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Periodic Orbits of Quantised Restricted Three-Body Problem

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Abstract: In this paper, perturbed third-body motion is considered under quantum corrections to analyse the existence of periodic orbits. These orbits are studied through two approaches to identify the first (second) periodic-orbit types. The essential conditions are given in order to prove that the circular (elliptical) periodic orbits of the *rotating Kepler problem* (RKP) can continue to the perturbed motion of the third body under quantum corrections where a massive primary body has excessive gravitational force over the smaller primary body. The primaries moved around each other in circular (elliptical) orbits, and the mass ratio was assumed to be sufficiently small. We prove the existence of the two types of orbits by using the terminologies of Poincaré for quantised perturbed motion.

Keywords: periodic-orbit continuation; quantised three-body problem; first-kind periodic orbits; second-kind periodic orbits

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1. Introduction

Among the easiest nonintegrable mathematical systems in dynamical astronomy is the so-called *planer circular restricted three-body problem* (PCRTBP). This simple model has many applications in the study of stellar dynamics and the dynamics of motion inside the solar system. For example, this model was used for studying the motion of exoplanets surrounding one star or a binary star system in stellar systems [1]. In particular, it was used for the problem of studying habitability in exoplanets [2–4]. This model was also used in several space missions for analyse the motion of different spacecraft between two planets or in the Earth–Moon system; see [5,6].

Many authors studied the different perturbations of PCRTBP due to zonal harmonic terms, oblateness, radiation pressure; see, for instance, [7–17]. In particular, in [18,19], the authors showed that there are periodic orbits that are symmetric of PCRTBP in a synodic (or rotating) reference frame that can be analytically continued into ones of full three-body problems where the mass assigned to the third body is considered very small with respect to that of the other two bodies.

Following [20–22], here, we consider quantum perturbation in PCRTBP, which is called *the planar circular quantised restricted three-body problem* (PCQRTBP). In [23,24], Poincaré found the first (second) periodic-orbit type for PCRTBP. Our goal is to show that these periodic orbits also extend to PCQRTBP. The first and second kinds of periodic orbits (FKPO and SKPO) at the PCRTBP are related orbits to the circular (elliptic) periodic orbits of the RKP. In the perturbed circular type of three-body problems, the two larger bodies rotate in circular orbit about their own centre of mass, and on the infinitesimal body, the Newtonian gravitational forces of the two larger bodies act together with a small perturbing force.

The rotating frame is employed to study PCQRTBP, such that the primaries take fixed locations on the x axis [25–28].

One of the most considerable problems in the theory of modern physics is incompatibility between quantum mechanics and general relativity, which permits a perturbative appearance that is nonrenormalisable to the theory of quantum gravity. On the other hand, quantum mechanics and its offspring, effective field theory, give extremely effective descriptions of all usual nongravitational phenomena, of which the outcomes coincide between predictive and experimental observations [22,29]. Furthermore, the theory of general relativity is elegant and brilliant, and has had successful tests within the solar system [30–32]. Regardless of the lack of an overarching theoretical frame to include the success of both quantum and general relativity theories, the prediction of quantum theory can be used in nonrenormalisable theories by applying the effective field theory technique [33–37].

Periodic orbits play a considerable role in exploring the nonlinear behaviour of dynamical systems. In mathematical physics, celestial mechanics, quantum mechanics, and in most branches of the mathematical sciences, particularly in dynamical system analysis, the orbit of motion is composed of a point set connected by the system evaluation function. This can be explored as the enveloped phase space subset via the system path in the framework of a specified set from initial conditions through the system's evolution. As the path of the phase space could perform unique estimations or calculations for any proposed set of phase space variables, the intersection of different orbit sets in the phase space is impossible. Thus, the orbital set of any dynamical system is a partition for the phase space. To explore the structures of periodic orbits by utilising topological techniques is one of the objectives of modern dynamical systems theory [38].

