Exotic supermultiplets in six dimensions : symmetries, quantisations and dynamics

Yi Zhang

THESE DE DOCTORAT

NNT : 2021UPASP107

Exotic supermultiplets in six dimensions : symmetries, quantisations and dynamics

Supermultiplets exotiques en six dimensions : symétries, quantifications et dynamique

Thèse de doctorat de l'université Paris-Saclay

École doctorale n° 564 : physique en Île-de-France (PIF)
Spécialité de doctorat : Physique
Graduate School : Physique, Référent : Faculté des sciences d'Orsay

Thèse préparée dans l'Institut de physique théorique (Université Paris-Saclay, CNRS, CEA), sous la direction de Ruben MINASIAN, Directeur de Recherches

Thèse soutenue à Paris-Saclay, le 20 octobre 2021, par

Yi Zhang

Composition du jury

Iosif Bena Checheur CEA, CNRS/CEA, Université Paris Saclay, IPhT	Président
Chris Hull Professeur des Universités, Imperial College London, Département de Physique	Rapporteur
Marc Henneaux Professeur des Universités, Université Libre de Bruxelles, Département de Physique et Professeur au Collège de France	Rapporteur
Axel Kleinschmidt Scientifique senior, Institut Max-Planck de physique gravitationnelle (Institut Albert-Einstein, Potsdam)	Examinateur
Leron Borsten Chercheur associé, Université Heriot-Watt d'Édimbourg, Département de mathématiques	Examinateur
Ruben Minasian Directeur de Recherches, CNRS/CEA, Université Paris Saclay, IPhT	Directeur de these

ÉCOLE DOCTORALE
Physique en
Île-de-France (PIF)

Titre : Supermultiplets exotiques en six dimensions : symétries, quantifications et dynamique
Mots clés : Supermultiplets exotiques, Dualités gravitationnelles, Théorie des champs exceptionnelle, Quantifications, Dynamique, Anomalies

Résumé : Parmi les représentations autorisées de la supersymétrie étendue en six dimensions, il existe des multiplets chiraux "exotiques" contenant des tenseurs mixtes de spin deux au lieu d'un graviton conventionnel. En particulier, le multiplet $\mathcal{N} = (4, 0)$ contient un graviton exotique à quatre indices, et il a été conjecturé qu'une théorie en interaction basée sur ce multiplet apparaît dans la limite à couplage fort de la théorie M compactifiée sur T^6. La première partie originale de cette thèse présente une étude algébrique de ces multiplets et leur plongement possible dans le cadre de la "théorie des champs exceptionnelle" ; un résultat important est que les impulsions six-dimensionnelles ne correspondent pas à une section conventionnelle de l'espace-temps. Compactifiés sur un cercle, ces multiplets donnent lieu aux mêmes degrés de liberté que ceux de la supergravité en cinq dimensions avec le même nombre de supersymétries. Cependant, en considérant les anomalies et la génération de couplages de Chern-Simons, nous avons des raisons de douter que leur dynamique reproduit celle des supergravités en cinq dimensions. Nous proposons une réalisation différente, similaire à celle de la théorie F : les espaces-temps fibrés par un tore T^3 de volume fixe y jouent un rôle important et suggèrent que la dynamique de la supergravité n'émergerait qu'en compactifiant vers trois dimensions. Ces multiplets exotiques contiennent aussi des champs de tenseurs-spineurs antisymétriques de rang deux. Dans la dernière partie de cette thèse, la quantification de champs de tenseurs-spineurs antisymétriques $\psi^{\alpha}_{\mu_1 \ldots \mu_p}$ généraux, de rang et en dimension arbitraires, est effectuée en utilisant le formalisme des antichamps de Batalin et Vilkovisky. Comme dans le cas du gravitino ($p = 1$), un fantôme dynamique de Nielsen-Kallosh apparaît dans les jauges Gaussiennes contenant un opérateur différentiel. L'apparition de ce "troisième fantôme" est décrite dans le formalisme BV pour une théorie de jauge réductible arbitraire. Finalement, le spectre de fantômes est utilisé en conjonction avec le théorème de l'indice d'Atiyah et Singer pour calculer les anomalies gravitationnelles de ces champs.

Title : Exotic supermultiplets in six dimensions : symmetries, quantisations and dynamics
Keywords : Exotic supermultiplets, Exceptional field theory, Gravitational dualities, Quantisation, Dynamics, Anomalies

Abstract : Among the allowed representations of extended supersymmetry in six dimensions there are exotic chiral multiplets that, instead of a graviton, contain mixed-symmetry spin-2 tensor fields. Notably, the $\mathcal{N} = (4, 0)$ multiplet has a four index exotic graviton and it was conjectured that an interacting theory based on this multiplet could arise as a strong coupling limit of M theory compactified on T^6. We present an algebraic study of these multiplets and their possible embedding into the framework of exceptional field theory, finding in particular that the six-dimensional momenta do not correspond to a conventional spacetime section. When compactified on a circle, the six-dimensional multiplets give rise to the same degrees of freedom as five-dimensional supergravity theories with the same number of supersymmetries. However, by considering anomalies and the generation of Chern-Simons couplings, we find reason to doubt that their dynamics will agree with the five-dimensional gravity theories. We propose an alternative picture, similar to F-theory, in which particular fixed-volume T^3-fibered spacetimes play a central role, suggesting that only on compactification to three-dimensions will one make contact with the dynamics of supergravity. In these exotic multiplets, there are also rank two antisymmetric tensor-spinors. In the last part of the thesis, we perform the quantisation of general antisymmetric tensor-spinors $\psi^{\alpha}_{\mu_1 \ldots \mu_p}$ using the Batalin-Vilkovisky field-antifield formalism for any p and in arbitrary dimensions. Just as for the gravitino ($p = 1$), an extra propagating Nielsen-Kallosh ghost appears in quadratic gauges containing a differential operator. The appearance of this "third ghost" is described within the BV formalism for arbitrary reducible gauge theories. We then use the resulting spectrum of ghosts and the Atiyah-Singer index theorem to compute gravitational anomalies.

To my beloved parents, for their unconditional love, support and encouragement.

Acknowledgements

First of all, I would like to express my deepest gratitude to my thesis supervisor, Ruben Minasian. The whole journey of my PhD has not been easy, and there are ups and downs. During all the time, Ruben was always on my side and always tried to guide me. Whenever I was in trouble, his advice and comfort was the best antidote. Our pleasant conversations are not limited to scientific research but also involve all aspects of life. Without him, the goal of my PhD would not have been realized.

I am particularly thankful for the consistent support and help from my parents. As someone who dropped out of medicine and switched to theoretical physics, I never heard a single word of complaint from them about my choices. They always chose to believe in me, encourage me and support me. This has been invaluable to me.

I am also in debt to my two other collaborators, Charles Strickland-Constable and Victor Lekeu. Both gave me tremendous amount of help in carrying out the research projects. I have my special thanks to Victor, who helped me a lot while I was writing my dissertation and also read through it meticulously.

During my stay in Paris, I am grateful to those who have also given me academic advice and who have been willing to listen to me, they are Costas Bachas, Iosif Bena and Guillaume Bossard. I am also grateful to the directors and secretary of Institut de Physique Théorique (IPhT) and other colleagues who make the heavy administrative work went smoothly, which allows me to focus better on my research.

I would like to thank Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut) at Golm, Potsdam for its hospitality. I wish all the best for the friends there. I owe special thanks to Franz Ciceri, André Coimbra, Anamaria Font, Jorjadze, George, Hadi Godazgar, Axel Kleinschmidt, Nikita Misuna, Hermann Nicolai and Stefan Theisen.

I would like to acknowledge Federico Bonetti, Leron Borsten, Marc Henneaux, Chris Hull and Daniel Waldram for useful discussions throughout my doctoral research.

