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Gauss–Bonnet effects in $f(R, \Sigma, T)$ gravity

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Abstract We investigate the cosmological dynamics of Parameterized Absolute Parallelism (PAP) geometry within the framework of modified $f(R, \Sigma, T)$ gravity. By generalizing the Gauss–Bonnet invariant $\mathcal{G}_b$ using the PAP curvature tensor $B^{\alpha}{}_{\mu\nu\sigma}$, we incorporate both Riemannian curvature and torsion effects into a unified framework. The modified field equations are derived for a spatially flat FLRW background, and exact analytical solutions for the Hubble parameter, scale factor, energy density, and pressure are obtained under a constant equation-of-state parameter ω. Our analysis shows that the PAP deformation parameter b and matter–torsion coupling η significantly modify cosmic expansion, the effective equation of state, and curvature diagnostics. Furthermore, we explore the role of $\mathcal{G}_b$ in torsion-driven Gauss–Bonnet inflation and demonstrate how PAP corrections alter the slow-roll dynamics, enabling sustained inflation even for flat potentials. A detailed stability analysis reveals the critical regions in the (b, η) parameter space separating stable and unstable cosmological phases. This unified approach recovers General Relativity, teleparallel gravity, and minimally coupled PAP models as limiting cases, offering new insights into early- and late-time acceleration driven by torsion and higher-curvature effects.

1 Introduction

The Gauss–Bonnet theorem, a pinnacle of differential geometry, finds profound and unexpected applications in modern theoretical physics, particularly in extending Einstein's theory of General Relativity (GR). In its native four-dimensional spacetime, the Gauss–Bonnet term is a topological invariant; it describes a global, unchanging property of space's shape (like its number of "holes") and thus has no dynamic influence on gravity [1]. Einstein's original equations remain untouched. However, its significance erupts in modified gravity (4D): by creatively coupling the Gauss–Bonnet term to an additional field (e.g., a scalar field), physicists can induce novel effects even in our four-dimensional universe, providing compelling alternative mechanisms for cosmic inflation (driving the universe's rapid initial expansion without fine-tuned inflationary fields) and dark energy [2] (explaining the observed accelerated expansion of the cosmos, offering an alternative to the cosmological constant). In essence, while silent in classical Einsteinian gravity, the Gauss–Bonnet term is a crucial actor in the most advanced attempts to resolve the deepest puzzles of the universe, from the Big Bang to the interiors of black holes [3]. The effects or theory of Gauss–Bonnet are an extension of Einstein's GR that appear in higher dimensions ($D > 4$). Its quadratic term (the Gauss–Bonnet term) is a topological geometric construct (a topological invariant) in four dimensions and therefore does not affect the dynamics of four-dimensional spacetime. However, in string theories and cosmological models with higher dimensions (such as the Randall–Sundrum model), this term becomes dynamic and important: it prevents singularities in primordial black holes, provides mechanisms for cosmic inflation and accelerated expansion without the need for a cosmological constant, and modifies the relationships between the mass and spin of a black hole (no-hair theorem). To understand the Gauss–Bonnet effect, one must first understand the journey from geometry to relativity: differential geometry describes how surfaces and spaces curve. One of the most important results is the Gauss–Bonnet theorem, which connects the local geometry of surfaces (their curvature) with their global topology (their number of holes, e.g.,

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a sphere has zero holes, a doughnut has one). The paper titled by "from geometry to cosmology: a pedagogical review of inflation in curvature, torsion, and extended gravity theories" [4], this pedagogical review offers a comprehensive exploration of inflationary cosmology within a wide spectrum of modified gravity theories, moving beyond Einstein's General Relativity. It systematically examines models based on curvature (e.g., f(R), f(G)), torsion (e.g., f(T)), Einstein–Cartan), and non-metricity (e.g., f(Q)), as well as scenarios where scalar fields couple to various geometric invariants. The review also covers exotic frameworks like mimetic gravity, non-local theories, and string-inspired models. Using dynamical systems and Bayesian methods, the work compares the predictions of these models such as the spectral index ns and tensor-to-scalar ratio r against current data from Planck and BICEP/Keck, and forecasts from future missions like Lite BIRD and CMB-S4. Special attention is given to post-inflationary processes such as reheating, primordial black hole formation, and gravitational wave signatures. The strong observational viability of geometrically motivated models like Starobinsky f(R) inflation, scalar-Gauss–Bonnet theories, and f(Q) gravity. The role of future observations in discriminating between models, especially through precision measurements of r, non-Gaussianity, and stochastic gravitational waves. The need for theoretical improvements in perturbation theory, reheating mechanisms, and UV completions for many modified gravity frameworks. The value of Bayesian and machine-learning approaches in navigating the high-dimensional landscape of inflationary models. This concise overview reflects the article's integrative and forward-looking perspective, emphasizing both theoretical richness and the growing potential for empirical validation.

In other words, it is a bridge between the local and global properties of space. Simplified formula: For a closed two-dimensional surface, the theorem states that the integral over the entire surface of the Gaussian Curvature K equals 2π multiplied by the Euler Characteristic (χ) of the surface, a constant number that depends on the number of holes (a topological invariant). That is:

$$\int K\, dA = 2\pi\chi. \tag{1}$$

Einstein's GR: Its core idea is that matter and energy curve the fabric of spacetime, and this curvature is what we perceive as gravity. Einstein's equation is a differential equation that connects spacetime curvature (described by the Einstein tensor $G_{\mu\nu}$) to its matter/energy content (the Stress–Energy tensor $T_{\mu\nu}$) [5]

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2}$$

This Einstein tensor is derived from the Ricci curvature $R_{\mu\nu}$ and the Scalar curvature R.

In GR, we deal with four-dimensional spacetime. The Gauss–Bonnet (GB) term is a specific combination of curvature quantities, known as the Gaussian curvature of higher-dimensional spacetime. Its formula is [2,6–8]

$$\mathcal{G} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{3}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann Curvature. The most important property: In four-dimensional spacetime (3+1 dimensions), the GB term is a topological invariant. It means that the integral $\int \mathcal{G}\sqrt{-g}\, d^4x$ over the entire closed four-dimensional spacetime is a constant number that depends only on the global topology of spacetime (like the Euler characteristic), and not on the local geometry or how the curvature changes from point to point. The integral $\int \mathcal{G}\sqrt{-g}\, d^4x$ is a topological invariant, its variation with respect to the metric (which describes the geometry of spacetime) is zero. Therefore, if we add this term to the Einstein–Hilbert action (from which we derive Einstein's equations), it will not contribute to the equations of motion. That is, Einstein's equations remain unchanged in 4 dimensions. This integral is proportional to the Euler characteristic of the underlying manifold. Although topological in GR, the GB term acquires dynamical significance in higher-curvature theories of gravity, providing insights into early-universe inflation, dark energy, and cosmic singularity avoidance [2,9,10]. The quest to understand the fundamental forces governing the evolution of our universe remains one of the most profound challenges in theoretical physics. While Einstein's GR has been the cornerstone of modern cosmology, successfully describing phenomena from gravitational lensing to the precession of Mercury's orbit, it faces significant challenges in explaining the universe's accelerated expansion both in its primordial (inflation) and contemporary (dark energy) phases [11,12]. The cosmological constant (Λ) in the standard ΛCDM model, though empirically successful, is plagued by theoretical fine-tuning problems, most notably the staggering discrepancy between its observed value and predictions from quantum field theory [13,14]. This has catalyzed the development of *modified gravity theories*, which extend GR by introducing higher-order curvature invariants, non-minimal couplings, or additional geometric degrees of freedom. Among these, *Gauss–Bonnet* (GB) gravity has emerged as a particularly compelling framework [2,15]. In its native four-dimensional form, the GB term (see Eq. (3)) is a topological invariant and thus does not contribute to the dynamics of pure GR [7]. However, when coupled to a dynamical scalar field, it induces significant modifications to the gravitational field equations, offering elegant mechanisms for inflation, dark energy, and the avoidance of spacetime singularities [16,17]. Concurrently, there has been a resurgence of interest in the role of *torsion* in gravity. *Teleparallel Gravity* (TEGR), where gravity is attributed to torsion rather than curvature, provides an alternative but equivalent formulation of GR [18]. More recently, its extensions, such as $f(T)$ gravity, have been explored to drive cosmic acceleration [19]. A powerful synthesis of these ideas is found in the *Parameterized Absolute Parallelism* (PAP) geometry, which introduces a continuous parameter b that interpolates between pure Riemannian geometry ($b = 0$, GR), pure teleparallelism ($b = 1$, TEGR), and a hybrid regime $0 < b < 1$ where both curvature and torsion coexist [20,21]. This geometric flexibility makes PAP an ideal framework for constructing unified cosmological models. The profound challenge of the cosmological constant problem – the staggering discrepancy between the observed value of dark energy and predictions from quantum field theory – serves as a primary catalyst for this work. This fine-tuning issue, coupled with the conceptual "why now?" problem, compels the search for compelling alternatives to the standard ΛCDM paradigm. Our approach is founded on the premise that the accelerating phases of the universe, both primordial and contemporary, may not stem from an *ad hoc* vacuum energy but could instead be driven by the intrinsic properties of spacetime itself. We develop a geometric alternative within the framework of Parameterized Absolute Parallelism (PAP) geometry, where acceleration emerges dynamically from the interplay between generalized curvature and torsion. Crucially, we extend this framework by incorporating a generalized Gauss–Bonnet invariant, $\mathcal{G}_b$, which becomes dynamically active in four dimensions through its non-minimal coupling to the torsion-rich PAP structure. This term is posited to provide a potent source of negative pressure at high curvatures, offering a mechanism to resolve initial singularities and drive inflation without invoking ultra-steep scalar potentials. Furthermore, the torsion in our model is not merely a mathematical artifact but is endowed with physical significance, often linked to the spin density of matter. This motivates the matter–torsion coupling parameter η in our $f(R, \Sigma, T)$ construction, framing it as an interaction between the geometry of spacetime and the intrinsic angular momentum of the cosmic fluid, which could yield significant effects in the high-density environments of the early universe. Ultimately, the physical goal of the PAP framework is to provide a unified geometric description of gravity – a single, continuous formalism that seamlessly interpolates between a curvature-dominated regime, describing large-scale structure, and a torsion-dominated regime, offering a novel characterization of gravitational energy. This unification allows for a cosmos that can inhabit different geometric phases throughout its evolution, providing a rich and versatile foundation for addressing the most pressing puzzles in modern cosmology.

