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QUARTER INTEGER RESONANCE BY SEXTUPOLES

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I. It is generally stated that the betatron oscillation resonances of the type $n_x \nu_x + n_y \nu_y = k$ (n_x, n_y, k = integers) are due to n-th derivatives of the magnetic field ($\partial^n B/\partial x^n$, $\partial^n B/\partial y^n$, etc.), where $n = |n_x| + |n_y| - 1$, provided that the radius of curvature and the betatron period are both large compared with magnet lengths. For example, a sextupole field (n=2) produces third-integer resonances ($3\nu_x = k$, $\nu_x \pm 2\nu_y = k$, etc.) and an octupole field (n=3) gives rise to quarter-integer resonances ($4\nu_x = k$, $2\nu_x \pm 2\nu_y = k$, etc.). It is therefore natural that, when quarter-integer resonances were detected in the main ring at 8 GeV (injection), the analysis was attempted to understand the phenomena in terms of the octupole field which exists in quadrupoles.[1] However, the magnitude of the octupole field necessary to explain the resonance was almost an order of magnitude larger compared to the result of magnetic field measurement. Furthermore, it has been observed that quarter-integer resonances are rather sensitive to the setting of correction sextupoles which are used to suppress the nearby third-integer resonances $3\nu_x = 60, 61$ and $\nu_x + 2\nu_y = 60, 61$.

A similar problem was discussed during the recent PEP Summer Study in connection with the long-term beam stability in PEP and ISABELLE.[2] In both cases, superconducting magnets will be used for the proton ring so that a fairly strong sextupole field is unavoidable. Presumably, correction sextupoles will be installed around the ring to compensate for the chromaticity (momentum dependence of tunes) and to eliminate harmonic components of the sextupole field that can drive nearby third-integer resonances. Numerical studies at BNL demonstrated that, under this kind of arrangement, the beam size grows when the tune is near a quarter-integer value.[3]

There is nothing mysterious about sextupole field being a cause of quarter-integer resonances. It has been known, at least theoretically, for many years.[4] The statement on the relation between the order of a resonance, $|n_x| + |n_y|$, and the order of the field derivative, n, is valid when one retains only the so-called "driving" term and the "phase-independent" term in the Hamiltonian and ignores all other terms which are of oscillating form. By means of a canonical transformation, one can show that these oscillating terms not only modify the original resonance but can also drive many other resonances as well when the tune satisfies a resonance condition. While the original (first-order) driving term is proportional to a specific harmonic component of the field, driving terms arising from oscillating terms in the original Hamiltonian are proportional to the product of two (or more) harmonic components so that their effects are usually negligible. On the other hand, if the first-order resonance is very weak (because of correction elements or the tune is too far away from the resonance value), second-order resonances can cause a serious growth in beam size when the tune is just "right" for one of these resonances. In the main ring, the horizontal tune is $20.2 \sim 20.3$. One can suppress the resonance $3\nu_x = 61$ by correction sextupoles with the proper 61st harmonic component. However, the product of n-th and $(81 \pm n)$th harmonic components can drive the resonance $4\nu_x = 81$ if the tune is very close to 20.25. A combination like 54th and 27th, 60th and 21st, etc. may be especially dangerous since all (6n)th harmonic components of the sextupole field are intrinsic in the main ring.

Unfortunately, the picture is much more complicated in reality. There are small but finite amounts of nonlinear fields with $n \geq 3$. For a sufficiently large amplitude of the betatron oscillation, many different resonances contribute and their effects may overlap in a complicated manner. There is as yet no general analytical treatment for this "stochastic" situation.[5] Second-order effects of sextupole field must be partially responsible for driving the quarter-integer resonance in the main ring but it is unlikely that they are the sole cause.[6]

The main purpose of this note is simply to emphasize the known fact, that sextupole fields can drive quarter-integer resonances when the second-order effect is taken into account. In section II, relevant formulas are given only to the extent that general characteristics of the second-order effects may be seen. The real main ring field is not amenable to an analytical treatment since there is no simple relation among amplitudes and phases of harmonic components. Therefore, in section III, the formalism is applied to two simple models with δ-function sextupoles. Hopefully these models simulate the essential feature of the main ring. The prediction based on this analysis is then compared with numerical results.

II. Since detailed, step-by-step constructions of the formalism are available elsewhere[4], a very limited case of a linear machine with sextupole fields alone will be discussed here. Horizontal-vertical coupling resonances arising from the sextupole field are not considered. The betatron oscillation of a particle is described by a Hamiltonian $H(\psi, I;\phi)$ in which the independent variable ϕ is the normalized betatron oscillation phase

$$\phi = \int_0^s \frac{ds}{\nu\beta(s)} \tag{1}$$

and the canonical variables (ψ, I), which are called the action-angle variables, are related to the transverse coordinate x of the particle by

$$x = \sqrt{\beta} \ \sqrt{2I} \quad \sin(\psi) \tag{2}$$

$$dx/ds = \frac{1}{\sqrt{\beta}} \sqrt{2I} \ \langle\cos(\psi) - \alpha \sin(\psi)\rangle \tag{3}$$