Finally, I would like to thank all the friends that I have spent time with in Paris, Potsdam and Berlin, including Jing Cao, Lilian Chabrol, Yifan Chen, Peng Cheng, Senwen Deng, Zhihao Duan, Xiao Fan, Gwenaël Ferrando, Hao Fu, Dongsheng Ge, Mengzi Huang, Zihao Huang, Grégoire Josse, Meng Kou, Songyuan Li, An Qu, Jiaxin Qiao, Yichen Qin, Keyu Wang, Heng Li, Junkang Li, Xiyuan Liu, Xuan Liu, Daheng Min, Yi Pan, Jian Qian, Zicheng Qian, Steven Schulz, Ran Sun, Yijun Wan, Qun Wang, Yiqiao Wang, Shengquan Xiang, Tianshuo Xu, Fan Yang, Kang Yang, Chao Zhang, Haowen Zhang, Zhiyuan Zhang, Zeyu Zhao, Zechuan Zheng, Jihong Zhu, Tunan Zhu and Deliang Zhong ..., the list can be endless and I have no way of naming them all. It is them who made my entire PhD less tedious and more enriching, and our interactions have become some of the best memories of my life.

Contents

Abstract

Among the allowed representations of extended supersymmetry in six dimensions there are exotic chiral multiplets that, instead of a graviton, contain mixed-symmetry spin-2 tensor fields. Notably, the $\mathcal{N} = (4,0)$ multiplet has a four index exotic graviton and it was conjectured that an interacting theory based on this multiplet could arise as a strong coupling limit of M theory compactified on T^6. We present an algebraic study of these multiplets and their possible embedding into the framework of exceptional field theory, finding in particular that the six-dimensional momenta do not correspond to a conventional spacetime section. When compactified on a circle, the six-dimensional multiplets give rise to the same degrees of freedom as five-dimensional supergravity theories with the same number of supersymmetries. However, by considering anomalies and the generation of Chern-Simons couplings, we find reason to doubt that their dynamics will agree with the five-dimensional gravity theories. We propose an alternative picture, similar to F-theory, in which particular fixed-volume T^3-fibered spacetimes play a central role, suggesting that only on compactification to three-dimensions will one make contact with the dynamics of supergravity. In these exotic multiplets, there are also rank two antisymmetric tensor-spinors. In the last part of the thesis, we perform the quantisation of general antisymmetric tensor-spinors $\psi^{\alpha}_{\mu_1 \dots \mu_p}$ using the Batalin-Vilkovisky field-antifield formalism for any p and in arbitrary dimensions. Just as for the gravitino ($p = 1$), an extra propagating Nielsen-Kallosh ghost appears in quadratic gauges containing a differential operator. The appearance of this "third ghost" described within the BV formalism for arbitrary reducible gauge theories. We then use the resulting spectrum of ghosts and the Atiyah-Singer index theorem to compute gravitational anomalies.

Résumé

Parmi les représentations autorisées de la supersymétrie étendue en six dimensions, il existe des multiplets chiraux "exotiques" contenant des tenseurs mixtes de spin deux au lieu d'un graviton conventionnel. En particulier, le multiplet $\mathcal{N} = (4, 0)$ contient un graviton exotique à quatre indices, et il a été conjecturé qu'une théorie en interaction basée sur ce multiplet apparaît dans la limite à couplage fort de la théorie M compactifiée sur T^6. La première partie originale de cette thèse présente une étude algébrique de ces multiplets et leur plongement possible dans le cadre de la "théorie des champs exceptionnelle"; un résultat important est que les impulsions six-dimensionnelles ne correspondent pas à une section conventionnelle de l'espace-temps. Compactifiés sur un cercle, ces multiplets donnent lieu aux mêmes degrés de liberté que ceux de la supergravité en cinq dimensions avec le même nombre de supersymétries. Cependant, en considérant les anomalies et la génération de couplages de Chern-Simons, nous avons des raisons de douter que leur dynamique reproduit celle des supergravités en cinq dimensions. Nous proposons une réalisation différente, similaire à celle de la théorie F: les espaces-temps fibrés par un tore T^3 de volume fixe y jouent un rôle important et suggèrent que la dynamique de la supergravité n'émergerait qu'en compactifiant vers trois dimensions. Ces multiplets exotiques contiennent aussi des champs de tenseurs-spineurs antisymétriques de rang deux. Dans la dernière partie de cette thèse, la quantification de champs de tenseurs-spineurs antisymétriques $\psi^{\alpha}_{\mu_1 \dots \mu_p}$ généraux, de rang et en dimension arbitraires, est effectuée en utilisant le formalisme des antichamps de Batalin et Vilkovisky. Comme dans le cas du gravitino ($p = 1$), un fantôme dynamique de Nielsen-Kallosh apparaît dans les jauges Gaussiennes contenant un opérateur différentiel. L'apparition de ce "troisième fantôme" est décrite dans le formalisme BV pour une théorie de jauge réductible arbitraire. Finalement, le spectre de fantômes est utilisé en conjonction avec le théorème de l'indice d'Atiyah et Singer pour calculer les anomalies gravitationnelles de ces champs.

Introduction

It has been proposed that a strong coupling limit of five-dimensional quantum $\mathcal{N} = 8$ supergravity in which the Planck length becomes infinite could give a six-dimensional superconformal phase of M-theory [1–3]. Moreover, for the free theory this limit has been argued to be given by a six-dimensional theory with maximal $\mathcal{N} = (4, 0)$ supersymmetry. This theory is conformal and hence has no length scales. When put on a circle, the compactification scale R becomes the five-dimensional Planck scale. Clearly, understanding such a limit would require radically new ideas and these would be important for our overall understanding of the gravitational physics of M-theory. In recent years, there has been a revival of interest in this area, producing many interesting developments and new approaches [4–19].

However, regardless of the implications for M theory, at the level of supermultiplets, the (free) multiplet with $\mathcal{N} = (4, 0)$ supersymmetry certainly exists [20] and has 32 supersymmetries and 32 conformal supersymmetries. Its dimensional reduction has the same degrees of freedom and the same field content as the maximal supergravity in five dimensions. The latter theory has $E_{6(6)}$ global symmetry, and in addition to the graviton has 27 vector and 42 scalar fields, as well as eight gravitini and 48 spin 1/2 fermions. It has been suggested that the former has the same $E_{6(6)}$ symmetry, such that the fields appear in similar representations. Instead of gravity (rank two symmetric field) it has a rank four tensor gauge field with the symmetries of the Riemann tensor. Due to self-duality constraints on its double field strength this field has five degrees of freedom (just like the five-dimensional graviton) and its dimensional reduction gives conventional linearised gravity in five dimensions [1]. Similarly, instead of 27 five-dimensional vectors, the $(4, 0)$ multiplet has 27 self-dual tensors.[1] In either case there are 27×3 degrees of freedom. The 48 spin 1/2 fermions simply become chiral fermions in six dimensions. Finally, the eight gravitini (vector-spinor fields) are replaced by eight[2] "exotic gravitini" $\psi_{\mu\nu}$, spinor-valued two-forms with self-duality constraint on their field strength.[3]

In fact, the $(4, 0)$ multiplet is not the only exotic six-dimensional theory. There exists also a $(3, 1)$ multiplet, where the self-duality constraints are partial, and from examining the scalar degrees of freedom one might guess that the symmetry governing the theory is $F_{4(4)}$. The multiplet has a rank 3 self-dual tensor field, and 28 scalars which could lie in the tangent space to the symmetric space $F_{4(4)}/\, Sp(2) \times Sp(6)$. However, the 14 vector fields and 12 self-dual tensors only form the **26** representation of $F_{4(4)}$ when combined together. This suggests that in fact only the R-symmetry group $Sp(2) \times Sp(6)$ (and not the full $F_{4(4)}$) would be a true symmetry. This could make one suspicious as to whether $E_{6(6)}$ would be a true symmetry of the $\mathcal{N} = (4, 0)$ theory, and we will see some indications that it may indeed not be. As these symmetries do not follow directly from the supermultiplets, but appear only in the construction of the associated theories, the absence of a complete construction of the $(4, 0)$ theory means that one cannot be sure. However, a simple argument in favour of the $E_{6(6)}$ symmetry is that the scalars of the 5d maximal supergravity are all lifted to scalars in 6d. Thus naively one would expect the 5d transformations of them also to lift to 6d. The fermionic fields of the $(3, 1)$ multiplet comprise two exotic gravitini, six standard gravitini of negative chirality, 28 spin 1/2 fermions of positive chirality and 14 spin 1/2 fermions of

[1] In our conventions, the six-dimensional $(2, 0)$ gravity multiplet has five anti-self-dual tensor fields, while the $(2, 0)$ tensor multiplets have self-dual tensors.

