Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 909 (2016) 584–606

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Super-renormalizable or finite Lee–Wick quantum gravity

Leonardo Modesto

Center for Field Theory and Particle Physics and Department of Physics, Fudan University, 200433 Shanghai, China

Received 17 May 2016; received in revised form 30 May 2016; accepted 2 June 2016

Available online 8 June 2016

Editor: Stephan Stieberger

Abstract

We propose a class of multidimensional higher derivative theories of gravity without extra real degrees of freedom besides the graviton field. The propagator shows up the usual real graviton pole in $k^2 = 0$ and extra complex conjugates poles that do not contribute to the absorptive part of the physical scattering amplitudes. Indeed, they may consistently be excluded from the asymptotic observable states of the theory making use of the Lee–Wick and Cutkosky, Landshoff, Olive and Polkinghorne prescription for the construction of a unitary S-matrix. Therefore, the spectrum consists of the graviton and short lived elementary unstable particles that we named "anti-gravitons" because of their repulsive contribution to the gravitational potential at short distance. However, another interpretation of the complex conjugate pairs is proposed based on the Calmet's suggestion, i.e. they could be understood as black hole precursors long established in the classical theory. Since the theory is CPT invariant, the conjugate complex of the micro black hole precursor can be interpreted as a white hole precursor consistently with the 't Hooft complementarity principle. It is proved that the quantum theory is super-renormalizable in even dimension, i.e. only a finite number of divergent diagrams survive, and finite in odd dimension. Furthermore, turning on a local potential of the Riemann tensor we can make the theory finite in any dimension. The singularity-free Newtonian gravitational potential is explicitly computed for a range of higher derivative theories. Finally, we propose a new super-renormalizable or finite Lee–Wick standard model of particle physics.

E-mail addresses: lmodesto@fudan.edu.cn, lmodesto1905@icloud.com.

http://dx.doi.org/10.1016/j.nuclphysb.2016.06.004

1. Introduction

We propose a "local" multidimensional gravitational theory compatible with renormalizability at perturbative level in addition to Lee–Wick [1] and Cutkosky, Landshoff, Olive and Polkinghorne unitarity [2] (CLOP). This work is a generalization of the theory recently proposed in [3,4]. In the last four years a weakly nonlocal action principle for gravity has been extensively studied to make up for the shortcomings of the quantization of the Einstein–Hilbert action [5–9]. Research records show that Krasnikov in 1988 and Kuz'min in 1989 proposed a similar theory [10] following Efimov's studies in nonlocal interacting quantum field theory [11]. Afterwords Tomboulis extended to gauge interactions the Kuz'min ideas and in 1996 proposed a class of weakly nonlocal super-renormalizable gauge and gravitational theories [12–14]. You may also refer to [15,16] about other excellent contributions in nonlocal theories. Recently in [17] it has been definitely proved that the theory is actually finite in any dimension when a local potential of the Riemann tensor is added. In [18] has been proposed and extensively studied a finite generalization of the nonlocal theory for gauge interactions proposed for the first time by Tomboulis [12]. However, the price to pay is that the classical action is weakly nonlocal, although the asymptotic polynomial behavior makes the theory very similar to any local higher derivative theory for all that concerns the divergent contributions to the quantum effective action.

In this paper we want to expand and specialize the seminal paper [20] about a general local super-renormalizable gravitational theory capitalizing what we learned in quasi-polynomial or weakly nonlocal theories. Actually, many results can be exported directly to the theory here proposed making a proper replacement of the *nonlocal form factor* in [17] with the *local form factor* that we are going to properly define later in this paper.

The theory here proposed fulfills a synthesis of minimal requirements: (i) Einstein–Hilbert action should be a good approximation of the theory at a much smaller energy scale than the Planck mass; (ii) the theory has to be super-renormalizable or finite at quantum level; (iii) the theory has to be unitary, with no other real poles in the propagator in addition to the graviton; if we require other poles neither real nor complex, then the theory will prove non-polynomial or weakly nonlocal. The outcome of previous studies is a nonlocal classical theory of gravity perturbatively super-renormalizable at quantum level. On the footprint of the nonlocal action we propose here a "local" theory that holds the same properties, but showing up extra complex conjugate poles besides the graviton.

Studies of higher derivative theories date back to quadratic gravity proposed in 1977 by Stelle [19]. This theory is renormalizable and asymptotically free, but unfortunately it violates unitarity showing up a real ghost state in the spectrum. In this paper we go behind the Stelle's action introducing a finite number of extra higher derivative operators to make the theory even more convergent: super-renormalizable or finite. However, we do not blindly introduce all the possible operators to a fixed order in the number of derivatives of the metric tensor. We actually consider a class of local theories that avoid extra real poles in the propagator. Looking at the above list of requirements (i)–(iii), the news with respect to the previous work on non-polynomial theories sits in the third point. We indeed do not exclude the possibility of complex conjugate mass poles, which do not prevent us from constructing a unitary local theory of gravity in the Lee–Wick formalism [1]. Lee and Wick argued that, as long as all ghost degrees of freedom in the interacting theory have complex energies, one obtains a unitary theory by constraining the physical subspace to be exactly the one for the states that have real energy. In gravity we end up with a classical theory with an extended spectrum in which the graviton is free to propagate on long distances while a bunch of other virtual elementary particles can only intrinsically live for a short amount of timeblandly

[30]. It is well known that in quantum electro-dynamics a photon can get converted into e^+e^- pairs, or more complicated channels, only when it interacts with matter, but when radiated into a "perfect vacuum" it will travel on indefinitely distances as a stable particle. In field theory this is described by a gauge independent pole at $k^2 = 0$ in the transverse photon propagator, which fixes the photon free field equations to $\Box A_\mu = 0$. By the contrast, we here have a finite number of short lived particles (named "anti-gravitons") that rapidly convert themselves into gravitons. The dispersion relation for these particles must show off a finite lifetime through gauge independent complex poles in the propagator, and the free equations of motion are $[\Box + (A + iB)]\phi = 0$ (where $A, B \in \mathbb{R}$). In particle physics a Lee–Wick extension of the standard model has been proposed to avoid quadratic divergences in the Higgs mass and hence no hierarchy puzzle [22]. In this theory, and generalizations, the classical action has a real ghost pole that, at one loop, is shifted out the real axes into a complex ghost pair. In gravity a similar Lee–Wick unitarization of the Stelle's theory [19] was evoked in [25–27] to remove the real ghost from the asymptotically free quadratic gravity [28]. Indeed, at one loop the real ghost pole splits into a pair of complex conjugate poles. In this paper we go beyond four derivatives and following the seminal papers [29,31] we propose a theory in which a finite number of complex conjugate poles, or unstable particles, are already present in the classical action [32]. Let us give here a taste of the theory,[1]

$$S_{\rm SR} = -\int d^D x \sqrt{|g|}\, 2\kappa_D^{-2}\Big[\mathbf{R} - 2\Lambda_{\rm cc} + \mathbf{G}\boldsymbol{\gamma}(\Box)\mathbf{Ric} + \mathbf{V}\Big], \tag{1}$$

where $\boldsymbol{\gamma}(\Box)$ is a polynomial (of the d'Alembertian operator $\Box$) constructed so as to avoid extra real poles in the propagator besides the graviton, $\mathbf{G}$ is the Einstein tensor, $\mathbf{Ric}$ is the Ricci tensor, and $\mathbf{V}$ is a potential at least cubic in the curvature tensor.

At classical level the solutions are stable when Lee and Wick appropriate boundary conditions are imposed [1,2]. More recently a mathematically well defined prescription has been defined in [33]. However, microcausality is violated.

2. The theory

The class of theories we are going to propose can be read out from the "non-polynomial" theories recently introduced and extensively studied in [5,12,17]. We here focus on a general local action compatible with unitarity [1,31] and super-renormalizability or finiteness,

$$\mathcal{L}_{\rm g} = -2\kappa_D^{-2}\sqrt{g}\,\left[\mathbf{R} + \mathbf{R}\,\gamma_0(\Box)\mathbf{R} + \mathbf{Ric}\,\gamma_2(\Box)\mathbf{Ric} + \mathbf{Riem}\,\gamma_4(\Box)\mathbf{Riem} + \mathbf{V}\right], \tag{2}$$

where we have distinguished the operators linear and quadratic in the curvature tensor from the higher in curvature operators. We can rewrite the theory making use of a more compact notation introducing a tensorial form factor, namely

$$\begin{aligned}
\mathcal{L}_{\rm g} &= -2\kappa_D^{-2}\sqrt{g}\,(\mathbf{R} + \mathbf{Riem}\,\boldsymbol{\gamma}(\Box)\mathbf{Riem} + \mathbf{V}) \\
&\equiv -2\kappa_D^{-2}\sqrt{g}\left(R + R_{\mu\nu\rho\sigma}\,\gamma(\Box)^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}\,R^{\alpha\beta\gamma\delta} + \mathbf{V}\right) \\
&\equiv -2\kappa_D^{-2}\sqrt{g}\big\{\mathbf{R} + R_{\mu\nu\rho\sigma}\,[g^{\mu\rho}g^{\alpha\gamma}g^{\nu\sigma}g^{\beta\delta}\gamma_0(\Box) + g^{\mu\rho}g^{\alpha\gamma}g^{\nu\beta}g^{\sigma\delta}\gamma_2(\Box) \\
&\quad + g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta}\gamma_4(\Box)]R_{\alpha\beta\gamma\delta} + \mathbf{V}\big\}.
\end{aligned} \tag{3}$$

[1] Definitions and notations. The definitions used in this paper are: the metric tensor $g_{\mu\nu}$ has signature $(+-\ldots-)$; the curvature tensor $R^\mu_{\nu\sigma\rho} = -\partial_\rho \Gamma^\mu_{\nu\sigma} + \ldots$, the Ricci tensor $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$, and the curvature scalar $R = g^{\mu\nu}R_{\mu\nu}$. We also use the notation $\mathcal{R}$ for the Riemann tensor when the indexes are suppressed.