Other symbols are standard in the treatment of betatron motions. With the integrated field strength of each sextupole $(B''\ell)_i$, the Hamiltonian is

$$H(\psi,I;\phi) = \nu I + (2I)^{3/2} \sum_k \sum_m A_{km} \cos(k\psi - m\phi + a_{km}) \tag{4}$$

where $k = 1$ and 3, $m = -\infty \sim \infty$ and

$$A_{3m}\, e^{ia_{3m}} = (1/48\pi)(1/B\rho) \sum_i \beta_i^{3/2} (B''\ell)_i\, e^{i(m\phi_i + \pi/2)}, \tag{5}$$

$$A_{1m} = 3A_{3m}\,, \quad a_{1m} = a_{3m} + \pi. \tag{6}$$

In the absence of sextupole field ($A_{km} = 0$),

$$\psi = \int (\partial H/\partial I)\, d\phi = \nu\phi$$

so that each sextupole term in the Hamiltonian is approximately given by

$$\sim A_{km} \cos\langle(k\nu - m)\phi + a_{km}\rangle.$$

The term with $|k\nu - m| << 1$ is called the driving term for the resonance $\nu = m/k$ and, in the first-order approximation, all other oscillatory terms are ignored as their effects will not increase with the number of revolutions. In the main ring, when the tune is close to 20 1/3, the driving term is the one with k = 3 and m = 61. If the tune is very close to 20 1/4, there is no driving term* in the Hamiltonian so that one would expect the beam to behave as if the machine were entirely linear.

One must proceed to the next order of approximation to see that this is not the case. This can be done by means of a canonical transformation to new variables Θ and J. The generating function is

$$S(\psi,\ J;\ \phi) = \psi J + (2J)^{3/2}\ \Sigma\frac{A_{km}}{m - k\nu}\sin(k\psi - m\phi + a_{km}) \tag{7}$$

and

$$I = \partial S/\partial\psi = J + (2J)^{3/2}\ \Sigma\frac{kA_{km}}{m - k\nu}\cos(k\psi - m\phi + a_{km}), \tag{8}$$

$$\Theta = \partial S/\partial J = \psi + (2J)^{1/2}\ \Sigma\frac{3A_{km}}{m - k\nu}\sin(k\psi - m\phi + a_{km}). \tag{9}$$

The new Hamiltonian is

$$K(\Theta,\ J;\ \phi) = H(\psi,\ I;\ \phi) + \partial S/\partial\phi =$$

$$= \nu J + \langle(2I)^{3/2} - (2J)^{3/2}\rangle \sum_{k,m} A_{km}\cos(k\psi - m\phi + a_{km}). \tag{10}$$

The new Hamiltonian contains, in addition to the linear term νJ, an infinite number of terms proportional to $J^{n/2}$, n = 4, 5, ... The lowest-order term proportional to J^2 represents an octupole-like characteristic. If one started with an

*One may still regard the term with k=3 and m=61 as the driving term since it has the weakest dependence on ϕ. However, the stable area in this approximation comes out to be generally much larger than the area occupied by the field, clearly a meaningless result.

octupole field instead of a sextupole field, the original Hamiltonian H would have $(2I)^2$ terms in place of $(2I)^{3/2}$ terms. In the lowest-order approximation,

$$K(\theta,\ J;\ \phi) = \nu J + (3/2)(2J)^2 \sum_{k,m} \sum_{j,n} \frac{kA_{km}A_{jn}}{m - k\nu} \times$$

$$\times\{\cos <(j-k)\theta - (n-m)\phi + a_{jn} - a_{km}> + + \cos <(j+k)\theta - (n+m)\phi + a_{jn} + a_{km}>\} \qquad (11)$$

Since j, k = 1 or 3, possible resonances are of the type 2ν = integer, 4ν = integer and 6ν = integer. The Hamiltonian also contains phase-independent terms (j=k and n=m), another characteristic of octupole field, that are responsible for the dependence of the tune on the oscillation amplitude. Phase-independent terms tend to limit the growth of the amplitude to a finite value creating islands of stable regions outside the central stable area. On the other hand, for a given value of ν, the central stable area is reduced by these terms so that they are not necessarily beneficial.[7]

Keeping only phase-independent terms and driving terms for $4\nu = N$ when the tune is close to N/4, one finds

$$K = \nu J + (3/2)(2J)^2 \sum \sum \frac{kA_{km}^2}{m - k\nu}$$

$$+ (3/2)(2J)^2 \sum_k \sum_m \frac{kA_{km}A_{jn}}{m - k\nu} \cos(4\theta - N\phi + a_{km} + a_{jn}) \qquad (12)$$

with j = 4 − k and n = N − m. Aside from the obvious fact that driving terms are now proportional to products of two harmonic components $A_{km}A_{jn}$, there is an important difference between the original Hamiltonian H and the new Hamiltonian K. In K, the amplitude of driving term (and the magnitude of phase-independent term) depends strongly on the value of ν whereas it is entirely independent of ν in H. In terms of new variables Θ and J, one can express x and dx/ds by the following relations:

$$x = \sqrt{\beta}\,\sqrt{2I}\,\sin(\psi) \equiv \sqrt{\beta}\,\xi \tag{13}$$

$$dx/ds = (1/\sqrt{\beta})\,\sqrt{2I}\,\langle\cos(\psi) - \alpha\sin(\psi)\rangle$$

$$\equiv (1/\sqrt{\beta})(\eta - \alpha\xi) \tag{14}$$

$$\xi = X + \sum_m \frac{3A_m}{m-\nu}\langle 2XY\cos(a_m) + (3Y^2 + X^2)\sin(a_m)\rangle$$
$$- \sum_m \frac{3A_m}{m-3\nu}\langle 2XY\cos(a_m) + (Y^2 - X^2)\sin(a_m)\rangle \tag{15}$$

$$\eta = Y - \sum_m \frac{3A_m}{m-\nu}\langle(3X^2 + Y^2)\cos(a_m) + 2XY\sin(a_m)\rangle$$
$$+ \sum_m \frac{3A_m}{m-3\nu}\langle(Y^2 - X^2)\cos(a_m) - 2XY\sin(a_m)\rangle \tag{16}$$

where

$$A_m \equiv A_{3m}\ (= A_{1m}/3);\ a_m \equiv a_{3m} (= a_{1m} - \pi), \tag{17}$$

$$X \equiv \sqrt{2J}\,\sin(\Theta) \text{ and } Y \equiv \sqrt{2J}\,\cos(\Theta). \tag{18}$$

