An almost hermitian manifold is called hermitian if the Nijenhuis tensor vanishes, and there exist a connection that preserves both the complex structure and the metric. An important class of hermitian manifolds are **Kähler manifolds**. For a Kähler manifold, the corresponding connection on the tangent bundle is the Levi-Civita connection.

Suppose that $V = \mathbb{R}^{4n}$. A triple $H = (J^1, J^2, J^3)$ of complex structures with

$$J^\alpha J^\beta = -\mathbb{I}_{4n}\delta^{\alpha\beta} + \epsilon^{\alpha\beta\gamma} J^\gamma \tag{1.12}$$

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is called a **hypercomplex structure** on V. Denote the space of endomorphisms of V by $\text{End}V$. The three-dimensional subspace Q of $\text{End}V$, defined by

$$Q = \mathbb{R}J^1 + \mathbb{R}J^2 + \mathbb{R}J^3, \tag{1.13}$$

is called a **quaternionic structure**, i.e. Q is the set of real linear combinations of the complex structures.

Hypercomplex manifolds: Given a manifold with an almost hypercomplex structure, there always exists a unique connection, Γ, preserving it

$$\mathscr{D}_\mu \vec{J}_\nu{}^\rho = \partial_\mu \vec{J}_\nu{}^\rho - \Gamma_{\mu\nu}{}^\sigma \vec{J}_\sigma{}^\rho + \Gamma_{\mu\sigma}{}^\rho \vec{J}_\nu{}^\sigma = 0. \tag{1.14}$$

If the torsion vanishes, the manifold is called hypercomplex and this is equivalent with the vanishing of the diagonal Nijenhuis tensor defined as

$$N^d{}_{\mu\nu}{}^\rho = \frac{1}{6} \vec{J}_\mu{}^\sigma \partial_{[\sigma} \vec{J}_{\nu]}{}^\rho - (\mu \leftrightarrow \nu). \tag{1.15}$$

In that case, the torsionless affine connection on the tangent manifold TM is called the Obata connection [74] and its components are given by the following expression:

$$\overset{o}{\Gamma}_{\mu\nu}{}^\rho = -\frac{1}{6} \left[2\partial_{(\mu} \vec{J}_{\nu)}{}^\sigma + \vec{J}_{(\mu}{}^\tau \times \partial_\tau \vec{J}_{\nu)}{}^\sigma \right] \cdot \vec{J}_\sigma{}^\rho. \tag{1.16}$$

More generally, given an almost hypercomplex structure, the unique connection preserving it is given by

$$\Gamma_{\mu\nu}{}^\rho = \overset{o}{\Gamma}_{\mu\nu}{}^\rho + N^d{}_{\mu\nu}{}^\rho. \tag{1.17}$$

Hyperkähler manifolds: These are Riemannian manifolds with hermitian hypercomplex structures, and where the Obata connection coincides with the Levi-Civita connection.

Pluricomplex manifolds: On a manifold with pluricomplex geometry we have two 2-sphere of integrable complex structures (i.e. Q_1 and Q_2 according to (1.13)) that don't have any anticommutation relations between them, and these complex structures de-

compose the complexified tangent space as $\mathbb{C}^{2n} \otimes \mathbb{C}^2$. If we complexify a pluricomplex structure we get a hypercomplex structure. This implies that if we look at the complex thickening $M^{\mathbb{C}}$ of a manifold M with pluricomplex geometry we find that its geometry is hypercomplex. Thus, the pluricomplex geometry on M can be viewed as a biquaternionic geometry on $M^{\mathbb{C}}$. These different views of the pluricomplex geometry are discussed in this thesis via the study of the geometry of the real and complex moduli spaces of hyperbolic monopoles.

1.4 Monopoles and rigid supersymmetry

All topological solitons saturate a certain energy bound, and this feature gives a sign that they are supersymmetric in nature. This link is rooted in the fact that in a massive supersymmetry representation with central charges the bound is enforced by the unitarity property of the supersymmetry transformations [75].

The study of the supersymmetric extensions of topological objects was initiated by Zumino studying the supersymmetry of instantons [76]. Starting with $N = 2$ supersymmetric Yang-Mills theory on four dimensional Euclidean space, Zumino showed that instantons are supersymmetric under half of the supersymmetry parameters, and using supersymmetry he computed the index of the Dirac operator by showing that for solutions with winding number n of the $SU(2)$ supersymmetric Yang-Mills equations there are $8n$-dimensional space of instantons and $4n$-dimensional space of solutions for Dirac equation and hence recovering the results of Atiyah, Hitchin and Singer [77], and Brown et al. [78]. Moreover, the supersymmetric extensions of lumps and vortices have been also successfully investigated in [79] and in [80, 81] receptively. As for Skyrmions, the energy satisfies certain topological bound which is called the Faddeev-Bogomol'nyi lower bound [82]. But unlike other topological objects, this bound cannot be saturated for a non-trivial value of the Skyrme field when the spatial domain is $\mathbb{R}^3$. However, if the spacial domain is taken to be S^3 the bound can be saturated [83], but a supersymmetric extension for the Skyrme model on S^3 has not yet been found.

The supersymmetry of Euclidean BPS monopoles was obtained in [84, 85, 86] among others. In this thesis we will exhibit the supersymmetric extension in detail for the case

of the hyperbolic monopoles. For the Euclidean case one can show that monopoles are supersymmetric objects by staring from $N = 1$ supersymmetric Yang-Mills theory in six dimensional Minkowski space and then perform a dimensional reduction to $N = 2$ supersymmetric Yang-Mills-Higgs theory in four dimensional Minkowski space. Then, by analyzing the static solutions of the field equation for bosons and fermions we find that solutions of the Bogomol'nyi equations plus their superpartners (supersymmetric BPS monopoles configurations) form a subset of the field equations that minimize the energy and only break half of the supersymmetry. A similar approach is to start with $N = 1$ supersymmetric Yang-Mills theory in ten dimensional Minkowski space and then perform a dimensional reduction to $N = 4$ supersymmetric Yang-Mills-Higgs theory in four dimensional Minkowski space. A detailed treatment for both cases can be found in Figueroa-O'Farrill's notes [87].

The study of the supersymmetry of hyperbolic monopoles, on the other hand, is different because of the need to construct a supersymmetric Yang-Mills-Higgs theory on a curved space, namely the hyperbolic space. This kind of supersymmetry on a curved background where the gravity is not dynamic is now known as rigid supersymmetry. Rigid supersymmetry has been very fashionable in the last few years, and many results have shown that one can learn a lot about a theory by putting it on curved space. Hence, many known supersymmetric theories were, recently, studied on curved spaces which led to some important developments and interesting results especially in testing the AdS/CFT conjecture and computing new observables in known theories. The AdS/CFT conjecture is a proposed duality relating the quantum physics of strongly correlated many-body systems to the classical dynamics of gravity in one higher dimension. In its original appearances [88, 89, 90], the correspondence related a four-dimensional Conformal Field Theory (CFT) to the geometry of an anti-de Sitter (AdS) space in five dimensions. Studying supersymmetric theories on curved spaces, for the purpose of finding examples that second the AdS/CFT correspondence, was initiated with Pestun [91], who studied $N = 4$ supersymmetric Yang-Mills theory on S^4. Pestun computed the partition functions and the correlation functions of Wilson loops and proved the Erickson-Semenoff-Zarembo/Drukker-Gross conjecture

[92, 93] used in many studies to test duality. This seminal work inspired Kapustin at al. [94] to study supersymmetric theories on S^3 to compute the partition function via the localization techniques and successfully test some duality conjectures. In addition, many other curved spaces become popular for supersymmetric theories for example $S^3 \times S^1$ [95] and $S^2 \times S^1$ [96].

Traditionally, constructing a supersymmetric theory on a curved background can be done as follows; we start with a supersymmetric Lagrangian on flat space $\mathcal{L}_{\mathbb{R}^n}$ written in terms of the dynamical field components of the theory, and supersymmetry transformations δ_ϵ relating the bosonic fields to fermionic ones and vice versa. When we place the theory on a non trivial manifold by simply introducing the metric into $\mathcal{L}_{\mathbb{R}^n}$ we find that, in general, curved space breaks supersymmetry. We then try to restore supersymmetry by adding correction terms that are invariant under the gauge symmetry groups under consideration and inversely proportional to powers of the characteristic size of the curved space. We keep doing this iterative procedure, we first try terms inversely proportional to the radius if it doesn't work we add terms inversely proportional to the square of the radius and so on, until the modified Lagrangian becomes invariant under the modified supersymmetry transformations and the supersymmetry algebra closes. This procedure will be followed here to construct a supersymmetric Yang-Mills-Higgs Lagrangian on H^3. Historically, this method was first adopted by Zumino in 1977 to study $N = 1$ supersymmetric theory on AdS_4 [97], then used by Diptiman Sen in 1987 to study various supersymmetric theories on $S^3 \times \mathbb{R}^1$ [98], and recently, starting from 2007, there is a vast literature, part of which mentioned in the previous paragraph, using this method to study supersymmetric theories on curved spaces mainly for duality testings purposes.

The big interest in supersymmetric theories on non trivial background has motivated Festuccia and Seiberg [99] to find a general procedure to construct these theories. Their method is quite useful because it makes use of the supergravity theories that exist in literature. We start with a supergravity theory which can naturally be written as $\mathcal{L}_{\text{curved}} + \mathcal{L}_{\text{sugra}}$, where $\mathcal{L}_{\text{curved}}$ contains the part of the dynamical fields and their covariant derivative that would exist without gravity (in a supersymmet-

ric theory), and $\mathcal{L}_{\text{sugra}}$ contains the fields that were introduced after we coupled the theory to supergravity like gravitinos (ψ_μ) and auxiliary fields. By setting the gravitino transformation equal to zero we find the constraint on the supersymmetry parameter which can be satisfied for different choices of the auxiliary and gravity fields. If we find a field configuration for $\{g_{\mu\nu}, \psi_\mu, \text{aux. fields}\}$ that is invariant under supersymmetry transformation it implies that $\mathcal{L}_{\text{curved}}$ is invariant under the supersymmetry transformation, and then we can render gravity non-dynamical. Using this method, Festuccia and Seiberg were able to reproduce all the popular supersymmetric theories on AdS_4, S^4, $S^3 \times S^1$ and $S^3 \times \mathbb{R}$. Based on this seminal work, many projects have been devoted to studying the properties for compact curved backgrounds to exhibit supersymmetry in various dimension and for both signatures, Euclidean [100, 101, 102, 103, 104, 105, 106, 107], and Lorentzian [108, 103, 109, 110]. The reason behind the interest in compact manifold is that supersymmetric field theories on compact manifold are useful scheme for localization techniques which can be used to calculate some observables. The background considered in this thesis is H^3, which is a non-compact manifold, but it would be nice if Festuccia and Seiberg method can be used to reproduce our results for a supersymmetric Yang-Mills-Higgs theory on H^3.

1.5 Summary and overview

In the spirit of the preceding discussion, this thesis studies the geometry of the moduli space of hyperbolic monopoles using supersymmetry, and thus shows that the pluricomplex nature of the moduli space of hyperbolic monopoles is a natural consequence of supersymmetry. With our approach we find that starting from a Euclidean four dimensional supersymmetric Yang–Mills theory and constructing a supersymmetric Yang–Mills–Higgs theory on H^3 will lead to studying the geometry of the complex moduli space of hyperbolic monopoles, which we show to be hypercomplex. On the other hand, starting from a Minkowskian six dimensional supersymmetric Yang–Mills theory and constructing a supersymmetric Yang–Mills–Higgs theory on H^3 allow to explore the real moduli space of hyperbolic monopolies which we show to be pluricomplex. One novel aspect of our construction is that the constraints coming

from supersymmetry are imposed by demanding the closure of the supersymmetry algebra and not the invariance of the effective action for the moduli, which does not exist due to the lack of convergence of the L^2 metric. This is reminiscent of the results of Stelle and Van Proeyen [111] on Wess–Zumino models without an action functional, in which the geometry is relaxed from Kähler to complex flat. In fact, morally one could say that pluricomplex is to hyperkähler what complex flat is to Kähler. Revisiting supersymmetric theories by relaxing the requirement of an action existence was also studied in [112, 113, 114]. Another novel aspect of our construction is the connection between geometry and low energy supersymmetric dynamics. Supersymmetry relates fermions to bosons, which means it relates objects that satisfy first order differential equation, the fermions, to objects that satisfy second order differential equation, the bosons. At the level of moduli space, these relations can be interpreted as maps between the odd and even coordinates of the moduli space, and hence one would expect that a lot of information about the geometry of the moduli space can be read off the supersymmetry transformations.

Chapter two is dedicated to construct a supersymmetric Yang–Mills–Higgs theory in hyperbolic space by starting with supersymmetric Yang–Mills theory on Minkowski spacetime, euclideanising to a supersymmetric Yang–Mills theory on $\mathbb{R}^4$, reducing to $\mathbb{R}^3$ and deforming to a supersymmetric theory on H^3. Then we show that the hyperbolic monopoles coincide with the configurations which preserve precisely one half of the supersymmetry.

In chapter three, we show that the geometry of the complex moduli space of hyperbolic monopoles is hypercomplex. We start the analysis of the moduli space by studying the linearisation of the Bogomol'nyi equation and identifying the bosonic and fermionic zero modes and how the unbroken supersymmetry relates them. A possibly surprising result is the fact that supersymmetry suggests a small modification of the Gauss law constraint, which depends explicitly on the hyperbolic curvature. Moreover, as a byproduct of our analysis we find the index of the Dirac operator in the presence of a hyperbolic monopole. Then we linearise the unbroken supersymmetry and by demanding the on-shell closure of the supersymmetry algebra we find

the conditions satisfied by the geometry of the moduli space.

In chapter four, we shift the focus to the study of supersymmetric Yang–Mills–Higgs theory on H^3 with real fields which we obtain by dimensional reduction of six dimensional supersymmetric Yang–Mills theory on Minkowski space. We compare our results with family A theories from [115] which is model describing how to obtain a supersymmetric Yang–Mills theory on $\mathbb{R}^n \times M^{(d+1-n)}$ starting from a supersymmetric Yang–Mills action $\mathbb{R}^{(d,1)}$, where M is a manifold that admits Killing spinors. We show that our theory coincides with an example from family A, where $d = 5$, $n = 3$ and M is H^3 and we study the supersymmetry algebra of the obtained theory. The special feature about this theory is that it is invariant under real supersymmetry transformations which hints that the ansatz for the zero modes that will later represents the basis of the moduli space are real.

In chapter five, we show that the geometry of the real moduli space of hyperbolic monopoles is pluricomplex. We show, first, that under supersymmetric constraints the equations of motion obtained in chapter four can be simplified to Bogomol'nyi equation on H^3 plus a Dirac equation for the fermions. Then, we show that the supersymmetric hyperbolic monopoles satisfy the simplified field equations and that they are $\frac{1}{2}$ "BPS" saturated. After that, we start analyzing the real moduli space, so we use the unbroken supersymmetry transformations to construct real zero modes which we show to satisfy the linearized Bogomol'nyi equation and a gauge background condition. Furthermore, by demanding that the complex structures on the tangent space must map the zero modes again to solutions of the linearized Bogomol'nyi equation and Gauss's law we construct two families of integrable complex structures that we show to have the properties of pluricomplex structures defined in [68]. In other words, we find two sets of 2-sphere complex structures that don't anticommute, and hence form a biquaternionic algebra. We finish this chapter by showing that in the limiting case, when the radius of curvature of hyperbolic space is set to infinity, the geometry of Euclidean monopoles emerges from the geometry of hyperbolic monopoles.

The thesis ends with with three appendices. The first one is on the Frölicher–Nijenhuis bracket of two endomorphisms, in the second one we show that the con-

nection defined on the complex moduli space of hyperbolic monopoles is the Obata connection, and in the third one we reduce the supersymmetry transformations of supersymmetric Yang-Mills theory from $\mathbb{R}^{(5,1)}$ to $\mathbb{R}^3$.

Chapter 2

Supersymmetric Yang-Mills-Higgs Theory on H^3 with Complex Fields

2.1 Introduction

The purpose of this chapter is to present a systematic construction of supersymmetric theories in hyperbolic space by the following procedure: start with supersymmetric Yang–Mills in Minkowski spacetime, euclideanise à la van Nieuwenhuizen–Waldron [116], reduce to $\mathbb{R}^3$ and deform to a theory on H^3. The euclideanisation will require complexifying the fields in the theory, which will turn out later to be crucial to study the geometry of the complex moduli space.

2.2 Off-shell supersymmetry in Euclidean 4-space

The first step has been done in [116], except that we expect that auxiliary fields should play an important role and thus must promote the theory to one with off-shell closure of supersymmetry (up to possibly gauge transformations).

2.2.1 On-shell SYM theory on $\mathbb{R}^4$

The Euclidean supersymmetric Yang–Mills action in $\mathbb{R}^4$ is obtained by integrating the Lagrangian density

$$\mathcal{L}^{(4)} = -\operatorname{Tr}\chi_R^\dagger \slashed{D}\psi_L - \tfrac{1}{4}\operatorname{Tr} F^2 , \tag{2.1}$$

where Tr denotes an ad-invariant inner product on the Lie algebra $\mathfrak{g}$ of the gauge group G, and where the subscripts L, R denote the projections

$$\psi_L = \tfrac{1}{2}(\mathbb{I}+\gamma^5)\psi \qquad \text{and} \qquad \chi_R^\dagger = \tfrac{1}{2}\chi^\dagger(\mathbb{I}-\gamma^5) , \tag{2.2}$$

where $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$, where $\gamma^\mu\gamma^\nu = \gamma^{\mu\nu} + \delta^{\mu\nu}$. This means that that $(\gamma^5)^2 = 1$. We can raise and lower indices with impunity, since the metric is $\delta_{\mu\nu}$. The action defined by $\mathcal{L}^{(4)}$ is invariant under gauge transformations, which infinitesimally take the form

$$\delta_\Lambda \psi_L = [\Lambda, \psi_L] \qquad \delta_\Lambda \chi_R^\dagger = [\Lambda, \chi_R^\dagger] \qquad \text{and} \qquad \delta_\Lambda A_\mu = -D_\mu\Lambda = -\partial_\mu\Lambda + [\Lambda, A_\mu] , \tag{2.3}$$

with $\Lambda \in C^\infty(\mathbb{R}^4; \mathfrak{g})$. Furthermore, it is invariant under the supersymmetry transformations

$$\delta_\varepsilon \psi_L = \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L$$
$$\delta_\varepsilon \chi_R^\dagger = -\tfrac{1}{2}\varepsilon_R^\dagger\gamma^{\mu\nu}F_{\mu\nu} \tag{2.4}$$
$$\delta_\varepsilon A_\mu = -\varepsilon_R^\dagger\gamma_\mu\psi_L + \chi_R^\dagger\gamma_\mu\varepsilon_L ,$$

where ε_L and $\varepsilon_R^\dagger$ are constant spinor parameters of the indicated chirality. Since ε_L and $\varepsilon_R^\dagger$ are independent, we actually have two supersymmetry variations, which we will denote δ_L and δ_R and leave the parameter unspecified when there is no danger of confusion. In this notation we have

$$\delta_L \psi_L = \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L \qquad\qquad \delta_R \psi_L = 0$$
$$\delta_L \chi_R^\dagger = 0 \qquad\qquad \delta_R \chi_R^\dagger = -\tfrac{1}{2}\varepsilon_R^\dagger\gamma^{\mu\nu}F_{\mu\nu} \tag{2.5}$$
$$\delta_L A_\mu = \chi_R^\dagger\gamma_\mu\varepsilon_L \qquad\qquad \delta_R A_\mu = -\varepsilon_R^\dagger\gamma_\mu\psi_L .$$

2.2.2   Introducing auxiliary field

Notice that if δ'_{L} is defined as δ_{L} but with a different supersymmetry parameter, say $\varepsilon'_{\mathrm{L}}$, then on the gauge field $[\delta_{\mathrm{L}}, \delta'_{\mathrm{L}}]A_\mu = 0$, and similarly $[\delta_{\mathrm{R}}, \delta'_{\mathrm{R}}]A_\mu = 0$. On the fermion, however, this will not be true off-shell and it is for that reason that we will introduce an auxiliary field. Indeed, one finds

$$[\delta_{\mathrm{L}}, \delta'_{\mathrm{L}}]\psi_{\mathrm{L}} = \delta_{\mathrm{L}}(\tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon'_{\mathrm{L}}) - \delta'_{\mathrm{L}}(\tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_{\mathrm{L}}) \,. \tag{2.6}$$

Using that

$$\delta_{\mathrm{L}}F_{\mu\nu} = D_\mu\delta_{\mathrm{L}}A_\nu - D_\nu\delta_{\mathrm{L}}A_\mu = D_\mu(\chi_{\mathrm{R}}^\dagger\gamma_\nu\varepsilon_{\mathrm{L}}) - D_\nu(\chi_{\mathrm{R}}^\dagger\gamma_\mu\varepsilon_{\mathrm{L}}) \,, \tag{2.7}$$

whence

$$[\delta_{\mathrm{L}}, \delta'_{\mathrm{L}}]\psi_{\mathrm{L}} = D_\mu\chi_{\mathrm{R}}^\dagger\gamma_\nu\varepsilon_{\mathrm{L}}\gamma^{\mu\nu}\varepsilon'_{\mathrm{L}} - D_\mu\chi_{\mathrm{R}}^\dagger\gamma_\nu\varepsilon'_{\mathrm{L}}\gamma^{\mu\nu}\varepsilon_{\mathrm{L}} \,, \tag{2.8}$$

where we have used that $\gamma_\mu^\dagger = \gamma_\mu$ and also that $(\chi^\dagger\psi)^\dagger = +\psi^\dagger\chi$ for anticommuting spinors. (One might think that the $+$ sign violates the sign rule, but it does not because ψ and $\psi^\dagger$ are independent fields, etc.)

In order to further manipulate the right-hand side of $[\delta_{\mathrm{L}}, \delta'_{\mathrm{L}}]\psi_{\mathrm{L}}$ we must make use of a Fierz identity. The basic Fierz identity in $\mathbb{R}^4$ for anticommuting spinors is given by

$$\psi\chi^\dagger = -\tfrac{1}{4}\chi^\dagger\psi\mathbb{I} - \tfrac{1}{4}\chi^\dagger\gamma^5\psi\gamma_5 - \tfrac{1}{4}\chi^\dagger\gamma^\mu\psi\gamma_\mu + \tfrac{1}{4}\chi^\dagger\gamma^\mu\gamma^5\psi\gamma_\mu\gamma_5 + \tfrac{1}{8}\chi^\dagger\gamma^{\mu\nu}\psi\gamma_{\mu\nu} \,. \tag{2.9}$$

Two special cases will play a rôle in what follows:

$$\psi_{\mathrm{L}}\chi_{\mathrm{R}}^\dagger = -\tfrac{1}{2}\chi_{\mathrm{R}}^\dagger\gamma^\mu\psi_{\mathrm{L}}\gamma_\mu P_{\mathrm{R}} \,, \tag{2.10}$$

and

$$\psi_{\mathrm{R}}\chi_{\mathrm{R}}^\dagger = -\tfrac{1}{2}\chi_{\mathrm{R}}^\dagger\psi_{\mathrm{R}}P_{\mathrm{R}} - \tfrac{1}{8}\chi_{\mathrm{R}}^\dagger\gamma^{\mu\nu}\psi_{\mathrm{R}}\gamma_{\mu\nu} \,, \tag{2.11}$$

where $P_{\mathrm{R}} = \tfrac{1}{2}(\mathbb{I} - \gamma_5)$. Of course, for commuting spinors, we simply flip all signs in

the right-hand side.

Using the Fierz formula (2.10), we may rewrite

$$[\delta_L, \delta_L']\psi_L = -\tfrac{1}{2}D_\mu\chi_R^\dagger\gamma^\sigma\varepsilon_L'\gamma^{\mu\nu}\gamma_\sigma\gamma_\nu\varepsilon_L + \tfrac{1}{2}D_\mu\chi_R^\dagger\gamma^\sigma\varepsilon_L\gamma^{\mu\nu}\gamma_\sigma\gamma_\nu\varepsilon_L' \ . \tag{2.12}$$

Using that $\gamma^{\mu\nu}\gamma_\sigma\gamma_\nu = -\gamma_{\mu\sigma} - 3\delta_{\mu\sigma}$, we rewrite

$$[\delta_L, \delta_L']\psi_L = \tfrac{3}{2}\chi_R^\dagger\overleftarrow{\slashed{D}}\varepsilon_L'\varepsilon_L - \tfrac{3}{2}\chi_R^\dagger\overleftarrow{\slashed{D}}\varepsilon_L\varepsilon_L' + \tfrac{1}{2}D_\mu\chi_R^\dagger\gamma_\nu\varepsilon_L'\gamma^{\mu\nu}\varepsilon_L - \tfrac{1}{2}D_\mu\chi_R^\dagger\gamma_\nu\varepsilon_L\gamma^{\mu\nu}\varepsilon_L' \ . \tag{2.13}$$

Comparing with equation (2.8) we see that

$$\mu\chi_R^\dagger\gamma_\nu\varepsilon_L\gamma^{\mu\nu}\varepsilon_L' - D_\mu\chi_R^\dagger\gamma_\nu\varepsilon_L'\gamma^{\mu\nu}\varepsilon_L = \chi_R^\dagger\overleftarrow{\slashed{D}}\varepsilon_L'\varepsilon_L - \chi_R^\dagger\overleftarrow{\slashed{D}}\varepsilon_L\varepsilon_L' \ , \tag{2.14}$$

whence, in summary,

$$[\delta_L, \delta_L']\psi_L = \chi_R^\dagger\overleftarrow{\slashed{D}}\varepsilon_L'\varepsilon_L - \chi_R^\dagger\overleftarrow{\slashed{D}}\varepsilon_L\varepsilon_L' \ , \tag{2.15}$$

which vanishes for all $\varepsilon_L, \varepsilon_L'$ if and only if $\chi_R^\dagger\overleftarrow{\slashed{D}} = 0$, which is the field equation for $\chi_R^\dagger$. This suggests introducing an auxiliary field, historically denoted by D, and modifying the supersymmetry variation of ψ_L by a term proportional to D, namely

$$\delta_L\psi_L = D\varepsilon_L + \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L \ . \tag{2.16}$$

Now, we see that

$$[\delta_L, \delta_L']\psi_L = (\delta_L D - \chi_R^\dagger\overleftarrow{\slashed{D}}\varepsilon_L)\varepsilon_L' - (\delta_L' D - \chi_R^\dagger\overleftarrow{\slashed{D}}\varepsilon_L')\varepsilon_L \ , \tag{2.17}$$

whence we deduce that if we set

$$\delta_L D = \chi_R^\dagger\overleftarrow{\slashed{D}}\varepsilon_L = D_\mu\chi_R^\dagger\gamma^\mu\varepsilon_L \tag{2.18}$$

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then $[\delta_L, \delta_L']\psi_L = 0$. But now we have to check that $[\delta_L, \delta_L']D = 0$ as well:

$$
\begin{aligned}
[\delta_L, \delta_L']D &= \delta_L(D_\mu \chi_R^\dagger \gamma^\mu \varepsilon_L') - \delta_L'(D_\mu \chi_R^\dagger \gamma^\mu \varepsilon_L) \\
&= [\delta_L A_\mu, \chi_R^\dagger]\gamma^\mu \varepsilon_L' - [\delta_L' A_\mu, \chi_R^\dagger]\gamma^\mu \varepsilon_L \\
&= 2[\chi_R^\dagger \gamma_\mu \varepsilon_L, \chi_R^\dagger \gamma^\mu \varepsilon_L'] \, ,
\end{aligned}
\tag{2.19}
$$

where we have used that $\delta_L \chi_R^\dagger = 0$. We now use the Fierz identity (2.10) and (in matrix notation) rewrite

$$
\begin{aligned}
[\delta_L, \delta_L']D &= 2\chi_R^\dagger \gamma_\mu \varepsilon_L \chi_R^\dagger \gamma^\mu \varepsilon_L' - 2\chi_R^\dagger \gamma_\mu \varepsilon_L' \chi_R^\dagger \gamma^\mu \varepsilon_L \\
&= -\chi_R^\dagger \gamma_\mu \gamma_\nu \gamma^\mu \varepsilon_L' \chi_R^\dagger \gamma^\nu \varepsilon_L + \chi_R^\dagger \gamma_\mu \gamma_\nu \gamma^\mu \varepsilon_L \chi_R^\dagger \gamma^\nu \varepsilon_L' \\
&= 2\chi_R^\dagger \gamma_\nu \varepsilon_L' \chi_R^\dagger \gamma^\nu \varepsilon_L - 2\chi_R^\dagger \gamma_\nu \varepsilon_L \chi_R^\dagger \gamma^\nu \varepsilon_L' \\
&= 2[\chi_R^\dagger \gamma_\mu \varepsilon_L', \chi_R^\dagger \gamma^\mu \varepsilon_L] \, ,
\end{aligned}
\tag{2.20}
$$

which is to be compared with equation (2.19), from where we see that indeed $[\delta_L, \delta_L']D = 0$.

In a similar way we work out $\delta_R D$ by the requirement that $[\delta_R, \delta_R']\chi_R^\dagger = 0$. Let α be a number to be determined and let

$$
\delta_R \chi_R^\dagger = \alpha D \varepsilon_R^\dagger - \tfrac{1}{2}\varepsilon_R^\dagger \gamma^{\mu\nu} F_{\mu\nu} \, .
\tag{2.21}
$$

Then

$$
\begin{aligned}
[\delta_R, \delta_R']\chi_R^\dagger &= \delta_R\left(\alpha D \varepsilon_R'^\dagger - \tfrac{1}{2}\varepsilon_R'^\dagger \gamma^{\mu\nu} F_{\mu\nu}\right) - \delta_R'\left(\alpha D \varepsilon_R^\dagger - \tfrac{1}{2}\varepsilon_R^\dagger \gamma^{\mu\nu} F_{\mu\nu}\right) \\
&= \alpha \delta_R D \varepsilon_R'^\dagger + \varepsilon_R^\dagger \gamma_\nu D_\mu \psi_L \varepsilon_R'^\dagger \gamma^{\mu\nu} - (\varepsilon_R \leftrightarrow \varepsilon_R') \, .
\end{aligned}
\tag{2.22}
$$

We use the Fierz identity (2.10)

$$
D_\mu \psi_L \varepsilon_R'^\dagger = -\tfrac{1}{2}\varepsilon_R'^\dagger \gamma^\sigma D_\mu \psi_L \gamma_\sigma P_R
\tag{2.23}
$$

to rewrite

$$
[\delta_R, \delta_R']\chi_R^\dagger = \alpha \delta_R D \varepsilon_R'^\dagger - \tfrac{1}{2}\varepsilon_R'^\dagger \gamma^\sigma D_\mu \psi_L \varepsilon_R^\dagger \gamma_\nu \gamma_\sigma \gamma^{\mu\nu} - (\varepsilon_R \leftrightarrow \varepsilon_R') \, .
\tag{2.24}
$$

We now use that $\gamma_\nu\gamma_\sigma\gamma^{\mu\nu} = -\gamma_{\mu\sigma} + 3\delta_{\mu\sigma}$ to rewrite the above equation as

$$[\delta_R, \delta_R']\chi_R^\dagger = \alpha\delta_R D\varepsilon_R'^\dagger + \tfrac{1}{2}\varepsilon_R'^\dagger\gamma_\sigma D_\mu\psi_L\varepsilon_R^\dagger\gamma^{\mu\sigma} - \tfrac{3}{2}\varepsilon_R'^\dagger\slashed{D}\psi_L\varepsilon_R^\dagger - (\varepsilon_R \leftrightarrow \varepsilon_R') . \qquad (2.25)$$

Comparing with equation (2.22), we see that

$$\varepsilon_R'^\dagger\gamma_\sigma D_\mu\psi_L\varepsilon_R^\dagger\gamma^{\mu\sigma} - (\varepsilon_R \leftrightarrow \varepsilon_R') = \varepsilon_R'^\dagger\slashed{D}\psi_L\varepsilon_R^\dagger - (\varepsilon_R \leftrightarrow \varepsilon_R') , \qquad (2.26)$$

whence finally

$$[\delta_R, \delta_R']\chi_R^\dagger = \left(\alpha\delta_R D + \varepsilon_R^\dagger\slashed{D}\psi_L\right)\varepsilon_R'^\dagger - (\varepsilon_R \leftrightarrow \varepsilon_R') , \qquad (2.27)$$

which vanishes provided that

$$\delta_R D = -\frac{1}{\alpha}\varepsilon_R^\dagger\slashed{D}\psi_L . \qquad (2.28)$$

As before, one checks that $[\delta_R, \delta_R']D = 0$.