In this paper, we build upon these foundations by generalizing the $\mathcal{G}_b$ within the PAP framework. We construct a novel topological scalar from the full PAP curvature tensor $B_{\mu\nu\sigma}{}^{\alpha}$, which encapsulates contributions from both standard Riemannian curvature and torsion. We then embed this construction into the broader context of $f(R, \Sigma, T)$ gravity, where R is the Ricci scalar, Σ is a torsion scalar, and T is the trace of the energy–momentum tensor, introducing a direct coupling between matter and torsion via a parameter η [22]. This approach allows us to explore a unified geometric origin for the universe's inflationary and late-time accelerating phases. Scientists have found clever ways to make the GB term effective even in 4 dimensions. The most common method is to couple it to a scalar field. The action takes the form [2]:

$$S = \int [R + f(\phi)\,\mathcal{G} + \cdots]\,\sqrt{-g}\,d^4x, \tag{4}$$

where $f(\phi)$ is a function of the scalar field ϕ. In this case, when varying the action with respect to the metric, the term $f(\phi)\mathcal{G}$ does not vanish because it is no longer "pure" but coupled to another dynamic field. This leads to significantly modified field equations. These modifications have profound and exciting physical implications: In standard GR, the center of a black hole is a "singularity" where density and spacetime curvature become infinite, and the laws of physics break down. The GB term, due to its quadratic nature in curvature, acts as a "repulsive" force at very high curvatures, which can prevent the formation of a singularity or replace it with a "non-singular core."

Cosmic Inflation: The GB term coupled to a scalar field (called Gauss–Bonnet Inflation models) can provide a natural mechanism for the period of early cosmic inflation – the rapid exponential expansion of the universe – without needing to introduce a special inflation field with very fine-tuned conditions. Modifications brought by the GB term to the dynamics of cosmic expansion can mimic the effect of dark energy, i.e., the observed acceleration in the current expansion of the universe. This offers an attractive alternative to the cosmological constant (Λ), which faces theoretical problems (such as the problem of its unnaturally small value). Gauss–Bonnet effects are a prime example of how deep mathematics (differential geometry and topology) can provide powerful tools for expanding our understanding of fundamental physics (gravity and the universe). While they are silent in our traditional four-dimensional world, they speak loudly and influentially in higher-dimensional scenarios and modified gravity models, offering potential solutions to the universe's deepest mysteries, from singularities to cosmic acceleration [6]. Observations from *Planck* [12], *WMAP* [23], DES [24], and SDSS-IV [25] confirm that our Universe evolved through distinct phases an early inflationary epoch, a matter-dominated era, and the current phase of accelerated expansion. While the ΛCDM model explains many features successfully, it suffers from theoretical challenges, such as the cosmological constant problem and the Hubble tension [26]. This has motivated the exploration of alternative frameworks, including $f(R)$ gravity [9,27], $f(T)$ gravity [19,28], and $f(R, \mathcal{G})$ theories [17,29]. In particular, modified GB models have been widely studied to explain late-time acceleration [30–32], inflationary scenarios, and the avoidance of finite-time singularities [16]. PAP geometry [20,33] generalizes Riemannian geometry by introducing torsion via a contortion tensor $\gamma^{\alpha}{}_{\mu\nu}$, controlled by a dimensionless parameter b. This framework unifies different geometric regimes

- $b = 0$ standard Riemannian geometry, recovering GR,
- $b = 1$ absolute parallelism (teleparallel gravity) with zero curvature but non-trivial torsion.

Such flexibility makes PAP geometry a promising platform for addressing early-universe inflation, dark energy, and modified gravitational dynamics [18,34,35].

In this work, we extend the GB invariant to PAP geometry by defining a generalized topological scalar $\mathcal{G}_b$, constructed from the modified curvature tensor $B^{\alpha}{}_{\mu\nu\sigma}$, which captures both Riemannian and torsional contributions. This construction is embedded into the $f(R, \Sigma, T)$ gravity framework, where Σ encodes torsion effects and T is the trace of the energy–momentum tensor. This combined setup allows us to explore

1. how torsion and matter-torsion couplings modify the expansion history,
2. the role of the generalized GB term $\mathcal{G}_b$ in resolving cosmological singularities,
3. a unified framework for early-time inflation, matter domination, and late-time acceleration.

The novelty of our work lies in obtaining exact analytical solutions for the Hubble parameter, scale factor, energy density, and pressure in PAP cosmology with GB corrections. By comparing our results to GR, teleparallel gravity, and minimally coupled PAP models, we demonstrate that torsion-induced effects and higher-curvature terms can dynamically mimic dark energy, modify the effective equation of state, and drive accelerated expansion without invoking exotic matter.

The paper is organized as follows Sect. 2 introduces PAP geometry and the generalized GB invariant. Section 3 formulates the field equations within the $f(R, \Sigma, T)$ framework. Section 4 derives analytical FLRW solutions and discusses their implications. Section 5 explores torsion-driven inflation and stability analysis. Finally, Sect. 6 summarizes our results and highlights potential observational tests.

2 Geometric structure

In this section, we follow the formalism of the Basic Building Blocks (BB) introduced in [36], which represent the elementary geometric quantities from which all other objects in the framework are derived. For the present analysis, we start from the setting of Absolute Parallelism (AP) geometry as the underlying structure and then extend it to its generalized version, the PAP geometry [36].

An AP space (M, λ^i) is defined on an n-dimensional differentiable manifold M, where each point is characterized by n independent coordinates x^{ν} $(\nu = 0, 1, \ldots, n-1)$. At each point, there exist n linearly independent vector fields λ^i $(i = 0, 1, \ldots, n-1)$ with n coordinate components each. These vector fields, collectively referred to as the BB of AP geometry, serve as the foundational objects from which the entire geometric structure is built. Consequently, the total number of vector components amounts to n^2. The contravariant components λ_i^{μ} are defined as the normalized cofactors of λ_{μ}^{i} in the determinant $\det(\lambda_{\mu}^{i})$, satisfying the following orthogonality and completeness relations [37]

$$\lambda_i^{\mu}\lambda_{\nu}^{i} = \delta_{\nu}^{\mu}, \tag{5}$$

$$\lambda_i^{\nu}\lambda_{\nu}^{j} = \delta_i^{j}, \tag{6}$$

where δ_{ν}^{μ} denotes the Kronecker delta. Latin indices $(i, j, \ldots)$ label vector numbers and appear only as subscripts, while the Einstein summation convention applies to repeated indices. Using these vector fields, we define the associated Riemannian metric tensor as

$$g_{\mu\nu} = \lambda_{\mu}^{i}\lambda_{\nu}^{i}, \tag{7}$$

with its inverse given by

$$g^{\mu\nu} = \lambda_i^{\mu}\lambda_i^{\nu}. \tag{8}$$

The canonical linear connection of AP geometry is defined by [36]

$$\Gamma^{\alpha}_{\mu\nu} \overset{\text{def}}{=} \lambda_i^{\alpha}\,\partial_{\nu}\lambda_{\mu}^{i}, \tag{9}$$

where the comma indicates partial differentiation. PAP geometry generalizes this connection by introducing a parameter b, which controls the contribution of torsion

$$\nabla^{\mu}{}_{\nu\rho} = \left\{ {}^{\mu}_{\nu\rho} \right\} + b\,\gamma^{\mu}_{\nu\rho}, \tag{10}$$

where $\left\{ {}^{\mu}_{\nu\rho} \right\}$ are the Christoffel symbols (Levi-Civita connection) constructed from the metric, and $\gamma^{\mu}_{\nu\rho}$ is the contortion tensor defined as

$$\gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \left\{ {}^{\alpha}_{\mu\nu} \right\}. \tag{11}$$

The parameter b thus interpolates smoothly between the purely Riemannian case $(b = 0)$ and the standard absolute parallelism limit $(b = 1)$. The torsion tensor is defined as twice the antisymmetric part of the AP connection

$$\Lambda^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}. \tag{12}$$

The curvature tensor associated with the PAP connection can be expressed as

$$B^{\alpha}{}_{\mu\nu\sigma} = \nabla^{\alpha}_{\mu\sigma,\nu} - \nabla^{\alpha}_{\mu\nu,\sigma} + \nabla^{\epsilon}_{\mu\sigma}\nabla^{\alpha}_{\epsilon\nu} - \nabla^{\epsilon}_{\mu\nu}\nabla^{\alpha}_{\epsilon\sigma}$$
$$= R^{\alpha}{}_{\mu\nu\sigma} + b\, Q^{\alpha}{}_{\mu\nu\sigma}, \quad (13)$$

where $R^{\alpha}{}_{\mu\nu\sigma}$ is the Riemann curvature tensor calculated from the Levi-Civita connection, and $Q^{\alpha}{}_{\mu\nu\sigma}$ depends solely on the contortion tensor $\gamma^{\mu}_{\nu\rho}$.