The Hamiltonian K is still a function of the independent variable ϕ so that it is not a constant of the motion. This ϕ dependence is eliminated by a transformation from (Θ, J) to a rotating system $(\Theta_1 \equiv \Theta - N\,\phi/4, J)$ and the corresponding Hamiltonian is

$$K_2\ (\Theta_, J) = \Delta J + (2J)^2 < S_o + S_1 \cos(4\Theta_1 + a)> \tag{19}$$

where

$$\Delta \equiv \nu - N/4, \tag{20}$$

$$S_o \equiv (3/2) \sum_{k,m} \frac{kA_{km}^2}{m - k\nu}, \tag{21}$$

$$S_1 \cos(4\Theta_1 + a)$$

$$\equiv (3/2) \sum_{k,m} \frac{kA_{km}A_{jn}}{m - k\nu} \cos(4\Theta_1 + a_{km} + a_{jn}). \tag{22}$$

$$(j=4-k,\ n=N-m)$$

Note that there is no difference between Θ and Θ_1 at $\phi = 0$, $8\pi/N$, $16\pi/N$, etc. At other locations, the phase space picture in (X,Y) space can be found by simply rotating (clockwise) the picture at $\phi = 0$ by $N\phi/4$.

III. Once the Hamiltonian $K_2(\Theta_1, J)$ is obtained, the procedure of finding stable and unstable fixed points together with resulting separatrices is well known. One draws a phase space diagram (K_2 = constant) in (X, Y) space*, rotates it by $N\phi/4$ if necessary and finds the corresponding diagram in (x, dx/ds) space by means of (13) - (16). Unfortunately, this is not easy to do for the main ring. In principle one must know all A_{km} and a_{km} for the sextupole field in order to perform the necessary summations in (15), (16), (21) and (22). One might keep only a few dominant terms ($|m - k\nu|$ small)

* $X = \sqrt{2J} \sin \Theta_1$, $Y = \sqrt{2J} \cos \Theta_1$.

in the summation but the resulting accuracy would be hard to estimate. Therefore, two simple cases have been studied to get some feelings on the importance of the higher order effects and to compare the analytical prediction with numerical results. In both cases, sextupoles are of the δ-function form and they are arranged at an equal interval around the ring. Summations are then possible without dropping any term.

Case A

Three sextupoles of strength $B''\ell$ each are located at $\phi = \pi/3$, π, and $5\pi/3$ where $\beta = \beta_s$. Because of the three-fold symmetry, A_{km} vanishes unless m is a multiple of 3. From (5) and (6),

$$A_{3m} = A_{1m}/3 = (\beta_s^{3/2}/16\pi)\ \frac{B''\ell}{B\rho} \equiv A, \qquad (23)$$

$$a_{3m} = a_{1m} - \pi = \pi/2 \text{ for } m/3 = \text{even}$$

$$= -\ \pi/2 \text{ for } m/3 = \text{odd}. \qquad (24)$$

The tune of the machine is assumed to be near 20.25 and the resonance one is interested in is $4\nu = 81$ ($N = 81$). From (21) and (22),*

$$S_o = (3\pi/2)A^2 < -\ \cot(\varepsilon\pi) + 3\ \cot(\varepsilon'\pi) >, \qquad (25)$$

$$* \sum_n \frac{1}{n \pm a} = \mp\ (\pi/2)\ \tan\ (a\pi/2) \text{ for n=positive and negative \underline{odd} integers,}$$

$$= \pm\ (\pi/2)\ \cot\ (a\pi/2) \text{ for n=0, positive and negative \underline{even} integers.}$$

$$S_1 = (3\pi/2)A^2 < 3\cot(\varepsilon\pi) - \cot(\varepsilon'\pi)>, \tag{26}$$

where $\varepsilon \equiv (\nu - 20)$ and $\varepsilon' \equiv (21 - \nu)/3$. Note that $a_{km} + a_{jn} = \pi$ in (22) so that one can put $a = 0$. From (15) and (16),

$$\xi = X + \pi A < (Y^2 - X^2)/\sin(\varepsilon\pi) - (X^2 + 3Y^2)/\sin(\varepsilon'\pi)>, \tag{27}$$

$$\eta = Y + 2\pi AXY < 1/\sin(\varepsilon\pi) + 1/\sin(\varepsilon'\pi)> . \tag{28}$$

Both phase-independent terms S_o and the amplitude of driving term S_1 in the Hamiltonian K_2 are a function of the tune. In particular, $S_o > S_1$ for $\nu > 20.25$ while $S_o < S_1$ for $\nu < 20.25$.

Unstable fixed points are at $\Theta_1 = 0$, $\pi/2$, π, and $3\pi/2$ with

$$(2J) = -\frac{\Delta}{4(S_o + S_1)} . \tag{29}$$

They exist only for $\Delta < 0$ ($\nu < 20.25$) since $J > 0$ for real values of x and dx/ds. Stable fixed points are located at $\Theta_1 = \pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$ with

$$(2J) = \frac{\Delta}{4(S_1 - S_o)} . \tag{30}$$

Because of the tune dependence of S_o and S_1, these stable points do not exist either for $\nu > 20.25$ or for $\nu < 20.25$. Therefore, in the second-order approximation, the analytical prediction is

$\nu > 20.25$	no fixed points
$\nu < 20.25$	four unstable fixed points which coalesce to $J = 0$ ($x = dx/ds = 0$) at $\nu = 20.25$.