[2] We count the four quaternionic fields as eight complex fields and will use similar counting throughout.

[3] Like in much of the literature, the fields in $(4, 0)$ and $(3, 1)$ multiplets that do not appear in ordinary gravity or matter multiplets, but have direct counterparts, i.e. like eight spinor-valued two forms in $(4, 0)$ vs eight gravitini in $(2, 2)$, will be labeled as "exotic". Due to its properties, for the exotic graviton in $(4, 0)$ multiplet the self-dual Weyl (SDW) label will also be used.

negative chirality. The exotic and conventional gravitini reduce to give the eight standard gravitini in five dimensions, while the spin $1/2$ fermions of either chirality simply reduce to five dimensional spin $1/2$ fields.

Finally, the exotic fields can appear in multiplets with less supersymmetry. These can be constructed via the usual representation-theoretic arguments. An alternative is to consider the decomposition of the maximally supersymmetric multiplets. For example, as we shall discuss, the $(4,0)$ multiplet decomposes into an exotic $(2,0)$ gravity multiplet as well as 4 exotic $(2,0)$ gravitino multiplets and 5 $(2,0)$ tensor multiplets. This decomposition is very similar to the decomposition of the maximal $(2,2)$ six-dimensional supergravity. This can be decomposed into $(2,0)$ multiplets: one gravity, 4 gravitino and 5 tensors.[4]

One useful perspective on these multiplets is given by the fact that they can be seen as square or product theories [4, 8, 10], in analogy to the linearised maximal supergravity in six dimensions, i.e. the $(2,2)$ theory being the square of the six-dimensional super Yang-Mills. In the same vein, the $(4,0)$ multiplet can be seen as a square of $(2,0)$ tensor multiplets, while the $(3,1)$ theory - as a product of a $(2,0)$ multiplet with a $(1,1)$ vector one. Similar product structures appear in the exotic theories with less supersymmetry. While much of the interest in double copy constructions comes from the computation of amplitudes in perturbation theory [21–23] (see [24] for a review) there have also been developments in off-shell field theoretical realisations [5–8, 25–27] and the construction of classical solutions [28–32]. Unfortunately in our case of interest, the strongly coupled theory has no perturbative expansion and there may also be no classical limit with interactions, limiting the direct usefulness of these constructions.

Algebraic aspects

Two main questions that preoccupy us in this thesis concern the algebraic symmetry-based reasons behind the existence of the the exotic multiplets and the possibility of probing the existence of interacting forms of these exotic theories (as well as their existence on non-flat spaces). Some of the arguments here can be made for both $(4,0)$ and $(3,1)$ multiplets, and some are specific only to $(4,0)$.

Much of the algebraic discussion in chapter 2 takes place in the context of the U-duality groups and their relation to the corresponding superalgebras. In particular, we will use the language of generalised geometry [33–35] and exceptional field theory [36–38], discussing the charges appearing in the supersymmetry algebra as generalised vectors in a generalised tangent space which transforms as a linear representation under the relevant U-duality group. In order to avoid encountering infinite dimensional duality algebras, we will work with dimensional splits of the theories considering three external dimensions separately from the rest.

As we will discuss in section 2.1, all supersymmetry algebras with 32 supercharges arise from a particular superalgebra $\mathcal{A}$ (with bosonic subalgebra $\mathfrak{sl}(32,\mathbb{R}) \ltimes \mathbb{R}^{528}$) by restricting $\mathfrak{sl}(32,\mathbb{R})$ to different $\mathfrak{spin}(d-1,1)$ subalgebras. For example, one can obtain the superalgebras of $11d$, type IIA and type IIB supergravities from this prescription. On performing a dimensional split, decomposing say $spin(9,1) \to spin(3,1) \times spin(6)$ in type IIA or IIB, one can see how the resulting $spin(6)$ group would act on the charges appearing in the generalised tangent space of the supergravity theory on the internal Euclidean signature part. In this way, merely requiring the chiralities of the fermions present in type IIA and type IIB implies that one requires $E_{d(d)}$-inequivalent "sections" (in the language of exceptional field theory) of the generalised tangent space to correspond to the physical momenta in spacetime for the two theories. For the particular case of type IIA vs type IIB, these inequivalent sections (or inequivalent embeddings of the general linear group into the U-duality group) have been discussed extensively in the literature [33, 35, 36, 39]. A similar discussion of sections for half-maximal supersymmetry can be found in [40], where it was concluded that inequivalent sections gave the $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supergravities in six dimensions (the former section extending to type I in ten dimensions).

Similarly, one can explore what happens if one instead requires $\mathcal{N} = (4,0)$ supersymmetry in six-dimensions from the decomposition. We examine the intersection of the relevant $Spin(5,1)$ group with the generalised spin group $Spin(2,1) \times SO(16)$. Under the common subgroup $Spin(2,1) \times$

[4] It is not hard to verify that even if individual multiplets are chiral the whole combination is not - for every chiral fermion or self-dual field there is another with the opposite chirality or anti-self-duality.

$Spin(3)$ we observe how the charges in the generalised tangent space are grouped into irreducible representations of the $Spin(3)$ factor and of the $SL(3,\mathbb{R}) \subset SL(9,\mathbb{R}) \subset E_{8(8)}$ which contains it. This reveals a very different behaviour to the normal situation in generalised geometry or exceptional field theory.

The root of this difference lies partly in the fact that in $E_{8(8)}$, the charges appearing in the supersymmetry algebra do not span the full **248** representation in which the generalised vector transforms, but rather only the **120** part under its $SO(16)$ subgroup. Under the direct embedding into the **248**, the momentum charges do not satisfy the section condition, even in standard supergravity.

The $Spin(3)$ triplet of momentum charges of the $(4,0)$ supersymmetry algebra thus embed into the generalised vector as a triplet of $SO(3)$, which consists of two of the momenta that would be present in the conventional reduction of five-dimensional supergravity to three dimensions, plus part of the dual graviton charge, much as expected from [1]. However, under the $SL(3,\mathbb{R})$ subgroup containing this $SO(3)$, these three charges are combined with five others to form an octuplet. Ordinarily, in supergravity one would expect them rather to be contained in a subspace of the sum of two triplets, a space in which one could identify an $SL(3,\mathbb{R})$ triplet solving the section condition. Here, this is not the case, and there is no such section. Further, this $SL(3,\mathbb{R})$ subgroup is related to that of $\mathcal{N}=(2,2)$ supergravity by a transformation in $SL(9,\mathbb{R}) \subset E_{8(8)}$, so any such section would be equivalent to the standard one anyway.