3 GB invariant in PAP geometry

In four-dimensional Riemannian geometry, the GB invariant is defined as [7]:

$$\mathcal{G}[R] = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \quad (14)$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor, $R_{\mu\nu}$ the Ricci tensor, and R the Ricci scalar. In standard four-dimensional GR, $\mathcal{G}[R]$ is a topological invariant: its variation with respect to the metric vanishes, $\delta\mathcal{G}/\delta g^{\mu\nu} = 0$, so the GB term does not contribute to the field equations. Its integral over a closed manifold is given by [8]:

$$\int \sqrt{-g}\,\mathcal{G}[R]\, d^4x = 32\pi^2\,\chi(M), \quad (15)$$

where $\chi(M)$ is the Euler characteristic. In PAP geometry [20, 33], curvature and torsion coexist. The total PAP curvature tensor $B^{\alpha}{}_{\mu\nu\sigma}$ can be decomposed as

$$B_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + b\, Q_{\mu\nu\rho\sigma}, \quad (16)$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann curvature of the Levi-Civita connection, $Q_{\mu\nu\rho\sigma}$ arises from the contortion, and b is the PAP torsion-control parameter. The generalized GB invariant in PAP geometry is then

$$\mathcal{G}_b = B_{\mu\nu\rho\sigma}B^{\mu\nu\rho\sigma} - 4B_{\mu\nu}B^{\mu\nu} + B^2, \quad (17)$$

where

$$B_{\mu\nu\rho\sigma}B^{\mu\nu\rho\sigma} = (R_{\mu\nu\rho\sigma} + bQ_{\mu\nu\rho\sigma})(R^{\mu\nu\rho\sigma} + bQ^{\mu\nu\rho\sigma})$$
$$= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 2b\, R_{\mu\nu\rho\sigma}Q^{\mu\nu\rho\sigma} + b^2 Q_{\mu\nu\rho\sigma}Q^{\mu\nu\rho\sigma} \quad (18)$$

with $B_{\mu\nu} = B^{\alpha}{}_{\mu\alpha\nu}$ and $B = g^{\mu\nu}B_{\mu\nu}$. Substituting the decomposition yields:

$$\mathcal{G}_b = \mathcal{G}[R] + 2b\Big(R_{\mu\nu\rho\sigma}Q^{\mu\nu\rho\sigma} - 4R_{\mu\nu}Q^{\mu\nu} + RQ\Big)$$
$$+ b^2\Big(Q_{\mu\nu\rho\sigma}Q^{\mu\nu\rho\sigma} - 4Q_{\mu\nu}Q^{\mu\nu} + Q^2\Big). \quad (19)$$

The primary motivation for constructing $\mathcal{G}_b$ within the PAP geometry, as opposed to pure Weitzenböck (TEGR) or symmetric teleparallel (STEGR) geometries, is to create a unified framework that inherently incorporates a smooth transition between curvature and torsion. This is not merely a mathematical exercise but is driven by a key physical question: what are the observable consequences of a spacetime where both Riemannian curvature and torsion are fundamental and dynamically active components? In highly symmetric FLRW tetrads, this reduces to the simpler relation:

$$\mathcal{G}_b = (1-b)^2\,\mathcal{G}[R]. \quad (20)$$

For metric-compatible PAP connections, the Chern–Weil theorem implies that $\mathcal{G}_b$ remains a bulk topological density [29]. However, torsion induces additional boundary contributions via a Chern–Simons 3-form $\Theta_3(\gamma, b)$:

$$\int_M \sqrt{-g}\,\mathcal{G}_b\, d^4x = 32\pi^2\chi(M) - \int_{\partial M}\Theta_3(\gamma, b), \quad (21)$$

where $\gamma^{\mu}_{\nu\rho}$ is the contortion tensor. PAP geometry also naturally incorporates other topological invariants, notably the Nieh–Yan term [38]:

$$N = d(e^a \wedge T_a) = T^a \wedge T_a - e^a \wedge e^b \wedge R_{ab}, \quad (22)$$

where e^a are tetrad 1-forms and T^a the torsion 2-form.

In the teleparallel limit ($b = 1$), the PAP curvature vanishes, $B^{\alpha}{}_{\mu\nu\sigma} = 0$, so $\mathcal{G}_b = 0$, and the Nieh–Yan invariant becomes the dominant topological contribution. Thus, the generalized GB invariant in PAP geometry unifies curvature- and torsion-driven topological effects, smoothly connecting GR, teleparallel gravity, and torsion-dominated cosmologies.

4 Cosmological dynamics and exact analytical solutions

In this section, we investigate how PAP geometry provides a consistent framework for modeling cosmological dynamics, particularly in the presence of torsion and spin–torsion couplings. For cosmological applications, we adopt a spatially flat FLRW metric,

$$ds^2 = dt^2 - A(t)^2\left(dx^2 + dy^2 + dz^2\right). \quad (23)$$

The line element above corresponds to the spatially flat FLRW metric ($k = 0$), with $A(t)$ denoting the scale factor. In the limit $b = 0$ it reduces exactly to the standard FLRW metric of GR. This ansatz ensures that PAP geometry preserves the large-scale FLRW symmetries (homogeneity and isotropy) while incorporating torsion effects absent in GR. The components of the PAP curvature tensor for this structure are [39]

$$B^1_{010} = (1-b)\,\frac{\ddot{A}}{A}, \quad B^1_{001} = (b-1)\,\frac{\ddot{A}}{A},$$
$$B^2_{020} = (1-b)\,\frac{\ddot{A}}{A}, \quad B^2_{002} = (b-1)\,\frac{\ddot{A}}{A}, \quad (24)$$
$$B^3_{030} = (1-b)\,\frac{\ddot{A}}{A}, \quad B^3_{003} = (b-1)\,\frac{\ddot{A}}{A}.$$

From these, the PAP Ricci tensor and scalar follow as

$$B_{00} = (1-b)\, R_{00} = -3(1-b)\left(\dot{H} + H^2\right), \tag{25}$$

$$B_{ij} = (1-b)\, R_{ij} = (1-b)\left(\dot{H} + 3H^2\right) g_{ij}, \tag{26}$$

$$B = (1-b)\, R = -6(1-b)\left(\dot{H} + 2H^2\right), \tag{27}$$

showing explicitly how the parameter b modulates the curvature contributions. The PAP structure preserving homogeneity and isotropy leads to the simplified expressions for the curvature quantities found in the original text (Eqs. (24)–(27)). Notably, the generalized GB scalar becomes:

$$\mathcal{G}_b = \mathcal{G}[B] = (1-b)^2\, \mathcal{G}[R] = 24(1-b)^2\, H^2\left(\dot{H} + H^2\right). \tag{28}$$

This result is pivotal: it shows how the torsion parameter b directly modulates the strength of the higher-curvature term, vanishing entirely in the teleparallel limit $b = 1$. For $b = 0$, this reduces to the standard GR result

$$\mathcal{G}_b\Big|_{b=0} = 24\, H^2\left(\dot{H} + H^2\right),$$

while in the teleparallel limit ($b = 1$), we find $\mathcal{G}_b = 0$, reflecting the vanishing Riemann curvature in this regime. Thus, PAP geometry naturally interpolates between GR and teleparallelism, introducing a controlled torsion contribution that influences the dynamics without spoiling isotropy or homogeneity. These structures form the foundation for the cosmological field equations and higher-curvature effects analyzed in the following sections. Starting from the action [11,29]

$$I = \frac{1}{16\pi}\int \sqrt{-g}\left(f(R, \Sigma, T) + \ell_m\right) d^4x, \tag{29}$$

Within the $f(R, \Sigma, T) = R + \Sigma + 2\eta T$ model, the cosmological background dynamics are governed by the modified FLRW equations. For a spatially homogeneous and isotropic universe, these reduce to the Friedmann and acceleration relations given in Eqs. (30) and (31), respectively.

$$3(1-b)^2\frac{\dot{a}^2}{a^2} = 8\pi\rho + \eta(3\rho - P), \tag{30}$$

$$3(1-b)\frac{\ddot{a}}{a} = 4\pi(\rho + 3P) + 4\eta P, \tag{31}$$

where ρ and P denote the energy density and pressure, respectively. By solving Eqs. (30) and (31) simultaneously, we find

$$\rho = \frac{(1-b)}{4\Delta}\left[4(3\pi + \eta)(1-b)\frac{\dot{a}^2}{a^2} - \eta\frac{\ddot{a}}{a}\right], \tag{32}$$

$$P = -\frac{(1-b)}{4\Delta}\left[4\pi(1-b)\frac{\dot{a}^2}{a^2} + (8\pi + 3\eta)\frac{\ddot{a}}{a}\right], \tag{33}$$

with

$$\Delta = 8\pi^2 + 6\pi\eta + \eta^2. \tag{34}$$

Assuming a barotropic equation of state $P = \omega\rho$, the parameter ω becomes

$$\omega = -\frac{4\pi(1-b)\frac{\dot{a}^2}{a^2} + (8\pi + 3\eta)\frac{\ddot{a}}{a}}{(12\pi + 4\eta)(1-b)\frac{\dot{a}^2}{a^2} - \eta\frac{\ddot{a}}{a}}. \tag{35}$$

From Sect. 4, the generalized GB scalar is

$$\mathcal{G}_b = 24(1-b)^2 H^2\left(\dot{H} + H^2\right), \tag{36}$$

which naturally enters higher-curvature corrections and dynamical dark energy models. Including a GB coupling ξ, the effective energy density and pressure become

$$\rho_{\text{eff}} = \rho - \frac{\xi}{\kappa^2}\mathcal{G}_b,$$
$$P_{\text{eff}} = P + \frac{\xi}{\kappa^2}\left(\mathcal{G}_b - 4(1-b)^2 H^2\frac{d\xi}{dt}\right). \tag{37}$$

For a constant $P = \omega\rho$, define

$$D = (8\pi + 3\eta) - \eta\omega, \quad C = 4\pi(1 + 3\omega) + 4\eta\omega. \tag{38}$$

Using Eq. (30), we express the density as

$$\rho = \frac{3(1-b)^2}{D}H^2. \tag{39}$$

Substituting this into Eq. (31) and eliminating ρ, we obtain

$$\dot{H} = -KH^2, \quad K = 1 + \frac{(1-b)C}{D}. \tag{40}$$

where K is a key parameter controlling the expansion dynamics. Integrating yields

$$H(t) = \frac{H_0}{1 + KH_0(t - t_0)},$$
$$a(t) = a_0\Big[1 + KH_0(t - t_0)\Big]^{1/K}. \tag{41}$$

These solutions reveal several critical insights:

1. **Torsion-Driven Acceleration:** Even for ordinary matter $\omega \geq 0$, a sufficiently large value of b can make $K < 1$, producing an accelerated expansion without any exotic dark energy component. This highlights torsion as a geometric driver of cosmic acceleration.
2. **Role of Matter-Torsion Coupling** η: The coupling parameter η directly affects the effective energy density and pressure. A positive η delays the dilution of ρ, mimicking the effect of a dark energy fluid, while a negative η enhances it.
3. **Unification of Limits:** Our framework seamlessly recovers standard GR, TEGR, and minimal PAP cosmology as