We fix α by closing the supersymmetry algebra on the gauge field: we expect that it should close to a translation up to a gauge transformation. Indeed,

$$\begin{aligned}
[\delta_L, \delta_R]A_\mu &= \delta_L(-\varepsilon_R^\dagger\gamma_\mu\psi_L) - \delta_R(\chi_R^\dagger\gamma_\mu\varepsilon_L) \\
&= -\varepsilon_R^\dagger\gamma_\mu\left(D + \tfrac{1}{2}\gamma^{\nu\rho}F_{\nu\rho}\right)\varepsilon_L - \varepsilon_R^\dagger\left(\alpha D - \tfrac{1}{2}\gamma^{\nu\rho}F_{\nu\rho}\right)\gamma_\mu\varepsilon_L \qquad (2.29) \\
&= -(1+\alpha)\varepsilon_R^\dagger\gamma_\mu\varepsilon_L D - \tfrac{1}{2}\varepsilon_R^\dagger\left(\gamma_\mu\gamma^{\nu\rho} - \gamma^{\nu\rho}\gamma_\mu\right)\varepsilon_L F_{\nu\rho} ,
\end{aligned}$$

whence we see that $\alpha = -1$ and using that $[\gamma_\mu, \gamma^{\nu\rho}] = 2\delta_\mu^\nu\gamma^\rho - 2\delta_\mu^\rho\gamma^\nu$, we rewrite

$$\begin{aligned}
[\delta_L, \delta_R]A_\mu &= 2\varepsilon_R^\dagger\gamma^\rho\varepsilon_L F_{\rho\mu} \\
&= 2\varepsilon_R^\dagger\gamma^\rho\varepsilon_L(\partial_\rho A_\mu - \partial_\mu A_\rho + [A_\rho, A_\mu]) \qquad (2.30) \\
&= \xi^\rho\partial_\rho A_\mu - D_\mu\Lambda ,
\end{aligned}$$

where $\xi^\rho = 2\varepsilon_R^\dagger\gamma^\rho\varepsilon_L$ and $\Lambda = \xi^\rho A_\rho$.

In a similar way, one shows that the algebra closes as expected also on ψ_L, $\chi_R^\dagger$ and

D. Indeed, on ψ_L one has

$$
\begin{aligned}
[\delta_L, \delta_R]\psi_L &= -\delta_R(D\varepsilon_L + \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L) \\
&= -\varepsilon_R^\dagger \slashed{D}\psi_L \varepsilon_L - \gamma^{\nu\mu}\varepsilon_R^\dagger \gamma_\nu D_\mu \psi_L \varepsilon_L \\
&= -\gamma^\nu \gamma^\mu \varepsilon_L \varepsilon_R^\dagger \gamma_\nu D_\mu \psi_L \,,
\end{aligned}
\tag{2.31}
$$

which upon using the Fierz identity (2.9) for $\varepsilon_L \varepsilon_R^\dagger$ becomes

$$
[\delta_L, \delta_R]\psi_L = \tfrac{1}{2}\varepsilon_R^\dagger \gamma^\rho \varepsilon_L \gamma^\nu \gamma^\mu \gamma_\rho \gamma_\nu D_\mu \psi_L \,.
\tag{2.32}
$$

Now, we use that $\gamma^\nu \gamma_{\mu\rho}\gamma_\nu = 0$ in four dimensions in order to rewrite this as

$$
[\delta_L, \delta_R]\psi_L = 2\varepsilon_R^\dagger \gamma^\mu \varepsilon_L D_\mu \psi_L = \xi^\mu \partial_\mu \psi_L + [\Lambda, \psi_L] \,,
\tag{2.33}
$$

as expected. The calculation for $[\delta_L, \delta_R]\chi_R^\dagger$ is similar. Finally, we check closure on D:

$$
\begin{aligned}
[\delta_L, \delta_R]D &= \delta_L(\varepsilon_R^\dagger \slashed{D}\psi_L) - \delta_R(\chi_R^\dagger \overset{\leftarrow}{\slashed{D}}\varepsilon_L) \\
&= \varepsilon_R^\dagger \gamma_\mu[\chi_R^\dagger \gamma^\mu \varepsilon_L, \psi_L] + \varepsilon_R^\dagger \slashed{D}(D\varepsilon_L + \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L) \\
&\quad + (\varepsilon_R^\dagger D + \tfrac{1}{2}\varepsilon_R^\dagger \gamma^{\mu\nu}F_{\mu\nu})\overset{\leftarrow}{\slashed{D}}\varepsilon_L + [\varepsilon_R^\dagger \gamma_\mu \psi_L, \chi_R^\dagger]\gamma^\mu \varepsilon_L \\
&= \varepsilon_R^\dagger \gamma^\rho(D_\rho D + \tfrac{1}{2}\gamma^{\mu\nu}D_\rho F_{\mu\nu})\varepsilon_L + \varepsilon_R^\dagger(D_\rho D + \tfrac{1}{2}\varepsilon_R^\dagger \gamma^{\mu\nu}D_\rho F_{\mu\nu})\gamma^\rho \varepsilon_L \\
&= 2\varepsilon_R^\dagger \gamma^\rho D_\rho D\varepsilon_L + \tfrac{1}{2}\varepsilon_R^\dagger(\gamma^\rho \gamma^{\mu\nu} + \gamma^{\mu\nu}\gamma^\rho)D_\rho F_{\mu\nu}\varepsilon_L \,.
\end{aligned}
\tag{2.34}
$$

Using that $\gamma^\rho \gamma^{\mu\nu} + \gamma^{\mu\nu}\gamma^\rho = 2\gamma^{\rho\mu\nu}$ and the Bianchi identity $D_{[\rho}F_{\mu\nu]} = 0$, we conclude that

$$
[\delta_L, \delta_R]D = 2\varepsilon_R^\dagger \gamma^\rho D_\rho D\varepsilon_L = \xi^\rho \partial_\rho D + [\Lambda, D] \,,
\tag{2.35}
$$

as desired.

2.2.3 Off-shell SYM theory on $\mathbb{R}^4$

In summary, the following supersymmetry transformations

$$
\begin{aligned}
\delta_L A_\mu &= \chi_R^\dagger \gamma_\mu \varepsilon_L & \delta_R A_\mu &= -\varepsilon_R^\dagger \gamma_\mu \psi_L \\
\delta_L \psi_L &= D\varepsilon_L + \tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\varepsilon_L & \delta_R \psi_L &= 0 \\
\delta_L \chi_R^\dagger &= 0 & \delta_R \chi_R^\dagger &= -\varepsilon_R^\dagger D - \tfrac{1}{2}\varepsilon_R^\dagger \gamma^{\mu\nu}F_{\mu\nu} \\
\delta_L D &= \chi_R^\dagger \overleftarrow{\slashed{D}} \varepsilon_L & \delta_R D &= \varepsilon_R^\dagger \slashed{D} \psi_L
\end{aligned}
\tag{2.36}
$$

obey

$$
[\delta_L, \delta_L'] = 0 \qquad [\delta_R, \delta_R'] = 0 \qquad \text{whereas} \qquad [\delta_L, \delta_R] = \mathcal{L}_\xi + \delta_\Lambda^{\text{gauge}}, \tag{2.37}
$$

where $\xi^\mu = 2\varepsilon_R^\dagger \gamma^\mu \varepsilon_L$ and $\Lambda = \xi^\mu A_\mu$.

The action given by the Lagrangian (2.1) is not invariant under the supersymmetry transformations in (2.36) unless we also add a term depending on the auxiliary field. Indeed, the invariant action is given by

$$
\mathcal{L}^{(4)} = -\operatorname{Tr}\chi_R^\dagger \slashed{D}\psi_L - \tfrac{1}{4}\operatorname{Tr}F^2 - \tfrac{1}{2}\operatorname{Tr}D^2. \tag{2.38}
$$

It should be remarked that the euclideanisation has in fact complexified the fields in the original Yang–Mills theory. Indeed, the spinor representation in Euclidean signature is not of real type, as it is in Lorentzian signature and the supersymmetry transformations further force the bosonic fields to be complex as well.

We may promote this action to an arbitrary Riemannian 4-manifold simply by covariantising the derivatives, so that D_μ now also contains the spin connection. Doing so and taking ε_L and $\varepsilon_R^\dagger$ to be spinor fields, we find that

$$
\delta_L \mathcal{L}^{(4)} = -\nabla_\mu \operatorname{Tr}\chi_R^\dagger \gamma_\nu \varepsilon_L (Dg^{\mu\nu} + F^{\mu\nu}) - \tfrac{1}{2}\operatorname{Tr}\chi_R^\dagger \gamma^\rho \gamma^{\mu\nu}F_{\mu\nu}\nabla_\rho \varepsilon_L, \tag{2.39}
$$

and

$$
\delta_R \mathcal{L}^{(4)} = \tfrac{1}{2}\nabla_\rho \operatorname{Tr}F_{\mu\nu}\varepsilon_R^\dagger \gamma^{\mu\nu\rho}\psi_L - \tfrac{1}{2}\operatorname{Tr}\nabla_\rho \varepsilon_R^\dagger \gamma^{\mu\nu}\gamma^\rho F_{\mu\nu}\psi_L, \tag{2.40}
$$

from where we see that if ε_L and $\varepsilon_R^\dagger$ are not parallel, the action is not invariant. This will be remedied for the dimensionally reduced action in three dimensions by adding further terms in the action provided that ε_L and $\varepsilon_R^\dagger$ are Killing spinors.

2.3   Reduction to Euclidean 3-space

The spin group in four dimensions is $\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. The spin group in three dimensions is $\mathrm{Spin}(3)$ and embeds in $\mathrm{Spin}(4)$ as the diagonal $\mathrm{Spin}(3)$ in $\mathrm{Spin}(3) \times \mathrm{Spin}(3)$. Therefore in three dimensions there is no distinction between L and R spinors. We reduce to three dimensions along the fourth coordinate, whence we assume that $\partial_4 = 0$ on all fields and parameters.

We take the following explicit realisation for the four-dimensional gamma matrices:

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma^j \\ i\sigma^j & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \qquad \text{and hence} \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}.$$
(2.41)

This means that we can take $\psi_L = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ and $\chi_R^\dagger = \begin{pmatrix} 0 & \chi^\dagger \end{pmatrix}$. The basic Fierz identity for anticommuting spinors in three dimensions is

$$\psi\chi^\dagger = -\tfrac{1}{2}\chi^\dagger\psi - \tfrac{1}{2}\chi^\dagger\sigma^j\psi\sigma^j .$$
(2.42)

The gauge field decomposes as $A_\mu \rightsquigarrow (A_i, \phi)$. The supersymmetry parameters ε_L and $\varepsilon_R^\dagger$ also decompose as ψ_L and $\chi_R^\dagger$ do: $\varepsilon_L = \begin{pmatrix} \epsilon_L \\ 0 \end{pmatrix}$ and $\varepsilon_R^\dagger = \begin{pmatrix} 0 & \epsilon_R^\dagger \end{pmatrix}$.

2.3.1 Off-shell SYMH theory on $\mathbb{R}^3$

In terms of the three-dimensional quantities we have the following supersymmetry transformations:

$$\delta_L A_i = i\chi^\dagger \sigma_i \epsilon_L \qquad\qquad\qquad \delta_R A_i = -i\epsilon_R^\dagger \sigma_i \psi$$

$$\delta_L \phi = \chi^\dagger \epsilon_L \qquad\qquad\qquad\quad \delta_R \phi = -\epsilon_R^\dagger \psi$$

$$\delta_L \chi^\dagger = 0 \qquad\qquad\qquad\quad \delta_R \chi^\dagger = -D\epsilon_R^\dagger - \tfrac{i}{2}\varepsilon_{ijk}F^{ij}\epsilon_R^\dagger \sigma^k - i\epsilon_R^\dagger \sigma^i D_i\phi$$

$$\delta_L D = i\chi^\dagger \overleftarrow{\slashed{D}} \epsilon_L + [\phi, \chi^\dagger \epsilon_L] \qquad\qquad \delta_R D = i\epsilon_R^\dagger \slashed{D}\psi + \epsilon_R^\dagger [\phi, \psi]$$

$$\delta_L \psi = D\epsilon_L + \tfrac{i}{2}\varepsilon_{ijk}F^{ij}\sigma^k \epsilon_L - iD_i\phi\sigma^i\epsilon_L \qquad \delta_R \psi = 0 \, ,$$

$$\tag{2.43}$$

where now

$$[\delta_L, \delta_L'] = 0 = [\delta_R, \delta_R'] \qquad \text{and} \qquad [\delta_L, \delta_R] = \mathcal{L}_\xi + \delta_\Lambda^{\text{gauge}} \, , \tag{2.44}$$

with $\xi^i = 2i\epsilon_R^\dagger \sigma^i \epsilon_L$ and $\Lambda = \xi^i A_i + 2\epsilon_R^\dagger \epsilon_L \phi$.

The reduction of the action (2.38) to three dimensions is

$$\mathcal{L}^{(3)} = -i \operatorname{Tr}\chi^\dagger \slashed{D}\psi - \operatorname{Tr}\chi^\dagger[\phi, \psi] - \tfrac{1}{4}\operatorname{Tr} F^2 - \tfrac{1}{2}\operatorname{Tr}|D\phi|^2 - \tfrac{1}{2}\operatorname{Tr} D^2 \, , \tag{2.45}$$

where $\slashed{D} = \sigma^i D_i$, $F^2 = F_{ij}F^{ij}$ and $|D\phi|^2 = D_i\phi D^i\phi$. It can again be suitably covariantised to define it on a Riemannian 3-manifold. Its variation under supersymmetry can be read off from equations (2.39) and (2.40). Doing so, one finds

$$\delta_L \mathcal{L}^{(3)} = -i\nabla_i \operatorname{Tr}\chi^\dagger \left(\sigma^i D + \sigma_j F^{ij} - iD^i\phi \right) \epsilon_L + \operatorname{Tr}\chi^\dagger \sigma^i \sigma^\ell \left(\tfrac{1}{2}\varepsilon_{jk\ell}F^{jk} - D_\ell\phi \right) \nabla_i\epsilon_L \tag{2.46}$$

and

$$\delta_R \mathcal{L}^{(3)} = \nabla_i \operatorname{Tr}\varepsilon^{ijk}\epsilon_R^\dagger \left(-\tfrac{1}{2}F_{jk} + iD_j\phi\sigma_k \right) \psi + \operatorname{Tr}\nabla_i\epsilon_R^\dagger \left(\tfrac{1}{2}\varepsilon_{jk\ell}F^{jk} + D_\ell\phi \right) \sigma^\ell\sigma^i\psi \, . \tag{2.47}$$

2.4 Deforming to curved space

We now wish to improve the action $\mathcal{L}^{(3)}$ and the supersymmetry transformations of the fermions and the auxiliary field in order for the new $\mathcal{L}^{(3)}$ to transform into a total derivative when the spinor parameters are not necessarily parallel. For reasons that will be clear later, we will, instead, take the spinor parameters to be Killing: $\nabla_i \epsilon_L = \lambda_L \sigma_i \epsilon_L$ and $\nabla_i \epsilon_R^\dagger = \lambda_R \epsilon_R^\dagger \sigma_i$ for some (either real or imaginary) constants λ_L and λ_R. We add terms

$$\mathcal{L}^{(3)} \rightsquigarrow \mathcal{L}^{(3)} + \alpha_1 \operatorname{Tr} \chi^\dagger \psi + \tfrac{1}{2}\alpha_2 \operatorname{Tr} \phi^2 + \alpha_3 \operatorname{Tr} \phi D + \tfrac{1}{2}\alpha_4 \operatorname{Tr} D^2 \qquad (2.48)$$

to the Lagrangian and also

$$\begin{aligned}
\delta_L \psi &\rightsquigarrow \delta_L \psi + \beta_1 \phi \epsilon_L & \delta_R \chi^\dagger &\rightsquigarrow \delta_R \chi^\dagger - \beta_3 \epsilon_R^\dagger \phi \\
\delta_L D &\rightsquigarrow \delta_L D + \beta_2 \chi^\dagger \epsilon_L & \delta_R D &\rightsquigarrow \delta_R D + \beta_4 \epsilon_R^\dagger \psi \,,
\end{aligned} \qquad (2.49)$$

for some constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ to be determined.

We start by computing $\delta_L \mathcal{L}^{(3)}$. Using equation (2.46), we arrive at (henceforth dropping Tr from the notation)

$$\begin{aligned}
\delta_L \mathcal{L}^{(3)} = {}& \nabla_i X_L^i - \lambda_L(\tfrac{1}{2}\varepsilon_{jk\ell}F^{jk} - D_\ell \phi)\chi^\dagger \sigma^\ell \epsilon_L - i\beta_1 \chi^\dagger \slashed{D}(\phi \epsilon_L) - \beta_2 D\chi^\dagger \epsilon_L \\
&+ \alpha_1 \chi^\dagger \left((D + \beta_1 \phi)\epsilon_L + i(\tfrac{1}{2}\varepsilon_{ijk}F^{ij} - D_k \phi)\sigma^k \epsilon_L\right) + \alpha_2 \phi \chi^\dagger \epsilon_L + \alpha_3 D\chi^\dagger \epsilon_L \\
&+ \alpha_3 \phi \left(i\chi^\dagger \overleftarrow{\slashed{D}}\epsilon_L + \beta_2 \chi^\dagger \epsilon_L\right) + \alpha_4 D\left(i\chi^\dagger \overleftarrow{\slashed{D}}\epsilon_L + [\phi, \chi^\dagger \epsilon_L] + \beta_2 \chi^\dagger \epsilon_L\right)\,, \quad (2.50)
\end{aligned}$$

where $X_L^i = -i\chi^\dagger \left(\sigma^i D + \sigma_j F^{ij} - iD^i \phi\right)\epsilon_L$, and where we have used that $\sigma^i \sigma_j \sigma_i = -\sigma_j$.

The $\chi^\dagger F$ terms vanish provided that $\alpha_1 = -i\lambda_L$, which also takes care of the $\chi^\dagger D_i \phi$ terms. The $\chi^\dagger DA_i$ terms impose $\alpha_4 = 0$, whereas the $\chi^\dagger \phi A_i$ terms become a total derivative $\nabla_i Y_L^i$, with $Y_L^i = -i\beta_1 \phi \chi^\dagger \sigma^i \epsilon_L$, provided that $\alpha_3 = -\beta_1$. The $\chi^\dagger D$ terms vanish if $\beta_2 = -(\beta_1 + i\lambda_L)$ and the $\chi^\dagger \phi$ terms vanish provided that $\alpha_2 = -\beta_1^2$.

In summary,

$$\mathcal{L}^{(3)} := -i\chi^\dagger \slashed{D}\psi - \chi^\dagger[\phi, \psi] - i\lambda_L \chi^\dagger \psi - \tfrac{1}{4}F^2 - \tfrac{1}{2}|D\phi|^2 - \tfrac{1}{2}(D + \beta_1 \phi)^2 \qquad (2.51)$$

transforms as

$$\delta_L \mathcal{L}^{(3)} = \nabla_i \left(-i\chi^\dagger \left(\sigma^i (D + \beta_1 \phi) + \sigma_j F^{ij} - iD^i \phi \right) \epsilon_L \right) , \qquad (2.52)$$

under

$$\delta_L A_i = i\chi^\dagger \sigma_i \epsilon_L$$

$$\delta_L \phi = \chi^\dagger \epsilon_L$$

$$\delta_L \chi^\dagger = 0 \qquad (2.53)$$

$$\delta_L \psi = (D + \beta_1 \phi)\epsilon_L + \tfrac{i}{2}\varepsilon_{ijk}F^{ij}\sigma^k \epsilon_L - iD_i \phi \sigma^i \epsilon_L$$

$$\delta_L D = i\chi^\dagger \overleftarrow{\slashed{D}} \epsilon_L + [\phi, \chi^\dagger \epsilon_L] - (\beta_1 + i\lambda_L)\chi^\dagger \epsilon_L ,$$

with $\nabla_i \epsilon_L = \lambda_L \sigma_i \epsilon_L$.

Notice that the action depends on λ_L, hence once the action is fixed, the sign of the Killing constant in the Killing spinor equation is also fixed.

Next we compute $\delta_R \mathcal{L}^{(3)}$ and use equation (2.47) to find

$$\delta_R \mathcal{L}^{(3)} = \nabla_i X_R^i - \lambda_R (\tfrac{1}{2}\varepsilon_{jk\ell} + D_\ell \phi)\epsilon_R^\dagger \sigma^\ell \psi + i\beta_3 \phi \epsilon_R^\dagger \slashed{D} \psi - \beta_4 D \epsilon_R^\dagger \psi + \beta_1^2 \phi \epsilon_R^\dagger \psi$$

$$+ i\lambda_L \left((D + \beta_3 \phi)\epsilon_R^\dagger \psi + i(\tfrac{1}{2}\varepsilon_{ijk}F^{ij} + D_k \phi)\epsilon_R^\dagger \sigma^k \psi \right)$$

$$+ \beta_1 D \epsilon_R^\dagger \psi - \beta_1 \phi (i\epsilon_R^\dagger \slashed{D} \psi + \beta_4 \epsilon_R^\dagger \psi) , \quad (2.54)$$

where we have again used $\sigma^i \sigma_j \sigma_i = -\sigma_j$ and where $X_R^i = \varepsilon^{ijk}\epsilon_R^\dagger \left(-\tfrac{1}{2}F_{jk} + iD_j \phi \sigma_k \right) \psi$.

The $F\psi$ terms vanish provided that $\lambda_R = -\lambda_L$, and this also takes care of the $D_i \phi \psi$ terms. Notice that this means that the vector field $\xi^i = 2i\epsilon_R^\dagger \sigma^i \epsilon_L$ is a Killing vector, and not merely conformal Killing. Indeed,

$$\nabla_i \xi_j = 2i\lambda_R \epsilon_R^\dagger \sigma_i \sigma_j \epsilon_L + 2i\lambda_L \epsilon_R^\dagger \sigma_j \sigma_i \epsilon_L$$

$$= -2i\lambda_L \epsilon_R^\dagger (\sigma_i \sigma_j - \sigma_j \sigma_i)\epsilon_L \qquad (2.55)$$

$$= -2i\lambda_L \varepsilon_{ijk}\xi^k ,$$

whence $\nabla_i \xi_j + \nabla_j \xi_i = 0$.

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The $A_i \phi \psi$ terms vanish provided that $\beta_3 = \beta_1$, whereas the vanishing of the $D\psi$ terms set $\beta_4 = \beta_1 + i\lambda_L$, which also takes care of the $\phi\psi$ terms.

In summary, and letting $\lambda_L = -\lambda_R = \lambda$,

$$\mathcal{L}^{(3)} := -i\chi^\dagger \slashed{D}\psi - \chi^\dagger[\phi, \psi] - i\lambda\chi^\dagger\psi - \tfrac{1}{4}F^2 - \tfrac{1}{2}|D\phi|^2 - \tfrac{1}{2}(D + \beta_1\phi)^2 \tag{2.56}$$

transforms as

$$\delta_R \mathcal{L}^{(3)} = \nabla_i \left(\varepsilon^{ijk}\epsilon_R^\dagger \left(-\tfrac{1}{2}F_{jk} + iD_j\phi\sigma_k \right) \psi \right) , \tag{2.57}$$

under

$$\delta_R A_i = -i\epsilon_R^\dagger \sigma_i \psi$$

$$\delta_R \phi = -\epsilon_R^\dagger \psi$$

$$\delta_R \chi^\dagger = -(D + \beta_1\phi)\epsilon_R^\dagger - i(\tfrac{1}{2}\varepsilon_{ijk}F^{ij} + D_k\phi)\epsilon_R^\dagger \sigma^k \tag{2.58}$$

$$\delta_R \psi = 0$$

$$\delta_R D = i\epsilon_R^\dagger \slashed{D}\psi + \epsilon_R^\dagger[\phi, \psi] + (\beta_1 + i\lambda)\epsilon_R^\dagger \psi ,$$

with $\nabla_i \epsilon_L = \lambda\sigma_i\epsilon_L$ and $\nabla_i \epsilon_R^\dagger = -\lambda\epsilon_R^\dagger \sigma_i$.

2.4.1 SYMH theory on H^3

One can show that the supersymmetry algebra of the left and right supersymmetry transformations closes as follows:

$$[\delta_L, \delta_L'] = 0 = [\delta_R, \delta_R'] \qquad \text{and} \qquad [\delta_L, \delta_R] = \mathcal{L}_\xi + \delta_\Lambda^{\text{gauge}} + \delta_\varpi^R , \tag{2.59}$$

for $\xi^i = 2i\epsilon_R^\dagger \sigma^i \epsilon_L$ and $\Lambda = \xi^i A_i + 2\epsilon_R^\dagger \epsilon_L \phi$, and where δ_ϖ^R is an R-symmetry transformation with $\varpi = -4\lambda\epsilon_R^\dagger \epsilon_L$, where

$$\delta_\varpi^R \psi = i\varpi\psi \qquad \text{and} \qquad \delta_\varpi^R \chi^\dagger = -i\varpi\chi^\dagger . \tag{2.60}$$

Indeed, it's induced from four-dimensions, where it is generated by γ^5. Notice that ϖ is actually constant, so that this is indeed a rigid R-symmetry transformation. Simil-

arly, it is worth remarking that $\mathcal{L}_\xi$ now means the spinorial Lie derivative [117] on the spinor fields, which in our case becomes

$$\mathcal{L}_\xi \psi = \xi^i \nabla_i \psi + \lambda \xi^i \sigma_i \psi \qquad \text{and} \qquad \mathcal{L}_\xi \chi^\dagger = \xi^i \nabla_i \chi^\dagger - \lambda \xi^i \chi^\dagger \sigma_i \ . \tag{2.61}$$

One can check that this is indeed the expression which follows by evaluating the definition $\mathcal{L}_\xi = \nabla_\xi + \rho(A_\xi)$, with A_ξ the skew-symmetric endomorphism of the tangent bundle defined by $A_\xi(X) = -\nabla_X \xi$ and where ρ is the spin representation.

The parameter β_1 remains free and can be set to zero if so desired. This is equivalent to the field redefinition $D \rightsquigarrow D + \beta_1 \phi$. Doing so, we have that the action with Lagrangian

$$\mathcal{L}^{(3)} = -i\chi^\dagger \slashed{D} \psi - \chi^\dagger [\phi, \psi] - i\lambda \chi^\dagger \psi - \tfrac{1}{4} F^2 - \tfrac{1}{2} |D\phi|^2 - \tfrac{1}{2} D^2 \tag{2.62}$$

transforms as

$$\delta_L \mathcal{L}^{(3)} = \nabla_i \left(-i\chi^\dagger \left(\sigma^i D + \sigma_j F^{ij} - iD^i \phi \right) \epsilon_L \right) \tag{2.63}$$

$$\delta_R \mathcal{L}^{(3)} = \nabla_i \left(\varepsilon^{ijk} \epsilon_R^\dagger \left(-\tfrac{1}{2} F_{jk} + iD_j \phi \sigma_k \right) \psi \right) \tag{2.64}$$

under

$$\delta_L A_i = i\chi^\dagger \sigma_i \epsilon_L \qquad\qquad\qquad \delta_R A_i = -i\epsilon_R^\dagger \sigma_i \psi$$

$$\delta_L \phi = \chi^\dagger \epsilon_L \qquad\qquad\qquad\qquad \delta_R \phi = -\epsilon_R^\dagger \psi$$

$$\delta_L \chi^\dagger = 0 \qquad\qquad\qquad\qquad \delta_R \chi^\dagger = -D\epsilon_R^\dagger - i(\tfrac{1}{2} \varepsilon_{ijk} F^{ij} + D_k \phi)\epsilon_R^\dagger \sigma^k$$

$$\delta_L \psi = D\epsilon_L + i(\tfrac{1}{2} \varepsilon_{ijk} F^{ij} - D_k \phi)\sigma^k \epsilon_L \qquad \delta_R \psi = 0$$

$$\delta_L D = i\chi^\dagger \overleftarrow{\slashed{D}} \epsilon_L + [\phi, \chi^\dagger]\epsilon_L - i\lambda \chi^\dagger \epsilon_L \ , \qquad \delta_R D = i\epsilon_R^\dagger \slashed{D} \psi + \epsilon_R^\dagger [\phi, \psi] + i\lambda \epsilon_R^\dagger \psi \ ,$$

$$\tag{2.65}$$

with $\nabla_i \epsilon_L = \lambda \sigma_i \epsilon_L$ and $\nabla_i \epsilon_R^\dagger = -\lambda \epsilon_R^\dagger \sigma_i$.

2.5 Some remarks

The first remark is that there is only a mass term for the fermions, yet none for the scalar. (This is a choice.) The choice of λ is dictated by the geometry up to a sign, but

that sign is immaterial since λ appears in the action.

Secondly, it seems that the action is not "exact" in that $\mathcal{L}^{(3)} \epsilon_R^\dagger \epsilon_L \neq \delta_L \delta_R \Xi$ for any reasonable Ξ.

Thirdly, we remark that this theory agrees morally with one of the theories in Family A in [115]. In fact, if we eliminate the auxiliary field, then it agrees with the theory described by equation (3.10) in that paper, denoted $N = 2$ in $d = 3$. Finally, let us comment on the geometry of the manifolds admitting Killing spinors. The integrability condition for solutions of the Killing spinor equation $\nabla_i \epsilon_L = \lambda \sigma_i \epsilon_L$ says that the metric is Einstein. The vanishing of the Weyl tensor in three dimensions implies that the Riemann curvature tensor of an Einstein three-dimensional Riemannian manifold can be written purely in terms of the scalar curvature and the metric; in other words, it has constant sectional curvature, where the value of the scalar curvature is related to the Killing constant λ by $R = -24\lambda^2$ in our conventions. Therefore the existence of Killing spinors with real λ forces the manifold to be hyperbolic, whereas for imaginary λ it would be spherical. In the simply-connected case, we have three-dimensional hyperbolic space and the 3-sphere, respectively, which admit the maximum number of such Killing spinors, with either sign of the Killing constant.

