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Te Kura Mātai Tatauranga

Black holes, Horizons, Cosmology, and the Memory Effect

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Abstract

The future of theoretical physics is unclear. Two large areas that fall under the umbrella of theoretical physics are cosmology and quantum gravity. Modern cosmology is relatively a much younger field than quantum gravity, and both of these fields require further developments of general relativity. In this thesis we do not hope to resolve the problems facing modern cosmology or theories of quantum gravity. Rather, we will conduct original research into aspects of general relativity that may be used in the future to aid the development and testing of theories of cosmology and quantum gravity.

It is our view that the largest problem facing astrophysics and cosmology stem from the existence of the dark sector of the Universe. The implication here being that more than ninety percent of the energy density of the Universe is "missing in action" and seemingly consists of dark energy and dark matter. Furthermore, it is apparent that there exist conceptual flaws in our understanding of observational concepts such as expansion versus motion and observer biases. To this end, we investigate the standard spacetime metric used in cosmology, the Friedmann–Lemaître-Roberston–Walker (FLRW) metric in a peculiar coordinate system — the Painlevé–Gullstrand coordinates. In this coordinate system (slicing), space is no longer expanding, rather, the galaxies are receding from each other. We hope this will aid in the understanding of expansion, motion, curvature, and observer bias with future work. We further investigate the possibility of black holes in cosmology being *directly* coupled to the accelerated expansion of the Universe — in other words, black holes as a source for dark energy. However, we show that this is highly implausible.

Relatively recently it has been postulated that the near black hole horizon limit may be a regime where quantum gravity effects become relevant i.e., quantum gravity may not be restricted to near the Planck scale. We investigate a curious model of black and white holes that shows how one may transition into the other over a finite period of time. This is research conducted in the near horizon limit of the Schwarzschild black hole. We introduce a time dependent function into the usual Schwarzschild black hole spacetime (leaving this new spacetime not a simple coordinate transformed version of the original). This function allows the black hole to transition into a white hole. Importantly, the action for this transition can be shown to be zero, meaning it can be added to the Feynman path integral at no cost.

Finally, we move to investigating the black hole memory effect. During the last decade, there has been an interesting connection made between the Bondi–Metzner–Sachs (BMS) group — an infinite dimensional group of symmetries found at null infinity — and the gravitational memory effect. In particular, it was shown that the passage of a gravitational wave that alters a Schwarzschild

black hole is seen as a supertranslation of the spacetime at null infinity. We extend these calculations to the Kerr and Kerr–Newman black holes. Hence, showing that there may be a way to verify the abstract mathematical ideas predicated on the BMS group by detection of the memory effect in future observations. It is our hope that when future gravitational wave detectors such as the laser-interferometer-space-antenna (LISA) are launched, research conducted in this thesis may shed light on how the memory may relate to black holes in their asymptotic & near horizon limits to aid our understanding of the nature of quantum gravity.

Acknowledgements

First and foremost, I must thank my supervisor, Professor Matt Visser. I think, as most PhDs typically go, Matt and I were unsure where these 3 years would lead. I wanted my PhD experience to be about learning new things in general relativity, and I could not have asked for a better supervisor for this. With his extremely (freakishly) vast knowledge in all that is general relativity, I have learned a lot. That isn't all, however. I also learned a lot from our bizarre, off topic discussions ranging from wormholes, P.L., to UFOs, and American politics. Thank you, Matt.

Secondly, I would like to thank both of the ladies at home. My fiancé, Jess and the little pup, Sylvie. Sylvie, while she cannot hope to understand the fabric of spacetime, has emotionally supported me over the last three years more than I can put into words. Jess, you have supported me through everything and I am forever grateful. There were far too many ups and downs in the last three years, but in the toughest of moments, you've always been there to remind me that physics is my passion and I should *at least* finish my PhD. Though the future is not certain, I know you'll be there to support me through it all. That being said, if there are any typos left in this thesis, they're your fault.

Further thanks are due to my parents, Deepa and Ruchir, who have always supported me with my education and academic journey. I must also thank my auntie Shikha, her partner Amol, and my grandparents, Renu & Suresh.

Finally, I would like to thank all my friends that I have made during my PhD; Chris, Marco, Morag, Mark, Daniel, Jago, Malcolm, and Ellen. All of whom have played an important part in the completion of this thesis. The Canterbury group - spending time with you all in Hobart provided so much more hope and enthusiasm for the rest of my PhD. The Journal club gang - meeting you all gave me reason to show up to campus and I have learned a lot from all our useful *and* useless conversations.

In many ways, the end of a PhD, to me, symbolises the end of a journey. A Journey that I have been working towards for over a decade — ever since I fell in love with theoretical physics as a highschool student. I have been incredibly lucky to have been supervised and mentored by two people who, while having slightly different views, do not fear to ask fundamental questions about our understanding of the Universe. Thank you, Matt and David.

Contents

Chapter 1

Introduction

In 1915 Einstein published his theory of general relativity (GR), a theory in which gravity is no longer a force – as in Newton's theory – rather, gravity is an emergent property of the geometry of spacetime [1]. As Wheeler quoted: In Einstein's theory, "*matter tells spacetime how to curve and spacetime tells matter how to move*".

The first experimental verification of GR came shortly after its development: it successfully explained the precession of the perihelion of Mercury[1] [2] and the curving of starlight due to the gravitational field of the sun[2] [4]. General relativity went on to provide a foundation for gravitational redshift [5] which was qualitatively described by Einstein in 1907 [6]. Furthermore, in 1964, Shapiro [7] predicted the existence of the Shapiro time delay predicated on general relativity. In recent times, the detection of gravitational waves with the Laser Interferometer Gravitational-Wave Observatory (LIGO) [8] was another confirmation of GR being the 'correct' theory of gravity. While GR has provided a description of gravity to high precision on small scales (when compared to the size of the Universe), its applicability on large scales is not thoroughly tested and is a point of debate.

Along with Einstein's famous mass-energy relation and general relativity, some may consider this the dawn of modern theoretical physics. This claim may indeed seem bold as *theoretical physics* has been a field of interest for centuries, if not millennia. After all, by definition, all one requires is a theory — *A system of ideas which is intended to explain something.* Many of these early theories were well approximated & explained by calculus and relatively simple relations from experimental data[3]. This is in stark contrast to modern theories

[1] In fact, the perihelion of Mercury was a *retrodiction* of GR as data collected over the previous century indicated there was more to gravity than Newton's theory.

[2] The bending of starlight was in fact, already predicted from Newtonian gravity [3] but the quantitative result here was out by roughly a factor of two. General relativity provided a much more accurate quantitative prediction.

[3] Of course, there is a comment that could be made here about the current state of theoretical physics and experimental data.

such as GR where gravity itself is explained via a four–dimensional manifold, something we cannot see, but is in some sense[4] *there*. These theories now require high-precision instruments to verify, are not free of immense statistical biases, and/or conceptual hurdles. All of this is to say, the true beginning of modern theoretical physics (in my opinion) is associated with the realisation that geometry and deep mathematical structures are required to explain the Universe.

Einstein was, of course, not the only pioneer of modern theoretical physics, there were many pioneers of what we consider modern physics during the 20$^{\text{th}}$ century. However, despite many advances across all areas of theoretical physics, the profound realisation that deep and beautiful mathematical structures were "required" to explain the Universe caused many of the problems we face in theory today. It became a want, *a need*, for everything to 'fit nicely'. This led us into the realm of 'unfalsifiable' theories, a deviation from the scientific method in the name of elegance. Hubris, after all, is part of the human condition.

Two key problems began to emerge as we progressed towards the 21$^{\text{st}}$ century.

i Despite many attempts, general relativity, a classical theory, has not been quantised, holding back our progress towards a *Grand Unified Theory*. This has led to the "'holy grail" of physics — a theory of quantum gravity. While there have been a plethora of theories proposed, many remain in the realm of "mathematical speculations" and have not been observationally verified. This thesis and the papers it contains does not address or attempt to solve quantum theory directly. However, we conduct original research of black holes in their near horizon limits. In recent years more of the community has begun to suspect that this limit may admit observation evidence of quantum gravity phenomenon, hence our interest in this regime.

ii Our observations of the Universe — which founded the modern theory of cosmology — began to point towards a 'mysterious' dark sector of the Universe that makes up more than 90 percent of the energy density of the Universe at the current epoch. This dark sector of course, is comprised of dark energy and dark matter — both of which do not interact via any of the fundamental forces, except gravity. Once again, this thesis does not directly address a new theory of cosmology or how we may solve the problem of the dark sector. However, we present a unique slicing of spacetime in the hope that with further developments we may be able to deviate from the standard model.

A theory of Quantum gravity has been sought after for at least a hundred years. What is so intrinsically difficult about quantising gravity? We have many theories that remain unsatisfactory to date as none of our state-of-the-

[4]Clearly, Maxwell's equations were also a set of equations that were hinting at something hidden behind the veil, but the true mathematical depth here was yet to be discovered.

art tools to probe the quantum realm have produced results that can prove (and in some cases disprove) any current theory to be correct. From a pure theoretical perspective, the problem is likely more fundamental. In GR, gravity is encoded in the very geometry of spacetime. All of its successes and spectacular predictions emerge from this encoding. To create[5] a theory of gravity that was about geometry, Einstein required a new language to describe all of classical physics, he required Riemannian geometry. Spacetime was now represented as a 4–dimensional manifold equipped with a metric with sharp light cones that are deterministic, and matter? Matter was now represented by tensor fields. Quantum mechanics, however, is a probabilistic theory. How does one even begin to create an even newer language - a sort of "qauntised Riemannian geometry". Furthermore, if one can even quantise the metric, what of the light cones which would now, intrinsically, not be sharp and deterministic.

Evidently, this has proved to be a monumental task — something the greatest minds over the last century have not been able to resolve. Out of this search, theories such as Kaluza–Klein emerged. Kaluza–Klein theory showed that electromagnetism could be unified with gravity, at the cost of an extra dimension. String theory built upon Kaluza–Klein and then there were more "minimally modified" theories such as loop quantum gravity. The failure of these ideas has not gone unnoticed of course. Many have critiqued these theories and their continued pursuit, for instance, Lee Smolin's "The Trouble with Physics" [9], Peter Woit's "Not even Wrong" [10], or Sabine Hossenfelder's "Lost in Maths" [11]. It is perhaps important to realise, however, that a lot of progress was made during the time these theories were developed. For instance, the standard model of particle physics emerged in the 20th century as a result of another mathematical construct, gauge theories.

In this ongoing, seemingly impossible quest for quantum gravity, one of the most useful theoretical playgrounds appears to be black holes. From our current understanding, black holes contain curvature singularities[6], which manifestly require new physics. These singularities may reveal underlying symmetries[7] of 'the full theory' of quantum gravity or suggest how evolution could remain well-defined through a singularity. (This is the case, for instance in certain timelike singularities in string theory.) It turns out that examination of the black hole horizon may contain clues towards the next step for general relativity — as implied by (but not limited by) black hole thermodynamics [13]. Indeed, in the increasingly popular Anti–de Sitter/conformal field theory (AdS/CFT) cor-

[5]We shall not delve into the topic of whether "physical theories are created or found" here. . .

[6]The event horizon (the surface of "no return") and singularities are both topics of debate. In particular, event horizons may not be exist, rather, they should be understood as long lived *apparent horizons*. Furthermore, in recent years, there has been a small argument put forward by Roy Kerr [12] regarding the nature of singularities and Penrose's singularity theorems.

[7]Yet another discussion could be had here regarding the nature of symmetries in physics and our immense love of pursuing them.

respondence one finds a set of symmetry generators, and their commutation relations admit a rich algebra that can be related to black hole thermodynamics. This is something we shall discuss briefly, when discussing near horizon physics, the black hole memory effect, and the Bondi–van der Burg–Metzner–Sachs (BMS) group.

Much like quantum gravity, modern cosmology has also faced many problems, albeit for an arguably shorter time. The current standard model of cosmology, Λ Cold Dark Matter (ΛCDM), is based on the Friedmann–Lemaître–Roberston–Walker (FLRW) solution to the Einstein equations, describing a expanding spacetime that is isotropic and homogeneous[8]. One arrives at the Universe being isotropic because of our observations of the Cosmic Microwave Background (CMB). The CMB is the earliest light in the Universe that we can observe, and from our place in the cosmos, it does appear to be isotropic to one part in ten-thousand[9]. Therefore, the CMB, as we see it, suggests that at the point of decoupling — when the Universe had cooled enough such that light could escape the 'hot, dense mess' — the Universe was in fact, close to isotropic. Coupling isotropy with the *Copernican principle*; the notion that, we, on Earth, do not occupy a 'privileged' place in the Universe, gives us homogeneity. Putting all these pieces together, we arrive at the *cosmological principle* which states: "on sufficiently large scales the spatial distribution of matter is isotropic *and* homogeneous".

Of course, we have not discussed where the *accelerated expansion* part of the standard model comes from, let alone the expanding part. The first evidence of an expanding Universe came from the observation of extragalactic nebulae by Edwin Hubble. These observations showed a positive trend between the distance of these nebulae and their radial velocities — suggesting the Universe is expanding [14]. After which in 1998 Riess et al. [15] followed by Perlmutter et al. in 1999 [16], fit astrophysical data to the FLRW model and found something unexpected — the Universe was not only expanding, the expansion was also accelerating. The surveys which serve as the foundation for this were observing type 1a supernovae (SNe1a). The SNe1a observed appeared to be fainter than predicted by the FLRW model. Therefore, the cosmological constant which had been omitted since the early 20th century made a return, becoming part of the standard model. The cosmological constant in modern times, is associated with "dark energy" — a repulsive negative pressure, opposing gravity, that drives the expansion of the Universe at late times. To date, the ΛCDM model — with the addition of standard perturbation theory and Newtonian N-body numerical simulations — has explained most of our cosmological observations.

There have been, however, a growing number of tensions in the past two

[8]It is useful to point out that 'isotropic' here means that the Universe is observed to be the 'same' in every direction. Homogeneity, is a stronger condition that states that something is observed to be isotropic from every point in the Universe.

[9]Many subtleties have been omitted here for ease of reading.

decades between the predictions of the ΛCDM model and observations. These tensions include the 'lack of power' at the largest scales in the CMB power spectrum, and the recent 3.7σ tension between local measurements of the Hubble parameter [17, 18] compared to the inferred value from the CMB [19]. Furthermore, there is a growing tension with the lack of 'direct observation' of the 'dark sector' of the Universe. These tensions may be — as most would lead one to believe — due to of insufficient precision, or systematic errors.

While cynicism is an easy hole to fall into, there are groups around the world who are looking beyond the standard model of cosmology. One contribution to this debate, Wiltshire's timescape model [20, 21, 22], claims that the expansion is not actually accelerating. Rather, our perception of this accelerated expansion is more of a fundamental issue associated with how one calibrates time parameters in the presence of cosmological backreaction. Wiltshire's group has also made steps towards reducing the need for dark matter in galaxies [23] and strong lensing [24]. Only time will tell if the dark sector of the Universe is fundamental or if it is an emergent property of our observations and biases.

In this thesis we will discuss a variety of topics surrounding black holes. Namely, black holes in cosmology, their near horizon limits, and their interaction with gravitational waves. All the ideas explored in this thesis aim to form theory that will aid observation in the coming decades. We will begin in Chapter 2 by giving a brief overview of the $3+1$ formalism in general relativity. We will then introduce a unusual slicing for cosmological spacetimes — The Painlevé–Gullstrand slicing. In this slicing, we will see that space is no longer expanding, rather fluid elements (commonly, thought of as galaxies) are receding away from each other. This will potentially provide a natural framework for the question of what is "motion and what is expansion". This chapter will investigate the FLRW spacetime and discuss how all of the symmetry generators transform under this choice of slicing. We will then provide a sort-of catalogue of cosmologies in this slicing. Namely, de Sitter space, the Kottler spacetime, and the McVittie spacetime.

In Chapter 3 we shall investigate a new series of articles which claim that black holes couple with the expansion of the Universe. The proposed mechanism for this is *black holes leaking dark energy into the Universe.* We discuss how black holes coupling to the large scale dynamics of the Universe is implausible due to the truly immense separation of scales. We then use various exact solutions of black holes embedded in expanding spacetimes to show there is no correlation on theoretical grounds. For this, we use similar slicings of spacetime as we did in chapter 1, building on the ideas of expansion versus motion.

In Chapter 4 we discuss (Schwarzschild) black and white holes. In particular, we will show that by the introduction of a function depending solely on the radial coordinate, r, one can obtain a static black and white hole in horizon-penetrating coordinates. Secondly, we will move to the near-horizon form of these spacetimes and show that a clear distinction can be made between a "black and white" horizon. We further introduced a function of time as well. This spacetime will now describe a 'black-to-white hole bounce'. Finally, the action of this bounce in the transition region will be investigated in order to discuss how quantum physics would be affected by this 'bounce'.

In Chapter 5 the black hole gravitational memory effect is explored. The memory effect has been shown to be one vertex of the infrared triangle [25]; the other two being asymptotic symmetries and soft theorems. The infrared triangle is a figurative triangle that illustrates how these three aspects of physics — which previously seemed disconnected — in recent years have been shown to be mathematically related. The black hole memory effect in this scheme illustrates that when a gravitational wave strikes a black hole, the spacetime is left permanently altered. Amazingly, the linear order approximation of this change is seen as a BMS supertranslation at null infinity, $\mathcal{I}^+$. This effect has been discussed in the literature for close to ten years; for instance see refs [26, 27, 28]. However, the supertranslations of the Kerr solution have not been previously calculated. We will do exactly this in Chapter 5 — compute the supertranslated Kerr spacetime and discuss how the asymptotic charges are changed due to a gravitational wave. In years to come we expect the gravitational memory effect to be detectable and so further development of this formalism with exact solutions such as the Kerr spacetime may prove vital.

In Chapter 6 we compute the memory effect for a more general black hole, the Kerr–Newman spacetime. With the presence of an electromagnetic field (the gauge field), the memory effect becomes slightly more interesting. A similar investigation was undertaken by Donnay et.al [28] for the Reisnner–Nordström solution. In their paper it was found that there is a permanent change in the gauge field as well as the spacetime metric; this is also the case for the Kerr-Newman spacetime. We further bring the Kerr–Newman spacetime into its extremal near horizon form to examine the memory effect as seen from an observer near the horizon. Following the calculations from refs [28, 29], we find that there is a non-trivial change in the electromagnetic charge generator on the horizon. This implies the existence of soft electric hair that is implanted from the passage of a gravitational wave. The effect of gravitational waves on the horizon charges & the electromagnetic field and its ties to the AdS/CFT correspondence may prove to aid our understanding of quantum gravity.

Chapter 2

Cosmology in Painlevé–Gullstrand coordinates

Cosmology is most typically analyzed using standard co-moving coordinates, in which the galaxies are (on average, up to presumably small peculiar velocities) "at rest", while "space" is expanding. This, however, is merely a specific coordinate choice; and it is important to realise that for certain purposes other, (sometimes *radically* different), coordinate choices might also prove useful and informative, but without changing the underlying physics. Specifically, herein we shall consider the $k = 0$ spatially flat FLRW cosmology but in Painlevé–Gullstrand coordinates — these coordinates are very explicitly *not* co-moving: "space" is now no longer expanding, although the distance between galaxies is still certainly increasing.

This particular coordinate/slicing choice, therefore, further provides a natural way of addressing the difference between (peculiar) motion versus expansion in cosmology in astrophysics. Whether space is expanding or the galaxies are receding — the physical redshift we observe is the same. Since space expanding is a more cosmology based concept and galaxy motion is more of an astrophysical concept, Painlevé–Gullstrand coordinates provide a middle ground for these two fields.

Working in these Painlevé–Gullstrand coordinates provides an alternate viewpoint on standard cosmology, the symmetries thereof, and also makes it somewhat easier to handle cosmological horizons. With a longer view, we hope that investigating these Painlevé–Gullstrand coordinates might eventually provide a better framework for understanding large deviations from idealised FLRW spacetimes. We illustrate these issues with a careful look at the Kottler and McVittie spacetimes.

Coordinate freedom in general relativity is an extremely powerful tool; but a very subtle one that took almost 45 years for most of the general relativity community to fully internalize. A judicious choice of coordinates can often make some aspect of the physics easy and obvious, but may make other aspects of the physics more obscure. On the other hand, no coordinate choice, (no matter how obtuse), can actually change the underlying physics. For instance, at a purely theoretical level, locally geodesic and Riemann normal coordinate systems greatly simplify manipulations leading to the Bianchi identities. At a more physical level, locally geodesic and Riemann normal coordinate systems greatly simplify analysis and understanding of the Einstein equivalence principle. See any of a vast number of relevant textbooks for more details on these issues [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41].

In this chapter we will explore some unusual coordinate choices in cosmology. While typically in a cosmological setting one uses comoving coordinates, tied to the average Hubble flow, this is by no means a necessary choice. Choosing non-comoving coordinates, (specifically, a cosmological variant of the Painlevé–Gullstrand coordinates) will simplify some aspects of the discussion, while (apparently) making other aspects more complicated, but without changing the underlying physics. For relevant background on Painlevé–Gullstrand coordinates see references [42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55].

Explicitly choosing a cosmological variant of the Painlevé–Gullstrand coordinates will allow us to eliminate the expansion of "space". The price we pay here is that typical galaxies will now be represented by "moving" Eulerian observers — the distance between galaxies will still be increasing, there will still be a Hubble flow. Furthermore, in these Painlevé–Gullstrand coordinates the light cones are "tipped over" so that "faster-than-light" with respect to non-expanding "space" is not the same as "faster-than-light" with respect to the locally defined light-cones. This provides an alternative viewpoint on the Hubble expansion, one that some cosmologists might be more comfortable with. We carefully consider the symmetries of FLRW spacetime, the crucial difference between apparent horizons and causal horizons, and as an example of large deviations from FLRW consider several versions of the Kottler and McVittie spacetimes. Our conventions will be those of Misner–Thorne–Wheeler [32].

2.1 Preliminaries: Definition of a Foliation

Any spacetime $(\mathcal{M}, \boldsymbol{g})$ that is globally hyperbolic can be *foliated* by a family of spacelike hypersurfaces, Σ_t. We define a foliation or slicing by supposing there exists a scalar field, $\tilde{t}$ on $\mathcal{M}$ (which has non vanishing gradient), such that each hypersurface is a level surface of $\tilde{t}$.

$$\forall t \in \mathbb{R}, \quad \Sigma_t := \{p \in \mathcal{M}, \tilde{t}(p) = t\}. \tag{2.1}$$

Since the gradient of $\tilde{t}$ does not vanish, the Σ_t are non-intersecting:

$$\Sigma_t \cap \Sigma_{t'} = \varnothing \quad \text{for} \quad t \neq t'. \tag{2.2}$$

Each hypersurface, Σ_t is called a *slice* of the *foliation.* Generally, we assume the hypersurfaces to be spacelike and thus the foliation covers $\mathcal{M}$:

$$\mathcal{M} = \bigcup_{t \in \mathbb{R}} \Sigma_t. \tag{2.3}$$

Foliation Kinematics

The kinematics of a foliation are determined by the 3-dimensional slices, Σ_t, the infinitesimal neighbouring slice, $\Sigma_{t+\mathrm{d}t}$ and the 4-dimensional space that fills the space between the slices. Misner, Thorne, and Wheeler, [56] and Alcubierre [57] discuss physical notions that are required to give the chosen foliation structure a sense of "rigidity". The physical notions are:

- A notion of how to measure proper distances given by the metric, h_{ij} . This is often called the induced metric on the hypersurfaces.
- The *lapse* function which defines a notion of proper time between slices.
- The relative velocity of observers travelling normal to the slices (Eulerian observers) and the worldlines corresponding to constant spatial coordinates. This is given by the shift vector, $\boldsymbol{\beta}$.

Terms such as "velocity" and "observer" are used here simply for physical motivation, they are not intrinsically required.

Eulerian Observers

The idea of Eulerian observers is fundamental to the 3+1 splitting of spacetime. We can regard $\boldsymbol{n}$, the normal vector to the hypersurfaces (see Figure 2.2) as the 4–velocity of an Eulerian observer. The worldlines of Eulerian observers are obviously orthogonal to the hypersurfaces Σ_t. One may physically interpret this as meaning that the spacelike hypersurface, Σ_t, is *locally* the surface of simultaneity of the Eulerian observers.

Lapse Function

Recall that the normal vector to Σ_t, $\boldsymbol{n}$, which is timelike and future-directed must be collinear to the vector $\vec{\boldsymbol{\nabla}}t$. Hence we will write

$$\boldsymbol{n} := -N\,\vec{\boldsymbol{\nabla}}t, \tag{2.4}$$

with

$$N := \left(\frac{-1}{\vec{\boldsymbol{\nabla}}t\cdot\vec{\boldsymbol{\nabla}}t}\right)^{1/2}. \tag{2.5}$$

The minus sign here is chosen so that $\boldsymbol{n}$ is future-oriented. Furthermore, the value of N ensures that $\boldsymbol{n}$ is a timelike unit vector with norm $= -1$. The scalar field N is the *lapse function*, coined by Wheeler in 1964 [58]. By construction we also have $N > 0$, i.e., the lapse function never vanishes for a 'regular' foliation, or equivalently,

$$\underline{\boldsymbol{n}} = -N\,\mathbf{d}t. \tag{2.6}$$

To properly understand the physical interpretation of the lapse function, let us introduce the normal evolution vector:

$$\boldsymbol{m} := N\boldsymbol{n}, \tag{2.7}$$

i.e., it has the properties

$$\boldsymbol{m}\cdot\boldsymbol{m} = -N^2 \quad \text{and} \quad \boldsymbol{\nabla}_{\boldsymbol{m}}\,t = m^\mu\,\nabla_\mu\,t = 1. \tag{2.8}$$

A consequence of this last property is that the hypersurface $\Sigma_{t+\delta t}$ can be obtained from the previous hypersurface, Σ_t, by the 'small displacement' $\boldsymbol{m}\,\delta t$. In particular, one can show if p corresponds to a point with the coordinate position, $\boldsymbol{x}$, then

$$t(p') = t(\boldsymbol{x} + \boldsymbol{m}\,\delta t) = t(p) + \delta t\,. \tag{2.9}$$

The last equality shows $p' \in \Sigma_{t+\delta t}$. Hence we say the vector $\boldsymbol{m}\,\delta t$ 'carries Σ_t into $\Sigma_{t+\delta t}$'. This notion is perfectly described by the *Lie derivative*[1] as the Lie derivative is associated directly with generating diffeomorphisms between manifolds (in this case, hypersurfaces). We describe the action of the Lie derivative of the curves and tangent vectors of Σ_t along $\boldsymbol{m}$ as 'evolving the hypersurface along the normal direction'. This justifies the name "normal evolution vector".

Finally, to understand the role of the lapse function better, let us consider two events on a worldline of some Eulerian observer. Let t be the time coordinate of the event $p \in \Sigma_t$ and $t + \delta t$ the 'time' of $p' \in \Sigma_t$ (refer to Figure 2.1 for an illustration). We note that the proper time between these two events, $\delta\tau$

[1] For an understanding of why the Lie derivative is natural for describing this scenario, the reader may refer to Appendix B of ref [36].

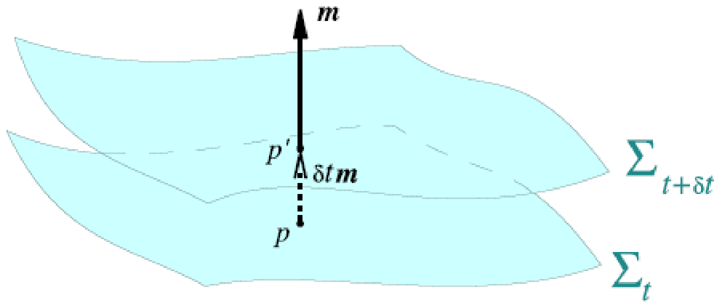

Figure 2.1: Diagram of a point in Σ_t and $\Sigma_{t+\delta t}$ from [59]. The hypersurface Σ_t evolves into $\Sigma_{t+\delta t}$ by the Lie derivative along $\boldsymbol{m}$. The point $p' \in \Sigma_{t+\delta t}$ is determined by $p \in \Sigma$ by the change in $\boldsymbol{m}$ over some time, δt i.e., by a displacement $\boldsymbol{m}\,\delta t$. The length of this displacement is the change in proper time, $\delta\tau$, for an Eulerian observer following the worldline connecting p and p'.

(measured by the Eulerian observer) is given by the metric length of the timelike vector linking these two events:

$$\begin{aligned} \delta\tau &= \sqrt{-\boldsymbol{g}(\boldsymbol{m},\boldsymbol{m})}\,\delta t \\ &= N\,\delta t. \end{aligned} \tag{2.10}$$

This justifies the name "lapse function" given to N. N relates the time coordinate which labels the slices of the foliation to the physical time, τ measured by an Eulerian observer. Without the notion of observers, the lapse function is said to determine how far consecutive slices are from each other in the slice-orthogonal time direction at each point.

Shift Vector

To define a shift vector, $\boldsymbol{\beta}$, we require the notion of coordinates on our spacetime manifold. We introduce the natural basis, $\boldsymbol{\partial}_\mu = (\boldsymbol{\partial}_t, \boldsymbol{\partial}_i)$ of the tangent place, $\mathcal{T}_p\left(\mathcal{M}\right)$ associated with the coordinates, x^μ. The vector which we usually refer to as the 'time vector', $\boldsymbol{\partial}_t$, has the same properties as $\boldsymbol{m}$. In particular, the tangent vectors on $\mathcal{T}_p\left(\Sigma_t\right)$ can evolve along either $\boldsymbol{\partial}_t$ or $\boldsymbol{m}$ and the difference is given by a *shift* in reference coordinates. The two vectors only coincide if the spatial coordinates x^i are such that the x^i = constant lines are orthogonal to $\Sigma_{\,t}$. The difference between $\boldsymbol{\partial}_t$ and $\boldsymbol{m}$ was also coined the *shift vector* by Wheeler in 1964 [58] and is denoted by β:

$$\boldsymbol{\beta} := \boldsymbol{\partial}_t - \boldsymbol{m} = \boldsymbol{\partial}_t - N\boldsymbol{n}. \tag{2.11}$$

For an illustration of this difference, one may refer to Figure 2.2. Note that the shift vector is tangent to the hypersurface as $\boldsymbol{n}\cdot\boldsymbol{\beta} = 0$. One can think of the shift vector as generating spatial diffeomorphisms relating points between successive slices [60].

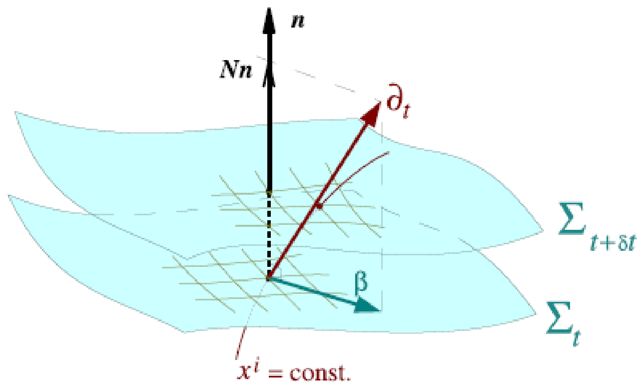

Figure 2.2: Illustration of the shift vector, $\boldsymbol{\beta}$ from [59]. The coordinates (x^i) on Σ_t define the 'time vector', $\boldsymbol{\partial}_t$ by the x^i = constant lines. The shift vector is the difference between the time vector and $\boldsymbol{m}$, therefore, the difference between the spacetime coordinates x^α, and the x^i = constant lines.

3+1 Splitting of the Metric

The components of the metric tensor, $\boldsymbol{g}$, on $\mathcal{M}$ with respect to the coordinates x^μ are defined as

$$\boldsymbol{g} = g_{\mu\nu}\,\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu. \tag{2.12}$$

We can therefore compute each component by using

$$g_{\mu\nu} = \boldsymbol{g}\left(\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu\right). \tag{2.13}$$

Using (2.11) we find

$$g_{00} = \boldsymbol{g}(\boldsymbol{\partial}_t, \boldsymbol{\partial}_t) = \boldsymbol{\partial}_t \cdot \boldsymbol{\partial}_t = -N^2 + \boldsymbol{\beta}\cdot\boldsymbol{\beta}. \tag{2.14}$$

Similarly we have[2]

$$g_{0i} = (\boldsymbol{m} + \boldsymbol{\beta}) \cdot \partial_i = \beta_i, \tag{2.15}$$

since $\boldsymbol{m} \cdot \boldsymbol{\partial}_i = 0$ by definition. Finally, the spatial part of the metric must be the induced metric[3],

$$g_{ij} = h_{ij}. \tag{2.16}$$

Collecting all of these components together we have[4]

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_k\,\beta^k & \beta_j \\ \beta_i & h_{ij} \end{pmatrix} \tag{2.17}$$

[2]Note that we have used only Latin indices for the scalar product of the shift vector as it is tangent to the constant time hypersurfaces, meaning there is no time component

[3]All Latin indices (spatial indices) are raised and lowered by the induced metric on the spatial slices.

[4]Interestingly, if we consider the evolution of the 3-metric, $\boldsymbol{h}$, by taking the Lie derivative (denotes by$\mathcal{L}$) along $\boldsymbol{m}$ we find:

$$\mathcal{L}_{\boldsymbol{m}} h_{\mu\nu} = -2N\mathcal{K}_{\mu\nu}.$$

This relationship means that the extrinsic curvature can also be thought of as a measure of how the induced metric evolves in time.

or,

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -N^2\,\mathrm{d}t^2 + h_{ij}\,(\mathrm{d}x^i + \beta^{\,i}\mathrm{d}t)(\mathrm{d}x^j + \beta^{\,j}\mathrm{d}t). \tag{2.18}$$

The inverse metric in matrix form is[5]

$$g^{\,\mu\nu} = \frac{1}{N^2}\begin{pmatrix} -1 & \beta^{\,j} \\ \beta^{\,i} & N^{\,2}\,h^{ij} - \beta^{\,i}\,\beta^{\,j} \end{pmatrix}. \tag{2.19}$$

2.2 Painlevé–Gullstrand Slicing

In this chapter, we will discuss Cosmology in Painlevé–Gullstrand coordinates. While one may see the relevant literature [42, 43, 44, 45, 46, 47, 49, 48, 50, 51, 52, 53, 54, 55], we will present a brief definition to provide a self-contained experience.

2.2.1 Strong Painlevé–Gullstrand Coordinates

We shall label any coordinate system as being in *strong* Painlevé–Gullstrand form if the spacetime line element can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left|\mathrm{d}\vec{x} - \vec{\beta}\,\mathrm{d}t\right|^2. \tag{2.20}$$

I.e., the metric can be written in the following form:

$$g_{\mu\nu} = \left[\begin{array}{c|c} -1 + \beta_k\,\beta^{\,k} & \beta_i \\ \hline \beta_j & \delta_{ij} \end{array}\right]. \tag{2.21}$$

Equivalently, for the inverse metric

$$g^{\mu\nu} = \left[\begin{array}{c|c} -1 & -\beta^i \\ \hline -\beta^j & \delta^{ij} - \beta^i\,\beta^j \end{array}\right]. \tag{2.22}$$

From this abstract mathematical definition, it is not entirely obvious what quality we are looking for in our spacetimes. It is in fact that the spatial slices are flat, shown explicitly by the δ_{ij} in the space-space component of the metric (2.21).

2.2.2 Weak Painlevé–Gullstrand Coordinates

We shall say that a coordinate system is of *weak* Painlevé–Gullstrand form if the spacetime line element can be written as

$$\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + \left|\mathrm{d}\vec{x} - \vec{\beta}\,\mathrm{d}t\right|^2. \tag{2.23}$$

[5] One may notice that while $g_{ij} = h_{ij}$, $g^{\,ij} \neq h^{\,ij}$ in general. They are, however, equal in the case of vanishing shift.

I.e., the metric can be cast in the form

$$g_{\mu\nu} = \left[\begin{array}{c|c} -N^2 + \beta^k \beta_k & -\beta_i \\ \hline -\beta_j & \delta_{ij} \end{array} \right]. \tag{2.24}$$

Equivalently, for the inverse metric

$$g^{\mu\nu} = \left[\begin{array}{c|c} -1/N^2 & -\beta^i/N^2 \\ \hline -\beta^j/N^2 & \delta^{ij} - \beta^i \, \beta^j/N^2 \end{array} \right]. \tag{2.25}$$

The difference here being that, now, the lapse is no longer restricted to being unity. In other words, we recover (2.18) and (2.19) with h_{ij} restricted to spatially flat slices only.