A computer program has been used to find fixed points for a given configuration of sextupoles. Parameters chosen for this case are: $B''\ell = 50$ kG/m, $B\rho = 296.5$ kG-m (injection) and $\beta_s = 100$ m. With this choice, harmonic components for $m = 6k (k = \pm 1, \pm 2, \ldots)$ are approximately the same as what exist in the main ring when no harmonic correction is applied. The average term, $m = 0$, is much larger than this in the main ring but correction sextupoles for the chromaticity reduce it to the same level or less. Components with $m = 3k$ $(k = \pm 1, \pm 3, \ldots)$ are due to the fluctuation in $B''\ell$ from magnet to magnet and they are 10 - 15% of what is used in this model. Also, there should be no simple phase relation among harmonics so that the choice of $B''\ell = 50$ kG/m is clearly an overestimate. There are, of course, other harmonic components $(m = \pm 1, \pm 2, \pm 4, \ldots)$ as well in the main ring but they are not considered in this model. The analytical prediction is compared with numerically computed values of three unstable fixed points in Table 1. Agreements are very good when the tune is not too far away from 20.25. As predicted analytically, there is no fixed point for $\nu > 20.25$. For a large value of $(20.25 - \nu)$, fixed points are located at long distances from the origin (large values of J) and the neglected higher order terms become important. Higher order terms are also responsible for the appearance of outer stable fixed points which are not predicted analytically but are found numerically for $\nu < 20.25$. Although they too coalesce to the origin at ν = 20.25, the "speed" of the coalescence is much slower than that of unstable points. As a consequence, while the central stable area shrinks as ν approaches 20.25 from below and disappears beyond 20.25, four islands of outer stable area around stable points stick out like four petals of a flower and remain that way even beyond 20.25. Since there is no fixed point beyond 20.25, particle trajectories in (x,dx/ds) space are all around the origin but they are substantially distorted because of the lingering of these shapes. Numerically obtained fixed points are plotted in Figure 1 and sometrajectories near ν = 20.25 are shown in Figure 2. Scales in Figure 2 for x and dx/ds are chosen such that trajectories are all circular if $B''\ell = 0$ (or $B'' = 0$ but the first-order effect only). The serious nature of the second-order effects is clearly seen here.

Case B

In order to make the model somewhat more realistic, six sextupoles are distributed around the ring at an equal interval with the strength

$$B''\ell = -\ 25.\ (1 + f)\,\mathrm{kG/m} : \phi = \pi/6,\ 5\pi/6,\ 3\pi/2;$$

$$= -\ 25.\ (1 - f)\,\mathrm{kG/m} : \phi = \pi/2,\ 7\pi/6,\ 11\pi/6.$$

From (5) and (6)

$$\begin{aligned} A_{3m} = A_{1m}/3 &= (\beta_s^{3/2}/8\pi)(25/B\rho) \equiv A \\ &\quad \text{for } m = 0,\ \pm\ 6,\ \pm\ 12,\ \text{etc.,} \\ &= fA \text{ for } m = \pm\ 3,\ \pm\ 9,\ \text{etc.,} \\ &= 0 \text{ for } m = \pm 1,\ \pm\ 2,\ \pm\ 4,\ \text{etc.} \end{aligned} \qquad (31)$$

$$a_{3m} = -\pi/2 \text{ for } m/6 = 0 \text{ or an even integer,}$$
$$= \pi/2 \text{ for } m/6 = \text{an odd integer,}$$
$$= 0 \text{ for } (m - 3)/6 = 0 \text{ or an even integer,}$$
$$= \pi \text{ for } (m - 3)/6 = \text{an odd integer;}$$
$$a_{1m} = a_{3m} + \pi. \qquad (32)$$

By taking $f = 0.15$, one simulates the sextupole field in the main ring fairly well as far as the amplitude of each harmonic component is concerned.* Phase relations are of course impossible to simulate. Dropping terms of the order f^2, one finds

$$S_o = (3\pi A^2/4) < \cot(\varepsilon\pi/2) + 3\tan(\varepsilon'\pi/2) >, \qquad (33)$$

$$S_1 = (3\pi fA^2/2) < 3\cot(\varepsilon\pi) - \cot(\varepsilon'\pi) >, \qquad (34)$$

$$\varepsilon \equiv \nu - 20, \qquad \varepsilon' \equiv (21 - \nu)/3,$$

and

$$K_2(\Theta_1, J) = \Delta J - (2J)^2 < S_o + S_1 \sin(4\Theta_1) >, \qquad (35)$$

$$\Delta \equiv \nu - 81/4.$$

Note that, in (22), $a_{km} + a_{jn} = 3\pi/2$ and the driving term is proportional to $\sin(4\Theta_1)$ instead of $\cos(4\Theta_1)$. From (15) and (16),

$$\xi = X + (\pi A/2) < (X^2 - Y^2)/\sin(\varepsilon\pi/2) -$$
$$- (X^2 + 3Y^2)/\cos(\varepsilon'\pi/2) >$$
$$- \pi fAXY < 1/\cos(\varepsilon\pi/2) + 1/\sin(\varepsilon'\pi/2) >, \qquad (36)$$

*Contributions from $m \neq 3k$ in the main ring are again ignored here. They are of the order f^2.