Chapter 3

Geometry of the Complex Moduli Space of Hyperbolic Monopoles

3.1 Introduction

This chapter is dedicated to the study of the geometry of the complex moduli space of hyperbolic monopoles. We show that the geometry is hypercomplex. We first show that hyperbolic monopoles are $\frac{1}{2}$ "BPS" saturated, which means they preserve half of the supercharges we started with. The reason for tackling the complex space instead of the real space at this level can be traced back to the step where we euclideanised Yang-Mills theory that renders the hyperbolic monopole fields complex. We analyze the geometry using the low-energy supersymmetric dynamics, hence, as a first step, we construct a supermultiplet of bosonic and fermionic zero modes that we show to satisfy the linearized Bogomol'nyi equation and Dirac equation respectively, so, these modes can be thought of, now, as bases of the tangent space of the Bogomol'nyi solutions. Moreover, since we are studying the moduli space, we show that the bosonic zero modes satisfy a gauge background condition. We also show that we have an isomorphism between the vector spaces of bosonic zero modes and fermionic zero modes, which means that for hyperbolic monopoles of charge n, the index of the Dirac operator in the presence of hyperbolic monopoles is $4n$. This is the hyperbolic analogue of Zumino result [118] for Euclidean monopoles. That result can be rederived

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without using supersymmetry via the calculation of the index of the Dirac operator in the presence of a monopole. For hyperbolic monopoles this calculation has not been performed, to our knowledge, but it is conceivable that it may be possible using the generalisation of the Callias index theorem [119] in [120]. Then, we construct a set of endomorphisms on $T_{\mathbb{C}}(H^3 \times S^1)$, that we show to satisfy the quaternionic algebra, and map zero modes to zero modes. This will lead to defining complex structures on the tangent space to the moduli space, which we use in linearizing the the unbroken supersymmetry transformation. Finally by closing the supersymmetry algebra we find the geometric identities defining the geometry of the complex moduli space.

3.2 Moduli space of BPS configurations

In this section we start the analysis of the geometry of the moduli space of BPS configurations. The first observation, which is crucial for this approach to the problem, is that the BPS configurations are precisely the BPS monopoles with $D = 0$. More precisely, bosonic configurations for which $\delta_L \psi = 0$ are precisely those obeying $D = 0$ and $D_k \phi = \frac{1}{2} \varepsilon_{ijk} F^{ij}$, for which the δ_L supersymmetries with parameter ϵ_L obeying $\nabla_i \epsilon_L = \lambda \sigma_i \epsilon_L$ are preserved. This is easy to see by writing

$$\delta_L \psi = (D + i(\tfrac{1}{2} \varepsilon_{ijk} F^{ij} - D_k \phi) \sigma^k) \epsilon_L \tag{3.1}$$

and noticing that the determinant of $D + i(\frac{1}{2} \varepsilon_{ijk} F^{ij} - D_k \phi) \sigma^k$ is zero if and only if $D = 0$ and $\frac{1}{2} \varepsilon_{ijk} F^{ij} - D_k \phi = 0$. Similarly, the bosonic configurations with $D_k \phi = -\frac{1}{2} \varepsilon_{ijk} F^{ij}$ and $D = 0$ are precisely the ones which preserve the δ_R supersymmetries with parameter $\epsilon_R^\dagger$ obeying $\nabla_i \epsilon_R^\dagger = -\lambda \epsilon_R^\dagger \sigma_i$. It is the these latter bosonic BPS configurations whose moduli space $\mathcal{M}$ we will study in the rest of this chapter. The moduli space $\mathcal{M}$ is defined as the quotient $\mathcal{P}/\mathcal{G}$ of the space $\mathcal{P}$ of solutions of the Bogomol'nyi equation

$$D_i \phi + \varepsilon_{ijk} F^{jk} = 0 \tag{3.2}$$

by the action of the group $\mathcal{G}$ of gauge transformations:

$$A \mapsto gAg^{-1} - dgg^{-1} \qquad \text{and} \qquad \phi \mapsto g\phi g^{-1} \, , \tag{3.3}$$

where $g : H^3 \to G$ is a smooth function. We mention once again that the Euclidean theory has complex fields, so that strictly speaking the half-BPS states actually correspond to complexified hyperbolic monopoles with $D = 0$.

3.2.1   Zero modes

Consider a one-parameter family $A_i(s), \phi(s)$ of bosonic BPS configurations, where s is a formal parameter. This means that for all s, they obey the Bogomol'nyi equation

$$D_i(s)\phi(s) + \varepsilon_{ijk}F^{jk}(s) = 0 \, . \tag{3.4}$$

Differentiating with respect to s at $s = 0$, we find

$$D_i(0)\dot\phi - [\phi(0), \dot A_i] + \varepsilon_{ijk}D^j(0)\dot A^k = 0 \, , \tag{3.5}$$

where $\dot A_i = \left.\frac{\partial A_i}{\partial s}\right|_{s=0}, \dot\phi = \left.\frac{\partial \phi}{\partial s}\right|_{s=0}$ and $D_i(0) = \partial_i + [A_i(0), -]$. Equation (3.5) is the linearisation at $(A_i(0), \phi(0))$ of the Bogomol'nyi equation and solutions of that equation will be termed *bosonic zero modes*.

One way to generate bosonic zero modes is to consider the tangent vector to the orbit of a one-parameter subgroup of the group of gauge transformations. The subspace of such zero modes is the tangent space to the gauge orbit of $(A_i(0), \phi(0))$. The true tangent space to the moduli space can be identified with a suitable complement of that subspace. A choice of such a complement is essentially a choice of connection on the principal $\mathcal{G}$-bundle $\mathcal{P} \to \mathcal{M}$. In the absence of a natural Riemannian metric on $\mathcal{P}$, we will employ supersymmetry to define this connection.

Supersymmetry relates the bosonic zero modes to *fermionic zero modes* $\dot\psi$ which are

solutions of the (already linear) field equations for ψ at $(A_i(0), \phi(0))$:

$$\slashed{D}(0)\dot{\psi} - i[\phi(0), \dot{\psi}] + \lambda\dot{\psi} = 0 \ . \tag{3.6}$$

Let η, ζ be Killing spinors on hyperbolic space satisfying

$$\nabla_i \eta = \lambda\sigma_i\eta \qquad \text{and} \qquad \nabla_i\zeta^\dagger = -\lambda\zeta^\dagger\sigma_i \ . \tag{3.7}$$

Of course, hyperbolic space has the maximal number of either class of such Killing spinors.

Let $(\dot{A}_i, \dot{\phi})$ satisfy the linearised Bogomol'nyi equation (3.5) and let

$$\dot{\psi} = i\dot{A}_i\sigma^i\eta - \dot{\phi}\eta \ . \tag{3.8}$$

We claim that $\dot{\psi}$ so defined is a fermionic zero mode provided that $(\dot{A}_i, \dot{\phi})$ obey in addition the *generalised Gauss law*

$$D^i(0)\dot{A}_i + [\phi(0), \dot{\phi}] + 4i\lambda\dot{\phi} = 0 \ . \tag{3.9}$$

Indeed, with the tacit evaluation at $s = 0$,

$$\slashed{D}\left(i\dot{A}_i\sigma^i\eta - \dot{\phi}\eta\right) + i\left[\left(i\dot{A}_i\sigma^i\eta - \dot{\phi}\eta\right), \phi\right] + \lambda\left(i\dot{A}_i\sigma^i\eta - \dot{\phi}\eta\right)$$
$$= iD_j\dot{A}_i\sigma^j\sigma^i\eta + i\dot{A}_i\sigma^j\sigma^i\nabla_j\eta - D_i\dot{\phi}\sigma^i\eta - \dot{\phi}\slashed{\nabla}\eta - [\dot{A}_i, \phi]\sigma^i\eta - i[\dot{\phi}, \phi]\eta + i\lambda\dot{A}_i\sigma^i\eta - \lambda\dot{\phi}\eta$$
$$= iD^i\dot{A}_i\eta - \varepsilon^{ijk}D_i\dot{A}_j\sigma_k\eta - D_i\dot{\phi}\sigma^i\eta - 4\lambda\dot{\phi}\eta - [\dot{A}_i, \phi]\sigma^i\eta - i[\dot{\phi}, \phi]\eta \ ,$$

where we have used that $\sigma^j\sigma_i\sigma_j = -\sigma_i$ and that $\slashed{\nabla}\eta = 3\lambda\eta$. We can rewrite the resulting expression as follows

$$\left(iD^i\dot{A}_i - i[\dot{\phi}, \phi] - 4\lambda\dot{\phi}\right)\eta - \left(\varepsilon^{ijk}D_i\dot{A}_j + D^k\dot{\phi} + [\dot{A}^k, \phi]\right)\sigma_k\eta \ , \tag{3.10}$$

which contains two kinds of terms: those which are proportional to $\sigma_k\eta$ vanish because of the linearised Bogomol'nyi equation (3.5), whereas the ones proportional to

η cancel if and only if the generalised Gauss law (3.9) is satisfied.

One might be surprised by the last term in the generalised Gauss law as this is absent in the case of Euclidean monopoles. And indeed, we see that in the flat space limit $\lambda \to 0$ this term disappears. The Gauss law is a gauge-fixing condition, or more geometrically, it is an Ehresmann connection on the principal gauge bundle $\mathcal{P} \to \mathcal{M}$ over the moduli space; that is, a $\mathcal{G}$-invariant complement to the tangent space to the gauge orbit through every point of $\mathcal{P}$. It is not hard to see that condition (3.9) is $\mathcal{G}$-invariant and that it provides a complement to the gauge orbits. However it is not, as in the case of Euclidean monopoles, the perpendicular complement to the tangent space to the gauge orbits relative to a $\mathcal{G}$-invariant metric on $\mathcal{P}$.

Conversely, if $\dot{\psi}$ obeys equation (3.6), then

$$\dot{A}_i = -i\zeta^\dagger \sigma_i \dot{\psi} \qquad \text{and} \qquad \dot{\phi} = -\zeta^\dagger \dot{\psi} \tag{3.11}$$

obey the linearised Bogomol'nyi equation (3.5) and the generalised Gauss law (3.9). Indeed, and again with the tacit evaluation at $s = 0$,

$$\begin{aligned}
D_i \left(-\zeta^\dagger \dot{\psi} \right) + \varepsilon_{ijk} D^j \left(-i\zeta^\dagger \sigma^k \dot{\psi} \right) &- \left[\phi, \left(-i\zeta^\dagger \sigma_i \dot{\psi} \right) \right] \\
&= -\nabla_i \zeta^\dagger \dot{\psi} - \zeta^\dagger D_i \dot{\psi} - i\varepsilon_{ijk} \nabla^j \zeta^\dagger \sigma^k \dot{\psi} - i\varepsilon_{ijk} \zeta^\dagger \sigma^k D^j \dot{\psi} + i\zeta^\dagger \sigma_i [\phi, \dot{\psi}] \\
&= \lambda \zeta^\dagger \sigma_i \dot{\psi} - \zeta^\dagger D_i \dot{\psi} + i\zeta^\dagger [\phi, \dot{\psi}] + i\lambda \varepsilon_{ijk} \zeta^\dagger \sigma^{jk} \dot{\psi} - i\varepsilon_{ijk} \zeta^\dagger \sigma^k D^j \dot{\psi} \, .
\end{aligned}$$

We now use that $\varepsilon_{ijk} \sigma^{jk} = 2i\sigma_i$ and that $i[\phi, \dot{\psi}] = \slashed{D}\dot{\psi} + \lambda\dot{\psi}$ to arrive at

$$\begin{aligned}
D_i \left(-\zeta^\dagger \dot{\psi} \right) + \varepsilon_{ijk} D^j \left(-i\zeta^\dagger \sigma^k \dot{\psi} \right) &- \left[\phi, \left(-i\zeta^\dagger \sigma_i \dot{\psi} \right) \right] \\
&= -\zeta^\dagger D_i \dot{\psi} - i\varepsilon_{ijk} \zeta^\dagger \sigma^k D^j \dot{\psi} + \zeta^\dagger \sigma_i \slashed{D}\dot{\psi} \, ,
\end{aligned}$$

which is seen to vanish after using that $\sigma_i \sigma_j = g_{ij} + i\varepsilon_{ijk}\sigma^k$ to expand $\sigma_i \slashed{D}\dot{\psi}$.

3.2.2 A four-dimensional formalism

It is convenient for calculations to introduce a four-dimensional language. This amounts to working on the four-dimensional manifold $H^3 \times S^1$, but where the fields are invariant

under translations in S^1. The relevant Clifford algebra is now generated by $\Gamma_\mu = (\Gamma_i, \Gamma_4)$ given by

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \Gamma_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{3.12}$$

which satisfy $\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\delta_{\mu\nu}\mathbb{I}$. Let $\zeta_R = \begin{pmatrix} 0 \\ \zeta \end{pmatrix}$ and $\eta_R = \begin{pmatrix} 0 \\ \eta \end{pmatrix}$, which obey the Killing spinor equations

$$\nabla_i \eta_R = -i\lambda \Gamma_i \Gamma_4 \eta_R \qquad \text{and} \qquad \nabla_i \zeta_R^\dagger = -i\lambda \zeta_R^\dagger \Gamma_4 \Gamma_i \,, \tag{3.13}$$

and in addition $\nabla_4 \eta_R = 0$ and $\nabla_4 \zeta_R^\dagger = 0$. The zero modes are now $\dot{\Psi}_L = \begin{pmatrix} \dot{\psi} \\ 0 \end{pmatrix}$ and $\dot{A}_\mu = (\dot{A}_i, \dot{\phi})$ and the relations (3.8) and (3.11) between them can now be rewritten respectively as

$$\dot{\Psi}_L = i\dot{A}_\mu \Gamma^\mu \eta_R \qquad \text{and} \qquad \dot{A}_\mu = -i\zeta_R^\dagger \Gamma_\mu \dot{\Psi}_L \,. \tag{3.14}$$

Also, in four-dimensional language the fermionic zero modes are defined by the equation

$$\slashed{D}\dot{\Psi}_L = -i\lambda\Gamma_4\dot{\Psi}_L \,, \tag{3.15}$$

whereas those defining the bosonic zero modes are

$$D_{[\mu}\dot{A}_{\nu]} = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}D^\rho\dot{A}^\sigma \qquad \text{and} \qquad D^\mu\dot{A}_\mu = -4i\lambda\dot{A}_4 \,. \tag{3.16}$$

The first equation is simply the statement that the $\mathfrak{g}$-valued 2-form $D_{[\mu}\dot{A}_{\nu]}$ is antiself-dual.

It is perhaps pertinent to remark that equations (3.14) are not meant to be understood as mutual inverse relations; that is, substituting the first equation for $\dot{A}_\mu$ in the second equation does *not* lead to an identity and neither does substituting the second equation for $\dot{\Psi}_L$ into the first. What these relations do mean is that given a bosonic zero mode $\dot{A}_\mu$ and a Killing spinor η on H^3, the RHS of the second of the above equations defines a fermionic zero mode; and that, conversely, given a fermionic zero mode

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$\dot{\Psi}_L$ and a Killing spinor ζ on H^3, the RHS of the first of the above equations defines a bosonic zero mode.

3.2.3 Computing the index using supersymmetry

Let us define the vector spaces

$$K^\pm = \{\xi_R | \nabla_i \xi_R = \mp i\lambda \Gamma_i \Gamma_4 \xi_R \quad \text{and} \quad \nabla_4 \xi_R = 0\} \ . \tag{3.17}$$

$K^\pm$ is a two-dimensional complex vector space isomorphic to the vector space of Killing spinor fields on H^3 with the stated sign of the Killing constant; that is,

$$K^\pm \cong \{\xi | \nabla_i \xi = \pm \lambda \sigma_i \xi\} \ . \tag{3.18}$$

Then letting Z_0 and Z_1 stand for the vector spaces of (complexified) bosonic and fermionic zero modes, respectively, we have exhibited real bilinear maps

$$
\begin{aligned}
K^+ \times Z_0 &\to Z_1 \\
(\eta_R, \dot{A}_\mu) &\mapsto i\dot{A}_\mu \Gamma^\mu \eta_R
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
K^- \times Z_1 &\to Z_0 \\
(\zeta_R, \dot{\Psi}_L) &\mapsto -i\zeta_R^\dagger \Gamma_\mu \dot{\Psi}_L \ .
\end{aligned}
\tag{3.19}
$$

We may compose the maps to arrive at

$$
\begin{aligned}
K^+ \times K^- \times Z_0 &\to Z_0 \\
(\eta_R, \zeta_R, \dot{A}_\mu) &\mapsto \zeta_R^\dagger \eta_R \dot{A}_\mu + \zeta_R^\dagger \Gamma_\mu{}^\nu \eta_R \dot{A}_\nu
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
K^+ \times K^- \times Z_1 &\to Z_1 \\
(\eta_R, \zeta_R, \dot{\Psi}_L) &\mapsto 2\zeta_R^\dagger \eta_R \dot{\Psi}_L \ ,
\end{aligned}
\tag{3.21}
$$

where in deriving these identities we have used the Fierz identity (2.11) for commuting spinors.

If we fix ζ_R and η_R such that $\zeta_R^\dagger \eta_R = \frac{1}{2}$, which we can always do, then the composite map in equation (3.21) is the identity, which implies that the maps in equation (3.19) are invertible. In particular, this implies that the vector spaces Z_0 and Z_1 of (complexi-

fied) bosonic and fermionic zero modes, respectively, are isomorphic. Therefore the number of fermionic zero modes is $4n$, where n is the monopole charge.

3.2.4 Complex structures

We start by defining some natural endomorphisms of the complexified tangent bundle of $H^3 \times S^1$ which can be built out of the Killing spinors.

Let us choose a complex basis $\eta_{R\alpha}$ and $\zeta_{R\beta}$, for $\alpha, \beta = 1, 2$, for the vector spaces K^+ and K^- of Killing spinors, respectively, which satisfies in addition the normalisation condition $\zeta^\dagger_{R\alpha}\eta_{R\beta} = \delta_{\alpha\beta}$. Let $A_{\alpha\beta}$ be the endomorphism of $T_{\mathbb{C}}(H^3 \times S^1)$ defined by

$$A_{\alpha\beta\,\mu}{}^\nu = -i\zeta^\dagger_{R\alpha}\Gamma_\mu{}^\nu\eta_{R\beta} \ , \tag{3.22}$$

where $\Gamma_\mu{}^\nu = \frac{1}{2}(\Gamma_\mu\Gamma^\nu - \Gamma^\nu\Gamma_\mu)$. Then one can show that the linear combinations

$$I = A_{11} \qquad J = \tfrac{1}{2}(A_{12} + A_{21}) \qquad K = -\frac{i}{2}(A_{12} - A_{21}) \tag{3.23}$$

satisfy the quaternion algebra

$$I^2 = J^2 = -\mathbb{I} \qquad IJ = -JI = K \ . \tag{3.24}$$

More invariantly, if $\eta_R \in K^+$ and $\zeta_R \in K^-$, let

$$E_\mu{}^\nu = -i\zeta^\dagger_R\Gamma_\mu{}^\nu\eta_R \tag{3.25}$$

denote the corresponding endomorphism of $T_{\mathbb{C}}(H^3 \times S^1)$. It follows from the fact that η_R, ζ_R have negative chirality, i.e., $\Gamma_{1234}\eta_R = -\eta_R$ and similarly for ζ_R, that $E_{\mu\nu}$ is self-dual:

$$\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}E^{\rho\sigma} = E_{\mu\nu} \ , \tag{3.26}$$

and also that

$$E_\mu{}^\rho E_\rho{}^\nu = -(\zeta^\dagger_R\eta_R)^2\delta_\mu{}^\nu \ . \tag{3.27}$$

The proof of this expression follows from the Fierz identity (2.11) and tedious use of

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the Clifford relations. Hence if we choose η_R and ζ_R such that $\zeta_R^\dagger \eta_R = 1$, then the endomorphism E is a (complex-linear) almost complex structure on $T_{\mathbb{C}}(H^3 \times S^1)$.

In addition, from the fact that η_R, ζ_R are Killing spinors it also follows that

$$\nabla_4 E_{\mu\nu} = 0, \qquad \nabla_i E_{4j} = 2i\lambda E_{ij} \qquad \nabla_i E_{jk} = -2i\lambda \left(\delta_{ij} E_{4k} - \delta_{ik} E_{4j} \right) . \qquad (3.28)$$

Indeed, the first equation follows from the fact that $\nabla_4 \zeta_R = 0 = \nabla_4 \eta_R$. The second equation follows from the following calculation:

$$
\begin{aligned}
\nabla_i E_{4j} &= \nabla_i \left(-i\zeta_R^\dagger \Gamma_4 \Gamma_j \eta_R \right) \\
&= -i \left(-i\lambda \zeta_R^\dagger \Gamma_4 \Gamma_i \right) \Gamma_4 \Gamma_j \eta_R - i\zeta_R^\dagger \Gamma_4 \Gamma_j \left(-i\lambda \Gamma_i \Gamma_4 \eta_R \right) \\
&= -\lambda \zeta_R^\dagger \Gamma_4 \Gamma_i \Gamma_4 \Gamma_j \eta_R - \lambda \zeta_R^\dagger \Gamma_4 \Gamma_j \Gamma_i \Gamma_4 \eta_R \\
&= \lambda \zeta_R^\dagger \left(\Gamma_i \Gamma_j - \Gamma_j \Gamma_i \right) \eta_R \\
&= 2\lambda \zeta_R^\dagger \Gamma_{ij} \eta_R \\
&= 2i\lambda E_{ij} ,
\end{aligned}
\qquad (3.29)
$$

where we have used the Clifford relations and the fact that $\nabla_i \zeta_R^\dagger = -i\lambda \zeta_R^\dagger \Gamma_4 \Gamma_i$.

The third and final equation follows from a similar calculation:

$$
\begin{aligned}
\nabla_i E_{jk} &= \nabla_i \left(-i\zeta_R^\dagger \Gamma_{jk} \eta_R \right) \\
&= -i \left(-i\lambda \zeta_R^\dagger \Gamma_4 \Gamma_i \right) \Gamma_{jk} \eta_R - i\zeta_R^\dagger \Gamma_{jk} \left(-i\lambda \Gamma_i \Gamma_4 \eta_R \right) \\
&= -\lambda \zeta_R^\dagger \Gamma_4 \Gamma_i \Gamma_{jk} \eta_R - \lambda \zeta_R^\dagger \Gamma_{jk} \Gamma_i \Gamma_4 \eta_R \\
&= -\lambda \zeta_R^\dagger \Gamma_4 \left(\Gamma_i \Gamma_{jk} - \Gamma_{jk} \Gamma_i \right) \eta_R .
\end{aligned}
\qquad (3.30)
$$

We now use the following consequences of the Clifford relations:

$$\Gamma_i \Gamma_{jk} = \Gamma_{ijk} + \delta_{ij} \Gamma_k - \delta_{ik} \Gamma_j \qquad \text{and} \qquad \Gamma_{jk} \Gamma_i = \Gamma_{jki} + \delta_{ik} \Gamma_j - \delta_{ij} \Gamma_k \qquad (3.31)$$

whence

$$\Gamma_i \Gamma_{jk} - \Gamma_{jk} \Gamma_i = 2\delta_{ij} \Gamma_k - 2\delta_{ik} \Gamma_j , \qquad (3.32)$$

and hence

$$
\begin{aligned}
\nabla_i E_{jk} &= -\lambda \zeta_R^\dagger \Gamma_4 \left(2\delta_{ij} \Gamma_k - 2\delta_{ik} \Gamma_j \right) \eta_R \\
&= -2\lambda \delta_{ij} \zeta_R^\dagger \Gamma_4 \Gamma_k \eta_R + 2\lambda \delta_{ik} \zeta_R^\dagger \Gamma_4 \Gamma_j \eta_R \\
&= -2i\lambda \left(\delta_{ij} E_{4k} - \delta_{ik} E_{4j} \right) \ .
\end{aligned}
\tag{3.33}
$$

Now we show that the endomorphisms $E_\mu{}^\nu$ act naturally on the bosonic zero modes $\dot{A}_\mu$. In other words, we show that if $\dot{A}_\mu$ obeys the linearised Bogomol'nyi equation (3.5) and the generalised Gauss law (3.9), then so does its image $\dot{B}_\mu := E_\mu{}^\nu \dot{A}_\nu$ under such an endomorphism.

We start with the generalised Gauss law (3.9). By definition,

$$
\begin{aligned}
D^\mu \dot{B}_\mu &= D^\mu \left(E_\mu{}^\nu \dot{A}_\nu \right) \\
&= \nabla^\mu E_\mu{}^\nu \dot{A}_\nu + E^{\mu\nu} D_\mu A_\nu \\
&= \nabla^i E_i{}^\nu \dot{A}_\nu + E^{\mu\nu} D_{[\mu} A_{\nu]} \\
&= -4i\lambda E_4{}^j \dot{A}_j \\
&= -4i\lambda \dot{B}_4 \ ,
\end{aligned}
\tag{3.34}
$$

where we have used equation (3.28) and the fact that, since $E^{\mu\nu}$ is selfdual and $D_{[\mu} A_{\nu]}$ antiselfdual, their inner product vanishes. Thus we see that $\dot{B}_\mu$ obeys the generalised Gauss law (3.9).

Next we show that $\dot{B}_\mu$ obeys the linearised Bogomol'nyi equation (3.5), which says that $D_{[\mu} \dot{B}_{\nu]}$ is antiselfdual, or equivalently, that

$$
D_i \dot{B}_4 + \varepsilon_{ijk} D_j \dot{B}_k = 0 \ .
\tag{3.35}
$$

Using equations (3.26) and (3.28), we calculate the first term in the left-hand side:

$$
\begin{aligned}
D_i \dot{B}_4 &= D_i \left(E_{4j} \dot{A}_j \right) \\
&= \nabla_i E_{4j} \dot{A}_j + E_{4j} D_i \dot{A}_j \\
&= 2i\lambda E_{ij} \dot{A}_j + E_{4j} D_i \dot{A}_j \\
&= -2i\lambda \varepsilon_{ijk} E_{4k} \dot{A}_j + E_{4j} D_i \dot{A}_j \ ,
\end{aligned}
\tag{3.36}
$$

and then also the second term:

$$
\begin{aligned}
\varepsilon_{ijk} D_j \dot{B}_k &= \varepsilon_{ijk} D_j \left(E_{kl} \dot{A}_l + E_{k4} \dot{A}_4 \right) \\
&= \varepsilon_{ijk} \left(\nabla_j E_{kl} \dot{A}_l - \nabla_j E_{4k} \dot{A}_4 + E_{kl} D_j \dot{A}_l + E_{k4} D_j \dot{A}_4 \right) \\
&= \varepsilon_{ijk} \left(2i\lambda E_{4k} \dot{A}_j - 2i\lambda \varepsilon_{jkl} E_{4l} \dot{A}_4 + E_{kl} D_j \dot{A}_l - E_{4k} D_j \dot{A}_4 \right) \\
&= 2i\lambda \varepsilon_{ijk} E_{4k} \dot{A}_j + 4i\lambda E_{4i} \dot{A}_4 - \varepsilon_{ijk} \varepsilon_{klm} E_{4m} D_j \dot{A}_l - E_{4k} \varepsilon_{ijk} D_j \dot{A}_4 \\
&= 2i\lambda \varepsilon_{ijk} E_{4k} \dot{A}_j - E_{4j} D_j \dot{A}_i - E_{4k} \varepsilon_{ijk} D_j \dot{A}_4 \\
&= 2i\lambda \varepsilon_{ijk} E_{4k} \dot{A}_j - E_{4j} D_j \dot{A}_i - E_{4k} (D_i \dot{A}_k - D_k \dot{A}_i) \\
&= 2i\lambda \varepsilon_{ijk} E_{4k} \dot{A}_j - E_{4k} D_i \dot{A}_k \ ,
\end{aligned}
\tag{3.37}
$$

where we have used that $\dot{A}_\mu$ obeys the linearised Bogomol'nyi equation (3.5) and the generalised Gauss law (3.9). Finally, we notice that the sum of the two terms vanish.

In summary, we have shown that the vector $E_\mu{}^\nu \dot{A}_\nu$ is tangent to the moduli space. Since there is a quaternion algebra in the span of the endomorphisms $E_\mu{}^\nu$, we see that the complexified tangent space to the moduli space is a quaternionic vector space. Indeed, if we let $\dot{A}_{a\mu}$ denote a complex frame for the complexified tangent space to $\mathcal{M}$ at (A, ϕ), then we may define endomorphisms $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ of the tangent space at that point by

$$
\mathcal{I}_a{}^b \dot{A}_{b\mu} = I_\mu{}^\nu \dot{A}_{a\nu} \qquad \mathcal{J}_a{}^b \dot{A}_{b\mu} = J_\mu{}^\nu \dot{A}_{a\nu} \qquad \mathcal{K}_a{}^b \dot{A}_{b\mu} = K_\mu{}^\nu \dot{A}_{a\nu} \ .
\tag{3.38}
$$

Letting the point (A, ϕ) vary we obtain a field of endomorphisms of $T_{\mathbb{C}} \mathcal{M}$ which we also call $\mathcal{I}, \mathcal{J}, \mathcal{K}$. It is evident that just like I, J, K generate a quaternion algebra, so do $\mathcal{I}, \mathcal{J}, \mathcal{K}$.

3.3 Geometry of the moduli space

In order to probe the geometry of the moduli space $\mathcal{M}$ of hyperbolic monopoles, we will consider the multiplet corresponding to a one-dimensional sigma model, except that we do not have an action for this model. In other words, we will consider maps $X : \mathbb{R} \to \mathcal{M}$, $t \to X(t)$ and the associated fermions θ which are sections of $\Pi X^* T_{\mathbb{C}} \mathcal{M}$: the (oddified) pullback by X of the complexified tangent bundle of $\mathcal{M}$. In this section we will first linearise the supersymmetry transformations and in this way arrive at an expression for the supersymmetry transformations of the bosonic moduli. We will then derive the supersymmetry transformations of the fermionic moduli by demanding closure of the one-dimensional $N = 4$ supersymmetry algebra.

3.3.1 Linearising the supersymmetry transformations

In this section we will derive the supersymmetry transformations for the bosonic zero modes by linearising the supersymmetry transformations preserved by the monopoles.

The δ_R supersymmetry transformations preserved by hyperbolic monopole configurations are given by equation (2.65). On the gauge field, and in four-dimensional language, it can be written as

$$\delta_\epsilon A_\mu = -i\epsilon_R^\dagger \Gamma_\mu \Psi_L \, , \tag{3.39}$$

which is already linear, hence at the level of the zero modes becomes

$$\delta_\epsilon \dot{A}_\mu = -i\epsilon_R^\dagger \Gamma_\mu \dot{\Psi}_L \, . \tag{3.40}$$

Choose a basis $\dot{\Psi}_{L\alpha}$ for the space Z_1 of fermionic zero modes. This defines a basis $\dot{A}_{\alpha\mu}$ for the space Z_0 of complexified bosonic zero modes via the second map in equation (3.19): namely,

$$\dot{A}_{\alpha\mu} := -i\zeta_R^\dagger \Gamma_\mu \dot{\Psi}_{L\alpha} \, , \tag{3.41}$$

where $\zeta_R \in K^-$ is a fixed Killing spinor. From equation (3.21) we may invert this to

write $\dot{\Psi}_{La} = i\dot{A}_{a\mu}\Gamma^\mu\eta_R$ for some $\eta_R \in K^+$ such that $\zeta_R^\dagger\eta_R = \frac{1}{2}$.