2.2.3 Conformal Painlevé–Gullstrand

Lastly, we shall say that a coordinate system is of *conformal* Painlevé–Gullstrand form if the spacetime line element is conformal to (either strong or weak versions of) the Painlevé–Gullstrand line element. Either

$$\mathrm{d}s^2 = \Omega^2 \left\{ -\mathrm{d}t^2 + \left| \mathrm{d}\vec{x} - \vec{\beta} \, \mathrm{d}t \right|^2 \right\}, \tag{2.26}$$

or

$$\mathrm{d}s^2 = \Omega^2 \left\{ -N^2 \, \mathrm{d}t^2 + \left| \mathrm{d}\vec{x} - \vec{\beta} \, \mathrm{d}t \right|^2 \right\}. \tag{2.27}$$

With all the preliminaries out of the way, we will move to applying this information to cosmological spacetimes.

2.3 Spatially flat FLRW cosmology

Observational evidence points to the spatially flat $k = 0$ FLRW cosmology as being an excellent *zeroth-order* approximation to the very large-scale structure of spacetime — beyond the scale of statistical homogeneity [30, 31, 37, 32]. As discussed, however, there are growing tensions in cosmology that are perhaps reaching a tipping point. Therefore, one ultimately might be interested in investigating *large* non-perturbative deviations from FLRW cosmology [61, 62, 63, 64, 65, 66, 67, 68]. For now we shall focus on the idealised case of exact FLRW spacetime. This is simply because we wish to develop an interesting catalogue of spacetimes that communicate with the standard model cosmologists first and foremost. Standard presentations of FLRW spacetime can be found in many places, see for instance refs [30, 31, 32, 33, 34, 35, 36, 37, 39, 38, 40, 41]. Let us start by considering several useful coordinate systems.

2.3.1 Standard comoving coordinates

Spherical polar version

The most common presentation of the spatially flat $k = 0$ FLRW cosmology is in terms of the explicit line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\}, \tag{2.28}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2$. In these coordinates the t coordinate is the physical time measured by a fiducial observer of normalized 4-velocity $V^a = (1, 0, 0, 0)$, so that $V_a = (-1, 0, 0, 0)$. The purely radial ingoing and outgoing light rays are described by the time-dependent opening angle

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right| = \frac{1}{a(t)}. \tag{2.29}$$

In these coordinates the $t = (constant)$ spatial slices are 3-flat but *expanding*

$$\mathrm{d}s_3^2 = a(t)^2\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\}. \tag{2.30}$$

For the spatially flat case $k = 0$ one has a *choice* as to whether the coordinate r is dimensionless while the scale factor a has units of length, or *vice versa.* For nonzero spatial curvature, if one sets $k = \pm 1$ then one is forced to take the coordinate r to be dimensionless, while the scale factor a has units of length. We shall make the same choice in the spatially flat $k = 0$ case.

Cartesian version

One could equally well use comoving Cartesian coordinates for the spatial slices

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\}. \tag{2.31}$$

Or equivalently,

$$\mathrm{d}s_3^2 = a(t)^2\{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\}. \tag{2.32}$$

Doing so will not change the physics, just the presentation. For instance the light cones are now described by the time-dependent opening angle

$$\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right| = \frac{1}{a(t)}\,. \tag{2.33}$$

For the fiducial observers we still have the normalized 4-velocity $V^a = (1, 0, 0, 0)$, so that $V_a = (-1, 0, 0, 0)$.

2.3.2 Conformal time coordinate

Define a conformal time coordinate by

$$\eta(t) = \int_0^t \frac{\mathrm{d}\bar{t}}{a(\bar{t})}\,. \tag{2.34}$$

Note that with our conventions the scale factor a has units of distance so that the conformal time is dimensionless. One can formally invert this definition to obtain $t(\eta)$, and thereby implicitly define $a(\eta) = a(t(\eta))$.

Spherical polar version

Using conformal time we can re-cast the line element as

$$\mathrm{d}s^2 = a(\eta)^2\{-\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\}. \tag{2.35}$$

This choice of coordinate system makes manifest the fact that $k = 0$ FLRW spacetime is conformally flat (the Weyl tensor is identically zero).

This conformal time coordinate has the technical advantage that the radial ingoing and outgoing light rays are now particularly simple

$$\left|\frac{\mathrm{d}r}{\mathrm{d}\eta}\right| = 1. \tag{2.36}$$

In contrast, the proper time (clock time) measured by a fiducial observer, now with normalized 4-velocity $V^a = \frac{1}{a(\eta)}\,(1, 0, 0, 0)$, becomes more complicated. Note that for the related co-vector one now has $V_a = a(\eta)\,(-1, 0, 0, 0)$. For the proper time one has

$$\tau(\eta) = \int_0^\eta a(\bar{\eta})\,\mathrm{d}\bar{\eta}. \tag{2.37}$$

This is a common theme of coordinate freedom — coordinates can often be chosen to make *some* formulae simpler, (in this case, the light cones), at the cost of complicating *other* formulae (in this case, the proper time).

Cartesian version

We could equally use comoving Cartesian coordinates for the spatial slices and re-write (2.35) as

$$\mathrm{d}s^2 = a(\eta)^2\{-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\}. \tag{2.38}$$

The light cones are now particularly simple

$$\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\eta}\right| = 1. \tag{2.39}$$

This simplified light cone structure makes the causal structure in these conformal coordinates particularly easy to deal with. For the fiducial observers we again have both $V^a = \frac{1}{a(\eta)}\,(1,0,0,0)$ and $V_a = a(\eta)\,(-1,0,0,0)$.

2.3.3 Painlevé–Gullstrand coordinates

We shall now introduce the cosmological Painlevé–Gullstrand coordinate systems. (For relevant background discussion see references [42, 43, 44, 45, 46, 47, 49, 48, 50, 51, 52, 53, 54, 55].)

Spherical polar version

Metric: Starting from the standard line element (2.28), let us now make the time-dependent coordinate transformation $\bar{r} = a(t)\,r$. Then $\bar{r}$ is a Schwarzschild radial coordinate, based on the notion of area, since the area of a 2-sphere of coordinate radius $\bar{r}$ is simply $4\pi\,\bar{r}^2$. (Consequently, these are sometimes called "area coordinates".)

Furthermore

$$\mathrm{d}\bar{r} = a(t)\,\mathrm{d}r + r\,\dot{a}(t)\,\mathrm{d}t = a(t)\,\mathrm{d}r + H(t)\,\bar{r}\,\mathrm{d}t, \tag{2.40}$$

where $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter. Therefore,

$$a(t)\,\mathrm{d}r = \mathrm{d}\bar{r} - H(t)\,\bar{r}\,\mathrm{d}t. \tag{2.41}$$

Consequently in these coordinates the line element becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \{[\mathrm{d}\bar{r} - H(t)\,\bar{r}\,\mathrm{d}t]^2 + \bar{r}^2\mathrm{d}\Omega^2\}. \tag{2.42}$$

That is

$$\mathrm{d}s^2 = -(1 - H(t)^2\,\bar{r}^2)\,\mathrm{d}t^2 - 2H(t)\,\bar{r}\,\mathrm{d}\bar{r}\,\mathrm{d}t + \{\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\Omega^2\}. \tag{2.43}$$

Note that the line element only contains the scale factor *implicitly*, via the Hubble parameter $H(t)$. Furthermore, in these new coordinates the $t =$ (*constant*) spatial slices are again 3-flat, but are now *non-expanding*

$$\mathrm{d}s_3^2 = \mathrm{d}\bar{r}^2 + \bar{r}^2 \mathrm{d}\Omega^2. \tag{2.44}$$

Adopting ADM terminology, as shown in section 2.1 (or see for instance refs [69, 70]), all the non-trivial aspects of the $k = 0$ FLRW spacetime geometry have now been pushed into the shift vector, $\beta_i = g_{0i} = (-H(t)\,\bar{r}, 0, 0)$. The lapse function is still unity, one still has $N^2 = -g^{tt} = 1$. Coordinate systems of this type are called Painlevé–Gullstrand coordinates [42, 43, 44, 45, 46, 47]. Very many, (but certainly not all), physically interesting spacetimes can be put into this Painlevé–Gullstrand form. For example: all of the Schwarzschild spacetime [49, 50], most of the Reissner–Nordström spacetime (the region $r > \frac{Q^2}{2m}$), all of the Lense–Thirring spacetime [52, 53, 54, 55], all spherically symmetric spacetimes (at least locally) can be recast in this form;[6] but not the Kerr or Kerr–Newman spacetimes [71, 72].

In these Painlevé–Gullstrand coordinates there is manifestly an *apparent* horizon, (where $g_{tt} = 0$), at the Hubble radius $\bar{r}_{\mathrm{Hubble}} = 1/H(t)$. Additionally, the fiducial Eulerian (geodesic) observers have covariant 4-velocity $V_a = (-1, 0, 0, 0)$, which now corresponds to the contravariant 4-velocity $V^a = (1,\ H(t)\,\bar{r}, 0, 0)$. So a typical galaxy (ignoring peculiar velocities) is certainly "moving" in this coordinate system. While "space" is now non-expanding, the Hubble flow is explicit, with $V^r = H(t)\,\bar{r}$.

The radial ingoing and outgoing light rays are now described by

$$\left| \frac{\mathrm{d}\bar{r}}{\mathrm{d}t} - H(t)\,\bar{r} \right| = 1. \tag{2.45}$$

That is

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} = H(t)\,\bar{r} \pm 1, \tag{2.46}$$

whereas a typical galaxy (vanishing peculiar velocity) is moving with 3-velocity

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} = H(t)\,\bar{r}, \tag{2.47}$$

which safely lies inside the light cone.

Tetrad: A suitable co-tetrad is easily read off from the line element:

$$e^{\hat{t}}{}_a = (1, 0, 0, 0); \quad e^{\hat{r}}{}_a = (-H\bar{r}, 1, 0, 0); \quad e^{\hat{\theta}}{}_a = (0, 0, \bar{r}, 0); \quad e^{\hat{\phi}}{}_a = (0, 0, 0, \bar{r}\sin\theta). \tag{2.48}$$

[6] In spherical symmetry the only obstructions to the global existence of Painlevé–Gullstrand coordinates are the possible existence of wormhole throats, (since then the area radial coordinate cannot be monotone), and/or negative Misner–Sharp quasi-local mass [46], (since then the shift vector is forced to become imaginary).

The corresponding tetrad is then given by the timelike leg

$$e_{\hat{t}}{}^{a} = V^a = (1, H(t)\,\bar{r}, 0, 0); \tag{2.49}$$

and the particularly simple spatial triad

$$e_{\hat{r}}{}^{a} = (0,1,0,0); \quad e_{\hat{\theta}}{}^{a} = \left(0, 0, \frac{1}{\bar{r}}, 0\right); \quad e_{\hat{\phi}}{}^{a} = \left(0, 0, 0, \frac{1}{\bar{r}\sin\theta}\right). \tag{2.50}$$

It is easy to check that with $\eta_{\hat{m}\hat{n}} = \mathrm{diag}\{-1,1,1,1\}$ one has (as expected):

$$g_{ab} = \eta_{\hat{m}\hat{n}} \;\; e^{\hat{m}}{}_{a} \;\; e^{\hat{n}}{}_{b}; \qquad \eta_{\hat{m}\hat{n}} = g_{ab} \;\; e_{\hat{m}}{}^{a} \;\; e_{\hat{n}}{}^{b}. \tag{2.51}$$

A brief computation yields the orthonormal components of the Riemann tensor

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = -H^2 - \dot{H} = -\frac{\ddot{a}}{a}; \qquad R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = H^2. \tag{2.52}$$

The Weyl tensor is (as expected) identically zero, while for the Einstein and Ricci tensors one has

$$G_{\hat{t}\hat{t}} = 3H^2; \qquad G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -3H^2 - 2\dot{H}; \tag{2.53}$$

and

$$R_{\hat{t}\hat{t}} = -3H^2 - 3\dot{H} = -3\,\frac{\ddot{a}}{a}; \qquad R_{\hat{r}\hat{r}} = R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = 3H^2 + \dot{H}. \tag{2.54}$$

The Ricci scalar is $R = 12H^2 + 6\dot{H}$.

These orthonormal components (that is, components in the basis defined by the orthonormal tetrad) are identical (as they should be) to the standard orthonormal components defined in comoving coordinates. The cosmological Friedmann equations will be unaffected. The fiducial observers, with 4-velocity $V^a = (1, H(t)\,\bar{r}, 0, 0)$ are geodesic.

Cartesian version

One could also construct a Cartesian version of Painlevé–Gullstrand coordinates.

Metric: Define

$$\bar{x} = \bar{r}\,\sin\theta\,\cos\phi; \qquad \bar{y} = \bar{r}\,\sin\theta\,\sin\phi; \qquad \bar{z} = \bar{r}\,\cos\theta. \tag{2.55}$$

Then $\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}$, and our spherical polar Painlevé–Gullstrand version of $k = 0$ FLRW spacetime,

$$\mathrm{d}s^2 = -(1 - H(t)^2\,\bar{r}^2)\,\mathrm{d}t^2 - 2H(t)\,\bar{r}\,\mathrm{d}\bar{r}\,\mathrm{d}t + \{\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\Omega^2\}, \tag{2.56}$$

now becomes

$$\begin{aligned}\mathrm{d}s^2 = -(1 - H(t)^2\,\{\bar{x}^2 + \bar{y}^2 + \bar{z}^2\})\,\mathrm{d}t^2 - 2H(t)\,\{\bar{x}\,\mathrm{d}\bar{x} + \bar{y}\,\mathrm{d}\bar{y} + \bar{z}\,\mathrm{d}\bar{z}\}\,\mathrm{d}t \\ + \{\mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2 + \mathrm{d}\bar{z}^2\}.\end{aligned} \tag{2.57}$$

That is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left\{[\mathrm{d}\bar{x} - H(t)\,\bar{x}\,\mathrm{d}t]^2 + [\mathrm{d}\bar{y} - H(t)\,\bar{y}\,\mathrm{d}t]^2 + [\mathrm{d}\bar{z} - H(t)\,\bar{z}\,\mathrm{d}t]^2\right\}. \tag{2.58}$$

In 3-vector notation the line element is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left[\mathrm{d}\vec{\bar{x}} - H(t)\,\vec{\bar{x}}\,\mathrm{d}t\right]^2. \tag{2.59}$$

The light cones are now simply

$$\left|\frac{\mathrm{d}\vec{\bar{x}}}{\mathrm{d}t} - H(t)\,\vec{\bar{x}}\right| = 1. \tag{2.60}$$

That is

$$\frac{\mathrm{d}\vec{\bar{x}}}{\mathrm{d}t} = H(t)\,\vec{\bar{x}} + \hat{n}, \tag{2.61}$$

where $\hat{n}$ is an arbitrary unit vector in 3-space.

Since a typical galaxy (zero peculiar velocity) is moving with 3-velocity

$$\frac{\mathrm{d}\vec{\bar{x}}}{\mathrm{d}t} = H(t)\,\vec{\bar{x}}, \tag{2.62}$$

the Hubble flow lies safely inside the light cones.

Tetrad: A suitable co-tetrad is easily read off from the line element:

$$e^{\hat{t}}{}_a = (1,0,0,0)\,; \quad e^{\hat{x}}{}_a = (-H\bar{x},1,0,0)\,;$$
$$e^{\hat{y}}{}_a = (-H\bar{y},0,1,0)\,; \quad e^{\hat{z}}{}_a = (-H\bar{z},0,0,1) \tag{2.63}$$

The corresponding tetrad is thus:

$$e_{\hat{t}}{}^a = (1, H\bar{x}, H\bar{y}, H\bar{z}) \quad e_{\hat{x}}{}^a = (0,1,0,0)\,;$$
$$e_{\hat{y}}{}^a = (0,0,1,0)\ ; \quad e_{\hat{z}}{}^a = (0,0,0,1)\ . \tag{2.64}$$

Note how simple the spatial triad now is: $e_{\hat{i}}{}^j = \delta_i{}^j$. A brief computation yields the orthonormal components of the Riemann tensor

$$R_{\hat{t}\hat{i}\hat{t}\hat{j}} = -\{H^2 + \dot{H}\}\,\delta_{ij} = -\frac{\ddot{a}}{a}\,\delta_{ij}; \qquad R_{\hat{i}\hat{j}\hat{k}\hat{l}} = H^2\{\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\}. \tag{2.65}$$

The Weyl tensor is (as expected) still identically zero, while for the Einstein and Ricci tensors one has

$$G_{\hat{t}\hat{t}} = 3H^2; \qquad G_{\hat{i}\hat{j}} = -\{3H^2 + 2\dot{H}\}\delta_{ij}; \tag{2.66}$$

and

$$R_{\hat{t}\hat{t}} = -3\dot{H} - 3H^2 = -3\,\frac{\ddot{a}}{a}; \qquad R_{\hat{i}\hat{j}} = \{\dot{H} + 3H^2\}\delta_{ij}. \tag{2.67}$$

The Ricci scalar is still $R = 6\dot{H} + 12H^2$.

These orthonormal components (that is, components in the basis defined by the orthonormal tetrad) are identical (as they should be) to the standard orthonormal components defined in the usual comoving coordinates. Consequently the cosmological Friedmann equations will be unaffected. The fiducial observers, with 4-velocity $V^a = (1,\ H(t)\,\bar{x}, H(t)\,\bar{y}, H(t)\,\bar{z})$ are geodesic.

2.3.4 Summary

Cosmological Painlevé–Gullstrand coordinates, (appropriate to $k = 0$ FLRW spacetime), have some very nice features. Three-space is flat and non-expanding — but the price one pays for this is that the light cones are "tipped over" and that the galaxies are "moving" with respect to "space".

The Hubble flow is then very explicit

$$\frac{\mathrm{d}\vec{\bar{x}}}{\mathrm{d}t} = H(t)\,\vec{\bar{x}}. \tag{2.68}$$

The light cones are characterized by

$$\frac{\mathrm{d}\vec{\bar{x}}}{\mathrm{d}t} = H(t)\,\vec{\bar{x}} + \hat{n}, \qquad |\hat{n}| = 1. \tag{2.69}$$

There is as always a "conservation of difficulty" inherent in any coordinate choice; since the underling physics cannot change.

2.4 Symmetries of spatially flat FLRW: Explicit, partial, and hidden

The FLRW spacetime possesses a number of explicit symmetries (associated with Killing vectors) and hidden symmetries (associated with Killing tensors and Killing–Yano 2-forms). These symmetries are often more obvious in appropriately chosen coordinates. We present several examples below.

2.4.1 Killing vectors

The explicit symmetries of FLRW spacetime are associated with the rotational and translational Killing vectors.

Spherical symmetry

The 2-sphere S^2, with metric $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$, can be shown to have an over-complete set of linearly dependent (rotational) Killing vectors. They can most easily be chosen to be (see for instance [36, page 139]):

$$R_1 = -\sin\phi\,\partial_\theta - \frac{\cos\phi}{\tan\theta}\,\partial_\phi; \qquad R_2 = \cos\phi\,\partial_\theta - \frac{\sin\phi}{\tan\theta}\,\partial_\phi; \qquad R_3 = \partial_\phi; \tag{2.70}$$

and are subject to the constraint

$$(\cos\phi\,\tan\theta)R_1 + (\sin\phi\,\tan\theta)R_2 + R_3 = 0. \tag{2.71}$$

It is easy to check that these three vectors all satisfy Killing's equation, that is $[R_{\{1,2,3\}}]_{(a;b)} = 0$.

Note that R_3 is particularly simple; and has the obvious physical interpretation of corresponding to a translation in the azimuthal ϕ coordinate; a rotation around the poles located at $\theta \in \{0,\pi\}$. In counterpoint R_1 and R_2 at first look a little more complicated, but there is no substantial difference; they correspond to rotations around the points $(\theta = \pi/2; \phi \in \{0,\pi\})$ and $(\theta = \pi/2; \phi \in \{\pi/2, 3\pi/2\})$ respectively. (These are the points where the Killing vectors R_1 and R_2 vanish.) These Killing vectors defined on S^2 can then be bootstrapped without alteration into the generic spherically symmetric 3-space: $\mathrm{d}s^2 = g_{rr}(r)\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$.

Specifically, flat 3-space in Cartesian coordinates, with line element given by $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, is also spherically symmetric and exhibits an over-complete set of linearly dependent (rotational) Killing vectors:

$$R_1 = y\,\partial_z - z\,\partial_y; \qquad R_2 = z\,\partial_x - x\,\partial_z; \qquad R_3 = x\,\partial_y - y\,\partial_x; \tag{2.72}$$

subject to the constraint

$$x\,R_1 + y\,R_2 + z\,R_3 = 0. \tag{2.73}$$

This presentation makes manifest the intimate relationship between the (rotational) Killing vectors and the angular momentum operators of quantum mechanics. (For some specific purposes we see that Cartesian coordinates are clearly superior to spherical polar coordinates.) These Killing vectors can then be bootstrapped into the (3+1) dimensional FLRW spacetime; in any of the various coordinate systems discussed above.

Spatial translation symmetry

The FLRW spacetimes also possess 3 linearly independent spatial translation Killing vectors. For the $k = 0$ FLRW spacetime in standard comoving Cartesian coordinates, where one has $\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}x^2\}$, these spatial translation Killing vectors are simply

$$T_1 = \partial_x; \qquad T_2 = \partial_y; \qquad T_3 = \partial_z. \tag{2.74}$$

However, since $\bar{x}^i = a(t)x^i$, and we want to find the translation Killing vectors for the Painlevé–Gullstrand form of FLRW

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left\{[\mathrm{d}\bar{x} - H(t)\,\bar{x}\,\mathrm{d}t]^2 + [\mathrm{d}\bar{y} - H(t)\,\bar{y}\,\mathrm{d}t]^2 + [\mathrm{d}\bar{z} - H(t)\,\bar{z}\,\mathrm{d}t]^2\right\}, \tag{2.75}$$

we observe that

$$\frac{\partial}{\partial x^i} = \frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial}{\partial \bar{x}^a} = \frac{\partial \bar{x}^j}{\partial x^i}\frac{\partial}{\partial \bar{x}^j} + \left.\frac{\partial t}{\partial x^i}\right|_{\bar{x}}\frac{\partial}{\partial t} = a(t)\frac{\partial}{\partial \bar{x}^i}\,. \tag{2.76}$$

So in the Painlevé–Gullstrand Cartesian coordinate system the space translation Killing vectors are

$$T_1 = a(t)\,\partial_{\bar{x}}; \qquad T_2 = a(t)\,\partial_{\bar{y}}; \qquad T_3 = a(t)\,\partial_{\bar{z}}. \tag{2.77}$$

If one wishes instead to use comoving spherical polar coordinates then the spatial translation Killing vectors appear to be somewhat less intuitive

$$T_1 = \partial_x \;=\; \frac{\partial x^a}{\partial x}\frac{\partial}{\partial x^a} = \sin\theta\,\cos\phi\,\partial_r + \frac{\cos\theta\,\cos\phi}{r}\,\partial_\theta - \frac{\sin\phi}{r\sin\theta}\,\partial_\phi\,; \tag{2.78}$$

$$T_2 = \partial_y \;=\; \frac{\partial x^a}{\partial y}\frac{\partial}{\partial x^a} = \sin\theta\,\sin\phi\,\partial_r + \frac{\cos\theta\,\sin\phi}{r}\,\partial_\theta + \frac{\cos\phi}{r\sin\theta}\,\partial_\phi\,; \tag{2.79}$$

$$T_3 = \partial_z \;=\; \frac{\partial x^a}{\partial z}\frac{\partial}{\partial x^a} = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\,\partial_\theta\,. \tag{2.80}$$

That is: while one can certainly use spherical polar coordinates to describe the spatial translations, after all, it's just a coordinate change. It is, therefore, unsurprising that the relevant Killing vectors then (superficially) appear to be somewhat more complicated.

Similarly if one wishes to use Painlevé–Gullstrand spherical polar coordinates then the spatial translation Killing vectors are

$$T_1 = a(t)\,\partial_{\bar{x}} = a(t)\left\{\sin\theta\,\cos\phi\,\partial_{\bar{r}} + \frac{\cos\theta\,\cos\phi}{\bar{r}}\,\partial_\theta - \frac{\sin\phi}{\bar{r}\sin\theta}\,\partial_\phi\right\}; \tag{2.81}$$

$$T_2 = a(t)\,\partial_{\bar{y}} = a(t)\left\{\sin\theta\,\sin\phi\,\partial_{\bar{r}} + \frac{\cos\theta\,\sin\phi}{\bar{r}}\,\partial_\theta + \frac{\cos\phi}{\bar{r}\sin\theta}\,\partial_\phi\right\}; \tag{2.82}$$

$$T_3 = a(t)\,\partial_{\bar{z}} = a(t)\left\{\cos\theta\,\partial_{\bar{r}} - \frac{\sin\theta}{\bar{r}}\,\partial_\theta\right\}. \tag{2.83}$$

In short, for some purposes the use of spherical polar coordinates is less useful than one might hope.

Time translation not-quite symmetry

Since the FLRW spacetime is explicitly time dependent there is no Killing vector for time translations — however, one does have the next best thing — a conformal Killing vector for time translations. Specifically the timelike co-vector $T^\flat = -a(t)\,\mathrm{d}t$, that is $T_a = -(a(t), 0, 0, 0)$, which in comoving coordinates has vector components $T^a = a(t)\;(1, 0, 0, 0)$, and in Painlevé–Gullstrand coordinates has vector components $T^a = a(t)(1,\, H\bar{x}, H\bar{y}, H\bar{z})$, is a conformal Killing vector which satisfies[7]

$$\mathcal{L}_T\, g = \dot{a}(t)\; g\,. \tag{2.84}$$

Explicitly

$$T_{(a;b)} = \dot{a}(t)\; g_{ab}\,. \tag{2.85}$$

This is enough to guarantee a conservation law for affinely parameterized null geodesics

$$a(t)\,\frac{\mathrm{d}t}{\mathrm{d}\lambda} = (\text{constant}). \tag{2.86}$$

The existence of this timelike conformal Killing vector is ultimately the reason why the locally measured energy of freely propagating photons is proportional to the inverse of the scale factor

$$E(t)\; a(t) = (\text{constant}). \tag{2.87}$$

Equivalently, this timelike conformal Killing vector guarantees that the locally measured wavelength of freely propagating photons is proportional to the scale factor

$$\lambda(t) \propto a(t). \tag{2.88}$$

The existence of this timelike conformal Killing vector in FLRW spacetimes is often not emphasized or explained in pedagogical presentations, but is central to understanding photon propagation over cosmological distances.

[7] recall that $\mathcal{L}$ is the Lie derivative.

2.4.2 Killing tensors

In addition to the obvious Killing vectors (corresponding to rotations and spatial translations), and the trivial Killing tensors that one can build out of the metric and the Killing vectors, the $k = 0$ FLRW geometry possesses two (non-trivial) Killing tensors. These satisfy the 3-index version of Killing's equation $K_{(ab;c)} = 0$.

Spherical symmetry

Due to spherical symmetry there is a non-trivial Killing tensor (see for instance [53]) which in comoving spherical polar coordinates takes the form

$$(K_\Omega)_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b = a(t)^4 r^4 \{\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\} = (a(t)^2 r^2\, \mathrm{d}\theta)^2 + (a(t)^2 r^2 \sin\theta\, \mathrm{d}\phi)^2. \tag{2.89}$$

In components

$$(K_\Omega)_{ab} = a(t)^2 r^2 \{g_{ab} + \nabla_a t\, \nabla_b t - a(t)^2\, \nabla_a r\, \nabla_b r\}. \tag{2.90}$$

Using the Painlevé–Gullstrand $\bar{r}$ coordinate, where $\bar{r} = a(t) r$, one simply has

$$(K_\Omega)_{ab} = \bar{r}^2 \{g_{ab} + \nabla_a t\, \nabla_b t - (\nabla_a \bar{r} - H(t)\, \bar{r}\, \nabla_a t)\, (\nabla_b \bar{r} - H(t)\, \bar{r}\, \nabla_b t)\}. \tag{2.91}$$

Then in Painlevé–Gullstrand, spherical polar coordinates a brief calculation yields

$$(K_\Omega)_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b = \bar{r}^4 \{\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\} = (\bar{r}^2\, \mathrm{d}\theta)^2 + (\bar{r}^2 \sin\theta\, \mathrm{d}\phi)^2. \tag{2.92}$$

Furthermore in Painlevé–Gullstrand Cartesian coordinates one can write

$$(K_\Omega)_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \bar{y}^2 + \bar{z}^2 & -\bar{x}\bar{y} & -\bar{x}\bar{z} \\ 0 & -\bar{x}\bar{y} & \bar{x}^2 + \bar{z}^2 & -\bar{y}\bar{z} \\ 0 & -\bar{x}\bar{z} & -\bar{y}\bar{z} & \bar{x}^2 + \bar{y}^2 \end{bmatrix}, \tag{2.93}$$

that is

$$(K_\Omega)_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b = (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)(\mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2 + \mathrm{d}\bar{z}^2) - (\bar{x}\, \mathrm{d}\bar{x} + \bar{y}\, \mathrm{d}\bar{y} + \bar{z}\, \mathrm{d}\bar{z})^2. \tag{2.94}$$

All four of these different coordinate representations of the angular Killing tensor K_Ω carry the same mathematical and physical information. It is easy to check that in any of these situations K_Ω satisfies the 3-index version of Killing's equation $[K_\Omega]_{(ab;c)} = 0$.

Spatial translation symmetry

There is also a non-trivial Killing tensor associated with uniformity of the spatial slices Σ. See for instance [36, page 344]. In comoving coordinates this takes the simple form

$$(K_\Sigma)_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b = a(t)^4\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\} = a(t)^4\{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\}. \tag{2.95}$$

This can also be written as

$$(K_\Sigma)_{ab} = a(t)^2(g_{ab} + \nabla_a t\, \nabla_b t). \tag{2.96}$$

Equivalently

$$(K_\Sigma)_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b = a(t)^2\left\{\mathrm{d}s^2 + \mathrm{d}t^2\right\}. \tag{2.97}$$

Phrased in this way it is clear what happens in Painlevé–Gullstrand coordinates. First, in Painlevé–Gullstrand spherical polar coordinates, from equation (2.42) one has

$$(K_\Sigma)_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b = a(t)^2\left\{(\mathrm{d}\bar{r} - H(t)\bar{r}\mathrm{d}t)^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right\}. \tag{2.98}$$

In contrast, in Painlevé–Gullstrand Cartesian coordinates, from equation (2.58) one has

$$(K_\Sigma)_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b = a(t)^2\Big\{[\mathrm{d}\bar{x} - H(t)\,\bar{x}\,\mathrm{d}t]^2 + [\mathrm{d}\bar{y} - H(t)\,\bar{y}\,\mathrm{d}t]^2 + [\mathrm{d}\bar{z} - H(t)\,\bar{z}\,\mathrm{d}t]^2\Big\}. \tag{2.99}$$

It is easy to check that in any of these situations K_Σ satisfies the 3-index version of Killing's equation $[K_\Omega]_{(ab;c)} = 0$.

2.4.3 Killing–Yano tensor

A Killing–Yano 2-form $Y_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b$ satisfies the differential equation $Y_{a(b;c)} = 0$. Thus, if we define $K_{ab} = Y_{ae}\, g^{ef}\, Y_{fb}$, then (with indices between vertical bars not being included in the symmetrization process) we have

$$K_{(ab;c)} = Y_{(a|e|;c}\, g^{ef}\, Y_{|f|b)} + Y_{(a|e}\, g^{ef}\, Y_{f|b;c)} = 0 + 0 = 0. \tag{2.100}$$

That is, the existence of a Killing–Yano 2-form implies the existence of a 2-index Killing tensor.

Specifically, the existence of the 2-index Killing tensor $(K_\Omega)_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b$ that is associated with spherical symmetry is related to the existence of a Killing–Yano 2-form $(Y_\Omega)_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b$. In comoving spherical polar coordinates

$$(Y_\Omega)_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b = a(t)^3 r^3 \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi = \frac{(a(t)^2 r^2\, \mathrm{d}\theta) \wedge (a(t)^2 r^2 \sin\theta\, \mathrm{d}\phi)}{a(t) r}. \tag{2.101}$$

In our Painlevé–Gullstrand spherical polar coordinates one simply has

$$(Y_\Omega)_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b = \bar{r}^3 \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi = \frac{(\bar{r}^2\, \mathrm{d}\theta) \wedge (\bar{r}^2 \sin\theta\, \mathrm{d}\phi)}{\bar{r}}. \tag{2.102}$$

The Killing–Yano tensor is colloquially referred to as the square root of the Killing tensor: $K_{ad} = Y_{ab}\, g^{bc}\, Y_{cd}$. Because the Killing–Yano tensor is a 2-form, represented by an anti-symmetric matrix, if it is nonzero it can only have rank 2 or rank 4; which then forces the associated Killing tensor to either have rank 2 or rank 4. Since the Killing tensor associated with uniformity of the spatial slices is manifestly rank 3, that particular Killing tensor will not have an associated Killing–Yano 2-form.

2.4.4 Summary

FLRW spacetimes possess significant symmetry structure. The spatial and rotational Killing vectors are the most obvious symmetries, but they are far from the only symmetries. The timelike conformal Killing vector can be viewed as an approximate symmetry, one that still leads to a conservation law for null geodesics.

More subtle are the "hidden" symmetries encoded in the non-trivial Killing tensors and the Killing–Yano tensor. The specific choice of coordinate system can make some of these symmetries manifest, at the cost of making other symmetries less obvious.

2.5 Cosmological horizons

Cosmological horizons can be quite tricky to properly define and interpret [73]. While event horizons are mathematically 'clean' concepts, and their use underlies many of the singularity theorems, there is a precise technical sense in which any physical observer (represented by a finite-size finite-duration laboratory) cannot ever, *even in principle*, detect an event horizon [74]. The point is that event horizons are *teleological*, and defining them requires one to back-track from the trump of doom[8]. (Quasi-local horizons are much better behaved in this regard; quasi-local horizons can be detected using finite-size finite-duration laboratories.) In the words of Stephen Hawking [75] (applied in the context of black hole physics):

> "The absence of event horizons means that there are no black holes — in the sense of regimes from which light can't escape to infinity. There are however apparent horizons which persist for a period of time."

Similar, related but distinct, issues arise in cosmology. One must be very careful to distinguish quasi-local horizons from causal horizons.

[8]Of course, this is a much more grand trump of doom than just the end of days on Earth!

2.5.1 Apparent horizon (Hubble sphere)

Consider the 2-sphere located at $\bar{r}(t)$, with area

$$S(t) = 4\pi\,\bar{r}(t)^2, \tag{2.103}$$

and ask how this area evolves as the 2-sphere expands or contracts at the speed of light

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} = H(t)\,\bar{r} \pm 1. \tag{2.104}$$

Then for outgoing light rays

$$\dot{S}_+(t) = 8\pi\bar{r}\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\right)_+ = 8\pi\bar{r}\,(H(t)\,\bar{r}+1) > 0, \tag{2.105}$$

while for ingoing light rays

$$\dot{S}_-(t) = 8\pi\bar{r}\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\right)_- = 8\pi\bar{r}\,(H(t)\,\bar{r}-1)\,. \tag{2.106}$$

Note that $\dot{S}_-(t)$ changes sign at $\bar{r}(t) = H(t)^{-1}$. That is, an apparent horizon is present at the Hubble sphere $\bar{r}_{\mathrm{Hubble}}(t) = H(t)^{-1}$ (sometimes called the "speed of light sphere").

One could also work in comoving coordinates where

$$S(t) = a(t)^2\,4\pi\,r(t)^2, \tag{2.107}$$

and

$$\dot{S}_\pm(t) = 8\pi\left\{a(t)\dot{a}(t)r(t)^2 + a(t)^2 r(t)\dot{r}_\pm(t)\right\} = S(t)\left\{H \pm \frac{1}{a(t)r(t)}\right\}. \tag{2.108}$$

There is again an apparent horizon when $\dot{S}_- = 0$, at the same physical location where

$$\bar{r}_{\mathrm{Hubble}}(t) = a(t)\,r_{\mathrm{Hubble}}(t) = H(t)^{-1}. \tag{2.109}$$

This apparent horizon is emphatically not a causal horizon; there is no obstruction to crossing an apparent horizon.[9] If one wishes to work in SI units, reinstating the speed of light, then

$$\bar{r}_{\mathrm{Hubble}}(t) = a(t)\;r_{\mathrm{Hubble}}(t) = \frac{c}{H(t)}. \tag{2.110}$$

[9]Despite claims sometimes made in the literature, the Hubble radius is *not* "the distance light travels since the Big Bang".