The theory consists of a kinetic local operator quadratic in the curvature, three polynomials $\gamma_0(\Box)$, $\gamma_2(\Box)$, $\gamma_4(\Box)$, and a local potential $\mathbf{V}$ made of the following three sets of operators,

$$\mathbf{V} = \sum_{j=3}^{\mathrm{N}+2}\sum_{k=3}^{j}\sum_{i} c_{k,i}^{(j)} \left(\nabla^{2(j-k)}\mathcal{R}^k\right)_i + \sum_{j=\mathrm{N}+3}^{n+\mathrm{N}+1}\sum_{k=3}^{j}\sum_{i} d_{k,i}^{(j)} \left(\nabla^{2(j-k)}\mathcal{R}^k\right)_i$$
$$+ \sum_{k=3}^{n+\mathrm{N}+2}\sum_{i} s_{k,i} \left(\nabla^{2(n+\mathrm{N}+2-k)}\,\mathcal{R}^k\right)_i,$$

where the third set of operators are called killers because they are crucial in making the theory finite in any dimension. Λ is an invariant mass scale and the indices, $c_{k,i}^{(j)}$, $d_{k,i}^{(j)}$, $s_{k,i}$ are running or not coupling constants, while the tensorial structure has been neglected. The last set of operators with front coefficients $s_{k,i}$ are technically called "killers" and are crucial in making the theory finite. The capital N is defined to be the following function of the spacetime dimension D: $2\mathrm{N}+4=D$, while n is a positive integer, i.e. $n \in \mathbb{N}^+$. Moreover, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant box operator. The polynomials $\gamma_i(\Box)$ are:

$$\gamma_2(\Box) = -\sum_{i=1}^{\mathrm{N}+n+1}\frac{a_i}{\Lambda^2}z^{i-1} - 4\gamma_4\,,$$
$$\gamma_0(\Box) = \frac{D-2}{4(D-1)}\sum_{i=1}^{\mathrm{N}+n+1}\frac{b_i}{\Lambda^2}z^{i-1} + \frac{D}{4(D-1)}\sum_{i=1}^{\mathrm{N}+n+1}\frac{a_i}{\Lambda^2}z^{i-1} + 4\gamma_4\,, \tag{4}$$

where $z := -\Box/\Lambda^2$, $n \in \mathbb{N}^+$, and we implicitly introduced the following two polynomials,

$$P(z) = 1 + \sum_{i=1}^{\mathrm{N}+n+1} a_i z^i\,, \qquad Q(z) = 1 + \sum_{i=1}^{\mathrm{N}+n+1} b_i z^i\,, \qquad n \in \mathbb{N}^+\,. \tag{5}$$

The reason of this particular choice of the polynomials $\gamma_i(\Box)$ will be clear in the next subsection when we explicitly evaluate the propagator for the theory (2).

2.1. Propagator

Splitting the spacetime metric $g_{\mu\nu}$ into the flat Minkowski background and the fluctuation $h_{\mu\nu}$ defined by $g_{\mu\nu} = \eta_{\mu\nu} + \kappa_D\, h_{\mu\nu}$, we can expand the action (2) to the second order in $h_{\mu\nu}$. The outcome of this expansion together with the usual harmonic gauge fixing term reads [40] $\mathcal{L}_{\rm lin} + \mathcal{L}_{\rm GF} = 1/2 h^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}\, h^{\rho\sigma}$, where the operator $\mathcal{O}$ is made of two terms, one coming from the linearization of (2) and the other from the following gauge-fixing term, $\mathcal{L}_{\rm GF} = \xi^{-1}\partial^\nu h_{\mu\nu}\omega(-\Box_\Lambda)\partial_\rho h^{\rho\mu}$ ($\omega(-\Box_\Lambda)$ is a weight functional [19,39]). The d'Alembertian operator in $\mathcal{L}_{\rm lin}$ and the gauge fixing term must be conceived on the flat spacetime. Inverting the operator $\mathcal{O}$ [40], we find the two-point function in the harmonic gauge ($\partial^\mu h_{\mu\nu} = 0$),

$$\mathcal{O}^{-1} = \frac{\xi(2P^{(1)} + \bar{P}^{(0)})}{2k^2\,\omega(k^2/\Lambda^2)} + \frac{P^{(2)}}{k^2 P(k^2/\Lambda^2)} - \frac{P^{(0)}}{k^2 Q(k^2/\Lambda^2)\,(D-2)}\,. \tag{6}$$

We omitted the tensorial indexes for the propagator $\mathcal{O}^{-1}$ and the projectors $\{P^{(0)}, P^{(2)}, P^{(1)}, \bar{P}^{(0)}\}$ [40,41] are given in the footnote.[2] We also have replaced $-\Box \to k^2$ in the linearized action.

In our construction the polynomials $P(z)$ and $Q(z)$ can only show up complex conjugate poles and are chosen to satisfy the condition $P(0) = Q(0) = 1$. The complex conjugate solutions of $P(z) = 0$ and $Q(z) = 0$ are ghostlike, but they do not contribute to the absorptive part of physical scattering amplitudes and may consistently be excluded from the asymptotic observable states of the theory making use of the Lee–Wick prescription for the construction of a unitary S-matrix over the physical subspace [1,31]. The theory is also classically stable when Lee and Wick appropriate boundary conditions are imposed [2,33].

2.2. Four-dimensional theory

In $D = 4$, assuming $P(z) = Q(z)$ and introducing a potential consisting only of two killer operators quartic in the curvature, the theory simplifies to

$$\mathcal{L} = -2\kappa_4^{-2}\Bigg[-2\Lambda_{\rm cc} + R - G_{\mu\nu}\sum_{i=0}^{n}\frac{a_i}{\Lambda^2}(-\Box_\Lambda)^i R^{\mu\nu} + s_1 R^2\,\Box^{n-2}R^2 + s_2 R_{\mu\nu}R^{\mu\nu}\,\Box^{n-2}R_{\rho\sigma}R^{\rho\sigma}\Bigg]. \tag{8}$$

In the search for a finite theory of quantum gravity, the most economic one is obtained for $n = 3$ and $a_i = 0$ for $i = 0, 1, 2$ while $a_3 = 1$ in (5), namely

$$S_{\rm F} = \int d^4x\sqrt{-g}\,2\kappa_4^{-2}\left[-R + 2\Lambda_{\rm cc} - s_0\,G_{\mu\nu}\Box^3 R^{\mu\nu} - s_1 R^2\Box R^2 - s_2 R^2_{\mu\nu}\Box R^2_{\rho\sigma}\right], \tag{9}$$

where $s_0 = 1/\Lambda^8$. If we are happy with super-renormalizability we can study the following minimal action,

$$S_{\rm SR} = \int d^4x\sqrt{|g|}\,2\kappa_4^{-2}\Big[-R + 2\Lambda_{\rm cc} - s_0 G_{\mu\nu}\Box R^{\mu\nu} - \sum_i(\mathbf{Riem}^3)_i\Big], \tag{10}$$

where now $s_0 = 1/\Lambda^4$, and the sum is over all possible invariants cubic in the Riemann tensor (6 independent operators [34][3]). More details about the finiteness will be given later in section 5.

---

[2] Projectors:

$$P^{(2)}_{\mu\nu,\rho\sigma}(k) = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}\,,$$
$$P^{(1)}_{\mu\nu,\rho\sigma}(k) = \frac{1}{2}\left(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}\right),$$
$$P^{(0)}_{\mu\nu,\rho\sigma}(k) = \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}\,,\quad \bar{P}^{(0)}_{\mu\nu,\rho\sigma}(k) = \omega_{\mu\nu}\omega_{\rho\sigma}\,,\quad \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\,,\quad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}\,. \tag{7}$$

[3] At the cubic order in the Riemann tensor the basis of curvature invariants consists of eight members [34], namely

$$R^3,\quad R R_{\mu\nu}R^{\mu\nu},\quad R_{\nu\alpha}R^{\nu}_{\mu}R^{\alpha\mu}, \tag{11}$$
$$R_{\nu\alpha}R_{\mu\beta}R^{\nu\mu\alpha\beta},\quad R R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta},\quad R_{\nu\alpha}R^{\nu}_{\beta\gamma\epsilon}R^{\beta\gamma\epsilon\alpha},\quad R_{\mu\nu\alpha\beta}R^{\mu\nu}_{\gamma\epsilon}R^{\alpha\beta\gamma\epsilon},\quad R_{\mu\nu\alpha\beta}R^{\mu\alpha}_{\gamma\ \epsilon}R^{\nu\gamma\beta\epsilon}, \tag{12}$$

but only three out of the five Riemann terms in (12) are independent in $D = 4$.

3. Complex conjugate poles and unitarity

We hereby study the propagator for the two minimal four dimensional theories proposed in (9) and (10). Since $P(z) = Q(z)$, the denominator of the propagator consists of the product of the monomial k^2 times the polynomial $P(k^2/\Lambda^2)$. Therefore, we have the usual graviton massless pole with the same tensorial structure already found in the Einstein–Hilbert action, plus other complex conjugate poles resulting from the particular choice for the polynomial $P(z)$.

For the theory (10) the polynomial is $P(-\Box_\Lambda) = 1 + (-\Box_\Lambda)^2 = 1 + k^4/\Lambda^4$ and the propagator in (6), leaving out the gauge dependent terms, decomposes in

$$\mathcal{O}^{-1} = \frac{1}{k^2 P(\frac{k^2}{\Lambda^2})} \underbrace{\left(P^{(2)} - \frac{P^{(0)}}{D-2} \right)}_{\rm TS} = \frac{|\eta|^4 \, {\rm TS}}{k^2(k^2-\eta^2)(k^2-\eta^{*2})}$$

$$= \left(\frac{1}{k^2 + i\epsilon} + \frac{c^2}{k^2-\eta^2} + \frac{c^{*2}}{k^2 - \eta^{*2}} \right) {\rm TS} \,, \tag{13}$$

with

$$1 + c^2 + c^{*2} = 0 \,, \quad c^2\eta^2 + c^{*2}\eta^{*2} = 0 \,, \quad \eta^2 = -i\Lambda^2 \,, \quad \eta^{*2} = i\Lambda^2 \,. \tag{14}$$

For the theory (9) the polynomial is $P(-\Box_\Lambda) = 1 + (-\Box_\Lambda)^4 = 1 + k^8/\Lambda^8$ and the propagator decomposes in

$$\mathcal{O}^{-1} = \frac{1}{k^2 P(k^2/\Lambda^2)} {\rm TS} = \frac{|\eta_1|^4 |\eta_2|^4}{k^2(k^2-\eta_1^2)(k^2-\eta_1^{*2})(k^2-\eta_2^2)(k^2-\eta_2^{*2})} {\rm TS}$$

$$= \left[\frac{1}{k^2+i\epsilon} + \frac{c_1^2}{k^2-\eta_1^2} + \frac{c_1^{*2}}{k^2-\eta_1^{*2}} + \frac{c_2^2}{k^2-\eta_2^2} + \frac{c_2^{*2}}{k^2-\eta_2^{*2}} \right] {\rm TS} \,, \tag{15}$$

with complex masses square

$$\eta_1^2 = e^{i\frac{\pi}{4}}\Lambda^2 \text{ and } \eta_2^2 = -e^{i\frac{\pi}{4}}\Lambda^2. \tag{16}$$

For the sake of simplicity, we considered $p(z) = 1 + z^2$ and $p(z) = 1 + z^4$, and we found the above particular values for the masses η_1 and η_2 (16). However, following Ref. [31] we can show that the group velocity v_g for the particles with complex masses is smaller than or equal to the light velocity iff the following condition are satisfied,

$$\begin{aligned} & {\rm Re}(\eta^2) \geqslant 0 \,, \quad {\rm Re}(\eta_1^2) \geqslant 0 \,, \quad {\rm Re}(\eta_2^2) \geqslant 0 \,, \\ & v_g = \frac{|\vec{p}|}{\sqrt{2}} \frac{\sqrt{\sqrt{(\vec{p}^2 + {\rm Re}(\eta^2))^2 + ({\rm Im}(\eta^2))^2} + \vec{p}^2 + {\rm Re}(\eta^2))}}{\sqrt{(\vec{p}^2 + {\rm Re}(\eta^2))^2 + ({\rm Im}(\eta^2))^2}} . \end{aligned} \tag{17}$$