Nonetheless, we go on to examine the decomposition of the generalised vector and the adjoint of $E_{8(8)}$ under $SL(3,\mathbb{R}) \times E_{6(6)}$, noting that if we had enhanced $SL(3,\mathbb{R})$ to $GL(3,\mathbb{R})$ as one would usually in standard supergravity, this would break the $E_{6(6)}$ commutant to $SO(5,5)$. We then look at these decompositions and attempt to apply the naive algebraic prescription (usually imagined only in the context of supergravity – see e.g. [35] for a discussion) to extract the field content of a parent six-dimensional theory. We find that, with suitable identifications, this matches exactly what one would expect from the $\mathcal{N}=(4,0)$ multiplet, though questions remain over whether one must decompose under $SO(3) \subset SL(3,\mathbb{R})$ and $Sp(8) \subset E_{6(6)}$ in order to make these identifications. Indeed, the algebraic construction of the generalised Lie derivative in flat space appears to reproduce a formula for the gauge transformation of the exotic graviton, which reassures us that our identification of the spacetime directions inside the generalised tangent space, together with the fields and charges, is somewhat correct.

h-theories

Of course, one can wonder if there is more to these multiplets than simply their algebraic properties. They stand out as multiplets with highest-spin ≤ 2 which do not appear in standard supergravity theories, their decompositions under sub-superalgebras and compactifications or their matter multiplets. We shall present arguments that the fact that the conjectured $(4,0)$ symmetry group $E_{6(6)}$ has an $SL(3,\mathbb{R})$ commutant inside the three-dimensional symmetry group $E_{8(8)}$ serves not only as a helpful technical tool, but is closely connected to the very existence of the six-dimensional theory with $E_{6(6)}$ symmetry. Correspondingly, the symmetry groups for exotic $(2,0)$ and $(1,0)$ symmetry groups have $SL(3,\mathbb{R})$ commutants inside the symmetry groups of three-dimensional theories with 16 and 8 supercharges respectively.

In general, the exceptional $E_{d(d)}$ groups have $GL(n,\mathbb{R})$ commutants inside bigger $E_{d+n(d+n)}$ groups. This is essentially by construction: the lower dimensional theories with maximal supersymmetry are obtainable from the higher dimensional ones after a torus T^n compactification. Finding other decompositions of $E_{d+n(d+n)}$ might be useful as a technical tool, but is of very little consequence as far as higher-dimensional theories are concerned. For other decompositions $G_d \times H_n \subseteq E_{d+n(d+n)}$, there is no (known) maximally supersymmetric theory (or multiplet) in $D = 11-d$ dimensions with symmetry H_n. For example the existence of the subgroup $SL(2,\mathbb{R}) \times E_{7(7)} \subseteq E_{8(8)}$ has no implications for five-dimensional physics, as there is no maximal five-dimensional theory with symmetry group $E_{7(7)}$.

In this sense, assuming that the $\mathcal{N}=(4,0)$ theory really has $E_{6(6)}$ symmetry, we see that $E_{6(6)}$, $SL(3,\mathbb{R})$ and $E_{8(8)}$ form a unique triple for maximally supersymmetric theories. As mentioned, less-supersymmetric counterparts of this triple exist with $SL(3,\mathbb{R})$ always playing a central role. For concreteness we shall be concentrating on the maximally supersymmetric case. Given that the

$SL(3,\mathbb{R})/SO(3)$ coset is the moduli space of flat metrics on T^3 of fixed volume, this suggests a way of thinking about the $(4,0)$ multiplet analogous to F-theory [41]. A solution of three-dimensional supergravity with five non-constant scalars parametrising the coset, can be thought of as a solution of a six-dimensional theory with the left-over $E_{6(6)}$ symmetry, i.e. the $(4,0)$ theory on a T^3-fibered manifold satisfying certain conditions. Moreover, using results from earlier work on "U-fold" torus fibrations [42], it can be shown that the geometrical information can be repackaged and presented in a form of a self-dual Weyl (SDW) tensor field, and differential conditions on the six-dimensional space upon linearisation can be reduced to the equations of motion for the SDW field. The details of this constructions which we call h-theory can be found in section 4. A novel feature of this construction is that both the geometry and the SDW field on it are constructed out of the physical scalar degrees of freedom in three-dimensions. Our analysis also has no propagating fields along the directions of the torus, similarly to the situation in F-theory where there are no momenta in the auxiliary T^2 directions. This intriguing picture would thus suggest that the $(4,0)$ theory is not really six-dimensional, as the physical states are not charged under the additional momenta.

It has been observed in [2] that due to the four-dimensional symmetry group $E_{7(7)}(\mathbb{Z})$ not having an $E_{6(6)}(\mathbb{Z}) \times SL(2,\mathbb{Z})$ subgroup, the $SL(2,\mathbb{Z})$ duality expected from six-dimensional description would act non-trivially on the graviton leading possibly to a modification of supergravity. Our picture suggests a more conservative possibility, inspired by the relations between F-theory, 11-dimensional supergravity and type IIB. We should not think of recovering the four-dimensional supergravities from T^2 reduction of the exotic $(4,0)$ theory any more than we expect a direct reduction of F-theory on a circle to yield the 11-dimensional supergravity, or of M-theory being simply reduced to IIB. Instead, when M-theory is put on a two-torus one can take the so called F-theory limit that decompactifies to ten-dimensions while while retaining the $SL(2,\mathbb{Z})$, i.e. yields the type IIB theory. The limit holds also from M-theory on an elliptically fibered manifold, in which case the decompactification yields type IIB on the base of the elliptic fibration. So the idea is to consider the three-dimensional maximal supergravity, i.e. the $(4,0)$ theory on a fixed volume T^3 in decompactification limits. Denoting the radii of circles in T^3 by r_1, r_2, r_3 and setting the $\mathrm{Vol}(T^3) = 1$, up to numerical factors one has $r_1 = 1/r_2r_3$. One can take $r_2, r_3 \to \infty$ and hence $r_1 \to 0$, i.e. decompactify two dimensions. The path

$$E_{8(8)} \supseteq SL(3,\mathbb{R})\times E_{6(6)} \supseteq SL(2,\mathbb{R})\times\mathbb{R}^+\times E_{6(6)} \hookrightarrow GL(2,\mathbb{R})\times E_{6(6)} \xrightarrow{r_2,r_3\to\infty} E_{6(6)} \quad \text{in D=5}$$

results in a five-dimensional theory with $E_{6(6)}$ symmetry, i.e. the ordinary five-dimensional supergravity. Another option is $r_2, r_3 \to 0$ and hence $r_1 \to \infty$, i.e. decompactify a single dimension. The path now is

$$E_{8(8)} \supseteq SL(3,\mathbb{R}) \times E_{6(6)} \supseteq SL(2,\mathbb{R}) \times \mathbb{R}^+ \times E_{6(6)} \hookrightarrow SL(2,\mathbb{R}) \times E_{7(7)} \xrightarrow{r_1\to\infty} E_{7(7)} \quad \text{in D=4.}$$

This explains the appearance of both five-dimensional $E_{6(6)}$ and four-dimensional $E_{7(7)}$ in the decompactification limits of three-dimensional maximal supergravity. As everything else relating to the embedding of $SL(3,\mathbb{Z})$ in three-dimensional duality group, these chains continue to hold for theories with 16 and eight supercharges. Calling the symmetry group G, we first note that $G^{\text{exotic}}_{D=6} = G_{D=5}$ and that $SL(3,\mathbb{R}) \times G^{\text{exotic}}_{D=6} \subseteq G_{D=3}$ as well as $SL(2,\mathbb{R}) \times G_{D=4} \subseteq G_{D=3}$. The decompactifications to ordinary supergravities in four and five dimensions now work as in the maximally supersymmetric case.