2.5.2 Particle horizon (causal horizon)

In contrast the particle horizon is a causal horizon determined by how far an outward moving light ray could move from its source (or equivalently how far an incoming light ray could move towards its reception point). For definiteness let us assume the light ray is emitted at time $t = 0$ at location $\bar{r} = 0$, then one is interested in solving the differential equation

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} = H(t)\,\bar{r} + 1. \tag{2.111}$$

Equivalently

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} - \frac{\dot{a}}{a}\,\bar{r} = a\,\frac{\mathrm{d}(\bar{r}/a)}{\mathrm{d}t} = 1, \tag{2.112}$$

where

$$\mathrm{d}(\bar{r}/a) = \frac{\mathrm{d}t}{a}, \tag{2.113}$$

This has the obvious solution (t_* being the time of the Big Bang when $a(t_*) = 0$)

$$\frac{\bar{r}(t)}{a(t)} = r(t) = \int_{t_*}^{t} \frac{\mathrm{d}t}{a(t)} = \eta(t). \tag{2.114}$$

Equivalently

$$\bar{r}_{\text{particle}}(t) = a(t)\; r_{\text{particle}}(t) = a(t)\;\eta(t). \tag{2.115}$$

This particle horizon is by construction a causal horizon. Note that the particle horizon has a very simple representation in terms of the conformal time coordinate.

2.5.3 Summary

The apparent horizon (Hubble sphere) and particle horizon are distinct concepts, and can occur at radically different locations:

$$\bar{r}_{\text{Hubble}}(t) = a(t)\,r_{\text{Hubble}}(t) = \frac{c}{H(t)}, \tag{2.116}$$

versus

$$\bar{r}_{\text{particle}}(t) = a(t)\,r_{\text{particle}}(t) = a(t)\int_{t_*}^{t} \frac{\mathrm{d}t}{a(t)} = a(t)\,\eta(t). \tag{2.117}$$

In particular, since the Hubble sphere is not a causal horizon, one should *not* attempt to apply "causality" arguments to the Hubble sphere.

What is always true based on dimensional analysis is that

$$\bar{r}_{\text{particle}}(t) = \bar{r}_{\text{Hubble}}(t) \times (\text{dimensionless number}). \tag{2.118}$$

However, there is absolutely no reason for this dimensionless number to be of order unity. In fact in the presence of cosmological inflation, whether it be exponential inflation, $a(t) \sim \exp(H_{\text{inflation}}\, t)$, or power law inflation, $a(t) \sim t^n$ with $n \in (0, 1]$, the integral $\eta(t) = \int_{t_*}^{t} \frac{\mathrm{d}t}{a(t)}$ formally diverges, pushing the Big Bang out to negative infinity in conformal time, $\eta_* \to -\infty$, while pushing the particle horizon out to positive infinity, $\bar{r}_{\text{particle}} \to +\infty$. Even if cosmological inflation switches on and off at some finite time, the particle horizon can be made arbitrarily large compared to the Hubble radius.

2.6 de Sitter spacetime

The de Sitter spacetime is most typically presented in static coordinates:

$$\mathrm{d}s^2 = -\left(1 - H^2\bar{r}^2\right)\mathrm{d}\bar{t}^2 + \frac{\mathrm{d}\bar{r}^2}{1 - H^2\bar{r}^2} + \bar{r}^2\mathrm{d}\Omega^2. \tag{2.119}$$

For this line element the Einstein tensor is $G_{ab} = -(3H^2)g_{ab}$, corresponding to a pure cosmological constant. Using the coordinate transformation

$$\bar{t} = t + \int \frac{H\bar{r}}{1 - H^2\bar{r}^2}\,\mathrm{d}\bar{r} = t + \frac{\ln(1 - H^2\bar{r}^2)}{2H}, \tag{2.120}$$

we can cast de Sitter spacetime into Painlevé–Gullstrand form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + [\mathrm{d}\bar{r} - H\bar{r}\,\mathrm{d}t]^2 + \bar{r}^2\mathrm{d}\Omega^2. \tag{2.121}$$

Finally, to make it abundantly clear that de Sitter spacetime is just a special case of FLRW spacetime, consider the specific coordinate transformation $\bar{r} = r\,e^{Ht}$, so that $\mathrm{d}\bar{r} = e^{Ht}(\mathrm{d}r + Hr\mathrm{d}t)$, and use this to recast the Painlevé–Gullstrand form of the de Sitter spacetime in the comoving form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}\left\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right\}. \tag{2.122}$$

Let us now generalize this discussion, first to the Kottler (Schwarzschild-de Sitter) spacetime, (which already presents a few subtleties), and then to the more complex and subtle McVittie spacetime.

2.7 Kottler spacetime

2.7.1 Standard form of Kottler

The Kottler (Schwarzschild–de Sitter) spacetime is most typically presented in static coordinates [76]:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{\bar{r}} - H^2\bar{r}^2\right)\mathrm{d}\bar{t}^2 + \frac{\mathrm{d}\bar{r}^2}{1 - \frac{2m}{\bar{r}} - H^2\bar{r}^2} + \bar{r}^2\mathrm{d}\Omega^2. \tag{2.123}$$

For this line element the Einstein tensor is $G_{ab} = -(3H^2)g_{ab}$, corresponding to pure cosmological constant, plus a central "point mass"[10]. The fiducial observers are in this situation best taken to be integral curves of the timelike Killing vector, and so are described by the non-geodesic 4-velocity field

$$V^a = \frac{1}{\sqrt{1-2m/\bar{r}-H^2\bar{r}^2}}\,(1,0,0,0)\,; \quad V_a = \sqrt{1-2m/\bar{r}-H^2\bar{r}^2}\,(-1,0,0,0). \tag{2.124}$$

Here the 4-acceleration for this set of fiducial observers is

$$A^a = V^b\nabla_b V^a = \left(0,\,\frac{m}{\bar{r}^2} - H^2\bar{r}\,,0,0\right). \tag{2.125}$$

2.7.2 Five variant forms of Kottler

Under suitable coordinate changes, we first present three alternative Painlevé–Gullstrand-like formulations of the Kottler spacetime:

- Using the coordinate transformation

  $$\bar{t} = t + \int \frac{H\bar{r}}{\sqrt{1-2m/\bar{r}}(1-2m/\bar{r}+H^2\bar{r}^2)}\mathrm{d}\bar{r}\,, \tag{2.126}$$

  we cast the metric into the form

  $$\mathrm{d}s^2 = -\left(1-\frac{2m}{\bar{r}}\right)\mathrm{d}t^2 + \frac{[\mathrm{d}\bar{r} - H\bar{r}\,\sqrt{1-2m/\bar{r}}\,\mathrm{d}t]^2}{1-\frac{2m}{\bar{r}}} + \bar{r}^2\mathrm{d}\Omega^2, \tag{2.127}$$

  which we can also write as

  $$\mathrm{d}s^2 = -\left(1-\frac{2m}{\bar{r}}\right)\mathrm{d}t^2 + \left[\frac{\mathrm{d}\bar{r}}{\sqrt{1-2m/\bar{r}}} - H\bar{r}\,\mathrm{d}t\right]^2 + \bar{r}^2\mathrm{d}\Omega^2. \tag{2.128}$$

  This form of the metric neatly disentangles the local physics, (depending only on the point mass m), from the cosmological physics (depending only on the Hubble parameter H). Specifically, as $m \to 0$ this becomes de Sitter space in Painlevé–Gullstrand form (2.121), whereas if $H \to 0$ this becomes Schwarzschild in standard form.

  The fiducial observers (4-orthogonal to the spatial slices, so $V^\flat \propto \mathrm{d}t$) are in this situation described by the non-geodesic 4-velocity field

  $$V^a = \left(\frac{1}{\sqrt{1-2m/\bar{r}}}, Hr, 0, 0\right); \qquad V_a = \sqrt{1-2m/\bar{r}}\,(-1,0,0,0). \tag{2.129}$$

  Here the 4-acceleration is

  $$A^a = V^b\nabla_b V^a = \left(0,\,\frac{m}{\bar{r}^2}\,,0,0\right). \tag{2.130}$$

[10] While most physicists would likely call this mass a point mass, the more mathematical relativists would likely view the mass concentrated in the singularities on the boundary of the spacetime.

- Using the coordinate transformation

$$\bar{t} = t + \int \frac{\sqrt{2m/\bar{r}}}{\sqrt{1 - H^2\bar{r}^2}\,(1 - 2m/\bar{r} + H^2\bar{r}^2)} \mathrm{d}\bar{r}\,, \tag{2.131}$$

we have another *partial* Painlevé–Gullstrand form

$$\mathrm{d}s^2 = -\left(1 - H^2\bar{r}^2\right)\mathrm{d}t^2 + \frac{[\mathrm{d}\bar{r} - \sqrt{2m/\bar{r}}\,\sqrt{1 - H^2\bar{r}^2}\,\mathrm{d}t]^2}{1 - H^2\bar{r}^2} + \bar{r}^2\mathrm{d}\Omega^2, \tag{2.132}$$

which we can also write as

$$\mathrm{d}s^2 = -\left(1 - H^2\bar{r}^2\right)\mathrm{d}t^2 + \left[\frac{\mathrm{d}\bar{r}}{\sqrt{1 - H^2\bar{r}^2}} - \sqrt{2m/\bar{r}}\;\mathrm{d}t\right]^2 + \bar{r}^2\mathrm{d}\Omega^2, \tag{2.133}$$

As $m \to 0$ this becomes de Sitter in static form (2.119), whereas if $H \to 0$ this becomes Schwarzschild in Painlevé–Gullstrand form.

The fiducial observers (4-orthogonal to the spatial slices, so $V^\flat \propto \mathrm{d}t$) are in this situation described by the non-geodesic 4-velocity field

$$V^a = \left(\frac{1}{\sqrt{1 - H^2\bar{r}^2}}, \sqrt{\frac{2m}{\bar{r}}}, 0, 0\right); \qquad V_a = \sqrt{1 - H^2\bar{r}^2}\,(-1, 0, 0, 0). \tag{2.134}$$

Here the 4-acceleration is

$$A^a = V^b\nabla_b V^a = \left(0,\; H^2\bar{r}\,, 0, 0\right). \tag{2.135}$$

- Using the coordinate transformation

$$\bar{t} = t + \int \frac{\sqrt{2m/\bar{r} + H^2\bar{r}^2}}{1 - 2m/\bar{r} - H^2\bar{r}^2}\;\mathrm{d}\bar{r}\,, \tag{2.136}$$

we have the *full* Painlevé–Gullstrand form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left[\mathrm{d}\bar{r} - \sqrt{2m/\bar{r} + H^2\bar{r}^2}\;\mathrm{d}t\right]^2 + \bar{r}^2\mathrm{d}\Omega^2. \tag{2.137}$$

As $m \to 0$ this becomes de Sitter in Painlevé–Gullstrand form (2.121), whereas if $H \to 0$ this becomes Schwarzschild in Painlevé–Gullstrand form. The fiducial observers (4-orthogonal to the spatial slices, so $V^\flat \propto \mathrm{d}t$) are in this situation described by the geodesic 4-velocity field

$$V^a = \left(1, \sqrt{2m/\bar{r} + H^2\bar{r}^2}, 0, 0\right); \qquad V_a = (-1, 0, 0, 0). \tag{2.138}$$

It is easy to check that the 4-acceleration is zero: $A^a = V^b\nabla_b V^a = 0$.

These four line elements (2.123)-(2.128)-(2.133)-(2.137) are all equally valid slicings of the Kottler spacetime; in all cases the Einstein tensor is $G_{ab} = -(3H^2)g_{ab}$, corresponding to pure cosmological constant (in the presence of a central point mass). Depending on one's choice of slicing, one could make different choices of fiducial observer, focussing on different aspects of the physics.

Finally to make it abundantly clear that Kottler spacetime is just a special case of Schwarzschild embedded in a specific FLRW (de Sitter) spacetime, consider the coordinate transformation $\bar{r} = re^{Ht}$, so that $\mathrm{d}\bar{r} = e^{Ht}(\mathrm{d}r + Hr\mathrm{d}t)$, and use this to recast the Painlevé–Gullstrand form of the Kottler spacetime (2.137) into the not entirely obvious comoving form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}\left\{\left[\mathrm{d}r + \left(Hr - \sqrt{2me^{-3Ht}/r + H^2r^2}\right)\,\mathrm{d}t\right]^2 + r^2\mathrm{d}\Omega^2\right\}. \tag{2.139}$$

Expanding, we have

$$\begin{aligned}\mathrm{d}s^2 \;=\; &-\left\{1 - e^{2Ht}\left(Hr - \sqrt{2me^{-3Ht}/r + H^2r^2}\right)^2\right\}\mathrm{d}t^2 \\ &-2\left(Hr - \sqrt{2me^{-3Ht}/r + H^2r^2}\right)\mathrm{d}r\mathrm{d}t + e^{2Ht}\left\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right\}.\end{aligned} \tag{2.140}$$

It is relatively easy to explicitly check that the Einstein tensor is still $G_{ab} = -3H^2 g_{ab}$.

In this form the connection between Kottler spacetime and spatially flat $k = 0$ FLRW is manifest since the limit $m \to 0$ simply yields

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}\left\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right\}. \tag{2.141}$$

The fiducial observers for (2.139) or (2.140) are described by the geodesic 4-velocity field

$$V^a = \left(1, -\left[Hr - \sqrt{2me^{-3Ht}/r + H^2r^2}\right], 0, 0\right); \qquad V_a = (-1, 0, 0, 0). \tag{2.142}$$

It is relatively easy to check that the 4-acceleration is zero: $A^a = V^b\nabla_b V^a = 0$.

In the same manner we can convert the (2.128) form of the Kottler spacetime into a distinct not entirely obvious comoving form

$$\begin{aligned}\mathrm{d}s^2 = -\left(1 - \frac{2me^{-Ht}}{r}\right)\mathrm{d}\bar{t}^2 + e^{2Ht}&\left\{\frac{\left(\mathrm{d}r + Hr\left[1 - \sqrt{1 - 2me^{-Ht}/r}\right]\mathrm{d}t\right)^2}{1 - \frac{2me^{-Ht}}{r}}\right\} \\ &+ e^{2Ht}\left\{r^2\mathrm{d}\Omega^2\right\}.\end{aligned} \tag{2.143}$$

It is relatively easy to explicitly check that the Einstein tensor is still $G_{ab} = -3H^2 g_{ab}$. Furthermore, as $m \to 0$ one recovers (2.122) the comoving slicing of the de Sitter spacetime.

The fiducial observers 4-orthogonal to the spatial slices are in this situation described by the non-geodesic 4-velocity field

$$V^a = \frac{\left(1,\ Hr\left[1-\sqrt{1-2me^{-Ht}/r}\right],0,0\right)}{\sqrt{1-2me^{-Ht}/r}};$$
$$V_a = \sqrt{1-2me^{-Ht}/r}\ (-1,0,0,0)\,. \tag{2.144}$$

Here the 4-acceleration is

$$A^a = V^b\nabla_b V^a = \left(0,\ \frac{me^{-3Ht}}{\tilde{r}^2},0,0\right). \tag{2.145}$$

2.7.3 Summary

Whereas the Kottler (Schwarzschild–de Sitter) spacetime is most commonly presented in static coordinates (2.123), it can with a little work be converted into Painlevé–Gullstrand form (2.128)-(2.133)-(2.137), and therefore into comoving coordinates — as per (2.139)–(2.140) and (2.143) above. While finding the required coordinate transformations is relatively straightforward, the process is not entirely obvious.

2.8 McVittie spacetime

The McVittie spacetime [77, 78, 79, 80] is a perfect fluid spacetime that is as close as one can get to modelling a Schwarzschild black hole embedded in an arbitrary FLRW spacetime.

2.8.1 Traditional form of McVittie spactime

It is traditional to work in isotropic coordinates, where for $k=0$ the McVittie line element is given by the equivalent of [77]:

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m}{2a(t)\tilde{r}}}{1+\frac{m}{2a(t)\tilde{r}}}\right)^2 \mathrm{d}t^2 + \left(1+\frac{m}{2a(t)\tilde{r}}\right)^4 a(t)^2\{\mathrm{d}\tilde{r}^2+\tilde{r}^2\mathrm{d}\Omega^2\}. \tag{2.146}$$

- For $a(t)=1$ this is Schwarzschild spacetime in isotropic coordinates.
- For $m=0$ this is a generic spatially flat $k=0$ FLRW spacetime.
- While not entirely obvious, for $a(t)=e^{Ht}$ this is indeed Kottler (Schwarzschild-de Sitter) spacetime in disguise.

In these coordinates the fiducial observer (4-orthogonal to the spatial slices) has 4-velocity

$$V^a = \left(\frac{1+\frac{m}{2a(t)\tilde{r}}}{1-\frac{m}{2a(t)\tilde{r}}}\right)(1,0,0,0); \tag{2.147}$$

and the unit radial vector is

$$R^a = \left(1 + \frac{m}{2a(t)\tilde{r}}\right)^{-2} \frac{1}{a(t)}\,(0,1,0,0). \tag{2.148}$$

Straightforward computation yields the orthonormal stress-energy components. The density is particularly simple,

$$\rho = \frac{3}{8\pi}\frac{\dot{a}^2}{a^2} = \frac{3}{8\pi}H^2, \tag{2.149}$$

whereas the pressure is slightly more complicated

$$\begin{aligned} p &= \frac{1}{8\pi}\left\{-\frac{2a\ddot{a}+\dot{a}^2}{a^2} - \frac{4m/[2a\tilde{r}]}{1-m/[2a\tilde{r}]}\;\frac{a\ddot{a}-\dot{a}^2}{a^2}\right\} \\ &= \frac{1}{8\pi}\left\{-3H^2 - 2\dot{H} - \frac{4m/[2a\tilde{r}]}{1-\frac{m}{2a\tilde{r}}}\;\dot{H}\right\} \\ &= \frac{1}{8\pi}\left\{-3H^2 - 2\,\frac{1+\frac{m}{2a\tilde{r}}}{1-\frac{m}{2a\tilde{r}}}\;\dot{H}\right\}. \end{aligned} \tag{2.150}$$

All other components of the stress-energy are zero. Note that the energy density is identically that of FLRW, while the pressure asymptotes to that of FLRW. In view of the fact that there is a non-zero pressure gradient the fiducial observers, being in this situation defined by the fluid flow, will now not be geodesic. In fact the fiducial observers have 4-acceleration

$$A^a = V^b\nabla_b V^a = \frac{m}{a^2\bar{r}^2(1+\frac{m}{2a\tilde{r}})^3(1-\frac{m}{2a\tilde{r}})}\;R^a, \tag{2.151}$$

and satisfy the Euler equation of fluid equilibrium

$$(\rho+p)A^a = -\left(g^{ab}+V^aV^b\right)\nabla_b p. \tag{2.152}$$

2.8.2 McVittie spacetime in Schwarzschild radial coordinates

Kaloper–Kleban–Martin [78] rewrite the McVittie line element by defining the Schwarzschild radial coordinate $\bar{r}$ by

$$\bar{r} = \left(1+\frac{m}{2a(t)\tilde{r}}\right)^2 a(t)\tilde{r}, \tag{2.153}$$

and transforming the line element into the equivalent of

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{\bar{r}}\right)\mathrm{d}t^2 + \frac{[\mathrm{d}\bar{r} - \sqrt{1-2m/\bar{r}}\;H(t)\bar{r}\;\mathrm{d}t]^2}{1-2m/\bar{r}} + \bar{r}^2\mathrm{d}\Omega^2. \tag{2.154}$$

Let us rewrite this as

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{\bar{r}}\right)\mathrm{d}t^2 + \left[\frac{\mathrm{d}\bar{r}}{\sqrt{1-2m/\bar{r}}} - H(t)\bar{r}\mathrm{d}t\right]^2 + \bar{r}^2\mathrm{d}\Omega^2. \tag{2.155}$$

This form of the metric again neatly disentangles the local physics, (depending only on the point mass m), from the cosmological physics, (depending only on the Hubble parameter $H(t)$, which is now allowed to be time-dependent). Specifically, setting $H(t) \to H$ yields equation (2.128), one of the representations of Kottler spacetime, while setting $m \to 0$ yields equation (2.42) one of the Painlevé–Gullstrand representations of $k = 0$ FLRW spacetime.

The Eulerian observer has 4-velocity

$$V_a = \sqrt{1 - 2m/\bar{r}}\,(-1, 0, 0, 0); \qquad V^a = \left(\frac{1}{\sqrt{1 - 2m/\bar{r}}}, H(t)\bar{r}, 0, 0\right); \tag{2.156}$$

and the unit radial vector is

$$R^a = \sqrt{1 - 2m/\bar{r}}\,(0, 1, 0, 0); \qquad R_a = \left(-H(t)\bar{r}, \frac{1}{\sqrt{1 - 2m/\bar{r}}}, 0, 0\right). \tag{2.157}$$

The density and pressure are now

$$\rho = \frac{3H(t)^2}{8\pi}; \qquad p = -\rho + \frac{\dot{H}(t)}{4\pi\sqrt{1 - 2m/\bar{r}}}. \tag{2.158}$$

The fiducial observers have non-zero 4-acceleration

$$A^a = \left(0, \frac{m}{\bar{r}^2}, 0, 0\right), \tag{2.159}$$

and satisfy the Euler equation of fluid equilibrium

$$(\rho + p)A^a = -\left(g^{ab} + V^a V^b\right)\nabla_b p. \tag{2.160}$$

2.8.3 McVittie spacetime in comoving radial coordinates

Now set $\bar{r} = a(t)r$, so that $\mathrm{d}\bar{r} = a(t)\,(\mathrm{d}r + H(t)r\mathrm{d}t)$. Then

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{a(t)r}\right)\mathrm{d}t^2 + \left[\frac{a(t)(\mathrm{d}r + H(t)r\mathrm{d}t)}{\sqrt{1 - \frac{2m}{a(t)r}}} - H(t)a(t)r\mathrm{d}t\right]^2 + a(t)^2 r^2 \mathrm{d}\Omega^2. \tag{2.161}$$

We, therefore, obtain a comoving form of the McVittie spacetime

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{a(t)r}\right)\mathrm{d}t^2 + a(t)^2\left\{\left[\frac{\left(\mathrm{d}r + H(t)r\left[1 - \sqrt{1 - \frac{2m}{a(t)r}}\right]\mathrm{d}t\right)^2}{1 - \frac{2m}{a(t)r}}\right] + r^2\mathrm{d}\Omega^2\right\}. \tag{2.162}$$

- For $a(t) = 1$, so that $H(t) = 0$, this is Schwarzschild spacetime in standard coordinates.

- For $a = e^{Ht}$, so that $H(t) \to H$, this is equivalent to the (2.143) representation of Kottler spacetime.
- For $m = 0$ this is standard $k = 0$ FLRW spacetime in comoving coordinates.

The natural Eulerian observer (the closest you can get to defining the Hubble flow) is specified by the unit 4-vector

$$V^a = \frac{\left(1, -H(t)r\left[1 - \sqrt{1 - \frac{2m}{a(t)r}}\right], 0, 0\right)}{1 - \frac{2m}{a(t)r}}. \tag{2.163}$$

This corresponds to the covector

$$V_a = \sqrt{1 - \frac{2m}{a(t)r}}\,(-1, 0, 0, 0). \tag{2.164}$$

The unit radial 4-vector is

$$R^a = \frac{\sqrt{1 - \frac{2m}{a(t)r}}}{a(t)}\;(0, 1, 0, 0). \tag{2.165}$$

This corresponds to the covector

$$R_a = \frac{a(t)}{\sqrt{1 - \frac{2m}{a(t)r}}}\left(H(t)r\left[1 - \sqrt{1 - \frac{2m}{a(t)r}}\right], 1, 0, 0\right). \tag{2.166}$$

In the appropriate orthonormal basis the energy density is still

$$\rho = \frac{3}{8\pi}\frac{\dot{a}^2}{a^2} = \frac{3}{8\pi}H(t)^2 \tag{2.167}$$

while the pressure now becomes

$$p \;=\; \frac{1}{8\pi}\left\{-3H(t)^2 + 2\,\frac{\dot{H}(t)}{\sqrt{1 - \frac{2m}{a(t)r}}}.\right\} \tag{2.168}$$

All other components of the stress-energy are zero. The fiducial observers have non-zero 4-acceleration

$$A^a = \left(0, \frac{m}{r^2 a(t)^3}, 0, 0\right), \tag{2.169}$$

and satisfy the Euler equation of fluid equilibrium

$$(\rho + p)A^a = -\left(g^{ab} + V^a V^b\right)\nabla_b p. \tag{2.170}$$

2.8.4 McVittie spacetime in (conformal) Painlevé–Gullstrand form

On quite general grounds, (since McVittie spacetime is spherically symmetric, does not possess any wormhole throats, and has a non-negative Misner–Sharp quasi-local mass), a (full) Painlevé–Gullstrand form for McVittie spacetime must exist [51]. However, as Faraoni has pointed out [80, 46], that (full) Painlevé–Gullstrand form depends on a quite messy (and implicit) integrating factor, over which one has little to no control; making the (full) Painlevé–Gullstrand form completely explicit seems a formidable task. Fortunately, there is an intermediate step, a *conformal* Painlevé–Gullstrand form, that is much easier to make fully explicit.

Start with McVittie spacetime in traditional form:

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m}{2a(t)\tilde{r}}}{1+\frac{m}{2a(t)\tilde{r}}}\right)^2 \mathrm{d}t^2 + \left(1+\frac{m}{2a(t)\tilde{r}}\right)^4 a(t)^2\{\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2\}. \qquad (2.171)$$

Define $\bar{r} = a(t)\tilde{r}$. Then as usual $\mathrm{d}\tilde{r} = \mathrm{d}(\bar{r}/a) = (\mathrm{d}\bar{r} - H(t)\,\bar{r}\,\mathrm{d}t)/a$, and so

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m}{2\bar{r}}}{1+\frac{m}{2\bar{r}}}\right)^2 \mathrm{d}t^2 + \left(1+\frac{m}{2\bar{r}}\right)^4 \{[\mathrm{d}\bar{r} - H(t)\,\bar{r}\,\mathrm{d}t]^2 + \bar{r}^2\mathrm{d}\Omega^2\}. \qquad (2.172)$$

This is not quite of Painlevé–Gullstrand form; but it is *conformal* to Painlevé–Gullstrand form:

$$\mathrm{d}s^2 = \left(1+\frac{m}{2\bar{r}}\right)^4 \left\{-\left(\frac{[1-\frac{m}{2\bar{r}}]^2}{[1+\frac{m}{2\bar{r}}]^6}\right)\mathrm{d}t^2 + \{[\mathrm{d}\bar{r} - H(t)\,\bar{r}\,\mathrm{d}t]^2 + \bar{r}^2\mathrm{d}\Omega^2\}\right\}. \qquad (2.173)$$

Note the spatial slices are conformally flat, and both the conformal factor and lapse function are time independent — the only time-dependence has now been isolated in the Hubble parameter $H(t)$. Straightforward computation yields the temporal and radial legs of the tetrad

$$V^a = \frac{1+\frac{m}{2r}}{1-\frac{m}{2r}}\left(1,\ H(t)r, 0, 0\right); \qquad R^a = \left(0,\ \frac{1}{1+\frac{m}{2r}}, 0, 0\right). \qquad (2.174)$$

The orthonormal stress-energy components are:

$$\rho = \frac{3}{8\pi}H(t)^2; \qquad p = \frac{1}{8\pi}\left\{-3H(t)^2 - 2\frac{1+\frac{2m}{\bar{r}}}{1-\frac{m}{2\bar{r}}}\ \dot{H}(t)\right\}. \qquad (2.175)$$

All other components of the stress-energy are zero. As required this is a perfect fluid, and as $H(t) \to H$, so that $\dot{H} = 0$, one recovers the isotropic form of Kottler spacetime.

2.8.5 Summary

We have now extracted 4 equivalent forms of the McVittie spacetime — the traditional (2.146), the Schwarzschild variant (2.154)–(2.155), a comoving variant (2.162), and finally a conformally Painlevé–Gullstrand (2.173) variant.

2.9 Discussion

Overall, we have seen that coordinate freedom in cosmology can be used to repackage and reorganize standard cosmological models in multiple different ways. This repackaging and reorganization can often simplify *some* aspects of the physics, while making *other* aspects seem more (apparently) complex.

- We have explored three specific ways of rewriting the generic $k = 0$ FLRW cosmologies; equations (2.28), (2.35), (2.42), and their Cartesian versions (2.31), (2.31), (2.57), there are many others. The three we have explored either make the Hubble flow simple, or make the light cones simple, or make the spatial slices simple. But there is no free lunch; the underlying physics is invariant.
- We have similarly considered three versions of de Sitter space, (2.119), (2.121), (2.122); either making the spacetime manifestly static, or making the spatial slices flat, or making the connection to generic FLRW manifest.
- For the Kottler spacetime we have developed six different line elements, (2.123), (2.127), (2.132), (2.137), (2.139), (2.143); focussing on different aspects of the physics. One either makes the spacetime manifestly static, or has three ways to make the spatial slices relatively simple, or has two ways to make the connection to generic FLRW manifest. There are yet other possibilities that one might explore.
- For the McVittie spacetime we presented four different line elements, (2.146), (2.154), (2.162), (2.173), two of which seem to be novel. The traditional version (2.146) is spatially isotropic, but every nonzero metric component is explicitly time dependent. The Schwarzschild version (2.154) sets three metric components to be time independent, and tightly constrains the time dependence of the remaining terms. The "comoving" line element (2.162) makes the connection with generic $k = 0$ FLRW manifest, while the *conformally* Painlevé–Gullstrand version (2.173) makes the spatial slices time independent and eliminates explicit occurrences of the scale factor $a(t)$ in favour of the Hubble parameter $H(t)$.

Perhaps the most bizarre feature of the above discussion is that one can apparently eliminate the expansion of the universe with a suitable choice of coordinates; of course there is then a *different* price to pay — the light cones then "tip over" and one must be *much* more careful when deciding which trajectories are now to be regarded as "superluminal". Specifically, the variant presentations of the Kottler and McVittie line elements give one a much better handle on how to merge the gravitational field of a non-perturbative localized compact object with the Hubble flow of an asymptotically FLRW cosmology. Finally we point out that familiarity with these variant coordinate systems is also helpful in understanding the symmetries (both explicit and hidden); and in demystifying the horizon structure. Ultimately we would be interested in extending these ideas to generic non-perturbative deviations from FLRW cosmology.

Chapter 3

Black Holes Embedded in FLRW Cosmologies

There has recently been some considerable interest expressed in a highly speculative model of black hole evolution — allegedly by a postulated direct coupling between black holes and cosmological expansion *independently of accretion or mergers*. We wish to make several cautionary comments in this regard. At least three exact solutions corresponding to black holes embedded in a FLRW background are known, (Kottler, McVittie, Kerr-de Sitter), and they show no hint of this claimed effect.Therefore, implying that this claimed effect (if it exists at all) is certainly nowhere near ubiquitous.

The dark sector of the Universe poses an immense problem for our current understanding of physics. Dark matter is constrained by many observations — dating back to Lord Kelvin who presented his theory on this elusive type of matter in 1884 [81]. Dark matter was further constrained by galaxy rotation curves [82, 83, 84]. While there has been some progress made recently in relaxing this constraint by using "full GR" [23] there are still many observations that constrain dark matter[1].

Dark energy supposedly makes up a much larger percentage of the energy density of the Universe at the current epoch. This dark energy drives the accelerated expansion of the Universe that we observe. While dark energy may constitute a larger percentage of the Universe, it remains undetected and there is no accepted source of dark energy. As present, there is a minority (slowly, but surely this minority is growing) opinion in the community [2] that dark energy simply does not exist and "can be done away with". This, perhaps started with Buchert [85] with the realisation that averaging the Einstein equations

[1]Not to mention, dark matter has somewhat also become a particle physics problem now; Intertwining both regimes of physics.

[2]Certain readers may not be happy with this statement. Yes, this situation is far more nuanced than stated. It is in fact true that more and more tensions in the theoretical physics and cosmology community are arising and some are beginning to accept that, perhaps, our foundations are not correct.

"properly" results in an extra term(s) in the "Friedmann equations" — the backreaction terms. Wiltshire then built upon this in 2007 with the timescape model [20, 21], wherein he explores more fundamental ideas of how we calibrate clocks. Only time will tell if we can do away with dark energy entirely.

Regardless of our stance on dark energy and the perceived accelerated expansion of the Universe, recently some rather bold and unusual claims have been made regarding how black holes might *directly* interact with the overall FLRW cosmological expansion [86]. (See also the somewhat earlier closely related references [87, 88, 89] which developed the theoretical framework for these claims.)

Key parts of the claims made in ref [86] were that:

- "The Kerr black hole solution is ... *provisional* as its behavior at infinity is incompatible with an expanding universe.".
- "Black hole models with realistic behavior at infinity predict that the gravitating mass of a black hole can increase with the expansion of the universe *independently of accretion or mergers...*" .
- "The redshift dependence of the mass growth implies that, at $z \leq 7$, *black holes contribute an effectively constant cosmological energy density* to Friedmann's equations.".

There are a number of significant problems with these claims:

- The truly enormous "separation of scales" that is observed to occur between galactic dynamics and cosmological dynamics makes all such claims grossly implausible. (More on this specific point below.)
- There are at least three exact solutions to the Einstein equations that embed black holes in expanding universes, (Kottler, McVittie, and Kerr–de Sitter), and in those known exact solutions the claimed effect simply does not occur.
- The underlying theoretical framework [87, 88, 89] adopted in ref [86] appears to be deeply flawed [90]. (One key issue here is that the cosmological mass fraction sequestered in black holes simply does *not* lead to an equal but opposite pressure; a "black hole gas" mimics "dust", it does not mimic "dark energy".) Several other authors have made related cautionary comments [91, 92].
- An independent observational analysis [93] strongly excludes the claimed effect at $\sim 3\sigma$, and is compatible with zero effect at $\sim 1\sigma$. (The technical difficulty with making this bound even tighter lies in guaranteeing that the observational sample is free of false positives. This could be due to the possible growth of superficially quiescent black holes actually being driven due to some unaccounted for variant of the usual processes of *accretion* and/or *mergers*.)

- Several other independent observational and/or numerical analyses similarly disfavour the existence of the claimed effect, see for instance [94, 95, 96, 97, 98].

In this chapter we will concentrate on the general relativistic aspects of the situation. We will pay particular attention both to physically relevant approximations, and to the known exact theoretical solutions. We will argue that based on the known exact solutions there is simply no physical reason to expect the claimed effect to occur, and good physics reasons to reject the claimed effect.

3.1 Separation of scales

We start the discussion by pointing out that there is a truly enormous separation of scales between galactic black hole physics and cosmological physics. Even the heaviest known galactic black holes have masses only of order 3×10^{10} solar masses, corresponding to a Schwarzschild radius $\lesssim 10^{-3}$ parsec. In contrast, the cosmological homogeneity scale is typically taken to be of order $\gtrsim 10^8$ parsecs[3], and the Hubble scale is even larger, of order 10^{10} parsec. There simply is no plausible mechanism for directly coupling milli-parsec black hole physics to giga-parsec cosmological physics. (For related comments see refs [99, 100].)

What is much more plausible is to directly couple the observed black hole candidates found in most spiral galaxy cores to matter in their immediate environment — the galactic cores and galaxies in which they reside. This of course implies black hole evolution due to the utterly standard processes of *accretion* and/or *mergers*, which is exactly what the authors of ref [86] are claiming to side-step.

More quantitatively, even in the absence of an explicit exact solution to the Einstein equations, we can argue as follows: Any attempt at inserting a black hole into a FLRW cosmology will at the very least involve two separate mass scales — m the mass of the central black hole, and $\rho_{\text{FLRW}}\, r^3$, the FLRW contribution to the mass contained in a ball of radius r. Combining these two quantities defines a natural distance scale

$$r_* = \sqrt[3]{\frac{m}{\rho_{\text{FLRW}}}}. \tag{3.1}$$

At distances $r < r_*$ black hole physics dominates, at distances $r > r_*$ the FLRW cosmology dominates. We will see this natural transition-distance scale crop up repeatedly in the discussion below.