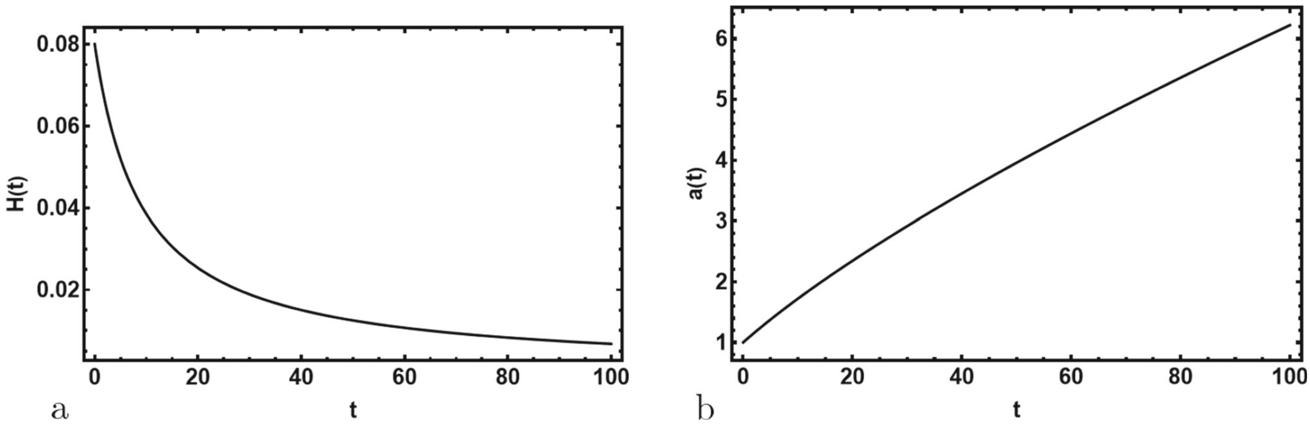

Fig. 1 **a** Hubble parameter $H(t)$ vs t. **b** Scale factor $a(t)$ vs t

specific limits of the parameters (b, η), demonstrating its generality.

The acceleration and deceleration parameters are

$$\frac{\ddot{a}}{a} = (1-K)H^2, \quad q = K - 1. \tag{42}$$

Additionally, the GB invariant evolves as

$$\mathcal{G}_b(t) = 24(1-b)^2(1-K)H^4. \tag{43}$$

Limiting cases

- **GR limit** $(b = 0, \eta = 0, \alpha = 1)$ $K = \frac{3}{2}(1+\omega)$, recovering the standard Friedmann dynamics.
- **Teleparallel limit** $(b = 1)$ $\mathcal{G}_b = 0$ and curvature effects vanish, leaving torsion-dominated evolution.
- **Minimal coupling** $(\eta = 0)$ The model reduces to a pure PAP gravity framework.

Physical implications

This framework shows that

- Torsion modifies the cosmic expansion rate through the PAP parameter b.
- Matter-torsion coupling η can mimic dark-energy-like behavior.
- The GB term $\mathcal{G}_b$ acts as a higher-curvature driver of cosmic acceleration.

This unification naturally interpolates between GR, teleparallel gravity, and PAP-dominated cosmology, providing a rich structure for exploring late-time acceleration and possible avoidance of future singularities. Now, we investigate the background cosmological evolution in the PAP framework within the $f(R, \Sigma, T)$ gravity model. Figure 1 shows the evolution of the Hubble parameter $H(t)$ and the scale factor $a(t)$ for the analytical solution derived in Eq. (41). These plots illustrate how the parameter b and the coupling η modify the cosmic expansion rate compared to the standard GR limit. Figure 1a: This graph shows the evolution of the Hubble parameter, $H(t)$, which measures the rate of the universe's expansion at any given time t. The formula for $H(t)$ is derived from the first Friedmann equation. The graph shows that $H(t)$ is a decreasing function of time. It starts at a very high value in the distant past and asymptotically approaches zero in the far future. Physical Interpretation: Early Universe $(t \to 0)$, the Hubble parameter is extremely large. This signifies a period of incredibly rapid expansion, characteristic of the very early stages of the universe, including epochs like inflation or the initial hot Big Bang. Late Universe $(t \to \infty)$, the expansion rate slows down over time, approaching zero. This indicates that the driving influence of the energy density ρ diminishes as the universe expands and dilutes.

Figure 1b: This graph shows the evolution of the scale factor, $a(t)$, which quantifies the relative size of the universe. A value of $a(t) = 1$ is usually set to represent the current size. The scale factor is an increasing function of time, confirming that the universe is expanding. The exact shape of the curve depends on the value of ω. The curve shows a universe that begins with a size of zero at $t = 0$ (the Big Bang singularity) and expands forever.

Figure 2a: This graph shows the energy density $\rho(t)$ of the dominant fluid in the universe (e.g., matter, radiation, or dark energy) as a function of time t. It is derived from the solutions of the Friedmann equations. The energy density is a strictly decreasing, convex function that starts at a very high value and rapidly decays towards zero as time progresses. Physical Interpretation: Big Bang Singularity $(t \to 0)$, the energy density approaches infinity at the very beginning $t = 0$. This is a hallmark of the initial singularity in standard Big Bang cosmology. The primary reason for the decrease in density

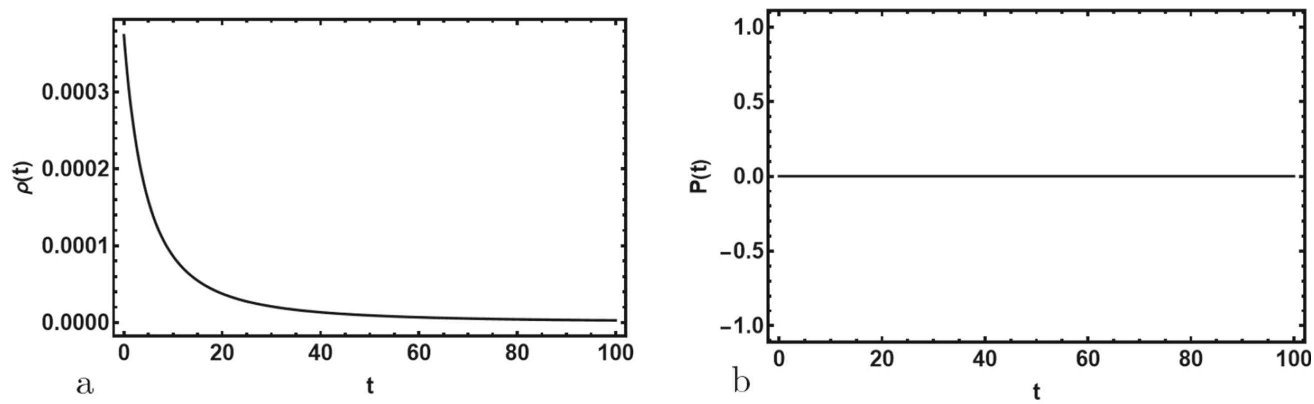

Fig. 2 **a** Energy density $\rho(t)$ vs t. **b** Pressure vs t

is the expansion of the universe, as described by the scale factor $a(t)$. As the universe expands, the volume increases, causing the energy density to dilute. The steep, convex decay shown in this figure is characteristic of a matter- or radiation-dominated universe, where dilution is significant. It is not consistent with a dark-energy-dominated universe, where the density would remain constant (a flat line).

Figure 2b: This graph shows the pressure $P(t)$ exerted by the cosmic fluid. The title confirms it is calculated directly from the energy density using the relation $P(t) = \omega\,\rho(t)$ for a constant value of ω. The pressure curve has the exact same shape as the energy density curve but is scaled by the factor ω. The y-axis shows both positive and negative values, indicating the sign of pressure depends on the value of ω. Physical Interpretation: Positive Pressure (if $\omega > 0$). If the fluid is radiation ($\omega = 1/3$) or matter (effectively $\omega = 0$, so $P \approx 0$), the pressure would be positive or zero. Positive pressure does work on the expansion, acting as a force that decelerates the universe (gravitational attraction). Negative Pressure (if $\omega < 0$). If the fluid is dark energy (e.g., $\omega \approx -1$), the pressure would be negative. Negative pressure (or tension) is the key property of dark energy that causes the expansion of the universe to accelerate.

Figure 3a: This graph shows the evolution of the GB invariant, $\mathcal{G}_b$, a fundamental geometric quantity in higher-dimensional gravity theories and string theory. The GB scalar is a positive, decreasing function of time. It starts at a very high value and rapidly decays towards zero. Physical Interpretation: $\mathcal{G}_b$ is a measure of the curvature of spacetime. Its high initial value indicates a universe with extreme geometric curvature and high energy at the beginning (near the Big Bang singularity). As the universe expands, it flattens, and its curvature decreases. The rapid decay of $\mathcal{G}_b$ directly reflects this "flattening" process. Its behavior is tightly coupled to the Hubble parameter $H(t)$ and its derivative $\dot{H}$, both of which decrease as the universe ages.

Figure 3b: This graph shows the deceleration parameter, $q(t)$, which quantifies whether the expansion of the universe is accelerating or decelerating. The caption states it is calculated as $q = K - 1$, where K is a constant for a fixed equation-of-state parameter ω. The deceleration parameter is a constant, positive value throughout cosmic time in this model. A positive deceleration parameter means the expansion of the universe is decelerating. This is the expected behavior for a universe dominated by ordinary matter and/or radiation, whose gravitational attraction acts to slow down the expansion.

Finally, Fig. 3 shows two key curvature diagnostics the generalized GB scalar $\mathcal{G}_b(t)$ and the deceleration parameter q. The GB term captures higher-curvature effects in PAP geometry, while q indicates the accelerating or decelerating nature of cosmic expansion.

