Existence of anti-Newtonian solutions in fourth-order gravity

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Abstract. We use the covariant consistency analysis formalism to show the existence of a class of perfect-fluid cosmological solutions, known as *anti-Newtonian universes*, in the context of modified gravity. There are no known linearized solutions in General Relativity, but the integrability conditions for such models in $f(R)$ gravity are presented.

1. Introduction

The phenomenological discovery that the Universe is currently undergoing accelerated expansion has attracted numerous theoretical alternatives to General Relativity (GR). Among such alternatives are fourth-order theories of gravity. Generally obtained by including higher-order curvature invariants in the Hilbert-Einstein action, or by making the action nonlinear in the Ricci curvature R or contain terms involving combinations of derivatives of R, these models attempt to address the shortcomings of GR in the infrared (IR) and ultraviolet (UV) ranges, i.e., very low and very high energy scales. The simplest, and perhaps most studied, of these models are the $f(R)$ models of gravitation. One advantage of such models is that they have enough freedom to explain early-universe inflation [1], late-time acceleration [2] and the cosmic evolution history in between [3, 4, 5]. As such freedom does not come without a cost, i.e., viable models should be constrained to conform with observational predictions and physically motivated mathematical restrictions, rigorous consistency analyses of the linearized field equations in such theories need to be made. Such initiatives have been undertaken in recent works [6, 7, 8, 9] for some classes of cosmological solutions with an underlying Friedman-Lemaître-Robertson-Walker (FLRW) background.

In GR, irrotational dust spacetimes have been studied as potential models for the description of gravitational collapse and late-time cosmic structure formation. The locally free gravitational field in such spacetimes is covariantly described by the gravito-electric and gravito-magnetic tensors, E_{ab} and H_{ab} , which, respectively, are responsible for the tidal tensor in the classical Newtonian gravitational theory and gravitational radiation in relativistic theories. Purely gravito-electric irrotational dust spacetimes are usually referred to as *Newtonian-like* or *quasi-Newtonian* universes, whereas those classes of irrotational dust universes with purely gravito-magnetic Weyl tensor are called *anti-Newtonian* universes. Consistency analyses of the general relativistic propagation and constraint equations arising from imposing external restrictions such as shear-free assumptions do generally show that such models suffer from severe integrability conditions [10, 11, 12, 13, 14] leading to the conclusion that there are no anti-Newtonian spacetimes that are linearized perturbations of FLRW universes. Our present work shows that fourth-order gravitational theories, that generally assume non-vanishing anisotropic pressure and heat-flux terms, allow such solutions.

2. Linearized anti-Newtonian field equations

If we generalize the Hilbert-Einstein action by making the geometric contribution to the Lagrangian a generic function of the Ricci scalar, R, we obtain the $f(R)$ gravitational action

$$\mathcal{A} = \tfrac{1}{2}\int d^4x\sqrt{-g}\left[f(R) + 2\mathcal{L}_m\right] \ , \tag{1}$$

which, upon the application of the variation principle, yields the generalized Einstein field equations [1]

$$f' G_{ab} = T^m_{ab} + \tfrac{1}{2}(f - Rf')g_{ab} + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f' \ . \tag{2}$$

Here $\mathcal{L}_m$ is Lagrangian for standard matter whose energy-momentum tensor (EMT) is given by

$$T^m_{ab} = \mu_m u_a u_b + p_m h_{ab} + q^m_a u_b + q^m_b u_a + \pi^m_{ab} \ , \tag{3}$$

where μ_m, p_m, q^m_a and π^m_{ab} are matter energy density, pressure, heat flux and anisotropic pressure respectively of matter, u^a is a normalized 4-velocity vector of fundamental observers, and h_{ab} is used to define the fully orthogonally projected covariant derivative operator $\tilde{\nabla}$. Eq. (2) shows that the standard Einstein field equations get modified due to the addition of the extra geometric terms ($f(R)$ and its derivatives) to the matter on the right hand side. These extra terms can be considered to make up the *curvature fluid* contributions to the EMT whose energy density, pressure, heat flux and anisotropic pressure are given by

$$\mu_R = \frac{1}{f'}\left[\tfrac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right] \ , \tag{4}$$