[3]This is usually called the *statistical scale of homogeneity* (SSH) and estimates thereof are most often based on the galaxy-galaxy 2–point correlation method.

3.2 Exact solutions in general relativity

There are at least three well-known *exact* solutions of Einstein's equations for black holes embedded in expanding FLRW universes:

- Schwarzschild–de Sitter (Kottler);
- Schwarzschild–FLRW (McVittie);
- Kerr–de Sitter.

Note that de Sitter spacetime is just a special case of FLRW, which, *in appropriate coordinates*, corresponds to exponential expansion $a(t) = e^{Ht}$, with constant Hubble parameter H. Furthermore, in the standard framework of ΛCDM cosmology, the universe at the current epoch is believed to already be cosmological constant dominated. Therefore, it follows that de Sitter space is an excellent approximation to both the near-current-epoch and future expansion history of the universe. Allowing for a completely arbitrary expansion history for the scale factor $a(t)$, (as in the Schwarzschild–FLRW (McVittie) spacetime discussed below), while it would be "nice to have", is not really critical for purposes of the current discussion.

3.2.1 Schwarzschild–de Sitter (Kottler)

Let us start from Schwarzschild–de Sitter (Kottler) spacetime presented in its most common form, in static (t, r) coordinates [101]:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r} - H^2r^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r} - H^2r^2} + r^2\mathrm{d}\Omega^2. \tag{3.2}$$

This coordinate system makes it obvious that at small r one recovers the Schwarzschild solution, and that the mass m of the central black hole is not changing. In Kottler spacetime the natural distance scale (in physical units) reduces to

$$r_* = \sqrt[3]{\frac{m}{\rho_{\text{FLRW}}}} \longrightarrow \sqrt[3]{\frac{mc^2}{H^2}}, \tag{3.3}$$

and can be identified as the radius of the OSCO, (the *outermost stable circular orbit*) [76, 102]. For $r \ll r_*$ the physics is black-hole dominated, for $r \gg r_*$ the physics is cosmological-constant dominated.

Now this particular static slicing makes the physical interpretation in terms of an exponentially expanding FLRW spacetime not entirely obvious. Therefore, we are required to make a few coordinate transformations to make this fully explicit. To proceed with the discussion, we first substitute

$$\mathrm{d}t = \mathrm{d}\bar{t} + \frac{\sqrt{\frac{2m}{r} + H^2 r^2}}{1 - \frac{2m}{r} - H^2 r^2}\, \mathrm{d}r, \tag{3.4}$$

to go to $(\bar{t}, r)$ Painlevé–Gullstrand coordinates, (as shown in Chapter 2)

$$\mathrm{d}s^2 = -\mathrm{d}\bar{t}^2 + \left[\mathrm{d}r - \sqrt{\frac{2m}{r} + H^2 r^2}\, \mathrm{d}\bar{t}\right]^2 + r^2 \mathrm{d}\Omega^2. \tag{3.5}$$

Second, we now set $r = e^{H\bar{t}}\, \bar{r}$, so that $\mathrm{d}r = e^{H\bar{t}}\, [\mathrm{d}\bar{r} + \bar{r} H \mathrm{d}\bar{t}]$. Then in these new $(\bar{t}, \bar{r})$ coordinates, which is exactly (2.140)

$$\mathrm{d}s^2 = -\mathrm{d}\bar{t}^2 + e^{2H\bar{t}} \left[\left(\mathrm{d}\bar{r} + \left\{H\bar{r} - \sqrt{\frac{2m\, e^{-3H\bar{t}}}{\bar{r}} + H^2 \bar{r}^2}\right\} \mathrm{d}\bar{t}\right)^2 + \bar{r}^2 \mathrm{d}\Omega^2\right]. \tag{3.6}$$

Note that at large distances

$$\begin{aligned} \left\{H\bar{r} - \sqrt{\frac{2me^{-3H\bar{t}}}{\bar{r}} + H^2\bar{r}^2}\right\} &= H\bar{r}\left\{1 - \sqrt{1 + \frac{2me^{-3H\bar{t}}}{H^2\bar{r}^3}}\right\} \\ &= -\frac{m}{H\bar{r}^2}\, e^{-3Ht} + \mathcal{O}\left(\frac{1}{\bar{r}^5}\right). \end{aligned} \tag{3.7}$$

Thus at large $\bar{r}$

$$\mathrm{d}s^2 = -\mathrm{d}\bar{t}^2 + e^{2H\bar{t}} \left[\mathrm{d}\bar{r}^2 + \bar{r}^2 \mathrm{d}\Omega^2\right] + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right), \tag{3.8}$$

making it obvious that the Schwarzschild–de Sitter (Kottler) black hole is embedded in an exponentially expanding FLRW universe.

We emphasize that in this specific example there simply is no coupling between the mass parameter m and the cosmological parameter H; they are independent constants.

3.2.2 Schwarzschild–FLRW (McVittie)

The Schwarzschild–FLRW (McVittie) spacetime metric [77, 78, 79, 80] describes an eternal black hole that has been part of the universe ever since the Big Bang — if in contrast one wants to describe a black hole that forms from stellar collapse, then a *segment* of the Schwarzschild–FLRW (McVittie) spacetime should be used — only for describing the quiescent period after the initial collapse and ringdown.

McVittie spacetime can be represented in any of the following four completely equivalent forms, as shown in Chapter 2 [103]:

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m}{2a(t)\tilde{r}}}{1+\frac{m}{2a(t)\tilde{r}}}\right)^2 \mathrm{d}t^2 + \left(1+\frac{m}{2a(t)\tilde{r}}\right)^4 a(t)^2\{\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2\}. \tag{3.9}$$

$$\mathrm{d}s^2 = \left(1+\frac{m}{2\bar{r}}\right)^4 \left\{-\left(\frac{[1-\frac{m}{2\bar{r}}]^2}{[1+\frac{m}{2\bar{r}}]^6}\right)\mathrm{d}t^2 + \{[\mathrm{d}\bar{r} - H(t)\bar{r}\mathrm{d}t]^2 + \bar{r}^2\mathrm{d}\Omega^2\}\right\}. \tag{3.10}$$

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{\hat{r}}\right)\mathrm{d}t^2 + \left[\frac{\mathrm{d}\hat{r}}{\sqrt{1-2m/\hat{r}}} - H(t)\hat{r}\mathrm{d}t\right]^2 + \hat{r}^2\mathrm{d}\Omega^2. \tag{3.11}$$

$$\begin{aligned}\mathrm{d}s^2 = &-\left(1-\frac{2m}{a(t)r}\right)\mathrm{d}t^2 \\ &+ a(t)^2\left\{\left[\frac{\left(\mathrm{d}r + H(t)r\left[1-\sqrt{1-\frac{2m}{a(t)r}}\right]\mathrm{d}t\right)^2}{1-\frac{2m}{a(t)r}}\right] + r^2\mathrm{d}\Omega^2\right\}.\end{aligned} \tag{3.12}$$

All four of these coordinate systems use the same time coordinate t, and also the same angular coordinates $\{\theta, \phi\}$, while we have used coordinate freedom of general relativity to adopt differing radial coordinates $\{\tilde{r}, \bar{r}, \hat{r}, r\}$. (The relevant coordinate transformations connecting these differing radial coordinates are explicitly presented in ref [103].)

In all four of these coordinate systems the energy density is determined by finding the time-like eigenvector of the stress-energy, and is easily calculated to be [103]:

$$\rho = \frac{3}{8\pi}H(t)^2. \tag{3.13}$$

The pressure is determined by the space-like eigenvectors of the stress energy and is more subtle: Depending on which of the coordinate systems (3.9)–(3.12) one adopts one finds the superficially differing but physically equivalent results, as discussed in Chapter 2 [103]:

$$p = -\rho - \frac{1}{4\pi}\frac{1+\frac{m}{2a(t)\tilde{r}}}{1-\frac{m}{2a(t)\tilde{r}}}\dot{H}(t); \tag{3.14}$$

$$= -\rho - \frac{1}{4\pi}\frac{1+\frac{m}{2\bar{r}}}{1-\frac{m}{2\bar{r}}}\dot{H}(t); \tag{3.15}$$

$$= -\rho - \frac{1}{4\pi}\frac{1}{\sqrt{1-2m/\hat{r}}}\dot{H}(t); \tag{3.16}$$

$$= -\rho - \frac{1}{4\pi}\frac{1}{\sqrt{1-\frac{2m}{a(t)r}}}\dot{H}(t). \tag{3.17}$$

At large distances, in all four of these coordinate systems, one recovers the standard spatially flat ($k = 0$) FLRW result:

$$p \to -\rho - \frac{\dot{H}(t)}{4\pi}. \tag{3.18}$$

Turning now to the explicit representations of the spacetime metric, at suitably large distances, $a(t)\tilde{r} \gg m$, the line element (3.9) implies

$$\mathrm{d}s^2 \approx -\mathrm{d}t^2 + a(t)^2\{\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2\}, \tag{3.19}$$

which clearly is ($k = 0$) FLRW with arbitrary scale factor $a(t)$.

On the other hand at suitably small distances, $\bar{r}H(t) \ll 1$, the line element (3.10) implies

$$\mathrm{d}s^2 \approx \left(1 + \frac{m}{2\bar{r}}\right)^4 \left\{ -\left(\frac{[1 - \frac{m}{2\bar{r}}]^2}{[1 + \frac{m}{2\bar{r}}]^6}\right) \mathrm{d}t^2 + \{\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\Omega^2\} \right\}. \tag{3.20}$$

This is just Schwarzschild spacetime in isotropic coordinates. The mass of the central black hole is simply m, a time-independent constant. Note there is no mass flux onto the central black hole; there is no accretion. That is, there simply is no direct coupling between the mass parameter m and the cosmological parameter $H(t) = \dot{a}(t)/a(t)$; they are independent quantities. As previously noted, the only even slightly tricky part of the analysis was in setting up the coordinate transformations used to make these properties manifest.

3.2.3 Kerr–de Sitter

Rotating black holes are much more subtle than their non-rotating counterparts. The basic asymptotically flat Kerr spacetime was first discovered some 60 years ago in 1963, see reference [104]. Further discussion can be found in [105] and [106, 107, 108, 109, 110], and more recently in references [111, 112, 113, 114, 115, 116, 117, 118].

The Kerr–de Sitter (KdS) geometry is even more subtle than Kerr, and was first obtained by Carter some 10 years later in 1973; still some 50 years ago, see refs [119, 120]. The Kerr–de Sitter geometry represents an eternal rotating black hole embedded in de Sitter spacetime. For a recent easily accessible discussions see reference [121], and even more recently see [122, 123].

For a black hole formed from stellar collapse, one should certainly wait until after the initial collapse and ringdown, until the black hole is quiescent. One should also wait until the universe is old enough to be cosmological constant dominated — as is now expected to be the situation in the current epoch. That is, the Kerr–de Sitter geometry should be a good approximation to rotating black holes in the current epoch. (This point is implicit in the discussion of ref [121].)

The metric for the Kerr–de Sitter spacetime is most typically presented in stationary coordinates [119, 120]:

$$
\begin{aligned}
ds^2 = &- \frac{(r^2+a^2)(1-\frac{\Lambda}{3}r^2) - 2mr}{r^2+a^2\cos^2\theta}\left[\frac{\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi}{1+\frac{1}{3}\Lambda a^2}\right]^2 \\
&+ \sin^2\theta\left[\frac{1+\frac{1}{3}\Lambda a^2\cos^2\theta}{r^2+a^2\cos^2\theta}\right]\left[\frac{a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi}{1+\frac{1}{3}\Lambda a^2}\right]^2 \\
&+ (r^2+a^2\cos^2\theta)\left[\frac{\mathrm{d}r^2}{(r^2+a^2)(1-\frac{\Lambda}{3}r^2)-2mr} + \frac{\mathrm{d}\theta^2}{(1+\frac{1}{3}\Lambda a^2\cos^2\theta)}\right].
\end{aligned}
\tag{3.21}
$$

(Warning: Here a is the spin parameter $a = J/m$, not the FLRW scale factor $a(t)$.) In this spacetime, the cosmological constant is related to the Hubble *constant* by $\Lambda = 3H^2$. A perhaps mildly surprising aspect of this line element is the presence of the constant $1+\frac{1}{3}\Lambda a^2 = 1+H^2a^2$ in several strategic places.

To be able to efficiently use computer algebra packages, it is more beneficial to have this metric in a fully expanded form, and to eliminate the trigonometric functions. We therefore re-write the line element in the following form:

$$
\begin{aligned}
g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu = &-\left[\frac{\Delta_r - \Delta_\theta\,a^2(1-\chi^2)}{\rho^2\,\Xi^2}\right]\mathrm{d}t^2 + \frac{\rho^2}{\Delta_r}\,\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta\,(1-\chi^2)}\,\mathrm{d}\chi^2 \\
&+ \frac{(1-\chi^2)}{\rho^2\,\Xi^2}\Big[\Delta_\theta\,(r^2+a^2)^2 - \Delta_r\,a^2\,(1-\chi^2)\Big]\,\mathrm{d}\phi^2 \\
&- \frac{2a(1-\chi^2)}{\rho^2\,\Xi^2}\Big[\Delta_\theta\,(r^2+a^2) - \Delta_r\Big]\,\mathrm{d}t\mathrm{d}\phi\,.
\end{aligned}
\tag{3.22}
$$

Here

$$
\begin{aligned}
&\chi = \cos\theta; \\
&\Delta_r = r^2 + a^2 - 2mr + \frac{\Lambda}{3}r^2\,(r^2+a^2); \\
&\Delta_\theta = 1 + \frac{\Lambda}{3}a^2\cos^2\theta = 1 + \frac{\Lambda}{3}a^2\chi^2; \\
&\rho^2 = r^2 + a^2\cos^2\theta = r^2 + a^2\chi^2; \\
&\Xi = 1 + \frac{\Lambda}{3}a^2\,.
\end{aligned}
\tag{3.23}
$$

The Kerr-de Sitter spacetime is a Λ-vacuum solution of the Einstein field equations, an Einstein manifold, and therefore satisfies

$$
R_{\mu\nu} = -\Lambda g_{\mu\nu}; \qquad G_{\mu\nu} = +\Lambda g_{\mu\nu}. \tag{3.24}
$$

Using, for example, `sagemath` or `Maple`, we may easily check this is in fact true.

We must also check that the Weyl tensor is nonzero, and that the Weyl scalar, $C^{\mu\nu\alpha\beta}C_{\mu\nu\alpha\beta}$ is position-dependent: Indeed

$$C^{\mu\nu\alpha\beta}C_{\mu\nu\alpha\beta} = -\frac{m^2(a^2\chi^2+4ar\chi+r^2)\,(a^2\chi^2-4ar\chi+r^2)\,(r^2-a^2\chi^2)}{(r^2+a^2\chi^2)^6}, \tag{3.25}$$

which depends on both r and χ. Furthermore, the Kretschmann scalar, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ is also non-zero and position-dependent. Lastly, due to the Kerr–de Sitter spacetime no longer being a pure vacuum solution to the Einstein equations, we expect the difference between the Kretschmann scalar and Weyl scalar to be non-zero (and position-independent). We find

$$R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - C^{\mu\nu\alpha\beta}C_{\mu\nu\alpha\beta} = \frac{8}{3}\Lambda^2\,. \tag{3.26}$$

When $\Lambda = 0$ — where the Kerr metric is recovered — the difference is zero.

We will subsequently look at the asymptotic large-distance behaviour and verify that in a suitable coordinate system the cosmological constant Λ can be reinterpreted in terms of a constant Hubble parameter H (with $\Lambda = 3H^2$), and an exponentially growing scale factor $a(t) = e^{Ht}$. For now let us focus on a number of internal consistency checks for the Kerr–de Sitter spacetime.

3.3 Extended consistency checks for Kerr–de Sitter

In this section we shall check that the Kerr-de Sitter spacetime does in fact (under suitable circumstances) reduce to the Kerr, Kottler, and de Sitter spacetimes as required.

3.3.1 Kerr spacetime

We first investigate the $\Lambda = 0$ limit of the KdS metric given in (3.22), resulting in the Kerr spacetime. This results in a vacuum solution to the Einstein equations, hence, providing the basis for a variety of consistency checks for the KdS spacetime.

When $\Lambda = 0$ we obtain the line element

$$\begin{aligned} g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu = &-\left[\frac{\bar{\Delta}_r - a^2(1-\chi^2)}{\rho^2}\right]\mathrm{d}t^2 + \frac{\rho^2}{\bar{\Delta}_r}\,\mathrm{d}r^2 + \frac{\rho^2}{(1-\chi^2)}\,\mathrm{d}\chi^2 \\ &+ \frac{(1-\chi^2)}{\rho^2}\Big[(r^2+a^2)^2 - \bar{\Delta}_r\,a^2\,(1-\chi^2)\Big]\,\mathrm{d}\phi^2 \\ &- \frac{2a(1-\chi^2)}{\rho^2}\Big[(r^2+a^2) - \bar{\Delta}_r\Big]\,\mathrm{d}t\mathrm{d}\phi\,, \end{aligned} \tag{3.27}$$

where $\bar{\Delta}_r = r^2 + a^2 - 2mr$.

To check this is in fact the Kerr spacetime, we compute the curvature quantities such as the Ricci tensor, Weyl tensor, Kretschmann scalar, and Weyl scalar. For a vacuum solution we expect the Ricci tensor to be zero and, therefore, the Riemann tensor and Weyl tensor to be equal. We verify this using `sagemath/Maple`. Furthermore, we find that the Riemann tensor and Weyl tensor are equal, non zero and position dependent. Lastly, the Weyl scalar and Kretschmann scalar are equal (as expected).

3.3.2 Kottler Spacetime

In the $a \to 0$ limit of the KdS spacetime, we recover the Kottler (Schwarzschild–de Sitter) spacetime. Firstly

$$\Delta_r \to \hat{\Delta}_r = r^2 - 2mr - \frac{\Lambda r^4}{3} = r^2\left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right). \tag{3.28}$$

Then

$$(\mathrm{d}s^2)_{Kottler} = -\frac{\hat{\Delta}_r}{r^2}\,[\mathrm{d}t]^2 + (1-\chi^2)r^2\,[\mathrm{d}\phi]^2 + r^2\left[\frac{\mathrm{d}r^2}{\hat{\Delta}_r} + \frac{\mathrm{d}\chi^2}{(1-\chi^2)}\right]. \tag{3.29}$$

Rewritten, this becomes

$$(\mathrm{d}s^2)_{Kottler} = -\frac{\hat{\Delta}_r}{r^2}\mathrm{d}t^2 + \frac{r^2}{\hat{\Delta}_r}\mathrm{d}r^2 + r^2\left[\frac{\mathrm{d}\chi^2}{(1-\chi^2)} + (1-\chi^2)\mathrm{d}\phi^2\right]. \tag{3.30}$$

That is,

$$\begin{aligned}(\mathrm{d}s^2)_{Kottler} \;=\; & -\left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2} \\ & + r^2\left[\frac{\mathrm{d}\chi^2}{(1-\chi^2)} + (1-\chi^2)\mathrm{d}\phi^2\right]. \end{aligned} \tag{3.31}$$

This metric is evidently the Kottler spacetime [101] in standard (t, r) coordinates (and not entirely standard (χ, ϕ) coordinates). We may now perform the same consistency checks on this spacetime as we did in the KdS case. We expect similar results, as it is no longer a pure vacuum solution and corresponds to pure cosmological constant. We again find

$$R_{\mu\nu} = \Lambda\, g_{\mu\nu}\,; \qquad G_{\mu\nu} = -\Lambda\, g_{\mu\nu}\,. \tag{3.32}$$

The curvature quantities such as the Riemann tensor and Weyl tensor are not equal, they are again non-zero and position-dependent.

The Kretschmann scalar is

$$R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = \frac{8}{3}\Lambda^2 + \frac{48m^2}{r^6}\,, \tag{3.33}$$

and the Weyl scalar is

$$C^{\mu\nu\alpha\beta}C_{\mu\nu\alpha\beta} = \frac{48m^2}{r^6}\,. \tag{3.34}$$

The difference is simply

$$R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - C^{\mu\nu\alpha\beta}C_{\mu\nu\alpha\beta} = \frac{8}{3}\,\Lambda^2\,, \tag{3.35}$$

which is the same result as in the KdS case. This is to be expected since in the KdS case, the difference did not depend on the angular momentum.

3.3.3 de Sitter Spacetime

The last parameter we shall set to zero is the mass of the black hole, $m \to 0$. The only change in the metric components is that now

$$\Delta_r \to \tilde{\Delta}_r = r^2 + a^2 - \frac{\Lambda r^2}{3}(r^2+a^2) = (r^2+a^2)\left(1 - \frac{1}{3}\Lambda a^2\right). \tag{3.36}$$

The KdS line element now reduces to

$$\begin{aligned} g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = &-\left[\frac{\tilde{\Delta}_r - \Delta_\theta\, a^2(1-\chi^2)}{\rho^2\,\Xi^2}\right]\mathrm{d}t^2 + \frac{\rho^2}{\tilde{\Delta}_r}\,\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta\,(1-\chi^2)}\,\mathrm{d}\chi^2 \\ &+ \frac{(1-\chi^2)}{\rho^2\,\Xi^2}\left[\Delta_\theta\,(r^2+a^2)^2 - \tilde{\Delta}_r\, a^2\,(1-\chi^2)\right]\mathrm{d}\phi^2 \\ &- \frac{2a(1-\chi^2)}{\rho^2\,\Xi^2}\left[\Delta_\theta\,(r^2+a^2) - \tilde{\Delta}_r\right]\mathrm{d}t\mathrm{d}\phi\,. \end{aligned} \tag{3.37}$$

Though not entirely obvious, this is actually de Sitter space in (rotating) oblate spheroidal coordinates.

In this $m \to 0$ limit, it is easy to check that the Weyl tensor is zero (and, therefore, the Weyl scalar will be zero too). The Kretschmann scalar is found to be

$$R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = \frac{8}{3}\,\Lambda^2\,. \tag{3.38}$$

Note, that (as expected) this is (trivially) the *difference* of the Kretschmann scalar and Weyl scalar, as for the KdS and Kottler cases.

One may now go one step further and perform an explicit coordinate transformation on the line element (3.37) to obtain the "standard" form of the de Sitter metric. Using the explicit coordinate transformation given in ref [121]

$$\begin{aligned} &T = \frac{t}{\Xi}; \\ &\Phi = \phi - \frac{a\Lambda t}{3\Xi}; \\ &y\cos\Theta = r\chi; \\ &y^2 = \frac{1}{\Xi}\left[r^2\Delta_\theta + a^2(1-\chi^2)\right], \end{aligned} \tag{3.39}$$

one can show that (3.37) reduces to

$$g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -(1-\frac{\Lambda}{3}y^2)\,\mathrm{d}T^2 + \frac{1}{1-\frac{\Lambda}{3}y^2}\,\mathrm{d}y^2 + y^2\mathrm{d}\Theta^2 + y^2\sin^2\Theta\,\mathrm{d}\Phi^2\,. \quad (3.40)$$

This is in fact the standard form for de Sitter space, presented in terms of the coordinates (T, y, Θ, Φ), which we could simply re-name (t, r, θ, ϕ) if desired.

Performing two further coordinate transformations allows us to cast this metric into a form where — explicitly — space is exponentially expanding. First, we transform the time coordinate according to

$$T = \tilde{t} + \int \frac{Hy}{1-H^2y^2}\mathrm{d}y = \tilde{t} + \frac{\ln(1-H^2y^2)}{2H}\,, \quad (3.41)$$

resulting in the Painlevé–Gullstrand (2.121) form of de Sitter space:

$$g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}\tilde{t}^{\,2} + [\,\mathrm{d}y - Hy\,\mathrm{d}t\,]^2 + y^2\mathrm{d}\Omega^2\,. \quad (3.42)$$

Secondly, we transform the radial coordinate according to

$$y = \tilde{r}e^{Ht}\,, \quad (3.43)$$

resulting in de Sitter space in comoving coordinates:

$$g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}\tilde{t}^{\,2} + e^{\,2H\tilde{t}}\{\,\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2\,\}\,. \quad (3.44)$$

Therefore, it is apparent that, as desired, the $m \to 0$ limit of Kerr–de Sitter is indeed the exponentially growing FLRW spacetime.

3.4 Asymptotic behaviour of Kerr-de Sitter

The claim that the mass of a black hole grows as a function of time has been proven to be false thus far. In subsection 3.2.1 we have shown that for the Schwarzschild–de Sitter (Kottler) spacetime the mass of the central black hole is simply m, a time independent constant. In sub-section 3.2.2 we obtained the same result for McVittie (Schwarzschild–FLRW) spacetime. We shall now show that this also holds for the Kerr–de Sitter black hole by considering the asymptotic behaviour of the Kerr–de Sitter spacetime.

3.4.1 Small r expansion

Let us begin with 'small' r, i.e., $|\Lambda| r^2 \ll 1$, while keeping $r > a$. It is perhaps obvious that we should expect — when close to the central black hole — the metric to be of the form "Kerr + perturbation". For the following analysis we shall use the binomial expansion in r. We note that in this limit:

$$\begin{aligned} \frac{1}{\Xi^2} &\approx 1 - \frac{2}{3}\Lambda a^2\,; \\ \frac{1}{\Delta_\theta} &\approx 1 - \frac{1}{3}\Lambda a^2 \chi^2\,. \end{aligned} \tag{3.45}$$

Component by component we explicitly find

$$\begin{aligned} (g_{tt})_{KdS} &\approx (g_{tt})_{Kerr} - \frac{2}{3}\Lambda a^2\,(g_{tt})_{Kerr} + \mathcal{O}\big(\Lambda a^2\big) = (g_{tt})_{Kerr}\Big(1 + \mathcal{O}\big(\Lambda a^2\big)\Big); \\ (g_{rr})_{KdS} &\approx (g_{rr})_{Kerr} + \frac{1}{3}\Lambda r^2\,(g_{rr})_{Kerr} + \mathcal{O}\big(\Lambda r^2\big) = (g_{rr})_{Kerr}\Big(1 + \mathcal{O}\big(\Lambda r^2\big)\Big); \\ (g_{\theta\theta})_{KdS} &\approx (g_{\theta\theta})_{Kerr} - \frac{1}{3}\Lambda a^2 \chi^2\,(g_{\theta\theta})_{Kerr} + \mathcal{O}\big(\Lambda a^2 \chi^2\big) \\ &\qquad\qquad = (g_{\theta\theta})_{Kerr}\Big(1 + \mathcal{O}\big(\Lambda a^2 \chi^2\big)\Big); \\ (g_{\phi\phi})_{KdS} &\approx (g_{\phi\phi})_{Kerr} - \frac{2}{3}\Lambda a^2\,(g_{\phi\phi})_{Kerr} + \mathcal{O}\big(\Lambda r^2\big) = (g_{\phi\phi})_{Kerr}\Big(1 + \mathcal{O}\big(\Lambda r^2\big)\Big); \\ (g_{\phi t})_{KdS} &\approx (g_{\phi t})_{Kerr} - \frac{2}{3}\Lambda a^2\,(g_{\phi t})_{Kerr} + \mathcal{O}\big(\Lambda r^2\big) = (g_{\phi t})_{Kerr}\Big(1 + \mathcal{O}\big(\Lambda r^2\big)\Big)\,. \end{aligned} \tag{3.46}$$

Note that the g_{rr} term is not merely a straightforward binomial expansion in r. Rather, we use the fact that in the region of interest

$$(g_{rr})_{KdS} = \frac{\rho^2}{(r^2 + a^2)(1 - H^2 r^2) - 2mr} \approx \frac{\rho^2}{r^2 + a^2 - 2mr}\,\frac{1}{1 - H^2 r^2}\,, \tag{3.47}$$

which is true as we can safely neglect $\mathcal{O}(r^3)$ terms. Finally, since $\chi \in [-1, 1]$ and we have assumed $a < r$, all the individual components of the Kerr–de Sitter metric may be written as

$$(g_{\mu\nu})_{KdS} = (g_{\mu\nu})_{Kerr}\Big(1 + \mathcal{O}\big(\Lambda r^2\big)\Big)\,. \tag{3.48}$$

Consequently at small distances (meaning $|\Lambda| r^2 \ll 1$) Kerr–de Sitter reduces to Kerr as expected — with a constant unchanging mass parameter m, and no sign of any direct coupling between the de Sitter expansion and the central black hole.

3.4.2 Large r expansion

For the large r expansion we assume $r \gg m$ (we also assume $m > a$ to avoid naked singularities). However, we do not want r to become cosmologically enormous, we still want to keep $|\Lambda| r^2 \lesssim 1$. (If $\Lambda > 0$ one certainly does not want to go past the cosmological horizon at $r_C \approx 1/\sqrt{\Lambda}$. In counterpoint, if $\Lambda < 0$, corresponding to an asymptotically anti-de Sitter space, there is simply no need to go past $r \sim 1/\sqrt{|\Lambda|}$ to detect cosmological physics.)

As we are a suitably large (but not too large) distance away from the black hole, one would expect the metric to be of the form "de Sitter + perturbation". For all of the metric components except the g_{rr} component, we may easily separate out the mass terms and then expand about large r:

$$
\begin{aligned}
(g_{tt})_{KdS} &= (g_{tt})_{dSO} + \frac{2mr}{\rho^2 \Xi^2} \approx (g_{tt})_{dSO} + \frac{2m}{r} + \mathcal{O}\left(\frac{m}{r^3}\right); \\
(g_{\theta\theta})_{KdS} &= (g_{\theta\theta})_{dSO} \approx (g_{\theta\theta})_{dSO} + \mathcal{O}\left(\frac{2m}{r}\right); \\
(g_{\phi\phi})_{KdS} &= (g_{\phi\phi})_{dSO} + \frac{2mra^2(1-\chi^2)^4}{\rho^2 \Xi^2} \frac{1}{\Xi^2} \\
&\approx (g_{\phi\phi})_{dSO} + \frac{2m}{r} \frac{1}{\Xi^2} a^2 (1-\chi^2)^4 + \mathcal{O}\left(\frac{2m}{r^3}\right); \\
(g_{\phi t})_{KdS} &= (g_{\phi t})_{dSO} + \frac{2mra(1-\chi^2)^2}{\rho^2 \Xi^2} \frac{1}{\Xi^2} \\
&\approx (g_{\phi t})_{dSO} + \frac{2m}{r} \frac{1}{\Xi^2} a (1-\chi^2)^2 + \mathcal{O}\left(\frac{2m}{r^3}\right).
\end{aligned}
\tag{3.49}
$$

Here, the subscript dSO is used to make it explicit that this is a component of the de Sitter metric *in oblate spheroidal coordinates*. The $(g_{rr})_{KdS}$ component is most easily dealt with by writing:

$$
\begin{aligned}
(g_{rr})_{KdS} &= (g_{rr})_{dSO} - \frac{2mr\,(r^2 + a^2\chi^2)}{(r^2+a^2)^2\,(1-H^2r^2)^2} \\
&\approx (g_{rr})_{dSO} - \frac{2m}{r} \frac{1}{(1-H^2r^2)^2} \left[1 + \mathcal{O}\left(\frac{a^2}{r^2}\right)\right].
\end{aligned}
\tag{3.50}
$$

Therefore, we may write the Kerr–de Sitter metric expanded about large r ($r \gg m$, but r not cosmologically large, $|\Lambda| r^2 \lesssim 1$) as

$$
(g_{\mu\nu})_{KdS} = (g_{\mu\nu})_{dSO} + \mathcal{O}\left(\frac{2m}{r}\right). \tag{3.51}
$$

As required, when approaching large (but not too large) r the Kerr–de Sitter spacetime asymptotically approaches de Sitter space. (Which, as we have already seen, after suitable coordinate transformations can be recast in terms of an exponentially growing scale factor $a(t) = \exp(Ht)$.)

We emphasize (again) that in this specific Kerr–de Sitter example there simply is no coupling between the mass parameter m and the cosmological parameter H; they are independent constants.

3.5 Kerr–FLRW spacetime?

While we have seen that Kerr–de Sitter, corresponding to specifically exponential expansion at asymptotic spatial infinity, can be written down explicitly in a not too complicated form, we know of no equivalent result for Kerr–FLRW for a general scale factor $a(t)$. There is a reason for this: in Kerr's original article [104] he asked whether it would be possible to find a (perfect fluid) interior solution for what is now called Kerr spacetime. This is a question that still remains open after 60 years. Only partial results are known, in terms of anisotropic non-perfect fluids and other anisotropic sources [124, 125, 126]. Finding an exact Kerr–FLRW spacetime would be tantamount to finding a time-dependent perfect fluid exterior solution to the Kerr black hole — which would be at least as hard as the still unsolved problem of finding a perfect fluid interior solution.

However, as mentioned in section 3.1, the largest known galactic black holes have masses of order $3 \times 10^{10}\ m_{\odot}$. This corresponds to a Schwarzschild radius $\lesssim 10^{-3}$ parsec, whereas the statistical scale of homogeneity is of order $\gtrsim 10^{8}$ parsecs. Therefore, having a solution that *asymptotes* to a perfect fluid on scales such that the FLRW solution is applicable is certainly *good enough*.

Furthermore, observational evidence strongly suggests that the universe is currently cosmological constant dominated, so the relevant FLRW spactime, now and for the foreseeable future, is de Sitter. Therefore, the Kerr-de Sitter solution is, for all practical purposes, certainly *good enough*.

3.6 Black hole internal structure?

As part of the plausibility argument for entertaining a possible direct black-hole/ cosmology coupling, ref [86] suggested that this might have something to do with an assumed non-trivial internal structure for black holes. Specifically, was dark energy *inside* the black hole slowly being released? Several authors have tried to make this idea more precise. While certainly there is widespread agreement that *regular black holes* and more generally black holes with a non-vacuum interior are of interest [127, 128, 129, 130, 131, 132, 133, 134, 135], there is much less agreement as to whether such black hole variants directly couple to the cosmology they are embedded in. Most investigations suggest there is no such direct coupling [136, 137, 138, 139]. The few investigations that suggest there is such an effect yield predictions that are quantitatively and qualitatively at variance [140] with the original proposal of reference [86].

3.7 Discussion

Starting from three relatively well-known exact solutions to the Einstein equations, (Kottler, McVittie, Kerr-de Sitter) all of which successfully embed black holes in a suitable FLRW background, we have seen that these exact solutions exhibit no evidence of any "direct coupling" between the black hole mass and the cosmological expansion. Furthermore, several purely phenomenological investigations have similarly failed to find evidence for any "direct coupling" between the black hole mass and the cosmological expansion.

Indeed the enormous separation of scales between milli-parsec black hole physics and giga-parsec cosmological physics renders any such "direct coupling" (*independently of accretion or mergers*) grossly implausible. While we understand the want and need to explain where dark energy comes from, this overwhelming force that drives the accelerated expansion of the Universe, black holes, simply cannot be *it*. We, therefore, urge extreme caution and care when mooting such ideas.

Chapter 4

Black-to-White Bounce and Near-Horizon Physics

Black and white holes play remarkably contrasting roles in general relativity *versus* observational astrophysics. While there is observational evidence for the existence of compact objects that are "cold, dark, and heavy", which thereby are natural candidates for black holes, the theoretically viable time-reversed variants — the "white holes" — have nowhere near the same level of observational support. In this chapter we shall explore the theoretical possibility that the connection between black and white holes is much more intimate than commonly appreciated.

We shall first construct "horizon penetrating" coordinate systems that differ from the standard curvature coordinates only in a small near-horizon region. Thereby, emphasizing that ultimately the distinction between black and white horizons depends only on near-horizon physics. We shall then construct an explicit model for a "black-to-white transition" where all of the nontrivial physics is confined to a compact region of spacetime. This is a finite-duration finite-thickness, (in principle arbitrarily small), region straddling the naïve horizon. Moreover we shall show that it is possible to arrange the "black-to-white transition" to have zero action — so that it will not be subject to destructive interference in the Feynman path integral. This then raises the very intriguing possibility that astrophysical black holes might be interpretable in terms of a quantum superposition of black and white horizons — a "gray" horizon.