Another observation which suggests that we do not think of the theory as truly six-dimensional comes from consideration of higher rank dualities. Considering the conjectured Kac-Moody symmetries $E_{8+n(8+n)}$ for $n = 1, 2, 3$, we might expect to find that the $SL(3,\mathbb{R})$ commutant of $E_{6(6)}$ is extended to $SL(3+n,\mathbb{R})$. However, this is not the case. In particular, the $SL(3,\mathbb{R})\times SL(3,\mathbb{R})\times E_{6(6)}$ that we consider in our dimensional split (into three external dimensions, three internal dimensions and an internal $E_{6(6)}$ symmetry) does not extend to an $SL(6,\mathbb{R})\times E_{6(6)}$ subgroup inside E_{11}.[5] However, there is a $Spin(1,5)$ subalgebra of KE_{11} corresponding to the decomposition of the 32 component spinor representation into 4 spinors of the same chirality in six dimensions, so that E_{11} does appear to accommodate the multiplet at the level of the superalgebra. The fact that the

[5] We thank Guillaume Bossard for explaining these features of E_{11} to us.

relevant $SL(6,\mathbb{R})$ subgroup fails to exist indicates (unsurprisingly) that there is no six-dimensional gravity for this multiplet and potentially that the theory is not truly six-dimensional.[6]

Chern-Simons couplings and anomalies

To provide further support to this picture, we include other arguments suggesting that the naive reduction of the $(4,0)$ theory on S^1 or T^2 might not produce the dynamics of supergravity in five or four dimensions. We will also find similar statements for the $(3,1)$ theory.

Firstly, we consider the generation of the topological Chern-Simons interactions present in five-dimensional maximal supergravity [43]

$$S_{\text{CS}} = \int k_{\Lambda\Sigma\Delta}\; A^\Lambda \wedge F^\Sigma \wedge F^\Delta \tag{1}$$

where $k_{\Lambda\Sigma\Delta}$ is the cubic $E_{6(6)}$-invariant and the Λ, Σ, Δ are $E_{6(6)}$ indices running from 1 to 27. This interaction does not involve the metric and does not admit linearisation. By supersymmetry, failure to generate it would indicate that the equations derived from the rank three and four tensor fields will not agree with those of gravity beyond linearised level. Similar calculations have been carried out, notably in the context of theories with eight supercharges, where is was shown how triangle diagrams with massive KK modes coming from the chiral six-dimensional fields in the loop generate five-dimensional Chern-Simons terms [44–47]. An important point here is that while KK modes of six-dimensional fields are involved, the calculation itself is carried out in five dimensions. As we show in section 3.2, under reasonable assumptions, only the reduction of the six-dimensional supergravity generates (1) consistent with the $E_{6(6)}$ cubic invariant.

Since the KK modes considered here come from chiral six-dimensional fields, the above calculation is closely related to six-dimensional anomalies and index theorems. Since the exotic multiplets feature chiral fields, questions about anomalies arise naturally. One may object that these are formulated in the flat space, and only upon reduction does (linearised) five-dimensional gravity and diffeomorphism symmetry appear. The five degrees of freedom carried by the SDW field are to be thought of as excitations of a five-dimensional metric, so that one does not expect six-dimensional diffeomorphism symmetry, but rather exotic symmetries that give rise to five-dimensional diffeomorphisms.

In general, diffeomorphism invariance is a critical property for quantum supergravity theories. It corresponds to the conservation of the energy momentum tensor at the quantum level and can be checked via one-loop computations with the external states being gravitons. At the same time, anomalies corresponding breakdown of diffeomorphisms can also be interpreted as the anomalous transformation of the path integral measure of chiral fields under diffeomorphism transformations of the space-time. Diffeomorphism anomalies are equivalent to anomalies for local Lorentz symmetry up to local, non-polynomial counterterms (see e.g. [48]). Thus, regardless of considerations of diffeomorphism symmetry, it makes sense to ask whether the non-gravitational $(4,0)$ theory is invariant under local Lorentz transformations on arbitrary background six-dimensional manifolds. This question can be answered by computing the gravitational anomalies in the conventional sense.

We find that the exotic fields of the $(4,0)$ theory lie inside the domains of certain Dirac operators, in much the same way that self-dual p-forms are found inside the signature complex (see e.g. [49]). This fact is intimately related to the exotic multiplets arising as products of matter multiplets, and is very similar to the treatment of self-dual p-forms as part of a bispinor field. As we shall see, for the exotic fields we simply have to take higher powers of the spinor representations. The explicit calculations can be found in section 3.1, with further details in appendix B.2. The conclusion is that both $(4,0)$ and $(3,1)$ multiplets have non-vanishing anomalies. In a way, the decomposition of the maximally supersymmetric multiplets mentioned above gives a heuristic explanation to this. The ordinary $(2,0)$ multiplets - gravity (GM) , gravitino (GoM) and tensor (TM) - while all chiral, have fields of different chirality appearing in them, so that a particular combination of them even becomes a non-chiral theory.[7] On the contrary, the exotic multiplets have maximally aligned chiralities so that a cancellation naively appears much less likely, and indeed does not happen.

[6] One slight difference between our picture and that of F-theory is that while there is no $SL(12,\mathbb{R})$ inside E_{11}, there is also no twelve-dimensional spin group or momentum charge.

[7] In fact all three multiplets have proportional anomaly polynomials: $I_{\text{T}M} = \frac{1}{4} I_{\text{Go}M} = -\frac{1}{21} I_{\text{G}M}$.

Quantisation and anomalies of antisymmetric tensor-spinors

Apart from the bosonic exotic fields, fermionic two-forms appear in the exotic $\mathcal{N} = (4,0)$ and $\mathcal{N} = (3,1)$ maximally supersymmetric multiplets in six dimensions [20].

The study of fermionic two-forms in six dimensions can be brought to a more general set up, namely, fermionic p-forms in arbitrary dimension D. Consider fermionic fields of the form $\psi^{\alpha}_{\mu_1\mu_2\dots\mu_p}$, where α is a spinor index and the μ_i are spacetime indices, which are totally antisymmetric in their spacetime indices:

$$\psi^{\alpha}_{\mu_1\mu_2\dots\mu_p} = \psi^{\alpha}_{[\mu_1\mu_2\dots\mu_p]} \,. \tag{2}$$

The free action for such a field in flat spacetime is a direct generalisation of the Rarita-Schwinger action for a fermionic one-form ψ^{α}_{μ} and reads [50, 51]

$$S_0[\psi] = -(-1)^{\frac{p(p-1)}{2}} \int d^D x \; \bar{\psi}_{\mu_1\mu_2\dots\mu_p}\, \gamma^{\mu_1\mu_2\dots\mu_p\nu\rho_1\rho_2\dots\rho_p}\, \partial_\nu \psi_{\rho_1\rho_2\dots\rho_p} \,. \tag{3}$$

This action is invariant under some reducible gauge symmetries, i.e. with "gauge-for-gauge" transformations. They are

$$\delta\psi = d\Lambda^{(p-1)} \,, \quad \delta\Lambda^{(p-1)} = d\Lambda^{(p-2)} \,, \quad \dots \,, \quad \delta\Lambda^{(1)} = d\Lambda^{(0)} \tag{4}$$

in differential form notation (with a spectator spinor index). Here, each parameter $\Lambda^{(k)}$ is an antisymmetric tensor-spinor of rank k. This reducibility introduces well-known subtleties upon quantisation, which we will tackle using the powerful Batalin-Vilkovisky (BV) field-antifield formalism [52, 53].

Gravitational anomalies for these exotic multiplets are computed in chapter 3 (the results were published in [15]), but some assumptions were required since the precise ghost structure for the fermionic two-form was unknown at the time. One of the goals of this part is to fill that gap. Another, more remote motivation for looking at these types of fields comes from considerations of dual gravity [1,3,54,55], where (in the linearised regime) the graviton is dualised to a $[D-3,1]$-type mixed-symmetry tensor. A supersymmetric, manifestly covariant model in which this field finds a partner is still lacking, however, and a fermionic p-form field would be the natural candidate (see [56] for an early attempt at dualising fermionic fields, and [57, 58] for related considerations in the prepotential formalism).