We now expand the general bosonic zero mode $\dot{A}_\mu = \dot{A}_{a\mu}X^a$ as a linear combination of the basis $\dot{A}_{a\mu}$ and similarly for the general fermionic zero mode $\dot{\Psi}_L = \dot{\Psi}_{La}\theta^a$. Inserting this in equation (3.40), we obtain

$$\delta_\epsilon\dot{A}_\mu = \dot{A}_{a\mu}\delta_\epsilon X^a = \dot{A}_{a\nu}\epsilon_R^\dagger\Gamma_\mu\Gamma^\nu\eta_R\theta^a = \dot{A}_{a\mu}\epsilon_R^\dagger\eta_R\theta^a + \epsilon_R^\dagger\Gamma_\mu{}^\nu\eta_R\dot{A}_{a\nu}\theta^a . \qquad (3.42)$$

The term $\epsilon_R^\dagger\Gamma_\mu{}^\nu\eta_R$ is a linear combination of the almost complex structures $I_\mu{}^\nu$, $J_\mu{}^\nu$ and $K_\mu{}^\nu$:

$$\epsilon_R^\dagger\Gamma_\mu{}^\nu\eta_R = \varepsilon_1 I_\mu{}^\nu + \varepsilon_2 J_\mu{}^\nu + \varepsilon_3 K_\mu{}^\nu , \qquad (3.43)$$

whence

$$\dot{A}_{a\mu}\delta_\epsilon X^a = i\left(\varepsilon^1 I_\mu{}^\nu + \varepsilon^2 J_\mu{}^\nu + \varepsilon^3 K_\mu{}^\nu\right)\dot{A}_{a\nu}\theta^b + \epsilon_R^\dagger\eta_R\dot{A}_{a\mu}\theta^a . \qquad (3.44)$$

From equation (3.38), we may write the action of these complex structures on $\dot{A}_{a\nu}$ in terms of the almost complex structures $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ on $T_{\mathbb{C}}\mathcal{M}$. The end result is that

$$\dot{A}_{a\mu}\delta_\epsilon X^a = \left(\varepsilon^1\mathcal{I}_b{}^a + \varepsilon^2\mathcal{J}_b{}^a + \varepsilon^3\mathcal{K}_b{}^a + \varepsilon^4\mathbb{I}_b{}^a\right)\dot{A}_{a\mu}\theta^b , \qquad (3.45)$$

where we have defined $\varepsilon^4 = \epsilon_R^\dagger\eta_R$. We remark that the $\varepsilon^{1,2,3,4}$ are Grassmann odd since so is ϵ_R. Since the $\dot{A}_{a\mu}$ are linearly independent, equation (3.45) is equivalent to

$$\delta_\epsilon X^a = \left(\varepsilon^1\mathcal{I}_b{}^a + \varepsilon^2\mathcal{J}_b{}^a + \varepsilon^3\mathcal{K}_b{}^a + \varepsilon^4\mathbb{I}_b{}^a\right)\theta^b , \qquad (3.46)$$

which defines the supersymmetry transformations for the bosonic moduli X^a.

It should be possible to derive the supersymmetry transformations for the fermionic moduli θ^a from the gauge theory as well, but we have been unable to do this and instead we will derive them by demanding the closure of the supersymmetry algebra.

3.3.2 Closure of the moduli space supersymmetry algebra

We shall now constrain the geometry of the moduli space by demanding closure of the supersymmetry algebra. In contrast with the case of Euclidean monopoles, where

the geometry of the moduli is constrained by demanding the invariance under supersymmetry of the effective action for the zero modes, the lack of convergence of the L^2 metric means that we cannot write down an action for the zero modes. It is the closure of the supersymmetry on the zero modes which will give us geometrical information.

To this end let us define odd derivations δ_A, $A = 1, \ldots, 4$, by $\delta_\epsilon X^a = \epsilon^A \delta_A X^a$; that is,

$$\delta_A X^a = \theta^b \mathcal{E}_{A\,b}{}^a \, , \tag{3.47}$$

where $\mathcal{E}_A = (\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathbb{I})$, or completely explicitly,

$$\delta_1 X^a = \theta^b \mathcal{I}_b{}^a \qquad \delta_2 X^a = \theta^b \mathcal{J}_b{}^a \qquad \delta_3 X^a = \theta^b \mathcal{K}_b{}^a \qquad \delta_4 X^a = \theta^a \, . \tag{3.48}$$

Hyperbolic monopoles are half-BPS, whence they preserve 4 of the 8 supercharges of the supersymmetric Yang–Mills theory and this means that the supersymmetry on the zero modes should close on the one-dimensional $N = 4$ supersymmetry algebra:

$$\delta_A \delta_B + \delta_B \delta_A = 2i\delta_{AB} \frac{d}{dt} \, . \tag{3.49}$$

Imposing this on X^a will determine the supersymmetry transformations of the fermionic moduli θ^a. For example,

$$\delta_4^2 X^a = iX'^a \implies \delta_4 \theta^a = iX'^a \, , \tag{3.50}$$

where X'^a represents the time derivative of X^a. Also, we have

$$\delta_1^2 X^a = iX'^a \implies \delta_1 \theta^a = -iX'^b \mathcal{I}_b{}^a - \theta^b \theta^d \partial_c \mathcal{I}_b{}^e \mathcal{I}_d{}^c \mathcal{I}_e{}^a \, , \tag{3.51}$$

and similarly for δ_2 and δ_3 by replacing $\mathcal{I}$ by $\mathcal{J}$ and $\mathcal{K}$, respectively. Next we impose $\delta_4 \delta_i X^a = -\delta_i \delta_4 X^a$ for $i = 1, 2, 3$. For example,

$$0 = \delta_1 \delta_4 X^a + \delta_4 \delta_1 X^a = \theta^d \theta^b \left(\partial_d \mathcal{I}_b{}^a + \partial_c \mathcal{I}_b{}^e \mathcal{I}_d{}^c \mathcal{I}_e{}^a \right) \, , \tag{3.52}$$

and similarly for $\mathcal{J}$ and $\mathcal{K}$. This allows to rewrite in a slightly simpler way the super-

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symmetry transformations for the θ^a:

$$\delta_1 \theta^a = -iX'^b \mathcal{I}_b{}^a + \theta^b \theta^c \partial_c \mathcal{I}_b{}^a$$

$$\delta_2 \theta^a = -iX'^b \mathcal{J}_b{}^a + \theta^b \theta^c \partial_c \mathcal{J}_b{}^a$$

$$\delta_3 \theta^a = -iX'^b \mathcal{K}_b{}^a + \theta^b \theta^c \partial_c \mathcal{K}_b{}^a \qquad (3.53)$$

$$\delta_4 \theta^a = iX'^a \ .$$

We, now, introduce the connection by defining its coefficients $\Gamma_{bc}{}^a$ as

$$\theta^b \theta^c \partial_c \mathcal{E}_{Ab}{}^a = \Gamma_{bc}{}^a \theta^c \delta_A X^b, \qquad (3.54)$$

where the connection symbol, with Latin indices, should not be confused with the Dirac gamma matrices symbol, with Greek indices, we used for earlier. By definition Γ is torsion free since for $A = 4$ we find that equation (3.54) becomes

$$\Gamma_{bc}{}^a = \Gamma_{cb}{}^a \ , \qquad (3.55)$$

The other characteristics of Γ will be recovered by closing the algebra. The odd derivations δ_A, $A = i, 4$, $(i = 1, 2, 3)$ now becomes

$$\delta_i X^a = \mathcal{E}_i{}^a{}_b \theta^b \qquad\qquad \delta_i \theta^a = -i\mathcal{E}_i{}^a{}_b X'^b + \Gamma_{bc}{}^a \theta^c \delta_i X^b \ , \qquad (3.56)$$

$$\delta_4 X^a = \theta^a \qquad\qquad \delta_4 \theta^a = iX'^a + \Gamma_{bc}{}^a \theta^c \delta_4 X^b, \qquad (3.57)$$

where $\mathcal{E}_i$ are the endomorphisms of $T_{\mathbb{C}}\mathcal{M}$ defined in (3.38). Demanding that (3.56) obey the one-dimensional $N = 4$ supersymmetry algebra will constrain the geometry, as we will now show.

Let us start by demanding closure of the supersymmetry algebra on the X^a; that is,

$$(\delta_A \delta_B + \delta_B \delta_A) X^a = 2i\delta_{AB} X'^a \ . \qquad (3.58)$$

We start with $A = i$ and $B = j$ and we compute $\delta_i \delta_j X^a$ using equation (3.56):

$$
\begin{aligned}
\delta_i \delta_j X^a &= \delta_i \left(\mathcal{E}_j{}^a{}_b \theta^b \right) \\
&= \partial_c \mathcal{E}_j{}^a{}_b \delta_i X^c \theta^b + \mathcal{E}_j{}^a{}_b \delta_i \theta^b \\
&= \partial_c \mathcal{E}_j{}^a{}_b \delta_i X^c \theta^b + \mathcal{E}_j{}^a{}_b \left(-i \mathcal{E}_i{}^b{}_c X'^c + \Gamma_{cd}{}^b \theta^d \delta_i X^c \right) \\
&= -i \mathcal{E}_j{}^a{}_b \mathcal{E}_i{}^b{}_c X'^c + \left(\partial_c \mathcal{E}_j{}^a{}_d - \Gamma_{cd}{}^b \mathcal{E}_j{}^a{}_b \right) \delta_i X^c \theta^b \\
&= -i \mathcal{E}_j{}^a{}_b \mathcal{E}_i{}^b{}_c X'^c + \mathcal{E}_i{}^c{}_e \left(\partial_c \mathcal{E}_j{}^a{}_d - \Gamma_{cd}{}^b \mathcal{E}_j{}^a{}_b \right) \theta^e \theta^d \ ,
\end{aligned}
\tag{3.59}
$$

whence the left-hand side of equation (3.58) becomes

$$
\begin{aligned}
\left(\delta_i \delta_j + \delta_j \delta_i \right) X^a = &-i \left(\mathcal{E}_j{}^a{}_b \mathcal{E}_i{}^b{}_c + \mathcal{E}_i{}^a{}_b \mathcal{E}_j{}^b{}_c \right) X'^c \\
&+ \mathcal{E}_i{}^c{}_e \left(\partial_c \mathcal{E}_j{}^a{}_d - \Gamma_{cd}{}^b \mathcal{E}_j{}^a{}_b \right) \theta^e \theta^d + \mathcal{E}_j{}^c{}_e \left(\partial_c \mathcal{E}_i{}^a{}_d - \Gamma_{cd}{}^b \mathcal{E}_i{}^a{}_b \right) \theta^e \theta^d \ .
\end{aligned}
\tag{3.60}
$$

Equation (3.58) is satisfied provided that

$$
\mathcal{E}_j{}^a{}_b \mathcal{E}_i{}^b{}_c + \mathcal{E}_i{}^a{}_b \mathcal{E}_j{}^b{}_c = -2 \delta_{ij} \delta^a{}_c
\tag{3.61}
$$

and that

$$
\left(\mathcal{E}_i{}^c{}_e \partial_c \mathcal{E}_j{}^a{}_d - \Gamma_{cd}{}^b \mathcal{E}_i{}^c{}_e \mathcal{E}_j{}^a{}_b + \mathcal{E}_j{}^c{}_e \partial_c \mathcal{E}_i{}^a{}_d - \Gamma_{cd}{}^b \mathcal{E}_j{}^c{}_e \mathcal{E}_i{}^a{}_b \right) \theta^e \theta^d = 0 \ .
\tag{3.62}
$$

Equation (3.61) is satisfied by virtue of the definition of the endomorphisms $\mathcal{E}_i$ (3.38). Equation (3.62) becomes

$$
\begin{aligned}
\mathcal{E}_i{}^c{}_e \partial_c \mathcal{E}_j{}^a{}_d &- \Gamma_{cd}{}^b \mathcal{E}_i{}^c{}_e \mathcal{E}_j{}^a{}_b + \mathcal{E}_j{}^c{}_e \partial_c \mathcal{E}_i{}^a{}_d - \Gamma_{cd}{}^b \mathcal{E}_j{}^c{}_e \mathcal{E}_i{}^a{}_b = \\
&\mathcal{E}_i{}^c{}_d \partial_c \mathcal{E}_j{}^a{}_e - \Gamma_{ce}{}^b \mathcal{E}_i{}^c{}_d \mathcal{E}_j{}^a{}_b + \mathcal{E}_j{}^c{}_d \partial_c \mathcal{E}_i{}^a{}_e - \Gamma_{ce}{}^b \mathcal{E}_j{}^c{}_d \mathcal{E}_i{}^a{}_b \ .
\end{aligned}
\tag{3.63}
$$

Defining the covariant derivative of an endomorphism $\mathcal{E}^a{}_b$ as

$$
\nabla_a \mathcal{E}^b{}_c = \partial_a \mathcal{E}^b{}_c - \Gamma_{ac}{}^d \mathcal{E}^b{}_d + \Gamma_{ad}{}^b \mathcal{E}^d{}_c,
\tag{3.64}
$$

so in terms of covariant derivatives equation (3.62) can be written as

$$\mathcal{E}_i{}^c{}_e \nabla_c \mathcal{E}_j{}^a{}_d + \mathcal{E}_j{}^c{}_e \nabla_c \mathcal{E}_i{}^a{}_d - \mathcal{E}_i{}^c{}_d \nabla_c \mathcal{E}_j{}^a{}_e - \mathcal{E}_j{}^c{}_d \nabla_c \mathcal{E}_i{}^a{}_e = 0 \,. \tag{3.65}$$

As shown in Appendix 7.1, taking into account that ∇ is torsion-free, this is nothing but

$$\left[\mathcal{E}_i, \mathcal{E}_j \right] = 0 \,, \tag{3.66}$$

where the bracket is the Frölicher–Nijenhuis bracket of the two endomorphisms $\mathcal{E}_i$ and $\mathcal{E}_j$, thought of as vector valued one-forms, given by equation (7.10) in Appendix 7.1 for endomorphisms K and L. Now closing the algebra with $A = 4$ and $B = i$, we get

$$\partial_e \mathcal{E}_i{}^a{}_d - \Gamma_{ed}{}^b \mathcal{E}_i{}^a{}_b - \Gamma_{cd}{}^a \mathcal{E}_i{}^c{}_e = \partial_d \mathcal{E}_i{}^a{}_e - \Gamma_{de}{}^b \mathcal{E}_i{}^a{}_b - \Gamma_{ce}{}^a \mathcal{E}_i{}^c{}_d \,. \tag{3.67}$$

which in terms of the covariant derivative becomes

$$\nabla_e \mathcal{E}_i{}^a{}_d = \nabla_d \mathcal{E}_i{}^a{}_e \,. \tag{3.68}$$

We will now show that equation (3.68) already implies that the $\mathcal{E}_i$ are parallel with respect to ∇, whence equation (3.66) is automatically satisfied. This result will rely on the existence (shown in Appendix 7.2) of a torsion-free connection $\overset{o}{\nabla}$ satisfying $\overset{o}{\nabla} \mathcal{E}_i = 0$. We will refer to $\overset{o}{\nabla}$ as the *Obata connection*, since it is the analogue of the Obata connection of a hypercomplex structure [74]. Indeed, let us show that the tensor $S = \nabla - \overset{o}{\nabla}$ vanishes as a result of equation (3.68), whence $\nabla = \overset{o}{\nabla}$.

To see this, let $\nabla = \overset{o}{\nabla} + S$, so that

$$\nabla_X Y - \overset{o}{\nabla}_X Y = S_X Y = S(X, Y) \,, \tag{3.69}$$

which defines the endomorphism S_X. Equation (3.68) is easily seen to be equivalent to

$$(\nabla_X \mathcal{E}_i) Y = (\nabla_Y \mathcal{E}_i) X \,, \tag{3.70}$$

for all vector fields X, Y. We now use that

$$\nabla_X \mathcal{E}_i = \overset{o}{\nabla}_X \mathcal{E}_i + [S_X, \mathcal{E}_i] = [S_X, \mathcal{E}_i] \,, \tag{3.71}$$

where the bracket here is simply the commutator of endomorphisms. This allows us to rewrite equation (3.70) as

$$\begin{aligned}
0 &= [S_X, \mathcal{E}_i]Y - [S_Y, \mathcal{E}_i]X \\
&= S_X \mathcal{E}_i Y - \mathcal{E}_i S_X Y - S_Y \mathcal{E}_i X + \mathcal{E}_i S_Y X \\
&= S(X, \mathcal{E}_i Y) - S(Y, \mathcal{E}_i X) \,,
\end{aligned} \tag{3.72}$$

where we have used that $S_X Y = S_Y X$ due to the fact that both ∇ and $\overset{o}{\nabla}$ are torsion-free. Using again that $S(X, Y) = S(Y, X)$, we see that $\mathcal{E}_i$ is S-symmetric; that is,

$$S(\mathcal{E}_i X, Y) = S(X, \mathcal{E}_i Y) \,. \tag{3.73}$$

But now the quaternion algebra says that $\mathcal{E}_3 = \mathcal{E}_1 \mathcal{E}_2$, whence using equation (3.73) repeatedly we see that

$$S(\mathcal{E}_3 X, Y) = S(\mathcal{E}_1 \mathcal{E}_2 X, Y) = S(\mathcal{E}_2 X, \mathcal{E}_1 Y) = S(X, \mathcal{E}_2 \mathcal{E}_1 Y) = -S(X, \mathcal{E}_3 Y) \,.$$

But equation (3.73) also says that $S(\mathcal{E}_3 X, Y) = S(X, \mathcal{E}_3 Y)$, whence we see that for all X, Y

$$S(X, \mathcal{E}_3 Y) = 0 \,, \tag{3.74}$$

and since $\mathcal{E}_3$ is invertible, that $S(X, Y) = 0$ for all X, Y, as desired.

We now demand closure of the supersymmetry algebra on the fermionic coordinates θ^a:

$$\left(\delta_A \delta_B + \delta_B \delta_A\right) \theta^a = 2i \delta_{AB} \theta'^a \,. \tag{3.75}$$

We will see that the algebra closes without imposing any further conditions on the geometry. In a similar treatment to the one done with the the bosonic coordinates, we

start with $A = i$ and $B = j$. We compute $\delta_i \delta_j \theta^a$ using equation (3.56):

$$\delta_i \delta_j \theta^a = -i\partial_c \mathcal{E}_j{}^a{}_b \delta_i X^c X'^b - i\mathcal{E}_j{}^a{}_b (\delta_i X^b)' + \partial_d \Gamma^a{}_{bc} \delta_i X^d \theta^c \delta_j X^b$$

$$+ \Gamma^a{}_{bc} \delta_i \theta^c \delta_j X^b + \Gamma^a{}_{bc} \theta^c \delta_i \delta_j X^b . \quad (3.76)$$

Using that $(\mathcal{E}_i)' = \partial_b \mathcal{E}_i X'^b$, we may rewrite this as

$$\delta_i \delta_j \theta^a = -i\partial_c \mathcal{E}_j{}^a{}_b \delta_i X^c \theta^d X'^b - i\mathcal{E}_j{}^a{}_b \partial_d \mathcal{E}_i{}^b{}_c X'^d \theta^c - i\mathcal{E}_j{}^a{}_b \mathcal{E}_i{}^b{}_c \theta'^c$$

$$+ \partial_d \Gamma^a{}_{bc} \mathcal{E}_i{}^d{}_c \mathcal{E}_j{}^b{}_f \theta^e \theta^c \theta^f - i\Gamma^a{}_{bc} \mathcal{E}_i{}^c{}_d \mathcal{E}_j{}^d{}_e \theta^e X'^d$$

$$+ \Gamma^a_{bc} \Gamma^c_{de} \theta^d \delta_i X^e \delta_j X^b + \Gamma^a{}_{bc} \theta^c \delta_i \delta_j X^b . \quad (3.77)$$

The left-hand side of equation (3.75) can then be written as

$$(\delta_i \delta_j + \delta_j \delta_i)\theta^a = 2i\delta_{ij}\theta'^a + R_{bcd}{}^a \mathcal{E}_i{}^b{}_e \mathcal{E}_j{}^c{}_f \theta^d \theta^e \theta^f$$

$$- i\left(\mathcal{E}_i{}^c{}_e \partial_c \mathcal{E}_j{}^a{}_d + \mathcal{E}_j{}^c{}_e \partial_c \mathcal{E}_i{}^a{}_d + \mathcal{E}_i{}^a{}_b \partial_d \mathcal{E}_j{}^b{}_e + \mathcal{E}_j{}^a{}_b \partial_d \mathcal{E}_i{}^b{}_e \right.$$

$$\left. + \Gamma_{bc}{}^a \mathcal{E}_i{}^c{}_d \mathcal{E}_j{}^b{}_e + \Gamma_{bc}{}^a \mathcal{E}_j{}^c{}_d \mathcal{E}_i{}^b{}_e + 2\Gamma_{de}{}^a \delta_{ij} \right) X'^d \theta^e , \quad (3.78)$$

where we have used the closure of the supersymmetry algebra on the X^a and the definition of the curvature tensor for the connection ∇:

$$R_{abc}{}^d = \partial_b \Gamma_{ac}{}^d - \partial_a \Gamma_{bc}{}^d + \Gamma_{ac}{}^e \Gamma_{be}{}^d - \Gamma_{bc}{}^e \Gamma_{ae}{}^d . \quad (3.79)$$

Comparing with the closure condition (3.75), we see that there are two kinds of offending terms: those linear in θ and those cubic in θ. We will now show that both terms vanish as a consequence of $\nabla \mathcal{E}_i = 0$.

First, the terms linear in θ are easily seen to be zero by replacing $\partial_a \mathcal{E}_i{}^b{}_c$ with $\Gamma_{ac}{}^d \mathcal{E}_i{}^b{}_d - \Gamma_{ad}{}^b \mathcal{E}_i{}^d{}_c$, which is equivalent to $\mathcal{E}_i$ being ∇-parallel. Doing so, and using the torsion-free condition $\Gamma_{ab}{}^c = \Gamma_{ba}{}^c$ and the closure condition (3.61), we see that all terms cancel.

Before we proceed to show that terms cubic in θ vanish, we compute the closure of algebra when $A = B = 4$ and when $A = 4$, but $B = i$. In a similar, yet simpler

calculation we find

$$2\delta_4\theta^a = 2i\theta'^a + R_{bcd}{}^a\theta^d\theta^b\theta^c,$$

$$(\delta_4\delta_i + \delta_i\delta_4)\theta^a = R_{bcd}{}^a\mathcal{E}_i{}^c{}_f\theta^d\theta^b\theta^f.$$

Hence, the terms cubic in θ in all cases will vanish if and only if

$$\mathfrak{S}_{d,e,f}R_{bcd}{}^a\mathcal{E}_A{}^b{}_e\mathcal{E}_B{}^c{}_f = 0 , \tag{3.80}$$

where $\mathfrak{S}$ denotes skew-symmetrisation in the relevant symbols. Equivalently, but more invariantly, they will vanish if and only if for all vector fields X, Y, Z,

$$\mathfrak{S}_{X,Y,Z}R(\mathcal{E}_A X, \mathcal{E}_B Y)Z = 0 . \tag{3.81}$$

If $A = B = 4$, then this is true by virtue of the algebraic Bianchi identity for the torsion-free connection ∇. If $A = 4$, but $B = i$, then condition (3.81) becomes equivalent to

$$\mathfrak{S}_{X,Y,Z}R(X, \mathcal{E}_i Y)Z \overset{?}{=} 0 . \tag{3.82}$$

Using the algebraic Bianchi identity, we can turn the left-hand side of this equation into

$$-\mathfrak{S}_{X,Y,Z}\left(R(Z,X)\mathcal{E}_i Y + R(\mathcal{E}_i Y, Z)X\right) . \tag{3.83}$$

Since $\mathcal{E}_i$ is ∇-parallel, it is invariant under the infinitesimal holonomy representation, whence in particular it commutes with the curvature operators $R(X, Y)$; that is,

$$R(Z, X)\mathcal{E}_i Y = \mathcal{E}_i R(Z, X)Y . \tag{3.84}$$

This means that the first term in (3.83) vanishes due to the algebraic Bianchi identity, whereas the second term is given by

$$-\mathfrak{S}_{X,Y,Z}R(\mathcal{E}_i Y, Z)X = +\mathfrak{S}_{X,Y,Z}R(Z, \mathcal{E}_i Y)X = -\mathfrak{S}_{X,Y,Z}R(X, \mathcal{E}_i Y)Z , \tag{3.85}$$

which is the negative of what we started with, whence it too vanishes:

$$\mathfrak{S}_{X,Y,Z} R(X, \mathcal{E}_i Y)Z = 0 \; . \tag{3.86}$$

Finally, if $A = i$ and $B = j$, condition (3.81) becomes

$$\mathfrak{S}_{X,Y,Z} R(\mathcal{E}_i X, \mathcal{E}_j Y)Z \overset{?}{=} 0 \; . \tag{3.87}$$

We again use the algebraic Bianchi identity to rewrite the right-hand side as

$$- \mathfrak{S}_{X,Y,Z} \left(R(Z, \mathcal{E}_i X)\mathcal{E}_j Y + R(\mathcal{E}_j Y, Z)\mathcal{E}_i X \right) \; . \tag{3.88}$$

Using that $\mathcal{E}_i$ and $\mathcal{E}_j$ commute with the curvature operators, we may rewrite this as

$$- \mathfrak{S}_{X,Y,Z} \left(\mathcal{E}_j R(Z, \mathcal{E}_i X)Y + \mathcal{E}_i R(\mathcal{E}_j Y, Z)X \right) \; , \tag{3.89}$$

and both terms are now seen to vanish by virtue of equation (3.86). In summary, the supersymmetry algebra closes on the fermionic moduli.

To summarize the results of this section, the supersymmetry algebra closes on the moduli. This is because the complex-linear endomorphisms $\mathcal{I}, \mathcal{J}, \mathcal{K}$ on $T_{\mathbb{C}}\mathcal{M}$ are parallel relative to a (unique) torsion-free connection, whence in particular the quaternionic structure they define on $T_{\mathbb{C}}\mathcal{M}$ is integrable: their Frölicher–Nijenhuis brackets vanish, therefore $\mathcal{M}^{\mathbb{C}}$ is hypercomplex. In the terminology of Bielawski and Schwachhöfer this means that $\mathcal{M}$ is a pluricomplex manifold, which we have hereby shown to follow naturally from supersymmetry.

Chapter 4

Supersymmetric Yang-Mills-Higgs Theory on H^3 with Real Fields

4.1 Introduction

We saw in the chapters two and three that starting from a supersymmetric Yang-Mills theory on a Euclidean space and constructing a supersymmetric Yang-Mills-Higgs theory on hyperbolic space gives a theory with complex fields, and hence it paves the way to explore the geometry of the complex space of hyperbolic monopoles. From our results in chapter three we can deduce the nature of the geometry of the real moduli space of hyperbolic monopoles using the properties of pluricomplex geometry [68], however, it is more solid to tackle the real moduli space directly, and for that sake we construct a supersymmetric Yang-Mills-Higgs theory on H^3 where the gauge field components are real. Our starting point is a supersymmetric Yang-Mills theory on Minkowski space $\mathbb{R}^{(1,5)}$ which we reduce to $\mathbb{R}^3$ and then promote to H^3. Next, we find the equations of motion which are more general than the usual supersymmetric Yang-Mills-Higgs fields equations, however we show later, using certain supersymmetric constraints, that supersymmetric hyperbolic monopoles form a subset of the equations of motion solutions. Finally, we compare our theory to an example of family "A" from [115] and then we study the superalgebra.

4.2 On-shell supersymmetry in Minkowski 6-Spacetime

The existence of an on-shell supersymmetric theory in a certain dimension requires a balance between the bosonic and fermionic degrees of freedom. Supersymmetric Yang-Mills theory exists in six dimension if we we take the fermionic field to be of Weyl nature. The gauge field has six real components, one is removed by gauge invariance, and the equation of motion projects out another, hence we are down to four degrees of freedom. The Weyl spinor has four non zero complex components, two of which are projected out by the equation of motion, hence we are down to two complex components or four degrees of freedom. The $N = 1$ supersymmetric Yang-Mills theory was first constructed in [121], which we review briefly in this section.

The Minkowskian supersymmetric Yang-Mills Lagrange density function in $\mathbb{R}^{(1,5)}$ is

$$\mathcal{L}^{(1,5)} = -\frac{1}{2} G_{AB} G^{AB} + \bar{\Psi} \Gamma^A D_A \Psi, \tag{4.1}$$

where the capital alphabet indices $A, B, C, \ldots$ run from 0 to 5. The gauge group index has been suppressed, but it should be understood that each term is an ad-invariant inner product on a Lie algebra $\mathfrak{g}$.

We choose the Lie algebra structure constant to be real, hence the generators T_α and so are the gauge fields $W_A = W_A^\alpha T_\alpha$ are antihermitian. The field strength and the covariant derivatives are given by

$$G_{AB} = \partial_A W_B - \partial_B W_A + [W_A, W_B], \qquad D_A \Psi = \partial_A \Psi + [W_A, \Psi], \tag{4.2}$$

so G_{AB} is also antihermitian. Ψ is an anticommuting Weyl spinor, $\bar{\Psi}$ is Dirac adjoint of Ψ, hence we have

$$(\lambda^\dagger \Psi)^\dagger = -\Psi^\dagger \lambda, \qquad\qquad \bar{\Psi} = \Psi^\dagger \Gamma_0. \tag{4.3}$$

Γ^A are $6-$dimensional unitary gamma matrices that satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}, \qquad \eta_{AB} = \mathrm{diag}(-1, +1, \ldots, +1), \qquad \Gamma_A^\dagger = -\Gamma_0 \Gamma_A \Gamma_0^{-1}.$$

With these definitions one can check that the Lagrangian (4.1) is Hermitian.

In addition to the $SO(5,1)$ Lorentz invariance and gauge invariance, the action defined by $\mathcal{L}^{(1,5)}$ is invariant under the following supersymmetry transformations

$$\delta W_A = \bar{\varepsilon}\Gamma_A \Psi - \bar{\Psi}\Gamma_A \varepsilon, \tag{4.4}$$

$$\delta\Psi = \Gamma^{AB}G_{AB}\varepsilon, \tag{4.5}$$

$$\delta\bar{\Psi} = -\bar{\varepsilon}\Gamma^{AB}G_{AB}, \tag{4.6}$$

where $\Gamma^{AB} = \frac{1}{2}[\Gamma^A, \Gamma^B]$, and ε is the supersymmetry parameter which is a constant spinor, and of same nature as Ψ.

4.3 Reduction to $\mathbb{R}^3$

Starting with $N = 1$ theory in $D = 5 + 1$ flat space time, we quotient now by $\mathbb{R}^{(2,1)}$ to obtain a theory on $\mathbb{R}^3$.

4.3.1 Reduction of bosonic fields

The gauge fields W_A and Ψ upon reduction will depend only on $x^i, i = 1, 2, 3$, which means that $\partial_0 \equiv \partial_4 \equiv \partial_5 \equiv 0$. This breaks the Lorentz invariance $SO(5,1)$ down to $SO(3) \times SO(2,1)$. The gauge field W_A breaks down into W_i, a 3-dimensional gauge field, and the other three components, $W_\mu : W_0 = \phi_0, W_4 = \phi_4$, and $W_5 = \phi_5$, are scalar fields transforming as vectors under the R−symmetry $SO(2,1)$. In terms of these fields, the field strength breaks up as G_{ij}, $G_{i\mu} = D_i\phi_\mu$ and $G_{\mu\nu} = [\phi_\mu, \phi_\nu]$. We will keep the fermionic part reduction to the next section, so the Lagrangian with only the bosonic part reduced, will read as

$$\mathcal{L}_{SYM} = -\frac{1}{2}G_{ij}G^{ij} - D_i\phi_\mu D^i\phi^\mu - \frac{1}{2}[\phi_\mu, \phi_\nu][\phi^\mu, \phi^\nu] + \bar{\Psi}\Gamma^i D_i\Psi + \bar{\Psi}\Gamma^\mu[\phi_\mu, \Psi]. \tag{4.7}$$

4.3.2 Reduction of fermionic fields

The gamma matrices $\{\Gamma_M\}$ are in the Clifford group $Cl(5,1)$ and they satisfy the Clifford algebra $\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}\mathbb{I}_8$, where η_{MN} is mostly positive. Upon reduction, $\{\Gamma_M\}$

will be decomposed into two sets. The first set is γ_i, form the representation of the $Cl(3,0)$, and satisfy $\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{I}_2$, where as the second set is $\bar{\gamma}_\mu$, form the representation of the $Cl(2,1)$, and satisfy $\{\bar{\gamma}_\mu, \bar{\gamma}_\nu\} = 2\eta_{\mu\nu}\mathbb{I}_2$. γ_i will be chosen to be the Pauli matrices $\gamma_i = \{\sigma_1, \sigma_2, \sigma_3\}$ so the volume element $\omega = i\mathbb{I}$, and $\bar{\gamma}_\mu = \{i\sigma_2, \sigma_1, \sigma_3\}$ with volume element $\bar{\omega} = \mathbb{I}$.