The main findings can be summarized as follows

- **GR limit** ($b = 0, \eta = 0, \alpha = 1$) We recover the standard Friedmann equations with $a(t) \propto t^{2/[3(1+\omega)]}$ and $\mathcal{G}_b$ reducing to the classical GB invariant $\mathcal{G}[R]$.
- **Teleparallel limit** ($b \to 1$) The Riemann curvature vanishes, $\mathcal{G}_b \to 0$, and the dynamics are dominated by torsion, resulting in an effectively flat but torsion-driven cosmology.
- **Matter-torsion coupling** ($\eta \neq 0$) Positive η delays the dilution of ρ and P, favoring accelerated expansion, while negative η enhances the decay of energy density.
- **Torsion-driven acceleration** Even for $\omega \geq 0$ (dust or radiation), a sufficiently large b can make $q < 0$, yielding accelerated expansion without dark energy.

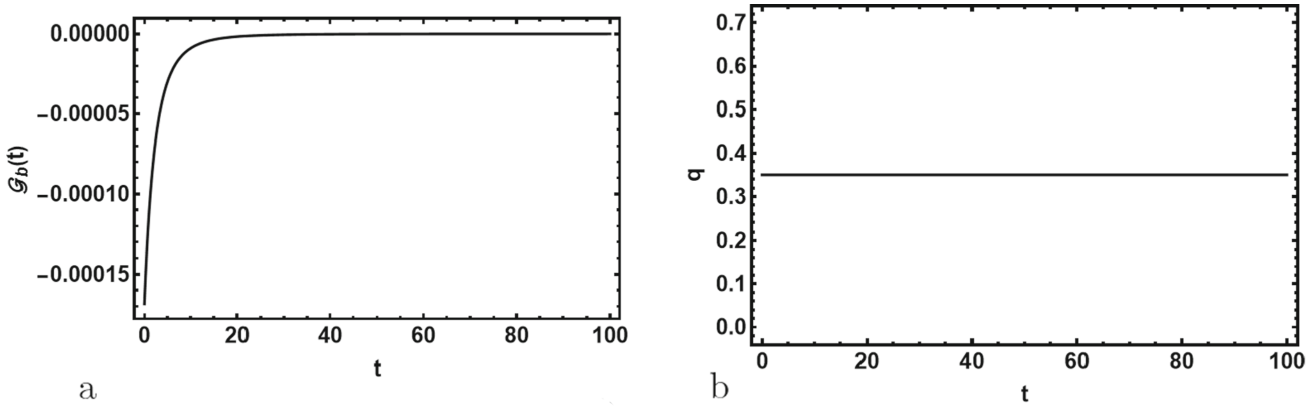

Fig. 3 **a** GB scalar $\mathcal{G}_b(t) = 24(1-b)^2 H^2(\dot{H} + H^2)$. **b** Deceleration parameter $q = K - 1$ (constant for fixed ω)

4.1 Torsion-driven inflation via GB coupling and stability

One of the most attractive features of the generalized GB invariant $\mathcal{G}_b$ in PAP geometry is its potential role in driving cosmic inflation. We consider a scalar field ϕ coupled to the GB term through a dynamical function $\xi(\phi)$, leading to the effective action

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa^2}R + \Sigma + 2\eta T - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \xi(\phi)\,\mathcal{G}_b\right]. \tag{44}$$

The resulting modified Friedmann and Klein–Gordon Eqs. (46) and (47) show that the torsion parameter b and the GB coupling $\xi(\phi)$ create an additional friction term in the scalar field's equation of motion. This allows for sustained slow-roll inflation even for potentials that would be too steep in standard GR, effectively flattening the potential dynamically [6]. Where $\kappa^2 = 8\pi G$, $V(\phi)$ is the scalar potential, and $\xi(\phi)$ encodes the non-minimal coupling between ϕ and the generalized GB invariant. For a spatially flat FLRW metric and $\mathcal{G}_b = 24(1-b)^2 H^2(\dot{H} + H^2)$, the modified Friedmann equation becomes

$$3H^2 = \kappa^2\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) + 24\kappa^2(1-b)^2 H^3\dot{\xi}, \tag{45}$$

where $\dot{\xi} = \frac{d\xi}{d\phi}\dot{\phi}$. The scalar field equation of motion is given by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + 24(1-b)^2 H^2(\dot{H} + H^2)\xi'(\phi) = 0, \tag{46}$$

where primes denote derivatives with respect to ϕ. In the slow-roll limit ($\dot{\phi}^2 \ll V$, $|\ddot{\phi}| \ll |3H\dot{\phi}|$), the system simplifies to

$$3H\dot{\phi} \simeq -V'(\phi) - 24(1-b)^2 H^4\xi'(\phi), \tag{47}$$
$$3H^2 \simeq \kappa^2 V(\phi). \tag{48}$$

This shows that torsion ($b \neq 0$) and the GB coupling $\xi(\phi)$ effectively modify the inflaton's dynamics, allowing sustained acceleration even for relatively flat potentials. Defining the effective slow-roll parameter

$$\epsilon_{\text{eff}} - \frac{\dot{H}}{H^2} = \frac{\kappa^2}{2H^2}\dot{\phi}^2 - 8\kappa^2(1-b)^2 H\dot{\xi}, \tag{49}$$

inflation occurs when $\epsilon_{\text{eff}} < 1$. Hence, even for simple potentials, torsion-induced contributions from $(1-b)^2\dot{\xi}$ can maintain $\epsilon_{\text{eff}} \ll 1$ over a prolonged period, extending the inflationary phase.

- When $b = 0$ and $\xi(\phi) = 0$, we recover standard inflation in GR.
- For $b \neq 0$ but $\xi(\phi) = 0$, torsion alone modifies the expansion dynamics.
- For $b \neq 0$ and $\xi(\phi) \neq 0$, PAP GB inflation provides a unified geometric mechanism for early accelerated expansion without requiring fine-tuned scalar potentials.

This framework connects the torsion parameter b, the coupling $\xi(\phi)$, and the GB term $\mathcal{G}_b$ to inflationary observables, offering clear pathways for testing PAP-based cosmology against data from CMB and large-scale structure surveys.

To demonstrate the inflationary dynamics driven by torsion and the generalized GB invariant $\mathcal{G}_b$, we consider a scalar field ϕ coupled to $\mathcal{G}_b$ via $\xi(\phi) = \xi_0\phi^2$ with a quartic potential $V(\phi) = \lambda\phi^4$. For this choice, the modified Friedmann and Klein–Gordon equations are

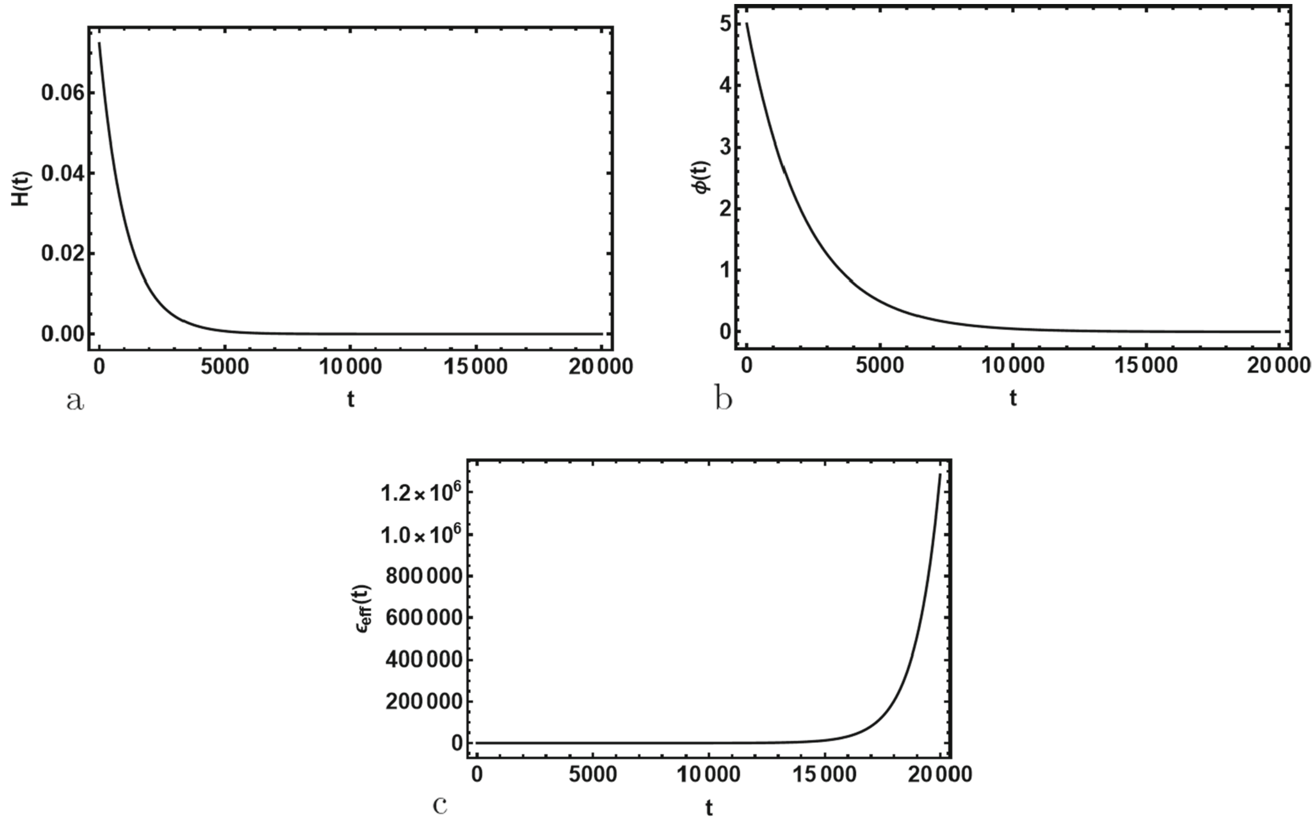

Fig. 4 **a** Evolution of $H(t)$ vs t. **b** Evolution of $\phi(t)$ vs t. **c** Slow-roll parameter $\epsilon_{\rm eff}(t)$ vs t

$$3H^2 = \kappa^2\left(\frac{1}{2}\dot{\phi}^2 + \lambda\phi^4\right) + 24\kappa^2(1-b)^2H^3\dot{\xi}, \tag{50}$$

$$\ddot{\phi} + 3H\dot{\phi} + 4\lambda\phi^3 + 48\xi_0(1-b)^2\phi H^2(\dot{H} + H^2) = 0. \tag{51}$$

We numerically solve Eqs. (50) and (51) using the initial conditions

$$\phi(0) = \phi_0, \quad \dot{\phi}(0) = 0, \quad H(0) = H_0,$$

and parameter values

$$\phi_0 = 5.0, \quad \lambda = 10^{-6}, \quad \xi_0 = 10^{-3}, \quad b = 0.3, \quad \kappa^2 = 8\pi.$$

For this setup, we monitor

$$\epsilon_{\rm eff} = -\frac{\dot{H}}{H^2}, \tag{52}$$

where inflation occurs while $\epsilon_{\rm eff} < 1$.