These inequalities are not met by the theory (9), but are perfectly satisfied by the minimal super-renormalizable theory (10). If we want (17) to be fulfilled for complex conjugate pairs with a strictly positive real part of the mass square, we have just to replace the polynomial in (13) with the following one,

$$P(k^2/\Lambda^2) = \frac{k^4 - k^2{\rm Re}(\eta^2) + |\eta|^4}{|\eta|^4} = \frac{\Box^2 + \Box\,{\rm Re}(\eta^2) + |\eta|^4}{|\eta|^4} \,, \tag{18}$$

and the theory reads

$$S'_{\rm SR} = \int d^4x \sqrt{|g|} 2\kappa_4^{-2} \Big[-R + \bar{\lambda} - G_{\mu\nu} \frac{\Box + {\rm Re}(\eta^2)}{|\eta|^4} R^{\mu\nu} - \sum_i (\mathbf{Riem}^3)_i \Big]. \tag{19}$$

We can also make (9) compatible with (17) by replacing the polynomial with the following one,

$$P(k^2/\Lambda^2) = \frac{k^8 + k^4 m_1^4 + k^4 m_2^4 + m_1^4 m_2^4}{\Lambda^8} = \frac{(-\Box)^4 + (-\Box)^2 m_1^4 + (-\Box)^2 m_2^4 + m_1^4 m_2^4}{\Lambda^8}, \tag{20}$$

where $m_1^4 m_2^4 = \Lambda^8$ and $m_1, m_2 \in \mathbb{R}$. The action now reads

$$S'_{\rm F} = \int d^4x \sqrt{-g}\, 2\kappa_4^{-2} \Big[-R + \bar{\lambda} - s_0\, G_{\mu\nu} (\Box^3 + \Box\, m_1^4 + \Box\, m_2^4) R^{\mu\nu} \\ - s_1 R^2 \Box R^2 - s_2 R^2_{\mu\nu} \Box R^2_{\rho\sigma} \Big], \tag{21}$$

where again $s_0 = 1/\Lambda^8$ and the poles are now located in: $\{-im_1^2, im_1^2, -im_2^2, im_2^2\}$. All the complex poles in (21) have group velocity zero (and real part of the mass square zero) like for the theory (10). We can get positive group velocity taking the following polynomial,

$$\begin{aligned} &P(k^2/\Lambda^2) \\ &= \Lambda^{-8} \left(k^8 - k^6 m_1^2 - k^6 m_2^2 + k^4 m_1^4 + k^4 m_2^4 + k^4 m_1^2 m_2^2 - k^2 m_1^2 m_2^4 - k^2 m_1^4 m_2^2 + m_1^4 m_2^4 \right) \\ &= \Lambda^{-8} \left(\Box^4 + \Box^3 m_1^2 + \Box^3 m_2^2 + \Box^2 m_1^4 + \Box^2 m_2^4 + \Box^2 m_1^2 m_2^2 + \Box m_1^2 m_2^4 + \Box m_1^4 m_2^2 \right) + 1, \end{aligned} \tag{22}$$

and the complex conjugate poles are now located in:

$$\left\{ \frac{1}{2}\left(1 - \sqrt{3}i\right) m_1^2,\ \frac{1}{2}\left(\sqrt{3}i + 1\right) m_1^2,\ \frac{1}{2}\left(1 - \sqrt{3}i\right) m_2^2,\ \frac{1}{2}\left(\sqrt{3}i + 1\right) m_2^2 \right\}. \tag{23}$$

Now we would illustrate in more detail the unitarity of the proposed actions. The theories under consideration are marked by pairs of complex conjugate poles. In (10) we have one pair of complex conjugate poles, while in (9) we have two complex conjugate poles, etc. We discarded extra real particles from the spectrum of the classical theory, but we allow for conjugate pairs of unstable and unphysical particles: "anti-gravitons". It is well known that, at least for a single pair of complex conjugate poles, a unitary S-matrix defined between physical asymptotic states exists [1,2]. The unphysical particles do not contribute to the absorptive part of the propagators (13) or (15) because they occur as complex conjugate pairs. We can easily check that the complex conjugate poles do not go on shell by taking the imaginary part of any one of the propagators above, namely

$$\mathrm{Im}(\mathcal{O}^{-1}(k)) = -\frac{\epsilon}{k^4 + \epsilon^2} \ \to\ -\pi\, \delta^4(k^2). \tag{24}$$

Since the incoming particles have real energy and momentum they can not produce on-shell intermediate states with complex mass. Therefore the complex poles do not destroy unitarity and their contribution to the scattering amplitudes is real. Indeed we can easily verify that the tree level exchange satisfies the optical theorem as a consequence of the energy–momentum conservation that generally follows from the definition of the S-matrix. From unitarity it follows that the imaginary part of the forward scattering amplitude, $\mathcal{M}$, must be a positive quantity (optical theorem). For example the inequality $2\,\mathrm{Im}[\mathcal{M}(2,2)] > 0$ is satisfied at tree level as a mere consequence of (24) [1,29–31]. Only the massless gravitons contribute to the imaginary part of the amplitude, while the anti-gravitons give contribution to the real part making it more convergent in the ultraviolet regime. Since (24) the tree-level unitarity for every propagators obtained in this section reads as follows,

$$2\,\mathrm{Im}\left\{T(k)^{\mu\nu}\mathcal{O}^{-1}_{\mu\nu,\rho\sigma}T(k)^{\rho\sigma}\right\} = 2\pi\,\mathrm{Res}\left\{T(k)^{\mu\nu}\mathcal{O}^{-1}_{\mu\nu,\rho\sigma}T(k)^{\rho\sigma}\right\}\Big|_{k^2=0} > 0\,, \tag{25}$$

where $T_{\mu\nu}(k)$ is the conserved energy–momentum tensor (see appendix A for more details about tree-level unitarity).

At quantum level the theory can be super-renormalizable or finite (see section 5 for more details). For the sake of simplicity and strictness, let us start considering the case of a finite theory. For this class of theories the beta functions are zero, we do not have to introduce counterterms, the propagator does not change (for what about divergences), and so the Lee–Wick unitarity is safe. However, we of course have finite contributions to the quantum effective action at any order in the loop expansion. Nevertheless, at perturbative level we typically have a slight displacement in the position of the complex conjugate poles or in the worst case a larger number of them up to infinity depending on the peculiar finite quantum nonlocal contributions to the effective action. Therefore, the unitarity Lee–Wick structure of the classical theory is likely preserved at quantum level.

For super-renormalizable theories we have logarithmic divergences and the running of the coupling constants comes along with the following nonlocal operators in $D = 4$,

$$\alpha_1\,\mathcal{R}\log\left(\frac{-\Box}{\mu^2}\right)\mathcal{R}. \tag{26}$$

In $D = 5$ the theory is finite and we expect the following contribution,

$$\alpha_2\,\mathcal{R}\sqrt{-\Box}\,\mathcal{R}. \tag{27}$$

Therefore, we end up with a quantum theory having the same structure of the initial classical theory. In the quantum action we can have a shift in the position or an increased number of the complex conjugate poles. However, we can easily treat them consistently with unitarity by applying again the Lee–Wick prescription. The quantum corrections to the spin two and/or spin zero inverse propagators (13) implied by (26) and (27) respectively read

$$k^2\left(1+\frac{k^4}{\Lambda^4}+\alpha_1\kappa_4^2\,k^2\,\log\frac{k^2}{\mu^2}\right) \quad \text{or} \quad k^2\left(1+\frac{k^4}{\Lambda^4}+\alpha_2\kappa_4^2\,k^2\sqrt{k^2}\right). \tag{28}$$

It is straightforward to check that the number of complex conjugate poles do not change, even though they are slightly moved out from the original classical position. Once again, unitarity is not affected by the quantum corrections.

For the special super-renormalizable theory (10) with propagator (13) there are only two complex conjugate poles, therefore, we can apply all the results derived in the paper [2]. In particular in [2] it is given a proof of perturbative unitarity compatible with Lorentz invariance exploring a large class of Feynman diagrams, while the *acausal* effects are fantastically small to be detected. In [24] it is given an explicit proof of the one-loop unitarity based on the CLOP [2] prescription to integrate in the complex energy plane. In [32] we can find a different Feynman $i\epsilon$ prescription suggested by the Hamiltonian analysis of the theory. Moreover, the formalism developed in

[35–37] for any higher derivative theory selects out one particular prescription regardless of the real or complex nature of the poles. Here we assume that the quantization of a general gravitational theory with an arbitrary number of complex conjugate poles has to be understood as a mere application of the procedure explained in section 2.1 of the report [37]. However, this particular prescription does not solve the very important ambiguity problem present in the Lee–Wick theory. This is a serious issue because the introduction of ambiguities at each order in the loop expansion is actually related to the problem of non-renormalizability that the higher derivative theory is supposed to solve. Therefore, the general foundations of the Lee–Wick theory and in particular the ambiguity referred above deserve to be further investigated in future work.

In short, the Lee–Wick unitarity reads as follows: the S-matrix is unitary in the physical subspace of real states (only gravitons in this section), while complex mass particles appear only as virtual states. Assuming the Lee–Wick [1] or [31] definitions, the S-matrix vanishes for all non-real initial and final states, while the unphysical complex states can appear only as virtual states. Therefore, the S-matrix is unitary as a mapping in the subspace of real physical states. The complex poles occur in a proper combination to cancel out the divergences that arise from the physical states. Once again, complex particles are consistently excluded from the asymptotic states preserving the usual unitarity notion in the subspace of real states. The Hamiltonian approach remains well defined in the indefinite metric Hilbert space [1,29]. The theory is unitary and Lorentz invariant, but microcausality is violated [1,2,38]. However, macrocausality is preserved because the Feynman propagator is convergent in the limit $|x_0| \to +\infty$, namely the propagator does not diverge for infinite separation time, as easy to see making the explicit integration in the energy complex plane [31]: $\left|\left|\langle 0|T(h_{\mu\nu}(x)h_{\rho\sigma}(y)|0\rangle\right|\right| < \infty$ for $|x_0| \to +\infty$.

4. Propagator in coordinates space and nonsingular gravitational potential

In this section we investigate the behavior of the two point function and of the gravitational potential at short distance.

Propagator in coordinates space — We can easily obtain the propagator in coordinate space by the Fourier transform of (6). Let us consider the super-renormalizable theory specified by the polynomial $P(z) = 1 + z^4$ and omit the tensorial structure in (6), then the propagator in coordinate space reads[4]

$$G(x-y) = \frac{1}{4\pi^2(x-y)^2} - \frac{G^{5,0}_{0,8}\left(\frac{(x-y)^8}{16777216}\middle| -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 0, \frac{1}{4}, \frac{1}{2}\right)}{(16\pi)^2}, \tag{29}$$

$$G(0) = \frac{\pi}{64\sqrt{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)^2\Gamma\left(\frac{5}{4}\right)}. \tag{30}$$

The first contribution in (29) is due to the massless graviton, while the second one to comes from the Lee–Wick complex particles that make finite the propagator in the coincidence limit $x \to y$. Indeed, for $x - y \to 0$ the two point function approaches the constant $G(0)$ displayed in (30). The high energy behavior of the two point function here derived has a universal character. Indeed, whatever the polynomial (or non-polynomial [5]) form factor is, the short distance limit always attains a constant value.