In the quantisation procedure of irreducible gauge theories (i.e., when there are no "gauge-for-gauge" transformations), quadratic gauges containing a differential operator give rise to a third propagating ghost. This was discovered firstly in supergravity in the quantisation of Rarita-Schwinger field, and the propagating third ghost is known as the Nielsen-Kallosh ghost [59, 60]. Later, the third ghost was derived again within the BV formalism [52] in a manifestly local way by Batalin and Kallosh in [61].

The "third ghost" for quadratic gauges appears also in the reducible case, as we prove in chapter 5. It should be emphasised that this statement is valid beyond the simple action and gauge symmetries for the fermionic p-form described above: we allow for non-abelian gauge algebras, on-shell closure, etc. These subtleties are all packaged in the explicit form of the 'minimal BV action' for the model at hand, which always exists and which we keep arbitrary.

The quantisation of free fermionic p-form fields, using the general results mentioned above, (in the words of [53]) is "like cracking nuts with a sledgehammer". Since fermionic fields satisfy first-order equations of motion and the action (3) is already in Hamiltonian form, the Hamiltonian quantisation methods of [62–64] would have been more economical. The third ghost has also been discussed in that formalism in reference [65]. However, the approach we use here has the advantage of preserving manifest covariance. This is done both in the usual delta-function gauge-fixing and in the Gaussian gauge-fixing where a generalised Nielsen-Kallosh ghost appears; propagators and BRST transformations are also discussed in both schemes. Explicit details are given only in the two-form case, but the generalisation to higher form degree poses no difficulty. We maintain manifest locality and covariance throughout.

We compute the gravitational anomaly of a chiral fermionic p-form in dimensions $D = 4m+2$. This is done using the ghost spectrum found in the quantisation and applying the Atiyah-Singer index theorem [66, 67], following the methods developed in the classic papers [68–71]. We describe the general procedure in detail and display the results in dimensions $D = 2$, 6 and 10 in tables 6.3, 6.4 and 6.5. An intriguing result is that in dimensions $D \geq 6$, the anomaly of a chiral fermionic p-form matches that of a $(D - p - 1)$-form; it would be very interesting to use this fact to attempt to build new anomaly-free models.

We should mention an important caveat related to the computation of the gravitational anomaly: to the best of our knowledge, there is currently no model that couples consistently a fermionic p-form to dynamical gravity. It can be hoped that this difficulty will be resolved in the future, perhaps by including (an infinite number of) other fields.[8] However, since the anomaly computations of section 6.4 are solely based on the ghost spectrum and not on the specific form of the action, we are confident that these results will survive such future developments.

Structure

The thesis is divided into three parts and it is organised as follows. The first part is an introductory chapter, we review the construction and structure of the exotic six-dimensional multiplets. This part also covers the discussion of the free exotic tensor fields appearing in these exotic multiplets.

The second part is based on the paper [15]. In chapter 2 we discuss how to relate the $\mathcal{N} = (4, 0)$ superalgebra to that of eleven-dimensional supergravity and how to interpret its charges in terms of $E_{8(8)}$ objects, within the framework of exceptional geometry. Chapter 3 section 3.1 contains the anomaly polynomials for the local Lorentz symmetry of exotic multiplets, which are found to be non-factorisable. We go on to show that there is no conventional mechanism to generate the Chern-Simons couplings of five-dimensional maximal supergravity from the circle compactification of the $\mathcal{N} = (4, 0)$ fields also in this chapter. In chapter 4 we present our construction of "h-theories" on T^3-fibered geometries, whose solutions are seen to match the linearised equations of motion of the exotic graviton.

Finally, in the third part, mainly based on [72], we focus on the antisymmetric tensor-spinors. In chapter 5, we firstly give a short review of the BV formalism, where the appearance of the "third ghost" is explained in irreducible theories. Then we show that the "third ghost" also appears in reducible theories. The last chapter turns to the application of BV quantisation to free fermionic p-form fields. With the ghost spectrum obtained from the BV formalism, the complete computation of gravitational anomalies is presented.

[8] Something even more exotic should happen in the $D = 6$, $\mathcal{N} = (4, 0)$ or $(3, 1)$ theories, if they exist, since they contain no metric at all. As discussed above, one should probably take the vanishing of the gravitational anomaly as a criterion selecting on which background manifolds these theories can be formulated consistently in certain regimes.

Synthèse en français

Il a été proposé qu'une limite de couplage fort de la supergravité quantique en cinq dimensions $\mathcal{N} = 8$ dans laquelle la longueur de Planck devient infinie pourrait donner une phase superconforme de la M-théorie en six dimensions. [1–3]. De plus, pour la théorie libre, cette limite est donnée par une théorie en six dimensions avec une supersymétrie maximale $\mathcal{N} = (4, 0)$. Cette théorie est conforme et n'a donc pas d'échelles de longueur. Lorsqu'elle est placée sur un cercle, l'échelle de compactification R devient l'échelle de Planck en cinq dimensions. Il est clair que la compréhension d'une telle limite nécessiterait des idées radicalement nouvelles et celles-ci seraient importantes pour notre compréhension globale de la physique gravitationnelle de la M-théorie. Ces dernières années, on a assisté à un regain d'intérêt pour ce domaine, qui a donné lieu à de nombreux développements intéressants et à de nouvelles approches [4–19].

Cependant, indépendamment des implications pour la théorie M, au niveau des supermultiplets, le multiplet (libre) avec supersymétrie $\mathcal{N} = (4, 0)$ existe certainement [20] et possède 32 supersymétries et 32 supersymétries conformes. Sa réduction dimensionnelle possède les mêmes degrés de liberté et le même contenu de champ que la supergravité maximale en cinq dimensions. Cette dernière théorie possède une symétrie globale $E_{6(6)}$ et, en plus du graviton, possède 27 champs vectoriels et 42 scalaires, ainsi que huit gravitini et 48 fermions de spin 1/2. Il a été suggéré que le premier possède la même symétrie $E_{6(6)}$, de sorte que les champs apparaissent dans des représentations similaires. Au lieu de la gravité (champ tensoriel symétrique de rang deux), elle possède un champ de jauge tensoriel de rang quatre avec les symétries du tenseur de Riemann. En raison des contraintes d'auto-dualité sur sa courbure, ce champ a cinq degrés de liberté (tout comme le graviton en cinq dimensions) et sa réduction dimensionnelle donne une gravité conventionnelle linéarisée en cinq dimensions [1]. De même, au lieu de 27 vecteurs en cinq dimensions, le multiplet $(4, 0)$ possède 27 tenseurs auto-duaux.[9] Dans les deux cas, il y a 27×3 degrés de liberté. Les 48 fermions de spin 1/2 deviennent simplement des fermions chiraux dans six dimensions. Enfin les huit gravitini (champs vecteurs-spineurs) sont remplacés par huit[10] "gravitini exotiques" $\psi_{\mu\nu}$, deux-formes spineurs avec une contrainte d'auto-dualité sur leurs courbures.[11]

En fait, le multiplet $(4, 0)$ n'est pas la seule théorie exotique en six dimensions. Il existe également un multiplet $(3, 1)$, où les contraintes d'auto-dualité sont partielles, et en examinant les degrés de liberté scalaires, on peut deviner que la symétrie qui régit la théorie est $F_{4(4)}$. Le multiplet possède un champ tenseur auto-dual de rang 3, et 28 scalaires qui pourraient se trouver dans l'espace tangent à l'espace symétrique $F_{4(4)}/\, Sp(2) \times Sp(6)$. Cependant, les 14 champs vectoriels et les 12 tenseurs auto-duaux ne forment que la représentation **26** de $F_{4(4)}$ lorsqu'ils sont combinés ensemble. Cela suggère qu'en fait, seul le groupe de R-symétrie $Sp(2) \times Sp(6)$ (et non la représentation complète $F_{4(4)}$) serait une vraie symétrie. Cela peut rendre suspicieux le fait que $E_{6(6)}$ soit une vraie symétrie de la théorie $\mathcal{N} = (4, 0)$, et nous verrons quelques indications que ce n'est peut-être pas le cas. Comme ces symétries ne découlent pas directement des supermultiplets, mais apparaissent seulement dans la construction des théories associées, l'absence d'une construction

[9]Dans nos conventions, le multiplet de gravité $(2, 0)$ en six dimensions possède cinq champs tenseurs anti-auto-duaux, tandis que les multiplets tensoriels $(2, 0)$ ont des tenseurs auto-duaux.