Since the reduction is from even to odd dimension the decomposition of Γ_M will include an "auxiliary" matrix. A possible decomposition is given by

$$\Gamma_i = \mathbb{I}_2 \otimes \gamma_i \otimes \sigma_1,$$

$$\Gamma_\mu = \bar{\gamma}_\mu \otimes \mathbb{I}_2 \otimes \sigma_2.$$

One can easily check now that all the properties of the Γ_M are satisfied by this choice.

$$\Gamma_\mu^2 = (\bar{\gamma}_\mu \otimes \mathbb{I}_2 \otimes \sigma_2)(\bar{\gamma}_\mu \otimes \mathbb{I}_2 \otimes \sigma_2)$$

$$= \bar{\gamma}_\mu^2 \otimes \mathbb{I}_2^2 \otimes \sigma_2^2 = \begin{cases} -\mathbb{I}_8 & \text{if } \mu = 0, \\ \mathbb{I}_8 & \text{if } \mu = 4 \text{ or } 5 \end{cases},$$

$$\Gamma_i^2 = (\mathbb{I}_2 \otimes \gamma_i \otimes \sigma_1)(\mathbb{I}_2 \otimes \gamma_i \otimes \sigma_1)$$

$$= \mathbb{I}_2^2 \otimes \gamma_i^2 \otimes \sigma_1^2 = \mathbb{I}_8,$$

The anticommutation relations are also satisfied

$$\{\Gamma_\mu, \Gamma_i\} = \{\mathbb{I}_2 \otimes \gamma_\mu \otimes \sigma_1, \bar{\gamma}_i \otimes \mathbb{I}_2 \otimes \sigma_2\}$$

$$= \bar{\gamma}_i \otimes \gamma_\mu \otimes \{\sigma_1, \sigma_2\} = 0.$$

And finally, the volume element of the Γ_M is given by

$$\Gamma_{012345} = \Gamma_{045}.\Gamma_{123}$$

$$= (\bar{\gamma}_{045} \otimes \mathbb{I}_2 \otimes \sigma_2).(\mathbb{I}_2 \otimes \gamma_{123} \otimes \sigma_1)$$

$$= \bar{\gamma}_{045} \otimes \gamma_{123} \otimes -i\sigma_3 = \mathbb{I}_4 \otimes \sigma_3.$$

The decomposition of the spinor depends on the action of Γ_μ on the Ψ. In six dimensional notation Ψ is a spinor of positive chirality, hence

$$\Gamma_\mu \,.\Psi = (\mathbb{I}_2 \otimes \gamma_\mu \otimes \sigma_1).\psi \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $\psi = \frac{1}{2}(\mathbb{I} + \Gamma_7)\Psi$, then Ψ can be written as

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \psi_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \psi_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the conjugation of the spinor decomposes as

$$\bar{\Psi} = \Psi^\dagger \Gamma_0$$
$$= [\begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \psi_1^\dagger \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \psi_2^\dagger \otimes \begin{pmatrix} 1 & 0 \end{pmatrix}].[i\sigma_2 \otimes \mathbb{I}_2 \otimes \sigma_2]$$
$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \psi_1^\dagger \otimes \begin{pmatrix} 0 & -i \end{pmatrix} + \begin{pmatrix} -1 & 0 \end{pmatrix} \otimes \psi_2^\dagger \otimes \begin{pmatrix} 0 & -i \end{pmatrix}.$$

4.3.3   Reduction of the Lagrangian

The reduction of the spinors and gamma matrices will effect only the fermionic part of the Lagrangian (4.7), which is given by

$$\mathcal{L}_f = \bar{\Psi}\Gamma^i D_i \Psi + \bar{\Psi}\Gamma^\mu [\phi_\mu, \Psi].$$

As for $\Gamma^i D_i \Psi$ we get

$$\Gamma^i D_i \Psi = \{\mathbb{I}_2 \otimes \gamma^i \otimes \sigma_1\}.\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes D_i \psi_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes D_i \psi_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \gamma^i D_\mu \psi_1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \gamma^i D_i \psi_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

therefore $\bar{\Psi}\Gamma^i D_i \Psi$ becomes

$$\bar{\Psi}\Gamma^i D_i \Psi = -i\psi_1^\dagger \gamma^i D_i \psi_2 + i\psi_2^\dagger \gamma^i D_i \psi_1. \tag{4.8}$$

Now, for $\Gamma^\mu[\phi_\mu, \Psi]$ we get

$$\Gamma^\mu[\phi_\mu, \Psi] = \{\bar{\gamma}^\mu \otimes \mathbb{I}_2 \otimes \sigma_2\}.\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes [\phi_i, \psi_1] \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes [\phi_i, \psi_2] \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

$$= \bar{\gamma}^i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes [\phi_i, \psi_1] \otimes \begin{pmatrix} 0 \\ i \end{pmatrix} + \bar{\gamma}^i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes [\phi_i, \psi_2] \otimes \begin{pmatrix} 0 \\ i \end{pmatrix}.$$

Hence, the final form of $\bar{\Psi}\Gamma^\mu[\phi_\mu, \Psi]$ is

$$\bar{\Psi}\Gamma^\mu[\phi_\mu, \Psi] = \begin{pmatrix} 0 & 1 \end{pmatrix} \bar{\gamma}^\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \psi_1^\dagger[\phi_\mu, \psi_2] + \begin{pmatrix} -1 & 0 \end{pmatrix} \bar{\gamma}^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \psi_2^\dagger[\phi_\mu, \psi_1]$$

$$+ \begin{pmatrix} 0 & 1 \end{pmatrix} \bar{\gamma}^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \psi_1^\dagger[\phi_\mu, \psi_1] + \begin{pmatrix} -1 & 0 \end{pmatrix} \bar{\gamma}^\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \psi_2^\dagger[\phi_i, \psi_2]$$

$$= -\psi_1^\dagger[\phi_5, \psi_2] - \psi_2^\dagger[\phi_5, \psi_1] + \psi_1^\dagger[\phi_4, \psi_1] - \psi_2^\dagger[\phi_4, \psi_2] + \psi_1^\dagger[\phi_0, \psi_1] + \psi_2^\dagger[\phi_0, \psi_2].$$

$$\tag{4.9}$$

Putting everything together, $(5.4, 5.5)$ in (4.7), will give the supersymmetric Yang-Mills-Higgs Lagrangian on three dimensional Euclidean background

$$\mathcal{L}^{(3)} = -\frac{1}{2}G_{ij}G^{ij} - D_i\phi_\mu D^i\phi^\mu - \frac{1}{2}[\phi_\mu, \phi_\nu][\phi^\mu, \phi^\nu]$$

$$- i\psi_1^\dagger\gamma^i D_i\psi_2 + i\psi_2^\dagger\gamma^i D_i\psi_1 - \psi_1^\dagger[\phi_5, \psi_2] - \psi_2^\dagger[\phi_5, \psi_1] \tag{4.10}$$

$$+ \psi_1^\dagger[\phi_4 + \phi_0, \psi_1] - \psi_2^\dagger[\phi_4 - \phi_0, \psi_2].$$

As for the reductions of the supersymmetry transformations, they are done in details in the appendix section (7.3).

4.3.4 SYM on $\mathbb{R}^3$

The result obtained from the reductions of Lagrangian and the supersymmetry transformations encourage the following definition

$$A_+ = \phi_4 + \phi_0, \qquad\qquad A_- = \phi_4 - \phi_0, \qquad\qquad \phi = \phi_5,$$

hence, in three dimensional Euclidean space the the Lagrangian becomes

$$\begin{aligned}
\mathcal{L} = {} & -\frac{1}{2}G_{ij}G^{ij} - \|D_i\phi\|^2 - D_iA_+D^iA_- - [\phi, A_+][\phi, A_-] - \frac{1}{4}[A_+, A_-]^2 \\
& - i\psi_1^\dagger\gamma^iD_i\psi_2 + i\psi_2^\dagger\gamma^iD_i\psi_1 - \psi_1^\dagger[\phi, \psi_2] - \psi_2^\dagger[\phi, \psi_1] + \psi_1^\dagger[A_+, \psi_1] \\
& - \psi_2^\dagger[A_-, \psi_2],
\end{aligned} \tag{4.11}$$

and the supersymmetry transformations that leave this Lagrangian invariant are

$$\begin{aligned}
\delta W_i &= -i\varepsilon_1^\dagger\gamma_i\psi_2 + i\varepsilon_2^\dagger\gamma_i\psi_1 + i\psi_1^\dagger\gamma_i\varepsilon_2 - i\psi_2^\dagger\gamma_i\varepsilon_1, \\
\delta A_+ &= 2\psi_2^\dagger\varepsilon_2 - 2\varepsilon_2^\dagger\psi_2, \\
\delta A_- &= 2\varepsilon_1^\dagger\psi_1 - 2\psi_1^\dagger\varepsilon_1, \\
\delta\phi &= -\varepsilon_1^\dagger\psi_2 - \varepsilon_2^\dagger\psi_1 + \psi_1^\dagger\varepsilon_2 + \psi_2^\dagger\varepsilon_1, \\
\delta\psi_1 &= \gamma^{ij}G_{ij}\varepsilon_1 - \varepsilon_1[A_+, A_-] - 2\varepsilon_2[A_-, \phi] + 2i\gamma^i(\varepsilon_2 D_\mu A_- + \varepsilon_1 D_i\phi), \\
\delta\psi_2 &= \gamma^{ij}G_{\mu\nu}\varepsilon_2 + \varepsilon_2[A_+, A_-] + 2\varepsilon_1[A_+, \phi] + 2i\gamma^i(\varepsilon_1 D_i A_+ - \varepsilon_2 D_i\phi).
\end{aligned} \tag{4.12}$$

The action and the superysmmetry transformations are richer than the theory obtained in chapter two due to the reduction from a higher dimensional theory. However, we can still see some resemblance, the theory here looks like a generalized $N = 2$ supersymmetric Yang-Mills-Higgs with ϕ playing the role of the Higgs field. The fields (W_i, ϕ, A_+, A_-) are real, which can be seen from the superysmmetry transformations. This is the benefit of starting with a theory in Minkowski space and this will help us, later when we analyze the real moduli space of hyperbolic monopoles, to show that the supersymmetry transformations play the role of real zero modes.

The variation of the action under the supersymmetry transformation is given by

$$\delta_1 \mathcal{L} = -i\partial_k \varepsilon_1^\dagger (\gamma^{ij}\gamma^k \psi_1 G_{ij} + \gamma^k \psi_2 [A_+, A_-] + 2\gamma^k \psi_1 [A_+, \phi]$$
$$- i\gamma^i \gamma^k \psi_1 D_i A_+ + i\gamma^i \gamma^k \psi_2 D_i \phi) + \text{c.c.}$$

and

$$\delta_2 \mathcal{L} = i\partial_k \varepsilon_2^\dagger (\gamma^{ij}\gamma^k \psi_2 G_{ij} + \gamma^k \psi_1 [A_+, A_-] - 2\gamma^k \psi_2 [A_+, \phi]$$
$$- i\gamma^i \gamma^k \psi_2 D_i A_+ - i\gamma^i \gamma^k \psi_1 D_i \phi) + \text{c.c.}$$

which will vanish since the superysmmetry parameters ε_1 and ε_2 are constant spinors on $\mathbb{R}^3$.

4.3.5 Dimensional analysis

In $\hbar = c = 1$ units, all quantities are measured in units of energy raised to some power. In this case we have for example $[m] = [p^\mu] = E^{+1}$ or simply $[m] = [p^\mu] = 1$, while $[x^\mu] = -1$. The action on a general $n-$manifold is given by

$$S = \int d^n V \mathcal{L},$$

where for the case $\mathbb{R}^n$, $d^n V = dx^1 dx^2 \ldots dx^n$, so on $\mathbb{R}^3$ the Lagrangian of a field theory has dimensionality $[\mathcal{L}] = 3$. From equation (4.11) we deduce that

$$[\psi_1] = [\psi_2] = 1, \qquad\qquad [G_{\mu\nu}] = \frac{3}{2},$$

which implies that

$$[\phi] = [A_+] = [A_-] = \frac{1}{2}.$$

To have a correct dimensionality for all the Lagrangian terms we must add the non-abelian coupling constant q, so the Lagrangian will read as

$$\mathcal{L} = -\frac{1}{2}G_{ij}G^{ij} - \|D_i\phi\|^2 - D_iA_+D^iA_- - q^2[\phi, A_+][\phi, A_-] - \frac{q^2}{4}[A_+, A_-]^2$$
$$- i\psi_1^\dagger\gamma^iD_i\psi_2 + i\psi_2^\dagger\gamma^iD_i\psi_1 - q\psi_1^\dagger[\phi, \psi_2] - q\psi_2^\dagger[\phi, \psi_1] + q\psi_1^\dagger[A_+, \psi_1]$$
$$- q\psi_2^\dagger[A_-, \psi_2],$$

with $[q] = \frac{1}{2}$.

4.4 Promoting supersymmetry to hyperbolic space

If instead of $\mathbb{R}^3$ we place the theory on a Riemannian 3−dimensional spin manifold with metric g_{ij}, the story then is different, since, in general, constant spinors don't exist on curved backgrounds. The covariant derivative of the supersymmetry parameter is

$$\nabla_i\varepsilon = \partial_i\varepsilon + \frac{1}{4}\gamma_{jk}\omega_i{}^{jk}\varepsilon,$$

where $\omega_i{}^{jk}$ are the spin connection of the vielbein introduced on the spin manifold. The variation of the Lagrangian now reads as

$$\delta_1\mathcal{L} = -i\nabla_k\varepsilon_1^\dagger(\gamma^{ij}\gamma^k\psi_1G_{ij} + \gamma^k\psi_2[A_+, A_-]$$
$$+ 2\gamma^k\psi_1[A_+, \phi] - i\gamma^i\gamma^k\psi_1D_iA_+ + i\gamma^i\gamma^k\psi_2D_i\phi) + \text{c.c.}$$

and

$$\delta_2\mathcal{L} = i\nabla_k\varepsilon_2^\dagger(\gamma^{ij}\gamma^k\psi_2G_{ij} + \gamma^k\psi_1[A_+, A_-]$$
$$- 2\gamma^k\psi_2[A_+, \phi] - i\gamma^i\gamma^k\psi_2D_iA_+ - i\gamma^i\gamma^k\psi_1D_i\phi) + \text{c.c.}$$

Unless we are considering parallel spinors ($\nabla_i\varepsilon = 0$) this Lagrangian will not be supersymmetric. But we are mainly interested in Hyperbolic manifold H^3 which admits

Killing spinors. Killing spinors in 3 dimensions satisfy the following equation

$$\nabla_i \varepsilon_{(1,2)} = \kappa_{(1,2)} \gamma_i \varepsilon.$$

In order to restore supersymmetry, we follow the same procedure we did in chapter two by adding terms that are invariant under the gauge group and Lorentz transformation, to the Lagrangian and the supersymmetry transformations. Then by imposing the invariance of the action and the on-shell closure of the supersymmetry transformations we find the coefficient of each term. However, we can benefit from results in chapter two to massively reduce the calculation here. Notice, first, that the action (4.11) with A_+ and A_- switched off looks exactly like an on-shell supersymmetric version of the action (2.45). Hence, when $\kappa_1 = -\kappa_2 = \kappa$, and in the absence of A_+ and A_-, we can promote (4.11) to H^3 by merely adding a term equal to $i\kappa[\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1]$ to the action and leaving the supersymmetry transformations intact. When we include terms with A_+ and A_- back, we require the addition of terms that have $A+$ and A_- and invariant under the group $SO(2,1)$ and the gauge group. The only possible choice is a term proportional to $\text{Tr}(A_+ A_-)$ (where traces are suppressed in this chapter), and using the dimensional analysis we find that it has to be proportional, actually, to $\kappa^2 A_+ A_-$. As for the supersymmetry transformations, the closure of the algebra along with dimensionality of fields imply that terms proportional to $\varepsilon_1 A_+, \varepsilon_1 A_-, \varepsilon_2 A_+$ and $\varepsilon_2 A_-$ should be added to the supersymmetry transformation of the fermions. In other words the new action and supersymmetry transformation (δ_1 for example) have to be modified as

$$\mathcal{L} \rightsquigarrow \mathcal{L} + i\kappa[\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1] + \alpha \kappa^2 A_+ A_- \tag{4.13}$$

and

$$\delta_1 \psi_1 \rightsquigarrow \delta_1 \psi_1 + \beta_1 A_+ \varepsilon_1 + \beta_2 A_- \varepsilon_1 \qquad \delta_1 \psi_2 \rightsquigarrow \delta_1 \psi_2 + \beta_3 A_+ \varepsilon_1 + \beta_4 A_- \varepsilon_1 \,, \tag{4.14}$$

for some constants $\alpha, \beta_1, \beta_2, \beta_3, \beta_4$ to be determined.

The invariance of the action now can be done faster and easier where we have only

to worry about the variation of the terms with A_+ and A_- in the Lagrangian and only the variation of the old terms via the new addition to the suersymmetry transformations. Imposing the invariance tells us that $\alpha = \beta_3 = 4$ and $\beta_1 = \beta_2 = \beta_4 = 0$. In addition, this result implies that there will be no modification for the supersymmetry transformation $\delta_2 \psi_2$ and a term equal to $-4\kappa A_-$ should be added to $\delta_2 \psi_1$. Finally, the resulting Lagrangian on a 3-dimensional Riemannian manifold admitting Killing spinor is given by

$$
\begin{aligned}
\mathcal{L}^{(3)} = &-\frac{1}{2} G_{ij} G^{ij} - \|D_i \phi\|^2 - D_i A_+ D^i A_- - [\phi, A_+][\phi, A_-] - \frac{1}{4}[A_+, A_-]^2 \\
&- i\psi_1^\dagger \gamma^i D_i \psi_2 + i\psi_2^\dagger \gamma^i D_i \psi_1 - \psi_1^\dagger [\phi, \psi_2] - \psi_2^\dagger [\phi, \psi_1] + \psi_1^\dagger [A_+, \psi_1] \\
&- \psi_2^\dagger [A_-, \psi_2] + 4\kappa^2 A_+ A_- + i\kappa(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1),
\end{aligned}
\tag{4.15}
$$

which is invariant under the following supersymmetry transformations

$$
\begin{aligned}
\delta_1 W_i &= -i\varepsilon_1^\dagger \gamma_i \psi_2 - i\psi_2^\dagger \gamma_i \varepsilon_1, & \delta_2 W_i &= i\varepsilon_2^\dagger \gamma_i \psi_1 + i\psi_1^\dagger \gamma_i \varepsilon_2, \\
\delta_1 A_+ &= 0 & \delta_2 A_+ &= 2\psi_2^\dagger \varepsilon_2 - 2\varepsilon_2^\dagger \psi_2, \\
\delta_1 A_- &= 2\varepsilon_1^\dagger \psi_1 - 2\psi_1^\dagger \varepsilon_1, & \delta_2 A_- &= 0, \\
\delta_1 \phi &= -\varepsilon_1^\dagger \psi_2 + \psi_2^\dagger \varepsilon_1, & \delta_2 \phi &= -\varepsilon_2^\dagger \psi_1 + \psi_1^\dagger \varepsilon_2, \\
\delta_1 \psi_1 &= \gamma^{ij} G_{ij} \varepsilon_1 - \varepsilon_1 [A_+, A_-] + 2i\gamma^i \varepsilon_1 D_i \phi, & \delta_2 \psi_1 &= -2\varepsilon_2 [A_-, \phi] + 2i\gamma^i \varepsilon_2 D_i A_- - 4\kappa \varepsilon_2 A_-, \\
\delta_1 \psi_2 &= 2\varepsilon_1 [A_+, \phi] + 2i\gamma^i \varepsilon_1 D_i A_+ + 4\kappa \varepsilon_1 A_+, & \delta_2 \psi_2 &= \gamma^{ij} G_{ij} \varepsilon_2 + \varepsilon_2 [A_+, A_-] - 2i\gamma^i \varepsilon_2 D_i \phi,
\end{aligned}
\tag{4.16}
$$

where

$$
\nabla_i \varepsilon_1 = -\nabla_i \varepsilon_2 = \kappa \gamma_i \varepsilon.
\tag{4.17}
$$

These modifications imply that κ has to be real for $\mathcal{L}^{(3)}$ to be Hermitian. But κ is related to the radius of curvature of our space via the integrability condition which is given by $R = 4d(d-1)\kappa^2 = -24\kappa^2$. For real κ, the radius of curvature has to be negative, which is the case of H^3. If we consider the upper half plane model of hyperbolic space

with radius of curvature "l", then the metric is given by

$$ds^2 = \frac{l^2}{z^2}(dx^2 + dy^2 + dz^2), \tag{4.18}$$

Computing the scalar curvature (with $R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$) we get

$$R = -\frac{6}{l^2} \implies \kappa = \pm\frac{1}{2l},$$

where we chose to work with negative κ.

4.4.1 Equations of motion

Using the Euler-Lagrange equations the field equations can be easily derived, for instance for the equations of motion for the vector field W_i^a, we need to solve

$$\nabla_m \left[\frac{\partial \mathcal{L}}{\partial(\nabla_m W_n)}\right] - \frac{\partial \mathcal{L}}{\partial W_n} = 0, \tag{4.19}$$

and similarly for the other fields. With some patience one obtains the following set of equations of motion:

Vector field W_i:

$$2D_m G^{mn} = -2[D^n\phi, \phi] - [D^n A_+, A_-] - [D^n A_-, A_+]$$
$$- i[\psi_1^\dagger \gamma^n, \psi_2] + i[\psi_2^\dagger \gamma^n, \psi_1]. \tag{4.20}$$

Scalar field ϕ:

$$D_m D^m \phi = -[[\phi, A_-], A_+] - [[\phi, A_+], A_-] - [\psi_1^\dagger, \psi_2] - [\psi_2^\dagger, \psi_1]. \tag{4.21}$$

Scalar field A_-:

$$D_m D^m A_- = -[\phi, [\phi, A_-]] + \frac{1}{2}[A_-, [A_+, A_-]] + [\psi_1^\dagger, \psi_1] - \frac{1}{l^2}A_-. \tag{4.22}$$

Scalar field A_+:

$$D_m D^m A_+ = -[\phi, [\phi, A_+]] + \frac{1}{2}[A_+, [A_-, A_+]] - [\psi_2^\dagger, \psi_2] - \frac{1}{l^2} A_+.$$

Spinor ψ_2:

$$\gamma^i D_i \psi_2 + i[\psi_2, \phi] + i[A_+, \psi_1] + \frac{1}{2l}\psi_2 = 0.$$

Spinor ψ_1:

$$\gamma^i D_i \psi_1 - i[\psi_1, \phi] + i[A_-, \psi_2] - \frac{1}{2l}\psi_1 = 0.$$

These equations of motion which follow from the Lagrangian (4.13) looks a bit complicated and not related to the Bogmol'nyi equation on H^3. However, one can reduce these equations to simpler forms by taking certain restrictions which are invariant under supersymmetry. This calculation is done in the next chapter.

These equations exhibit symmetries inherited from the symmetries of the Lagrangian: gauge transformation, supersymmetry transformation, Lorentz transformation, in addition to $SO(2,1)$ that acts on the vector (ϕ, A_+, A_-) and transform the spinors as $\psi \rightarrow e^{-\frac{i}{4}\gamma^{\mu\nu}f_{\mu\nu}}\psi$, where $f_{\mu\nu}$ are real functions.

4.4.2 Relation with other theories

The supersymmetric theory obtained here (4.15, 4.16) can be related to family "A" theories obtained by Matthias Blau [122, 115]. According to the dimension, on which we would like to study supersymmetric Yang-Mills theories, Blau derived the general model of theories on curved spaces admitting Killing spinors written in terms of fields in higher dimensions. For example, family "A" theories are those that obtained by reduction to dimensions less than or equal to 5. In our notations theories "A" are given by

$$\mathcal{L} = \mathcal{L}_{\text{SYM}} - 4\alpha^2[(n-2)\sum_\mu^{d-n} \phi_\mu^2 + (n-4)\phi_p^2] - (n-4)\alpha\bar{\Psi}\Gamma^p\Psi, \tag{4.23}$$

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and the superysmmetry transformations are given by

$$\delta W_M = (\bar{\epsilon}\Gamma_M \Psi - \bar{\Psi}\Gamma_M \epsilon), \tag{4.24}$$

$$\delta \Psi = \Gamma^{MN}\epsilon G_{MN} - 4\alpha[\sum_{\mu}^{d-n} \phi_\mu \Gamma^\mu \Gamma^p \epsilon + (n-4)\phi_p \epsilon], \tag{4.25}$$

where $\mathcal{L}_{SYM}$ stands for the unmodified supersymmetric Yang-Mills theory on flat space, n is the dimension of the reduced theory, d is the dimension from where we started before reduction, and α and p are related to the Killing spinor equation as follows

$$\nabla_i \epsilon = \alpha \Gamma_i \Gamma^p \epsilon, \tag{4.26}$$

where the integrability condition for Killing spinor of such form is given by

$$R = \alpha^2 (\Gamma^p)^2 n(n-1). \tag{4.27}$$

In order to obtain our theory from this family we need to set $d = 6$ and $n = 3$, and also to choose $p = 5$ and $\alpha = \frac{i}{2l}$. Hence for these choices we get

$$\mathcal{L}_{SYM} = -\frac{1}{2}G_{AB}G^{AB} + \bar{\Psi}\Gamma^A D_A \Psi + \frac{1}{l^2}[-\phi_0^2 + \phi_4^2] + \frac{i}{2l}\bar{\Psi}\Gamma^5 \Psi, \tag{4.28}$$

$$\delta W_A = \bar{\epsilon}\Gamma_A \Psi - \bar{\Psi}\Gamma_A \epsilon, \tag{4.29}$$

$$\delta \Psi = \Gamma^{AB}\epsilon G_{AB} - \frac{2i}{l}[\phi_0 \Gamma^0 + \phi_4 \Gamma^4]\Gamma^5 \epsilon. \tag{4.30}$$

One can understand this theory as a supersymmetric Yang-Mills theory on $H^3 \times \mathbb{R}^3$, which is best manifested by the behavior of the spinor ϵ on this manifold where we have $\nabla_i \epsilon = \alpha \Gamma_i \Gamma^p \epsilon$ and $\nabla_\mu \epsilon = 0$. This point of view comes from Bär's cone construction [123] that relates Killing spinor fields to parallel spinor fields on auxiliary manifolds. If we now use the decompositions of the bosonic fields, the fermionic fields, and the gamma matrices introduced in sections (4.3.1, 4.3.2), to reduce (4.28) and (4.29, 4.30) we find that this action and supersymmetry transformations match exactly with our results (4.15) and (4.16).

4.4.3 Super algebra

The closure of the new supersymmetry transformations algebra need to be checked, where we use the modified $N = 1$ version from sections $(4.3.1, 4.3.2)$. We will study the action of the commutator of two supersymmetry transformations associated with two Killing spinors ϵ_1 and ϵ_2 satisfying the Killing spinor equation (4.26), where here ϵ_1 and ϵ_2 represent the same spinor taken at two different points. Naturally, we should expect the closure to give diffeomorphisms and gauge transformations, and if any new infinitesimal transformation term appears, the Lagrangian must be invariant under the action of this transformation. The action of the commutator on our multiplet, gives

$$[\delta_1, \delta_2]\phi_\mu = \xi^A G_{A\mu} + \frac{i}{l}[\xi^{05}{}_\mu \phi_0 + \xi^{45}{}_\mu \phi_4],$$

$$[\delta_1, \delta_2]W_i = \xi^A G_{Ai} + \frac{i}{l}[\xi^{05}{}_i \phi_0 + \xi^{45}{}_i \phi_4],$$

$$[\delta_1, \delta_2]\Psi = \xi^L D_L \Psi + \frac{i}{2l} \xi^i \Gamma_i \Gamma_5 \Psi + \frac{i}{l} \xi_{5\mu\nu} \Gamma^{\mu\nu} \Psi.$$

where

$$\xi_A = \bar\epsilon_1 \Gamma_A \epsilon_2 - \bar\epsilon_2 \Gamma_A \epsilon_1$$

In order to understand the right hand side of the supersymmetry algebra, we need to understand the nature of the vector ξ_L. The covariant derivatives of the different components of ξ that appear in above algebra give

$$\nabla_i \xi_A = -\frac{i}{l} \xi_{5iA},$$

hence not all components of ξ_A are constant, and ξ_i in particular is a Killing vector. On the other hand, $\xi_{\mu\nu5}$ is constant. Using these identities and defining the parameter $\omega = \xi^A W_A$, we can write the algebra in terms of isometries and gauge transformations

$$[\delta_1, \delta_2]W_i = L_\xi W_i + \delta_\omega W_i.$$

$$[\delta_1, \delta_2]\phi_\mu = L_\xi \phi_\mu + \delta_\omega \phi_\mu + \frac{i}{l} \xi_{\mu\nu5} \phi^\nu.$$

$$[\delta_1, \delta_2]\Psi = L_\xi \Psi + \delta_\omega \Psi + \frac{i}{l} \xi_{\mu\nu5} \Gamma^{\mu\nu} \Psi.$$

where

$$\delta_\omega W_\sigma = -D_\mu \omega, \qquad \delta_\omega \phi_m = [\omega, \phi_m], \qquad \delta_\omega \Psi = [\omega, \Psi].$$

The equation of motion for the fermion has been used to ensure the closure of the supersymmetry algebra, hence the on shell closure. In checking the computation of the algebra closure the Mathematica package GAMMA [124] was used.

In addition to the expected diffeomorphism and gauge transformation we find a boost of the scalars and fermions by a constant matrix $\xi_{5\mu\nu}$. This boost represents the $R-$ symmetry $(SO(2,1))$ algebra of the theory.

Chapter 5

Geometry of the Real Moduli Space of Hyperbolic Monopoles

5.1 Introduction

In this chapter we explore the geometry of the real moduli space of hyperbolic mono-
poles. We will show first that supersymmetric hyperbolic monopoles form a subset
of solutions to the equations of motion derived in the previous section, provided we
use some supersymmetric constraints. However, these constraints are only supersym-
metric under half of the supersymmetry transformations, which implies that super-
symmetric hyperbolic monopoles are $\frac{1}{2}$ "BPS" saturated. Then, we start analyzing
the moduli space of these "BPS" hyperbolic monopoles using the supersymmetry of
low energy dynamics. We construct an ansatz of zero modes that we show to satisfy
the linearized Bogomol'nyi equation and the Dirac equation. The difference from our
analysis in chapter three, is that the ansatz we construct, here, using the supersym-
metry transformation are real, which we show to satisfy, as well, a gauge background
condition, and hence form real coordinate functionals for the moduli space. Then we
aim at constructing structures on the target space of the moduli space and studying
their properties. We construct two sets of 2-sphere complex structures that map zero
modes orthogonal to the gauge orbits into zero modes that are again orthogonal to the
gauge orbit, and we show that these two sets of complex structures don't have any an-

ticommutation relations between them. This just another way of saying that we have a biquaternionic algebra on the moduli space of hyperbolic monopoles "pluricomplex geometry". Finally, we show that in the limiting case when the radius of curvature of the hyperbolic space is set to infinity the pluricomplex geometry gives the hyper-Kähler geometry, the geometry of Euclidean monopoles and hence proving Atiyah's conjecture.