[4] In (29), $G^{mn}_{pq}(z|a_1, \dots, a_p; b_1, \dots, b_q)$ is the generalized Majer G function.

Gravitational potential — To address the problem of classical singularities we can begin by calculating the Newtonian gravitational potential. Given any propagator, the graviton solution of the linear equations of motion is:

$$h_{\mu\nu}(x) = \frac{\kappa_D}{2}\int d^D x' \mathcal{O}^{-1}_{\mu\nu,\rho\sigma}(x-x')T^{\rho\sigma}(x')$$
$$= \frac{\kappa_D}{2}\int d^D x' \int \frac{d^D k}{(2\pi)^D}\frac{e^{ik(x-x')}}{k^2 P(k^2/\Lambda^2)}\left(T_{\mu\nu} - \frac{\eta_{\mu\nu}}{D-2}T^\mu_\mu\right). \tag{31}$$

For a static source with energy tensor $T^\mu_\nu = \mathrm{diag}(M\,\delta^{D-1}(\vec{x}), 0, \dots, 0)$, the spherically symmetric solution reads

$$h_{\mu\nu}(r) = -\frac{\kappa_D M}{2}E_{\mu\nu}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{e^{-i\vec{k}\cdot\vec{x}}}{\vec{k}^2\, P(\vec{k}^2/\Lambda^2)}$$
$$= -\frac{\kappa_D M}{2}\frac{\pi^{\frac{D-3}{2}}}{(2\pi)^{D-2}}\frac{E_{\mu\nu}}{r^{D-3}}\int dp\, \frac{p^{D-4}{}_0\tilde{F}_1\left(\frac{D-1}{2}; -\frac{p^2}{4}\right)}{P(p^2/r^2\Lambda^2)}, \tag{32}$$

where ${}_0\tilde{F}_1(a;z) = {}_0F_1(a;z)/\Gamma(a)$ is the regularized hypergeometric confluent function. In (32), we also have introduced the variable $p = |\vec{k}|r$ and the matrix $E_{\mu\nu} = (D-2)^{-1}\mathrm{diag}(D-3, 1, \dots, 1)$. Using the graviton solution above in (31), (32) we can reconstruct all the components of the metric tensor and then we can get the spacetime line element for a spherically symmetric source. The gravitational potential is related to the h_{00} component of the graviton field by $\Phi = \kappa_D h_{00}/2$. Then, using (32) we get

$$\Phi(r) = -\frac{\kappa_D^2 M}{4}\frac{D-3}{D-2}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{e^{-i\vec{k}\cdot\vec{x}}}{\vec{k}^2\, P\left(\vec{k}^2/\Lambda^2\right)}$$
$$= -\frac{G_N M}{r^{D-3}}2\frac{D-3}{D-2}\frac{2^{4-D}}{\pi^{\frac{D-3}{2}}}\int dp\, \frac{{}_0\tilde{F}_1\left(\frac{D-1}{2}; -\frac{p^2}{4}\right)}{p^{4-D}\, P\left(p^2/r^2\Lambda^2\right)}. \tag{33}$$

For example, in $D = 4$, (33) simplifies to

$$\Phi(r) = -\frac{G_N M}{r}\frac{2}{\pi}\int_0^{+\infty} dp\, \frac{J_0(p)}{P\left(p^2/r^2\Lambda^2\right)}, \quad J_0(p) = \mathrm{sinc}(p) \equiv \frac{\sin(p)}{p}. \tag{34}$$

We are now ready to evaluate the gravitational potential for three different choices of the polynomial $P(z)$.

- For $P(z) = 1 + z$, which corresponds to a particular theory quadratic in the curvature, the potential reads

$$\Phi(r) = -\frac{m(1-e^{-\Lambda r})}{r}. \tag{35}$$

- For $P(z) = 1 + z^2$, which corresponds to a super-renormalizable gravitational theory, the potential reads

$$\Phi(r) = \Phi(r) = -\frac{m\left(1 - e^{-\frac{\Lambda r}{\sqrt{2}}}\cos\left(\frac{\Lambda r}{\sqrt{2}}\right)\right)}{r}. \tag{36}$$

- For $P(z) = 1 + z^4$, which eventually corresponds to a finite gravitational theory, the potential reads

$$\Phi(r) = \Phi(r) = -\frac{m\left(4 - e^{-\eta_1 r} - e^{-\eta_2^* r} - e^{\eta_2 r} - e^{\eta_1^* r}\right)}{4r}. \tag{37}$$

For the first choice $P(z) = 1 + z$ the potential is regular in $r = 0$ because of the real ghost pole in the propagator [42], for the second and third choice at short distance ($\sim 1/\Lambda$) the complex ghosts screen the anti-screening effect of the gravitons. For the case of $P(z) = 1 + z^4$ we can rewrite the potential in the following explicitly real form,

$$\Phi(r) = -\frac{m}{r} + \frac{m\, e^{-\Lambda r \sin\left(\frac{\pi}{8}\right)}\cos\left(\Lambda r \cos\left(\frac{\pi}{8}\right)\right)}{2r} + \frac{m\, e^{-\Lambda r \cos\left(\frac{\pi}{8}\right)}\cos\left(\Lambda r \sin\left(\frac{\pi}{8}\right)\right)}{2r}. \tag{38}$$

The reader can easily recognize the complex conjugate mass poles in the classical gravitational potential (37). They clearly play a crucial rule in making the potential singularity free in agreement with the Lee–Wick requirement for a consistent theory at classical and quantum level. This result is in agreement with the interpretation given in a previous work [42]. Actually this is a generalization of the result in [42] to a theory with complex conjugate poles.

In the same approximation we can reconstruct the metric for black hole or cosmological solutions [43–46]. Exact solutions can be found closely following the derivations in [47–51].

5. Quantum divergences

Let us then examine the ultraviolet behavior of the quantum theory and consider what kind of operators present in the action can generate divergences. In the high energy regime the graviton propagator in momentum space for the theory (2) schematically scales as

$$\mathcal{O}^{-1}(k) \sim \frac{1}{k^{2n+D}}. \tag{39}$$

Since the interactions have leading ultraviolet scaling k^{2n+D}, we find the following upper bound to the superficial degree of divergence in a D-dimensional spacetime,

$$\omega(G) = D - 2n(L-1)\,. \tag{40}$$

In (40) we used the topological relation between vertexes V, internal lines I and number of loops L: $I = V + L - 1$. Thus, if $n > D/2$ only 1-loop divergences survive in this theory, therefore, it is super-renormalizable. Only a finite number of constants is renormalized in the action (2), i.e. κ_D, $\bar{\lambda}$, a subset of the couplings $\{a_i, b_i\}$, and the finite number of couplings that multiply the operators $O(\mathcal{R}^3)$ in the first line of (2) up to $\mathcal{R}^{D/2}$.

Let us now expand on the one-loop divergences. The main divergent integrals contributing to the one-loop effective action have the following form,

$$\int \frac{d^D k}{(2\pi)^D} \left\{ \prod_{i=1}^{s} \frac{1}{(k+p_i)^{2m}} \right\} P(k)_{2sm}. \tag{41}$$

$P_{2sm}(k)$ is a polynomial function of degree $2ms$ in the momentum k (generally it also relies on the external momenta $\bar{p}_a$), $p_i = \sum_{a=1}^{i} \bar{p}_a$, and $m = n + \mathrm{N} + 2$ for the graviton field $h_{\mu\nu}$. We can write, as usual,

$$\prod_{i=1}^{s} \frac{1}{(k+p_i)^{2m}} \propto \int_0^1 \left(\prod_{i=1}^{s} x_i^{n-1} dx_i \right) \frac{\delta\left(1 - \sum_{i=1}^{s} x_i\right)}{[k'^2 + \mathcal{R}]^{ms}}, \quad k' = k + \sum_{i=1}^{s} x_i p_i ,$$

$$\mathcal{R} = \sum_{i=1}^{s} p_i^2 x_i - \left(\sum_{i=1}^{s} x_i p_i \right)^2 . \tag{42}$$

In (41), we move outside the convergent integral in x_i and we replace k' with k

$$\int \frac{d^D k}{(2\pi)^D} \frac{P'(k, p_i, x_i)_{2sm}}{(k^2 + \mathcal{R})^{ms}} . \tag{43}$$

Using Lorentz invariance and missing the argument x_i, we replace the polynomial $P'(k, p_i, x_i)_{2ms}$ with a polynomial of degree $m \times s$ in k^2, namely $P''(k^2, p_i)_{ms}$. Therefore, the integral (43) reduces to

$$\int \frac{d^D k}{(2\pi)^D} \frac{P''(k^2, p_i)_{ms}}{(k^2 + \mathcal{R})^{ms}} . \tag{44}$$

We can decompose the polynomial $P''(k^2, p_i)_{ms}$ in a product of external and internal momenta only to obtain the divergent contributions,

$$P''(k^2, p_i)_{ms} = \sum_{\ell=0}^{[D/2]} \alpha_\ell(p_i) k^{2ms-2\ell} \tag{45}$$

$$= k^{2ms}\alpha_0 + k^{2ms-2}\alpha_1(p_i) + k^{2ms-4}\alpha_2(p_i) + \dots . \tag{46}$$

Given the polynomial

$$P(z) = 1 + c_{n+\mathrm{N}+1} z^{n+\mathrm{N}+1} + c_{n+\mathrm{N}} z^{n+\mathrm{N}} + c_{n+\mathrm{N}-1} z^{n+\mathrm{N}-1} + \dots , \tag{47}$$

we find the following logarithmic divergences,

$$\sum_{\ell=0}^{[D/2]} \int \frac{d^D k}{(2\pi)^D} \frac{\alpha_\ell(p_i) k^{2ms-2\ell}}{(k^2 + \mathcal{R})^{ms}} =$$

$$= \sum_{\ell=0}^{[D/2]} \frac{i\alpha_\ell(p_i)(\mathcal{R})^{\frac{D}{2}-\ell}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(\ell - \frac{D}{2}\right)\Gamma\left(ms - \ell + \frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma(ms)}$$

$$\Longrightarrow \overbrace{\frac{1}{\epsilon}\beta_\lambda + \frac{1}{\epsilon}\beta_R R + \dots + \frac{1}{\epsilon}\beta_{\mathcal{R}^{D/2}} \mathcal{R}^{D/2}}^{\text{counterterms}} . \tag{48}$$

We schematically listed above the counterterms and explicitly introduced the ultraviolet cut-off ϵ in the dimensional regularization scheme.

We can express the one-loop counterterms in any dimension D explicitly displaying the number of derivatives acting on the metric tensor [20], namely

$$S_{\text{counterterms}} = \frac{1}{\epsilon}\sum_{j=0}^{D/2}\alpha_{2j}\Lambda^{D-2j}\int d^D x\sqrt{|g|}\mathcal{O}_{2j}(\partial_\lambda g_{\mu\nu}), \tag{49}$$

where $\mathcal{O}_{2j}(\partial_\lambda g_{\mu\nu})$ denotes the general covariant scalar operator containing $2j$ derivatives of the metric tensor $g_{\mu\nu}$, white the α_{2j} are dimensionless constants.