There are two different approaches to show that the circular orbits of the RKP are prolongated to FKPO and SKPO in the classical three-body problem. The first approach is based on Poincaré variables [39] and the associated proof for Theorem 2 of the present work. The second approach deals with the related multipliers for the circular periodic orbits (CPOs) [40] or the proof of Theorem 1. In the two proofs, we used the fact that the PCQRTBP had a first integral; then, the FKPO could be prolongated from the classical problem of three bodies to the perturbed one via quantum correction. However, the proofs of the two theorems are not found when the first integral is absent, and the prolongation remains an open problem for more investigations and studies. Lastly, we studied Theorem 2, the prolongation of the SKPO from the classical problem of three bodies to the perturbed one via quantum corrections.

In this work, we studied the prolongation of the so-called first (second) periodic-orbit type FKPO and SKPO to the perturbed three-body problem via quantum corrections. The existence of the two types of orbits is proven by using the terminologies of Poincaré for quantised perturbed motion when the mass ratio is small enough, and the perturbed third body has infinitesimal mass, but the primary bodies move about each other in circular (elliptical) orbits. In the literature, there are many papers studying the periodic orbits of other classes of the restricted three-body problem; see, for instance, [41–47].

2. Model Description

2.1. Equations of Motion for PCQRTBP

Let m_1, m_2 be the masses of the two primaries bodies, and m the mass of the infinitesimal body with its own relative positional vectors with respect to primaries m_1 and m_2 be $\boldsymbol{\rho}_1 = (\xi_1, \eta_1, \zeta_1)$, $\boldsymbol{\rho}_2 = (\xi_2, \eta_2, \zeta_2)$ and it positional vector with respect to the origin of inertial reference frame $O\xi\eta\zeta$ is $\boldsymbol{\rho} = (\xi, \eta, \zeta)$. We assumed that the infinitesimal body did not affect the motion of the primaries and was moving under their mutual gravitational forces where all three bodies move in the same plane. Thus, the vectorial motion equation of this body is given as follows:

$$m\ddot{\boldsymbol{\rho}} = \nabla\Gamma_I \tag{1}$$

where Γ_I is the potential function that takes the following form:

$$\Gamma_I(\rho_1, \rho_2) = m\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right) \tag{2}$$

The relative distances of the infinitesimal body are

$$\begin{aligned} \rho_1^2 &= (\xi - \xi_1)^2 + (\eta - \eta_1)^2 + (\zeta - \zeta_1)^2 \\ \rho_2^2 &= (\xi - \xi_2)^2 + (\eta - \eta_2)^2 + (\zeta - \zeta_2)^2 \\ \rho^2 &= \xi^2 + \eta^2 + \zeta^2 \end{aligned} \tag{3}$$

In this work, the restricted motion of the three bodies is the proposed model in order to analyse the periodic orbits of the perturbed motion of the third body, which had infinitesimal mass compared with the masses of the other two bodies, namely, the so-called primaries that move around each other in CPO. Furthermore, the motion of the third body occurs in the same plane as that of the two primaries.

Now, we consider nondimensional rotating frame $OXYZ$ that moves about the Z axis with angular velocity ω. However, to normalise the variable and use dimensionless coordinates, we also assumed that the separation primary distance, the total of their masses, and the constant of universal gravitation had a unity value, while the perturbed mean motion was equal to ω; see [20] for details. In this description, parameter $m_2 = \mu_2 = \mu$ was also assigned to the smaller primary body mass; hence, the mass of the massive body was equal to $m_1 = \mu_1 = 1 - \mu$, where $\mu = m_2/(m_1 + m_2) \in (0, 1/2)$. Then, in synodical coordinates, the position of the large body is $(x_1, y_1) = (-\mu_2, 0)$ and of the small one is $(x_2, y_2) = (\mu_1, 0)$, while the infinitesimal body occurs at (x, y).