Classical black holes are objects that — from a theoretical perspective — are very well understood within the standard framework of the theory of general relativity [37, 32, 40, 33, 41, 35, 36, 39].
Likewise, the observational [141, 142, 143, 144, 145] and phenomenological [146, 147, 148, 134, 149] situations are both increasingly well understood. The (mathematical) event horizon, or the physically more relevant long-lived apparent horizon [75, 74], is often dubbed "the point of no return" and is not really a problematic issue under suitable coordinate choices. However, one certainly

finds that the central singularity still causes many conceptual problems with our understanding of physics. One of the most prominent problems being the destruction of information as it approaches the singularity. Some of the theories that are put forward to resolve the information paradox are soft hairs that evaporate to null infinity (discussed in Chapter 5 and Chapter 6), and *white holes*. While we will not delve into the information paradox itself in this chapter, it is important to understand some of the motivation behind white holes. A representative selection of references includes [150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 79, 164, 165, 166, 167, 168, 169, 170].

White holes, as the name may suggest, are hypothesised to be the opposite of black holes; a "time reversed" black hole. Matter is radiated from the horizon instead of being absorbed thereby. There are many theories as to how white holes might form from black holes, most of which involve some sort of quantum mechanical effect. A representative selection of references includes [171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195].

One specific example of this phenomenon can be found in reference [171], where the authors discuss "gray" horizons — as hypothetical quantum superpositions of black and white horizons. Another example can be found in ref [174] where the authors hypothesise that black holes quantum tunnel into white holes once a black hole evaporates down to the Planck mass. Other theories, such as those proposed in refs [173, 180], involve modifying large wedges of the spacetime (typically all the way down to the central singularity) in order to have a black hole "bounce" to a white hole.

In this chapter we will propose simple and explicit fully *classical* models for a white hole, and in particular for a black-to-white hole transition.

- Firstly, starting from the standard (Hilbert) form of the Schwarzschild metric in curvature coordinates, we shall introduce a simple coordinate change, through a function depending solely on the radial coordinate, r. Specific choices of this function will result in a static black hole and white hole in horizon-penetrating coordinates — such as Painléve–Gullstrand, Kerr–Schild, and Eddington–Finkelstein coordinates.
- Secondly, we shall localize the required coordinate change to a compact near-horizon radial region, showing that both black and white holes can be cast into the standard manifestly static form outside of some compact radial region. Thus a clean distinction can be made between "black" and "white" horizons with minimal modifications to the standard (Hilbert) form of the Schwarzschild metric.
- Thirdly, we introduce a function of time to create a non-vacuum spacetime, one that is no longer static, and which describes a black to white hole "bounce"; with the "bounce" being confined to a compact (arbitrarily small) region of spacetime. Furthermore, an analysis of the action

in the transition region will be conducted, the radial null curves will be investigated, and various energy conditions will be checked. Finally, we shall connect the discussion to quantum physics by applying the Feynman functional integral approach.

Our approach will only require fine tuning of the Schwarzschild spacetime in a compact radial region *near the horizon*. Therefore, the entire spacetime outside of a small neighbourhood of $r = 2m$ will be that of the standard (Hilbert) form of Schwarzschild spacetime. This is achieved by the use of smooth bump functions that will not create discontinuities in the metric; and, therefore, the Christoffel symbols will not be discontinuous, and the Riemann tensor will not contain delta-function contributions.

4.1 Static black and white horizons: Global analysis

Firstly, we will introduce a particularly simple model for (static) black and white horizons, by performing some absolutely minimal modifications of standard textbook results. We begin with the Schwarzschild spacetime (in the usual Hilbert/curvature coordinates):

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - 2m/r} + r^2 d\Omega^2. \tag{4.1}$$

Using the following coordinate transformation,

$$t \to t + F(r); \qquad dt \to dt + f(r)dr, \tag{4.2}$$

results in the line element

$$ds^2 = -\left(1 - \frac{2m}{r}\right) (dt + f(r)dr)^2 + \frac{dr^2}{1 - 2m/r} + r^2 d\Omega^2. \tag{4.3}$$

Expanding, this implies

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2m}{r}\right) dt^2 - 2(1 - 2m/r)f(r)drdt \\ &+ \left[\frac{1}{1 - 2m/r} - (1 - 2m/r)f(r)^2\right] dr^2 + r^2 d\Omega^2. \end{aligned} \tag{4.4}$$

It is important to note that this line element is still Ricci flat, and so is merely the Schwarzschild geometry in disguise, for *arbitrary* $f(r)$.

Without any loss of generality, one may choose:

$$f(r) = \frac{h(r)}{1 - 2m/r}. \tag{4.5}$$

This then results in the line element

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 - 2h(r)drdt + \left[\frac{1 - h(r)^2}{1 - 2m/r}\right] dr^2 + r^2 d\Omega^2. \tag{4.6}$$

All of these line elements, for arbitrary $h(r)$, are just (coordinate) variants of the standard Schwarzschild spacetime — they are all Ricci-flat for *arbitrary* $h(r)$. For specific choices for the function $h(r)$ we obtain some particularly well known coordinate variants of the Schwarzschild spacetime.

4.1.1 Painléve–Gullstrand coordinates

Set $h(r) \to \pm\sqrt{2m/r}$, then

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 \mp 2\sqrt{2m/r}\, drdt + dr^2 + r^2 d\Omega^2. \tag{4.7}$$

Examining the radial null condition, $-dt^2 + \left(dr \mp \sqrt{2m/r}\, dt\right)^2 = 0$, we see that in this coordinate system the radial null curves are

$$\frac{dr}{dt} = \pm 1 \pm \sqrt{2m/r}, \tag{4.8}$$

where the signs are to be chosen independently.

- For a black hole we choose

$$\frac{dr}{dt} = \pm 1 - \sqrt{2m/r}, \tag{4.9}$$

  with $\frac{dr}{dt} \in \{0, -2\}$ at horizon crossing ($r = 2m$).
- In contrast for a white hole we choose

$$\frac{dr}{dt} = \pm 1 + \sqrt{2m/r}, \tag{4.10}$$

  with $\frac{dr}{dt} \in \{+2, 0\}$ at horizon crossing ($r = 2m$).

4.1.2 Kerr–Schild coordinates

Set $h(r) \to \pm 2m/r$, then

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 + \frac{2m}{r}(dt \pm dr)^2. \tag{4.11}$$

Examining the radial null condition, $-dt^2 + dr^2 + (2m/r)(dt \pm dr)^2 = 0$, in this coordinate system we find the radial null curves are either

$$\frac{dr}{dt} = \mp 1, \quad \text{or} \quad \frac{dr}{dt} = \pm 1 \mp \frac{4m}{r+2m}, \tag{4.12}$$

where the signs are to be chosen in a correlated manner.

- Thus, for a black hole we choose either

  $$\frac{dr}{dt} = -1 \quad \text{(ingoing)}, \qquad \text{or} \qquad \frac{dr}{dt} = 1 - \frac{4m}{r+2m} \quad \text{(“outgoing”)}, \tag{4.13}$$

  with $\frac{dr}{dt} \in \{-1, 0\}$ at horizon crossing ($r = 2m$).
- In contrast for a white hole we choose either

  $$\frac{dr}{dt} = 1 \quad \text{(outgoing)}; \qquad \text{or} \qquad \frac{dr}{dt} = -1 + \frac{4m}{r+2m} \quad \text{(“ingoing”)}, \tag{4.14}$$

  with $\frac{dr}{dt} \in \{1, 0\}$ at horizon crossing ($r = 2m$).

4.1.3 Eddington–Finkelstein null coordinates

Set $h(r) = \pm 1$, then

$$ds^2 = -(1 - 2m/r)dt^2 \mp 2dr dt + r^2 d\Omega^2 . \tag{4.15}$$

Depending on the choice of sign, $\pm$, one usually relabels $t \to u$ or $t \to v$.

- The ingoing Eddington–Finkelstein coordinates are typically given as

  $$ds^2 = -(1 - 2m/r)dv^2 + 2dv dr + r^2 d\Omega^2 . \tag{4.16}$$

  Examining the radial null condition, $[-(1 - 2m/r)dv + 2dr]dv = 0$, and noting that this quantity must be negative for timelike curves, we find the radial null curves are

  $$\frac{dr}{dv} = -\infty; \qquad \frac{dr}{dv} = \frac{1 - 2m/r}{2}. \tag{4.17}$$

  The ingoing Eddington–Finkelstein coordinates, therefore, represent a black hole with $\frac{dr}{dv} \in \{-\infty, 0\}$ at horizon crossing ($r = 2m$).
- The outgoing Eddington–Finkelstein coordinates are typically given as

  $$ds^2 = -(1 - 2m/r)du^2 - 2du dr + r^2 d\Omega^2 . \tag{4.18}$$

  Examining the radial null condition, $[-(1 - 2m/r)du - 2dr]du = 0$, and noting that this quantity must be negative for timelike curves, we find the radial null curves are

  $$\frac{dr}{du} = +\infty; \qquad \frac{dr}{du} = -\frac{1 - 2m/r}{2}. \tag{4.19}$$

  The outgoing Eddington–Finkelstein coordinates, therefore, represent a white hole with $\frac{dr}{du} \in \{+\infty, 0\}$ at horizon crossing ($r = 2m$).

4.1.4 Generic horizon-penetrating coordinates

From the above we see that all three of these coordinate systems, Painléve–Gullstrand, Kerr–Schild, and Eddington–Finkelstein provide three specific *examples* of horizon-penetrating coordinates. In each case, depending on whether one is in a black hole or a white hole configuration, one of the radial null geodesics remains frozen on the horizon. I.e., the coordinate velocity is zero — while the other crosses the horizon with a non-zero coordinate velocity.

Of course there are infinitely many other horizon-penetrating coordinates [196, 197, 198, 199, 200, 201], some of which we explore below, these three *examples* are just three of the most obvious ones. We can make the required coordinate transformations fully explicit by noting

$$F(r) = \int f(r)\, dr = \int \frac{h(r)}{1 - 2m/r}\, dr. \tag{4.20}$$

Then, for these three specific examples, we see

$$F_{PG}(r) = \pm \int \frac{\sqrt{2m/r}}{1 - 2m/r}\, dr = \pm 2\sqrt{2mr} \pm 2m \ln\left(\frac{1 - \sqrt{2m/r}}{1 + \sqrt{2m/r}}\right); \tag{4.21}$$

$$F_{KS}(r) = \pm \int \frac{2m/r}{1 - 2m/r}\, dr = \pm 2m \ln(r - 2m); \tag{4.22}$$

$$F_{EF}(r) = \pm \int \frac{1}{1 - 2m/r}\, dr = \pm r \pm 2m \ln(r - 2m). \tag{4.23}$$

These three functions all share the feature of being somewhat unpleasantly behaved near spatial infinity. Specifically, for these three coordinate systems one has (perhaps unexpectedly) to make unboundedly large alterations to the time coordinate near spatial infinity, where the gravitational field is weak. Such behaviour, while not fatal, is perhaps somewhat annoying. We shall first seek to ameliorate it by keeping the function $h(r)$ finite and localised to a compact region thereby keeping the function $f(r)$ integrable, and the function $F(r)$ bounded.

4.2 Static black and white horizons: Local analysis

We now let $h(r)$ be a bump function. At the horizon, pick $h(2m) = \pm 1$, with $h(r)$ being some finite smooth function of compact support. Then we have a version of the Schwarzschild line element presented with localised version of horizon penetrating coordinates. At $r = 2m$ there is either a black or white horizon depending on the *sign* of $h(2m)$. This line element goes to the standard Hilbert form of Schwarzschild at some finite r, (both large and small r). That is: $support\{h(r)\} \subseteq [r_<, r_>]$, with $2m \in (r_<, r_>)$. This is still a Ricci-flat coordinate transformed version of Schwarzschild:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 - 2h(r) dr dt + \left[\frac{1 - h(r)^2}{1 - 2m/r}\right] dr^2 + r^2 d\Omega^2. \qquad (4.24)$$

Note specifically that to get horizon-penetrating coordinates, (and so obtain either an explicitly black or explicitly white horizon) one only needs to adjust the coordinates in the immediate vicinity of the horizon. "Global" changes to the coordinates are by no means necessary.

We check the ingoing/outgoing null curves to verify that the coordinates are in fact horizon penetrating. We have

$$-\left(1 - \frac{2m}{r}\right) dt^2 - 2h(r) dr dt + \left[\frac{1 - h(r)^2}{1 - 2m/r}\right] dr^2 = 0. \qquad (4.25)$$

Therefore, rearranging to obtain a differential equation in $\mathrm{d}r/\mathrm{d}t$,

$$-\left(1 - \frac{2m}{r}\right)^2 - 2\left(1 - \frac{2m}{r}\right) h(r)\dot{r} + \left[1 - h(r)^2\right] \dot{r}^2 = 0. \qquad (4.26)$$

This is an easily solved quadratic for $\dot{r}$, leading to

$$\dot{r} = \mp \frac{1 - 2m/r}{1 \pm h(r)}. \qquad (4.27)$$

Depending on the (implicit) sign choice hiding in $h(2m) = \pm 1$, and the explicit sign choice $\pm$ multiplying $h(r)$, one of these null curves will be trapped at the horizon (with $\dot{r}_H = 0$). The other null curve crosses the horizon with a coordinate speed that is formally $0/0$, and so must be determined by using l'Hôpital's rule:

$$\dot{r}_H = \pm \frac{1}{2m\, h'(2m)}. \qquad (4.28)$$

Therefore, we find these are generically horizon-penetrating coordinates (At least *one* of the radial null curves has non-zero coordinate velocity at horizon crossing). The net amount by which we have to adjust the time coordinate to achieve this localised horizon-penetrating behaviour is

$$\Delta F = F(\infty) - F(0) = F(r_>) - F(r_<) = \int_{r_<}^{r_>} \frac{h(r)}{1-2m/r}\, dr \; = \int_{r_<}^{r_>} \frac{r\, h(r)}{r-2m}\, dr. \tag{4.29}$$

The naïve singularity at the horizon $r = 2m$ is an integrable singularity, so the net shift in the time coordinate is finite.

4.3 Black-to-white bounce: Compact transition region

We now wish to move away from consideration of static black and white holes, and explore a classical model of a black-to-white hole transition. To do so, we make the following change:

$$h(r) \to s(t)\; h(r). \tag{4.30}$$

This is no longer *just* a coordinate transformation. The spacetime is no longer Ricci-flat. Specifically, we consider the metric

$$\mathrm{d}s^2 = -(1-2m/r)\mathrm{d}t^2 - 2s(t)h(r)\mathrm{d}r\mathrm{d}t + \left[\frac{1-s(t)^2h(r)^2}{1-2m/r}\right]\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{4.31}$$

We again take $h(2m) = \pm 1$, and take $h(r)$ to be of compact support, i.e., $support\{h(r)\} \subseteq [r_<, r_>]$. Furthermore we shall also assume that $1 - s(t)^2$ is of compact support with $s(t) \to \pm 1$ at large $|t|$. In fact we shall take $s(+\infty) = \pm 1$ and $s(-\infty) = \mp 1$, since we want to enforce a sign flip in $s(t)$ to enforce a black-to-white transition. That is, $support\{1 - s(t)^2\} \subseteq [t_<, t_>]$. This in turn implies $support\{\dot{s}(t)\} \subseteq [t_<, t_>]$. We again emphasize: this geometry is not Ricci flat — it is no longer *just* a coordinate transformation.[1]

4.3.1 Einstein tensor

Since the spacetime is not just a coordinate transformation of the Schwarzschild metric, the Einstein tensor and Ricci tensor will now be non-zero. We calculate the Einstein tensor, (using Maple), its non-zero radial-temporal components are

$$G_{tt} = 0; \qquad\qquad G_{rr} = -\frac{2\dot{s}(t)h(r)}{r(1-2m/r)}; \tag{4.32}$$

while the orthonormal angular components are

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{d^2[s^2(t)]/dt^2\; h(r)^2}{2(1-2m/r)} + h'(r)\dot{s}(t) - \frac{(1-m/r)h(r)\dot{s}(t)}{r(1-2m/r)}. \tag{4.33}$$

[1]Somewhat similar constructions can be found in refs [179, 182].

The Ricci scalar is

$$R = -\frac{d^2[s^2(t)]/dt^2 \; h(r)^2}{(1-2m/r)} - 2h'(r)\dot{s}(t) + \frac{2(2-3m/r)h(r)\dot{s}(t)}{r(1-2m/r)}. \tag{4.34}$$

The Einstein tensor is of compact support — it is only non-zero where *both* $h(r)$ *and the derivatives* $\{\dot{s}(t), \ddot{s}(t)\}$ are non-zero. Note that both the metric determinant, $g = -r^4 \sin^2\theta$, and the volume element, $\sqrt{-g} = r^2 \sin\theta$, are independent of both $h(r)$ and $s(t)$.

4.3.2 Finite action for the bounce

The contribution to the action from the transition region is finite. First we note

$$S = \int \sqrt{-g}\; R \; \mathrm{d}^4x = \int \sqrt{-g}\; R\; \mathrm{d}^4 x = 4\pi \int r^2 \; R \; \mathrm{d}t\mathrm{d}r. \tag{4.35}$$

But the t integration yields

$$\int_{-\infty}^{+\infty} \left(\frac{\mathrm{d}^2[s^2(t)]}{\mathrm{d}t^2} \right) \mathrm{d}t = \left[\frac{\mathrm{d}[s^2(t)]}{\mathrm{d}t} \right]_{-\infty}^{+\infty} = 0 - 0 = 0, \tag{4.36}$$

and

$$\int_{-\infty}^{+\infty} \left(\frac{\mathrm{d}s(t)}{\mathrm{d}t} \right) \mathrm{d}t = [s(t)]_{-\infty}^{+\infty} = \pm 1 - (\mp 1) = \pm 2. \tag{4.37}$$

Therefore,

$$S = \pm 4\pi \int r^2 \left[4h'(r) + \frac{4(2-3m/r)h(r)}{r(1-2m/r)} \right] dr. \tag{4.38}$$

Now, integrating by parts in the radial coordinate yields

$$\int_{-\infty}^{+\infty} r^2 h'(r) \mathrm{d}r = \left[r^2 h(r) \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} 2rh(r)\mathrm{d}r = -\int_{-\infty}^{+\infty} 2rh(r)\mathrm{d}r\,. \tag{4.39}$$

Thus,

$$S = \pm 4\pi \int r \left[-8h(r) + \frac{4(2-3m/r)h(r)}{(1-2m/r)} \right] \mathrm{d}r. \tag{4.40}$$

After some algebra, this is explicitly:

$$S = \pm 16\pi m \int_{r_<}^{r_>} \frac{h(r)}{(1-2m/r)} \, \mathrm{d}r = \pm 16\pi m \int_{r_<}^{r_>} \frac{r\, h(r)}{(r-2m)} \, \mathrm{d}r\,. \tag{4.41}$$

(The naïve singularity at $r = 2m$ is again an integrable singularity.)

While the interpolating spacetime geometry is now dynamic —not static— the total action can be written in terms of the time-shift (4.29) at late and early times, (when the geometry is static), as

$$S = \pm 16\pi m \, \Delta F. \tag{4.42}$$

The reason the finiteness of the action is important is that finite-action configurations can easily contribute non-destructively to the Feynman path-integral. (The contributions of infinite action configurations tend to 'wash out' due to destructive interference.)

4.3.3 Zero action for the bounce

Perhaps unexpectedly, by making a suitable (symmetric) choice for $h(r)$ we can even drive the action of our black-to-white bounce to zero, not just keeping it finite.

For example: Take $r_> = 2m+\Delta$, and $r_< = 2m-\Delta$, and subsequently choose $h(r) = \pm(2m/r)B(|r-2m|)$; where $B(x)$ is a bump function with $B(0) = 1$ and $B(\Delta) = 0$; in this static case this leads to coordinates that are locally Kerr–Schild in the immediate vicinity of the horizon.

Then for the action of the black-to-white bounce, after integrating out the time dependence, from (4.41) we have:

$$S = \pm 16\pi m \int_{r_<}^{r_>} \frac{h(r)}{(1-2m/r)}\, dr = \pm 16\pi m \int_{2m-\Delta}^{2m+\Delta} \frac{(2m/r)B(|r-2m|)}{(1-2m/r)}\, dr \tag{4.43}$$

$$= \pm 32\pi m^2 \int_{2m-\Delta}^{2m+\Delta} \frac{B(|r-2m|)}{(r-2m)}\, dr = \pm 32\pi m^2 \int_{-\Delta}^{+\Delta} \frac{B(|z|)}{z}\, dz. \tag{4.44}$$

Here we have defined $z = r-2m$. This integral obviously vanishes by symmetry, but for clarity, being careful with the integrable singularity

$$S \propto \lim_{\epsilon\to 0}\left(\int_{-\Delta}^{-\epsilon} \frac{B(|z|)}{z}\, dz + \int_{\epsilon}^{\Delta} \frac{B(|z|)}{z}\, dz\right). \tag{4.45}$$

Thus,

$$S \propto \lim_{\epsilon\to 0}\left(\int_{\epsilon}^{\Delta} \frac{B(|z|)}{z}\, dz - \int_{\epsilon}^{\Delta} \frac{B(|z|)}{z}\, dz\right) = 0. \tag{4.46}$$

We may therefore conclude that one can even construct a *zero-action* compact support Lorentzian "bounce" that converts black holes to white holes (and *vice versa*). Note here, that in (4.45), we are taking the *standard* Cauchy principal value. One may argue that this integral does not in fact result in zero and will have a complex part added to it. This is the case if one uses 'Feynman's $i\epsilon$ prescription'[202] which is common in quantum field theories [203] but not so common in relativity.

4.3.4 Radial null curves

The radial null curves in this time dependent geometry are specified by

$$-(1-2m/r)\,\mathrm{d}t^2 - 2s(t)h(r)\mathrm{d}r\mathrm{d}t + \left[\frac{1-s(t)^2h(r)^2}{1-2m/r}\right]\mathrm{d}r^2 = 0. \tag{4.47}$$

That is

$$-(1-2m/r)^2 - 2s(t)h(r)(1-2m/r)\dot{r} + [1-s(t)^2h(r)^2]\dot{r}^2 = 0. \tag{4.48}$$

This is a simple quadratic for $\dot{r}$, implying

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\frac{(1-2m/r)}{[1\mp s(t)h(r)]}. \tag{4.49}$$

Unfortunately this ODE is not separable, and is not easy to solve.

The radial null tangent vectors to the null curves are of the form

$$k^a \propto (1, \dot{r}; 0, 0) = \left(1, \pm\frac{(1-2m/r)}{[1\mp s(t)h(r)]}; 0, 0\right). \tag{4.50}$$

In regions where $s(t)^2 = 1$, and using the fact that we always impose $h(2m) = 1$, one or the other of these radial null curves will be horizon penetrating. (In particular at early and late times, where $|s(t)| = 1$, one or the other of the null curves will penetrate the naïve horizon.)

During the bounce we can, for simplicity, assert $|s(t)| < 1$, and in fact $s(t)$ must, by construction, pass through zero. We can also for simplicity assert $|h(r)| \leq 1$, with equality only at the naïve horizon $r = 2m$. Under these conditions the denominator $1 \mp s(t)h(r)$ is always non-zero. Both incoming and outgoing null rays will be (temporarily) trapped at the naïve horizon, both with $\dot{r}_H = 0$ — at least until the end of the bounce — when, as per our analysis above, one or the other null curve can cross $r = 2m$ with non-zero coordinate velocity.

4.3.5 Energy conditions

While it is by now clear that the classical point-wise energy conditions of general relativity are not truly fundamental [204, 205, 206], (since they are all violated to one extent or another by quantum effects [207, 208, 209, 210, 211]) they are nevertheless extremely good diagnostics for detecting "unusual physics". This merits a very careful examination [212, 213, 214, 215]. The status of integrated energy conditions [216, 217, 218] and quantum inequalities is much more subtle [219]. In the current context it is most useful to focus on the null energy condition (NEC) and trace energy condition (TEC).

NEC: The condition for the null energy condition (NEC) to hold is $G_{ab}\, k^a\, k^b \geq 0$. The quantity $G_{ab}\, k^a\, k^b$ can be easily calculated for radial null curves, and in this case is:

$$G_{ab}\, k^a\, k^b \propto G_{rr} \left(\frac{(1-2m/r)^2}{[1 \mp s(t)h(r)]^2} \right) = -\frac{2\dot{s}(t)h(r)(1-2m/r)}{[1 \mp s(t)h(r)]^2}. \tag{4.51}$$

Since the denominator is non-negative we see

$$G_{ab}\, k^a\, k^b \propto -\dot{s}(t)\, h(r)\, (1-2m/r). \tag{4.52}$$

Regardless of the sign of $\dot{s}(t)$, or the sign of $h(2m)$, the product $\dot{s}(t)\ (1-2m/r)$ will certainly flip sign as one crosses the naïve horizon at $r = 2m$. Therefore, the NEC is definitely violated in parts of the black-to-white transition region. Furthermore, this automatically implies that the WEC, SEC, and DEC are also violated in parts of the black-to-white transition region.

TEC: The trace energy condition (TEC) is important mainly for historical reasons [204], though there is currently some resurgence of interest in this long-abandoned energy condition. (The TEC is useful for ordinary laboratory matter, but is already known to be violated by the equation of state for the material in the deep core of neutron stars, and in fact for any "stiff" system where $w \equiv p/\rho$ exceeds $1/\sqrt{3}$.)

The TEC asserts

$$g_{ab}\, T^{ab} = -(\rho - 3p) \leq 0. \tag{4.53}$$

For the Einstein tensor this becomes $g_{ab}\, G^{ab} \leq 0$, and for the Ricci scalar $R \geq 0$. But this would imply a positive semidefinite action, and we know that the black-to-white transition region is non-vacuum and can be chosen to have zero action. Therefore, there must certainly be regions in the compact black-to-white transition region where the TEC is violated.

ANEC: Analyzing the averaged null energy condition (ANEC) would require one to trace the null geodesics through the bounce region, and to unambiguously identify a suitable null affine parameter. Unfortunately, this is one of those situations where (despite recent progress [220]) these issues are still in the "too hard" basket.

Overall, we see that key point-wise energy conditions are definitely violated by the black-to-white bounce. This is an invitation to think carefully about the underlying physics.

4.4 Quantum implications

Despite considerable efforts, we do not as yet have a fully acceptable and widely agreed upon theory of quantum gravity. On the other hand, there are plausible and tolerably well accepted partial models — such as approximations based on semi-classical gravity (and quantized linearized weak-field gravity for that matter). One issue on which there is widespread agreement is the use of the Feynman functional integral formalism in the semi-classical regime.

One of the key features of the Feynman functional integral formalism is that quantum amplitudes are dominated by classical configurations (plus fluctuations). In the current context, the fact that we have found zero-action black-to-white bounces, combined with the fact that the usual classical vacuum (Schwarzschild) is also zero-action, implies that these configurations reinforce constructively. If the black-to-white bounces are to be quantum mechanically suppressed, such suppression will have to come from the quantum fluctuations, not from the leading order term.

This situation is somewhat reminiscent of the role played by instanton contributions to the QCD vacuum [221, 222, 223, 224]. There are significant differences, zero-action *versus* finite action, Lorentzian signature *versus* Euclidean signature — but crucial key features are similar. Indeed, the existence of localized zero-action configurations is not all that unusual, also occurring in flat Minkowski space classical field theories [225], though their implications have not been particularly well studied.

This *suggests* the possibility that astrophysical black holes (the "cold, dark, and heavy" objects detected by astronomers) might be in a quantum superposition of black hole and white hole states. For somewhat similar suggestions, differing in detail, see also [173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195]. Finally one could *speculate* that this is evidence in favour of quantum physics becoming dominant in near-horizon physics. This was, for many decades, (pre-2000 CE) a minority opinion within the general relativity community, as there was a broad but not universal consensus that quantum physics should only come into play in the deep core where curvature reaches Planck scale values. More recently (post-2000 CE) the situation is more nuanced.

One of the main counterweights to that prior (pre-2000 CE) consensus opinion is the "gravastar" model [129, 128, 226, 130, 131, 227, 228, 229, 230, 231, 232, 133], where quantum physics kicks in at/near the would-be horizon. Similarly for the "fuzzball" model, stringy physics [233, 234, 235, 236, 237, 238] kicks in at/near the would-be horizon. Furthermore, for the "firewall" proposal [239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249] something again happens at/near the would-be horizon. While these proposals typically severely impact on the spacetime geometry throughout the entire interior region, the novel construction we are dealing with affects only the near-horizon spacetime geometry.

4.5 Discussion

In this chapter we wanted to investigate if a simple and compelling classical model of a black-to-white hole transition could be found. We began by performing a simple coordinate transformation of the standard Schwarzschild metric by modifying the radial coordinate. This resulted in the line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - 2h(r)\mathrm{d}r\mathrm{d}t + \left[\frac{1-h(r)^2}{1-2m/r}\right]\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{4.54}$$

For specific choices of $h(r)$ this returns the Schwarzschild spacetime in other well known coordinates, such as the Painlevé–Gullstand, Kerr–Schild, and Eddington–Finkelstein coordinates. By imposing the restriction $h(2m) = \pm 1$ we showed that this line element can model a classical black or white hole where one or the other of the null curves are horizon penetrating with non-zero coordinate velocity

$$\dot{r}_H = \pm\frac{1}{2m\;h'(2m)}\,. \tag{4.55}$$

By choosing $h(r)$ to be of compact support, we demonstrated that we could confine the non-trivial aspects of black and white horizons to a compact radial region straddling the naïve horizon $r = 2m$.

By introducing a time-dependent function, $s(t)$, we then produced a simple classical model for a black-hole-to-white-hole transition. This spacetime, however, was no longer *just* a coordinate transformation of Schwarzschild spacetime. The introduction of $s(t)$ led to the following line element

$$\mathrm{d}s^2 = -(1-2m/r)\mathrm{d}t^2 - 2s(t)h(r)\mathrm{d}r\mathrm{d}t + \left[\frac{1-s(t)^2h(r)^2}{1-2m/r}\right]\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{4.56}$$

The non-static spacetime in these coordinates was found (at early and late times) to have horizon penetrating null curves with coordinate velocity

$$\dot{r}_H = \pm\frac{1}{2m\;h'(2m)}\,. \tag{4.57}$$

During the bounce itself the behaviour of the null curves is much trickier.

We further showed that the action in the transition region was *finite*,

$$S = 16\pi m\int_{r_<}^{r_>}\frac{h(r)}{(1-2m/r)}\,dr\,. \tag{4.58}$$

More importantly though, this action can be arranged to be zero by carefully choosing $h(r)$. This proves to be a significant result as this action could then be added to the Feynman path integral and have no impact on any quantum amplitudes.

For tractability and ease of exposition the current analysis has focussed on the Schwarzschild spacetime, though there is no real difficulty (apart from tedium) in working with the outer horizon of non-extremal Reissner–Nordström or indeed any spherically symmetric non-extremal black hole. Extremal black holes would seem to require a more subtle analysis. In a different direction, there are certainly purely technical issues arising in dealing with non-extremal Kerr and Kerr–Newman, a topic we hope to turn to in the future. We do not expect to encounter any fundamental issues with non-extremal Kerr and Kerr–Newman, but the extremal case is again likely to be problematic.

Chapter 5

The Kerr Memory Effect at Null Infinity

Over the last decade or so, various — seemingly disconnected — parts of physics have been shown to be mathematically connected in the infrared regime. The connections we will investigate here are one leg of the 'Infrared Triangle' [25] (see Figure 5.1). This leg illustrates the gravitational memory effect and asymptotic symmetries of null infinity are related. The infrared triangle may be a step forward in understanding quantum gravity. In particular, the memory effect will potentially be observable in the near future due to missions such as LISA [250]. Therefore, further development of these formalisms may lead to true observations of the nature of quantum gravity.

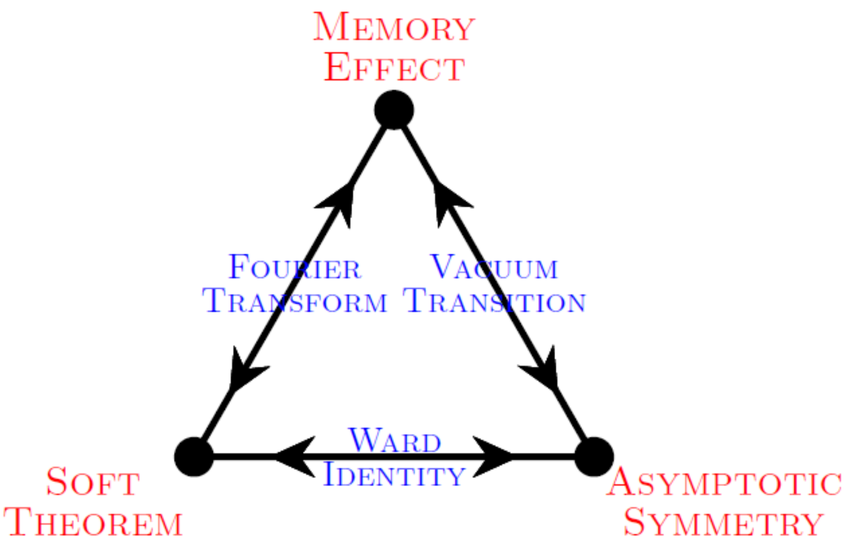

Figure 5.1: The infrared triangle from ref [25]: Three different areas of physics are realised as one in the infrared limit (at null infinity).

In this chapter we compute the memory effect due to a gravitational wave striking a Kerr black hole as seen by an observer at null infinity. This is done by working in Bondi–Sachs coordinates. It was shown by Hawking, Perry, and Strominger (HPS) that the memory effect due to a gravitational shockwave is seen as a pure BMS supertranslation from null infinity. Hence, it is of interest to compute the supertranslated Kerr solution in Bondi–Sachs coordinates. Finally, the gravitational wave is said to implant soft supertranslation hair on the event horizon of the black hole which carries superrotation charge. We will explicitly calculate the change in superrotation charge on the event horizon due to the supertranslation hair.

Since the observational discovery of gravitational waves nearly a decade ago by the Laser Interferometer Gravitational-Wave Observatory (LIGO) (and other detectors such as Virgo and KAGRA), many have wondered about detection of the gravitational memory effect — the permanent alteration of a system due to a transient gravitational wave (for instance, see Figure 5.2). The memory effect has been discussed in the literature since 1972, first introduced by Zel'dovich and Polnarev [251], then greatly expanded upon in the last few decades by Christodoulou and others [252, 253, 254, 255, 256, 257, 258]. In the past few years there has been a deep mathematical connection made between the gravitational memory effect and a set of infinite symmetries at null infinity [259, 260, 25, 27, 26]. These symmetries are associated to a set of transformations known as *supertranslations* and *superrotations*, collectively known as *supertransformations*. In fact, when two particles are left permanently displaced by a gravitational wave, the initial and final states are related by a supertranslation.

This group of infinite symmetries has been known of for nearly 60 years, first introduced by Bondi, van der Burg, Metzner and Sachs [261, 262, 263, 264] - known as the BMS group[1]. In recent years, there has been further research and development of the BMS algebra [265, 266, 267] and the charges associated with supertranslations and superrotations. These charges have led to a hope of better understanding the 'scattering problem' in general relativity [259, 25]. Furthermore, it seems that charges associated with supertransformations may play an important role in resolving — part of — the information loss problem [268, 26, 27]. This is addressed by asserting that gravitational waves implant soft supertranslation hair on the event horizons of black holes. The soft hair is then evaporated off to null infinity, thereby preserving information from past null infinity through to future null infinity. Therefore, detection of the memory effect may offer a better understanding of abstract mathematical ideas and how they may be physically realised. Unfortunately, the memory effect will likely not be observationally detected until after the Laser Interferometer Space An-

[1]Unfortunately, van der Burg's name is often forgotten.

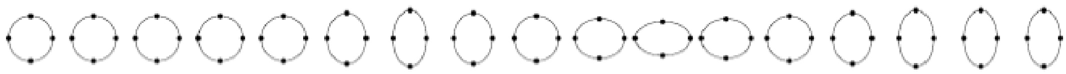

Figure 5.2: Particle configurations as a gravitational wave passes (time running left to right). The memory effect results in the final configuration differing from the initial. From ref [257]

tenna (LISA) [269, 250] is launched.

Here we will discuss the effects of a transient gravitational shockwave striking the Kerr black hole as seen from an observer at future null infinity[2]. We follow the calculations of HPS and others [26, 25, 270] who have discussed the effects of a transient shock wave striking Schwarzschild and Reissner Nordström black holes. These authors show that the deformation of a black hole due to a gravitational shockwave (the memory effect) is seen as a pure BMS supertranslation *from future null infinity*[3]. Hence, we will compute the explicit supertranslated Kerr solution in Bondi–Sachs coordinates.