Now we specify the above general analysis to our particular class of theories (9), (10), and (19). The three theories are super-renormalizable, but for the case of (9) we only have one loop divergences, while for (10) and (19) we also have divergences at two loops or three loops.

Given the particular choice of the polynomial $P(z)$ for the theory (9), the counterterms can only be proportional to R^2 and $R^2_{\mu\nu}$. Moreover, it is always possible to tune the front coefficients s_1 and s_2 for the quartic operators $\mathcal{R}^2\Box\mathcal{R}^2$ and to make zero the beta functions. This is due to the linearity in s_1 and s_2 of the beta functions β_{R^2} and $\beta_{R^2_{\mu\nu}}$. For the theory (19) we also expect the beta functions β_{R^2} and $\beta_{R^2_{\mu\nu}}$ to be zero for some special choice of the front coefficients s_1 and s_2. However, the beta function are probably quadratic in s_1 and s_2 and only an explicit computation could confirm this property. At two loops we can have counterterms proportional to the Ricci scalar (Einstein–Hilbert operator) or the cosmological constant, while at three loops we only have divergences proportional to the cosmological constant.

More details about the one-loop quantum action are given in the appendix B.

6. New Lee–Wick standard model

In the previous papers [17,18] a higher derivative and weakly nonlocal theory beyond the standard model of particle physics has been proposed. However, such theory is quasi-polynomial in many respects and it is straightforward to take into account the results in [17,18] to propose here a local higher derivative and super-renormalizable action for gauge interactions and matter. This is a forced extension beyond the standard model if we want to preserve super-renormalizability of the gravitational interactions after coupling to matter. Moreover, Lee–Wick gauge interactions turn out to be (super-)renormalizable or finite regardless of the spacetime dimension. Following the notation of section 2, the action for gauge bosons reads as follows,

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4g^2}\Big[F_{\mu\nu}P_g(\mathcal{D}^2_\Lambda)F^{\mu\nu} + \frac{s_g}{\Lambda^4}F^2(\mathcal{D}^2_\Lambda)^2F^2\Big], \tag{50}$$

where the polynomial $P_g(\mathcal{D}^2_\Lambda)$, as a function of the square of the gauge covariant derivative $\mathcal{D}$, must be chosen having only complex conjugate poles and the same asymptotic behavior as the analogue functions introduced for the pure gravity sector. For the fermionic and scalar sectors we achieve super-renormalizability with the following action,

$$\mathcal{L}_{\text{F}} = \sum_a^{N_f}\bar{\psi}_a\, i\not{\mathcal{D}}_a P_f(\mathcal{D}^2_\Lambda)\,\psi_a, \tag{51}$$

$$\mathcal{L}_{\text{H}} = (\mathcal{D}_\mu\Phi)^\dagger P_s(\mathcal{D}^2_\Lambda)(\mathcal{D}^\mu\Phi) - \mu^2\Phi^\dagger P_s(\mathcal{D}^2_\Lambda)\Phi - \lambda(\Phi^\dagger\Phi)^2.$$

$P_f(\mathcal{D}^2_\Lambda)$ and $P_s(\mathcal{D}^2_\Lambda)$ are again polynomial free of real ghosts. To achieve full finiteness of all running coupling constants we need few other local operators, which the interested reader can find in Refs. [17,18]. In contrast to the Lee–Wick standard model of particle physics previously proposed [21–23] where real ghosts move out the real axis at quantum level, here the complex conjugate poles are a feature of the classical theory. Moreover, the theory proposed in this section is super-renormalizable or finite at quantum level.

7. Conclusions and remarks

The class of local higher derivative gravitational theories studied in this paper have extra complex conjugate poles besides the standard massless graviton pole in the propagator. Theories with complex conjugate poles in the propagator seem to be well defined as shown in [1,2,29–31]. The extra unphysical particles associated with the new poles are not in the physical Hilbert space for the asymptotic states, but are forced to decay in ordinary gravitational degrees of freedom by the real energy conservation.

At quantum level the higher derivative operators make the theory super-renormalizable in any dimension. Indeed only one-loop up to three-loops divergences could be present depending on the particular set of higher derivative operators included in the action. However, for the case of a one-loop super-renormalizable theory, a local potential starting cubic in the Riemann tensor does not affect the propagator around the flat spacetime, but makes all the beta functions to vanish, and the theory turns out to be finite. Moreover, using dimensional regularization the theory is finite in odd dimension because there are no local one-loop counterterms with an odd number of derivatives in odd dimension. We here again show a D-dimensional minimal prototype theory,

$$\mathcal{L} = -2\kappa_D^{-2}\sqrt{|g|}\Big[R - s_0\, G_{\mu\nu}\Box^{n+\frac{D}{2}-2}R^{\mu\nu} + \sum_{i=1}^{n_K} s_i\,\mathcal{R}^{\frac{D}{2}}\,\nabla^{2n-4}\,\mathcal{R}^2\Big], \tag{52}$$

where $s_0 = (-1)^{n+(D-4)/2}/\Lambda^{2n+D-2}$, and the sum is over the minimal number "n_K" of killer operators we need to make the theory finite. Within the quantum field theory framework this theory preserves Lorentz and diffeomorphism invariance, and satisfies Lee–Wick unitarity in the subspace of real physical states. Furthermore, (52) is finite in odd dimension and super-renormalizable in even dimension for any choice of the parameters s_0, s_i. Moreover, for particular choice of the parameters s_i all the beta functions can be made to vanish in $D = 4$ and likely in any even dimension. Therefore the theory turns out to be finite.

In our theory tree-level unitarity is guaranty by the real energy conservation that comes together with the S-matrix definition. In other words the complex conjugate poles never go on shell and the optical theorem is satisfied on the real physical and Lorentz invariant subspace. At quantum level the CLOP prescription guarantees unitarity at least for the minimal super-renormalizable theory [2,24].

The singularities that plague the gravitational potential of Einstein gravity are here smeared out because of the soft behavior of the propagator at short distance. The complex conjugate particles contribute to overall cancel out the divergent contribution of the massless physical graviton field [42].

Let us further expand on the interpretation of complex conjugate poles. In the theory here proposed, by increasing the energy, gravity becomes stronger, but in the short distance limit $\ell << 1/\Lambda \sim 1/M_P$[5] gravity becomes weak again (constant gravitational potential and zero gravitational force) due to the anti-screening effect of the gravitons, which wins over the screening effect of the virtual anti-graviton particle pairs. In other words, getting closer to the mass the anti-screening effect of the surrounding gravitons diminishes, so the full contribution of this effect would be increasingly weak and the "effective mass" will decrease with decreasing distance. This is analog to what occurs in quantum chromo-dynamics where the quarks play the role of

[5] We have two scales in our theory G_N and the length scale $\ell = 1/\Lambda$, but here they are identified in order to simplify the discussion.

anti-gravitons and gluons the role of gravitons. Similarly, also the attractive gravitational force increases with the energy, but vanishes in the zero separation distance limit because of the repulsion due to the anti-gravitons. In the intermedium energy regime such unstable unphysical pairs, the anti-gravitons, are excited without to go on shell. This is reminiscent of classical radiation surrounded by a complete absorber. In a complete absorber, radiation has to be absorbed and no asymptotic photons exist. Therefore, a quantum theory with complex conjugate poles would not have the associated asymptotic particles, as a simple calculation of the absorptive part shows.

On the footprint of the Calmet's proposal [52–56] we can give here the following alternative (or maybe equivalent) interpretation to the complex conjugate poles: they actually are the mass and the width of light black holes precursors,

$$k_0^2 = \left(M_{\rm BH} - i \, \frac{\Gamma_{\rm BH}}{2} \right)^2 . \tag{53}$$

In our example (18) η^2 is identified with the above pole, while the complex conjugate leads to the acausal effects. In other words, our local theory describes the usual massless graviton and a finite number of micro black holes. This idea can be supported evaluating the classical equations of motion. Indeed, it has been shown in [58] that the Schwarzschild and Kerr black holes are exact solutions of the theory. Following the 't Hooft suggestion we could say that the theory here proposed is a kind of "unitarization" of the higher derivative Stelle's gravity throughout the explicit introduction of virtual black holes (see below the comparison with the Stelle's quadratic gravity at one loop).

In a nonlocal super-renormalizable theory we expect the same structure not at the classical level, but for the quantum action (for example, this can be read out of the finite *log* contributions to the quantum action [4]). However, here the number of complex conjugates poles is infinite allowing for a spectrum of arbitrary large black holes.

In Stelle theory we have the same phenomenon because the real ghost pole splits into two complex conjugate poles at quantum level. Again we can give the same interpretation and infer that at one-loop the spectrum of the quantum action is compatible with unitarity and the real ghost is converted into a pair of particles consisting of a black hole and the complex conjugate state [59–61].

Let us notice that the above interpretation is based on a one-loop computation, therefore it is only perturbative, and in Stelle theory we need to compute higher loop corrections to show the stability of the spectrum. However, in a super-renormalizable theory (convergent for $L > 1$) the beta functions are one-loop exact and the asymptotic freedom makes the interpretation likely correct at any perturbative order. Indeed, finite perturbative contributions to the quantum action can only slightly move the complex conjugates poles.

Following the van Tonder [62] argument, or suggestion, it comes natural (by CPT invariance of the theory) to interpret a full scattering process as the creation and evaporation of a black hole system. When two gravitons (or matter particles) scatter, a black hole is created, together with a white hole (the CPT conjugate solution), that in turn decays into two gravitons again (or matter particles) without ever appearing on-shell.

Since we have microcausality violation or more properly effective non-locality in time [62], the particles seen by the future observer are emitted from a point causally prior to the collision of the incoming matter (which happens at the singularity). This is indeed reminiscent of the way Hawking radiation originates causally prior to the singularity that absorbs the incoming matter in the black hole. The scattering process happens at the singularity, while the outgoing particles are earlier generated at the event horizon. On the other hand, the particles are emitted in the region near the singularity before the Hawking scattering process (CPT revers of the Hawking emission) can occur at the horizon.

The spacetime structure is obtained replacing the event horizon with a simply connected trapped surface achieved gluing together the black hole and white hole horizons.

This interpretation of complex conjugate pairs as describing black holes – white hole pairs seems compatible with the 't Hooft complementary principle [57] as a consequence of the CPT invariance of the theory.

Finally, we hope that the local action here proposed will stimulate cosmologists and people of the black hole community in starting looking for exact solutions and eventually infer about their stability. The minimal theory here proposed is "just six order" in derivatives of the metric, therefore a classic study of the theory can be fielded relatively easily.

Acknowledgements

A special thanks goes to Ilya L. Shapiro for drawing our attention to higher derivative theories involving complex conjugate poles. We also thank Damiano Anselmi and Leslaw Rachwal for asking numerous stimulating questions.