$$p_R = \frac{1}{f'}\left[\tfrac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \tfrac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R\right)\right] \ , \tag{5}$$

$$q^R_a = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \tfrac{1}{3}f''\Theta\tilde{\nabla}_a R\right] \ , \tag{6}$$

$$\pi^R_{ab} = \frac{f''}{f'}\left[\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\dot{R}\right] \ , \tag{7}$$

where Θ and σ_{ab} are the expansion scalar and the shear tensor obtained from the full covariant derivative of u^a as follows:

$$\nabla_a u_b = -A_a u_b + \tfrac{1}{3}h_{ab}\Theta + \sigma_{ab} + \varepsilon_{abc}\omega^c. \tag{8}$$

Here $A_a = \dot{u}_a$ and $\omega^a = \varepsilon^{abc}\tilde{\nabla}_b u_c$ are the 4-acceleration and vorticity vectors. The angular brackets in Eq. (7) denote orthogonal projections of vectors and tensors. The volume element for the 3-rest spaces orthogonal to u^a is defined by [2]

$$\varepsilon_{abc} = u^d\eta_{dabc} = -\sqrt{|g|}\delta^0_{[a}\delta^1_b\delta^2_c\delta^3_{d]}u^d \Rightarrow \varepsilon_{abc} = \varepsilon_{[abc]}, \ \varepsilon_{abc}u^c = 0, \tag{9}$$

[1] We have used the shorthands $f = f(R)$, $f' \equiv df/dR$, etc. Moreover, we use ∇, $\tilde{\nabla}$ and an overhead dot $\dot{}$ to denote the full covariant derivative, the covariant 3-spatial derivative and differentiation with respect to cosmic time t, respectively.

[2] In this work, round brackets (ab) indicate symmetrization over the indices a and b whereas square brackets $[ab]$ denote anti-symmetrization over these indices.

where η_{abcd} is the 4-dimensional volume element such that

$$\eta_{abcd} = \eta_{[abcd]} = 2\varepsilon_{ab[c}u_{d]} - 2u_{[a}\varepsilon_{b]cd}. \tag{10}$$

We define the covariant spatial divergence and curl of vectors and tensors as

$$\text{div}V = \tilde{\nabla}^a V_a \,, \qquad (\text{div}S)_a = \tilde{\nabla}^b S_{ab} \,, \tag{11}$$

$$curlV_a = \varepsilon_{abc}\tilde{\nabla}^b V^c \,, \quad curlS_{ab} = \varepsilon_{cd(a}\tilde{\nabla}^c S_{b)}{}^d \,. \tag{12}$$

Covariant spatial derivative operators acting on any scalar field ϕ obey the commutation relation

$$[\tilde{\nabla}_a\tilde{\nabla}_b - \tilde{\nabla}_b\tilde{\nabla}_a]\phi = 2\varepsilon_{abc}\omega^c\dot{\phi} \,, \tag{13}$$

and to linear order, the following differential relations apply:

$$[\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\nabla}_a - \tilde{\nabla}_b\tilde{\nabla}^2]\phi = \tfrac{1}{3}\tilde{R}\tilde{\nabla}_b\phi \,, \tag{14}$$

$$[\tilde{\nabla}^2\tilde{\nabla}_b - \tilde{\nabla}_b\tilde{\nabla}^2]\phi = \tfrac{1}{3}\tilde{R}\tilde{\nabla}_b\phi + 2\varepsilon_{dbc}\tilde{\nabla}^d(\omega^c\dot{\phi}) \,, \tag{15}$$

where $\tilde{R} = \frac{6K}{a^2} = 2\left(\mu - \frac{1}{3}\Theta^2\right)$ is the 3-curvature scalar. $K = -1, 0$ or 1 depending on whether the Universe is *open*, *flat* or *closed* and $a = a(t)$ is the cosmological scale factor. μ is the energy density of the total cosmic fluid. Moreover, the curl of the spatial gradient of a scalar field is given by

$$\varepsilon^{abc}\tilde{\nabla}_b\tilde{\nabla}_c\phi = 2\omega^a\dot{\phi} \,. \tag{16}$$