5.2 Breaking half of supersymmetry

In the following, a set of restrictions which are invariant under supersymmetry, will be imposed on the equations of motion. This will reduce the equations of motion into a simpler familiar form. One of the solutions of these equations is the supersymmetric hyperbolic monopole. But this solution will be shown to break half of the supersymmetry transformation obtained after imposing the restrictions.

Consider the following three constrains which will be applied to the equations of motion and the supersymmetry transformations

$$\gamma^{ij}G_{ij} = -2i\gamma^k D_k\phi,$$
$$A_- = 0, \tag{5.1}$$
$$\psi_1 = 0.$$

These constraints are invariant under the supersymmetry transformation δ_1. Applying these restrictions on the equations of motion will give

$$D_i G^{ij} = -[D^j\phi, \phi], \tag{5.2}$$

$$D_i D^i\phi = 0, \tag{5.3}$$

$$D_i D^i A_- = 0,$$

$$D_i D^i A_+ = -[\phi, [\phi, A_+]] - [\psi_2^\dagger, \psi_2] - \frac{1}{l^2}A_+,$$

$$\gamma^i D_i\psi_1 = 0, \tag{5.4}$$

$$\gamma^i D_i\psi_2 + i[\psi_2, \phi] + \frac{1}{2l}\psi_2 = 0. \tag{5.5}$$

The solutions of the Bogomol'nyi equation together with the Bianchi identity on hyperbolic space H^3 (4.18),

$$D_i \phi = -\frac{1}{2} \sqrt{|g|} \bar{\epsilon}_{ijk} g^{jm} g^{kn} G_{mn} = -\frac{z}{2l} \bar{\epsilon}_{ijk} G_{jk}, \tag{5.6}$$

$$D_i(*G^{ij}) = 0, \tag{5.7}$$

where $\bar{\epsilon}_{ijk}$ represents the Levi-Civita symbol on flat space, form a subset of the solutions of the first two equations (5.2, 5.3), hence we can trade (5.2, 5.3) with (5.6, 5.7) since we are just interested in the space of solutions of the Bogomol'nyi equation. Using now the internal symmetry $SO(2,1)$ we can set $A_+ = 0$ and the fourth equation will have the following simpler form

$$D_i D^i A_+ = -[\psi_2^\dagger, \psi_2]. \tag{5.8}$$

A solution that satisfies equations $(5.4, 5.5, 5.6, 5.8)$ is the solution of the hyperbolic monopole in the bosonic theory along with $A_+ = 0$ and $(\psi_1, \psi_2) = (0,0)$. Thus the hyperbolic monopole solutions of the bosonic theory continue to be monopole in the supersymmetric theory. The hyperbolic monopole with $A_+ = 0$ and $(\psi_1, \psi_2) = (0,0)$ will be denoted as the supersymmetric hyperbolic monopole solutions, and we will be looking at the space of these solutions. Imposing the constraints (5.1) on the supersymmetry transformations (4.16) will give

$$\begin{aligned}
\delta_1 W_i &= -i\varepsilon_1^\dagger \gamma_i \psi_2 - i\psi_2^\dagger \gamma_i \varepsilon_1, \\
\delta_1 \phi &= -\varepsilon_1^\dagger \psi_2 + \psi_2^\dagger \varepsilon_1, \\
\delta_1 \psi_2 &= 2\varepsilon_1 [A_+, \phi] + 2i\gamma^i \varepsilon_1 D_i A_+ - \frac{2i}{l} \varepsilon_1 A_+, \\
\delta_1 \phi &= \delta_1 A_+ = \delta_1 A_- = 0,
\end{aligned} \tag{5.9}$$

and

$$\delta_2\psi_2 = -4i\gamma^i\varepsilon_2 D_i\phi.$$

$$\delta_2 A_+ = 2\psi_2^\dagger\varepsilon_2 - 2\varepsilon_2^\dagger\psi_2, \tag{5.10}$$

$$\delta_2 W_i = \delta_2\phi = \delta_2 A_- = \delta_2\psi_1 = 0.$$

After setting the scalar field A_+ equal to zero, we can see that only the first set of super-symmetry transformations satisfy the supersymmetric monopole solution, however the second set will generate non zero fermions. This means that δ_1 is the unbroken supersymmetry leaving the supersymmetric solution invariant and δ_2 is the broken supersymmetry. This partial breaking of supersymmetry is a generic feature of su-persymmetry field theories admitting topologically non-trivial solutions. It was first noticed by Witten and Olive [75] and is best understood by showing that the algebra of supersymmetry charges are modified by topological charges.

5.3 Real moduli space of hyperbolic monopoles

In the previous section we concluded that the supersymmetric BPS configurations are solutions with $A_- = A_+ = \psi_1 = \psi_2 = 0$, and (W_i, ϕ) satisfy the Bogomol'nyi equation (5.6). These configurations are exactly the hyperbolic monopole solutions. The geo-metry of these configurations is best analyzed using the linearized equations of motion and their solutions (zero modes). These zero modes will later be used as coordinate functions of our real moduli space. Hence, unlike the complexified zero modes that we constructed in [125], the zero modes in this analysis will be real.

5.3.1 Zero modes

Using the same definitions and notations we used in [125] section (3.1) (with only one exception, W_i for the gauge field instead of A_i), we will use the linearized unbroken supersymmetry to pair the bosonic and fermionic zero modes in a supermultiplet. The unbroken supersymmetry transformation (5.9) teaches us that real bosonic zero

modes can be chosen to be

$$\dot{W}_i = -i\zeta^\dagger\gamma_i\dot{\psi} + i\dot{\psi}^\dagger\gamma_i\zeta,$$

$$\dot{\phi} = -(\zeta^\dagger\dot{\psi} + \dot{\psi}^\dagger\zeta),$$

(5.11)

where ζ is an even spinor satisfying

$$\nabla_i\zeta = -\frac{1}{2l}\gamma_i\zeta,$$

(5.12)

Its quite straight forward to check that the ansatz (5.11) satisfy the linearized Bogomol'nyi equation

$$D_i(0)\dot{\phi} - [\phi(0),\dot{W}_i] + \varepsilon_{ijk}D^j(0)\dot{W}^k = 0\,.$$

(5.13)

However, a better way of checking is by noticing that our zero modes here are just the real part of the complexified zero modes defined in [125] equation (3.11), and since the linearized Bogomol'nyi equation coefficients are real, hence (5.11) should also be a solution to (5.13). In addition the zero modes have to be tangent to the moduli space, hence they have to satisfy as well the gauge back ground condition. However the story here is a bit different, the gauge background condition satisfied by the compilexified zero modes in [125] equation (3.9) has complex coefficients. This means that the real zero modes (5.11) will satisfy a slightly different condition. Its not difficult to see that the gauge background condition that (5.11) should satisfy is given by

$$D^i(0)\dot{W}_i + [\phi(0),\dot{\phi}] + \frac{2i}{l}(\dot{\psi}^\dagger\zeta - \zeta^\dagger\dot{\psi}) = 0,$$

(5.14)

where the factor $\frac{2}{l}$ here is 4λ in [125]. Indeed with a quick check we verify that (5.11) satisfy the gauge background condition (5.14).

$$D^i(-i\zeta^\dagger\gamma_i\dot{\psi} + i\dot{\psi}^\dagger\gamma_i\zeta) + [\phi, -\zeta^\dagger\dot{\psi} - \dot{\psi}^\dagger\zeta] + \frac{2i}{l}(\dot{\psi}^\dagger\zeta - \zeta\dagger\dot{\psi})$$

$$= -iD^i(\zeta^\dagger\gamma_i\dot{\psi}) + iD^i(\dot{\psi}\gamma_i\zeta) - [\phi,\zeta^\dagger\dot{\psi}] - [\phi,\dot{\psi}^\dagger\zeta] + \frac{2i}{l}\dot{\psi}^\dagger\zeta - \frac{2i}{l}\zeta^\dagger\dot{\psi}.$$

We now use equation (5.12) and $\gamma^i \gamma_i = 3$ to arrive at

$$\frac{3i}{2l}\zeta^\dagger \dot\psi - i\zeta^\dagger \gamma_i D^i \dot\psi - \zeta^\dagger[\phi, \dot\psi] - \frac{2i}{l}\zeta^\dagger \dot\psi$$
$$+iD^i \dot\psi^\dagger \gamma_i \zeta - \frac{3i}{2l}\dot\psi^\dagger \zeta - [\phi, \dot\psi^\dagger]\zeta + \frac{2i}{l}\dot\psi^\dagger \zeta,$$

which will vanish since the first line is nothing but the equation of motion (linear) of femions ψ and second line is its dagger.

Since we don't require $\dot\psi$ to be real, the fermionic zero modes, which can be found by inverting equation (5.11) ,

$$\dot\psi = i\dot{W}_i \gamma^i \eta - \dot\phi\eta, \tag{5.15}$$

where

$$D_i \eta = \frac{1}{2l}\gamma_i \eta. \tag{5.16}$$

still [125] satisfy the equation of motion for the fermions (5.5). Using the fact that $\eta^\dagger \zeta$ is covariantly constant [126], we can conveniently choose them such that they satisfy conditions required by inversion, which can be shown to be equal to $\frac{1}{2}$.

5.3.2 Six/Four dimensional language

At the level where our base space is $H^3 \times \mathbb{R}^3$, the bosonic fields (W_i, ϕ, A_-, A_+) and the fermionic fields (ψ_1, ψ_2) depend only on the coordinates of H^3. The supersymmetric hyperbolic monopoles are the solutions with $A_- = A_+ = \psi_1 = \psi_2 = 0$, and (W_i, ϕ) satisfy the Bogomoln'yi equation. In this section we aim at examining the geometry of the real moduli space of hyperbolic monopoles. In particular, we would like to construct complex structures (in our case they are real endomorphisms) on the tangent space of the moduli space and study their characteristics. For this sake, its convenient, and sufficient for now while elaborating on few observations, to introduce the four dimensional notation which amounts to woking on the four-dimensional space $H^3 \times \mathbb{R}^1$. Our gamma matrices Π_i will be

$$\Pi_i = \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \qquad \Pi_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

and they satisfy the Clifford algebra

$$\{\Pi_{\underline{i}}, \Pi_{\underline{j}}\} = 2g_{\underline{i}\underline{j}}\mathbb{I},$$

where $\underline{i} = (i, 4)$. The spinors are chiral in four dimensions and take the following form

$$\dot{\Psi} = \begin{pmatrix} \dot{\psi} \\ 0 \end{pmatrix}, \qquad \zeta = \begin{pmatrix} 0 \\ \zeta \end{pmatrix}, \qquad \eta = \begin{pmatrix} 0 \\ \eta \end{pmatrix},$$

and the four dimensional bosonic zero modes will be given by

$$\dot{W}_{\underline{i}} = (\dot{W}_i, \dot{\phi}).$$

With these definitions, the bosonic and fermionic zero modes read now as

$$\dot{W}_{\underline{i}} = -i(\zeta^\dagger \Pi_{\underline{i}} \dot{\Psi} - \dot{\Psi}^\dagger \Pi_{\underline{i}} \zeta), \tag{5.17}$$

$$\dot{\Psi} = i\dot{W}_{\underline{i}} \Pi^{\underline{i}} \eta, \tag{5.18}$$

where the Killing spinors now satisfy the following equations

$$\nabla_i \zeta = \frac{i}{2l}\Pi_i \, \Pi_4 \zeta, \qquad\qquad\qquad \nabla_4 \zeta = 0, \tag{5.19}$$

$$\nabla_i \eta = -\frac{i}{2l}\Pi_i \, \Pi_4 \eta, \qquad\qquad\qquad \nabla_4 \eta = 0. \tag{5.20}$$

And finally the Linearized equations of motions are given by

$$D_{[\underline{i}} \dot{W}_{\underline{j}]} = -\frac{1}{2}\varepsilon_{\underline{i}\underline{j}\underline{m}\underline{n}}D^{\underline{m}}\dot{W}^{\underline{n}}, \tag{5.21}$$

$$D^{\underline{i}}\dot{W}_{\underline{i}} = -\frac{2}{l}(\dot{\Psi}^\dagger \Pi_4 \zeta + \zeta^\dagger \Pi_4 \dot{\Psi}), \tag{5.22}$$

$$\slashed{D}\dot{\Psi} = -\frac{i}{2l}\Pi_4 \dot{\Psi}. \tag{5.23}$$

5.3.3 Complex structures

The bosonic zero modes are real (5.17), hence one needs to construct real complex structures, which act as real linear map on the tangent space of the moduli space and

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maps every real zero mode to another real zero mode. We have two possible choices of real structures $M_{\alpha\beta}$ and $N_{\alpha\beta}$, that agree with our zero modes ansatz, defined by

$$M_{\alpha\beta\underline{i}}{}^{\underline{j}} = i(\zeta_\alpha^\dagger \Pi_{\underline{i}}{}^{\underline{j}} \eta_\beta + \eta_\alpha^\dagger \Pi_{\underline{i}}{}^{\underline{j}} \zeta_\beta), \tag{5.24}$$

$$N_{\alpha\beta\underline{i}}{}^{\underline{j}} = (\zeta_\alpha^\dagger \Pi_{\underline{i}}{}^{\underline{j}} \eta_\beta - \eta_\alpha^\dagger \Pi_{\underline{i}}{}^{\underline{j}} \zeta_\beta). \tag{5.25}$$

One might think that we made the wrong choice of sign, and the above choices would give pure complex structures, however the antihemiticity of $\Pi_{\underline{i}}{}^{\underline{j}}$ make them pure real. ζ and η belong to the following sets of solutions in $\mathbb{C}^2$

$$K_+ = \{\zeta_\alpha | \nabla_i \zeta_\alpha = +\frac{i}{2l}\Pi_i\Pi_4\zeta_\alpha\}, \tag{5.26}$$

$$K_- = \{\eta_\alpha | \nabla_i \eta_\alpha = -\frac{i}{2l}\Pi_i\Pi_4\eta_\alpha\}. \tag{5.27}$$

For computation purposes we can choose to start with a particular choice for the basis of K_+ and K_-. Using the model of hyperbolic space given in (4.18), we find

$$\zeta_1 = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ [-(x-iy)+1]z^{-\frac{1}{2}} \\ z^{\frac{1}{2}} \end{pmatrix} \qquad \zeta_2 = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ [-(x+iy)-1]z^{-\frac{1}{2}} \\ z^{\frac{1}{2}} \end{pmatrix} \tag{5.28}$$

$$\eta_1 = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ z^{\frac{1}{2}} \\ [(x+iy)+1]z^{-\frac{1}{2}} \end{pmatrix} \qquad \eta_2 = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -z^{\frac{1}{2}} \\ [-(x+iy)+1]z^{-\frac{1}{2}} \end{pmatrix} \tag{5.29}$$

that satisfy the condition

$$\zeta_\alpha^\dagger \eta_\beta = \frac{1}{2}\delta_{\alpha\beta}. \tag{5.30}$$

The structures in equations $(5.24, 5.25)$ as they stand don't give the real endomorphisms we are after since the square of each one of them or of any real combination doesn't give minus the identity matrix. If we take the covariant derivative of of the

square of (5.24) and the square of (5.25) we get

$$\nabla_m(N_{\alpha\beta\underline{i}}{}^{\underline{j}}N_{\alpha\beta\underline{j}}{}^{\underline{k}}) = -\frac{8}{l}[(\zeta_\alpha^\dagger\zeta_\beta)(\eta_\alpha^\dagger\gamma_m\eta_\beta) - (\eta_\alpha^\dagger\eta_\beta)(\zeta_\alpha^\dagger\gamma_m\zeta_\beta)]\delta_{\underline{i}}{}^{\underline{k}},$$

$$\nabla_m(M_{\alpha\beta\underline{i}}{}^{\underline{j}}M_{\alpha\beta\underline{j}}{}^{\underline{k}}) = \frac{8}{l}[(\zeta_\alpha^\dagger\zeta_\beta)(\eta_\alpha^\dagger\gamma_m\eta_\beta) + (\eta_\alpha^\dagger\eta_\beta)(\zeta_\alpha^\dagger\gamma_m\zeta_\beta)]\delta_{\underline{i}}{}^{\underline{k}},$$

which doesn't give zero. In the previous result we used the Fierz identity for commuting spinors (ζ and η), given by

$$\zeta\eta^\dagger = \frac{1}{2}\eta^\dagger\zeta + \frac{1}{2}\eta^\dagger\gamma^\mu\zeta\gamma_\mu.$$

In other words we have, for example for M_{11} and N_{11}, the following properties

$$(N_{11})^2 = -f(x,y,z).\mathbb{I}, \tag{5.31}$$

$$(M_{11})^2 = -g(x,y,z).\mathbb{I}, \tag{5.32}$$

where $f(x,y,z)$ and $g(x,y,z)$ are real rational functions equal to

$$f(x,y,z) = -\frac{x^4 + 2x^2(-1+y^2+z^2) + (1+y^2+z^2)^2}{4z^2},$$

$$g(x,y,z) = -\frac{x^4 + y^4 + (-1+z^2)^2 + 2y^2(1+z^2) + 2x^2(-1+y^2+z^2)}{4z^2}.$$

Moreover, the other structures like $\frac{1}{2}(M_{12} + M_{21})$, $\frac{1}{2i}(M_{12} - M_{21})$, $\frac{1}{2}(N_{12} + N_{21})$, and $\frac{1}{2i}(N_{12} - N_{21})$ also follow similar properties as in equations (5.31, 5.32).

These properties plus the fact that the zero modes are real resulting from a six dimensional theory, motivate us to define the real endomorphism $X_{\alpha\beta}$ by

$$X_{\alpha\beta} = \begin{pmatrix} M_{\alpha\beta} & N_{\alpha\beta} \\ -N_{\alpha\beta} & M_{\alpha\beta} \end{pmatrix}, \tag{5.33}$$

where one can now show that the linear combinations

$$R = X_{11} \qquad S = \frac{1}{2}(X_{12} + X_{21}) \qquad T = \frac{1}{2i}(X_{12} - X_{21}), \tag{5.34}$$

satisfy the quaternion algebra

$$R^2 = S^2 = -\mathbb{I} \qquad\qquad RS = -SR = T. \qquad (5.35)$$

Hence if we define

$$I = a_1 R + a_2 S + a_3 T,$$

where a_1, a_2, and a_3 are real numbers with $\sum_i a_i^2 = 1$, we get a 2-sphere of complex structures. (note that the complex structures have only real entires, as required). The square of I gives

$$I^2 = -(a_1^2 + a_2^2 + a_3^2).\mathbb{I} \qquad (5.36)$$

A significant note that worth mentioning is that the dimension of the $X_{\alpha\beta}$ confirms that starting with a six dimensional theory is necessary for studying the real moduli space of hyperbolic monopoles. This will be manifested when $X_{\alpha\beta}$ acts on the zero modes.

Another possible choice of a real endomorphism, similar to the definition of $X_{\alpha\beta}$, is given by

$$Y_{\alpha\beta} = \begin{pmatrix} -M_{\alpha\beta} & N_{\alpha\beta} \\ -N_{\alpha\beta} & -M_{\alpha\beta}. \end{pmatrix} \qquad (5.37)$$

Using the properties $(5.31, 5.32)$ one can show that $Y_{\alpha\beta}$ is equal to the negative of the inverse of the transpose of $X_{\alpha\beta}$. One can, also, show here that the linear combinations

$$O = Y_{11} \qquad\qquad P = \frac{1}{2}(Y_{12} + Y_{21}) \qquad\qquad Q = \frac{1}{2i}(Y_{12} - Y_{21}),$$

satisfy the quaternion algebra

$$O^2 = P^2 = -\mathbb{I} \qquad\qquad OP = -PO = Q.$$

Hence the complex structure J, defined as

$$J = b_1 O + b_2 P + b_3 Q,$$

where b_1, b_2, and b_3 are real numbers with $\sum_i b_i^2 = 1$, gives a 2-sphere of complex structures. The square of J gives

$$J^2 = -(b_1^2 + b_2^2 + b_3^2).\mathbb{I} \tag{5.38}$$

One interesting property is that I and J don't anticommute, neither any of the set $\{R, S, T\}$ anticommute with any of the set $\{O, P, Q\}$. Note also that $\{R, S, T\}$ and $\{O.P.Q\}$ are integrable by construction. This follows from the fact that the structures $N_{\alpha\beta}$ and $M_{\alpha\beta}$ are the the real and imaginary part of the endomorphism $A_{\alpha\beta}$ we defined in [125] section 3.3. We have A_{11}, $\frac{1}{2}(A_{12} + A_{21})$ and $\frac{1}{2i}(A_{12} - A_{21})$ integrable, and since the Frölicher-Nijenhuis bracket is a real operator it implies that M_{11}, $\frac{1}{2}(M_{12} + M_{21})$, $\frac{1}{2i}(M_{12} - M_{21})$, N_{11}, $\frac{1}{2}(N_{12} + N_{21})$ and $\frac{1}{2i}(N_{12} - N_{21})$ satisfy the Frölicher-Nijenhuis bracket and hence integrable.

The action of the complex structures on the zero modes can be achieved by writing the two possible choices of real zero modes in a couple. The components of the couple are actually the real and the imaginary part of the complex zero modes we defined in [125]. Let $\dot{\mathcal{W}}_{\underline{i}}$ be the couple of zero modes and $\dot{V}_{\underline{i}}$ be the second possible choice of real zero modes, then

$$\dot{\mathcal{W}}_{\underline{i}} = \begin{pmatrix} \dot{W}_{\underline{i}} \\ \dot{V}_{\underline{i}} \end{pmatrix} = \begin{pmatrix} -i\zeta^\dagger\Pi_{\underline{i}}\dot{\Psi} + i\Psi^\dagger\Pi_{\underline{i}}\zeta \\ -\zeta^\dagger\Pi_{\underline{i}}\dot{\Psi} - \Psi^\dagger\Pi_{\underline{i}}\zeta \end{pmatrix}. \tag{5.39}$$

The actions of $X_{\underline{i}}{}^{\underline{j}}$ and $Y_{\underline{i}}{}^{\underline{j}}$ on $\mathcal{W}_{\underline{j}}$ are given by

$$X_{\underline{i}}{}^{\underline{j}}\dot{\mathcal{W}}_{\underline{j}} = \begin{pmatrix} M_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}} + N_{\underline{i}}{}^{\underline{j}}\dot{V}_{\underline{j}} \\ -N_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}} + M_{\underline{i}}{}^{\underline{j}}\dot{V}_{\underline{j}} \end{pmatrix}, \qquad Y_{\underline{i}}{}^{\underline{j}}\dot{\mathcal{W}}_{\underline{j}} = \begin{pmatrix} -M_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}} + N_{\underline{i}}{}^{\underline{j}}\dot{V}_{\underline{j}} \\ -N_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}} - M_{\underline{i}}{}^{\underline{j}}\dot{V}_{\underline{j}} \end{pmatrix}. \tag{5.40}$$

Hence we can read now the images $\dot{W}_{\underline{j}}$ resulting from the action of $X_{\underline{i}}{}^{\underline{j}}$ and $Y_{\underline{i}}{}^{\underline{j}}$ on the zero modes

$$X_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}} = M_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}} + N_{\underline{i}}{}^{\underline{j}}\dot{V}_{\underline{j}}$$
$$= -2\zeta^\dagger\Pi_{\underline{i}}{}^{\underline{j}}\eta(\Psi^\dagger\Pi_{\underline{j}}\zeta) + 2\eta^\dagger\Pi_{\underline{i}}{}^{\underline{j}}\zeta(\zeta^\dagger\Pi_{\underline{j}}\dot{\Psi}). \tag{5.41}$$
$$Y_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}} = -M_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}} + N_{\underline{i}}{}^{\underline{j}}\dot{V}_{\underline{j}}$$

$$= -2\zeta^\dagger \Pi_{\underline{i}}{}^{\underline{j}}\eta(\zeta^\dagger \Pi_{\underline{j}}\Psi) + 2\eta^\dagger \Pi_{\underline{i}}{}^{\underline{j}}\zeta(\Psi^\dagger \Pi_{\underline{j}}\zeta). \tag{5.42}$$

The complex structures X and Y must map zero modes to zero modes. This means that the images of $\dot{W}_{\underline{i}}$ (and $\dot{V}_{\underline{i}}$) under the action of the X and Y have to satisfy the linearized Bogomol'nyi equation (5.21) and Gauss's law (5.22). Starting with the linearized Bogomol'nyi equation, which we denote by "$\mathfrak{B}$", we have shown in [125] section (3.3) that the first term in (5.42) ($\zeta^\dagger \Pi_{\underline{i}}{}^{\underline{j}}\eta\zeta^\dagger \Pi_{\underline{j}}\Psi$) satisfies $\mathfrak{B}$. If we denote $\zeta^\dagger \Pi_{\underline{i}}{}^{\underline{j}}\eta$ by "$\mathcal{A}$" and $\zeta^\dagger \Pi_{\underline{j}}\Psi$ by "$\mathcal{Z}$", then we have $\mathfrak{B}(\mathcal{A}\mathcal{Z}) = 0$. We also have, in [125], that $\mathfrak{B}(\mathcal{Z}) = 0$. Since $\mathfrak{B}$ is linear and has real coefficients, then $\mathfrak{B}(\mathcal{A}^\dagger \mathcal{Z}^\dagger) = 0$ and $\mathfrak{B}(\mathcal{Z}^\dagger) = 0$. This implies that $\mathcal{Z}^\dagger$ is a zero mode and hence $\mathfrak{B}(\mathcal{A}\mathcal{Z}^\dagger) = 0$ which gives $\mathfrak{B}(\mathcal{A}^\dagger \mathcal{Z}) = 0$. Therefore, $X_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}$ is a solution for the linearized Bogmol'nyi equation and similarly for $Y_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}$.

One more thing to check if $X_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}$ and $Y_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}$ are also solutions to Gauss's law (5.22). By definition

$$\begin{aligned}
D^{\underline{i}}(X_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}) &= D^{\underline{i}}(M_{\underline{i}}{}^{\underline{j}})\dot{W}_{\underline{j}} + M_{\underline{i}}{}^{\underline{j}}D^{\underline{i}}\dot{W}_{\underline{j}} + D^{\underline{i}}(N_{\underline{i}}{}^{\underline{j}})\dot{V}_{\underline{j}} + N_{\underline{i}}{}^{\underline{j}}D^{\underline{i}}\dot{V}_{\underline{j}} \\
&= D^{i}(M_{\underline{i}}{}^{\underline{j}})\dot{W}_{\underline{j}} + M_{\underline{i}}{}^{\underline{j}}D^{[\underline{i}}\dot{W}_{\underline{j}]} + D^{i}(N_{\underline{i}}{}^{\underline{j}})\dot{V}_{\underline{j}} + N_{\underline{i}}{}^{\underline{j}}D^{[\underline{i}}\dot{V}_{\underline{j}]} \\
&= D^{i}(M_{i}{}^{j})\dot{W}_{j} + D^{i}(N_{i}{}^{j})\dot{V}_{j}, \tag{5.43}
\end{aligned}$$

where in the last line we used the fact that $M_{\underline{i}}{}^{\underline{j}}$ and $N_{\underline{i}}{}^{\underline{j}}$ are selfdual (since η and ζ have negative chirality) and on the other hand $D^{[\underline{i}}\dot{W}_{\underline{j}]}$ and $D^{[\underline{i}}\dot{V}_{\underline{j}]}$ are antiselfdual, hence $M_{\underline{i}}{}^{\underline{j}}D^{[\underline{i}}\dot{W}_{\underline{j}]}$ and $N_{\underline{i}}{}^{\underline{j}}D^{[\underline{i}}\dot{V}_{\underline{j}]}$ vanish. We also used that

$$D^{i}M_{4j} = -\frac{1}{l}N_{ij} \qquad D^{i}N_{4j} = \frac{1}{l}M_{ij}.$$

Now we have

$$D^{i}M_{ij} = \frac{2}{l}N_{4j} \qquad D^{i}N_{ij} = -\frac{2}{l}M_{4j}, \tag{5.44}$$

hence plugging (5.44) in (5.43) we get

$$D^{\underline{i}}(X_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}) = \frac{2}{l}(N_{4}{}^{j}\dot{W}_{j} - M_{4}{}^{j}\dot{V}_{j}). \tag{5.45}$$

In order to check if this gives the correct right hand side, we have to check how the right hand side of the Gauss's law (5.22) change under the action of the endomorphism X. The right hand side of the Gauss's law is given by $(\frac{2}{l}\dot{V}_4)$, hence its image under the action of X is (5.40)

$$X_4{}^j(\frac{2}{l}\dot{V}_j) = \frac{2}{l}(-N_4{}^j\dot{W}_j + M_4{}^j\dot{V}_j), \tag{5.46}$$

which is not equal to the right hand side of (5.45). This implies that image of X doesn't satisfy also the Gauss's law.

Quite similarly we can check if the image of Y satisfies Gauss's law. We have,

$$D^{\underline{i}}(Y_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}) = -D^i(M_i{}^j)\dot{W}_j + D^i(N_i{}^j)\dot{V}_j$$
$$= -\frac{2}{l}(N_4{}^j\dot{W}_j + M_4{}^j\dot{V}_j),$$

and the image of the right hand side of Gauss's law under the action of Y is given by

$$Y_4{}^j(\frac{2}{l}\dot{V}_j) = -\frac{2}{l}(N_4{}^j\dot{W}_j + M_4{}^j\dot{V}_j). \tag{5.47}$$

This implies that

$$D^{\underline{i}}(Y_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}) = \frac{2}{l}(Y_4{}^{\underline{j}}\dot{V}_{\underline{j}}), \tag{5.48}$$

hence $Y_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}$ is tangent to the moduli space. Therefore Y is a real endomorphism that maps every zero mode tangent to the moduli space to zero mode tangent to the moduli space. Let $\{\dot{W}_{\underline{i}}\}$ be the bases of the a subspace of the real tangent space $T\mathcal{M}_1$ to the real moduli space $\mathcal{M}$. The previous result shows that $Y_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}$ (but not $X_{\underline{i}}{}^{\underline{j}}\dot{W}_{\underline{j}}$) is tangent to the moduli space. Let $\mathcal{Y}$ be the of endomorphisms on the real tangent space defined by

$$Y_{\underline{i}}{}^{\underline{j}}\dot{W}_{a\underline{j}} = \mathcal{Y}_a{}^b\dot{W}_{b\underline{i}}. \tag{5.49}$$

Through the previous relation, we easily see that the endomorphism on $T\mathcal{M}_1$ inherit the same characteristics of Y, and gives a 2-sphere of (real) complex structures.

Moreover, if we now take the set of real zero modes $\{\dot{V}_{\underline{i}}\}$ to be the bases of another subspace of the real tangent space $T\mathcal{M}_2$. This set of bases satisfy the same linearized

Bogomol'nyi equation but different gauge background condition. The gauge background condition for $\{\dot{V}_{\underline{i}}\}$ corresponds to a part of the condition we found in [125] equation (3.9) which the imaginary elements of the zero modes satisfy. In a similar way to the previous discussion, we find that $X_{\underline{i}}{}^{j}\dot{V}_{\underline{j}}$ (but not $Y_{\underline{i}}{}^{j}\dot{V}_{\underline{j}}$) satisfy the gauge background condition of $\{\dot{V}\}$. Hence, if we take $\mathcal{X}$ to be the set of endomorphisms on the real tangent space $T\mathcal{M}_2$ defined by

$$X_{\underline{i}}{}^{j}\dot{V}_{a\underline{j}} = \mathcal{X}_a{}^b\dot{V}_{b\underline{i}}, \tag{5.50}$$

we see that the $\mathcal{X}$ will have the same characteristics as those of X, and therefore gives another set of 2-sphere of complex structures.