We will begin by summarising the expanded Bondi–Sachs metric, asymptotic Killing vectors which generate symmetries that have associated charges, and the relationship between the memory effect and supertranslations. In section 5.2 we will discuss the Kerr solution in general Bondi–Sachs coordinates introduced by Fletcher and Lun [271] and put it in the Bondi–Sachs gauge. In section 5.3 with the Kerr solution in the Bondi–Sachs gauge we will find the supertranslated metric functions. Finally, in section 5.4 we will calculate the supertransformation charges associated to the supertranslation hair that is implanted on the Kerr black hole due to the gravitational shockwave.

5.1 The BMS Group

In 1962 Bondi, Metzner, van der Burg and Sachs (BMS) [261, 262, 264] were attempting to find a group of diffeomorphisms at null infinity which acted non-trivially on asymptotic data. BMS found an infinite dimensional group which contained the Poincaré group as a subgroup [25, 272, 273, 267]. This group has an infinite amount of generators known as supertransformations [267, 25, 27]. This was a surprising result as it meant that general relativity did not reduce to special relativity in the weak field limit and the symmetries found at null infinity were not those of the Poincaré group alone.

[2]It is worth mentioning that we do not have an effective operational answer for "where is null infinity", nor do we claim to. It is an interesting question to ask, however. How does one define an operational notion of null infinity and how "far one has to be from the source" to observe something that is being radiated off to null infinity.

[3]There are additional effects that are not seen from null infinity, and this will be discussed in subsection 5.1.4.

In physics — since the 1920s, most of the fundamental interactions can be described as 'gauge theories'. The transformations between gauges are called *gauge transformations*. These transformations form a *gauge group* (a specific type of Lie group), which has an associated Lie algebra of group generators. These group generators have an associated gauge field and the gauge field is used in the Lagrangian of a theory to ensure that the theory is gauge invariant. In general relativity, the gauge symmetry is *diffeomorphism invariance*.

In the literature, asymptotic gauge symmetry groups (AGS) are defined as the quotient group of *allowed gauge symmetries* and *trivial gauge symmetries* [25]. The 'need' for asymptotic symmetry groups arises from noting that Noether's theorem only applies to on-shell physics. The use of asymptotic symmetries allows one to generalise Noether's theorem to examine symmetries of spacetime that may occur off-shell [274]. In the case of BMS, the asymptotic boundary they wished to investigate was null infinity.

When investigating asymptotic symmetries we impose boundary conditions that reflect the nature of a spacetime at the boundary. These boundary conditions should be weak enough to allow all physically possible solutions to exist, but also strong enough that charges — i.e., globally conserved quantities — are finite and well defined [25, 272]. In general relativity, however, it is difficult to impose boundary conditions and therefore determine how a system *should* behave at the boundary. When considering spatial infinity, i^0, the AGS are the Poincaré group symmetries which consist of Lorentz transformations and spacetime translations. In terms of Noether's theorem, the ADM mass is defined by symmetries at i^0 while the Bondi mass is defined by symmetries at future null infinity, $\mathcal{I}^+$.

Definitions of the BMS group

In general relativity, there is no global conservation of 4-momentum. This is because the extended form of energy–momentum conservation,

$$\nabla_\nu T^{\mu\nu} = 0,$$

is not integrable. However, if a spacetime possesses a Killing vector, ξ^μ then the current, $J^\mu = T^{\mu\nu}\xi_\nu$ admits a conserved charge. Therefore, we say that the existence of a Killing vector field results in the existence of continuous symmetries which imply a conserved charge via Noether's theorem. Furthermore, Killing vectors of a spacetime for a group of symmetries. Under the Lie bracket, these Killing vectors or more generally, *symmetry generators*, generate a Lie algebra. For instance, in the case of Minkowski space, the global Killing vectors form the Poincaré group and the commutation relations of the Killing vectors generate the Poincaré algebra. The BMS group, is an infinite dimensional symmetry group formed by an infinite number of asymptotic Killing vectors or *supertransformations*. These Killing vectors generate the bms_4 algebra.

We can further denote the BMS group in abstract mathematical notation [25, 275],

$$\text{BMS} = S \rtimes \text{SO}(1,3).$$

Note that $\text{SO}(1,3)$ is a representation of the Lorentz group and S is the group of supertranslations. Here $\rtimes$ is the semi-direct product which simply means that elements of the BMS group are pairs of elements in the Lorentz transformation group, and the group of supertranslations. To compare this definition with something more familiar, we can define the Poincaré group as [267]

$$\text{Poincaré} = T_4 \rtimes \text{SO}(1,3).$$

Here T_4 is the group of spacetime translations.

5.1.1 The Bondi–Sachs Metric

Bondi, van der Burg, Metzner, and Sachs [261, 262, 264] wanted to define a concept of asymptotic flatness at null infinity. The falloffs needed to be restrictive enough that unphyiscal spacetimes — such as those with infinite energy — would be ruled out, yet not so restrictive such that physical spacetimes and gravitational waves would be ruled out. While the falloffs may differ in the literature, we will use the choice made by BMS [261, 262, 264, 25]:

$$\begin{aligned} &g_{uu} = -1 + \mathcal{O}(r^{-1}), \quad g_{ur} = -1 + \mathcal{O}(r^{-2}), \quad g_{uA} = \mathcal{O}(r^0), \\ &g_{AB} = r^2\gamma_{AB} + \mathcal{O}(r), \quad g_{rr} = g_{rA} = 0\,. \end{aligned} \tag{5.1}$$

The class of allowed asymptotic line elements for these falloffs is given by

$$\begin{aligned} ds^2 =& -\mathrm{d}u^2 - 2\,\mathrm{d}u\mathrm{d}r + r^2\gamma_{AB}\,\mathrm{d}\Theta^A\mathrm{d}\Theta^B \\ &+ \frac{2m_{\,\text{bondi}}}{r}\mathrm{d}u^2 + r\,C_{AB}\,\mathrm{d}\Theta^A\mathrm{d}\Theta^B + D^B C_{AB}\,\mathrm{d}u\,\mathrm{d}\Theta^A \\ &+ \frac{1}{16r^2}\Big\{C^{FD}C_{FD}\Big\}\,\mathrm{d}u\mathrm{d}r \\ &+ \frac{1}{r}\Big(\frac{4}{3}N_A + \frac{4u}{3}\partial_A m_{\,\text{bondi}} - \frac{1}{8}\partial_A\Big\{C^{FD}C_{FD}\Big\}\Big)\mathrm{d}u\,\mathrm{d}\Theta^A \\ &+ \frac{1}{4}\gamma_{AB}\Big\{C^{FD}C_{FD}\Big\}\,\mathrm{d}\Theta^A\mathrm{d}\Theta^B + \ldots \end{aligned} \tag{5.2}$$

where $\Theta^A \in \{\theta, \phi\}$ and the uppercase Latin indices run over θ, ϕ. D_A is the covariant derivative on the 2–sphere with respect to the 2–sphere metric, γ_{AB}. The function m_{bondi} is the Bondi mass aspect, which is in general a function of u and the angles, $\theta\,, \phi$. This can be used to obtain the Bondi mass after integrating m_{bondi} over the entire 2–sphere at null infinity. In the case of the Kerr spacetime, the Bondi mass is simply, M, the mass of the black hole. N_A is the angular moment aspect. Contracting N_A with the generator of rotations and integrating over the entire sphere is related to the total angular momentum of the spacetime. C_{AB} is another field which is symmetric and traceless ($\gamma^{AB}C_{AB} = 0$). The retarded time derivative of C_{AB} is in fact the Bondi news tensor,

$$N_{AB} := \partial_u C_{AB} \,. \tag{5.3}$$

The news tensor is the gravitational analogue of the Maxwell field strength and its square is proportional to the energy flux across $\mathcal{I}^+$ [25, 260].

It is important to note that N_A is defined slightly differently in various parts of the literature — usually depending on the asymptotic expansion (5.2). For instance, Comperé in refs [276, 277] has a decomposition that leads to N_A being defined as[4]

$$N_A := -\frac{3}{32}\partial_A\Big(C_{BC}C^{BC}\Big) - \frac{1}{4}C_{AB}D_C C^{AC}. \tag{5.4}$$

However, Strominger in ref [25] and Comperé in ref [260] uses the decomposition (5.2), which leads to N_A being defined as

$$\frac{2}{3}N_A - \frac{1}{16}\partial_A\Big(C_{BC}C^{BC}\Big) := g^{(1)}{}_{uA} \,. \tag{5.5}$$

Here, $g^{(1)}_{uA}$ corresponds to the r^{-1} expansion in g_{uA}. We will opt to use the second definition as the superrotation charge will not be changed. This can be seen explicitly in the case of the Kerr solution.

5.1.2 Asymptotic Killing Vector

Symmetries of a spacetime are associated to Killing vectors of that spacetime. Hence, before discussing the charges associated with symmetries in our spacetime, we must first briefly discuss the Killing vectors of our spacetime - or rather, the *asymptotic Killing vectors*. The most general Killing vector, $\boldsymbol{\xi}$, that preserves the metric (5.2) to leading order is [25]

$$\begin{aligned}\xi^\alpha\partial_\alpha := f\partial_u &+ \left[-\frac{1}{r}D^A f + \frac{1}{2r^2}C^{AB}D_B f + \mathcal{O}\Big(\frac{1}{r^3}\Big)\right]\partial_A \\ &+ \left[\frac{1}{2}D^2 f - \frac{1}{r}\left\{\frac{1}{2}D_A f D_B C^{AB} + \frac{1}{4}C^{AB}D_A D_B f\right\} + \mathcal{O}\Big(\frac{1}{r^2}\Big)\right]\partial_r \,.\end{aligned} \tag{5.6}$$

Here, f is a function of the angular coordinates (θ, ϕ) only and D^2 is the standard Laplacian on the 2–sphere. However, since the analysis conducted here is

[4]This corresponds to pure vacuum and the mass contribution is omitted.

only a calculation to linear order — as done by HPS [26], the Killing vector is truncated:

$$\xi^\alpha \partial_\alpha = f\,\partial_u + \frac{1}{2} D^2 f \partial_r - \frac{1}{r} D^A f \partial_A\,. \tag{5.7}$$

This does indeed beg the question of whether a second order analysis would still show that the memory effect — as seen from null infinity — is still a supertranslation with the Killing vector (5.6).

5.1.3 Associated Charges and Charge Conservation

It is well known that the symmetries of spacetime are associated to conserved charges via Noether's theorem. Before the discovery of the BMS group, the largest symmetry group was the Poincaré group, which has associated conserved charges such as energy and momentum. The BMS group, which is an infinite dimensional group of symmetries at null infinity also has charges associated to supertranslations and superrotations.

Supertranslation charge and its conservation is given by[5] [25]

$$Q_f^+ = \frac{1}{4\pi} \int_{\mathcal{I}^+} \mathrm{d}^2\Theta \, \sqrt{\gamma}\, f m_{\text{bondi}} = \frac{1}{4\pi} \int_{\mathcal{I}^-} \mathrm{d}^2\Theta \, \sqrt{\gamma}\, f m_{\text{bondi}} = Q_f^-\,. \tag{5.8}$$

Here f is a function of angular coordinates[6] and can be thought of as the generator of supertranslations. In general, the supertranslation charges will depends on advanced/retarded time. This is simply due to m_{bondi} — in general — depending on advanced/retarded time. This conservation 'law' is a statement about the total energy of the system.

[5]Note that integration is carried out over the 2–sphere at null infinity.

[6]Note that while f is often chosen to be a spherical harmonic function, it is not limited to be only a spherical harmonic.

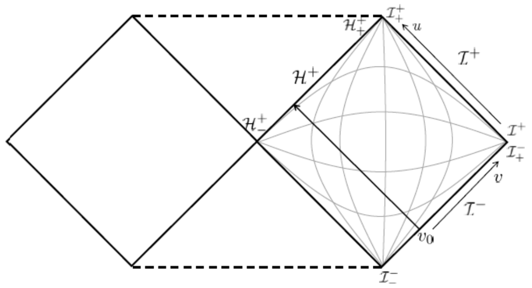

Figure 5.3: This Penrose diagram illustrates how charges are carried from past null infinity to the future. The gravitational wave travelling from v_0 on $\mathcal{I}^-$ strikes the horizon ($\mathcal{H}^+$) and the horizon charges are then evaporated off to the future, at $\mathcal{I}^+_+$. Thus, charges are preserved from the past to the future even with the presence of a black hole in the spacetime.

Superrotation charge and its conservation is given by[7]

$$Q_Y^+ = \frac{1}{8\pi}\int_{\mathcal{I}^+} \mathrm{d}^2\Theta\,\sqrt{\gamma}\,Y^A N_A = \frac{1}{8\pi}\int_{\mathcal{I}^-} \mathrm{d}^2\Theta\,\sqrt{\gamma}\,Y^A N_A = Q_Y^-, \tag{5.9}$$

where Y^A is an arbitrary vector field on the 2–sphere [8]. In the case that Y^A is one of the 6 global conformal Killing vectors on the 2–sphere (5.9) expresses conservation of ADM angular momentum and boost charges.

The notation used here indicates that *supertransformation* charge at future null infinity, $\mathcal{I}^+$ should *match* the supertransformation charge at past null infinity, $\mathcal{I}^-$. Furthermore, these *matching conditions* are a statement about having a well-posed scattering problem in general relativity (see Figure 5.3). The existence of these conserved charges is — at least in principle — verifiable with the gravitational memory effect, which may be detectable in the near future.

[7]This definition of superrotation charge follows from using (5.5) as our definition of N_A. In the case that one uses (5.4), superrotation charge is expressed via [266]

$$Q_Y = \frac{1}{16\pi}\int \mathrm{d}^2\Theta\,\sqrt{\gamma}\,Y^A\Big[2N_A + \frac{1}{16}\partial_A\Big(C_{BC}C^{BC}\Big)\Big]\,.$$

[8]As is the case for f, the *components* of Y^A are often chosen to be spherical harmonic functions, however, the components of Y^A are not limited to be spherical harmonics.

5.1.4 The Black Hole Memory Effect and Supertranslations

During the last decade there has been a lot of discussion regarding the gravitational memory effect and the direct correspondence of this effect with the BMS group — for instance, one may see refs [25, 26, 27, 260, 276, 278, 279, 280]. It was shown by HPS [26] that a deformation due to a shockwave defined via a impulse energy-momentum tensor of the form [9]

$$T_{vv} = \frac{\mu + T(z)}{4\pi r^2}\delta(v - v_0), \tag{5.10}$$

where μ is the monopole contribution of the shock wave and $T(z)$ characterises the angular profile of the wave is equivalent to a BMS supertranslation at null infinity (for a Schwarzschild black hole). Such a supertranslation is given by taking the Lie derivative of the spacetime metric — in this case, the Schwarzschild metric — along the asymptotic Killing vector[10],

$$\xi^\mu \partial_\mu = f\partial_v - \frac{1}{2}D^2 f\partial_r + \frac{1}{r}D^A f\partial_A\,. \tag{5.11}$$

However, as HPS [26] state, supertranslations only equate to part of the deformation a gravitational wave would produce when striking a black hole.

Intuitively, gravitational waves, which carry energy, should impart some of this energy to the black hole it strikes and it should alter the *mass* and/or (angular) momentum. In fact, it was shown by HPS [26] that there was a change in the mass of a Schwarzschild black hole due to the monopole contribution (μ) of the shockwave. Therefore, the full[11] "memory effect" due to such a shockwave was written as[12]

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} + \frac{2\mu}{r}\delta^v{}_\mu \delta^v{}_\nu\,. \tag{5.12}$$

Here $\delta g_{\mu\nu}$ refers to the permanent change in the spacetime due to a gravitational wave. Therefore, the ADM mass of the hairy Schwarzschild black hole would be $m = M + \mu$ after the gravitational shockwave strikes the black hole. However, while the mass of the black hole may change, the *Bondi mass* of the black hole — at least at linear order — does not change. This further emphasises that the memory effect is not *entirely captured by BMS supertransformations.* However, the black hole memory effect as seen from null infinity *is entirely captured by BMS supertranslations.*

[9]If one wishes to see the full details of this derivation, they may see page 137 of Strominger's lecture notes [25].

[10]Note the sign change here due to changing from retarded time to advanced time.

[11]The word *full* is used here, however, it is worth noting that this is only an analysis at linear order.

[12]The Heaviside terms here have been neglected as we are assuming the shockwave has already struck the black hole.

This discussion is important when analysing the calculations in this chapter. Applying the same gravitational shockwave (5.10) used in [26, 25, 270] to the Kerr solution is a non-trivial task and thus one does not know for certain what the other components to the memory effect will appear. A common suspicion within the community is that there may be another monopole contribution to the mass term that appears in the $g_{u\phi}$ component of the metric. This would imply a change in the angular momentum of the Kerr black hole due to a gravitational shockwave — which will not be seen at null infinity by observation of the superrotation charge[13].

[13]Recall that if one changes the ADM mass of a rotating black hole, this changes the ADM angular momentum. This would, therefore, change the behaviour at spatial infinity, i^0, but not null infinity, $\mathcal{I}$.

5.2 The Kerr Metric

We will now discuss the Kerr metric in general Bondi–Sachs coordinates. The Kerr solution was first introduced in generalised Bondi–Sachs coordinates by Fletcher and Lun [271]. They were interested in investigating gravitational radiation in the Kerr spacetime. As we are interested in the behaviour of this solution at large r, i.e., expanded in powers of $1/r$, we shall only present the asymptotic line element[14] found in the appendix of ref [271]:

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1-\frac{2M}{\tilde{r}}\right)\mathrm{d}u^2 - 2\left(1+\frac{a^2\cos^2\theta-\frac{1}{2}a^2}{\tilde{r}^2}\right)\mathrm{d}u\mathrm{d}\tilde{r} \\ &- 2\cos\theta\left(a-\frac{2aM+2a^2\sin\theta}{\tilde{r}}\right)\mathrm{d}u\mathrm{d}\theta - 2\left(\frac{2aM\sin^2\theta}{\tilde{r}}\right)\mathrm{d}u\mathrm{d}\phi \\ &+ \tilde{r}^2\left(1+\frac{2a\sin\theta}{\tilde{r}}+\frac{2a^2-3a^2\cos^2\theta}{\tilde{r}^2}\right)\mathrm{d}\theta^2 \\ &+ \tilde{r}^2\left(\sin^2\theta-\frac{2a\sin\theta\cos^2\theta}{\tilde{r}}+\frac{a^2-3a^2\cos^2\theta+3a^2\cos^4\theta}{\tilde{r}^2}\right)\mathrm{d}\phi^2+\dots \end{aligned} \tag{5.13}$$

The line element given in (5.13) does not, however, satisfy the Bondi-Sachs gauge. This gauge is reached by requiring the coordinate, r, to be the "luminosity distance" [261]. This is, however, misleading as the term luminosity distance means something quite different in cosmology. In fact, r, is simply chosen to be an areal coordinate that varies along null rays and satisfies $\det(g_{AB}) = r^4\sin^2\theta$. This is achieved by defining[15],

$$\tilde{r} := r + \frac{a}{2}\frac{\cos 2\theta}{\sin\theta} + \frac{a^2}{8}\left(4\cos 2\theta + \frac{1}{\sin^2\theta}\right)\frac{1}{r}\,. \tag{5.14}$$

With this radial coordinate the line element (5.13) may be recast into the

[14] We have chosen to not list the $g_{\theta\phi}$ term here since it is of order $\mathcal{O}(1/r)$. Comparing this to (5.2) it is clear that this term will be of subleading order and not used in the analysis.

[15] In fact, this is not the only reason this transformation is chosen. This coordinate transformation also leads to the trace of C_{AB} vanishing and the metric taking the form (5.2).

following form[16]:

$$
\begin{aligned}
\mathrm{d}s^2 = &-\left(1-\frac{2M}{r}\right)\mathrm{d}u^2 - 2\left(1-\frac{a^2}{16\sin^2\theta}\frac{1}{r^2}\right)\mathrm{d}u\mathrm{d}r \\
&-2\left(-\frac{a}{2}\frac{\cos\theta}{\sin^2\theta}-\frac{a\cos\theta}{4}\{8M+\frac{a}{\sin^3\theta}\}\frac{1}{r}\right)\mathrm{d}u\mathrm{d}\theta - 2\left(\frac{2aM\sin^2\theta}{r}\right)\mathrm{d}u\mathrm{d}\phi \\
&+\left(r^2+\frac{a}{\sin\theta}r+\frac{a^2}{2\sin^2\theta}\right)\mathrm{d}\theta^2+\left(r^2\sin^2\theta-a\sin\theta\, r+\frac{a^2}{2}\right)\mathrm{d}\phi^2+\ldots
\end{aligned}
\tag{5.15}
$$

By comparing the expanded form of Kerr solution in the Bondi-Sachs gauge in (5.15) with (5.2) we may read off N_A, C_{AB}, and $C_{AB}C^{AB}$ for the Kerr solution.

$$
\begin{aligned}
&C_{AB}C^{AB} \equiv \frac{2a^2}{\sin^2\theta}; \\
&C_{AB}\,\mathrm{d}x^A\mathrm{d}x^B \equiv \frac{a}{\sin\theta}\,\mathrm{d}\theta^2 - a\sin\theta\,\mathrm{d}\phi^2; \\
&N_\theta \equiv 3aM\cos\theta\,; \\
&N_\phi \equiv -3aM\sin^2\theta; \\
&m_{\,\mathrm{bondi}} \equiv M\,.
\end{aligned}
\tag{5.16}
$$

One may note here that the coordinate transformation (5.14) is singular when $\sin\theta = 0$. However, this is in fact only a coordinate singularity as one may confirm by checking both the Ricci and Kretschmann scalars. Both of these scalar invariants remain finite when $\sin\theta = 0$. Intuitively, one *may* see this as a result of using simple null geodesics — those with zero angular momentum about the *axis of symmetry* — to arrive at this particular version of the Kerr solution. Simple null geodesics were used by Fletcher and Lun [271] as the principal null directions of the Kerr solution do not form constant u hypersurfaces. This feature has been discussed in the literature and does limit the applicability of this metric in numerical studies — one may see refs [281, 282] for further discussions.

Furthermore, note that we are using retarded time instead of advanced time. The original calculations by HPS [26, 25] were done in advanced time. However, we are interested in the potential observation of the supertranslated Kerr black hole in the *future* of the black hole being struck by a gravitational wave[17].

[16]The g_{ur} component here differs from ref [266]. This is in fact a small error in Appendix D of their paper and can be verified by computing $\mathrm{d}\tilde{r}$. Furthermore, one may note that this component now correctly provides the scalar, $C^{AB}C_{AB}$.

[17]By following the calculations of Fletcher and Lun [271] changing from retarded time to advanced time will only require a sign change. Therefore, if one wishes to see charges *implanted* on the horizon, one can easily do so by changing the a sign in the relevant components.

5.3 Supertranslations of the Kerr Spacetime

Unlike the cases considered in refs [26, 270, 29] — which are non-rotating spacetimes — there are several other terms involved when calculating the diffeomorphisms of the Kerr solution in the form (5.13). All of these diffeomorphisms are explicitly given by

$$\delta g_{\mu\nu} := \mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu \xi^\alpha + g_{\mu\alpha}\partial_\nu \xi^\alpha \,, \tag{5.17}$$

where the $\boldsymbol{\xi}$ is once again the asymptotic Killing vector in (5.11). The supertranslated metric functions are found to be:

$$\begin{aligned}
\delta g_{uu} &= -M\frac{D^2 f}{r^2}\,;\\
\delta g_{ur} &= \frac{1}{r^2}\frac{a\cos\theta}{2\sin^2\theta}D^2 f\,;\\
\delta g_{u\theta} &= -\left(\partial_\theta f + \frac{1}{2}\partial_\theta D^2 f\right) + \frac{1}{r}\left(2M\partial_\theta f - \partial_\theta\left\{\frac{a\,\cos\theta}{2\sin^2\theta}\partial_\theta f\right\}\right)\,;\\
\delta g_{u\phi} &= -\left(\partial_\phi f + \frac{1}{2}\partial_\phi D^2 f\right) + \frac{1}{r}\left(2M\partial_\phi f - \frac{a\,\cos\theta}{2\sin^2\theta}\partial_\phi\partial_\theta f\right)\,;\\
\delta g_{\theta\theta} &= \left\{2{\partial_\theta}^2 f - D^2 f\right\}r - \frac{a}{\sin\theta}\left\{+\frac{1}{2}D^2 f + 2\frac{\cos\theta}{\sin\theta}\partial_\theta f - 2{\partial_\theta}^2 f\right\}\,;\\
\delta g_{\phi\phi} &= \left\{2{\partial_\phi}^2 f + 2\sin\theta\cos\theta\,\partial_\phi f - \sin^2\theta D^2 f\right\}r\\
&\qquad - \frac{a}{\sin\theta}\left\{\frac{1}{2}\sin^2\theta D^2 f + \cos\theta\,\partial_\phi f + 2{\partial_\phi}^2 f\right\}.
\end{aligned} \tag{5.18}$$

Comparing the supertranslated metric functions with refs [26, 270, 29] it is clear that upon setting $a = 0$ we recover the supertranslated Schwarzschild black hole[18]. One may note that the supertranslated C_{AB} field is the same for the Kerr solution and Schwarzschild. This is to be expected as in refs [25, 260] is it shown that there should be no change unless m_{bondi} is actually a function of retarded/advanced time.

The new metric functions are defined via

$$\bar{g}_{\mu\nu} := g_{\mu\nu} + \delta g_{\mu\nu}\,, \tag{5.19}$$

[18] The notation used in this thesis is more explicit when comparing to the referenced papers. This leads to the $g_{\phi\phi}$ supertranslation looking slightly different. This is, however, due to the fact that the covariant derivatives have been calculated explicitly and not left in the form $D_A(D_B f)$.

which are:

$$
\begin{aligned}
\bar{g}_{uu} &= -\left(1-\frac{2M}{r}-M\frac{D^2 f}{r^2}\right);\\
\bar{g}_{ur} &= -\left(1-\left\{\frac{a^2}{16\sin^2\theta}+\frac{8a\cos\theta}{16\sin^2\theta}D^2 f\right\}\frac{1}{r^2}\right);\\
\bar{g}_{u\theta} &= -\left\{-\frac{a\cos\theta}{2\sin^2\theta}+\partial_\theta f+\frac{1}{2}\partial_\theta D^2 f\right\}\\
&\quad -\left\{-\frac{a\cos\theta}{4}\{8M+\frac{a}{\sin^3\theta}\}-2M\partial_\theta f+\partial_\theta\left[\frac{a\cos\theta}{2\sin^2\theta}\partial_\theta f\right]\right\}\frac{1}{r};\\
\bar{g}_{u\phi} &= -\left(\partial_\phi f+\frac{1}{2}\partial_\phi D^2 f\right)+\left\{-2aM\sin^2\theta+2M\partial_\phi f-\frac{a\cos\theta}{2\sin^2\theta}\partial_\phi\partial_\theta f\right\}\frac{1}{r};\\
\bar{g}_{\theta\theta} &= r^2+\left\{\frac{a}{\sin\theta}+2\partial_\theta{}^2 f-D^2 f\right\}r\\
&\quad -\frac{a}{\sin\theta}\left\{-\frac{a}{2\sin\theta}+\frac{1}{2}D^2 f+2\frac{\cos\theta}{\sin\theta}\partial_\theta f-2\partial_\theta{}^2 f\right\};\\
\bar{g}_{\phi\phi} &= r^2\sin^2\theta+\left\{-a\sin\theta+2\partial_\phi{}^2 f+2\cos\theta\sin\theta\,\partial_\phi f-\sin^2\theta D^2 f\right\}r\\
&\qquad -\frac{a}{\sin\theta}\left\{-\frac{a\sin\theta}{2}+\frac{1}{2}\sin^2\theta D^2 f+\cos\theta\partial_\phi f+2\partial_\phi{}^2 f\right\}.
\end{aligned}
\tag{5.20}
$$

The supertranslated Kerr metric in section 5.3 is referred to as a "hairy black hole". The hair carried is soft supertranslation hair, which have corresponding charges. From the metric components, one can read off $C_{AB}C^{AB}$, C_{AB}, and N_A after the Kerr spacetime has been supertranslated.

$$
\begin{aligned}
C_{AB}C^{AB} &= \frac{2a^2}{\sin^2\theta}+\frac{16a\cos\theta}{\sin^2\theta}D^2 f;\\
C_{AB}\mathrm{d}x^A\mathrm{d}x^B &= \left(\frac{a}{\sin\theta}+2\partial_\theta{}^2 f-D^2 f\right)\mathrm{d}\theta^2\\
&\quad -\left(a\sin\theta-2\partial_\phi{}^2 f-2\frac{\cos\theta}{\sin\theta}\partial_\phi f-\sin^2\theta D^2 f\right)\mathrm{d}\phi^2;
\end{aligned}
\tag{5.21}
$$

$$
\begin{aligned}
N_\theta &= 3M\{a\cos\theta+\partial_\theta f\}+\frac{3}{2}a\partial_\theta\left\{\frac{\cos\theta}{\sin^2\theta}\left[D^2 f-\frac{1}{2}\partial_\theta f\right]\right\};\\
N_\phi &= 3M\{-a\sin^2\theta+\partial_\phi f\}+\frac{3}{2}a\partial_\phi\left\{\frac{\cos\theta}{\sin^2\theta}\left[D^2 f-\frac{1}{2}\partial_\theta f\right]\right\},
\end{aligned}
$$

and the supertranslated event horizon is now located at

$$(r_+)_f = r_+ + \frac{1}{2}D^2 f\,. \tag{5.22}$$

It is apparent that several terms in the supertranslated metric functions are once again singular when $\sin\theta = 0$. If one is to use this solution for numerical studies then it is likely that f must be restricted to very particular functions in order to eliminate this issue.

5.4 Charges

With the supertranslated spacetime — the hairy Kerr black hole — we may now discuss the superrotation charges. These charges are associated to the supertranslation hair which are evaporated off the event horizon to future null infinity.

5.4.1 Supertranslation Charge

As discussed by HPS [26, 25] supertranslation hair do not impart supertranslation charge. Recall that the supertranslation charge observed at future null infinity is defined as follows:

$$Q_f = \frac{1}{4\pi}\int_{\mathcal{I}^+} \mathrm{d}^2\Theta\,\sqrt{\gamma}\,f m_{\,\mathrm{bondi}}\,. \tag{5.23}$$

It is now clear that since the Bondi mass aspect, m_{bondi}, is not changed due to the *supertranslation alone* we will not have any supertranslation charge turned on by supertranslation hair. However, it is worth noting that in Donnay et al. [270] and ref [29] an analysis of the Schwarzschild and Kaluza–Klein spacetimes in the near horizon limit is conducted. This analysis shows that nontrivial supertranslation charge is turned on at the horizon due to the gravitational wave that is absent at null infinity. Furthermore, one may see ref [283] for an analysis relating near horizon displacement effects to supertransformation charges.

5.4.2 Superrotation Charge

Supertranslation hair does, however, carry superrotation charge. The superrotation charge that is measured at future null infinity is given by

$$Q_Y = \frac{1}{8\pi}\int_{\mathcal{I}^+} \mathrm{d}^2\Theta\,\sqrt{\gamma}\,Y^A N_A\,. \tag{5.24}$$

Using (5.21) we see that superrotation charge present at null infinity is

$$\begin{aligned} Q_{Y=Y^\theta} =& \frac{1}{8\pi}\int_{\mathcal{I}^+} \sqrt{\gamma}\,\mathrm{d}^2\Theta\,Y^\theta\,3Ma\cos\theta\,+ \\ & \frac{1}{8\pi}\int_{\mathcal{I}^+} \sqrt{\gamma}\,\mathrm{d}^2\Theta\,Y^\theta\left[\partial_\theta f + \frac{3}{2}a\partial_\theta\left\{\frac{\cos\theta}{\sin^2\theta}\left[D^2 f - \frac{1}{2}\partial_\theta f\right]\right\}\right], \end{aligned} \tag{5.25}$$

and

$$Q_{Y=Y^\phi} = \frac{1}{8\pi}\int_{\mathcal{I}^+} -\sqrt{\gamma}\,\mathrm{d}^2\Theta\, Y^\phi\, 3Ma\sin^2\theta + \frac{1}{8\pi}\int_{\mathcal{I}^+} \sqrt{\gamma}\,\mathrm{d}^2\Theta\, Y^\phi \left[\partial_\phi f + \frac{3}{2}a\partial_\phi\left\{\frac{\cos\theta}{\sin^2\theta}\left[D^2 f - \frac{1}{2}\partial_\phi f\right]\right\}\right]. \tag{5.26}$$

Here the first lines correspond to the bald Kerr black hole superrotation charges and can easily be recovered if the supertranslation function f vanishes. Furthermore, as one would expect when $a = 0$ we recover the superrotation charges of the hairy Schwarzschild black hole [26, 270, 29]. As shown by Barnich and Troessaert in [266], when Y^ϕ is the Killing vector, ∂_ϕ, (5.26) corresponds to conservation of angular momentum (for the bald Kerr black hole). For the hairy Kerr black hole, one may see that the zero-mode superrotation charge (when $f = 0$ and $Y^\phi = 1$) given by (5.26), does not change and will still correspond to conservation of angular momentum.

Detection of these charges still remains an open question. One first requires an *operational* notion of finite infinity. Secondly, we require a notion of what higher order charges would be observed as in our detectors. In theory, the supertranslation field should be detectable via classical tests of general relativity such as the bending of light [276, 284, 285]. However, as Comperé discusses in [276], one would need an array of detectors surrounding the central object in order to deduce the superrotation charges, thereby confirming the existence of the supertranslation field.

5.4.3 Supertransformation charges and the Memory effect

In subsection 5.1.4 we discussed that it has been shown the mass, M — in the case of the Schwarzschild spacetime — is changed by a factor of μ which is the monopole contribution to the shockwave. We discussed that in the case of Kerr, this shockwave may also change the mass term present in the g_{uA} terms. This would change the angular momentum of the Kerr black hole, as one may expect, from the passing of a gravitational wave. Since the mass of the black hole is changed, one may ask why there is no supertranslation charge found at null infinity or why the zero-mode of superrotation charge (angular momentum) is not changed.

Recall, however, that supertranslation and superrotation charges are only defined for $\mathcal{I}^+$ and $\mathcal{I}^-$. This, therefore, becomes a statement of what an observer at null infinity measures as the memory effect rather than what the memory effect may be for *all observers*. Indeed, it seems that the change in mass and, therefore, changes in momentum and/or angular momentum are not measurable at null infinity via the measurement of superrotation charges.

5.5 Discussion

We have studied the effects of a BMS supertranslation on the Kerr black hole in Bondi coordinates. This was first done by taking the asymptotic expansion of this solution which matches perfectly with the general BMS expansion of an asymptotically flat metric (5.2) after a coordinate change given by (5.14). In section 5.3 we then found the supertranslated metric functions — the hairy Kerr black hole — which were used to to determine the supertranslation and superrotation charges that may be found at null infinity in section 5.4.

We discussed the change in these supertransformation charges due to the supertranslation hair implanted on the Kerr black hole by the gravitational wave. It was shown that supertranslation charge was absent at null infinity. However, the supertranslation hair did in fact turn on superrotation charge that was detectable at null infinity, given by (5.25) and (5.26). We showed that the zero-mode of the superrotation charge remained unchanged at null infinity since any change in mass is not due to pure supertranslations. While detection of these charges still requires further technological *and* theoretical developments, calculations of these charges does hope to provide a better understanding of the scattering problem when astrophysical, rotating black holes are involved.

The near horizon limit of the extremal Kerr solution was not discussed. It was shown in refs [270, 29] that there was a non-trivial supertranslation charge turned on at the horizon for the hairy Schwarzschild and Kaluza–Klein spacetimes. Hence, the near horizon Kerr limit would be interesting to explore. Furthermore, the charged Kerr black hole — the Kerr–Newman solution — would be interesting to explore as it has been shown that the presence of a vector potential will lead to *soft electric hair* [27]. The Kerr–Newman memory effect in the asymptotic limit and the near-horizon limit will be explored in Chapter 6.