From the pervious description, Equations (1)–(3) in rotating frame are as follows:

$$m[\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega} \wedge \dot{\boldsymbol{\rho}} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \wedge \boldsymbol{\rho}] = \nabla\Gamma_R \tag{4}$$

$$\begin{aligned} \rho_1^2 &= (x + \mu)^2 + y^2 + z^2 \\ \rho_2^2 &= (x + \mu - 1)^2 + y^2 + z^2 \\ \rho^2 &= x^2 + y^2 + z^2 \end{aligned} \tag{5}$$

Γ_I is the total Newtonian gravitational potential exerted by m_1 and m_2. The potential under quantum corrections is

$$\Gamma_R = \Gamma_I + \Upsilon_1 + \Upsilon_2, \tag{6}$$

where Υ_1 and Υ_2 are the related quantum corrections to bodies m_1 and m_2, respectively, which are given in [21,22]:

$$\begin{aligned} \Upsilon_1 &= m m_1\left(\frac{Q_{11}}{\rho_1} + \frac{Q_{12}}{\rho_1^2}\right) \\ \Upsilon_2 &= m m_2\left(\frac{Q_{21}}{\rho_2} + \frac{Q_{22}}{\rho_2^2}\right) \end{aligned} \tag{7}$$

where

$$\begin{aligned} Q_{11} &= k_1(R_{m_1} + R_m) \\ Q_{12} &= Q_{22} = k_2(l_p)^2 \\ Q_{21} &= k_3(R_{m_2} + R_m) \end{aligned} \tag{8}$$

where R_{m_1}, R_{m_2}, and R_m are the radius lengths of gravitational masses m_1, m_2, and m, respectively, while n_1, n_2, k_1, k_2, and k_3 are dimensionless amounts that can be computed with Feynman–Diagram analysis, and ℓ denotes the Planck length. Thus, for the values of n_1 and n_2 in [22], $n_1 = -1$ and $n_2 = -127/(30\pi^2)$ were obtained, while in [21,29], they were corrected to $n_1 = 3$ and $n_2 = 41/(10\pi)$. c denotes the speed of light.

Furthermore, quantities Q_{11} and Q_{21} represent the effect of the post-Newtonian approximation; there are comprehensive studies on this effect or the so-called general relativistic effect in the framework of the restricted three-body problem [30,48,49]. Constants Q_{12} and Q_{22}, on the other hand, describe a quantum correction contribution.

In the framework of quantum corrections using Equations (4)–(8), the equations of motion of the infinitesimal body body in the rotating reference frame are as follows:

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} &= \Pi_x(x, y, z) \\ \ddot{y} + 2\omega\dot{x} &= \Pi_y(x, y, z) \\ \ddot{z} &= \Pi_z(x, y, z), \end{aligned} \tag{9}$$

where Π is the effective potential within the framework of quantum corrections that can be written as follows:

$$\Pi(x, y, z) = \frac{1}{2}\omega^2\left(x^2 + y^2\right) + \frac{\mu_1}{\rho_1}\left(1 + \frac{Q_{11}}{\rho_1} + \frac{Q_{12}}{\rho_1^2}\right) + \frac{\mu_2}{\rho_2}\left(1 + \frac{Q_{21}}{\rho_2} + \frac{Q_{22}}{\rho_2^2}\right), \tag{10}$$

Functions Π_x, Π_y, and Π_z are the first partial derivatives of potential function $\Pi(x, y, z)$ with respect to coordinate variables x, y, and z, respectively. Using Equations (9) and (10), the Jacobian integral is given as follows:

$$2\Pi(x, y, y) - \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = C_J. \tag{11}$$

Relation (11) can be used to identify the possible regions of motion, where C_J is the constant of integration, Jacobi constant, or the so-called first integral of motion.

2.2. Hamiltonian of PCQRTBP

From [40,50], we have that the Hamiltonian equations of the classical PCRTBP in synodical coordinates are given by the following Hamiltonian:

$$\varkappa = \varkappa(x, y, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + yp_1 - xp_2 - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}, \tag{12}$$

where (p_1, p_2) is the conjugate momentum of the infinitesimal body localised at (x, y), and distances among the third body and the primaries are identified with $\rho_1^2 = (x + \mu_2)^2 + y^2$ and $\rho_2^2 = (x - \mu_1)^2 + y^2$.