Results

Figure 4 shows the numerical solutions for $H(t)$, $\phi(t)$, and $\epsilon_{\rm eff}$. We find

- The Hubble parameter $H(t)$ remains nearly constant during an early de Sitter-like phase.
- The scalar field $\phi(t)$ slowly rolls down the potential due to the effective torsion-induced friction term.
- The slow-roll parameter $\epsilon_{\rm eff}$ stays below unity for an extended period, indicating sustained inflation.

Figure 4a: This graph shows the behavior of the Hubble parameter, which defines the expansion rate of the universe, during the inflationary epoch. The Hubble parameter starts at a very high value and undergoes a rapid, non-linear decrease before stabilizing to a nearly constant, positive value. The high initial H indicates an era of extremely rapid expansion, likely connected to the initial conditions of the universe or the moment the inflation field begins to dominate. The rapid decrease in H signifies the transition into the slow-roll regime. The inflation field's energy is initially rapidly converted into expansion, causing H to drop. Figure 4b: This graph tracks the value of the inflation scalar field ϕ over time. This field is the source of energy that drives the inflationary expansion. The inflation field ϕ is a monotonically decreasing function of time. It starts at a high value and slowly "rolls" down to a lower value. The smooth, gradual decrease of the field is the visual definition of the "slow-roll" mechanism. The field does not plummet rapidly but instead slowly evolves, ensuring that its potential energy density (which drives H) remains nearly constant for a long time. The initial high value of ϕ places it high up on its potential energy curve $V(\phi)$. As it slowly rolls down this potential, it converts its potential energy into the vacuum energy that fuels the exponential expansion seen in Fig. 4a. The end of inflation would occur when ϕ reaches the minimum of its potential and begins to oscillate, converting its energy into particles (reheating). Figure 4c: This figure depicts the evolution of a crucial parameter, $\epsilon_H(t)$, used to describe the inflationary epoch of the very early universe. The slow-roll parameter $\epsilon_H(t)$, specifically the Hubble slow-roll parameter, measures the rate of change of the Hubble parameter during inflation. The parameter starts at an astronomically high value (on the order of 10^5 to 10^6) at the very beginning of time $t \to 0$. It undergoes an incredibly steep and rapid decline by several orders of magnitude. After this precipitous drop, the parameter appears to stabilize and approach zero for the remaining duration of the plotted time. Figure 4c successfully visualizes the dynamical establishment of the inflationary state from a likely non-inflationary initial condition. The rapid decay of $\epsilon_H(t)$ from an extremely high value to a near-zero value demonstrates that this specific model universe undergoes a transition into a sustained phase of accelerated exponential expansion, fulfilling the primary requirement for solving the key problems of the standard Big Bang model. The plot's utility lies in showing that the model's parameters are chosen such that the slow-roll condition $\epsilon_H(t)$ is achieved and maintained for a significant period, which is the essential feature of any viable inflationary model. This is a sophisticated model where the coupling between a scalar field (inflation), the Gauss–Bonnet term, and torsion in the geometry conspires to produce a classic inflationary history: a slow-rolling field enabling a period of near-exponential expansion, as required by modern cosmology. In Fig. 4, we present a numerical illustration of torsion-driven GB inflation in PAP cosmology with ($b = 0.3$, $\lambda = 10^{-6}$, $\xi_0 = 10^{-3}$). Sustained inflation occurs while $\epsilon_{\rm eff} < 1$.

5 Stability analysis of the PAP-Gauss–Bonnet cosmological model

An important aspect of any modified gravity model is the stability of its cosmological solutions. In the PAP framework with the generalized GB invariant $\mathcal{G}_b$, the background evolution is governed by the analytic solution (41).We consider small perturbations $\delta H(t)$ around the background solution

$$H(t) \to H(t) + \delta H(t), \tag{53}$$

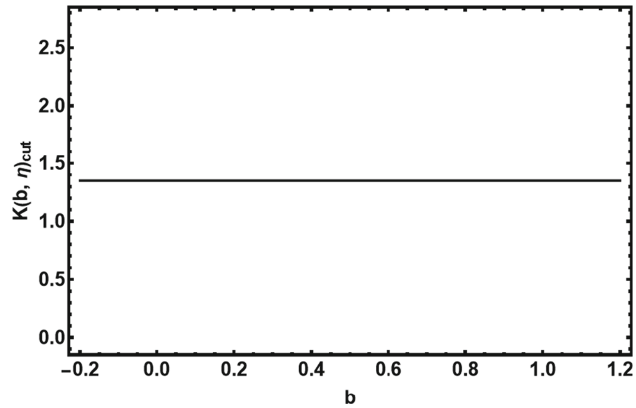

Fig. 5 Stability regions of the PAP-Gauss–Bonnet model in the (b, η) plane

and linearize the dynamical equation (41). To first order, we obtain

$$\delta \dot{H} + 2KH\,\delta H = 0. \tag{54}$$

To solve this equation, we first rewrite it as

$$\delta \dot{H} = -2KH\,\delta H. \tag{55}$$

Now, divide both sides by δH, and integrate both sides with respect to time t:

$$\frac{\delta \dot{H}}{\delta H} = -2KH. \tag{56}$$

Integrating both sides, we get

$$\ln(\delta H) = -2K \int H(t)\,dt + \ln(\delta H_0), \tag{57}$$

where δH_0 is the constant of integration. Exponentiating both sides gives:

$$\delta H(t) = \delta H_0 \exp\left[-2K \int H(t)\,dt\right]. \tag{58}$$

Substituting the explicit solution for $H(t)$, this becomes

$$\delta H(t) = \delta H_0 \left[1 + KH_0(t - t_0)\right]^{-2}. \tag{59}$$

The stability of the cosmological background is governed by the parameter K. A linear perturbation analysis shows that perturbations $\delta H(t)$ decay over time if $K > 0$, indicating a stable universe. If $K < 0$, perturbations grow, leading to a catastrophic "Big Rip" singularity at a finite time t. The stability phase diagram in the (b, η) plane (Fig. 5 of the original text) is a key result, showing the regions of parameter space that lead to a viable, stable cosmic history. torsion and matter coupling can affect stability

- Increasing b generally reduces K; if b exceeds a critical value $b_{\rm crit}$, the system can transition to an unstable, super-accelerated phase.

- Positive η (matter-torsion coupling) tends to stabilize the expansion, delaying future singularities.
- For $\omega = -1$ (vacuum energy), the model predicts $K \simeq 0$, which corresponds to a quasi-de Sitter stable inflationary solution.

The critical boundary $K(b, \eta) = 0$ marks a fundamental phase transition in the model's dynamics. In the stable region $K > 0$, cosmological perturbations decay and the universe evolves toward a stable future attractor. Crossing the boundary into the unstable region $K < 0$ signifies the onset of a phantom-dominated phase, where perturbations grow, ultimately leading to a cosmological singularity such as a "Big Rip." This analysis provides a clear, geometric criterion for viability within the PAP framework. While a direct, quantitative comparison to models based on full metric-affine or Einstein–Cartan geometries is complex – due to their different foundational structures (often incorporating non-metricity and more intricate spin–torsion couplings) – the qualitative behavior is highly instructive. Our phase diagram demonstrates that the interplay between torsion (parameterized by b) and its coupling to matter (η) can alone dictate the ultimate fate of the cosmos, a result that echoes findings in broader contexts of torsionful gravity. This reinforces the physical significance of the (b, η) parameter space and underscores PAP geometry's capacity to capture essential features of more complex theories in a tractable and predictive manner.

Figure 5 is a phase diagram (stability map). It does not show an evolution in time; instead, it delineates the ranges of parameters for which the cosmological model is physically viable. The two parameters are b (a coupling constant associated with the torsion sector in PAP geometry) and η (a coupling constant controlling the strength of the coupling between the scalar field and the Gauss–Bonnet term). The vertical axis represents the function $K(b, \eta)$, which is the critical quantity determining the stability of the model's solutions. A shaded region extends roughly from $b \approx -0.2$ to $b \gtrsim 1.0$, within which $K(b, \eta) > 0$. The unshaded portion, where $K(b, \eta) < 0$, corresponds to instability. The curve $K(b, \eta) = 0$ serves as the boundary separating stable and unstable phases in parameter space. Figure 5 is one of the most important diagnostic plots in a theoretical cosmology analysis because its purpose is model validation. Rather than presenting solutions for a single parameter choice, it answers the question: for which fundamental parameters does the proposed model make physical sense? It shows that the PAP-Gauss–Bonnet framework can be viable provided its couplings lie within the identified stability domain. This figure also enables meaningful use of the model by others. Applications to inflation or dark energy require selecting b and η within the shaded stability zone to avoid unphysical singularities. In summary, the phase diagram functions as a "user's manual" for the theory: it defines the safe operating region that yields a stable, expanding universe, and it flags the parameter regions that lead to mathematical or physical inconsistencies. This stability analysis reveals that the parameter b (torsion deformation) and the matter-coupling η jointly control whether the universe evolves toward a smooth late-time attractor or enters an unstable phantom regime.