$$\eta = Y + \pi AXY < 1/\cos(\varepsilon'\pi/2) - 1/\sin(\varepsilon\pi/2) >$$

$$+ (\pi fA/2) < (Y^2 - X^2)/\cos(\varepsilon\pi/2) +$$

$$+ (3X^2 + Y^2)/\sin(\varepsilon'\pi/2) >. \tag{37}$$

With f = 0.15, $S_o > S_1$ for $\nu < 20.25$ as well as for $\nu >$ 20.25. Unstable fixed points are at $\theta_1 = \pi/8$, $5\pi/8$, $9\pi/8$ and $13\pi/8$ with

$$(2J) = \frac{\Delta}{4(S_o + S_1)} . \tag{38}$$

These points exist only for $\Delta > 0$ ($\nu > 20.25$). Stable fixed points are at $\theta_1 = 3\pi/8$, $7\pi/8$, $11\pi/8$ and $15\pi/8$ with

$$(2J) = \frac{\Delta}{4(S_o - S_1)} . \tag{39}$$

They too exist only for $\nu > 20.25$ since $S_o > S_1$. The analytical prediction is:

$\nu < 20.25$	no fixed points
$\nu > 20.25$	four stable and four unstable fixed points, all of which coalesce to $J = 0$ at $\nu = 20.25$.

The situation is opposite to what happened for Case A. This conclusion is independent of the sign of $B''\ell$. Because of the small factor f in S_1, distance to unstable points and stable points are not much different from each other at a given value of ν and the distortion of particle trajectories should be less pronounced for this case. A series of stable and unstable fixed points are listed in

Table 2. Agreements of analytical values with numerical results are again generally good when the tune is close to 20.25. No fixed point was found by the computer program for $\nu < 20.25$. All fixed points are shown in Figure 3 and some trajectories for $\nu = 20.2505$ are given in Figure 4.

I am grateful to A.G. Ruggiero for calling my attention to this problem.

References

1. NAL Accelerator Experiment No. 39, April 9, 1973.

2. A.G. Ruggiero, private communication.

3. A.G. Ruggiero, private communication. The study was made by R. Chasman for ISABELLE. L. Smith and R.L. Gluckstern have also investigated the same problem. Ruggiero treated the problem analytically with a somewhat different approach than the one presented here.

4. See, for example, K.R. Symon, NAL FN-140, April 29, 1968; F.T. Cole, NAL TM-179, June 13, 1969. In a way, the present note is an unintentional substitute for the report which was promised by Cole to follow his TM note.

5. See A.G. Ruggiero, NAL FN-249, October 1972 for references related to this topic. A claim that a substantial progress has been made recently in this area does not seem to be universally accepted.

6. An experiment by R. Stiening was not conclusive in establishing the relative importance of sextupole field. He introduced a large 21st sextupole harmonic in the main ring

in order to see the coupling effect of 60th, which is large in the main ring, and 21st harmonic components on the resonance $4\nu_x = 81$. Unfortunately, he was unable to vary the phase of the 21st harmonic and the beam was rather insensitive to the change in the magnitude.

7. P.A. Sturrock, Annals of Physics 3, 113 (1958). See pp. 132 - 136 and pp. 151 - 155.

Table 1. Unstable fixed points for Case A

$\phi = 0,\ 8\pi/81,\ 16\pi/81, \ldots\ldots$

$\xi = x/\sqrt{\beta}$

$\eta = \sqrt{\beta}\ (dx/ds) + (\alpha/\sqrt{\beta})\ x$

All values are in 10^{-3} (meter)$^{1/2}$. For each ν, the first row is analytical and the second row is numerical results.

$(20.25 - \nu)$	ξ_1	η_1	ξ_2	ξ_3
0.3	− .884	5.74	4.72	−6.76
	−1.090	5.87	5.16	−7.34
0.02	− .628	4.75	4.06	−5.43
	− .718	4.81	4.30	−5.73
0.01	− .333	3.395	3.05	−3.74
	− .355	3.416	3.13	−3.84
0.005	− .171	2.414	2.24	−2.59
	− .177	2.421	2.27	−2.62
0.002	−.0695	1.532	1.462	−1.602
	−.0704	1.534	1.470	−1.610
0.0005	−.01752	.7672	.7496	−.7847
	−.01757	.7674	.7506	−.7858

$\eta_2 = \eta_3 = 0.\ ;\quad \xi_4 = \xi_1,\quad \eta_4 = -\eta_1\ .$

Table 2. Fixed points for Case B.

$$\phi = 0,\ 8\pi/81,\ 16\pi/81,\ \ldots\ldots$$

$$\xi = x/\sqrt{\beta}$$

$$\eta = \sqrt{\beta}\ (dx/ds) + (\alpha/\sqrt{\beta})\ x$$

All values are in 10^{-3} $(\text{meter})^{1/2}$. For each ν, the first row is analytical and the second row is numerical results. Only one series of unstable and stable fixed points out of four are given here.

	unstable		stable	
$(\nu - 20.25)$	ξ	η	ξ	η
0.03	1.34	7.93	9.07	3.83
	1.50	7.81	9.54	3.60
0.02	1.38	6.34	7.37	3.07
	1.47	6.31	7.67	2.99
0.01	1.22	4.40	5.18	2.14
	1.26	4.43	5.35	2.14
0.005	0.981	3.08	3.65	1.50
	1.00	3.13	3.76	1.53
0.0005	0.373	0.971	1.15	0.474
	0.380	0.990	1.18	0.488
0.0001	0.174	0.435	0.514	0.212
	0.178	0.443	0.528	0.219