[10]Nous comptons les quatre champs quaternioniques comme huit champs complexes et nous utiliserons un comptage similaire tout au long de l'ouvrage.

[11]Comme dans une grande partie de la littérature, les champs dans les multiplets $(4, 0)$ et $(3, 1)$ qui n'apparaissent pas dans les multiplets ordinaires de gravité ou de matière, mais qui ont des contreparties directes, c'est-à-dire comme huit deux-formes spineurs dans $(4, 0)$ contre huit gravitini dans $(2, 2)$, seront étiquetés comme "exotiques". En raison de ses propriétés, pour le graviton exotique dans le multiplet $(4, 0)$, on utilisera également l'étiquette "self-dual-Weyl" (SDW).

complète de la théorie $(4,0)$ signifie qu'on ne peut pas être sûr. Cependant, un argument simple en faveur de la symétrie $E_{6(6)}$ est que les scalaires de la supergravité maximale de $5d$ sont tous élevés en scalaires dans $6d$. Ainsi, naïvement, on pourrait s'attendre à ce que les transformations de $5d$ de ces scalaires soient également élevées à $6d$. Les champs fermioniques du multiplet $(3,1)$ comprennent deux gravitini exotiques, six gravitini standard de chiralité négative, 28 fermions de spin 1/2 de chiralité positive et 14 fermions de spin 1/2 de chiralité négative. Les gravitini exotiques et conventionnelles se réduisent pour donner les huit gravitini standard en cinq dimensions, tandis que les fermions de spin 1/2 de chiralité quelconque se réduisent simplement à des champs de spin 1/2 en cinq dimensions.

Enfin, les champs exotiques peuvent apparaître dans des multiplets avec moins de supersymétrie. Ceux-ci peuvent être construits via les arguments habituelles de la théorie des représentations. Une alternative est de considérer la décomposition des multiplets maximalement supersymétriques. Par exemple, comme nous le verrons, le multiplet $(4,0)$ se décompose en un multiplet exotique de gravité $(2,0)$ ainsi qu'en 4 multiplets exotiques de gravitino $(2,0)$ et 5 multiplets tenseurs $(2,0)$. Cette décomposition est très similaire à la décomposition de la supergravité maximale $(2,2)$ en six dimensions. Celle-ci peut être décomposée en multiplets $(2,0)$: une gravité, 4 gravitino et 5 tenseurs.[12]

Une perspective utile sur ces multiplets est donnée par le fait qu'ils peuvent être vus comme des théories carrées ou produits [4, 8, 10], en analogie avec la supergravité maximale linéarisée en six dimensions, c'est-à-dire que la théorie $(2,2)$ est le carré du super Yang-Mills en six dimensions. Dans la même veine, le multiplet $(4,0)$ peut être vu comme un carré de multiplets tenseurs $(2,0)$, tandis que la théorie $(3,1)$ - comme un produit d'un multiplet $(2,0)$ avec un multiplet vectoriel $(1,1)$. Des structures de produit similaires apparaissent dans les théories exotiques avec moins de supersymétrie. Alors qu'une grande partie de l'intérêt pour les constructions en "double copy" provient du calcul des amplitudes dans la théorie des perturbations [21–23] (voir [24] pour une revue), il y a également eu des développements dans les réalisations théoriques de champs off-shell [5–8, 25–27] et dans la construction de solutions classiques [28–32]. Malheureusement, dans le cas qui nous intéresse, la théorie fortement couplée n'a pas d'expansion perturbative et il se peut également qu'il n'y ait pas de limite classique avec les interactions, ce qui limite l'utilité directe de ces constructions.

Aspects algébriques

Deux questions principales qui nous préoccupent dans cette thèse concernent les raisons basées sur la symétrie algébrique derrière l'existence des multiplets exotiques et la possibilité de sonder l'existence de formes d'interactions de ces théories exotiques (ainsi que leur existence sur des espaces courbes). Certains des arguments présentés ici peuvent être avancés à la fois pour les multiplets $(4,0)$ et $(3,1)$, et d'autres ne sont spécifiques qu'à $(4,0)$.

Une grande partie de la discussion algébrique du chapitre 2 se déroule dans le contexte des groupes de U-dualité et de leur relation avec les superalgèbres correspondantes. En particulier, nous utiliserons le langue de la géométrie généralisée [33–35] et de la théorie des champs exceptionnelle [36–38], en discutant les charges apparaissant dans l'algèbre de supersymétrie comme des vecteurs généralisés dans un espace tangent généralisé qui se transforme en une représentation linéaire sous le groupe de U-dualité correspondant. Afin d'éviter de rencontrer des algèbres de dualité de dimension infinie, nous travaillerons avec des décompositions dimensionnelles des théories considérant trois dimensions externes séparément du reste.

Comme nous le verrons dans la section 2.1, toutes les algèbres de supersymétrie avec 32 supercharges proviennent d'une superalgèbre particulière $\mathcal{A}$ (avec une sous-algèbre bosonique $\mathfrak{sl}(32,\mathbb{R}) \ltimes \mathbb{R}^{528}$) en restreignant $\mathfrak{sl}(32,\mathbb{R})$ à différentes sous-algèbres $\mathfrak{spin}(d-1,1)$. Par exemple, on peut obtenir les superalgèbres de $11d$, les supergravités de type IIA et de type IIB à partir de cette prescription. En effectuant une décomposition dimensionnelle, en décomposant disons $spin(9,1) \to spin(3,1) \times spin(6)$ en type IIA ou IIB, on peut voir comment le groupe $spin(6)$ résultant agirait sur les charges apparaissant dans l'espace tangent généralisé de la théorie de supergravité sur la partie

[12] Il n'est pas difficile de vérifier que même si les multiplets individuelles sont chiraux, la combinaison entière ne l'est pas - pour chaque fermion chiral ou champ auto-dual, il en existe un autre avec la chiralité opposée ou l'anti-auto-dualité.

signature euclidienne interne. Ainsi, le simple fait d'exiger les chiralités des fermions présents dans le type IIA et le type IIB implique que l'on exige des "sections" (dans le cadre de la théorie des champs exceptionnelle) $E_{d(d)}$-inequivalentes de l'espace tangent généralisé pour correspondre aux impulsions physiques dans l'espace-temps pour les deux théories. Pour le cas particulier du type IIA vs type IIB, ces sections inéquivalentes (ou plongements inéquivalents du groupe linéaire général dans le groupe de U-dualité) ont été largement discutées dans la littérature [33,35,36,39]. Une discussion similaire des sections pour la supersymétrie demi-maximale peut être trouvée dans [40], où il a été conclu que des sections inéquivalentes donnaient les supergravités $\mathcal{N} = (1,1)$ et $\mathcal{N} = (2,0)$ en six dimensions (la première section s'étendant au type I en dix dimensions).

De même, on peut explorer ce qui se passe si l'on exige une supersymétrie de

$$\mathcal{N} = (4,0)$$

en six dimensions à partir de la décomposition. Nous examinons l'intersection du groupe pertinent $Spin(5,1)$ avec le groupe de spin généralisé $Spin(2,1) \times SO(16)$.Sous le sous-groupe commun $Spin(2,1) \times Spin(3)$, nous observons comment les charges dans l'espace tangent généralisé sont regroupées en représentations irréductibles du facteur $Spin(3)$ et du $SL(3,\mathbb{R}) \subset SL(9,\mathbb{R}) \subset E_{8(8)}$ qui le contient. Ceci révèle un comportement très différent de la situation normale en géométrie généralisée ou en théorie des champs exceptionnelle.