Appendix A. Tree-level unitarity

In quantum field theory unitarity means that the S-matrix meets

$$S^\dagger S = 1\,. \tag{54}$$

If we introduce the T-matrix defined by

$$S = 1 + iT, \tag{55}$$

then the unitarity condition (54) turns into

$$-i(T - T^\dagger) = T^\dagger T \tag{56}$$

We must consider the matrix elements of the above equation between all possible states. Let us examine the the matrix element between the initial state $|i\rangle$ and the final state $\langle f|$,

$$\begin{aligned}
&-i\left(\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle\right) = \langle f|T^\dagger \left(\sum_k |k\rangle\langle k|\right) T|i\rangle \\
&-i\left(T_{fi} - \left(T^\dagger\right)_{fi}\right) = \sum_k \left(T^\dagger\right)_{fk} T_{ki} \\
&-i\left(T_{fi} - T^*_{if}\right) = \sum_k T^*_{kf} T_{ki}\,.
\end{aligned} \tag{57}$$

Notice that we used the following definition for the scattering amplitude,

$$\langle f|T|i\rangle = (2\pi)^D \delta^D(p_i - p_f)\, T_{fi}. \tag{58}$$

For the forward scattering amplitude, namely $i = f$ (or in any theory invariant under the inversion $x^\mu \to -x^\mu$, so that $T_{if} = T_{fi}$), the previous equation (57) simplifies to the following,

$$2\,\mathrm{Im}\,T_{ii}=\sum_k T_{ik}^* T_{ik}>0 \quad (i=f) \qquad \text{or} \qquad 2\,\mathrm{Im}\,T_{if}=\sum_k T_{ik}^* T_{fk}>0 \quad (T_{if}=T_{fi})\,. \tag{59}$$

We now present a systematic study of the tree-level unitarity [40]. A general theory is well defined if "tachyons" and "ghosts" are absent, in which case the corresponding propagator has only first poles at $k^2-M^2=0$ with real masses (no tachyons) and with positive residues (no ghosts). Therefore, to test the tree-level unitarity of a multidimensional super-renormalizable local (or nonlocal) higher derivative gravity we couple the graviton to the external conserved stress–energy tensor $\Theta^{\mu\nu}$ and we examine the amplitude at the pole. When we introduce a general source, the linearized action including the gauge-fixing reads

$$\mathcal{L}_{h\Theta}=\frac{1}{2}h^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}h^{\rho\sigma}-g\,h_{\mu\nu}\Theta^{\mu\nu}. \tag{60}$$

The transition amplitude in momentum space is

$$\begin{aligned} &S^{(2)}=iT=(-i)^2g^2\,\Theta^{\mu\nu}\,i\Delta_{F\mu\nu,\rho\sigma}\,\Theta^{\mu\nu}\,,\\ &\langle 0|T_{\mathrm{Wick}}\left(h_{\mu\nu}(x')h_{\rho\sigma}(x)\right)|0\rangle=i\Delta_{F\mu\nu,\rho\sigma}(k)\equiv i\mathcal{O}^{-1}_{\mu\nu,\rho\sigma}(k)\,, \end{aligned} \tag{61}$$

where $S^{(2)}$ is the S-matrix at the second order in perturbation theory ($S=1+iT\approx 1+S^{(2)}$), and g is an effective coupling constant. To make the analysis explicit, we can expand the sources using the following set of independent vectors in the momentum space,

$$k^\mu=(k^0,\vec{k})\,,\ \tilde{k}^\mu=(k^0,-\vec{k})\,,\ \ \epsilon_i^\mu=(0,\vec{\epsilon})\,,\ i=1,\ldots,D-2\,, \tag{62}$$

where $\vec{\epsilon}_i$ are unit vectors orthogonal to each other and to $\vec{k}$. The symmetric stress–energy tensor reads

$$\Theta^{\mu\nu}=ak^\mu k^\nu+b\tilde{k}^\mu\tilde{k}^\nu+c^{ij}\epsilon_i^{(\mu}\epsilon_j^{\nu)}+d\,k^{(\mu}\tilde{k}^{\nu)}+e^i k^{(\mu}\epsilon_i^{\nu)}+f^i\tilde{k}^{(\mu}\epsilon_i^{\nu)}. \tag{63}$$

The conditions $k_\mu\Theta^{\mu\nu}=0$ and $k_\mu k_\nu\Theta^{\mu\nu}=0$ place constrains on the coefficients a,b,d,e^i,f^i.

In presence of the usual graviton pole and a finite sequence of complex conjugate poles, the Feynman propagator reads

$$i\Delta_F(k)=i\left[\frac{1}{k^2+i\epsilon}+\sum_n\left(\frac{c_n}{k^2-\eta_n^2}+\frac{c_n^*}{k^2-(\eta_n^2)^*}\right)\right]\left(P^{(2)}-\frac{P^{(0)}}{D-2}\right), \tag{64}$$

where for the sake of simplicity we omitted the tensorial structure in Δ_F and in the projectors. Plugging the above propagator in (61) we end up with the following expression for the amplitude,

$$\begin{aligned} S^{(2)}&=iT=(2\pi)^D\delta(P_i-P_f)\,i\,T_{if}=(-i)^2g^2\,\Theta^{\mu\nu}\,i\Delta_{F\mu\nu,\rho\sigma}\,\Theta^{\mu\nu}\\ &=(2\pi)^D\delta(P_i-P_f)\,i\\ &\quad\underbrace{\times(-i)^2\,\Theta^{\mu\nu}\left[\frac{1}{k^2+i\epsilon}+\sum_n\left(\frac{c_n}{k^2-\eta_n^2}+\frac{c_n^*}{k^2-(\eta_n^2)^*}\right)\right]\left(P^{(2)}-\frac{P^{(0)}}{D-2}\right)_{\mu\nu,\rho\sigma}\Theta^{\rho\sigma}}_{T_{if}}\,, \end{aligned} \tag{65}$$

where we explicitly pointed out the definition of T_{if} in the full amplitude, namely

$$T_{if} = (-i)^2\, \Theta^{\mu\nu} \left[\frac{1}{k^2 + i\epsilon} + \sum_n \left(\frac{c_n}{k^2 - \eta_n^2} + \frac{c_n^*}{k^2 - (\eta_n^2)^*} \right) \right] \left(P^{(2)} - \frac{P^{(0)}}{D-2} \right)_{\mu\nu,\rho\sigma} \Theta^{\rho\sigma} . \tag{66}$$

Introducing the spin-projectors and the conservation of the stress–energy tensor $k_\mu \Theta^{\mu\nu} = 0$ in (61), the imaginary part of T_{if} reads

$$\begin{aligned} 2\,\mathrm{Im}\, T_{if} = {} & 2\,\mathrm{Im}\,(-i)^2 g^2 \left\{ \Theta_{\mu\nu}\Theta^{\mu\nu} - \frac{\Theta_\mu^{\mu\,2}}{D-2} \right\} \left[\frac{k^2 - i\epsilon}{k^4 + \epsilon^2} + \sum_n \left(\frac{c_n}{k^2 - \eta_n^2} + \frac{c_n^*}{k^2 - (\eta_n^2)^*} \right) \right] \\ = {} & 2g^2 \left\{ \Theta_{\mu\nu}\Theta^{\mu\nu} - \frac{\Theta_\mu^{\mu\,2}}{D-2} \right\} \left[\frac{\epsilon}{k^4 + \epsilon^2} \right] \\ \to {} & 2g^2 \left\{ \Theta_{\mu\nu}\Theta^{\mu\nu} - \frac{\Theta_\mu^{\mu\,2}}{D-2} \right\} \pi\, \delta(k^2) . \end{aligned} \tag{67}$$

In the last step we used the following representation of the Dirac delta,

$$\frac{\epsilon}{k^4 + \epsilon^2} = \pi\, \delta(k^2) , \quad \epsilon \to 0^+ , \tag{68}$$

and the final result has to be understood in the space of distributions.

From (59) and (67) the tree-level unitarity requirement simplifies to

$$2\,\mathrm{Im} \left\{ \Theta(k)^{\mu\nu} \mathcal{O}^{-1}_{\mu\nu,\rho\sigma} \Theta(k)^{\rho\sigma} \right\} = 2\pi\ \mathrm{Res} \left\{ \Theta(k)^{\mu\nu} \mathcal{O}^{-1}_{\mu\nu,\rho\sigma} \Theta(k)^{\rho\sigma} \right\} \Big|_{k^2=0} > 0. \tag{69}$$

In particular, (67) can be recast in the following form,

$$2\pi \mathrm{Res}\ (\mathcal{A}) \big|_{k^2=0} = 2\pi g^2 \left[(c^{ij})^2 - \frac{(c^{ii})^2}{D-2} \right] \tag{70}$$

which is zero in $D = 3$ and positive for $D > 3$.

In the Lee–Wick theory the propagator shows extra complex conjugate poles and at the moment it is not obvious how to derive, if any, the usual Largest Time Equation. However, we can still analyze (59) for the case of individual graphs by cutting the diagrams. Energy–momentum conservation comes together with the S-matrix and must be satisfied by both sides of (59). Therefore, if we cut through normal particle propagators (in our case we only have the massless graviton) we have to replace the propagator with $\delta(k^2)$. If we cut through the Lee–Wick propagators, these just correspond to take the imaginary part of the sum in (64), and the imaginary part of the sum of complex conjugates poles vanishes. In particular, in $T^\dagger T$ we only have to sum over intermediate normal tree particle states. Therefore, the theory is unitary in the subspace of the real normal and stable particles as a consequence of the energy–momentum conservation and the presence of extra poles in the propagator that always come in complex conjugate pairs.

Since the S-matrix provides a one-to-one map from the past to the future in scattering experiments, the existence of a well-defined S-matrix is enough to show that there are no paradoxes in the scattering processes. Nevertheless, the theory manifests acausal effects at short distance [24].

Appendix B. Higher derivative quantum gravity

In this technical section we study a quite general local super-renormalizable quantum gravity. The results will be exported to any local super-renormalizable or finite Lee–Wick gravitational theory.