From Equations (9) and (10), the Hamiltonian of the PCQRTBP in rotating frame is controlled by

$$\varkappa = \frac{1}{2}(p_1^2 + p_2^2) + \omega^2(yp_1 - xp_1) - \frac{1-\mu}{\rho_1}\left(1 + \frac{Q_{11}}{\rho_1} + \frac{Q_{12}}{\rho_1^2}\right) - \frac{\mu}{\rho_2}\left(1 + \frac{Q_{21}}{\rho_2} + \frac{Q_{22}}{\rho_2^2}\right), \tag{13}$$

where

$$\begin{aligned} \omega^2 = &\quad 1 + 2n_1(R_{m_1} + R_{m_2}) + 3n_2\ell^2 = 1 + O(1/c^2), \\ Q_{11} = &\quad k_1(R_{m_1} + R_m) = O(1/c^2), \\ Q_{12} = &\quad Q_{22} = k_2\ell^2 = O(1/c^3), \\ Q_{21} = &\quad k_3(R_{m_2} + R_m) = O(1/c^2), \end{aligned}$$

3. Periodic Orbits of First Kind

Well-known techniques for analysing PCRTBP can be applied only to approximately determine the trajectory of the infinitesimal body when the massive primary body has excessive gravitational force over the smaller primary body. In planetary theory, the classical orbit of motion must be considered as a reference to identify perturbed body motions where the entire parameters of motion are well-known.

The variational path may be characterised as a disfigurement of the CPO of the smaller body due to the attractive force of the bigger body, and this is a proposed periodic solution for the perturbed problem (PCQRTBP). Poincaré stated that there are three different kinds of periodic solutions. In the first (FKPO), the eccentricity is very small without any appearance of inclination. In the second (FKPO), the eccentricity is finite and also without any appearance of inclination. In the third, the inclination has a real appearance. In this context, the FKPO arise when the perturbed third body has infinitesimal mass, the primary bodies move about each other in CPO, and the mass ratio is sufficiently small.

If we denote $\delta = 1/c^2$, Hamiltonian (13) can be written as follows:

$$\varkappa = \frac{1}{2}(p_1^2 + p_2^2) + yp_1 - xp_2 - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} + O(\delta). \tag{14}$$

Moreover, if mass μ of the second primary is sufficiently small, Hamiltonian (14) becomes

$$\varkappa = \frac{1}{2}(p_1^2 + p_2^2) + yp_2 - xp_2 - \frac{1}{\sqrt{x^2+y^2}} + O(\delta, \mu). \tag{15}$$

We must take care with $O(\delta, \mu)$, because it includes some terms that go to infinity in the proximity of the primary bodies; so, we must exclude a neighbourhood of the primaries. Hamiltonian (15) also coincides with the Hamiltonian of RKP when μ and δ tend to zero.

To explore the existence of the circular (elliptic) periodic orbits of the well-known RKP to the first (second) periodic-orbit type of the PCQRTBP, we can follow the classical approach below. We performed a change in a polar coordinate variable with the corresponding momentum as follows: $(x, y, \dot{x}, \dot{y}) \to (\varrho, \vartheta, p_\varrho, p_\vartheta)$ where

$$x = \varrho\cos\vartheta, \quad y = \varrho\sin\vartheta, \quad p_1 = p_\varrho\cos\vartheta - \frac{p_\vartheta}{\varrho}\sin\vartheta, \quad p_2 = p_\varrho\sin\vartheta + \frac{p_\vartheta}{\varrho}\cos\vartheta,$$

and Hamiltonian (15) is written as follows:

$$H(r, \vartheta, p_\varrho, p_\vartheta) = \frac{1}{2}\left(p_\varrho^2 + \frac{p_\vartheta^2}{\varrho^2}\right) - p_\vartheta - \frac{1}{\varrho} + O(\mu, \delta). \tag{16}$$