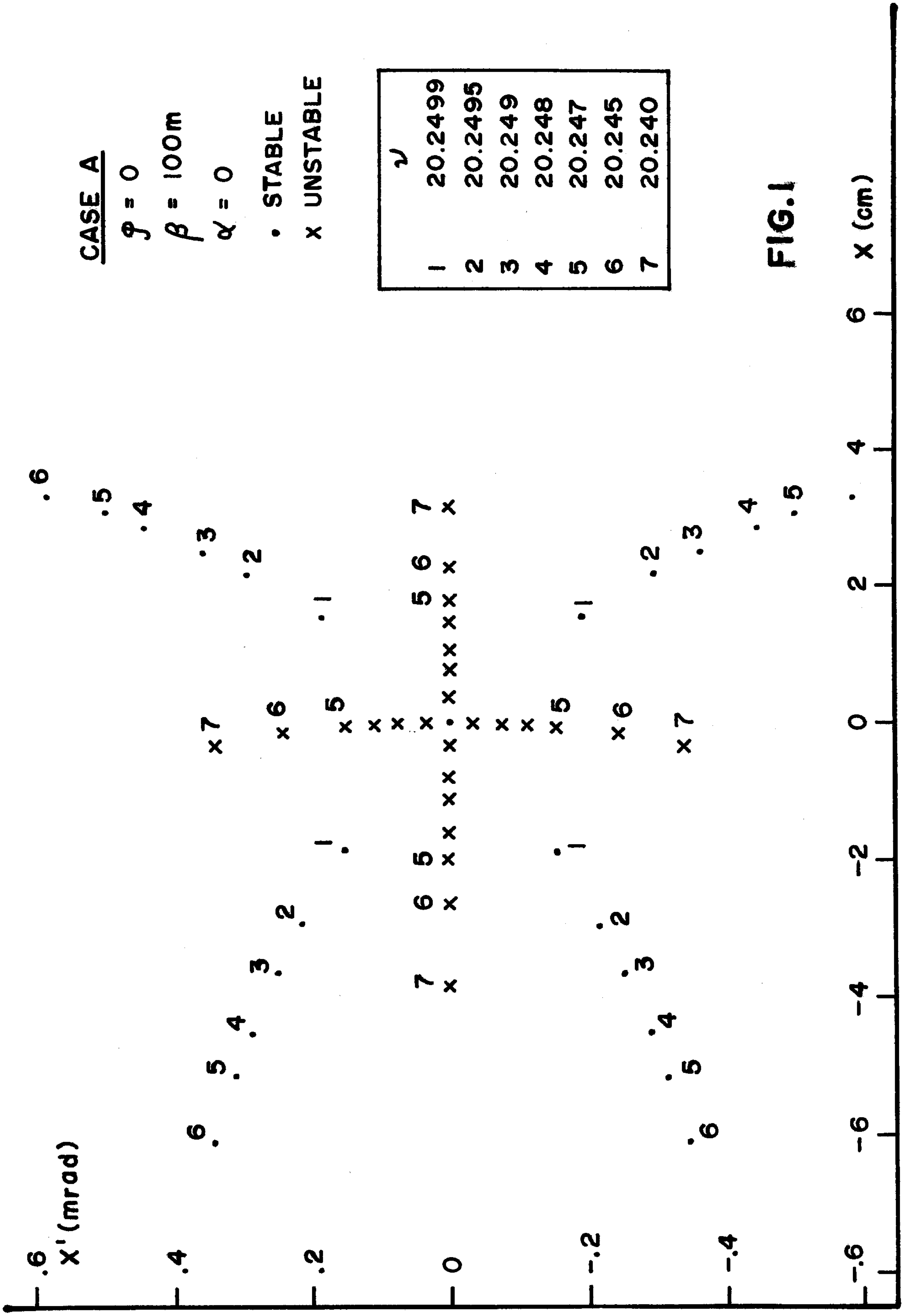
CASE A
ρ = 0
β = 100m
α = 0
• STABLE
x UNSTABLE
ν
1 20.2499
2 20.2495
3 20.249
4 20.248
5 20.247
6 20.245
7 20.240
X (cm)
X' (mrad)
-6 -4 -2 0 2 4 6
-.6 -.4 -.2 0 .2 .4 .6