Chapter 6

Kerr–Newman Memory effect and Near-Horizon Physics

6.1 Introduction

In this Chapter, we extend the calculations undertaken in Chapter 5. This is done by first bringing the Kerr–Newman spacetime into the Bondi–Sachs gauge by means of zero angular momentum null geodesics. We compute the memory effect produced at the black hole horizon by a transient gravitational shock wave, which from future null infinity is seen as a Bondi–Metzner–Sachs supertranslation. This results in a change of the supertransformation charges at infinity between the spacetime geometries defined by the black hole before, and after, the shockwave scattering. For an extremal Kerr–Newman black hole, we give the complementary description of this process in the near-horizon limit, as seen by an observer hovering above the horizon. This was not done in the Kerr case in the previous chapter. In this limit, we compute the supertransformation charges and compare them to those calculated at null infinity. We analyse the effect of these transformations on the electromagnetic gauge field and explore the self-interaction between this and the angular momentum of the black hole.

The Kerr–Newman spacetime [286] describes a rotating, charged Black Hole (BH) and represents the most general of the asymptotically Minkowskian, stationary BH solutions to the Einstein–Maxwell equations. It is a direct generalisation of the Kerr solution [287] for a chargeless, rotating BH. The Kerr solution is widely accepted as providing an accurate description of the exterior spacetime surrounding realistic BHs. In particular, the matching of its ray tracing predictions with the recent direct observations of Sagittarius A* and M87* further support its relevance [288, 289, 141, 290]. In spite of these successes, a more realistic representation of a BH would have to include the effects of its inherent electromagnetic charge, in principle varying over time due to the in-fall of charged matter. The Kerr–Newman solution represents a first step in this direction, as it provides the spacetime geometry for an intrinsically charged, stationary, rotating BH. As such, the study of this solution is of critical interest for understanding the dynamics and structure of physically realistic BHs.

Furthermore, a regime of interest is the Near Horizon (NH) limit of the Kerr–Newman spacetime, specifically the case of an extremal Kerr–Newman BH. Indeed, the study of the NH limit of classical BHs is of fundamental importance for the investigation of their geometry and topology, carrying consequences for any traditional approach to quantum gravity [291, 292]. Here, extremal BHs are highly relevant because even as semi-classical objects, they remain inert, since they do not emit any Hawking radiation [293, 294]. As such, they represent simple objects for investigating links between quantum physics and general relativity.

Additionally, the NH limit provides a framework for describing the gravitational shockwave scattering as seen by an observer hovering above the horizon. Therefore, it provides a complementary analysis to the memory effect study carried out at null infinity. For such cases as the Reisnner– Nordström and Kaluza–Klein BHs, the NH observer is known to measure a horizon superrotation after the scattering process has occurred — something absent at null infinity [28, 29]. Moreover, the passage of a gravitational shockwave imparts *soft electric hairs* on the horizon of charged BH, thus showing the interplay between the gravitational and electromagnetic fields. We will reproduce these calculations for the near horizon extremal Kerr–Newman BH. Furthermore, we will show that the interaction between angular momentum and the electromagnetic field is present even for the bald extremal Kerr–Newman solution.

In section 6.2 we put the Kerr–Newman metric in the Bondi-Sachs gauge. In section 6.3 we supertranslate the resulting spacetime, electromagnetic gauge field and discuss the physical implications of this procedure in the presence of charge. In Section 6.4 we explore NH physics for an extremal Kerr–Newman BH and relate the effect of outgoing gravitational radiation to null infinity with the respective modifications of the BH horizon. Section 6.5 presents a brief summary of the results and a discussion regarding future lines of research.

6.2 Kerr–Newman Spacetime in the Bondi–Sachs Gauge

The Kerr–Newman line element in Boyer–Lindquist coordinates $\left\{\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi}\right\}$ is

$$\mathrm{d}s^2 = -\left(\frac{\mathrm{d}\bar{r}^2}{\bar{\Delta}} + \mathrm{d}\bar{\theta}^2\right)\bar{\rho}^2 + (\mathrm{d}\bar{t} - a\sin^2\bar{\theta}\,\mathrm{d}\bar{\phi})^2\,\frac{\bar{\Delta}}{\bar{\rho}^2} - (\bar{A}^2\,\mathrm{d}\bar{\phi} - a\,\mathrm{d}\bar{t})^2\,\frac{\sin^2\bar{\theta}}{\bar{\rho}^2}, \quad (6.1)$$

with

$$\bar{\Delta}(\bar{r}) = \bar{r}^2 + a^2 - \Xi; \tag{6.2}$$

$$\bar{\Xi}(\bar{r}) = 2M\,\bar{r} - Q^2; \tag{6.3}$$

$$\bar{\rho}^2(\bar{r},\bar{\theta}) = \bar{r}^2 + a^2\cos^2\bar{\theta}; \tag{6.4}$$

$$\bar{A}^2(\bar{r}) = \bar{r^2} + a^2, \tag{6.5}$$

where M, a and Q are, the mass, angular momentum per unit mass, and electric charge respectively, of the Kerr–Newman black hole in geometrised units. We aim to cast (6.1) in the BS gauge. To do so, we first move from the Boyer–Lindquist coordinates to the general Bondi–Sachs (GBS) coordinates, in which the metric has to respect the constraints and then impose the falloffs (5.1) through a further coordinate transformation. Only once (6.1) is put into the BS gauge, a meaningful analysis of the asymptotic structure is then possible.

Analogous to the pioneering work of Fletcher & Lun on the Kerr metric [295] and the following expansion by Houque & Virmani to the Kerr–de Sitter solution [296], we begin by considering Zero Angular Momentum Null Geodesics[1] (ZANGs) in the Kerr–Newman spacetime in Boyer–Lindquist coordinates [2]. These are

$$\bar{\rho}^2\,\frac{\mathrm{d}\bar{t}}{\mathrm{d}\lambda} = \frac{\bar{\Sigma}^2}{\bar{\Delta}}E\,, \tag{6.6}$$

$$\bar{\rho}^4\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}\lambda}\right)^2 = \bar{B}^2\,E^2\,, \tag{6.7}$$

$$\bar{\rho}^4\left(\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\lambda}\right)^2 = \bar{\Omega}\,, \tag{6.8}$$

$$\bar{\rho}^2\,\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}\lambda} = \frac{a\bar{\Xi}}{\bar{\Delta}}E\,, \tag{6.9}$$

where λ is an affine parameter along the ZANGs, E is the constant of motion interpreted as the energy of the photons, and the remaining functions appearing in equations (6.6)–(6.9) are

$$\bar{\Sigma}^2(\bar{r},\bar{\theta}) = \bar{A}^4 - a^2\,\bar{\Delta}\sin^2\bar{\theta}\,, \tag{6.10}$$

$$\bar{B}^2(\bar{r}) = \bar{A}^4 - a^2\,\bar{X}^2\,\bar{\Delta}\,, \tag{6.11}$$

$$\bar{\Omega}(\bar{r},\bar{\theta}) = a^2\,E^2\,(\bar{X}^2 - \sin^2\bar{\theta})\,. \tag{6.12}$$

$\bar{X} = \bar{X}(\bar{r},\bar{\theta})$ is related to Carter's separation constant, K, by $K = a^2\,E^2\,\bar{X}^2$ [297]. Hence, it also results as a constant of geodesic motion

$$\frac{d}{d\lambda}\bar{X}(\bar{r},\bar{\theta}) = 0\,. \tag{6.13}$$

[1]Otherwise known as Zero Angular Momentum (null) Observers, null ZAMOs

[2]To maintain a consistent nomenclature with the existing literature, we resolve to use the same notation adopted by Fletcher & Lun and Houque & Virmani [295, 296].

When $Q = 0$, equations (6.6)–(6.9) reduce to equations (12)–(15) of [295], as it should be expected. Moreover, we point out that as for the pure Kerr case solution, the r.h.s. of (6.9) is a function of $\bar{r}$ alone. We can now proceed to write the Kerr–Newman metric in GBS coordinates. To do so, we start with the coordinate transformation

$$\bar{t} = \tilde{u} + J(\tilde{r}, \tilde{\theta})\,, \tag{6.14}$$
$$\bar{r} = \tilde{r}\,, \tag{6.15}$$
$$\bar{\theta} = \bar{\theta}(\tilde{r}, \tilde{\theta})\,, \tag{6.16}$$
$$\bar{\phi} = \tilde{\phi} + L(\tilde{r}, \tilde{\theta})\,, \tag{6.17}$$

where the functions $J(\tilde{r}, \tilde{\theta})$, $\bar{\theta}(\tilde{r}, \tilde{\theta})$ and $L(\tilde{r}, \tilde{\theta})$ are arbitrarily defined, at this stage. The coordinate transform is chosen in this manner as to preserve the simple form of the Killing vector fields in the new coordinate system

$$\partial_{\bar{t}} = \partial_{\tilde{u}}\,, \tag{6.18}$$
$$\partial_{\bar{\phi}} = \partial_{\tilde{\phi}}\,. \tag{6.19}$$
$$\tag{6.20}$$

We further impose that the integral curves of the ZANGs, in the new coordinates, are lines of constant $\{\tilde{v}, \tilde{\theta}, \tilde{\phi}\}$, i.e.

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\lambda} = 0\,, \tag{6.21}$$
$$\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}\lambda} = 0\,, \tag{6.22}$$
$$\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\lambda} = 0\,. \tag{6.23}$$

Applying the coordinate transformation (6.14)–(6.17) with conditions (6.21)–(6.23) to (6.6)–(6.9) gives

$$\frac{\partial J}{\partial \tilde{r}} = \frac{\bar{\Sigma}^2}{\bar{B}\bar{\Delta}}\,, \tag{6.24}$$
$$\frac{\partial L}{\partial \tilde{r}} = \frac{a\bar{\Xi}}{\bar{\Delta}\bar{B}}\,, \tag{6.25}$$
$$\left(\frac{\partial \bar{\theta}}{\partial \tilde{r}}\right)^2 = \frac{\bar{\Omega}}{\bar{B}^2 E^2}\,, \tag{6.26}$$

with

$$\frac{\mathrm{d}\tilde{r}}{\mathrm{d}\lambda} = \frac{\mathrm{d}\bar{r}}{\mathrm{d}\lambda} = \frac{\bar{B}E}{\bar{\rho}^2}\,. \tag{6.27}$$

The choice of the positive root for $\tilde{B}$, combined with (6.14) and (6.24), indicates that we are using a retarded time coordinate and that the ZANGs are outgoing rather than ingoing null geodesics. Furthermore, from (6.13) and (6.22) we deduce,

$$\bar{X} = \tilde{X}(\tilde{\theta})\,. \tag{6.28}$$

Since we have picked $\bar{r} = \tilde{r}$ we can take the square root of (6.26) and integrate to obtain

$$\int^{\bar{\theta}} \frac{\mathrm{d}\theta'}{\sqrt{X^2 - \sin^2\theta'}} = \pm \int^{\tilde{r}} \frac{a\,\mathrm{d}r'}{\tilde{B}(r')} =: \pm\alpha_X\,, \tag{6.29}$$

where

$$\alpha_X(\tilde{r}) = \int^{\tilde{r}} \frac{a\,\mathrm{d}r'}{\sqrt{(r'^2 + a^2)^2 - a^2 X^2 (r'^2 + a^2 - 2Mr' + Q^2)}}\,, \tag{6.30}$$

and

$$\frac{\mathrm{d}\alpha_X}{\mathrm{d}\tilde{r}} = \frac{a}{B(\tilde{r})}\,. \tag{6.31}$$

Here, when a is positive, α_X, is chosen to be a negative, monotonically increasing function.

To integrate (6.29) we notice that the l.h.s. is the Legendre incomplete integral of the first kind and hence defines the Jacobi elliptic sine (sn) function. Thus, we have

$$\sin\bar{\theta} = \begin{cases} \mathrm{sn}\left(\pm\alpha_{\mathrm{X}}\mathrm{X} + \mathrm{H}(\tilde{\theta}), \frac{1}{\mathrm{X}^2}\right) & X^2 > 1 \\ \tanh\left(\pm\alpha_X + \mathrm{H}(\tilde{\theta})\right) & X^2 = 1 \\ X\,\mathrm{sn}(\pm\alpha_{\mathrm{X}} + \mathrm{H}(\tilde{\theta}), \mathrm{X}^2) & \sin^2\bar{\theta} \leq X^2 < 1 \end{cases}\,, \tag{6.32}$$

where $\mathrm{H}(\tilde{\theta})$ is an arbitrary function of $\tilde{\theta}$. We now require $\bar{\theta} \to \tilde{\theta}$ for $\tilde{r} \to \infty$, that is, the two angular coordinates must match at large distances. Therefore, we obtain

$$\mathrm{H} = \begin{cases} \mathrm{sn}^{-1}(\sin\tilde{\theta}, \frac{1}{\mathrm{X}^2}) & X^2 > 1 \\ \tanh^{-1}(\sin\tilde{\theta}) & X^2 = 1 \\ \mathrm{sn}^{-1}\left(\frac{\sin\tilde{\theta}}{\mathrm{X}}, \mathrm{X}^2\right) & \sin^2\bar{\theta} \leq X^2 < 1 \end{cases}\,. \tag{6.33}$$

Finally, by requiring a fixed equatorial plane under the transformation of coordinates – $\bar{\theta} = \pm\pi/2 \longleftrightarrow \tilde{\theta} = \pm\pi/2$ – the case $X^2 = 1$ is selected[3]. This corresponds to choosing the simplest possible class of ZANGs with non-zero energy. Indeed, it forces both Carter's constant ($\mathcal{Q} = K - a^2E^2$) and the total angular momentum about the axis of symmetry to be zero.

Here, we must also stress an interesting difference between the coordinate systems built following this procedure for the Kerr, Kerr–de Sitter and Kerr–Newman spacetimes. For the latter, due to the presence of the charge term in the denominator of (6.30), the coordinate system is not well-defined over

[3]Henceforth, the subscript X is dropped from α_X.

the whole spacetime. Indeed, by analysing (6.30) for $X = 1$, we see that the coordinate chart develops a singularity at the real positive root of

$$P(r; M; a; Q) = (r^2 + a^2)^2 - a^2(r^2 + a^2 - 2Mr + Q^2)\,. \tag{6.34}$$

Therefore, in the Kerr–Newman case, the coordinate system will not be global, unlike in Kerr and Kerr–de Sitter. However, the coordinate singularity appears only below the outer horizon of the charged BH. Thus, the coordinate chart built using ZANGs can still be used in studying the asymptotic structure of the spacetime. Then, by using equations (6.32) and (6.33) we obtain

$$\tanh^{-1}(\sin\bar{\theta}) = \tanh^{-1}(\sin\tilde{\theta}) \pm \alpha\,, \tag{6.35}$$

with

$$\alpha(\tilde{r}) = -\int_{\tilde{r}}^{\infty} \frac{a\,\mathrm{d}r'}{\sqrt{r'^4 + a^2(r'^2 + r'2M - Q^2)}}\,. \tag{6.36}$$

From (6.35), and choosing the plus side in front of α, we directly deduce

$$\sin\bar{\theta} = \frac{D}{C}\,, \tag{6.37}$$

$$\cos\bar{\theta} = \frac{\cos\tilde{\theta}}{C\cosh\alpha}\,, \tag{6.38}$$

where

$$C = 1 + \tanh\alpha\sin\tilde{\theta}\,, \tag{6.39}$$

$$D = \tanh\alpha + \sin\tilde{\theta}\,. \tag{6.40}$$

From (6.37), (6.38), (6.39) and (6.40) we obtain

$$\frac{\partial\bar{\theta}}{\partial\tilde{r}} = \frac{\cos\tilde{\theta}}{C\cosh\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}\tilde{r}} = \frac{\cos\tilde{\theta}}{C\cosh\alpha}\frac{a}{B(\tilde{r})}\,, \tag{6.41}$$

$$\frac{\partial\bar{\theta}}{\partial\tilde{\theta}} = \frac{1}{C\cosh\alpha}\,. \tag{6.42}$$

Therefore, we have

$$\mathrm{d}\bar{t} = \mathrm{d}\tilde{u} + \frac{\tilde{\Sigma}^2}{\tilde{B}\tilde{\Delta}}\,\mathrm{d}\tilde{r} + g(\tilde{r},\tilde{\theta})\,\mathrm{d}\tilde{\theta}\,, \tag{6.43}$$

$$\mathrm{d}\bar{\phi} = \mathrm{d}\tilde{\phi} + \frac{a\tilde{\Xi}}{\tilde{B}\tilde{\Delta}}\,\mathrm{d}\tilde{r} + h(\tilde{r},\tilde{\theta})\,\mathrm{d}\tilde{\theta}\,, \tag{6.44}$$

$$\mathrm{d}\bar{\theta} = \frac{\cos\tilde{\theta}}{C\cosh\alpha}\frac{a}{B}\,\mathrm{d}\tilde{r} + \frac{1}{C\cosh\alpha}\,\mathrm{d}\tilde{\theta}\,, \tag{6.45}$$

where $g(\tilde{r},\tilde{\theta}) = \partial J(\tilde{r},\tilde{\theta})/\partial\tilde{\theta}$ and $h(\tilde{r},\tilde{\theta}) = \partial L(\tilde{r},\tilde{\theta})/\partial\tilde{\theta}$. To complete the coordinate transformation, we need to establish the function form of $g(\tilde{r},\tilde{\theta})$ and $h(\tilde{r},\tilde{\theta})$. From the condition

$$g_{\tilde{r}\tilde{\theta}} = 0\,, \tag{6.46}$$

we deduce the form of $g(\tilde{r},\tilde{\theta})$ as

$$g(\tilde{r},\tilde{\theta}) = \frac{a\cos\tilde{\theta}}{C^2\cosh^2\alpha}\,, \tag{6.47}$$

whilst from the integrability condition

$$\partial_{\tilde{\theta}}\partial_{\tilde{r}}L(\tilde{r},\tilde{\theta}) = \partial_{\tilde{r}}\partial_{\tilde{\theta}}L(\tilde{r},\tilde{\theta})\,, \tag{6.48}$$

we get

$$h(\tilde{r},\tilde{\theta}) = h(\tilde{\theta}). \tag{6.49}$$

Without losing any generality we are then free to choose

$$h(\tilde{\theta}) = 0. \tag{6.50}$$

Therefore, (6.24), (6.25), (6.41), (6.42), (6.47) and (6.50) completely define the correct coordinate transform – (6.14)-(6.17) – to cast the line element (6.1) into the GBS form

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1-\frac{\tilde{\Xi}}{\tilde{\rho}^2}\right)\mathrm{d}\tilde{u}^2 - 2\frac{\tilde{\rho}^2}{\tilde{B}}\,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{r} - 2\left(1-\frac{\tilde{\Xi}}{\tilde{\rho}^2}\right)\frac{a\cos\tilde{\theta}}{C^2\cosh^2\alpha}\mathrm{d}\tilde{u}\mathrm{d}\tilde{\theta} \\ &- 2\frac{a\tilde{\Xi}}{\tilde{\rho}^2}\left(\frac{D}{C}\right)^2\mathrm{d}\tilde{u}\mathrm{d}\tilde{\phi} + \left[\frac{\tilde{\rho}^2}{C^2\cosh^2\alpha} - \left(1-\frac{\tilde{\Xi}}{\tilde{\rho}^2}\right)\frac{a^2\cos^2\tilde{\theta}}{C^4\cosh^4\alpha}\right]\mathrm{d}\tilde{\theta}^2 \\ &- 2\frac{a^2\cos\tilde{\theta}}{C^2\cosh^2\alpha}\left(\frac{D}{C}\right)^2\frac{\tilde{\Delta}}{\tilde{\rho}^2}\mathrm{d}\tilde{\theta}\mathrm{d}\tilde{\phi} + \left(\frac{D}{C}\right)^2\frac{\tilde{\Sigma}^2}{\tilde{\rho}^2}\mathrm{d}\tilde{\phi}^2\,. \end{aligned} \tag{6.51}$$

Finally, to put the line element (6.51) into the BS gauge we apply the following coordinate transformation (5.14)[298]

$$\tilde{u} = u\,, \tag{6.52}$$

$$\tilde{\theta} = \theta\,, \tag{6.53}$$

$$\tilde{\phi} = \phi\,, \tag{6.54}$$

$$\tilde{r} = r + \frac{a}{2}\frac{\cos 2\theta}{\sin\theta} + \frac{a^2}{8}\left(4\cos 2\theta + \frac{1}{\sin^2\theta}\right)\frac{1}{r}\,. \tag{6.55}$$

At the expansion order of interest in r, we find the metric components to be[4]

[4]The calculations put forward in this chapter have been checked with the Mathemathica codes described in Appendix B.

$$
\begin{aligned}
g_{uu} &= -1 + \frac{2M}{r} - \frac{aM\csc\theta\cos 2\theta + Q^2}{r^2} + \mathcal{O}\left(r^{-3}\right)\ ;\\
g_{ur} &= -1 + \frac{a^2\csc^2\theta}{8r^2} + \frac{a^2\left(2M + a\cos 4\theta\csc\theta\right)}{2r^3} + \mathcal{O}\left(r^{-4}\right)\ ;\\
g_{u\phi} &= -\frac{2aM\sin^2\theta}{r} + \frac{a\sin\theta\left(3aM\cos 2\theta + 2aM + Q^2\sin\theta\right)}{r^2} + \mathcal{O}\left(r^{-3}\right)\ ;\\
g_{u\theta} &= \frac{1}{2}a\cot\theta\csc\theta + \frac{a\cos\theta\left(a\csc^3\theta + 8M\right)}{4r}\\
&\qquad - \frac{a\cot\theta\csc\theta\left(4a^2+Q^2\right)\cos 2\theta}{2r^2} -\\
&\frac{a\cot\theta\csc\theta\left(3a^2\cos 4\theta + 2\left(2a^2 - 7aM\sin\theta + 3aM\sin 3\theta + Q^2\right)\right)}{4r^2} + \mathcal{O}\left(r^{-3}\right)\ ;\\
g_{\theta\theta} &= r^2 + ar\csc\theta + \frac{1}{2}a^2\csc^2\theta + \frac{a^2\left(a\csc\theta\cos 4\theta + 8M\cos^2\theta\right)}{4r} + \mathcal{O}\left(r^{-2}\right)\ ;\\
g_{\theta\phi} &= -\frac{2a^2M\sin^2\theta\cos\theta}{r} + \frac{a^2\sin\theta\cos\theta\left(5aM\cos 2\theta + Q^2\sin\theta\right)}{r^2} + \mathcal{O}\left(r^{-3}\right)\ ;\\
g_{\phi\phi} &= r^2\sin^2\theta - ar\sin\theta + \frac{a^2}{2} + \frac{a^3\sin\theta\cot^2\theta(\cos 4\theta - 2\cos 2\theta)}{4r}\\
&\quad + \frac{a^2\sin\theta\left(4a\sin^4\theta - 5a\sin^2\theta + a + 2M\sin^3\theta\right)}{r} + \mathcal{O}\left(r^{-2}\right)\ .
\end{aligned}
\tag{6.56}
$$

Furthermore, we can compute the electromagnetic four-potential in the selected gauge. We start by considering the four-potential in Boyer–Lindquist coordinates

$$
A_\mu \mathrm{d}\bar{x}^\mu = \frac{\bar{r}Q}{\bar{\rho}^2}\mathrm{d}\bar{t} + a\frac{\bar{r}Q}{\bar{\rho}^2}\sin^2\bar{\theta}\,\mathrm{d}\bar{\phi}\,. \tag{6.57}
$$

Moving to GBS coordinates, we then find

$$
A_\mu \mathrm{d}\tilde{x}^\mu = \frac{\tilde{r}Q}{\tilde{\rho}^2}\mathrm{d}\tilde{u} + \frac{\tilde{r}Q}{\tilde{B}\tilde{\Delta}}(\tilde{r}^2+a^2)\mathrm{d}\tilde{r} + \frac{\tilde{r}Q}{\tilde{\rho}^2}\frac{a\cos\tilde{\theta}}{C^2\cosh^2\alpha}\mathrm{d}\tilde{\theta} + a\frac{\tilde{r}Q}{\tilde{\rho}^2}\left(\frac{D}{C}\right)^2\mathrm{d}\tilde{\phi}\,, \tag{6.58}
$$

where $\tilde{\rho} = \tilde{r}^2 + a^2 - a^2\left(D/C\right)^2$. Given that $A_{\tilde{r}}$ is solely a function of $\tilde{r}$, it can be set to zero via a classical $U(1)$ gauge transformation. Then, by moving to the BS gauge we find

$$
\begin{aligned}
A_\mu \mathrm{d}x^\mu &= \left(\frac{Q}{r} - \frac{aQ\cos 2\theta\csc\theta}{2r^2}\right)\mathrm{d}u\\
&\quad + \left(\frac{aQ\cos\theta}{r} + \frac{a^2Q\csc\theta(\cos\theta - 3\cos 3\theta)}{4r^2}\right)\mathrm{d}\theta\\
&\qquad + \left(\frac{aQ\sin^2\theta}{r} - \frac{a^2Q\sin\theta(3\cos 2\theta + 2)}{2r^2}\right)\mathrm{d}\phi + \mathcal{O}(r^{-3})\,.
\end{aligned}
\tag{6.59}
$$

Thus, we can now move to the evaluation of the asymptotic structure of the spacetime.

6.3 The Memory Effect at Null Infinity

The gravitational memory effect as seen by an observer at null infinity has been shown to be equivalent to a BMS supertranslation. Following the investigation of the Kerr memory effect at null infinity in Chapter 5 (in particular, (5.18)) [298], we now focus on the Kerr–Newman memory effect. The supertranslated metric functions are once again calculated via

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu},$$

where ξ is the asymptotic Killing vector, (5.7). We find

$$\delta g_{uu} = \frac{1}{r^3}\left\{ - Mr + Q^2 + \frac{aM(1-2\sin^2\theta)}{\sin\theta} \right\} D^2 f + \frac{aM}{r^3}\left\{ (-2+\cos 2\theta)\cot\theta\csc\theta \right\}\partial_\theta f + \mathcal{O}\big(r^{-4}\big)\,; \tag{6.60}$$

$$\delta g_{ur} = \frac{1}{r^2}\left\{ \frac{a\cos\theta}{2\sin^2\theta} \right\} D^2 f + \mathcal{O}\big(r^{-3}\big)\,; \tag{6.61}$$

$$\delta g_{u\theta} = -\left\{ \partial_\theta f + \frac{1}{2}\partial_\theta D^2 f \right\} + \frac{1}{r}\left\{ 2M\partial_\theta f - \partial_\theta\left(\frac{a}{2}\frac{\cos\theta}{\sin^2\theta}\partial_\theta f \right) \right\} - \frac{1}{r^2}\left\{ Q^2 \partial_\theta f \right\} + \frac{1}{r^2}\left\{ aM\csc\theta\cos 2\theta \right\}\partial_\theta f + \mathcal{O}(r^{-3})\,; \tag{6.62}$$

$$\delta g_{u\phi} = -\left\{ \partial_\phi f + \frac{1}{2}\partial_\phi D^2 f \right\} + \frac{1}{r}\left\{ 2M\partial_\phi f - \frac{a}{2}\frac{\cos\theta}{\sin^2\theta}\partial_\phi\partial_\theta f \right\} - \frac{1}{r^2}\left\{ Q^2 \partial_\phi f \right\} + \frac{1}{r^2}\left\{ aM\csc\theta\cos 2\theta \right\}\partial_\phi f + \mathcal{O}(r^{-3})\,; \tag{6.63}$$

$$\delta g_{\theta\theta} = \left\{ 2{\partial_\theta}^2 f - D^2 f \right\} r - \frac{a}{\sin\theta}\left\{ + \frac{1}{2}D^2 f + 2\frac{\cos\theta}{\sin\theta}\partial_\theta f - 2{\partial_\theta}^2 f \right\} + \mathcal{O}\big(r^{-1}\big)\,; \tag{6.64}$$

$$\delta g_{\phi\phi} = \left\{ 2{\partial_\phi}^2 f + 2\sin\theta\cos\theta\,\partial_\phi f - \sin^2\theta D^2 f \right\} r - \frac{a}{\sin\theta}\left\{ \frac{1}{2}\sin^2\theta D^2 f + \cos\theta\,\partial_\phi f + 2{\partial_\phi}^2 f \right\} + \mathcal{O}\big(r^{-1}\big)\,. \tag{6.65}$$

Additionally, the supertranslated gauge field components at null infinity are:

$$\delta A_u = -\frac{1}{2}\frac{Q}{r^2}D^2 f - \frac{1}{r^2}\csc\theta\Big(\cos 2\theta\cot\theta + 2\sin 2\theta\Big)\partial_\theta f + \mathcal{O}(r^{-3})\,; \tag{6.66}$$

$$\delta A_B = \frac{1}{2}D^2 f\partial_r A_B - \partial_\theta f\partial_\theta A_B \tag{6.67}$$

$$+\left(\frac{Q}{r} - \frac{aQ\cos 2\theta\csc\theta}{2r^2}\right)\partial_\theta f + A_C\partial_B D^C f + \mathcal{O}(r^{-3})\,. \tag{6.68}$$

As can be seen, the supertranslated gauge field has components which match the leading order parts of the original gauge field. Therefore, when an observer at null infinity measures the Maxwell field through $F_{\mu\nu}$, they will observe a difference in a bald Kerr–Newman spacetime and hairy Kerr–Newman spacetime.

Comparing the supertranslated metric components with (5.2) we find $C_{AB}C^{AB}$, C_{AB}, N_A, and m_{bondi}, after the impact of the gravitational wave[5].

$$C_{AB}C^{AB} = \frac{2a^2}{\sin^2\theta} + \frac{16a\cos\theta}{\sin^2\theta}D^2 f\,; \tag{6.69}$$

$$C_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B = \left(\frac{a}{\sin\theta} + 2\partial_\theta{}^2 f - D^2 f\right)\mathrm{d}\theta^2$$
$$- \left(a\sin\theta - 2\partial_\phi{}^2 f - 2\frac{\cos\theta}{\sin\theta}\partial_\phi f - \sin^2\theta D^2 f\right)\mathrm{d}\phi^2\,; \tag{6.70}$$

$$N_\theta = 3M\{a\cos\theta + \partial_\theta f\} + \frac{3}{2}a\partial_\theta\left\{\frac{\cos\theta}{\sin^2\theta}\left[D^2 f - \frac{1}{2}\partial_\theta f\right]\right\}; \tag{6.71}$$

$$N_\phi = 3M\{-a\sin^2\theta + \partial_\phi f\} + \frac{3}{2}a\partial_\phi\left\{\frac{\cos\theta}{\sin^2\theta}\left[D^2 f - \frac{1}{2}\partial_\theta f\right]\right\}; \tag{6.72}$$

$$m_{\text{bondi}} = M\,. \tag{6.73}$$

We are now in a position to discuss the supertranslation and superrotation charges that are implanted on the BH horizon, as seen by an observer at null infinity. As expected [27, 298], the scattering of a gravitational wave by the BH will not excite supertranslation charge. However, this process, equivalent

[5] There have been developments in the BMS group where authors have started investigating higher order terms in the expansion. For instance in refs [299, 282] there are higher order terms, such as 'E_{AB}' and 'F_{AB}' which appear in the g_{AB} expansion. However, these modifications are made for the inclusion of a cosmological constant. The relevance of these terms in our analysis and the effect these may have on charges that we observe at null infinity remains unclear and is perhaps an avenue for further research. Furthermore, the incorporation of these terms would likely also require tweaking of the transformation (5.14), similar to what is done in ref [282].

to a supertranslation at null infinity, will modify the superrotation charge. The superrotation charge that is measured at future null infinity is given by

$$Q_Y = \frac{1}{8\pi} \int_{\mathcal{I}^+} \mathrm{d}^2\Theta \sqrt{\gamma}\, Y^A N_A \,. \tag{6.74}$$

Using (6.69) we get

$$\begin{aligned} Q_{Y=Y^\theta} = & \frac{1}{8\pi} \int_{\mathcal{I}^+} \sqrt{\gamma}\, \mathrm{d}^2\Theta\, Y^\theta\, 3Ma\cos\theta + \\ & \frac{1}{8\pi} \int_{\mathcal{I}^+} \sqrt{\gamma}\, \mathrm{d}^2\Theta\, Y^\theta \left[\partial_\theta f + \frac{3}{2} a \partial_\theta \left\{ \frac{\cos\theta}{\sin^2\theta} \left[D^2 f - \frac{1}{2} \partial_\theta f \right] \right\} \right] , \end{aligned} \tag{6.75}$$

and

$$\begin{aligned} Q_{Y=Y^\phi} = & \frac{1}{8\pi} \int_{\mathcal{I}^+} -\sqrt{\gamma}\, \mathrm{d}^2\Theta\, Y^\phi\, 3Ma\sin^2\theta + \\ & \frac{1}{8\pi} \int_{\mathcal{I}^+} \sqrt{\gamma}\, \mathrm{d}^2\Theta\, Y^\phi \left[\partial_\phi f + \frac{3}{2} a \partial_\phi \left\{ \frac{\cos\theta}{\sin^2\theta} \left[D^2 f - \frac{1}{2} \partial_\phi f \right] \right\} \right] . \end{aligned} \tag{6.76}$$

The first terms in (6.75) and (6.76) correspond to the bald Kerr–Newman BH superrotation charges and can easily be recovered if the supertranslation function f vanishes. Furthermore, when $a = 0$ we recover the superrotation charges of the hairy Schwarzschild BH [26, 28, 29]. Moreover, as shown by Barnich and Troessaert in [266], when Y^ϕ is the Killing vector, ∂_ϕ, (6.76) corresponds to conservation of angular momentum for both the bald Schwarzschild and Kerr BH. For the hairy Kerr–Newman BH, one may see that the zero-mode superrotation charge (when $f = 0$ and $Y^\phi = 1$) given by (6.76), does not change and will still correspond to the conservation of angular momentum.

We note that the calculated charges are no different from those obtained for the Kerr solution, as seen in (5.26) and (5.25)[298]. Therefore, within the current framework, the expected memory effect at null infinity in these two spacetimes is indistinguishable. This follows from the electric charge, Q, appearing only at a higher order than r^{-1} in the expansion of the metric in the BS gauge. In our opinion, this result represents a clear drawback of the current, first-order framework. A higher-order approach is needed to distinguish fundamentally different spacetimes, such as the Kerr and Kerr–Newman solutions, and should therefore be pursued as an important milestone for the field [300].

Nonetheless, we point out that the presence of an electromagnetic field in the Kerr–Newman spacetimes gives a novel method to measure the scattering of a gravitational wave from the BH, via the change in the field. In particular, if such a change were to be detected and agree with our calculations, it could be considered as an indirect test for the presence of supertransformation charges. However, such a measurement clearly presents observational challenges.

6.4 Near Horizon Physics: Extremal Kerr–Newman

We now shift our attention to the NH form of the Kerr–Newman spacetime. In particular, contrary to the null infinity analysis, we show that the two spacetimes differ in their response to the scattering of a gravitational shockwave. Indeed, in the Kerr–Newman case, the gravitational wave excites supertransformation charges and implants soft, electric hair on the horizon, due to its interaction with the electromagnetic four-potential. To determine the charges that are implanted on the horizon, we must first find the NH metric components, and secondly, derive an expression for the electromagnetic gauge field in the NH limit. Chruściel [301] shows that the general form of a NH metric is given by

$$\mathrm{d}s^2 = -2R\kappa \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R + 2R\theta_A \mathrm{d}v\mathrm{d}x^A + \Omega_{AB}\mathrm{d}x^A\mathrm{d}x^B + \ldots\,, \tag{6.77}$$

where v is the advanced time, x^A are angular coordinates, θ_A, $\Omega_{AB} \equiv \Omega_a\, \gamma^a{}_{AB}$[6] are in principle arbitrary metric functions of v and x^A, and κ is the surface gravity. Note, that when dealing with an extremal horizon, the surface gravity vanishes, i.e., $\kappa = 0$. In the coordinate used in (6.77), the horizon is now located at $R = 0$ and the ellipsis are to denote terms that are $\mathcal{O}(R^2)$. Furthermore, we have the constraints

$$g_{RR} = 0\,, \quad g_{vR} = 1\,, \quad g_{AR} = 0\,. \tag{6.78}$$

Additionally, in analogy to refs [278, 28], we use the boundary conditions

$$g_{vv} = -2\kappa R + \mathcal{O}(R^2)\,, \quad g_{vA} = \theta_A R + \mathcal{O}(R^2)\,, \quad g_{AB} = \Omega_{AB} + \mathcal{O}(R)\,. \tag{6.79}$$

We can then find a set of asymptotic Killing vectors that preserve (6.78) and (6.79), generating an algebra consisting of *both* supertranslations and superrotations. The resulting Killing vectors are

$$\begin{aligned}\xi^\mu\partial_\mu = f\partial_v + \Big(Y^A - \partial_B f \int^R \mathrm{d}R' g^{AB}\Big)\partial_A \\ + \Big(Z(v,x^A) - R\partial_v f + \partial_A f \int^R \mathrm{d}R' g^{AB} g_{vB}\Big)\partial_R\,.\end{aligned} \tag{6.80}$$

As is the case in refs [278, 28], we find a vector, Y^A, that is a "constant" of integration that represents the horizon superrotations[7]. Then, using the

[6] Note the use of the *internal* index, a, here. This is required in the case of the NH Kerr–Newman metric as we can not use only one scaling factor for $\Omega_{\Theta\Theta}$ and $\Omega_{\Phi\Phi}$.