Using Hamiltonian Relation (16), the Hamiltonian equations of motion are governed by the following form:

$$\begin{aligned} \dot{\varrho} &= \mathcal{Q}_1(\varrho, \vartheta, p_\varrho, p_\vartheta) = p_\varrho, \\ \dot{\vartheta} &= \mathcal{Q}_2(\varrho, \vartheta, p_\varrho, p_\vartheta) = \frac{p_\vartheta}{\varrho^2} - 1, \\ \dot{p_\varrho} &= \mathcal{Q}_3(\varrho, \vartheta, p_\varrho, p_\vartheta) = \frac{p_\vartheta^2}{\varrho^3} - \frac{1}{\varrho^2} + O(\mu, \delta), \\ \dot{p_\vartheta} &= \mathcal{Q}_4(\varrho, \vartheta, p_\varrho, p_\vartheta) = O(\mu, \delta). \end{aligned} \tag{17}$$

Now, we suppose that the periodic solutions of System (17) with a ϖ period can be represented by functions $\varrho(t)$, $\vartheta(t)$, $p_\varrho(t)$ and $p_\vartheta(t)$ when $\delta = \mu = 0$. The prolongation of these solutions represents a quaternion that is composed of four differentiable and continuous functions everywhere in their domains. Functions have smoothness properties $(\varrho(t,\delta), \vartheta(t,\delta), p_\varrho(t,\delta), p_\vartheta(t,\delta))$, $\varpi(\delta)$. When $\delta > 0$ in the proximity of zero where

$(\varrho(t,0), \vartheta(t,0), p_\varrho(t,0), p_\vartheta(t,0)) = (\varrho(t), \vartheta(t), p_\varrho(t), p_\vartheta(t))$, $\varpi(0) = \varpi$, and $(\varrho(t,\delta), \vartheta(t,\delta),$ $p_\varrho(t,\delta), p_\vartheta(t,\delta))$ represent periodic orbits for System (17), and the length of its period is ϖ.

In this setting, the variational equation linked to ϖ-periodic orbit $\sigma(t,\delta) = (\varrho(t,\delta),$ $\vartheta(t,\delta), p_\varrho(t,\delta), p_\vartheta(t,\delta))$ has the following form:

$$\dot{\mathcal{K}} = \left(\frac{\partial(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4)}{\partial(\varrho, \vartheta, p_\varrho, p_\vartheta)} \Big|_{(\varrho,\vartheta,p_\varrho,p_\vartheta)=\sigma(t,\delta)} \right) \mathcal{K}. \tag{18}$$

The matrix $\mathcal{K}$ is a 4×4 matrix. By $\partial(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4)/\partial(\varrho, \vartheta, p_\varrho, p_\vartheta)$ represents the Jacobian matrix of function $(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4)$ with respect to variables $(\varrho, \vartheta, p_\varrho, p_\vartheta)$. The associated fundamental (monodromy) matrix of the ϖ-periodic orbit $\sigma(t,\delta)$ supplied us with solution $\mathcal{K}(\varpi, \delta)$, of (18) such that the identity matrix was $\mathcal{K}(0,\delta)$. The characteristic roots for fundamental matrix $\mathcal{K}(\varpi, 0)$ related to periodic solution $\sigma(t,\delta)$ are called the *multipliers of periodic orbits* (MPOs).

Periodic orbit $\sigma(t,\delta)$ has $+1$ as the eigenvalue of its linked monodromy matrix $\mathcal{K}(\varpi,\delta)$ all the time, and $+1$ with multiplicity 2 in the case of having a system with a first integral. We removed such trivial multipliers in order to work with the nontrivial ones, i.e., different from $+1$. If there are nontrivial multipliers, then the periodic orbit is *elementary*. An elementary periodic solution for $\delta = 0$ can be prolonged to the case of $\delta > 0$ if it is small enough; see Proposition 9.1.1 of [51].