FIG. I

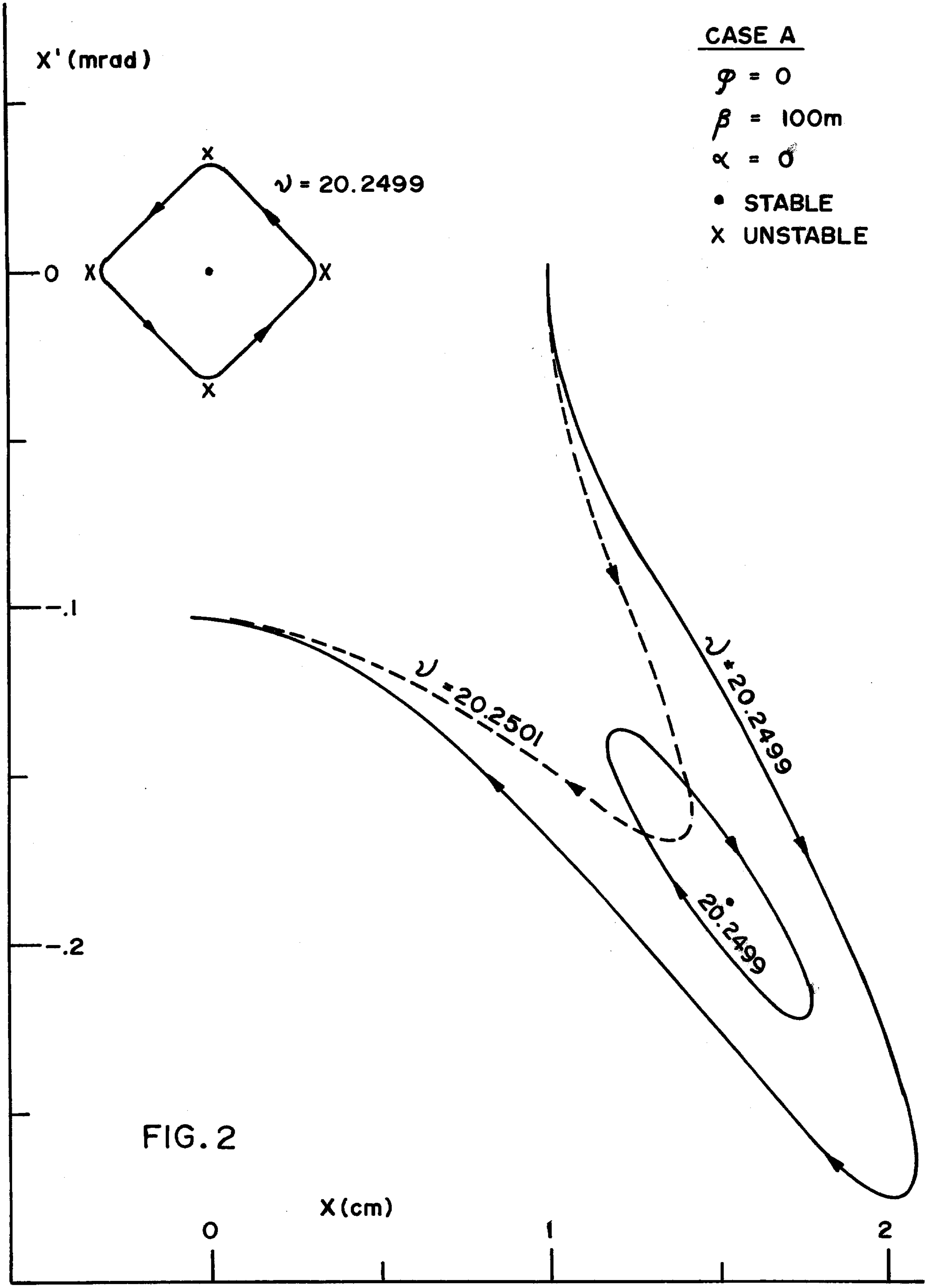
CASE A
φ = 0
β = 100m
α = 0
• STABLE
X UNSTABLE
X'(mrad)
ν = 20.2499
ν = 20.2501
ν = 20.2499
20.2499
0
-.1
-.2
X(cm)
0
1
2

FIG. 2

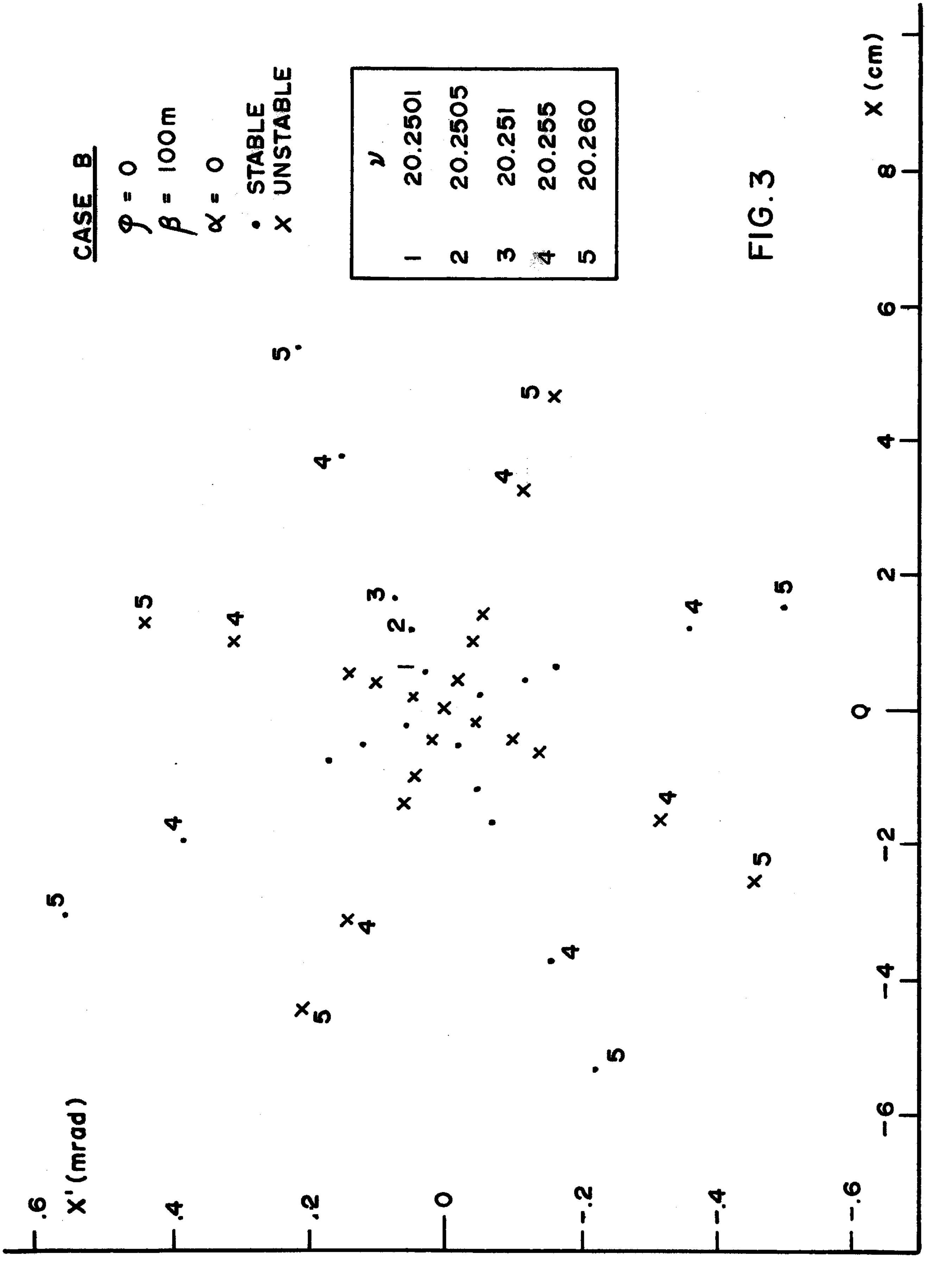
CASE B
$\varphi = 0$
$\beta = 100$m
$\alpha = 0$
• STABLE
X UNSTABLE
ν
1 20.2501
2 20.2505
3 20.251
4 20.255
5 20.260
X (cm)
X' (mrad)
-6
-4
-2
0
2
4
6
8
.6
.4
.2
-.2
-.4
-.6