Using the $(-+++)$ spacetime signature, the Riemann tensor is given by

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd}\Gamma^a_{ce} - \Gamma^f_{bc}\Gamma^a_{df} \tag{17}$$

together with the Christoffel symbols, Ricci tensor and Ricci scalar respectively given as

$$\Gamma^a_{bd} = \frac{1}{2}g^{ae}\left(g_{be,d} + g_{ed,b} - g_{bd,e}\right) \,, \qquad R_{ab} = g^{cd}R_{cadb} \,, \qquad R = R^a{}_a \,. \tag{18}$$

The Weyl conformal curvature tensor C_{abcd} is defined from the trace-free part of the Riemann tensor as

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3}g^{[a}{}_{[c}g^{b]}{}_{d]} \,, \tag{19}$$

and is usually split into its "gravito-electric" and "gravito-magnetic" parts, E_{ab} and H_{ab}, respectively given by

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} = \tfrac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d. \tag{20}$$

Now the *total* thermodynamic quantities of the matter-curvature fluid composition can be defined as

$$\mu \equiv \frac{\mu_m}{f'} + \mu_R \,, \quad p \equiv \frac{p_m}{f'} + p_R \,, \quad q_a \equiv \frac{q^m_a}{f'} + q^R_a \,, \quad \pi_{ab} \equiv \frac{\pi^m_{ab}}{f'} + \pi^R_{ab} \,. \tag{21}$$

Anti-Newtonian universes are special classes of irrotational dust spacetimes characterized by

$$p_m = 0 \,, \quad A_a = 0 \,, \quad q^m_a = 0 \,, \omega_a = 0 \,, \quad \pi^m_{ab} = 0 \,, \quad E_{ab} = 0 \,. \tag{22}$$

Putting all these together, the generalized propagation relations of the Einstein field equations for generic $f(R)$ gravitation describing anti-Newtonian universes can be given as [12, 15]

$$\dot{\mu}_m = -\mu_m \Theta \ , \tag{23}$$
$$\dot{\mu}_R = -(\mu_R + p_R)\Theta + \frac{\mu_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q^R_a \ , \tag{24}$$
$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) \ , \tag{25}$$
$$\dot{q}^R_a = -\tfrac{4}{3}\Theta q^R_a + \frac{\mu_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi^R_{ab} \ , \tag{26}$$
$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} + \tfrac{1}{2}\pi^R_{ab} \ , \tag{27}$$
$$\dot{\pi}^R_{ab} = 2\varepsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - (\mu + p_R)\,\sigma_{ab} - \tilde{\nabla}_{\langle a} q^R_{b\rangle} - \tfrac{1}{3}\Theta\pi^R_{ab} \ , \tag{28}$$
$$\dot{H}_{ab} = -\Theta H_{ab} + \tfrac{1}{2}\varepsilon_{cd\langle a}\tilde{\nabla}^c \pi^{R\ d}_{b\rangle} \ . \tag{29}$$

The role of Eqs. (23)-(29) is to make sure that the covariant variables on some initial hypersurface S_0 are uniquely determined. The corresponding constraint relations

$$(C^1)_a := \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + q^R_a = 0 \ , \tag{30}$$

$$(C^2)_{ab} := \varepsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^d - H_{ab} = 0 \ , \tag{31}$$
$$(C^3)_a := \tilde{\nabla}^b H_{ab} + \tfrac{1}{2}\varepsilon_{abc}\tilde{\nabla}^b q^c_R = 0 \ . \tag{32}$$

$$(C^4)_a := \tilde{\nabla}^b\pi^R_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\mu + \tfrac{2}{3}\Theta q^R_a = 0 \ , \tag{33}$$

put restrictions on the initial data to be specified and must remain satisfied on any hypersurface S_t for all comoving time t .