System (17) can be rewritten in the following form when $\delta = 0$:

$$\dot{\varrho} = p_\varrho, \quad \dot{\vartheta} = \frac{p_\vartheta}{\varrho^2} - 1, \quad \dot{p_\varrho} = \frac{p_\vartheta^2}{r^3} - \frac{1}{\varrho^2}, \quad \dot{p_\varrho} = 0.$$

It is clear that the first integral is represented by p_ϑ. Thereby, $p_\vartheta = \mathcal{C}$ for a specified value $\mathcal{C}^3 \neq 1$, and one obtains the so-called *circular periodic solution* (CPS) $\varrho = \mathcal{C}^2$, $p_\varrho = 0$ of period $|2\pi\mathcal{C}^3/(1-\mathcal{C}^3)|$. By computing the multipliers of CPS, we obtain $+1$ with multiplicity 2, and $\exp(\pm i2\pi/(1-\mathcal{C}^3)) \neq +1$ if $1/(1-\mathcal{C}^3)$ is not an integer. If the first integral of the perturbed system can be evaluated with Proposition 9.1.1 of [51], the obtained periodic solution can be prolongated, which leads to the following theorem:

Theorem 1. *If $\mathcal{C}^3 \neq 1$ and $1/(1-\mathcal{C}^3)$ do not belong to $\mathbb{Z}$, then periodic orbits of the circular type of RKP with angular momentum (AM) $\mathcal{C}$ can be prolongated to PCQRTBP.*

Now, we can deduce that, from Theorem 1, the CPO of the first type of the RKP can be continued to the quantised restricted three-body problem where the perturbed motion has its own first integral. However, this property can only be satisfied when the parameter mass ratio is small enough, and we must prevent the infinitesimal body from taking a location or pass through the position of the smaller primary for revoking the collision case.

4. Periodic Orbits of the Second Kind

The study of the periodic orbits of the second type in the classical problem of three bodies has a considerable literature in celestial mechanics. In [52,53], these orbits were found by using continuation analysis in the framework of Cartesian and Delaunay coordinates. In the previous section, we studied the periodic orbits of the first type. Here, we analyse the periodic orbits of the second type because they are a generalisation to the first type of periodic orbits and more interesting for the applications of real astronomical dynamical systems.

In the framework of multipliers are $+1$ for the elliptic type of periodic orbits for the RKP, along with the fact that they are not isolated at the energy level (see Proposition 8.5.2 of [51]). The method used previously to prolongate the circular periodic orbits from the RKP to the PCQRTBP does not work to extend elliptical periodic orbits (EPOs).

We used the so-called *Delaunay variables* $(L, G, Ł, \hbar)$ rather than polar coordinates $(\varrho, \vartheta, p_\varrho, p_\vartheta)$ via the following canonical transformation:

$$p_\varrho = \sqrt{-\frac{\hbar^2}{\varrho^2} + \frac{2}{\varrho} - \frac{1}{L^2}}, \quad p_\vartheta = \hbar, \quad L = \frac{t}{Ł^3}, \quad G = \vartheta - f,$$

t denotes the time variable. We obtained RKP when $\delta = 0$.

Delaunay variables are action-angle variables that are extremely important in the theory of perturbation because the equations of motion in using these variables are greatly convenient for applying numerical and asymptotic research techniques. These variables represent the action-angle variables of a Kepler problem; see [51,54]. Delaunay variables are related to parameters of Kepler elliptical motion, such that $Ł = \sqrt{a}$, where $\hbar = Ł\sqrt{1-e^2}$, L is the mean anomaly that is the angular distance that is measured from the pericentre, and g is the angle of the pericentre from the ascending node of the body to its periapsis and is measured in the motion direction.

In these new variables of the Delaunay type, Hamiltonian (16) takes the following form:

$$\varkappa(L, G, Ł, \hbar) = -\frac{1}{2Ł^2} - \hbar + O(\delta, \mu), \tag{19}$$

The corresponding Hamiltonian system is

$$\dot{L} = \frac{1}{Ł^3} + O(\delta, \mu), \quad \dot{Ł} = O(\delta, \mu), \quad \dot{G} = -1 + O(\delta, \mu), \quad \dot{\hbar} = O(\delta, \mu). \tag{20}$$

Delaunay coordinates only work in the vicinity of the phase space where there are elliptical periodic orbits of the rotating Kepler problem (EPORKP).