Let us start with the following general prototype for a local super-renormalizable action,

$$S_{\rm HD} = -2\kappa_D^{-2} \int d^D x \sqrt{|g|} \Big[-2\Lambda_{\rm cc} + R + \sum_{i=0}^{n+{\rm N}} \omega_{{\rm Ric},i}\, R_{\mu\nu}\, \Box^i R^{\mu\nu} + \sum_{i=0}^{n+{\rm N}} \omega_{{\rm R},i}\, R\, \Box^i R + \mathbf{V}(\mathbf{R}) \Big]. \tag{71}$$

In background field method the metric $g_{\mu\nu}$ is split into a background metric $\bar{g}_{\mu\nu}$ and a quantum fluctuation $h_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{72}$$

Sometimes below we will denote these metrics by g, $\bar{g}$ and h without writing covariant indices explicitly. However, we will denote the background metric again by g because we will not speak about the full metric g in the rest of this section. Since the theory is Diff. invariant we have to fix the gauge and in the quantization procedure we must introduce Faddeev–Popov (FP) ghosts. The gauge-fixing and FP-ghost actions read as follows,

$$\begin{aligned}
&S_{\rm gf} = \int d^D x \sqrt{|g|}\, \frac{1}{2}\, \chi_\mu\, C^{\mu\nu}\, \chi_\nu\,, \quad \chi_\mu = \nabla_\sigma h^\sigma_\mu - \beta_g \nabla_\mu h\,,\\
&C^{\mu\nu} = -\frac{1}{\alpha_g}\left(g^{\mu\nu}\Box + \gamma_g \nabla^\mu \nabla^\nu - \nabla^\nu \nabla^\mu\right) \Box_\Lambda^{{\rm N}+n},\\
&S_{\rm gh} = \int d^D x \sqrt{|g|} \left[\bar{C}_\alpha\, M^\alpha{}_\beta\, C^\beta + \frac{1}{2} b_\alpha C^{\alpha\beta} b_\beta \right],\\
&M^\alpha{}_\beta = \Box \delta^\alpha_\beta + \nabla_\beta \nabla^\alpha - 2\beta_g \nabla^\alpha \nabla_\beta.
\end{aligned} \tag{73}$$

In (73) we used the covariant gauge-fixing condition χ_μ with weight function $C^{\mu\nu}$ [20]. The standard (complex) FP-ghost and anti-ghost fields we denote by C^β and $\bar{C}_\alpha$ respectively. Due to the higher derivative character of our theory we are forced to introduce also a third (real-)ghost field [39], which we appoint b_α. The gauge-fixing parameters β_g and γ_g are dimensionless, while $[\alpha_g] = M^{4-D}$. We notice right here that in our theory the beta functions are independent of these gauge parameters (see [20] for a rigorous proof).

The partition function of the full quantum theory with the right functional measure compatible with BRST invariance reads

$$Z[g] = \int \mu(g,h) \prod_{\mu\leqslant\nu} \mathcal{D}h_{\mu\nu} \prod_\alpha \mathcal{D}\bar{C}_\alpha \prod_\beta \mathcal{D}C^\beta \prod_\gamma \mathcal{D}b_\gamma\, e^{i[S_{\rm HD}+S_{\rm gf}+S_{\rm gh}]}. \tag{74}$$

At one loop we can evaluate the functional integral explicitly and express the partition function as a product of functional determinants, namely

$$Z[g] = e^{i S_{\rm g}[g]} \left\{ {\rm Det} \left[\left. \frac{\delta^2 (S_{\rm HD}[g+h] + S_{\rm gf}[g+h])}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} \right|_{h=0} \right] \right\}^{-\frac{1}{2}} ({\rm Det}\, M^\alpha{}_\beta)\, ({\rm Det}\, C^{\mu\nu})^{\frac{1}{2}}.$$

By symbol $S_{\rm g}[g]$ we understand the classical functional of the gravitational action for the background metric g. To calculate the one-loop effective action we need to expand the action plus the gauge-fixing term to the second order in the quantum fluctuation $h_{\mu\nu}$

$$\hat{H}^{\mu\nu,\rho\sigma} = \left.\frac{\delta^2 S_{\rm HD}}{\delta h_{\mu\nu}\delta h_{\rho\sigma}}\right|_{h=0} + \left.\frac{\delta\chi_\delta}{\delta h_{\mu\nu}} C^{\delta\tau}\frac{\delta\chi_\tau}{\delta h_{\rho\sigma}}\right|_{h=0}. \tag{75}$$

Following [20] we can recast (75) for the simpler four-dimensional case in the following compact form

$$\begin{aligned}\hat{H}^{\mu\nu,\alpha\beta} = & \left(\frac{\omega_{\rm Ric}}{4} g^{\mu(\rho} g^{\nu)\sigma} - \frac{\omega_{\rm Ric}(\omega_{\rm Ric}+4\omega_{\rm R})}{16\omega_{\rm R}} g^{\mu\nu} g^{\rho\sigma}\right) \\ & \times \Big\{\delta^{\alpha\beta}_{\rho\sigma}\Box^{n+2} + V_{\rho\sigma}{}^{\alpha\beta,\lambda_1\ldots\lambda_{2n+2}}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n+2}} + \\ & + W_{\rho\sigma}{}^{\alpha\beta,\lambda_1\ldots\lambda_{2n+1}}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n+1}} + U_{\rho\sigma}{}^{\alpha\beta,\lambda_1\ldots\lambda_{2n}}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n}} + O(\nabla^{2n-1})\Big\},\end{aligned} \tag{76}$$

where $\delta^{\rho\sigma}_{\mu\nu} \equiv \delta^{(\rho}_{\mu}\delta^{\sigma)}_{\nu} = \frac{1}{2}\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} + \delta^{\sigma}_{\mu}\delta^{\rho}_{\nu}\right)$, and the tensors V, W and U depend on curvature tensors of the background metric and its covariant derivatives. In (76) the pre-factor in round brackets (called de Witt metric $\mathcal{G}^{\mu\nu,\rho\sigma}$) does not give any contribution to the divergences and, therefore, it can be omitted. The coefficients $\omega_{\rm R}$ and $\omega_{\rm Ric}$ (76) stay for $\omega_{{\rm R},n+{\rm N}}$ and $\omega_{{\rm Ric},n+{\rm N}}$ respectively. The tensor V is linear in a curvature tensor ($\mathcal{R}$), while the tensor U contains contributions quadratic in curvature ($\mathcal{R}^2$) and also terms with two covariant derivatives on one curvature ($\nabla^2\mathcal{R}$). We obtain expressions for U, V and W tensors by contracting the operator $\hat{H}^{\mu\nu,\alpha\beta}$ with the inverse de Witt metric and extracting at the end covariant derivatives. They have the canonical position of first matrix indices (two down followed by two up) thanks to the application of this metric in the field fluctuation space.

The one-loop effective action is defined by [20]

$$\Gamma^{(1)}[g] = -i\ln Z[g] = S_{\rm HD}[g] + \frac{i}{2}\ln{\rm Det}(\hat{H}) - i\ln{\rm Det}(\hat{M}) - \frac{i}{2}\ln{\rm Det}(\hat{C}). \tag{77}$$

Once the relevant contributions to the operator $\hat{H}$ are known we can apply the Barvinsky–Vilkovisky method [37] to extract the divergent part of $\ln{\rm Det}(\hat{H}^{\mu\nu,\alpha\beta})$. It is noteworthy that the formalism in [37] has been derived only in $D=4$ and its generalization to an arbitrary number of dimensions requires some extra efforts.

The explicit calculation of $\hat{H}$ in a D-dimensional spacetime goes beyond the scope of this paper and here we only offer the schematic tensorial structure in terms of the curvature tensors of the background metric and its covariant derivatives. For the action in (71), where we have only terms with maximal number of derivatives, here given by $2n+2{\rm N}+4$ and with front coefficients $\omega_{\rm Ric}$ and $\omega_{\rm R}$, the matrix in fully covariant form $H_{\mu\nu,\rho\sigma}$ consists solely of the terms coming from the vertices proportional to the *non-running* constants $\omega_{\rm Ric}$ and $\omega_{\rm R}$,

$$\begin{aligned}\hat{H}_{\mu\nu,\rho\sigma} = \mathcal{G}^{-1}_{\mu\nu,\alpha\beta}\Big(\Box^{n+{\rm N}+2} & + \underbrace{V^{\alpha\beta(n+{\rm N}+2)\,\lambda_1\ldots\lambda_{2n+2{\rm N}+2}}_{\rho\sigma}}_{\sim\mathcal{R}}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n+2{\rm N}+2}} \\ & + \underbrace{W^{\alpha\beta(n+{\rm N}+2)\,\lambda_1\ldots\lambda_{2n+2{\rm N}+1}}_{\rho\sigma}}_{\sim\nabla\mathcal{R}}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n+2N+1}} \\ & + \underbrace{U^{\alpha\beta(n+{\rm N}+2)\,\lambda_1\ldots\lambda_{2n+2{\rm N}}}_{4\,\rho\sigma}}_{\sim\mathcal{R}^2+\nabla^2\mathcal{R}}\nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n+2{\rm N}}}\end{aligned} \tag{78}$$

$$
\begin{aligned}
&+ \underbrace{U_{5\,\rho\sigma}^{\alpha\beta(n+\mathrm{N}+2)\,\lambda_1\ldots\lambda_{2n+2\mathrm{N}-1}}}_{\sim\nabla\mathcal{R}^2+\nabla^3\mathcal{R}} \nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n+2\mathrm{N}-1}} \\
&+ \underbrace{U_{6\,\rho\sigma}^{\alpha\beta(n+\mathrm{N}+2)\,\lambda_1\ldots\lambda_{2n+2\mathrm{N}-2}}}_{\sim\mathcal{R}^3+\nabla^2\mathcal{R}^2+\nabla^4\mathcal{R}} \nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n+2\mathrm{N}-2}} + \ldots \\
&+ \underbrace{U_{D\,\rho\sigma}^{\alpha\beta(n+\mathrm{N}+2)\,\lambda_1\ldots\lambda_{2n+2\mathrm{N}+4-D}}}_{\sim\mathcal{R}^{D/2}+\ldots+\nabla^{D-2}\mathcal{R}} \nabla_{\lambda_1}\cdots\nabla_{\lambda_{2n+2\mathrm{N}+4-D}} + O(\nabla^{2n+2\mathrm{N}+3-D})\Big). \qquad (79)
\end{aligned}
$$

We wrote above only terms giving rise to quantum divergences. We explicitly showed the relationship of the tensors

$$V^{(i)},\ W^{(i)},\ U_4^{(i)},\ U_5^{(i)},\ldots,U_D^{(i)}$$

(for the case $i = n + \mathrm{N} + 2$) to the background curvature tensors and its covariant derivatives. Employing the universal trace formulae of Barvinsky and Vilkovisky [37]

$$\mathrm{Tr}\ln\Box\Big|_{\mathrm{div}} \sim \frac{1}{\epsilon}\int d^Dx\sqrt{|g|}\left(\mathcal{R}^{\frac{D}{2}} + \nabla^2\mathcal{R}^{\frac{D}{2}-1} + \ldots + \nabla^{D-2}\mathcal{R}\right), \qquad (80)$$

$$\nabla^p\,\frac{1}{\Box^{\mathrm{N}+n+2}}\,\delta(x,y)\Big|_{\mathrm{div}}^{y\to x} \sim \frac{1}{\epsilon}\left(\mathcal{R}^{\frac{p}{2}-(n+\mathrm{N}+2)+\frac{D}{2}} + \ldots + \nabla^{p-2n-2\mathrm{N}-6+D}\mathcal{R}\right)$$
$$(p \leqslant 2n + 2\mathrm{N} + 4), \qquad (81)$$

we can derive the following divergent contribution to the effective action,

$$
\begin{aligned}
\Gamma^{(1)}_{\mathrm{div}} \sim -\frac{1}{\epsilon}\int d^Dx\sqrt{|g|}\Bigg[&\beta_{\bar\lambda} - 2\beta_{\kappa_D^{-2}}R + \sum_{i=0}^{\mathrm{N}}\left(\beta_{a_i}R\,\Box^i R + \beta_{b_i}R_{\mu\nu}\Box^i R^{\mu\nu}\right) \\
&+ \sum_{j=3}^{\mathrm{N}+2}\sum_{k=3}^{j}\sum_i \beta_{c^{(j)}_{k,i}}\left(\nabla^{2(j-k)}\mathcal{R}^k\right)_i\Bigg],
\end{aligned}
$$

where all the beta functions depend on the "non-running" constants $\omega_{\mathrm{Ric},i}$ or $\omega_{\mathrm{R},i}$.