[7] It is important to point out that one does not need a gravitational shockwave here to have "a superrotation/supertranslation charge". These aspects exist as a property of the asymptotic structure of the NH metric.

NH asymptotic Killing vector, we compute the general supertranslated metric functions; κ, θ_A, and Ω_{AB} subject to (6.79) [8].

$$\delta_\xi \kappa = \mathcal{L}_\xi \kappa = 0\,, \tag{6.81}$$
$$\delta_\xi \theta_A = \mathcal{L}_Y \theta_A + f\partial_v \theta_A - 2\kappa \partial_A f - 2\partial_v \partial_A f + \Omega^{BC}\partial_v \Omega_{AB} D_C f\,, \tag{6.82}$$
$$\delta_\xi \Omega_{AB} = f\partial_v \Omega_{AB} + \mathcal{L}_Y \Omega_{AB}\,. \tag{6.83}$$

To properly study the NH physics of a charged BH, we must also discuss the NH expansion of the gauge field. The Taylor expansion of the U(1) electromagnetic gauge field near $R = 0$ is given by [28]

$$A_v = A_v^{(0)} + RA_v^{(1)}(v, x^A) + \mathcal{O}(R^2), \tag{6.84}$$
$$A_B = A_B^{(0)}(x^A) + RA_B^{(1)}(v, x^A) + \mathcal{O}(R^2)\,, \tag{6.85}$$
$$A_R = 0. \tag{6.86}$$

Here $A_v^{(0)}$ is the Coulomb potential. In particular, we find that the supertranslated gauge field components take the form:

$$\delta_\xi A_v = 0, \tag{6.87}$$
$$\delta_\xi A_B = Y^C \partial_C A_B^{(0)} + A_C^{(0)} \partial_B Y^C + \partial_B U. \tag{6.88}$$

Where U is an arbitrary function of angular coordinates and is referred to as the *electromagnetic charge generator*, just as f is referred to as the generator of supertranslations.

We now discuss the NH extremal Kerr-Newman spacetime and provide the supertranslated metric functions. This will allow us to examine the effect of a gravitational shockwave — under the identification of supertranslations with the scattering of such waves by the BH — on the extremal horizon as seen by an observer near the horizon. This leads to a horizon superrotation that is absent at null infinity[9], similar to the Schwarzschild and Kaluza–Klein cases discussed in [28, 29] respectively.

6.4.1 Near Horizon Metric and Gauge Four-Potential

To derive the extremal NH Kerr–Newman metric, we begin by defining [301]

$$\bar{t} = \epsilon^{-1}\hat{t}\,, \tag{6.89}$$
$$\bar{r} = M + \epsilon\hat{r}\,, \tag{6.90}$$

[8]We correct a small mistake here that is present in ref [278]. This third equation now correctly states that the Lie derivative of Ω_{AB} is along $\mathbf{Y}$ and not ξ.

[9]In Chapter 5 we stated that there is a change in the *superrotation charges* at null infinity due to the *supertranslation*. However, in the near horizon case, (mathematically due to the boundary conditions) we note that there is also a superrotation that has associated *supertranslation charges* discussed in subsection 6.4.2.

$$\bar{\theta} = \hat{\theta}\,, \tag{6.91}$$

$$\bar{\phi} = \hat{\phi} + \epsilon^{-1}\frac{a}{r_0^2}\hat{t}\,, \tag{6.92}$$

where $r_0^2 = M^2 + a^2$. After taking the limit $\epsilon \to 0$, the metric becomes

$$ds^2 = \left(1 - \frac{a^2}{r_0^2}\sin^2\hat{\theta}\right)\left[-\frac{\hat{r}^2}{r_0^2}d\hat{t}^2 + \frac{r_0^2}{\hat{r}^2}d\hat{r}^2 + r_0^2 d\hat{\theta}^2\right] + r_0^2\sin^2\hat{\theta}\left(1 - \frac{a^2}{r_0^2}\sin^2\hat{\theta}\right)^{-1}\left[d\hat{\phi} + \frac{2\,a\,M}{r_0^4}\,r\,d\hat{t}\right]^2\,. \tag{6.93}$$

This metric is clearly singular on the horizon. Hence, we apply the following coordinate transform

$$\hat{t} = V - \frac{r_0^2}{r}\,, \tag{6.94}$$

$$\hat{r} = R\,, \tag{6.95}$$

$$\hat{\theta} = \Theta\,, \tag{6.96}$$

$$\hat{\phi} = \Phi - \frac{2Ma}{r_0^2}\log\left(\frac{\hat{r}}{r_0}\right)\,, \tag{6.97}$$

leading to the line element

$$\mathrm{d}s^2 = \frac{\left(r_0^2 - a^2\sin^2\Theta\right)}{r_0^2}\left[-\frac{R^2}{r_0^2}\mathrm{d}V^2 - 2\mathrm{d}V\mathrm{d}R + r_0^2\mathrm{d}\Theta^2\right] + \frac{r_0^4\sin^2\Theta}{r_0^2 - a^2\sin^2\Theta}\left[\mathrm{d}\Phi + \frac{2aM}{r_0^4}R\mathrm{d}V\right]^2\,, \tag{6.98}$$

which is regular for $R = 0$. We may now read off the metric functions in (6.77):

$$\kappa = 0; \tag{6.99}$$

$$\theta_\Theta = 0; \tag{6.100}$$

$$\theta_\Phi = \frac{2\,a\,M\,\sin^2\Theta}{r_0^2 - a^2\sin^2\Theta}; \tag{6.101}$$

$$\Omega_{\Theta\Theta} = r_0^2 - a^2\sin^2\Theta; \tag{6.102}$$

$$\Omega_{\Phi\Phi} = \frac{r_0^4\sin^2\Theta}{r_0^2 - a^2\sin^2\Theta}\,. \tag{6.103}$$

We must now bring the Kerr–Newman gauge potential into the form (6.84). Performing the same coordinate transformations as we did for the metric, we first find:

$$A_\mu\mathrm{d}\hat{x}^\mu = \frac{Q\left(M^2 - a^2\cos^2\hat{\theta}\right)}{r_0^2\left(M^2 + a^2\cos^2\hat{\theta}\right)}\hat{r}\,\mathrm{d}\hat{t} + \frac{Q\,a\,M\sin^2\hat{\theta}}{M^2 + a^2\cos^2\hat{\theta}}\mathrm{d}\hat{\phi}\,, \tag{6.104}$$

where we have eliminated the constant term $\epsilon^{-1}(MQ/r_0^2)$ in $A_{\hat{t}}$ through a $U(1)$ gauge transformation before taking the limit for small ϵ. With the final coordinate transformation (6.94), we obtain

$$A_\mu \mathrm{d}X^\mu = \frac{Q\,(M^2 - a^2\cos^2\Theta)}{r_0^2\,(M^2 + a^2\cos^2\Theta)} R\mathrm{d}V + \frac{Q\,a\,M\sin^2\Theta}{M^2 + a^2\cos^2\Theta}\mathrm{d}\Phi\,, \tag{6.105}$$

where we have renormalised $A_R = -(M^2 - a^2)(Q/r_0^2)R^{-1}$ with a further $U(1)$ gauge transform. Note, that the Coulomb potential does not appear here. However, because it is coordinate-independent, this can be added back in at any point without changing the Maxwell field. Moreover, as expected, we will see that the Coulomb potential will not appear in the expressions for surface charges.

6.4.2 Near Horizon Supertranslations and Charges

Bringing the asymptotic Killing vector, (5.7) to the NH limit for the extremal case – i.e., $M^2 = a^2 + Q^2$ – and supertranslating the NH Kerr–Newman spacetime (6.98) we find the following metric components

$$g_{VV} = -\frac{R^2}{r_0^2}\frac{\left(r_0^2 - a^2\sin^2\Theta\right)}{r_0^2} + \frac{r_0^4\sin^2\Theta}{r_0^2 - a^2\sin^2\Theta}\left(\frac{2aMR}{r_0^4}\right)^2\,, \tag{6.106}$$

$$g_{\Theta\Theta} = \left\{r_0^2 - a^2\sin^2\Theta\right\}\left\{1 - \frac{2}{r_+}\partial_\Theta^2 f\right\} + \frac{1}{r_+}a^2\sin 2\Theta\,\partial_\Theta f\,, \tag{6.107}$$

$$g_{\Phi\Phi} = \left\{\frac{1}{r_+}\frac{r_0^4}{r_0^2 - a^2\sin^2\Theta}\right\}\left\{r_+\sin^2\Theta - \frac{r_0^2\sin 2\Theta}{r_0^2 - a^2\sin^2\Theta}\partial_\Theta f - 2\partial_\phi^2 f\right\}\,, \tag{6.108}$$

$$g_{VR} = -\frac{r_0^2 - a^2\sin^2\Theta}{r_0^2}\,, \tag{6.109}$$

$$g_{V\Phi} = \left\{\frac{1}{r_+}\frac{2\,a\,M}{r_0^2 - a^2\sin^2\Theta}\right\}\left\{\sin^2\Theta - \frac{\sin 2\Theta\, r_0^2}{r_0^2 - a^2\sin^2\Theta}\partial_\Theta f - \partial_\Phi^2 f\right\}R\,, \tag{6.110}$$

$$g_{V\Theta} = \left\{\frac{1}{r_+}\frac{2\,a\,M}{r_0^2 - a^2\sin^2\Theta}\right\}\left\{2\cot\Theta\,\partial_\Phi f - \partial_\Theta\partial_\Phi f\right\}R\,, \tag{6.111}$$

$$g_{RR} = 0\,, \tag{6.112}$$

$$g_{R\Theta} = 0\,, \tag{6.113}$$

$$g_{R\Phi} = 0\,, \tag{6.114}$$

$$g_{\Theta\Phi} = 0\,. \tag{6.115}$$

We can now compare our results with the supertranslated extremal NH Kerr–Newman spacetime, see (6.93), and the general NH supertranslated metric functions, see (6.81)-(6.83). Since Ω_{AB} does not depend on retarded/advanced time, from (6.83) we find the corresponding horizon superrotation to be

$$Y_A = \frac{1}{M}D_A f\,. \tag{6.116}$$

Surprisingly, this does not differ from the horizon superrotation of a Schwarzschild BH found in [28]. Indeed, this is perhaps not expected as the Kerr class of solutions are already rotating. However, this could be intuitively understood by noticing that the correlation between the memory effect and supertranslations — in both regimes, null infinity, and NH — is only examined at linear order. In fact, one can show that if Ω_{AB} does not depend on advanced or retarded time, we will always have a horizon superrotation of this form — up to a factor which depends on the horizon radius.

The diffeomorphisms generated by asymptotic Killing vectors have associated horizon charges. The derivation of these charges stem from ref [274] and are also discussed in refs [28, 302]. The NH charges take the form:

$$\mathcal{Q}[X, Y^A, U] = \frac{1}{16\pi} \int \mathrm{d}\Theta \mathrm{d}\Phi \sin\Theta \, r_0^2 \left(2X - Y^A \theta_A - 4U A_V^{(1)} - 4A_B^{(0)} Y^B A_V^{(1)} \right) . \tag{6.117}$$

In the extremal case, it is apparent that the surface gravity vanishes and so too does the Hawking temperature [293, 294]. This leads to an interesting scenario in which the Hawking temperature is no longer the associated zero-mode for the first charge. In this case, this zero-mode (the supertranslation charge) is associated with the product of Bekenstein-Hawking entropy and the geometric temperature [28, 303]. The second term is analogous to the superrotation charge found at null infinity. The third term is due to the electromagnetic generator and the last term mixes the superrotation vector field with the gauge field.

Let the associated charges to X, Y^A, and U be $\mathcal{X}$, $\mathcal{Y}^A$, and $\mathcal{U}$ respectively. The associated zero-mode (bald) horizon charges are

$$\mathcal{X} = \mathcal{Q}[1, 0, 0] = \frac{r_0^2}{2} \,, \tag{6.118}$$

$$\mathcal{Y}^\Theta = \mathcal{Q}[0, Y^\Theta = 1, Y^\Phi = 0, 0] = 0 \,, \tag{6.119}$$

$$\mathcal{Y}^\Phi = \mathcal{Q}[0, Y^\Phi = 1, 0] = \frac{1}{16\pi} \int \mathrm{d}\Theta \mathrm{d}\Phi \sin\Theta \, r_0^2 \left(\theta_\Phi - 4A_\Phi A_V^{(1)} \right) , \tag{6.120}$$

$$\begin{aligned} \mathcal{U} = \mathcal{Q}[0, 0, 1] &= -\frac{1}{4\pi} \int \mathrm{d}\Theta \mathrm{d}\Phi \sin\Theta \, r_0^2 \left(A_V^{(1)} \right) \\ &= Q \left(1 - \frac{2M}{a} \arctan\left(\frac{a}{M} \right) \right) . \end{aligned} \tag{6.121}$$

The zero mode of $\mathcal{Y}^\Phi$ gives the angular momentum of the BH as measured by the hovering observer. As one may note, there is a contribution to this zero-mode from the gauge field which does not vanish. Therefore, we see a strong interaction between the electromagnetic gauge potential and the angular momentum of the BH, with the former influencing the latter for the chosen observer. As the gauge field vanishes, we retrieve the extremal NH Kerr solution, and the angular momentum depends solely on θ_Φ. The final charge, $\mathcal{U}$, the zero-mode charge corresponding to the electromagnetic charge generator, gives the total electric charge of the BH as measured in the NH limit. Here, the complementary effect is observed and the angular momentum of the BH effectively shields the intrinsic charge for the NH observers. Remarkably, these unexpected effects of the self-interactions between angular momentum and electric charge are not found via a null infinity analysis. Thus, they further indicate the importance in general relativity of studying the same phenomena using a plurality of observers. Lastly, one may verify that these charges do indeed agree with the extremal Reissner–Nordström horizon when $a \to 0$ as seen in ref [28].

We may also use (6.117) to determine the zero-mode of the NH charges of the supertranslated horizon. To do so, we compute

$$\theta_\Theta = \left\{ \frac{1}{M} \frac{2\,a\,M}{r_0^2 - a^2 \sin^2\Theta} \right\} \left\{ 2\cot\Theta\,\partial_\Phi f - \partial_\Theta\,\partial_\Phi f \right\} ; \tag{6.122}$$

$$\theta_\Phi = \left\{ \frac{1}{M} \frac{2\,a\,M}{r_0^2 - a^2 \sin^2\Theta} \right\} \left\{ \sin^2\Theta - \frac{\sin 2\Theta\, r_0^2}{r_0^2 - a^2\sin^2\Theta} \partial_\Theta f - \partial_\Phi^2 f \right\} ; \tag{6.123}$$

$$A_V^{(1)} = \frac{Q}{r_0^2} \left\{ \frac{(M^2 - a^2\cos^2\Theta)}{(M^2 + a^2\cos^2\Theta)} - \frac{1}{M} \partial_\Theta \frac{(M^2 - a^2\cos^2\Theta)}{(M^2 + a^2\cos^2\Theta)} \partial_\Theta f \right\} ; \tag{6.124}$$

$$A_B^{(0)} = \frac{Q\,a\,M\sin^2\Theta}{M^2 + a^2\cos^2\Theta} + \delta A_B \,, \tag{6.125}$$

$$\tag{6.126}$$

where δA_B is given below. Interestingly, we see that once the NH spacetime is supertranslated by the passage of a gravitational wave, then θ_Θ is no longer zero. However, even though the NH geometry is transformed, the zero-mode horizon charges remain unchanged. This is because we are setting the supertranslation generator, f, to zero[10] in all cases.

In the bald and supertranslated BHs we already see an interplay between the electromagnetic field and angular momentum. However, a further interaction between the gravitational and electromagnetic fields can be investigated by determining the change in the electromagnetic field generator due to the memory effect in the NH limit

$$U = \int \left(Y^C \partial_C A_B^{(0)} + A_C^{(0)} \partial_B Y^C - \delta_\xi A_B \right) \mathrm{d}x^B \,. \tag{6.127}$$

Here, Y^A is the horizon superrotation,

$$Y^A = \frac{1}{M} D^A f \,, \tag{6.128}$$

[10]In fact, f, can be expanded in Fourier modes which relate it linearly to X in the extremal case. Hence, when setting X to zero, we are also setting f to zero, and the zero-modes now correspond solely to the bald near-horizon geometry.

and

$$\delta_\xi A_B = -Q\,a \left\{ \partial_\Theta \left(\frac{\sin^2\Theta}{M^2 + a^2\cos^2\Theta} \right) \partial_\Theta f - \left(\frac{1}{M^2 + a^2\cos^2\Theta} \right) \partial_\Theta \partial_\Phi f \right\}. \tag{6.129}$$

This illustrates that the gravitational memory effect due to the passing of a gravitational wave is not only seen as a supertranslation from null infinity, but in the NH limit implants *soft electric hair* on the extremal Kerr–Newman horizon.

6.5 Discussion

Motivated by the rising relevance of the gravitational memory effect, in this chapter we have investigated the connection between the scattering of a gravitational shockwave by the Kerr–Newman black hole, as seen in the near-horizon region and in the far asymptotic region.

In ref [27] the authors showed that a transient gravitational shockwave modifies the black hole geometry in a way that can be interpreted as a BMS supertranslation at null infinity. Here, we have brought for the first time the Kerr–Newman black hole in the Bondi–Sachs gauge and computed the action of a BMS supertranslation on its asymptotic structure. In particular, we discussed the change in the supertransformation charges due to the supertranslation hair implanted on the Kerr–Newman black hole by the gravitational wave. We have shown that the supertranslation charge was absent at null infinity, whilst a superrotation charge is instead detectable. Furthermore, the zero-mode of the superrotation charge remains unchanged, as any change in mass cannot be captured by the action of pure BMS supertranslations.

Following the pioneering work of Donnay *et al* [302], we studied the gravitational memory effect in the near horizon limit of an extremal Kerr–Newman black hole. Surprisingly, we found that the corresponding horizon superrotation matches the one computed for non-rotating black holes. Moreover, we find that no non-trivial supertranslation charge is turned on at the horizon, due to the extremality of the black hole. Finally, we show that the scattering of the gravitational shockwave by the black hole implants soft electric hair on the horizon, via its interaction with the electromagnetic gauge field.

Some questions remain open and require further study. Indeed, we showed that a higher-order formalism is needed to properly capture the full properties of the spacetime when dealing with the memory effect. Consequently, a rigorous definition, and interpretation, of higher-order charges would be required. Furthermore, we have found a series of previously unexplored interactions between the gravitational and electromagnetic fields. To wit, the presence of electric charge invalidates the construction of the Bondi–Sachs coordinates as a global coordinate patch for the spacetime, failing below the horizon. Moreover, we have showed that the electric charge, and angular momentum, inferred for the black hole by a near horizon observer differ from what would be measured in the asymptotic region, on account of the interplay between these two quantities. This unexpected interaction between spin and charge requires further clarification, with a possible avenue of research leading to the study of a similar effect in higher dimensional, charged, rotating black holes.

Chapter 7

Conclusion

The future of cosmology and quantum gravity is unclear. Cosmology, in particular seems to be reaching a ‘tipping point’, where observations of different astrophysical events/objects are producing vastly different results. Without doubt, this is likely due to the over-simplification of the concordance model — ΛCDM, predicated on a spatially flat FLRW spacetime — which fails to address many fundamental questions. Furthermore, several decades have passed and there has been no direct detection of dark energy and dark matter. Not only has there been no detection, there has also not been any proposed mechanism for their production that has been replicated or observed. In this thesis we did not attempt to develop a new theory of cosmology or statistical method for observation, rather we addressed fundamental questions regarding observers.

In terms of quantum gravity, it has seemed that we are *on the edge* of discovering the ‘next big thing’ for the last (at least) fifty years. Despite many promising theories such as loop quantum gravity and string theory, we are still no closer to having observational evidence for how gravity behaves on quantum scales. While we did not discuss any theory of quantum gravity directly, we investigated the ‘black-to-white’ hole bounce and discussed the quantum implications of this. We further probed the memory effect of black holes at null infinity and in the near-horizon limit.

We began by introducing a coordinate system/slicing of spacetime that is new to cosmology — the Painlevé–Gullstrand coordinates/slicing. This was done, firstly in the spatially flat ($k = 0$) FLRW spacetime. It was shown that in this slicing, space is no longer expanding, however, the galaxies (fluid particles) are still receding from each other. In other words, the Hubble flow became very explicit in this choice of coordinates. We then proceeded by calculating all of the Killing vectors and Killing tensors in Painlevé–Gullstrand coordinates. This was done because Killing vectors and the symmetries they represent are a cornerstone of theoretical physics due to the conservation laws they are associated with. For purely cosmological spacetimes, we further considered three versions of de Sitter space, (2.119), (2.121), (2.122). These either made the spacetime static, the spatial slices flat, or made the connection to generic FLRW manifest.

For black holes embedded in cosmological spacetime we considered the Kottler spacetime and the McVittie spacetime. For Kottler, we developed six different line elements, (2.123), (2.127), (2.132), (2.137), (2.139), (2.143). These line elements focused on different aspects of 'the physics'. It was shown that one can either make the spacetime manifestly static, or make the spatial slices 'simple', or make the connection to a generic FLRW spacetime manifest. For the McVittie spacetime we presented four different line elements, (2.146), (2.154), (2.162), (2.173), two of which seem to be novel. The traditional version (2.146) is spatially isotropic, but every nonzero metric component is explicitly time dependent. The "comoving" line element (2.162) makes the connection with generic $k = 0$ FLRW manifest. While the *conformally* Painlevé–Gullstrand version (2.173) makes the spatial slices time independent and eliminates explicit occurrences of the scale factor $a(t)$ in favour of the Hubble parameter $H(t)$.

We then moved on to discussing an increasingly popular proposed mechanism for the production of dark energy [86, 87, 88, 89]. This idea was that black holes could grow independently of accretion or mergers due to dark energy production in their interiors. This is supposedly correlated with the accelerated expansion of our Universe. While an independent observational analysis had strongly excluded these claims at $\sim 3\sigma$ [93] we chose to investigate this claim on a purely theoretical basis. We started with three relatively well-known exact solutions to the Einstein equations (Kottler, McVittie, Kerr-de Sitter) all of which successfully embed black holes in suitable FLRW background. We have seen that these exact solutions exhibit no evidence of any "direct coupling" between the black hole mass and the cosmological expansion. While an embedding of the Kerr black hole in an asymptotically FLRW spacetime would have been ideal to show that the Kerr black hole also does not couple to cosmological expansion such an embedding has proven to be in the "too hard basket".

We further discussed the enormous scales of separation between milli-parsec black hole physics and giga-parsec cosmological physics. Even on a pure intuitive level, these scales render any coupling between black holes and cosmic evolution implausible. Despite our views on the dark sector of the Universe, we understand the need to explain the source(s) of dark energy within the current paradigm of cosmology. However, we showed that — on theoretical grounds — black holes simply cannot be this mysterious source.

In the next part of this thesis we moved to exploring pure black hole spacetimes. The aim of this research was to investigate novel ideas that may aid our understanding of the nature of quantum gravity and the graviton. We first investigated if a simple and compelling classical model of a black-to-white hole transition could be found. We began by performing a simple coordinate transformation of the standard Schwarzschild metric by modifying the radial coordinate. We showed that for specific choices of $h(r)$ in (4.6) the Schwarzschild spacetime — in other well known coordinates — was found; such as the Painlevé–Gullstand, Kerr–Schild, and Eddington–Finkelstein coordinates. By imposing the restriction $h(2m) = \pm 1$ we showed that this line element can model a classical black or white hole where one or the other of the null curves are horizon penetrating with nonzero coordinate velocity.

By further introducing a time-dependent function, $s(t)$, we then produced a simple classical model for a black-hole-to-white-hole transition. This spacetime, however, was no longer *just* a coordinate transformation of Schwarzschild spacetime. The non-static spacetime in these coordinates was found (at early and late times) to have horizon penetrating null curves with coordinate velocity. We found that the the action in the transition region was *finite*. More importantly, however, this action can be arranged to be zero by carefully choosing $h(r)$. This proved to be a significant result as this action could then be added to the Feynman path integral and have no impact on any quantum amplitudes. Furthermore, this investigation could lead one to *speculate* that quantum physics could become dominant in the near-horizon limit. This is contrary to the other more-universal consensus that quantum physics should only be relevant when curvature reaches the Planck scale. Evidently, there is room for future research here. We hope to expand these calculations to the Reissner–Nordström, Kerr, Kerr–Newman solutions and their extremal variants in the future.

Motivated by the rising relevance of the gravitational memory effect in future observational missions, we investigated the connection between the BMS group and the memory effect for black holes. We extended the current body of literature and pioneering work of Donnay *et al* [28, 302] by investigating the Kerr and Kerr–Newman memory effect. Hawking, Perry and Strominger in [27] showed that a transient gravitational shockwave modifies the black hole geometry in a way that can be interpreted as a BMS supertranslation at null infinity. In Chapter 5, we extended the supertranslation and charge calculations by HPS and Donnay *et al* to the Kerr solution. In order to do this, we first brought the Kerr solution into the Bondi–Sachs gauge. After supertranslating the spacetime we discussed the supertranslation and superrotation charges that one would observe at *operational* null infinity in order to verify the gravitational memory effect for the Kerr spacetime. This calculation showed that there was a non-trivial superrotation charge turned on at null infinity due to the supertranslation hair that was implanted on the horizon due to the passing of a gravitational shockwave. As one would expect, in the absence of rotation, i.e., $a \to 0$, the supertranslated spacetime and the associated charges were that of the Schwarzschild spacetime. We did not show explicitly that the gravitational wave induces a permanent change in the metric as is done for the Schwarzschild case; this is a calculation that is quite dense and will be an avenue for future research.

In Chapter 6 we moved the memory effect-BMS group investigation to the the Kerr–Newman spacetime. We brought, for the first time, the Kerr–Newman black hole in the Bondi–Sachs gauge and computed the action of a BMS supertranslation on its asymptotic structure. We again discussed the change in the supertransformation charges due to the supertranslation hair implanted on the Kerr–Newman black hole by the gravitational wave. As was the case for the Kerr spacetime, superrotation charge was shown to be detectable at null infinity, while the supertranslation charge was not. We showed that there should also be a change in the gauge field, leading to another method of detection for the memory effect — illustrating an interplay between gravitation and electromagnetism.

We further studied the gravitational memory effect and horizon charges in the near-horizon limit of an extremal Kerr–Newman black hole. Interestingly, The horizon charges of the bald extremal Kerr–Newman black hole showed that the angular momentum of the black hole depended on the gauge field. This further illustrated the interplay between gravitation and the electromagnetic field. Upon supertranslating the near-horizon extremal Kerr–Newman black hole we found that there is a also a horizon superrotation that has an associated supertranslation charge — which was absent at null infinity. Finally, we showed that the scattering of the gravitational shockwave by the black hole implants soft electric hair on the horizon, via its interaction with the electromagnetic gauge field.

As for all research, there are many questions and avenues for future study. Painlevé–Gullstrand coordinates have been investigated briefly in ref [304] for different cosmological spacetimes. In these spacetimes — those that have a global non-zero curvature — it was shown that there is no global notion of simultaneity. This raises interesting questions about observation of the CMB and time in the Universe. It would be of great interest to investigate inhomogenous spacetimes in this slicing, such as the Szekeres solution.

Research into the dark sector of the Universe is ongoing. In this thesis we simply addressed a proposal for dark energy that could not be a solution to our current concordance model. Of course, proving or disproving the existence of dark energy will take more than a simple discussion based on theory alone. With new observational missions such as Euclid, and ongoing tensions in the cosmology community, it is only a matter of time — we believe — before we enter a new era of cosmological research.

While the infrared triangle and the correspondence between the memory effect & the BMS group are fascinating, they are not free of conceptual and technical hurdles. As we discussed, the connection between the black hole memory effect and supertranslations of the BMS group is only established at linear order. Indeed, the memory effect at null infinity does not capture the entire 'memory effect' - it does not capture the change in Bondi mass. Beyond theoretical development, detection of supertransformation charges remains an open question. Even if all of these questions are addressed, we do not know exactly how one could take this research and use it to further develop a theory of quantum gravity, even though implicit assumptions of the graviton are made in the foundations of these calculations. Clearly, just as is the case for the last fifty years, the more interesting connections we find, the more questions arise...

In the end, there are many open ended questions to the research undertaken in this thesis. We hope that the theories further developed here will aid observation in the near future for both cosmology and quantum gravity.

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Appendix A

Painlevé–Gullstrand coordinates and related formulae

Here we present some collected formulae for easy reference.

A.1 Spatially flat FLRW

By suitable choice of coordinates spatially flat FLRW spacetime can be represented in any of the following six equivalent forms:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\}, \tag{A.1}$$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\}. \tag{A.2}$$

$$\mathrm{d}s^2 = a(\eta)^2\{-\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\}. \tag{A.3}$$

$$\mathrm{d}s^2 = a(\eta)^2\{-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\}. \tag{A.4}$$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \{[\mathrm{d}\bar{r} - H(t)\,\bar{r}\,\mathrm{d}t]^2 + \bar{r}^2\mathrm{d}\Omega^2\}. \tag{A.5}$$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left\{[\mathrm{d}\bar{x} - H(t)\,\bar{x}\,\mathrm{d}t]^2 + [\mathrm{d}\bar{y} - H(t)\,\bar{y}\,\mathrm{d}t]^2 + [\mathrm{d}\bar{z} - H(t)\,\bar{z}\,\mathrm{d}t]^2\right\}. \tag{A.6}$$

A.2 Kottler

By suitable choice of coordinates Kottler (Schwarzschild-de Sitter) spacetime can be represented in any of the following six equivalent forms:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{\bar{r}} - H^2\bar{r}^2\right)\mathrm{d}\bar{t}^2 + \frac{\mathrm{d}\bar{r}^2}{1 - \frac{2m}{\bar{r}} - H^2\bar{r}^2} + \bar{r}^2\mathrm{d}\Omega^2. \tag{A.7}$$

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{\bar{r}}\right)\mathrm{d}t^2 + \frac{[\mathrm{d}\bar{r} - H\bar{r}\,\sqrt{1 - 2m/\bar{r}}\,\mathrm{d}t]^2}{1 - \frac{2m}{\bar{r}}} + \bar{r}^2\mathrm{d}\Omega^2, \tag{A.8}$$

$$ds^2 = -\left(1 - H^2\bar{r}^2\right) dt^2 + \frac{[d\bar{r} - \sqrt{2m/\bar{r}}\,\sqrt{1-H^2\bar{r}^2}\,dt]^2}{1-H^2\bar{r}^2} + \bar{r}^2 d\Omega^2, \tag{A.9}$$

$$ds^2 = -dt^2 + \left[d\bar{r} - \sqrt{2m/\bar{r} + H^2\bar{r}^2}\ dt\right]^2 + \bar{r}^2 d\Omega^2. \tag{A.10}$$

$$ds^2 = -dt^2 + e^{2Ht}\left\{\left[dr + \left(Hr - \sqrt{2me^{-3Ht}/r + H^2r^2}\right) dt\right]^2 + r^2 d\Omega^2\right\}. \tag{A.11}$$

$$ds^2 = -\left(1 - \frac{2me^{-Ht}}{r}\right) d\bar{t}^2 + e^{2Ht}\left\{\frac{\left(dr + Hr\left[1 - \sqrt{1 - 2me^{-Ht}/r}\right] dt\right)^2}{1 - \frac{2me^{-Ht}}{r}} + r^2 d\Omega^2\right\}. \tag{A.12}$$

A.3 McVittie

By suitable choice of coordinates McVittie spacetime can be represented in any of the following four equivalent forms:

$$ds^2 = -\left(\frac{1 - \frac{m}{2a(t)\tilde{r}}}{1 + \frac{m}{2a(t)\tilde{r}}}\right)^2 dt^2 + \left(1 + \frac{m}{2a(t)\tilde{r}}\right)^4 a(t)^2\{d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\}. \tag{A.13}$$

$$ds^2 = -\left(1 - \frac{2m}{\bar{r}}\right) dt^2 + \left[\frac{d\bar{r}}{\sqrt{1 - 2m/\bar{r}}} - H(t)\bar{r}dt\right]^2 + \bar{r}^2 d\Omega^2. \tag{A.14}$$

$$ds^2 = -\left(1 - \frac{2m}{a(t)r}\right) dt^2 + a(t)^2\left\{\left[\frac{\left(dr + H(t)r\left[1 - \sqrt{1 - \frac{2m}{a(t)r}}\right] dt\right)^2}{1 - \frac{2m}{a(t)r}}\right] + r^2 d\Omega^2\right\}. \tag{A.15}$$

$$ds^2 = \left(1 + \frac{m}{2\bar{r}}\right)^4 \left\{-\left(\frac{[1 - \frac{m}{2\bar{r}}]^2}{[1 + \frac{m}{2\bar{r}}]^6}\right) dt^2 + \{[d\bar{r} - H(t)\bar{r}dt]^2 + \bar{r}^2 d\Omega^2\}\right\}. \tag{A.16}$$

Appendix B

Mathematica Codes for Chapter 6

We have developed three ancillary Mathematica files relevant to the analysis of Chapter 6 . A brief explanation on these files is provided below. The actual Mathematica files are available on the arXiv as supplementary material to the submission 2407.15289.

1. **KerrNewmanGeneralisedBondiSachsForm.nb**: this file writes the Kerr–Newmann metric into the generalised Bondi–Sachs form.

2. **KerrNewmanBondiGaugeComponentsExpansions.nb**: this file computes the Kerr–Newmann metric components expansions at null infinity in the generalised Bondi–Sachs form; transforms the metric into the Bondi–Sachs gauge and computes their asymptotic expansion.

3. **KerrNewman4PotentialNullInfinity.nb**: this file computes the electromagnetic four-potential in the Bondi–Sachs gauge.

Appendix C

List of Papers Relevant to this Thesis

This appendix contains a brief list of the papers that are included in this thesis and ones that are not[1].

Published papers included in this thesis

1. Chapter 2: Cosmology in Painlevé–Gullstrand Coordinates
   - https://arxiv.org/abs/2207.08375
   - JCAP 09 (2022) 030
   - Authors: Rudeep Gaur & Matt Visser
2. Chapter 3: Black holes embedded in FLRW cosmologies
   - https://arxiv.org/abs/2308.07374
   - Phys. Rev. D 110 (2024) 043529
   - Authors: Rudeep Gaur & Matt Visser
3. Chapter 4: Black holes, white holes, and near-horizon physics
   - https://arxiv.org/abs/2304.10692
   - JHEP 05 (2024) 172
   - Authors: Rudeep Gaur & Matt Visser

Unpublished papers included in this thesis

1. Chapter 5: The Kerr Memory Effect at Null Infinity
   - https://arxiv.org/abs/2403.07302
   - Status: Currently waiting with referee at JHEP
   - Author: Rudeep Gaur

[1] The format will be: Chapter number, title of paper, arxiv link, journal reference, authors.

2. Chapter 6: Kerr–Newman Memory Effect
    - https://arxiv.org/abs/2407.15289
    - Status: Currently waiting with referee at JHEP
    - Authors: Marco Galoppo, Rudeep Gaur, and Christopher Harvey-Hawes

Published papers not included in this thesis

This paper was written during the PhD and was a useful endeavour to expand my skill set but was not included in my thesis.

1. Defect Wormholes Are Defective
    - https://arxiv.org/abs/2407.15289
    - Universe 9 (2023) 10, 452
    - Authors: Joshua Baines, Rudeep Gaur, and Matt Visser

Beyond these papers, there are two or three that are works in progress. These expand upon chapter 2 and chapter 4. In particular, we have begun to investigate the effects that slicing (especially Painléve–Gullstrand slicing) has on backreaction and curvature. We also have very early calculations for black-to-white bounces for spacetimes beyond Schwarzschild.