In GR, the evolution equations (23)-(29) decouple from the gradient, divergence and curl terms, forming ordinary differential evolution equations. As a result, anti-Newtonian cosmologies in GR are said to be *silent* models because the flowlines emerging from the initial hypersurface S_0 evolve separately from each other [11, 12, 14]. Interestingly, however, the non-vanishing of the curvature anisotropic pressure and total heat flux in Eqs. (24), (26), (28) and (29) shows that *anti-Newtonian solutions in $f(R)$ gravity are not silent* models. The curl terms in Eqs. (29) and (32) vanish if we apply the differential identities (13) and (14), the definitions (6) and (7) and the constraint Eq. (30), together with the irrotational , *i.e.,* $\omega^a = 0$ spacetime assumption. A consequence of these results is that the evolution and constraint equations for the gravito-magnetic Weyl tensor simplify as

$$\dot{H}_{ab} = -\Theta H_{ab} \ , \tag{34}$$
$$(C^{3*})_a := \tilde{\nabla}^b H_{ab} = 0 \ . \tag{35}$$

The vanishing of the divergence of H_{ab} in the last equation is a necessary condition for gravitational radiation.

No new constraint equations have emerged as a result of the anti-Newtonian assumption (with the vanishing of the gravito-electric Weyl tensor). But in GR, this is not the case because Eq. (28) becomes a new constraint equation since the left-hand side of this equation identically vanishes. Integrability conditions arise as a result.

3. Integrability conditions in $f(R)$ gravity

Even if no new constraint equations emerge from the anti-Newtonian assumption, Eq. (33) is a modified constraint relation. If we take the curl of both sides of this equation and obtain an identity, we say that the constraint equation is *spatially consistent*. In fact, using our previous argument that the curl of a gradient (and hence also of a divergence) of an irrotational flow vanishes, we have

$$\begin{aligned} 0 &= \frac{f''}{f'}\varepsilon^{acb}\tilde{\nabla}_c\tilde{\nabla}^d\pi^R_{bd} - \tfrac{2}{3}\varepsilon^{acb}\tilde{\nabla}_c\tilde{\nabla}_b\mu + \tfrac{2}{3}\Theta\varepsilon^{acb}\tilde{\nabla}_c q^R_b \\ &= \omega^a\left\{\frac{f''}{f'}\left[\tfrac{2}{3}\tilde{R}\dot{R} - \tfrac{4}{3}\dot{R}\dot{\Theta}\right] - \tfrac{4}{3}\dot{\mu} - \left(\frac{\dot{R}f''}{f'^2} + \frac{2\Theta}{3f'}\right)\left[2\left(f'''\dot{R} - \tfrac{1}{3}\Theta\right)\dot{R} + 2f''\ddot{R}\right]\right\} = 0. \end{aligned} \quad (36)$$

Spatial consistency alone does not guarantee that the constraint equations are preserved under time evolution. A constraint equation is said to be *temporally consistent* if taking the time derivative of both sides of the equation results in an identity. Let us now study the evolution of Eq. (33):

$$\begin{aligned} 0 &= \left(\tilde{\nabla}^b\pi^R_{ab}\right)^{\cdot} - \tfrac{2}{3}\left(\tilde{\nabla}_a\mu\right)^{\cdot} + \tfrac{2}{3}\dot{\Theta}q^R_a + \tfrac{2}{3}\Theta\dot{q}^R_a \\ &= \tilde{\nabla}^b\dot{\pi}^R_{ab} - \tfrac{1}{3}\Theta\tilde{\nabla}^b\pi^R_{ab} - \tfrac{2}{3}\left(\tilde{\nabla}_a\dot{\mu} - \tfrac{1}{3}\Theta\tilde{\nabla}_a\mu\right) + \tfrac{2}{3}\dot{\Theta}q^R_a + \tfrac{2}{3}\Theta\dot{q}^R_a\,, \end{aligned} \quad (37)$$

where, in the second step, we have used the commutation relations

$$\left(\tilde{\nabla}_a\phi\right)^{\cdot} = \tilde{\nabla}_a\dot{\phi} - \tfrac{1}{3}\Theta\tilde{\nabla}_a\phi + \dot{\phi}A_a\,, \quad (38)$$

$$\left(\tilde{\nabla}_a S_{b\cdots}\right)^{\cdot} = \tilde{\nabla}_a\dot{S}_{b\cdots} - \tfrac{1}{3}\Theta\tilde{\nabla}_a S_{b\cdots} \quad (39)$$

of temporal and spatial derivatives for scalar and tensor fields, respectively. The last expression in Eq. (37) can be further expanded using Eqs. (29), (35), (30) and (33) for the divergences of H_{cd}, σ_{ab} and π^R_{ab} and making use of the linearized vectorial identity

$$\tilde{\nabla}^b\tilde{\nabla}_{\langle a}V_{b\rangle} = \tfrac{1}{2}\tilde{\nabla}^2V_a + \tfrac{1}{6}\tilde{\nabla}_a(\tilde{\nabla}^bV_b) + \tfrac{1}{6}\tilde{R}V_a\,. \quad (40)$$