FIG. 3

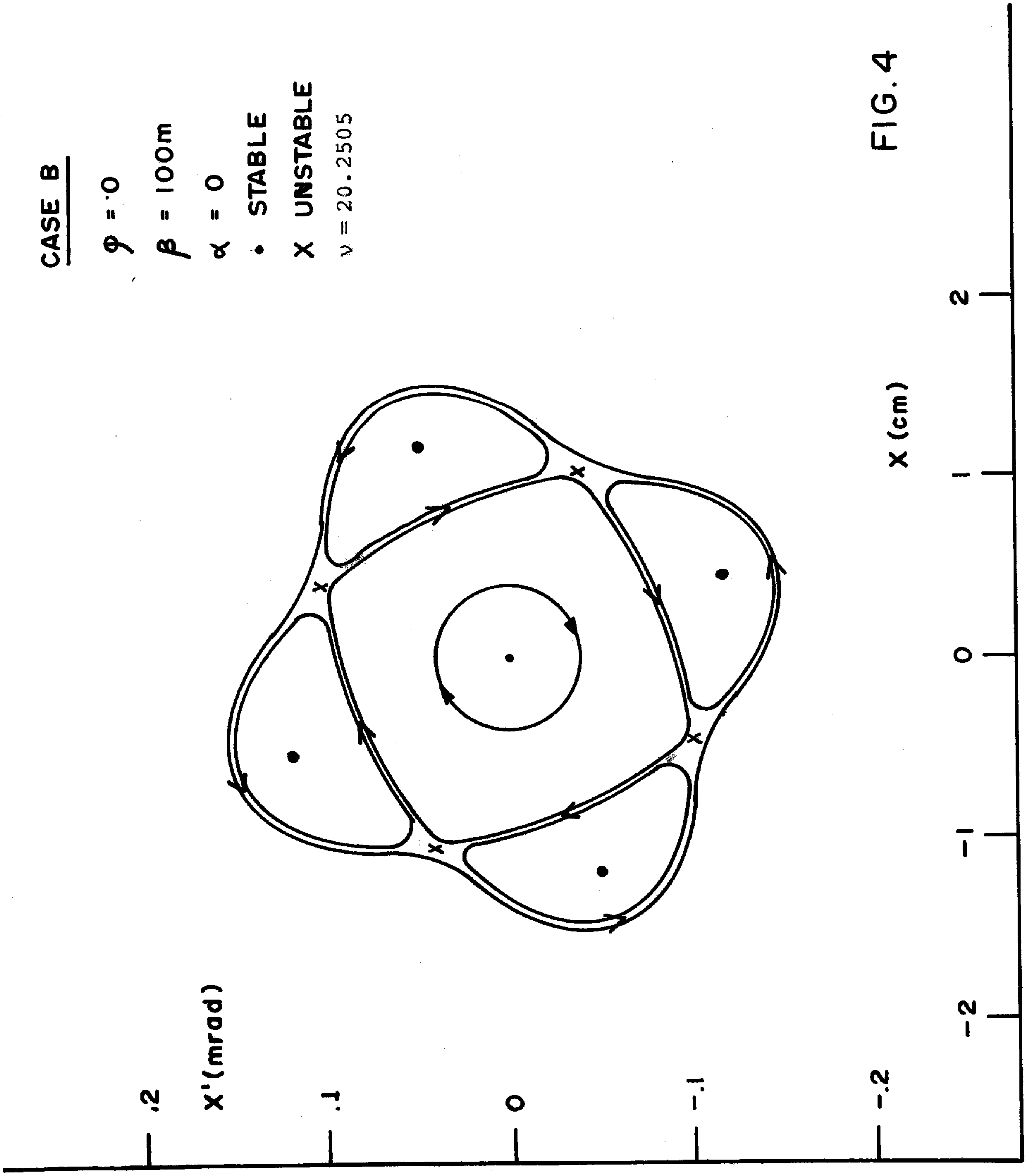
CASE B
$\varphi = 0$
$\beta = 100m$
$\alpha = 0$
• STABLE
X UNSTABLE
$\nu = 20.2505$
X (cm)
2
1
0
-1
-2
X'(mrad)
.2
.1
0
-.1
-.2

FIG. 4