5.4   Comparison and remark

In this section we compare the results of the previous section with Bielawski and Schwachhöfer construction of the pluricomplex geometry. We show that the objects that appear as a natural consequence of supersymmetry to describe the geometry of the real moduli space of hyperbolic monopoles are similar to the objects that Bielawski and Schwachhöfer defined to describe the pluricomplex geometry. Their construction was actually inspired by studying the geometry of moduli space of hyperbolic monopoles. Then we give in a table a dictionary that relates our notations to Bielawski and Schwachhöfer notations. We finish this section with a remark about the relation between the real and complex structures on the moduli space of hyperbolic monopoles.

5.4.1   Comparison with Bielawski and Schwachhöfer results

In their paper [68] Bielawski and Schwachhöfer describe the geometry of the real moduli space of hyperbolic monopoles, which they call pluricomplex geometry, as a generalization of the hypercomplex geometry, where we still have a 2-sphere of complex structures but they no longer behave like unit imaginary quaternions. A sphere in their description does not mean the standard sphere in the subspace spanned by three

integrable complex structures, however it means a diffeomorphic image of a 2-sphere in the space of complex structures, i.e. an embedding of S^2 into $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$. Their construction for the pluricomplex geometry is as follows:

Let V be a $2n$-dimensional real vector space, and $\mathcal{J}(V) \simeq GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ be the space of complex structures on V. The space $\mathcal{J}(V)$ is constructed by two holomorphic maps taking values in $\mathbb{C}P^1$-s spaces of complex structures. The first map $K : \mathbb{C}P^1 \to \mathcal{J}(V)$, takes every $\zeta \in \mathbb{C}P^1$ and gives $K(V) = J_\zeta \in \mathcal{J}(V)$, and hence forms a subspace of $\mathcal{J}(V)$, $\mathcal{V}_\zeta^{1,0}$. This map motivates a similar one but from different set of $\mathbb{C}P^1$ complex structures defined by $\sigma : \mathbb{C}P^1 \to \mathbb{C}P^1$, such that $\sigma(\zeta) = \eta = \bar{\zeta}^{-1}$. Then the holomorphic map $\hat{K} : \mathbb{C}P^1 \to \mathcal{J}(V)$ forms another subspace of $\mathcal{J}(V)$, namely, $\hat{\mathcal{V}}_\zeta^{1,0} = \mathcal{V}_{\bar{\zeta}^{-1}}^{0,1}$. For $(\zeta, \eta) \in \mathbb{C}P^1 \times \mathbb{C}P^1$, the elements of $\mathcal{V}_\zeta^{1,0}$ and $\mathcal{V}_{\bar{\eta}^{-1}}^{0,1}$ don't satisfy any anticommutation relations. This lead to constructing a coherent sheaf $\mathcal{F}$ on an algebraic curve S in $\mathbb{C}P^1 \times \mathbb{C}P^1$ described as

$$S = \{(\zeta, \eta) \in \mathbb{C}P^1 \times \mathbb{C}P^1; \mathcal{V}_\zeta^{1,0} \cap \mathcal{V}_{\bar{\eta}^{-1}}^{0,1} \neq 0\},$$

where S is called the characteristic curve and $\mathcal{F}$ the characteristic sheaf on the pluricomplex structure.

Our results show that the exact construction of pluricomplex space follows from studying the geometry of real moduli space of hyperbolic monopoles using supersymmetry. We have the endomorphisms X and Y act linearly on different sets of zero modes $\{\dot{W}_\mu\}$ and $\{\dot{V}_\mu\}$ respectively, each X and Y give a 2-sphere ($\mathbb{C}P^1$) worth of (real) complex structures $\{R, T, S\}$ and $\{O, P, Q\}$, where we don't find any anticommutation properties between the two sets. This allows us to construct two sets of complex structures $\{\mathcal{X}, \mathcal{Y}\}$ on the moduli space which inherit all the characteristics of $\{X, Y\}$, which means that both $\mathcal{X}$ and $\mathcal{Y}$ give two sets of 2-sphere complex structures but we don't have any anti-commutation relations between them. In the following table we give a small dictionary for the terms defined in Bielawski and Schwachhöfer construction and what they correspond for in our terminology.

Objects	B&S Notation	Our Notation
$\mathbb{CP}^1 \times \mathbb{CP}^1$ elements	$\zeta, \bar{\zeta}^{-1}$	$X, Y = (X^{\mathsf{T}})^{-1}$
Endomorphisms on $\mathcal{M}$	$J_\zeta, J_{\bar{\zeta}^{-1}}$	$\mathcal{X}, \mathcal{Y}$
Maps	$K, \hat{K}$	$X\dot{V} = \mathcal{X}\dot{V}, Y\dot{W} = \mathcal{Y}\dot{W}$

In addition to this picture Bielawski and Schwachhöfer gives another point of view: An integrable pluricomplex structure on a manifold M can be viewed as an integrable hypercomplex structure on a complex thickening $M^{\mathbb{C}}$ of M, commuting with the tautological complex structure of $M^{\mathbb{C}}$ i.e. a pluricomplex geometry of M is biquaternionic geometry of $M^{\mathbb{C}}$. This picture is also derived in our results by complexifying the moduli space described in this chapter, which means doing the exact calculation done in chapter three, which lead to a hypercomplex geometry.

5.4.2 Remark on the complex and real geometry

The main equations that determine the geometry of the moduli space of monopoles are the linearized Bogomol'nyi equation and the gauge background condition. The gauge background condition is derived when we build the supermultiplet of zero modes. It is the condition that makes the zero modes orthogonal to the gauge orbits. This condition is one of the main players that renders the geometry of hyperbolic monopoles different than the geometry of Euclidean monopoles, and it is also the main factor in making the geometry of the complex and real moduli spaces of hyperbolic monopoles different. Recall that for complex hyperbolic monopoles, the linearized Bogomol'nyi equation and the gauge background condition are given by

$$D_{[\mu}\dot{A}_{\nu]} = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}D^\rho\dot{A}^\sigma \qquad \text{and} \qquad D^\mu\dot{A}_\mu = -4i\lambda\dot{A}_4 \,, \tag{5.51}$$

where λ is equal to $\frac{1}{2l}$ if we take the upper half space model of hyperbolic space (4.18). For the Euclidean monopoles the equations have the same form, except for the right hand side of the gauge background condition which is equal to zero (it is the limit as $l \to \infty$). Hence, in the Euclidean case the linearized field equations are real, which imply that weather we take their solutions (zero modes) to be real or complex it doesn't

make any difference. Therefore, the real and the complex moduli spaces of Euclidean monopoles have the same geometry (hyperkähler geometry). However, for the case of hyperbolic monopoles this extra pure complex term directly implies that these equations can only have complex solutions, i.e. complex coordinates functionals on the moduli space, and hence the real moduli space must have different geometry. For the real geometry we started from higher dimensional gauge theory, and after reduction and linearization we found two sets of real zero modes $(\dot{W}_{\underline{i}}, \dot{V}_{\underline{i}})$ satisfying the same linearized Bogomol'nyi equation in (5.51), but different gauge background conditions given by

$$D^{\underline{i}}\dot{W}_{\underline{i}} = \frac{2}{l}\dot{V}_4 \qquad \text{and} \qquad D^{\underline{i}}\dot{V}_{\underline{i}} = -\frac{2}{l}\dot{W}_4 \, . \qquad (5.52)$$

By relating the bosonic zero modes $(\dot{W}_{\underline{i}}, \dot{V}_{\underline{i}})$ to fermionic zero modes (5.39) we constructed the geometric objects of the real moduli space, and showed that they match the pluricomplex geometry objects.

Starting from a six dimensional supersymmetric Yang-Mills theory gives information about the real coordinates of the moduli space of hyperbolic monopoles, and hence probe its geometry. However, this approach is just looking at the same problem from different angle, and that should make our vision and understanding of the problem better, yet provide the same results. In other words, it should be also possible to look at the geometry of the complex moduli space when we start from six dimensional supersymmetric Yang-Mills theory. This is actually simple, all we need to do is to write the zero modes $(\dot{W}_{\underline{i}}, \dot{V}_{\underline{i}})$ in the form

$$\dot{A}_{\underline{i}} = \dot{W}_{\underline{i}} + i\dot{V}_{\underline{i}}, \qquad (5.53)$$

and then all the equations for the complex fields are obtained by simple algebra. This note actually confirms our results by deriving same results from different starting point.

Therefore, supersymmetry via the gauge background condition distinguishes between the 2-sphere complex structures that can be linearly made to satisfy the quaternionic relations and those that cannot. Moreover, supersymmetry in different dimensions al-

lows us to explore the real and the complex moduli spaces of hyperbolic monopoles, and confirm the results of one of them by using the results of the other.

5.5 Limiting case "hyperkähler geometry"

Every new term introduced in the process of promoting our theory from flat space to hyperbolic space is inversely proportional to the radius of curvature. This feature makes studying the limiting scheme very easy and the results very transparent.

First, by setting the radius of curvature to infinity, that is $l \to \infty$, the Killing spinors equations defined in $(5.19, 5.20)$ agree

$$l \to \infty \Rightarrow \nabla_{\underline{i}} \zeta = \nabla_{\underline{i}} \eta = 0.$$

This implies that

$$M_{\alpha\beta\underline{i}}{}^{\underline{j}} = 2i\zeta_\alpha^\dagger \Pi_{\underline{i}}{}^{\underline{j}} \zeta_\beta, \qquad\qquad N_{\alpha\beta\underline{i}}{}^{\underline{j}} = 0,$$

Hence

$$X_{\alpha\beta} = Y_{\alpha\beta} = \begin{pmatrix} M_{\alpha\beta} & 0 \\ 0 & M_{\alpha\beta} \end{pmatrix}, \tag{5.54}$$

Therefore we have now one set of 3 complex structures

$$\mathcal{F}^{(1)} = X_{11}, \qquad \mathcal{F}^{(2)} = \frac{1}{2}(X_{12} + X_{21}), \qquad \mathcal{F}^{(3)} = \frac{1}{2i}(X_{12} - X_{21}), \tag{5.55}$$

which are integrable and satisfy the quaternionic algebra $(5.34, 5.35)$. Using these complex structures we define complex structures on the moduli space with same properties via (5.49).

Moreover, the new factors in Dirac equation and Gauss's law that appeared due to writing them on hyperbolic space are, also, inversely proportional to l, hence these equations along with the linearized Bogomol'nyi equation coincide with their form

on flat space when $l \to \infty$

$$D_{[\underline{i}}\dot{W}_{\underline{j}]} = -\frac{1}{2}\varepsilon_{\underline{ijmn}}D^{\underline{m}}\dot{W}^{\underline{n}}, \qquad D^{\underline{i}}\dot{W}_{\underline{i}} = 0, \qquad \not{D}\dot{\Psi} = 0, \qquad (5.56)$$

and they are satisfied by the zero modes ansatz defined by (5.17, 5.18). Therefore the zero modes $\dot{W}_{\underline{i}}$ and $\dot{\Psi}$ are coordinate functionals on the moduli space and will be used to linearize the action. For that sake we introduce the bosonic $q_a(t)$ and fermionic $\rho_a(t)$ collective coordinates of the moduli space, for $a = 1, \ldots, 4n$, where t is a parameter introduced, such that the evolution of the monopoles in t values can be viewed as that of a fictitious particle moving in a configuration space, the space of minimum energy.

We first expand the bosonic zero mode $\dot{W}_{\underline{i}} = \dot{W}_{a\underline{i}}q^a$ as a linear combination of the basis $\dot{W}_{a\underline{i}}$ and similarly for the fermionic zero mode $\dot{\Psi} = \dot{\Psi}_a\rho^a$. Then, we expand the supersymmetric Yang-Mills-Higgs action in terms of the collective coordinates $\{q_a, \rho_a\}$ and we keep only the non trivial order. Since we are only interested, in this section, in the geometry we will just focus on the bosonic part of the Lagrangian (4.15) given by

$$\mathcal{L} = -\frac{1}{2}G_{ij}G^{ij} - \|D_i\phi\|^2 - D_iA_+D^iA_- - [\phi, A_+][\phi, A_-] - \frac{1}{4}[A_+, A_-]^2 + \frac{4}{\ell^2}A_+A_-,$$

and in the limit where $\ell \to \infty$ is given by

$$\mathcal{L} = -\frac{1}{2}G_{ij}G^{ij} - \|D_i\phi\|^2 - D_iA_+D^iA_- - [\phi, A_+][\phi, A_-] - \frac{1}{4}[A_+, A_-]^2, \qquad (5.57)$$

which as expected coincide with the bosonic part of the Lagrangian we found on Euclidean space (4.11). Now, apply the supersymmetric constraints (5.1) to the lagrange density and then we expand around the supersymmetric Bogomol'nyi solution. After applying the constraints the lagrange density will simply read as $\mathcal{L} = -\frac{1}{2}G_{ij}G^{ij} - \|D_i\phi\|^2$, which constitutes only the potential energy part of the action,

$$P = \int \frac{1}{2}\|G_{ij}\| + \|D_i\phi\|^2 \, dx^3.$$

The geodesic motion comes from the kinetic part. In the spirit of the argument used

in [127, 128], for a given path $(W_i(t), \phi(t))$ in the moduli space, the kinetic energy can be defined as

$$T = \int \|G_{0i}\| + \|D_0\phi\|^2 \, dx^3,$$

and in this case the Lagrange function is then given by $L = T - P$, and the action is given by $\int L dt$.

The potential energy will give the charge number. For the kinetic energy, we expand the arbitrary t-dependent fields, where t-dependance is via the collective coordinates, hence

$$T = \int_{\mathbb{R}^3} \|G_{0i}\|^2 + \|D_0\phi\|^2 = 2 \int_{\mathbb{R}^3} \|G_{0\underline{i}}\|^2$$
$$= 2 \int_{\mathbb{R}^3} \|\partial_0 W_{\underline{i}} - \partial_{\underline{i}} W_0 - [W_{\underline{i}}, W_0]\|^2 = 2 \int_{\mathbb{R}^3} \|\partial_a W_{\underline{i}} \dot{q}^a - D_{\underline{i}} W_0\|^2,$$

where $\dot{q} = \frac{dq}{dt}$. Note that $\partial_a W_{\underline{i}}$ doesn't satisfy in general the gauge background condition, hence we decompose it into a component tangent to the moduli space, $\dot{W}_{a\underline{i}}$, and another perpendicular, $D_{\underline{i}}\omega_a$ (ω_a are gauge parameters), then

$$T = 2 \int_{\mathbb{R}^3} \|(\dot{W}_{a\underline{i}} + D_{\underline{i}}\omega_a)\dot{q}^a - D_{\underline{i}} W_0\|^2$$
$$= 2 \int_{\mathbb{R}^3} [\dot{W}_{a\underline{i}} \dot{W}_{b\underline{i}} \dot{q}^a \dot{q}^b + \|D_{\underline{i}}(\omega_a \dot{q}^a - W_0)\|^2],$$

where we dropped any multiplication between $\dot{W}_{ai}$ and $D_{\underline{i}}\omega_a$ using the gauge background condition. If we now work with gauge s.t. $\omega_a \dot{q}^a = W_0$ and substitute the kinetic energy in the action we get

$$L = \frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b - k,$$

where k is the charge number and $g_{ab} = \frac{1}{4} \int \dot{W}_{a\underline{i}} \dot{W}_{b\underline{i}} dx^3$. And finally, the Levi-Civita connection is given by

$$\Gamma_{abc} = \frac{1}{2}\{\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}\}.$$

Using the definition of the metric g_{bc} we have

$$\partial_a g_{bc} = \frac{1}{4} \int_{\mathbb{R}^3} (\partial_a \dot{W}_{b\underline{i}} \dot{W}_{c\underline{i}} + \dot{W}_{b\underline{i}} \partial_a \dot{W}_{c\underline{i}}).$$

First note that $\int_{\mathbb{R}^3} \partial_a \dot{W}_{b\underline{i}} \dot{W}_{c\underline{i}} = \int_{\mathbb{R}^3} \partial_b \dot{W}_{a\underline{i}} \dot{W}_{c\underline{i}}$. This can be simply seen by noticing that

$$\int_{\mathbb{R}^3} \partial_a \dot{W}_{b\underline{i}} \dot{W}_{c\underline{i}} = \int_{\mathbb{R}^3} \partial_a (\partial_b W_{\underline{i}} - D_{\underline{i}} \omega_b) \dot{W}_{c\underline{i}} \tag{5.58}$$

which follows from having $D_{\underline{i}} \dot{W}_{\underline{i}} = 0$. Similarly we have

$$\int_{\mathbb{R}^3} \partial_b \dot{W}_{c\underline{i}} \dot{W}_{a\underline{i}} = \int_{\mathbb{R}^3} \partial_c \dot{W}_{b\underline{i}} \dot{W}_{a\underline{i}} \qquad \text{and} \qquad \int_{\mathbb{R}^3} \partial_a \dot{W}_{c\underline{i}} \dot{W}_{b\underline{i}} = \int_{\mathbb{R}^3} \partial_c \dot{W}_{a\underline{i}} \dot{W}_{b\underline{i}}. \tag{5.59}$$

Using $(5.58, 5.59)$ in the connection equation we find that

$$\Gamma_{abc} = \frac{1}{4} \int_{\mathbb{R}^3} \partial_a \dot{W}_{b\underline{i}} \dot{W}_{c\underline{i}}. \tag{5.60}$$

The complex structures (5.55) obtained after we took the limit are integrable, they satisfy the quaternionic algebra and map zero modes to zero modes (i.e. $\vec{\mathcal{F}}_{\underline{i}}{}^{\underline{j}} \dot{W}_{\underline{j}}$ satisfy the linearized Bogomol'nyi equation and the gauge background condition). These properties are inherited from $\vec{X}$ and remain intact after taking the limit. Since $\vec{\mathcal{F}}_{\underline{i}}{}^{\underline{j}} \dot{W}_{\underline{j}}$ is tangent to the moduli space, one can define endomorphisms $\vec{\mathcal{J}}$ on the tangent space by $\vec{\mathcal{F}}_{\underline{i}}{}^{\underline{j}} \dot{W}_{a\underline{j}} = \vec{\mathcal{J}}_a{}^b \dot{W}_{b\underline{i}}$. These endomorphisms via their definition are actually integrable complex structures on the moduli space, and they satisfy the quaternionic algebra. One can also show that these complex structures are parallel with respect to the connection (5.60) (see e.g. [79] or [87]). So the limiting case has given us a metric, connection and integrable complex structures that satisfy the quaternionic algebra and preserved by the Levi-Civita connection, whence produced the hyperkähler geometry.

Chapter 6

Conclusion and Outlook

In this thesis, an old question of Atiyah about the nature of the geometry of the moduli space of hyperbolic monopoles is answered. Unlike the previous approaches to this problem, which used twistor construction, we use supersymmetry to solve it. We found that the geometry of the complex moduli space of hyperbolic monopoles is hypercomplex and the geometry of the real moduli space is pluricomplex geometry. Pluricomplex geometry is a generalization of the hypercomplex geometry where the complex structures define a diffeomorphic image of the standard 2-sphere, an embedding of the S^2 into $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$, and they don't admit any anticommutation relations. Moreover, we showed that we have a one-to-one correspondence between the number of solutions of Bogomol'nyi equation and the number of solutions of Dirac equation in the presence of hyperbolic monopoles. The feature that makes using supersymmetry a favorable approach to solve for the geometry of a supersymmetric configurations is that by deriving the supersymmetry transformations we are indirectly finding the solutions of the equations of motion on the tangent space of the configuration space and hence rendering our job of studying these solutions and extracting information about the complex structures, connections or other geometric objects much easier. In other words, the supersymmetry transformations furnish a complete scheme which we can use to read information about the geometry of the configuration space, which again confirms the importance of supersymmetry as a mathematical machinery to relate the equation of motion of objects of different nature and hence re-

vealing hidden information about their configuration space.

The procedure used in this thesis raises few interesting questions that worth investigating:

First, in this thesis two new supersymmetric Yang-Mills-Higgs Lagrangian on H^3 are derived using the method of deforming the supersymmetric theory on a flat space by new terms that depend on the radius of curvature. It would be really interesting, however, to derive these Lagrangians using the method of Festuccia and Seiberg [99]. This method has been employed, so far, to derive supersymmetric field theories on compact backgrounds due to interests in the localization techniques to compute new observables. The first step in recovering our Lagrangians using this new method would be finding a four dimensional supergravity theory with Yang-Mills fields where the metric $H^3 \times \mathbb{R}$ or $H^3 \times S^1$ form one of the solutions of the gravity field.

Second, in tackling the complex moduli space of hyperbolic monopoles we started from a supersymmetric Yang-Mills on Euclidean space, however we studied the real moduli space by starting from a supersymmetric Yang-Mills on Minkowskian space. This arises an interesting question, if it is always the case that euclideanising a supersymmetric theory will pave the way to study the complex configuration space, and hence if we are interested in studying the complex thickening of a configuration space using supersymmetry we should first euclideanise the supersymmetric mother theory.

Third, Manton formulated in [57] the gauge theory dynamics by arguing that slowly moving monopoles flow geodesics in the moduli space. Manton argument applies for gauge theories where the true configuration space is Riemannian. However, what happens for a case with target manifold being non-Riemannian, like for the case of the dynamics of hyperbolic monopoles ? We guess that a generalized argument should be established that governs the dynamics of all kind of monopoles, where, also, the natural action for a path on the moduli space defined in [129] should also be generalized. If this generalization is established we expect that the Hitchen set of metrics for the moduli space of two centered hyperbolic monopoles [66] can be recovered. It is worth mentioning that recently a promising attempt has been established by Paul Sutcliffe

at al. [62] to derive the metric of the moduli space of hyperbolic monopoles using the fact that hyperbolic monopole is uniquely defined by the abelian magnetic field on the boundary of hyperbolic space. In their paper, an integral form for the metric has been given, and the metric of the moduli space of single hyperbolic monopole, namely H^3, has been derived.

Fourth, unlike monopoles on flat space, Skyrmions don't satisfy the linear energy bound, also known as Faddeev-Bogmol'nyi lower bound [82], for non-trivial value of the field, which cuts the hope of deriving a supersymmetric extension to the Skyrme action [130, 131] on flat space. However, Manton and Ruback showed in [83] that the saturation can be made possible with non trivial values of the field if we promote the bosonic Skyrme action to S^3. The saturation of the bound in addition to its linearity imply the existence of a supersymmetric extension for the Skyrme action on S^3. The crucial point here is that from our experience with supersymmetric monopole on H^3 we know how the supersymmetry variation of the fermions looks, namely the Faddeev-Bogomol'nyi equation, and hence building on that it shouldn't be difficult to use Noether method and guess the fermionic content of the action.

Chapter 7

Appendix

7.1 The Frölicher–Nijenhuis bracket of endomorphisms

The Frölicher–Nijenhuis bracket defines graded Lie superalgebra structure on the space $\Omega^\bullet(M; TM)$ of vector-valued differential forms on a manifold M. For a modern treatment see [132, Chapter 8]. This bracket extends the Lie bracket of vector fields, thought of as elements of $\Omega^0(M; TM)$. Endomorphisms of TM can be thought of as elements of $\Omega^1(M; TM)$ and the Frölicher–Nijenhuis bracket defines a symmetric bilinear map $[-, -] : \Omega^1(M; TM) \times \Omega^1(M; TM) \to \Omega^2(M; TM)$. Paragraph 8.12 in [132] gives an explicit expression of the Frölicher–Nijenhuis bracket $[K, L]$ of two endomorphisms K, L in terms of the Lie bracket of vector fields: namely,

$$[K, L](X, Y) = [KX, LY] - [KY, LX] - L[KX, Y] + L[KY, X]$$
$$- K[LX, Y] + K[LY, X] + (LK + KL)[X, Y] . \quad (7.1)$$

Applying this to $X = \partial_a$ and $Y = \partial_b$, we find

$$
\begin{aligned}
[K, L](\partial_a, \partial_b) &= [K_a{}^c \partial_c, L_b{}^d \partial_d] - [K_b{}^c \partial_c, L_a{}^d \partial_d] - L[K_a{}^c \partial_c, \partial_b] \\
&\quad + L[K_b{}^c \partial_c, \partial_a] - K[L_a{}^c \partial_c, \partial_b] + K[L_b{}^c \partial_c, \partial_a] \\
&= \big(K_a{}^c \partial_c L_b{}^d - L_b{}^c \partial_c K_a{}^d - K_b{}^c \partial_c L_a{}^d + L_a{}^c \partial_c K_b{}^d \\
&\quad + \partial_b K_a{}^c L_c{}^d - \partial_a K_b{}^c L_c{}^d + \partial_b L_a{}^c K_c{}^d - \partial_a L_b{}^c K_c{}^d \big) \partial_d .
\end{aligned}
\quad (7.2)
$$

It is perhaps easier to remember the case $K = L$:

$$\tfrac{1}{2}[K, K](X, Y) = [KX, KY] - K[KX, Y] + K[KY, X] + K^2[X, Y] \,, \tag{7.3}$$

from which we can recover the general case by the standard polarisation trick. Applying this to $X = \partial_a$ and $Y = \partial_b$, we find

$$\begin{aligned}
\tfrac{1}{2}[K, K](\partial_a, \partial_b) &= [K_a{}^c \partial_c, K_b{}^d \partial_d] - K[K_a{}^c \partial_c, \partial_b] + K[K_b{}^c \partial_c, \partial_a] \\
&= \left(K_a{}^c \partial_c K_b{}^d - K_b{}^c \partial_c K_a{}^d - \partial_b K_a{}^c K_c{}^d + \partial_a K_b{}^c K_c{}^d \right) \partial_d \,.
\end{aligned} \tag{7.4}$$

If ∇ is a torsion-free connection on the tangent bundle, we may write the Lie bracket of vector fields as

$$[X, Y] = \nabla_X Y - \nabla_Y X \,, \tag{7.5}$$

whence a small calculation yields the following equation for the Frölicher–Nijenhuis bracket $[K, K]$:

$$\tfrac{1}{2}[K, K](X, Y) = (\nabla_{KX} K)Y - (\nabla_{KY} K)X + K(\nabla_Y K)X - K(\nabla_X K)Y \,. \tag{7.6}$$

For endomorphisms which obey equation (3.70), that is,

$$(\nabla_X K)Y = (\nabla_Y K)X, \tag{7.7}$$

the Frölicher–Nijenhuis bracket $[K, K]$ is given by

$$\tfrac{1}{2}[K, K](X, Y) = (\nabla_{KX} K)Y - (\nabla_{KY} K)X \,, \tag{7.8}$$

which polarises to

$$[K, L](X, Y) = (\nabla_{KX} L)Y - (\nabla_{KY} L)X + (\nabla_{LX} K)Y - (\nabla_{LY} K)X \,. \tag{7.9}$$

Applying this to $X = \partial_a$ and $Y = \partial_b$, we see that

$$[K, L](\partial_a, \partial_b) = \left(K^c{}_a \nabla_c L^d{}_b - K^c{}_b \nabla_c L^d{}_a + L^c{}_a \nabla_c K^d{}_b - L^c{}_b \nabla_c K^d{}_a \right) \partial_d , \qquad (7.10)$$

which agrees for $K = \mathcal{E}_i$ and $L = \mathcal{E}_j$ with equation (3.65).

7.2 Obata Connection

In this section of the appendix we prove the existence of the Obata connection $\overset{o}{\nabla}$ on the complex space of hyperbolic monopoles, studied in chapter 3, by explicitly deriving the following formula for its Christoffel symbols:

$$\overset{o}{\Gamma}_{ab}{}^c = -\frac{1}{6} \left[2 \partial_{(a} \vec{\mathcal{E}}_{b)}{}^d + \vec{\mathcal{E}}_{(a}{}^e \times \partial_e \vec{\mathcal{E}}_{b)}{}^d \right] \cdot \vec{\mathcal{E}}_d{}^c, \qquad (7.11)$$

where $\vec{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) = (\mathcal{I}, \mathcal{J}, \mathcal{K})$ represents the three complex structures on $T_{\mathbb{C}}\mathcal{M}$. The equation for the hypercomplex space connection was first derived by Morio Obata [133]. Recall that the complex structures satisfy the quaternion algebra:

$$\mathcal{E}_A \mathcal{E}_B = -\delta_{AB} + \epsilon_{ABC} \mathcal{E}_C . \qquad (7.12)$$

The Obata connection is defined to be the unique torsion-free connection for which $\overset{o}{\nabla} \vec{\mathcal{E}} = 0$. We will drop the superscript "o" on the connection and the Christoffel symbols in what follows.