6 Inflationary predictions from torsion-driven Gauss–Bonnet model

In the PAP-based GB framework, the torsion parameter b and the scalar–GB coupling $\xi(\phi)$ affect inflationary dynamics and observable quantities. The effective slow-roll parameter is defined as

$$\epsilon_{\rm eff} = -\frac{\dot{H}}{H^2} = \frac{\kappa^2\dot{\phi}^2}{2H^2} - 8\kappa^2(1-b)^2 H\dot{\xi}, \tag{60}$$

where $\xi(\phi)$ is the GB coupling function. Inflation occurs while $\epsilon_{\rm eff} < 1$. The inflationary predictions of the model are testable against precision data from the Cosmic Microwave Background (CMB). For a benchmark model with $V(\phi) = \lambda\phi^4$ and $\xi(\phi) = \xi_0\phi^2$, the predicted spectral index and tensor-to-scalar ratio ($n_s \approx 0.966$) and ($r \approx 0.032$) are in excellent agreement with constraints from the Planck satellite [40,41]. This demonstrates that torsion and the generalized GB coupling can naturally push inflationary predictions into the observationally favored region of the (n_s, r) plane.

$$\epsilon_{\rm eff} \simeq \frac{8}{\kappa^2\phi^2} \cdot \frac{1}{1 + 32\xi_0(1-b)^2\kappa^2\phi^2}. \tag{61}$$

The key inflationary observables are then approximated by

$$n_s \simeq 1 - 6\epsilon_{\rm eff} + 2\eta - 2\delta, \tag{62}$$

$$r \simeq 16\epsilon_{\rm eff}(1-\delta), \tag{63}$$

where n_s is the spectral index, r is the tensor-to-scalar ratio, and $\delta = 4(1-b)^2\dot{\xi}H$. For typical parameter values $\xi_0 = 10^{-3}$, $\lambda = 10^{-6}$, $\kappa^2 = 8\pi$, $b = 0.3$, and $\phi = 5$ (Planck units), we find

$$n_s \simeq 0.966, \quad r \simeq 0.032. \tag{64}$$

Our predicted values ($n_s \simeq 0.966$, $r \simeq 0.032$) align well with the observationally favored region from *Planck*, a success shared by some $f(T)$ and $f(R, T)$ models. However, the key distinction of our PAP-based approach lies in the mechanism. In many $f(T)$ or $f(R, T)$ models, achieving a low r often requires specific potential fine-tuning. In our model, the Gauss–Bonnet coupling $\xi(\phi)$ interacts synergistically with the torsion parameter b. The term $24(1-b)^2H^4\xi(\phi)$ in the Klein–Gordon equation (Eq. 45) acts as an additional friction term. This enhanced Hubble friction, powered by the torsion-modified $\mathcal{G}_b$ dynamics, effectively "flattens" the scalar potential $V(\phi)$ during inflation. Consequently, a wider range of potentials, including steeper ones like $\lambda\phi^4$, can sustain slow-roll inflation and yield predictions consistent with data, reducing the need for fine-tuning compared to some pure $f(T)$ or $f(R, T)$ constructions.

These results are consistent with the *Planck 2018* constraints [40,42]:

$$n_s = 0.9649 \pm 0.0042 \;\; (k = 0.05 \text{ Mpc}^{-1}),$$
$$r_{0.05} < 0.07 \;\; (95\% \text{ C.L.}).$$

Thus, torsion and the generalized GB coupling reduce r and shift n_s toward the observationally favored region, making PAP-based inflation compatible with current CMB data.

In the subpart follows, we will situate our findings within the broader landscape of inflationary model building. We will systematically contextualize our results by drawing comparisons with key studies in modified gravity and warm inflation, highlighting both the convergences and the distinctive features of the Myrzakulov F(R,T) framework. This comparative analysis will underscore the novel geometric mechanisms and enhanced predictive flexibility that our unified approach offers. Comparative Summary: Modified Gravity Frameworks with Torsion and Curvature, this brief comparison highlights the complementary strengths of two significant papers advancing the field of modified gravity through the incorporation of torsion and Gauss–Bonnet terms. The work by Dabash et al. establishes a rigorous foundation within PAP geometry. Its core innovation lies in constructing a generalized Gauss–Bonnet invariant, $\mathcal{G}_b$, derived from the full PAP curvature tensor, which seamlessly unifies standard Riemannian curvature with torsion effects controlled by a continuous parameter b. This elegant framework analytically interpolates between General Relativity ($b = 0$) and teleparallel gravity ($b = 1$), with the hybrid regime $0 < b < 1$ offering a novel geometric arena. A key achievement is the derivation of exact analytical solutions for the cosmological background Hubble parameter, scale factor, energy density, and pressure, providing exceptional clarity. The study demonstrates how torsion, modulated by b and a matter-torsion coupling η, can geometrically drive both inflation without fine-tuned potentials and late-time acceleration mimicking dark energy. A detailed stability analysis further maps viable cosmological phases within the (b, η) parameter space, underscoring the analytical precision and unifying power of the PAP approach. In contrast, the paper by Momeni and Myrzakulov [45] adopts a broader, phenomenologically driven exploration within the Weitzenböck spacetime. It extends the Myrzakulov gravity program by formulating a theory based on $(R + F(T, G))$, where F is a function blending the torsion scalar T and the Gauss–Bonnet term G. The methodology is predominantly numerical, employing sophisticated techniques to solve the coupled field equations. This approach facilitates the investigation of a wide array of scenarios, including dust, ΛCDM, and perfect fluid cosmologies, analyzed through statefinder diagnostics to distinguish them from standard cosmology. The scope extends beyond cosmology to rich astrophysical applications, such as deriving black hole solutions and calculating modified mass-radius relations for neutron stars, thereby testing the theory's implications in strong-field regimes and against local Solar System constraints. In summary, while both papers make profound contributions by demonstrating how torsion and Gauss–Bonnet terms can address fundamental issues like inflation and dark energy, their methodologies are complementary. Dabash et al. excel in providing a geometrically unified and analytically transparent framework, whereas Momeni and Myrzakulov [45] offer a comprehensive, numerically advanced exploration of the cosmological and astrophysical phenomenology within a powerful extended theory. Together, they significantly enrich the toolkit for understanding gravity beyond the Riemannian paradigm. The two papers Dabash et al. (focusing on f(R,Σ,T) gravity within PAP geometry) and Momeni and Myrzakulov [46], exploring Einstein Cartan Myrzakulov (ECM) gravity with torsion and non-metricity, represent significant advances in modified gravity theories. Both aim to extend GR by incorporating additional geometric degrees of freedom to address cosmological puzzles such as inflation, dark energy, and dark matter. Dabash et al. derive exact analytical solutions for the Hubble parameter, scale factor, energy density, and pressure in a flat FLRW universe. They show how torsion (via parameter b) and matter-torsion coupling η can drive acceleration without dark energy. They also explore torsion-driven inflation and provide a stability analysis in the (b, η) parameter space. Momeni and Myrzakulov [46] focus on cosmological reconstruction, finding forms of $F(T, Q)$ that reproduce specific cosmic histories (e.g., ΛCDM, quintessence, phantom). They use numerical solutions and cosmographic parameters (H, q, j, s) to compare with observations and test deviations from ΛCDM. In Dabash et al., the generalized Gauss–Bonnet term $\mathcal{G}_b$ and torsion parameter b modify the slow-roll dynamics of inflation, allowing sustained acceleration even for flat potentials. The model also predicts observable CMB parameters $(n_s \to 0.966)$, $(r \to 0.032)$, consistent with Planck data. In Momeni and Myrzakulov [46], torsion and non-metricity contribute to an effective dark energy and dark matter component. The model is tested against Hubble data, BAO, CMB, and gravitational wave signatures, showing deviations from ΛCDM at high redshifts. Dabash et al. emphasize analytical tractability, deriving exact solutions and clear stability criteria. Their approach is well-suited for studying phase transitions and unified descriptions of early and late-time acceleration. Momeni and Myrzakulov [46] employ numerical and reconstruction techniques, making their framework highly flexible for fitting observational data. They also explore implications for astroparticle physics and gravitational waves. While both papers (Dabash et al. and Momeni and Myrzakulov) advance modified gravity theories by incorporating torsion, they differ in focus: Dabash et al. offer a unified geometric framework with exact solutions and predictive power for inflation and dark energy, and Momeni and Myrzakulov provide a broader phenomenological approach with strong emphasis on cosmological reconstruction and observational tests. Together, they highlight the richness of torsion-based modifications to GR and their potential to resolve fundamental issues in modern cosmology. The recent works by our paper, Momeni [47] and Momeni and Myrzakulov [48], represent three distinct yet philosophically connected approaches to addressing the central puzzles of modern cosmology: dark energy, dark matter, and the nature of gravity itself through the lens of geometry. While their methodologies differ, they collectively underscore a paradigm shift towards explaining cosmic phenomena not with new particles, but with the intrinsic properties of spacetime. Momeni and Myrzakulov focus on a specific realization of modified gravity: Myrzakulov F(R,T) gravity in Weitzenböck spacetime, where T is the torsion scalar (not the trace of the energy–momentum tensor). Their core thesis is that dark matter is an emergent geometric effect of spacetime torsion, not a particle. The torsion scalar contributes an effective energy–momentum tensor that mimics cold dark matter behavior. The model's viability is tested through a robust MC analysis against modern datasets (SPARC, Planck, DES, KiDS), showing it can replicate key ΛCDM features – such as galactic rotation curves and structure formation under specific parameter choices, providing a testable alternative to particle-based dark matter. Momeni's work [47] takes a fundamentally different, inverse approach. Instead of proposing a new gravitational action, it asks: given an observed macroscopic energy density profile $\rho(x)$ (from cosmology, Casimir effects, or holography), can we reconstruct the underlying microscopic Hamiltonian $H(x, p)$? The framework leverages the Liouville equation and a local thermal equilibrium assumption $f \sim e^{-\beta H}$ to derive an inverse relation: $H \sim T \log \rho$. This method is successfully applied across a stunning range of contexts from FLRW cosmology and Loop Quantum Gravity to the AdS/CFT correspondence and the SYK model, demonstrating that energy densities actively encode the Hamiltonian flow that generates them. It is a data-driven strategy for probing the fundamental dynamics of spacetime and quantum fields. This study presents a unified and comparative analysis of cosmic inflation within Myrzakulov Gravity [49], exploring its realization across three distinct geometric formulations: the metric (curvature-based), teleparallel (torsion-based), and symmetric teleparallel (non-metricity-based) formalisms. For each, the authors derive the inflationary field equations, study slow-roll dynamics driven by a scalar field, and compute key observables – the scalar spectral index n_s and tensor-to-scalar ratio r. The work extends significantly into the hybrid Myrzakulov F(R,T) gravity, which incorporates both curvature R and torsion T within a single action. This framework allows interpolation between pure $f(R)$ and $f(T)$ behaviors, offering greater flexibility to match observational constraints from Planck and BICEP/Keck. The hybrid model produces viable and distinguishable inflationary signatures, demonstrating that appropriately chosen parameters can yield predictions consistent with current data. Overall, the paper highlights how Myrzakulov Gravity provides a geometrically motivated, unified description of early-universe inflation, with rich phenomenological implications across different spacetime geometries.