Birkhoff [55] observed that the Hamilton equations with Hamiltonian (12) are invariant due to variable transformation $(x, y, p_1, p_2, t) \to (x, -y, -p_1, p_2, -t)$. Then,if $(x(t), y(t), p_1(t), p_2(t))$ is an orbit of such a Hamiltonian system, $(x(-t), -y(-t), -p_1(-t), p_2(-t))$ is also an orbit of that system. In the event of the perpendicular crossing of the x axis from the orbits at the two locations of $t = 0$ and $t = \varpi/2$, the orbit is periodic with a ϖ period. So, a protocol to find periodic orbits is to search for orbits with two perpendicular crossings with the x axis. In [53], the authors applied ideas of Birkhoff for prolongating the EPORKP to the quantised problem (PCQRTBP). Hence, we examined if the proposed idea in [53] could be employed to extend the EPO from the RKP to the quantised problem.

There are two clear perpendicular intersections with the x axis in Delaunay's variables at the locations of

$$G(t) = i\pi, \quad L(t) = j\pi,$$

where i and j represent different integers. Here $g(t) = i\pi$ refers to the fact that the major axis of EPO coincides with the x axis, while $L(t) = j\pi$ indicates that the infinitesimal body occurs on the x axis at either the apoapsis or periapsis, where the motion of the third body is perpendicular to the x axis.

When $\delta = \mu = 0$, the motion of an elliptical orbit at initial time $t = 0$ is considered at apoapsis with a positive x axis. Thereby, one obtains

$$G(0) = -\pi, \qquad L(0) = \pi. \tag{21}$$

From Equation Set (20), the elliptical orbit of motion is determined with

$$\dot{L} = \frac{1}{Ł^3}, \quad \dot{Ł} = 0, \quad \dot{G} = -1, \quad \dot{\hbar} = 0. \tag{22}$$

Thus, the solutions of Equations (22) verifying the given conditions in (21) are

$$L(t) = \frac{t}{Ł_0^3} + \pi, \quad Ł(t) = Ł_0, \quad G(t) = -t - \pi, \quad \hbar(t) = \hbar_0, \tag{23}$$

where quantities $Ł_0$ and $\hbar_0$ are the integration constants.

Solution (23) represents an elliptical orbit with period $2\pi Ł_0^3$, and we considered the case of $2\pi Ł_0^3 = 2\pi\alpha/\beta$ with α and β coprime positive integers. Hence, we needed to continue Periodic Orbit (23) when $\delta > 0$, and mass ration $\mu > 0$ was small enough with a period $\varpi = 2\pi\alpha = 2\pi\beta Ł_0^3$. Therefore,

$$G(\varpi/2) = \mp(1+\alpha)\pi, \qquad l(\varpi/2) = (1+\beta)\pi,$$

and this orbit had two perpendicular intersections with the x axis. However, the motion at the small-body location $(1,0)$ had to be excluded to avoid collision cases.

In the context of $\delta > 0$ and $\mu > 0$ having small enough values, we searched for solutions of Equation System (20) with the following forms:

$$(L(t, Ł_0, \hbar_0; \delta, \mu), G(t, Ł_0, \hbar_0; \delta, \mu), Ł(t, Ł_0, \hbar_0; \delta, \mu), \hbar(t, Ł_0, \hbar_0; \delta, \mu)),$$

which had to verify the following initial conditions:

$$G(0, Ł_0, \hbar_0; \delta, \mu) = -\pi\,, \quad L(0, Ł_0, \hbar_0; \delta, \mu) = \pi,$$

such that

$$\begin{aligned} \varphi_1(t, Ł_0, \hbar_0; \delta, \mu) = &\quad G(t, Ł_0, \hbar_0; \delta, \mu) \pm (1+\alpha)\pi, \\ \varphi_2(t, Ł_0, \hbar_0; \delta, \mu) = &\quad L(t, Ł_0, \hbar_0; \delta, \mu) - (1+\beta)\pi, \end{aligned}$$

with $Ł_0$ near $(\alpha/\beta)^{1/3}$ and t near $\varpi/2$.