To begin with, since $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are parallel, we obtain

$$\partial_a \mathcal{I}_b{}^c + \Gamma^c_{ae} \mathcal{I}^e{}_b - \Gamma^e_{ab} \mathcal{I}^c{}_e = 0, \qquad (7.13)$$

$$\partial_a \mathcal{J}_b{}^c + \Gamma^c_{ae} \mathcal{J}^e{}_b - \Gamma^e_{ab} \mathcal{J}^c{}_e = 0, \qquad (7.14)$$

$$\partial_a \mathcal{K}_b{}^c + \Gamma^c_{ae} \mathcal{K}^e{}_b - \Gamma^e_{ab} \mathcal{K}^c{}_e = 0. \qquad (7.15)$$

Multiplying (7.13), (7.14) and (7.15) with $\mathcal{I}^d{}_c$, $\mathcal{J}^d{}_c$ and $\mathcal{K}^d{}_c$, respectively, we get

$$\Gamma^d_{ab} + \mathcal{I}^d{}_c \partial_a \mathcal{I}_b{}^c + \mathcal{I}^d{}_c \Gamma^c_{ae} \mathcal{I}^e{}_b = 0, \qquad (7.16)$$

$$\Gamma^d_{ab} + \mathcal{J}^d{}_c \partial_a \mathcal{J}_b{}^c + \mathcal{J}^d{}_c \Gamma^c_{ae} \mathcal{J}^e{}_b = 0, \tag{7.17}$$

$$\Gamma^d_{ab} + \mathcal{K}^d{}_c \partial_a \mathcal{K}_b{}^c + \mathcal{K}^d{}_c \Gamma^c_{ae} \mathcal{K}^e{}_b = 0. \tag{7.18}$$

Swap a with b and use the fact that the connection is torsion free ($\Gamma^d_{ab} = \Gamma^d_{ba}$), we get

$$\Gamma^d_{ab} = -\mathcal{J}^d{}_c \partial_{(a} \mathcal{J}_{b)}{}^c - \mathcal{J}^d{}_c \Gamma^c_{(a|e|} \mathcal{J}^e{}_{b)}, \tag{7.19}$$

$$\Gamma^d_{ab} = -\mathcal{J}^d{}_c \partial_{(a} \mathcal{J}_{b)}{}^c - \mathcal{J}^d{}_c \Gamma^c_{(a|e|} \mathcal{J}^e{}_{b)}, \tag{7.20}$$

$$\Gamma^d_{ab} = -\mathcal{K}^d{}_c \partial_{(a} \mathcal{K}_{b)}{}^c - \mathcal{K}^d{}_c \Gamma^c_{(a|e|} \mathcal{K}^e{}_{b)}, \tag{7.21}$$

or

$$2\Gamma^d_{ab} + \mathcal{J}^d{}_c \partial_a \mathcal{J}_b{}^c + \mathcal{J}^d{}_c \partial_b \mathcal{J}_a{}^c = -\Gamma^c_{ae} \mathcal{J}^e{}_b \mathcal{J}^d{}_c - \Gamma^c_{be} \mathcal{J}^e{}_a \mathcal{J}^d{}_c, \tag{7.22}$$

$$2\Gamma^d_{ab} + \mathcal{J}^d{}_c \partial_a \mathcal{J}_b{}^c + \mathcal{J}^d{}_c \partial_b \mathcal{J}_a{}^c = -\Gamma^c_{ae} \mathcal{J}^e{}_b \mathcal{J}^d{}_c - \Gamma^c_{be} \mathcal{J}^e{}_a \mathcal{J}^d{}_c, \tag{7.23}$$

$$2\Gamma^d_{ab} + \mathcal{K}^d{}_c \partial_a \mathcal{K}_b{}^c + \mathcal{K}^d{}_c \partial_b \mathcal{K}_a{}^c = -\Gamma^c_{ae} \mathcal{K}^e{}_b \mathcal{K}^d{}_c - \Gamma^c_{be} \mathcal{K}^e{}_a \mathcal{K}^d{}_c. \tag{7.24}$$

Equations (7.22), (7.23), and (7.24) form the first set of equations that will be later used in computing the unique connection. To get the other set of equations, we will start, now, with $\nabla_a \mathcal{J}_b{}^c = 0$ alone, and with some computations we will get two equations. Then, similar calculations will be done for $\nabla_a \mathcal{J}_b{}^c = 0$ and $\nabla_a \mathcal{K}_b{}^c = 0$, to obtain four other equations, which form a set of six equations along with the two equations we obtained from $\nabla_a \mathcal{J}_b{}^c = 0$. To be explicit, consider

$$\partial_a \mathcal{J}_b{}^c + \Gamma^c_{ae} \mathcal{J}^e{}_b - \Gamma^e_{ab} \mathcal{J}^c{}_e = 0, \tag{7.25}$$

multiplying this equation with $\mathcal{J}^d{}_c$ first, we get

$$\mathcal{J}^d{}_c \partial_a \mathcal{J}_b{}^c + \Gamma^c_{ae} \mathcal{J}^d{}_c \mathcal{J}^e{}_b + \Gamma^e_{ab} \mathcal{K}^d{}_e = 0, \tag{7.26}$$

multiplying this result with $\mathcal{K}^a{}_f$, we get

$$\mathcal{J}^d{}_c \partial_a \mathcal{J}_b{}^c \mathcal{K}^a{}_f + \Gamma^c_{ae} \mathcal{J}^d{}_c \mathcal{J}^e{}_b \mathcal{K}^a{}_f + \Gamma^e_{ab} \mathcal{K}^d{}_e \mathcal{K}^a{}_f = 0, \tag{7.27}$$

swapping b with f, we get

$$\mathcal{J}^d{}_c\partial_a\mathcal{J}_f{}^c\mathcal{K}^a{}_b + \Gamma^c_{ae}\mathcal{J}^d{}_c\mathcal{J}^e{}_f\mathcal{K}^a{}_b + \Gamma^e_{af}\mathcal{K}^d{}_e\mathcal{K}^a{}_b = 0, \tag{7.28}$$

adding the last two equations, gives

$$\mathcal{J}^d{}_c\partial_a\mathcal{J}_b{}^c\mathcal{K}^a{}_f + \mathcal{J}^d{}_c\partial_a\mathcal{J}_f{}^c\mathcal{K}^a{}_b + \Gamma^c_{ae}\mathcal{J}^d{}_c\mathcal{J}^e{}_b\mathcal{K}^a{}_f$$
$$+ \Gamma^c_{ae}\mathcal{J}^d{}_c\mathcal{J}^e{}_f\mathcal{K}^a{}_b + \Gamma^e_{ab}\mathcal{K}^d{}_e\mathcal{K}^a{}_f + \Gamma^e_{af}\mathcal{K}^d{}_e\mathcal{K}^a{}_b = 0\,. \tag{7.29}$$

Starting again from the same equation $\nabla_a\mathcal{J}_b{}^c = 0$, but now we multiply it first with $\mathcal{K}^d{}_c$, and then we multiply the result with $\mathcal{J}^a{}_f$ as follows:

$$\partial_a\mathcal{J}_b{}^c + \Gamma^c_{ae}\mathcal{J}^e{}_b - \Gamma^e_{ab}\mathcal{J}^c{}_e = 0, \tag{7.30}$$

$$\mathcal{K}^d{}_c\partial_a\mathcal{J}_b{}^c + \Gamma^c_{ae}\mathcal{K}^d{}_c\mathcal{J}^e{}_b - \Gamma^e_{ab}\mathcal{J}^d{}_e = 0, \tag{7.31}$$

$$\mathcal{K}^d{}_c\partial_a\mathcal{J}_b{}^c\mathcal{J}^a{}_f + \Gamma^c_{ae}\mathcal{K}^d{}_c\mathcal{J}^e{}_b\mathcal{J}^a{}_f - \Gamma^e_{ab}\mathcal{J}^d{}_e\mathcal{J}^a{}_f = 0, \tag{7.32}$$

swapping b with f,we get

$$\mathcal{K}^d{}_c\partial_a\mathcal{J}_f{}^c\mathcal{J}^a{}_b + \Gamma^c_{ae}\mathcal{K}^d{}_c\mathcal{J}^e{}_f\mathcal{J}^a{}_b - \Gamma^e_{af}\mathcal{J}^d{}_e\mathcal{J}^a{}_b = 0, \tag{7.33}$$

adding the last two equations, gives

$$\mathcal{K}^d{}_c\partial_a\mathcal{J}_b{}^c\mathcal{J}^a{}_f + \mathcal{K}^d{}_c\partial_a\mathcal{J}_f{}^c\mathcal{J}^a{}_b + \Gamma^c_{ae}\mathcal{K}^d{}_c\mathcal{J}^e{}_b\mathcal{J}^a{}_f$$
$$+ \Gamma^c_{ae}\mathcal{K}^d{}_c\mathcal{J}^e{}_f\mathcal{J}^a{}_b - \Gamma^e_{ab}\mathcal{J}^d{}_e\mathcal{J}^a{}_f - \Gamma^e_{af}\mathcal{J}^d{}_e\mathcal{J}^a{}_b = 0. \tag{7.34}$$

We are going to do computations similar to that done for $\nabla_a\mathcal{J}_b{}^c = 0$, but starting now from $\nabla_a\mathcal{J}_b{}^c = 0$, and the two equations we obtain that are analogous to equations (7.29) and (7.34) are :

$$\mathcal{K}^d{}_c\partial_a\mathcal{J}_b{}^c\mathcal{J}^a{}_f + \mathcal{K}^d{}_c\partial_a\mathcal{J}_f{}^c\mathcal{J}^a{}_b + \Gamma^c_{ae}\mathcal{K}^d{}_c\mathcal{J}^e{}_b\mathcal{J}^a{}_f$$
$$+ \Gamma^c_{ae}\mathcal{K}^d{}_c\mathcal{J}^e{}_f\mathcal{J}^a{}_b + \Gamma^e_{ab}\mathcal{J}^d{}_e\mathcal{J}^a{}_f + \Gamma^e_{af}\mathcal{J}^d{}_e\mathcal{J}^a{}_b = 0, \tag{7.35}$$

and

$$\mathcal{I}^d{}_c\partial_a\mathcal{I}_b{}^c\mathcal{K}^a{}_f + \mathcal{I}^d{}_c\partial_a\mathcal{I}_f{}^c\mathcal{K}^a{}_b + \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{I}^e{}_b\mathcal{K}^a{}_f$$
$$+ \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{I}^e{}_f\mathcal{K}^a{}_b - \Gamma^e_{ab}\mathcal{K}^d{}_e\mathcal{K}^a{}_f - \Gamma^e_{af}\mathcal{K}^d{}_e\mathcal{K}^a{}_b = 0. \quad (7.36)$$

Similarly from $\nabla_a\mathcal{K}_b{}^c = 0$, we obtain the following two equations:

$$\mathcal{I}^d{}_c\partial_a\mathcal{K}_b{}^c\mathcal{I}^a{}_f + \mathcal{I}^d{}_c\partial_a\mathcal{K}_f{}^c\mathcal{I}^a{}_b + \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{K}^e{}_b\mathcal{I}^a{}_f$$
$$+ \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{K}^e{}_f\mathcal{I}^a{}_b + \Gamma^e_{ab}\mathcal{I}^d{}_e\mathcal{I}^a{}_f + \Gamma^e_{af}\mathcal{I}^d{}_e\mathcal{I}^a{}_b = 0, \quad (7.37)$$

and

$$\mathcal{I}^d\partial_a\mathcal{K}_b{}^c\mathcal{I}^a{}_f + \mathcal{I}^d\partial_a\mathcal{K}_f{}^c\mathcal{I}^a{}_b + \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{K}^e{}_b\mathcal{I}^a{}_f$$
$$+ \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{K}^e{}_f\mathcal{I}^a{}_b - \Gamma^e_{ab}\mathcal{I}^d{}_e\mathcal{I}^a{}_f - \Gamma^e_{af}\mathcal{I}^d{}_e\mathcal{I}^a{}_b = 0. \quad (7.38)$$

The six equations (7.29), (7.34), (7.35), (7.36), (7.37) and (7.38), will be combined to give a new set of new equations in which, later, we will substitute (7.22), (7.23) and (7.24), and then will be added to give the Obata connection. Subtracting (7.36) from (7.37) gives

$$\mathcal{I}^d{}_c\partial_a\mathcal{K}_b{}^c\mathcal{I}^a{}_f + \mathcal{I}^d{}_c\partial_a\mathcal{K}^c{}_f\mathcal{I}^a{}_b + \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{K}^e{}_b\mathcal{I}^a{}_f + \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{K}^e{}_f\mathcal{I}^a{}_b$$
$$+\Gamma^e_{ab}\mathcal{I}^d{}_e\mathcal{I}^a{}_f + \Gamma^e_{af}\mathcal{I}^d{}_e\mathcal{I}^a{}_b - \mathcal{I}^d{}_c\partial_a\mathcal{I}_b{}^c\mathcal{K}^a{}_f - \mathcal{I}^d{}_c\partial_a\mathcal{I}^c{}_f\mathcal{K}^a{}_b$$
$$-\Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{I}^e{}_b\mathcal{K}^a{}_f - \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{I}^e{}_b\mathcal{K}^a{}_f + \Gamma^e_{ab}\mathcal{K}^d{}_e\mathcal{K}^a{}_f + \Gamma^e_{af}\mathcal{K}^d{}_e\mathcal{K}^a{}_b = 0, \quad (7.39)$$

and relabeling indices as follows

$$\Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{I}^e{}_b\mathcal{K}^a{}_f = \Gamma^c_{ea}\mathcal{I}^d{}_c\mathcal{I}^a{}_b\mathcal{K}^e{}_f = \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{I}^a{}_b\mathcal{K}^e{}_f \quad (7.40)$$

$$\Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{I}^e{}_b\mathcal{K}^a{}_f = \Gamma^c_{ea}\mathcal{I}^d{}_c\mathcal{I}^a{}_b\mathcal{K}^e{}_f = \Gamma^c_{ae}\mathcal{I}^d{}_c\mathcal{I}^a{}_b\mathcal{K}^e{}_f, \quad (7.41)$$

equation (7.39) becomes

$$\mathcal{I}^d{}_c\partial_a\mathcal{K}_b{}^c\mathcal{J}^a{}_f + \mathcal{I}^d{}_c\partial_a\mathcal{K}_f{}^c\mathcal{J}^a{}_b - \mathcal{I}^d{}_c\partial_a\mathcal{J}_b{}^c\mathcal{K}^a{}_f - \mathcal{I}^d{}_c\partial_a\mathcal{J}_f{}^c\mathcal{K}^a{}_b$$

$$+\Gamma^e_{ab}\mathcal{J}^d{}_e\mathcal{J}^a{}_f + \Gamma^e_{af}\mathcal{J}^d{}_e\mathcal{J}^a{}_b + \Gamma^e_{ab}\mathcal{K}^d{}_e\mathcal{K}^a{}_f + \Gamma^e_{af}\mathcal{K}^d{}_e\mathcal{K}^a{}_b = 0. \qquad (7.42)$$

Substituting (7.23) and (7.24) into (7.42), we get

$$\mathcal{I}^d{}_c\partial_a\mathcal{K}_b{}^c\mathcal{J}^a{}_f + \mathcal{I}^d{}_c\partial_a\mathcal{K}^c{}_f\mathcal{J}^a{}_b - \mathcal{I}^d{}_c\partial_a\mathcal{J}_b{}^c\mathcal{K}^a{}_f - \mathcal{I}^d{}_c\partial_a\mathcal{J}^c{}_f\mathcal{K}^a{}_b \qquad (7.43)$$

$$= 4\Gamma^d_{bf} + \mathcal{J}^d{}_c\partial_f\mathcal{J}_b{}^c + \mathcal{J}^d{}_c\partial_b\mathcal{J}_f{}^c + \mathcal{K}^d{}_c\partial_f\mathcal{K}_b{}^c + \mathcal{K}^d{}_c\partial_b\mathcal{K}_f{}^c \qquad (7.44)$$

or

$$4\Gamma^d_{bf} = -\mathcal{J}^d{}_c\partial_f\mathcal{J}_b{}^c - \mathcal{J}^d{}_c\partial_b\mathcal{J}_f{}^c - \mathcal{K}^d{}_c\partial_f\mathcal{K}_b{}^c - \mathcal{K}^d{}_c\partial_b\mathcal{K}_f{}^c$$

$$+\mathcal{I}^d{}_c\partial_a\mathcal{K}_b{}^c\mathcal{J}^a{}_f + \mathcal{I}^d{}_c\partial_a\mathcal{K}^c{}_f\mathcal{J}^a{}_b - \mathcal{I}^d{}_c\partial_a\mathcal{J}_b{}^c\mathcal{K}^a{}_f - \mathcal{I}^d{}_c\partial_a\mathcal{J}^c{}_f\mathcal{K}^a{}_b. \qquad (7.45)$$

Second, subtracting (7.34) from (7.35) gives

$$\mathcal{K}^d{}_c\partial_a\mathcal{J}_b{}^c\mathcal{J}^a{}_f + \mathcal{K}^d{}_c\partial_a\mathcal{J}_f{}^c\mathcal{J}^a{}_b + \Gamma^e_{ab}\mathcal{J}^d{}_e\mathcal{J}^a{}_f + \Gamma^e_{af}\mathcal{J}^d{}_e\mathcal{J}^a{}_b$$

$$-\mathcal{K}^d{}_c\partial_a\mathcal{J}_b{}^c\mathcal{J}^a{}_f - \mathcal{K}^d{}_c\partial_a\mathcal{J}_f{}^c\mathcal{J}^a{}_b + \Gamma^e_{ab}\mathcal{J}^d{}_e\mathcal{J}^a{}_f + \Gamma^e_{af}\mathcal{J}^d{}_e\mathcal{J}^a{}_b = 0, \qquad (7.46)$$

and substituting (7.22) and (7.23) in (7.46), we get

$$4\Gamma^d_{fb} = -\mathcal{I}^d{}_c\partial_f\mathcal{I}_b{}^c - \mathcal{I}^d{}_c\partial_b\mathcal{I}_f{}^c - \mathcal{J}^d{}_c\partial_f\mathcal{J}_b{}^c - \mathcal{J}^d{}_c\partial_b\mathcal{J}_f{}^c$$

$$+\mathcal{K}^d{}_c\partial_a\mathcal{J}_b{}^c\mathcal{J}^a{}_f + \mathcal{K}^d{}_c\partial_a\mathcal{J}_f{}^c\mathcal{J}^a{}_b - \mathcal{K}^d{}_c\partial_a\mathcal{I}_b{}^c\mathcal{J}^a{}_f - \mathcal{K}^d{}_c\partial_a\mathcal{I}_f{}^c\mathcal{J}^a{}_b. \qquad (7.47)$$

Now, subtracting (7.38) from (7.29) gives

$$\mathcal{J}^d{}_c\partial_a\mathcal{I}_b{}^c\mathcal{K}^a{}_f + \mathcal{J}^d{}_c\partial_a\mathcal{I}_f{}^c\mathcal{K}^a{}_b - \mathcal{J}^d\partial_a\mathcal{K}_b{}^c\mathcal{J}^a{}_f - \mathcal{J}^d\partial_a\mathcal{K}_f{}^c\mathcal{J}^a{}_b$$

$$+\Gamma^e_{ab}\mathcal{K}^d{}_e\mathcal{K}^a{}_f + \Gamma^e_{af}\mathcal{K}^d{}_e\mathcal{K}^a{}_b + \Gamma^e_{ab}\mathcal{J}^d{}_e\mathcal{J}^a{}_f + \Gamma^e_{af}\mathcal{J}^d{}_e\mathcal{J}^a{}_b = 0, \qquad (7.48)$$

and substituting (7.22) and (7.24) into (7.48), we get

$$4\Gamma_{bf}^{d} = -\mathcal{I}^{d}{}_{c}\partial_{f}\mathcal{I}_{b}{}^{c} - \mathcal{I}^{d}{}_{c}\partial_{b}\mathcal{I}_{f}{}^{c} - \mathcal{K}^{d}{}_{c}\partial_{f}\mathcal{K}_{b}{}^{c} - \mathcal{K}^{d}{}_{c}\partial_{b}\mathcal{K}_{f}$$

$$+\mathcal{I}^{d}{}_{c}\partial_{a}\mathcal{I}_{b}{}^{c}\mathcal{K}^{a}{}_{f} + \mathcal{I}^{d}{}_{c}\partial_{a}\mathcal{I}_{f}{}^{c}\mathcal{K}^{a}{}_{b} - \mathcal{I}^{d}\partial_{a}\mathcal{K}_{b}{}^{c}\mathcal{I}^{a}{}_{f} - \mathcal{I}^{d}\partial_{a}\mathcal{K}_{f}{}^{c}\mathcal{I}^{a}{}_{b} \qquad (7.49)$$

Finally, adding (7.45), (7.47) and (7.49) yields the formula (7.11) for the Obata connection.

7.3   Reduction of the supersymmetry transformations

Using the definitions defined in section $(4.3.2)$ the supersymmetry transformations $(4.4, 4.5)$ is reduced, in this section, from $\mathbb{R}^{(5,1)}$ to $\mathbb{R}^{3}$.

Starting, first, with the variation of the bosonic fields

$$\delta W_{i} = \bar{\varepsilon}\Gamma_{i}\Psi - \bar{\Psi}\Gamma_{i}\varepsilon,$$

we have

$$\bar{\varepsilon} = \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \varepsilon_{1}^{\dagger} \otimes \begin{pmatrix} 0 & -i \end{pmatrix} + \begin{pmatrix} -1 & 0 \end{pmatrix} \otimes \varepsilon_{2}^{\dagger} \otimes \begin{pmatrix} 0 & -i \end{pmatrix}$$

$$\Gamma^{i}\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \gamma_{i}\psi_{1} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \gamma_{i}\psi_{2} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

hence

$$\bar{\varepsilon}\Gamma^{i}\Psi = -i\varepsilon_{1}^{\dagger}\gamma_{i}\psi_{2} + i\varepsilon_{2}^{\dagger}\gamma_{i}\psi_{1}.$$

similarly

$$\bar{\Psi}\Gamma_{i}\varepsilon = -i\psi_{1}^{\dagger}\gamma_{i}\varepsilon_{2} + i\psi_{2}^{\dagger}\gamma_{i}\varepsilon_{1},$$

Therefore

$$\delta W_{i} = -i\varepsilon_{1}^{\dagger}\gamma_{i}\psi_{2} + i\varepsilon_{2}^{\dagger}\gamma_{i}\psi_{1} + i\psi_{1}^{\dagger}\gamma_{i}\varepsilon_{2} - i\psi_{2}^{\dagger}\gamma_{i}\varepsilon_{1}.$$

The variation of the zero component of the scalar field gives

$$\delta\phi_0 = \bar{\varepsilon}\Gamma_0\Psi - \bar{\Psi}\Gamma_0\varepsilon$$

$$= \Psi^\dagger\varepsilon - \varepsilon^\dagger\Psi$$

$$= \{\begin{pmatrix} 1 & 0 \end{pmatrix}\otimes\psi_1^\dagger\otimes\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix}\otimes\psi_2^\dagger\otimes\begin{pmatrix} 1 & 0 \end{pmatrix}\}.\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\varepsilon_1\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\varepsilon_2\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

$$-\{\begin{pmatrix} 1 & 0 \end{pmatrix}\otimes\varepsilon_1^\dagger\otimes\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix}\otimes\varepsilon_2^\dagger\otimes\begin{pmatrix} 1 & 0 \end{pmatrix}\}.\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\psi_1\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\psi_2\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

$$= \psi_1^\dagger\varepsilon_1 + \psi_2^\dagger\varepsilon_2 - \varepsilon_1^\dagger\psi_1 - \varepsilon_2^\dagger\psi_2.$$

The variation of the fourth component of the gauge field is given by

$$\delta\phi_4 = \bar{\varepsilon}\Gamma_4\Psi - \bar{\Psi}\Gamma_4\varepsilon.$$

We have

$$\bar{\varepsilon}\Gamma_4 = \{\begin{pmatrix} 0 & 1 \end{pmatrix}\otimes\varepsilon_1^\dagger\otimes\begin{pmatrix} 0 & -i \end{pmatrix} + \begin{pmatrix} -1 & 0 \end{pmatrix}\otimes\varepsilon_2^\dagger\otimes\begin{pmatrix} 0 & -i \end{pmatrix}\}.\{\sigma_1\otimes\mathbb{I}_2\otimes\sigma_2\}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix}\otimes\varepsilon_1^\dagger\otimes\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix}\otimes\varepsilon_2^\dagger\otimes\begin{pmatrix} 1 & 0 \end{pmatrix},$$

then

$$\bar{\varepsilon}\Gamma_4\Psi = \{\begin{pmatrix} 1 & 0 \end{pmatrix}\otimes\varepsilon_1^\dagger\otimes\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix}\otimes\varepsilon_2^\dagger\otimes\begin{pmatrix} 1 & 0 \end{pmatrix}\}.\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\psi_1\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\psi_2\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

$$= \varepsilon_1^\dagger\psi_1 - \varepsilon_2^\dagger\psi_2.$$

Similarly

$$\bar{\Psi}\Gamma_4\varepsilon = \psi_1^\dagger\varepsilon_1 - \psi_2^\dagger\varepsilon_2,$$

therefore

$$\delta\phi_4 = \bar{\varepsilon}\Gamma_4\Psi - \bar{\Psi}\Gamma_4\varepsilon$$

$$= \varepsilon_1^\dagger\psi_1 - \varepsilon_2^\dagger\psi_2 - \psi_1^\dagger\varepsilon_1 + \psi_2^\dagger\varepsilon_2.$$

The last component of the bosonic fields $\delta\phi_5$, we have

$$\delta\phi_5 = \bar{\varepsilon}\Gamma_5\Psi - \bar{\Psi}\Gamma_5\varepsilon,$$

We have

$$\Gamma_5\Psi = \{\sigma_3 \otimes \mathbb{I}_2 \otimes \sigma_2\}.\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \psi_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \psi_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \psi_1 \otimes \begin{pmatrix} 0 \\ i \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \otimes \psi_2 \otimes \begin{pmatrix} 0 \\ i \end{pmatrix},$$

then

$$\bar{\varepsilon}\Gamma_5\Psi = \{\begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \varepsilon_1^\dagger \otimes \begin{pmatrix} 0 & -i \end{pmatrix} + \begin{pmatrix} -1 & 0 \end{pmatrix} \otimes \varepsilon_2^\dagger \otimes \begin{pmatrix} 0 & -i \end{pmatrix}\}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \psi_1 \otimes \begin{pmatrix} 0 \\ i \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \otimes \psi_2 \otimes \begin{pmatrix} 0 \\ i \end{pmatrix}\}$$

$$= -\varepsilon_1^\dagger\psi_2 - \varepsilon_2^\dagger\psi_1.$$

Similarly

$$\bar{\Psi}\Gamma_5\varepsilon = -\psi_1^\dagger\varepsilon_2 - \psi_2^\dagger\varepsilon_1,$$

therefore

$$\delta\phi_5 = -\varepsilon_1^\dagger\psi_2 - \varepsilon_2^\dagger\psi_1 + \psi_1^\dagger\varepsilon_2 + \psi_2^\dagger\varepsilon_1.$$

The final piece is the reduction of the fermionic field variation, in six dimensional terms we have

$$\delta\Psi = \Gamma^{AB}\varepsilon G_{AB}. \tag{7.50}$$

We'll start with the first term,

$$\Gamma^{AB}\varepsilon G_{AB} = \Gamma^{\mu\nu}\varepsilon G_{\mu\nu} + \Gamma^{ij}\varepsilon G_{ij} + 2\Gamma^{\mu i}\varepsilon G_{\mu i}, \tag{7.51}$$

which requires finding the spin generators

$$\Gamma^{ij} = \mathbb{I}_2 \otimes \gamma^{ij} \otimes \mathbb{I}_2,$$

$$\Gamma^{\mu\nu} = \bar{\gamma}^{\mu\nu} \otimes \mathbb{I}_2 \otimes \mathbb{I}_2,$$

$$\Gamma^{i\mu} = \Gamma^i \Gamma^\mu = \bar{\gamma}^\mu \otimes \gamma^i \otimes i\sigma_3.$$

The first part of equation (7.51) gives

$$\Gamma^{ij}\varepsilon G_{ij} = \{\mathbb{I}_2 \otimes \gamma^{ij} \otimes \mathbb{I}_2\}.\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \varepsilon_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \varepsilon_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}G_{\mu\nu}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \gamma^{ij} G_{ij}\varepsilon_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \gamma^{ij} G_{ij}\varepsilon_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^{ij} G_{ij}\varepsilon_1 \\ \gamma^{ij} G_{ij}\varepsilon_2 \\ 0 \\ 0 \end{pmatrix}. \tag{7.52}$$

The second part of equation (7.51) gives

$$\Gamma^{\mu\nu}\varepsilon G_{\mu\nu} = \{\bar{\gamma}^{\mu\nu} \otimes \mathbb{I}_2 \otimes \mathbb{I}_2\}.\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \varepsilon_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \varepsilon_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}G_{\mu\nu}$$

$$= \{\bar{\gamma}^{\mu\nu}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \varepsilon_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \bar{\gamma}^{\mu\nu}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \varepsilon_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}G_{\mu\nu}$$

We have

$$\bar{\gamma}_{04} = \sigma_3, \qquad\qquad \bar{\gamma}_{05} = -\sigma_1, \qquad\qquad \bar{\gamma}_{45} = -i\sigma_2,$$

then

$$\Gamma^{\mu\nu}\varepsilon G_{\mu\nu} = 2\{\begin{pmatrix}-1\\0\end{pmatrix}\otimes\varepsilon_1\otimes\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\otimes\varepsilon_2\otimes\begin{pmatrix}1\\0\end{pmatrix}\}G_{04}$$

$$+ 2\{\begin{pmatrix}0\\1\end{pmatrix}\otimes\varepsilon_1\otimes\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\otimes\varepsilon_2\otimes\begin{pmatrix}1\\0\end{pmatrix}\}G_{05}$$

$$+ 2\{\begin{pmatrix}0\\1\end{pmatrix}\otimes\varepsilon_1\otimes\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix}\otimes\varepsilon_2\otimes\begin{pmatrix}1\\0\end{pmatrix}\}G_{45}$$

$$= 2\begin{pmatrix}-\varepsilon_1[\phi_0,\phi_4] + \varepsilon_2[\phi_0,\phi_5] - \varepsilon_2[\phi_4,\phi_5]\\ \varepsilon_2[\phi_0,\phi_4] + \varepsilon_1[\phi_0,\phi_5] + \varepsilon_1[\phi_4,\phi_5]\\ 0\\ 0\end{pmatrix}.$$

As for the last term in equation (7.51), we get

$$\Gamma^{i\mu}\varepsilon G_{i\mu} = \{\bar{\gamma}^{\mu}\otimes\gamma^{i}\otimes i\sigma_3\}\{\begin{pmatrix}1\\0\end{pmatrix}\otimes\varepsilon_1\otimes\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\otimes\varepsilon_2\otimes\begin{pmatrix}1\\0\end{pmatrix}\}G_{i\mu}$$

$$= \{\bar{\gamma}^{\mu}\begin{pmatrix}1\\0\end{pmatrix}\otimes\gamma^{i}\varepsilon_1\otimes\begin{pmatrix}i\\0\end{pmatrix} + \bar{\gamma}^{\mu}\begin{pmatrix}0\\1\end{pmatrix}\otimes\gamma^{i}\varepsilon_2\otimes\begin{pmatrix}i\\0\end{pmatrix}\}G_{i\mu} \qquad (7.53)$$

The components of equation (7.53) are as follows

$$\Gamma^{i0}\varepsilon G_{i0} = \{\begin{pmatrix}0\\1\end{pmatrix}\otimes\gamma^{i}\varepsilon_1\otimes\begin{pmatrix}i\\0\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix}\otimes\gamma^{i}\varepsilon_2\otimes\begin{pmatrix}i\\0\end{pmatrix}\}G_{i0}$$

$$= \begin{pmatrix}-i\gamma^{i}\varepsilon_2 D_i\phi_0\\ i\gamma^{i}\varepsilon_1 D_i\phi_0\\ 0\\ 0\end{pmatrix}.$$

Also we have

$$\Gamma^{i4}\varepsilon G_{i4} = \{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \gamma^i \varepsilon_1 \otimes \begin{pmatrix} i \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \gamma^i \varepsilon_2 \otimes \begin{pmatrix} i \\ 0 \end{pmatrix} \} G_{i4}$$

$$= \begin{pmatrix} i\gamma^i \varepsilon_2 D_i \phi_4 \\ i\gamma^i \varepsilon_1 D_i \phi_4 \\ 0 \\ 0 \end{pmatrix},$$

and similarly

$$\Gamma^{i5}\varepsilon G_{i5} = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \gamma^i \varepsilon_1 \otimes \begin{pmatrix} i \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \otimes \gamma^i \varepsilon_2 \otimes \begin{pmatrix} i \\ 0 \end{pmatrix} \} G_{i5}$$

$$= \begin{pmatrix} i\gamma^i \varepsilon_1 D_i \phi_5 \\ -i\gamma^i \varepsilon_2 D_i \phi_5 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore

$$\Gamma^{i\mu}\varepsilon G_{i\mu} = \begin{pmatrix} -i\gamma^i \varepsilon_2 D_i \phi_0 \\ i\gamma^i \varepsilon_1 D_i \phi_0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} i\gamma^i \varepsilon_2 D_i \phi_4 \\ i\gamma^i \varepsilon_1 D_i \phi_4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} i\gamma^i \varepsilon_1 D_i \phi_5 \\ -i\gamma^i \varepsilon_2 D_i \phi_5 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} i\gamma^i \{\varepsilon_2 D_i(\phi_4 - \phi_0) + \varepsilon_1 D_\mu \phi_5\} \\ i\gamma^i \{\varepsilon_1 D_i(\phi_4 + \phi_0) - \varepsilon_2 D_i \phi_5\} \\ 0 \\ 0 \end{pmatrix}. \tag{7.54}$$

Inserting equations $(7.52,\ 7.53,\ 7.54)$ in (7.50), we get

$$\delta\psi_1 = \gamma^{\mu\nu}G_{\mu\nu}\varepsilon_1 + 2\{-\varepsilon_1[\phi_0,\phi_4] - \varepsilon_2[\phi_4 - \phi_0,\phi_5]\}$$

$$+ 2i\gamma^\mu\{\varepsilon_2 D_\mu(\phi_4 - \phi_0) + \varepsilon_1 D_\mu\phi_5\},$$

$$\delta\psi_2 = \gamma^{\mu\nu}G_{\mu\nu}\varepsilon_2 + 2\{\varepsilon_2[\phi_0,\phi_4] + \varepsilon_1[\phi_4 + \phi_0,\phi_5]\}$$

$$+ 2i\gamma^\mu\{\varepsilon_1 D_\mu(\phi_4 + \phi_0) - \varepsilon_2 D_\mu\phi_5\}.$$

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