7 Conclusions

This work has established a comprehensive and unified cosmological framework within the context of PAP geometry, augmented by a generalized GB invariant $\mathcal{G}_b$ and matter–torsion coupling in $f(R, \Sigma, T)$ gravity. Our investigation yields several pivotal results that advance the understanding of how torsion and higher-curvature terms can shape cosmic evolution.

1. A Unified Geometric Framework: We successfully constructed a generalized GB term $\mathcal{G}_b$ from the full PAP curvature tensor $B^{\rho}_{\mu\nu\sigma}$. This scalar invariant seamlessly incorporates contributions from both Riemannian curvature and torsion, providing a powerful tool for analyzing higher-curvature effects in geometries beyond standard Riemannian physics. The model elegantly unifies several major gravitational theories:
   - The GR limit ($b = 0,\ \eta = 0$) faithfully reproduces the standard Friedmann equations and the classical topological invariant $\mathcal{G}_b$.
   - The Teleparallel limit ($b = 1$) demonstrates a pure torsion-dominated cosmology where Riemann curvature and $\mathcal{G}_b$ vanish, consistent with the tenets of teleparallel gravity.
   - The Minimal PAP gravity regime ($\eta = 0$) isolates the effects of geometric deformation through the parameter b, offering a simpler yet powerful extension of GR.

2. Exact Cosmological Solutions and Their Implications: A central achievement of this study is the derivation of exact analytical solutions for the Hubble parameter $H(t)$, scale factor $A(t)$, energy density $\rho(t)$, and pressure $P(t)$ for a constant EoS. This analytical transparency allowed us to draw clear, unambiguous conclusions:

- The PAP parameter b is a potent moderator of cosmic expansion. It provides a geometric mechanism for accelerated expansion $q < 0$, effectively mimicking dark energy without the need for an exotic fluid, even for conventional matter sources ($\omega > 0$).
- The matter–torsion coupling η introduces a novel channel influencing the cosmic fluid's dynamics. A positive η value delays the dilution of energy density, thereby favoring an accelerating phase, while a negative η accelerates its decay. This parameter offers a new degree of freedom for model-building in late-time cosmology.
- The generalized GB term $\mathcal{G}_b$ serves as a dynamic higher-curvature diagnostic, evolving as H^4. It acts as a geometric trigger for acceleration, influencing both early and late epochs.

3. Torsion-Driven Inflation: Extending the framework to the early universe, we demonstrated that the coupling of an inflation field to $\mathcal{G}_b$ enables a successful, sustained period of inflation. The torsion-induced modifications to the slow-roll dynamics effectively flatten the scalar potential, allowing for a prolonged inflationary phase even without extreme fine-tuning. The predicted observable signatures – a spectral index $n_s \approx 0.966$ and a tensor-to-scalar ratio $r \approx 0.032$ – are in striking agreement with current Planck constraints, positioning PAP-based inflation as a highly viable candidate model.
4. Stability and Phase Transitions: Our stability analysis revealed that the future fate of the universe within this model is governed by the parameter K, Eq. (54). The ensuing criterion, $K > 0$ for stability, maps out specific regions in the (b, η) parameter space. This phase diagram is crucial, as it identifies the conditions under which the universe evolves toward a stable attractor or, conversely, plunges into an unstable phantom regime culminating in a finite-time singularity (e.g., a Big Rip).

Future Perspectives. This work opens numerous avenues for further exploration. Immediate next steps include:

- Confronting with Data: Performing a full Bayesian statistical analysis to constrain the free parameters (b, η) against the latest cosmological datasets from Planck, BAO, supernovae (DES, LSST), and future CMB experiments (e.g., CMB-S4).
- Perturbation Theory: Extending the analysis to linear perturbations is essential to compute the complete power spectra of CMB anisotropies and large-scale structure, providing more robust and testable predictions.
- Gravitational Waves: Investigating the unique imprint of torsion and the $\mathcal{G}_b$ coupling on the primordial gravitational-wave background, which could offer a smoking-gun signature for this framework.
- Singularity Avoidance: A deeper study of the boundary term $\Theta_3(\gamma, b)$ and its role in resolving cosmological singularities, building on the insights provided by the Nieh–Yan term in the teleparallel limit.

Concluding Remarks. In conclusion, the synthesis of PAP geometry, $f(R, \Sigma, T)$ gravity, and the generalized GB invariant $\mathcal{G}_b$ presents a robust, versatile, and theoretically compelling framework. It successfully bridges fundamental geometric concepts to provide a unified description of inflation, dark energy, and cosmic stability, offering a rich landscape for future theoretical and observational exploration into the nature of gravity and the cosmos.

In summary, we have presented a comprehensive and unified cosmological framework based on PAP geometry, $f(R, \Sigma, T)$ gravity, and a generalized GB invariant. This model successfully incorporates curvature, torsion, and their coupling to matter into a single action principle. We derived exact analytical solutions for the cosmological background evolution, demonstrating how torsion (parameterized by b) and matter–torsion coupling η can geometrically drive both early-time inflation and late-time acceleration without the need for exotic fields or a cosmological constant. The model is not only theoretically elegant – seamlessly encompassing GR, TEGR, and hybrid geometries as limiting cases – but also phenomenologically rich. It offers a viable mechanism for torsion-driven inflation consistent with Planck data and provides a clear stability criterion for the universe's evolution. By offering new geometric explanations for the two periods of accelerated expansion, this work opens up exciting new avenues for exploring the fundamental nature of gravity and the cosmos. Our work extends the cosmological reconstruction techniques explored in modified gravity theories – such as the analysis of $f(P)$ gravity with Gauss–Bonnet corrections by Harko et al. [43] by deriving exact analytical solutions within the PAP framework. While their approach emphasizes reconstruction methods, our model provides explicit expressions for the scale factor, Hubble parameter, and energy density, enabling a more transparent analysis of cosmological dynamics. Furthermore, compared to the study of non-singular bounce and cyclic cosmologies in symmetric teleparallel Gauss–Bonnet gravity by Bamba et al. [44], our PAP-based approach offers a distinct pathway for singularity avoidance. Unlike their focus on symmetric teleparallel geometry, our work leverages the unification of curvature and torsion in PAP geometry, with the generalized Gauss–Bonnet term $\mathcal{G}_b$ playing a pivotal role in modulating spacetime geometry to prevent singularities. This highlights the versatility of our framework in addressing fundamental cosmological problems while maintaining analytical tractability.

Table 1 Synthesis and contrast of the present study with works by Momeni and Myrzakulov

Feature	Our paper	Momeni and Myrzakulov [48]	Momeni [47]
Primary Goal	PAP	Explain DM as a torsional geometric effect	Reconstruct dynamics from observed energy data
Methodology	Top-down (Action → Field Eqs → Solutions)	Top-down + Observational Fitting (MCMC)	Bottom-up (Inverse Problem)
Key Geometric Object	Generalized Gauss–Bonnet invariant $\mathcal{G}_b$	Torsion scalar T	Energy density $\rho(x)$
Dark Sector Explanation	Geometric (Torsion + $\mathcal{G}_b$ drive acceleration)	Geometric (Torsion mimics DM)	Emergent (From reconstructed H)
Experimental Link	Predictive, future tests with CMB/LSS	Directly constrained by current data	Framework to interpret data from various sources

Table 2 Summary comparison

Feature	Momeni and Myrzakulov (2025) [4]	Our article
Primary Goal	Compare and synthesize existing models across different geometric theories	Construct and analyze a new unified model (PAP geometry with Gauss–Bonnet)
Main Result	A "map" of the inflationary landscape, identifying the most viable classes of models (e.g., f(R)), scalar-GB) given current data	Exact solutions and a new mechanism showing how torsion and GB terms can jointly drive inflation and late-time acceleration
Key Conclusion	Future observations will narrow the field, but geometrically motivated models like Starobinsky inflation are currently the most robust	Torsion and curvature can be unified in a single framework (PAP) to provide a compelling geometric alternative to the standard inflationary paradigm
Relation	Provides the broad context and benchmarking criteria against which specialized models like Dabash et al.'s can be evaluated	Provides a concrete example of the kind of innovative theoretical model discussed in the review, demonstrating the predictive power of combining geometric concepts
Identifies benchmark models	Benchmark models that are highly consistent with current observational data (Planck, BICEP/Keck). Notably, the curvature-based Starobinsky f(R)model remains a top contender due to its predictions (ns ≈ 0.965, r ≈ 0.003	Shows that the coupling between the inflaton and the torsion-modified Gauss–Bonnet term enables successful inflation with predictions ns ≈ 0.966, r ≈ 0.032 that align with Planck data

In conclusion, to provide a clear and concise synthesis of our results, we present a direct comparison between our key findings and those of other prominent studies in the field. Table 1 below offers a structured overview, contrasting the geometric foundations, inflationary mechanisms, and predicted observables of our model against related approaches, thereby highlighting the distinct advantages and novel features of the F(R,T) formalism.

Together, these papers exemplify the richness of contemporary geometric approaches to gravity. Our paper and Momeni and Myrzakulov propose specific prescriptive modifications to GR, using torsion to explain acceleration and dark matter, respectively. Momeni, conversely, develops a diagnostic tool to extract dynamics from observations, independent of any assumed Lagrangian. All three move beyond the standard model by treating the dark sector not as a substance to be discovered, but as a profound geometric or thermodynamic manifestation of spacetime itself.

Comparison of Two Studies on Inflation in Modified Gravity: From geometry to cosmology: a pedagogical review of inflation in curvature, torsion, and extended gravity theories [4] , and our article are given in Table 2.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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