By applying implicit function theorem (IFT), we could find the required solutions if the following determinant value was not equal to zero, i.e.,

$$\det\left(\frac{\partial(\varphi_1, \varphi_2)}{\partial(t, Ł_0)}\right)\bigg|_{\delta=0} \neq 0,$$

on the elliptic periodic orbit. Since

$$\left|\frac{\partial(\varphi_1, \varphi_2)}{\partial(t, Ł_0)}\right|_{\delta=0} = \begin{vmatrix} \dfrac{\partial G}{\partial t} & \dfrac{\partial G}{\partial Ł_0} \\ \dfrac{\partial L}{\partial t} & \dfrac{\partial L}{\partial Ł_0} \end{vmatrix}_{\delta=0},$$

then, from (23), we obtain

$$\left|\frac{\partial(\varphi_1, \varphi_2)}{\partial(t, Ł_0)}\right|_{\delta=0} = \frac{3\varpi}{2Ł_0^4} \neq 0.$$

Thus, we obtain the next result:

Theorem 2. *Let α and β be different prime integers and $\varpi = 2\pi\alpha$. Then, the EPO with period ϖ for the RKP satisfying*

$$G(0) = -\pi\,, \qquad L(0) = \pi\,, \qquad Ł^3(0) = \alpha/\beta,$$

without crossing location $(1,0)$ can be continued to the PCQRTBP when $\delta > 0$ and $\mu > 0$ are small enough.

Our main results are proven through the proofs of Theorems 1 and 2. In the first theorem, the polar variables were used instead of the rectangular coordinates to explore the existence of the circular periodic orbits of the well-known RKP to the first periodic-orbit type of the quantised restricted three-body problem. In the second theorem, periodic orbits of the second type were constructed by using Delaunay's variables.The proofs of the two theorems were based on the mass ratio being small enough; in the first theorem, the motion at the location of the small body had to be excluded to avoid the collision case.

5. Conclusions

In the framework of the perturbed restricted three-body problem, the perturbed motion of the third body was considered under quantum corrections. Hence, the existence of periodic orbits was studied through two approaches to find the first (second) type of periodic orbits. The main results were stated through the proofs of Theorems 1 and 2. Thus, the essential conditions were given in order to prove that the CPORKP could be continued to the perturbed motion of the third body under quantum corrections when the massive primary body has excessive gravitational force over the smaller primary body. We showed that the first type of periodic orbits can arise when the perturbed third body has infinitely small mass, the primary bodies move about each other in circular orbit, and the mass ratio is sufficiently small. The terminologies of Poincaré were used to prove that the PCQRTBP had periodic orbits of first kind; see Theorem 1.

In the second approach, a canonical transformation using Delaunay's variables was used to study the existence of an EPO. This transformation enabled us to prove that the EPORKP could also be continued to the perturbed motion of the third body under quantum corrections by employing the same conditions as those of circular orbits; the mass ratio was very small, and the perturbed third body had infinitesimal mass, but the primary bodies moved about each other in elliptical orbit. Therefore, using the terminologies of Poincaré, the perturbed motion of the infinitesimal body could be continued with periodic orbits of the second type; see Theorem 2.

We conclude that the circular (elliptical) periodic orbits of the RKP coyld be continued to the perturbed motion of the third body via quantum corrections. The existence of the first (second) periodic-orbit type was proven for the quantised perturbed motion. The obtained results could be applied to the perturbed restricted three-body problem in stellar, solar, and planetary systems when the Newtonian potential of at least one body from the primaries is corrected with the quantum effect. Despite the first and second kinds of periodic orbits being able to be continued to classical and quantised motions when the mass ratio is very small, the Lagrangian point locations within the new pattern provide quantum corrections to the coordinates of Sun–Earth or Earth–Moon libration points [21,56]. In spite of the extreme smallness of the quantum corrections, there was no sign indicating that the qualitative characteristics of the restricted three-body problem in the framework of Newtonian potential remained unchanged. There is an extended study testing the effect of quantum-corrected gravity on solar and stellar systems [57].

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