The resulting equation reads

$$\Theta\dot{q}^R_a - \tfrac{1}{4}\tilde{\nabla}_a(\tilde{\nabla}^bq^R_b) + \tfrac{1}{4}\tilde{\nabla}^2q^R_a + \tfrac{1}{2}\left(\mu + \Theta^2\right)q^R_a + \Theta\tilde{\nabla}_a p_R + \tfrac{2}{3}\Theta\tilde{\nabla}_a\mu = 0\,, \quad (41)$$

and acts as the necessary condition for the consistent evolution of the constraints in anti-Newtonian cosmologies with $f(R)$ gravity as the underlying theory of gravitation. Interestingly, this equation reduces to the simple result

$$\Theta\tilde{\nabla}_a\mu_m = 0 \quad (42)$$

when we take the limiting case of GR, *i.e.*, when $f(R) = R$. This is because all the curvature contributions to the thermodynamical quantities automatically vanish in GR. Eq. (42) implies that *there are no expanding linearized anti-Newtonian solutions* in GR, for if $\Theta \neq 0$ (expanding spacetime), $\tilde{\nabla}_a\mu_m = 0$ implies an FLRW background or $\mu_m = 0$ (vacuum spacetime), which contradicts the dust-universe assumption of anti-Newtonian cosmologies.

Let us now make one more simplification to Eq. (41); to do that, we use Eq. (26) to substitute for $\dot{q}^R_a$ as well as Eq. (33). The resulting equation, after some algebraic manipulations, becomes

$$\tilde{\nabla}^2q^R_a - \tilde{\nabla}_a(\tilde{\nabla}^bq^R_b) + \tilde{R}q^R_a + \frac{4f''}{f'^2}\mu_m\Theta\tilde{\nabla}_aR = 0\,. \quad (43)$$

Two interesting conclusions can be drawn from this consistency relation:

- For flat universes ($K = 0 = \tilde{R}$) universes, upon using Eq. (15) the condition (43) reduces to
$$\frac{f''}{f'^2}\mu_m \Theta \tilde{\nabla}_a R = 0\ . \tag{44}$$
For a dust universe in $f(R)$, $\mu_m \neq 0$ and $f'' \neq 0$. As a result *flat, anti-Newtonian spacetimes in $f(R)$ gravity are either static ($\Theta = 0$) or spaces of constant Ricci curvature* ($\tilde{\nabla}_a R = 0$). No vacuum ($R = 0$) solution is allowed (this contradicts the dust-universe assumption).
- For closed and open universes ($K = \pm 1$), using the commutation (15) in (43) yields
$$\left[\frac{f''\mu_m\Theta}{f'} \mp \frac{2}{a^2}\left(\dot{R}f''' - \tfrac{1}{3}\Theta f''\right)\right]\tilde{\nabla}_a R \mp \frac{2f''}{a^2}\tilde{\nabla}_a \dot{R} = 0\ . \tag{45}$$
This means that, provided $f'' \neq 0$, *any dust solution of Eq. (45) describes an anti-Newtonian universe.* Note that the case $f'' = 0$ is equivalent to GR (with or without a cosmological constant) and is, therefore, not an anti-Newtonian solution.

4. Conclusion

A linearized covariant consistency analysis of dust universes with vanishing gravito-electric part of the Weyl tensor in $f(R)$ gravity has been explored. The integrability conditions of such models for generic $f(R)$ gravitation actions have been presented. The solutions for such integrability conditions describe the existence of anti-Newtonian cosmological universes, which are known not to exist as solutions of General Relativity.

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