References

[1] T.D. Lee, G.C. Wick, Phys. Rev. D 2 (1970) 1033;
T.D. Lee, G.C. Wick, Nucl. Phys. B 9 (1969) 209.
[2] R.E. Cutkosky, P.V. Landshoff, D.I. Olive, J.C. Polkinghorne, Nucl. Phys. B 12 (1969) 281.
[3] L. Modesto, I.L. Shapiro, Phys. Lett. B 755 (2016) 279, arXiv:1512.07600 [hep-th].
[4] I.L. Shapiro, Phys. Lett. B 744 (2015) 67, arXiv:1502.00106 [hep-th].
[5] L. Modesto, Phys. Rev. D 86 (2012) 044005, arXiv:1107.2403 [hep-th].
[6] L. Modesto, Astron. Rev. 8 (2) (2013) 4–33, arXiv:1202.3151 [hep-th];
L. Modesto, arXiv:1202.0008 [hep-th];
L. Modesto, arXiv:1206.2648 [hep-th].
[7] S. Alexander, A. Marciano, L. Modesto, Phys. Rev. D 85 (2012) 124030, arXiv:1202.1824 [hep-th].
[8] F. Briscese, A. Marciano, L. Modesto, E.N. Saridakis, Phys. Rev. D 87 (2013) 083507, arXiv:1212.3611 [hep-th].
[9] G. Calcagni, L. Modesto, Phys. Rev. D 91 (12) (2015) 124059, arXiv:1404.2137 [hep-th].
[10] N.V. Krasnikov, Teor. Mat. Fiz. 73 (2) (November 1986) 235–244.
[11] G.V. Efimov, Nonlocal Interactions, Nauka, Moscow, 1977 [in Russian];
G.V. Efimov, review paper [in Russian];
V.A. Alebastrov, G.V. Efimov, Commun. Math. Phys. 31 (1973) 1–24;
V.A. Alebastrov, G.V. Efimov, Commun. Math. Phys. 38 (1974) 11–28;
G.V. Efimov, Theor. Math. Phys. 128 (3) (2001) 1169–1175.

[12] E.T. Tomboulis, arXiv:hep-th/9702146v1.
[13] E.T. Tomboulis, Mod. Phys. Lett. A 30 (03n04) (2015) 1540005, http://dx.doi.org/10.1142/S0217732315400052.
[14] E.T. Tomboulis, Phys. Rev. D 92 (12) (2015) 125037, http://dx.doi.org/10.1103/PhysRevD.92.125037, arXiv: 1507.00981 [hep-th].
[15] G. Calcagni, J. High Energy Phys. 0605 (2006) 012, arXiv:hep-th/0512259;
G. Calcagni, M. Montobbio, G. Nardelli, Phys. Rev. D 76 (2007) 126001, arXiv:0705.3043 [hep-th];
G. Calcagni, G. Nardelli, Int. J. Mod. Phys. D 19 (2010) 329, arXiv:0904.4245 [hep-th];
G. Calcagni, G. Nardelli, Phys. Rev. D 82 (2010) 123518, arXiv:1004.5144 [hep-th];
G. Calcagni, M. Montobbio, G. Nardelli, Phys. Lett. B 662 (2008) 285, arXiv:0712.2237 [hep-th].
[16] M. Jaccard, M. Maggiore, E. Mitsou, Phys. Rev. D 88 (2013) 044033, arXiv:1305.3034 [hep-th];
S. Foffa, M. Maggiore, E. Mitsou, Phys. Lett. B 733 (2014) 76, arXiv:1311.3421 [hep-th];
Y. Dirian, S. Foffa, N. Khosravi, M. Kunz, M. Maggiore, J. Cosmol. Astropart. Phys. 1406 (2014) 033, arXiv: 1403.6068 [astro-ph.CO];
G. Cusin, J. Fumagalli, M. Maggiore, arXiv:1407.5580 [hep-th].
[17] L. Modesto, L. Rachwal, Nucl. Phys. B 889 (2014) 228, arXiv:1407.8036 [hep-th].
[18] L. Modesto, L. Rachwal, Nucl. Phys. B 900 (2015) 147, http://dx.doi.org/10.1016/j.nuclphysb.2015.09.006, arXiv: 1503.00261 [hep-th].
[19] K.S. Stelle, Phys. Rev. D 16 (1977) 953–969.
[20] M. Asorey, J.L. Lopez, I.L. Shapiro, Int. J. Mod. Phys. A 12 (1997) 5711–5734, arXiv:hep-th/9610006.
[21] B. Grinstein, D. O'Connell, M.B. Wise, Phys. Rev. D 77 (2008) 025012, arXiv:0704.1845 [hep-ph].
[22] M.B. Wise, Int. J. Mod. Phys. A 25 (2010) 587, arXiv:0908.3872 [hep-ph].
[23] R.S. Chivukula, A. Farzinnia, R. Foadi, E.H. Simmons, Phys. Rev. D 82 (2010) 035015, arXiv:1006.2800 [hep-ph];
R.S. Chivukula, A. Farzinnia, R. Foadi, E.H. Simmons, Phys. Rev. D 81 (2010) 095015, arXiv:1002.0343 [hep-ph].
[24] B. Grinstein, D. O'Connell, M.B. Wise, Phys. Rev. D 79 (2009) 105019, arXiv:0805.2156 [hep-th].
[25] E. Tomboulis, Phys. Lett. B 70 (1977) 361.
[26] E. Tomboulis, Phys. Lett. B 97 (1980) 77.
[27] I. Antoniadis, E.T. Tomboulis, Phys. Rev. D 33 (1986) 2756.
[28] E.S. Fradkin, Arkady A. Tseytlin, Phys. Lett. B 104 (1981) 377–381;
E.S. Fradkin, Arkady A. Tseytlin, Nucl. Phys. B 201 (1982) 469–491;
I.G. Avramidi, A.O. Barvinsky, Phys. Lett. B 159 (1985) 269–274;
J. Julve, M. Tonin, Nuovo Cimento B 46 (1978) 137–152.
[29] K. Jansen, J. Kuti, C. Liu, Phys. Lett. B 309 (1993) 119, arXiv:hep-lat/9305003.
[30] M.J.G. Veltman, Physica 29 (1963) 186.
[31] H. Yamamoto, Prog. Theor. Phys. 42 (1969) 707;
H. Yamamoto, Prog. Theor. Phys. 44 (1970) 272;
H. Yamamoto, K. Kudo, Prog. Theor. Phys. 53 (1975) 275.
[32] C. Liu, arXiv:0704.3999 [hep-ph].
[33] N. Barnaby, N. Kamran, J. High Energy Phys. 0802 (2008) 008, arXiv:0709.3968 [hep-th].
[34] S.A. Fulling, R.C. King, B.G. Wybourne, C.J. Cummins, Class. Quantum Gravity 9 (1992) 1151.
[35] G.A. Vilkovisky, Nucl. Phys. B 234 (1984) 125.
[36] A.O. Barvinsky, G.A. Vilkovisky, Phys. Lett. B 131 (1983) 313.
[37] Barvinsky, Vilkovisky, Phys. Rep. 119 (1) (1985) 1–74.
[38] A. van Tonder, Int. J. Mod. Phys. A 22 (2007) 2563, arXiv:hep-th/0610185;
A. van Tonder, arXiv:0810.1928 [hep-th].
[39] I.L. Buchbinder, S.D. Odintsov, I.L. Shapiro, Effective Action in Quantum Gravity, IOP Publishing Ltd, 1992.
[40] A. Accioly, A. Azeredo, J. Math. Phys. 43 (1) (January 2002).
[41] P. Van Nieuwenhuizen, Nucl. Phys. B 60 (1973) 478–492.
[42] L. Modesto, T.d.P. Netto, I.L. Shapiro, arXiv:1412.0740 [hep-th].
[43] L. Modesto, J.W. Moffat, P. Nicolini, Phys. Lett. B 695 (2011) 397–400, arXiv:1010.0680v3 [gr-qc];
P. Nicolini, arXiv:1202.2102 [hep-th].
[44] C. Bambi, D. Malafarina, L. Modesto, Phys. Rev. D 88 (2013) 044009, arXiv:1305.4790 [gr-qc].
[45] C. Bambi, D. Malafarina, L. Modesto, Eur. Phys. J. C 74 (2014) 2767, arXiv:1306.1668 [gr-qc].
[46] G. Calcagni, L. Modesto, P. Nicolini, Eur. Phys. J. C 74 (8) (2014) 2999, http://dx.doi.org/10.1140/epjc/s10052-014-2999-8, arXiv:1306.5332 [gr-qc].
[47] T. Biswas, A. Mazumdar, W. Siegel, J. Cosmol. Astropart. Phys. 0603 (2006) 009, arXiv:hep-th/0508194v2.
[48] A.S. Koshelev, arXiv:1302.2140 [astro-ph.CO].

[49] T. Biswas, A.S. Koshelev, A. Mazumdar, S.Yu. Vernov, J. Cosmol. Astropart. Phys. 1208 (2012) 024, arXiv:1206.6374 [astro-ph.CO].
[50] A.S. Koshelev, S.Yu. Vernov, Phys. Part. Nucl. 43 (2012) 666–668, arXiv:1202.1289 [hep-th].
[51] A.S. Koshelev, arXiv:1112.6410 [hep-th].
[52] X. Calmet, Mod. Phys. Lett. A 29 (2014) 1450204, arXiv:1410.2807 [hep-th].
[53] X. Calmet, D. Croon, C. Fritz, arXiv:1505.04517 [hep-th].
[54] X. Calmet, R. Casadio, arXiv:1509.02055 [hep-th].
[55] X. Calmet, Class. Quantum Gravity 32 (4) (2015) 045007, arXiv:1412.6270 [gr-qc].
[56] G. 't Hooft, in: Columbia 1992, Proceedings, Quantum Coherence and Reality, pp. 158–171, and Utrecht U. - THU-93-04 (93,rec.Mar.) 17 p.;
G. 't Hooft, in: Gravitation and Quantizations. Proceedings, 57th Session of the Les Houches Summer School in Theoretical Physics, NATO Advanced Study Institute, Les Houches, France, July 5–August 1, 1992.
[57] G. 't Hooft, Nucl. Phys. B 256 (1985) 727.
[58] Y.D. Li, L. Modesto, L. Rachwal, J. High Energy Phys. 1512 (2015) 173, http://dx.doi.org/10.1007/JHEP12(2015)173, arXiv:1506.08619 [hep-th].
[59] E. Tomboulis, Phys. Lett. B 70 (1977) 361.
[60] E. Tomboulis, Phys. Lett. B 97 (1980) 77.
[61] L. Smolin, Nucl. Phys. B 208 (1982) 439.
[62] A. van Tonder, arXiv:0810.1928 [hep-th].