L'origine de cette différence réside en partie dans le fait que dans $E_{8(8)}$, les charges apparaissant dans l'algèbre de supersymétrie ne engendrent pas la représentation **248** complète dans laquelle le vecteur généralisé se transforme, mais seulement la partie **120** sous son sous-groupe $SO(16)$. Sous le plongement direct dans la **248**, les charges de impulsion ne satisfont pas la condition de section, même en supergravité standard.

Le triplet $Spin(3)$ de charges de impulsion de l'algèbre de supersymétrie $(4,0)$ se plonge donc dans le vecteur généralisé comme un triplet de $SO(3)$, qui consiste en deux des impulsions qui seraient présentes dans la réduction conventionnelle de la supergravité cinq-dimensionnelle à trois dimensions, plus une partie de la charge du graviton dual, comme prévu par [1]. Cependant, sous le sous-groupe $SL(3,\mathbb{R})$ contenant ce $SO(3)$, ces trois charges sont combinées avec cinq autres pour former un octuplet. Normalement, en supergravité, on s'attendrait plutôt à ce qu'elles soient contenues dans un sous-espace de la somme de deux triplets, un espace dans lequel on pourrait identifier un triplet $SL(3,\mathbb{R})$ résolvant la condition de section. Ici, ce n'est pas le cas, et il n'y a pas de telle section. De plus, ce sous-groupe $SL(3,\mathbb{R})$ est lié à celui de la supergravité $\mathcal{N} = (2,2)$ par une transformation dans le $SL(9,\mathbb{R}) \subset E_{8(8)}$, de sorte que toute section de ce type serait de toute façon équivalente à la section standard.

Néanmoins, nous continuons à examiner la décomposition du vecteur généralisé et de l'adjoint de $E_{8(8)}$ sous $SL(3,\mathbb{R}) \times E_{6(6)}$, en notant que si nous avions renforcé $SL(3,\mathbb{R})$ en $GL(3,\mathbb{R})$ comme on le ferait habituellement en supergravité standard, cela changerait la commutante $E_{6(6)}$ en $SO(5,5)$. Nous examinons ensuite ces décompositions et tentons d'appliquer la prescription algébrique naïve (généralement imaginée uniquement dans le contexte de la supergravité – voir par exemple [35] pour une discussion) pour extraire le contenu du champ d'une théorie parente en six dimensions. Nous trouvons que, avec des identifications appropriées, cela correspond exactement à ce que l'on pourrait attendre du multiplet $\mathcal{N} = (4,0)$, bien que des questions subsistent quant à savoir si l'on doit décomposer sous $SO(3) \subset SL(3,\mathbb{R})$ et $Sp(8) \subset E_{6(6)}$ afin de faire ces identifications. En effet, la construction algébrique de la dérivée de Lie généralisée dans l'espace plat semble reproduire une formule pour la transformation de jauge du graviton exotique, ce qui nous rassure sur le fait que notre identification des directions de l'espace-temps à l'intérieur de l'espace tangent généralisé, ainsi que des champs et des charges, est plutôt correcte.

h-théories

Bien sûr, on peut se demander si ces multiplets n'ont pas d'autres caractéristiques que leurs simples propriétés algébriques. Ils se distinguent en tant que multiplets avec le plus haut spin ≤ 2 qui n'apparaissent pas dans les théories de supergravité standard, leurs décompositions sous les sous-superalgèbres et les compactifications ou leurs multiplets de matière. Nous présenterons des arguments selon lesquels le fait que le groupe de symétrie conjecturé de $(4,0)$, $E_{6(6)}$ possède un commutant $SL(3,\mathbb{R})$ dans le groupe de symétrie tridimensionnel $E_{8(8)}$ ne sert pas seulement d'outil technique utile, mais est étroitement lié à l'existence même de la théorie en six dimensions avec la symétrie $E_{6(6)}$. De manière correspondante, les groupes de symétrie des exotiques $(2,0)$ et $(1,0)$ ont des $SL(3,\mathbb{R})$ commutants à l'intérieur des groupes de symétrie des théories tridimensionnelles avec 16 et 8 supercharges respectivement.

En général, les groupes exceptionnels $E_{d(d)}$ ont des commutants $GL(n,\mathbb{R})$ à l'intérieur de groupes plus grands $E_{d+n(d+n)}$. Ceci est essentiellement par construction : les théories de dimension inférieure avec supersymétrie maximale peuvent être obtenues à partir des théories de dimension supérieure après une compactification sur le tore T^n. Trouver d'autres décompositions de $E_{d+n(d+n)}$ peut être utile en tant qu'outil technique, mais n'a que très peu de conséquences en ce qui concerne les théories de dimension supérieure. Pour les autres décompositions $G_d \times H_n \subseteq E_{d+n(d+n)}$, il n'existe pas de théorie (ou multiplet) maximalement supersymétrique (connue) en $D = 11-d$ dimensions avec une symétrie H_n. Par exemple, l'existence du sous-groupe $SL(2,\mathbb{R}) \times E_{7(7)} \subseteq E_{8(8)}$ n'a aucune implication pour la physique en cinq dimensions, car il n'existe aucune théorie maximale en cinq dimensions avec le groupe de symétrie $E_{7(7)}$.

Dans ce sens, en supposant que la théorie $\mathcal{N} = (4,0)$ possède réellement une symétrie $E_{6(6)}$, nous voyons que $E_{6(6)}$, $SL(3,\mathbb{R})$ et $E_{8(8)}$ forment un triple unique pour les théories maximalement supersymétriques. Comme nous l'avons mentionné, il existe des contreparties moins supersymétriques de ce triple avec $SL(3,\mathbb{R})$ jouant toujours un rôle central. Pour être plus concret, nous nous concentrerons sur le cas maximalement supersymétrique. Étant donné que le coset $SL(3,\mathbb{R})/SO(3)$ est l'espace modulaire des métriques plates sur T^3 de volume fixe, cela suggère une façon de penser au multiplet $(4,0)$ analogue à la F-théorie [41]. Une solution de supergravité tridimensionnelle avec cinq scalaires non constants paramétrant le coset, peut être considérée comme une solution d'une théorie six-dimensionnelle avec la symétrie restante $E_{6(6)}$, c'est-à-dire la théorie $(4,0)$ sur une variété fibrée de T^3 satisfaisant certaines conditions. De plus, en utilisant les résultats de travaux antérieurs sur les fibrations de torus "U-fold" [42], on peut montrer que l'information géométrique peut être reconditionnée et présentée sous la forme d'un champ tenseur auto-dual de Weyl (SDW), et les conditions différentielles sur l'espace six-dimensionnel après linéarisation peuvent être réduites aux équations du mouvement pour le champ SDW. Les détails de cette construction que nous appelons h-théorie peuvent être trouvés dans la section 4. Une nouvelle caractéristique de cette construction est que la géométrie et le champ SDW sont tous deux construits à partir des degrés de liberté scalaires physiques en trois dimensions. Notre analyse n'a pas non plus de champs de propagation le long des directions du tore, de manière similaire à la situation en F-théorie où il n'y a pas des impulsions dans les directions auxiliaires T^2. Cette image intrigante suggérerait donc que la théorie $(4,0)$ n'est pas vraiment six-dimensionnelle, car les états physiques ne sont pas chargés sous l'effet des impulsions supplémentaires.

Il a été observé dans [2] qu'en raison du fait que le groupe de symétrie quadridimensionnel $E_{7(7)}(\mathbb{Z})$ n'a pas de sous-groupe $E_{6(6)}(\mathbb{Z}) \times SL(2,\mathbb{Z})$, la dualité $SL(2,\mathbb{Z})$ attendue de la description six-dimensionnelle agirait de manière non triviale sur le graviton, ce qui pourrait conduire à une modification de la supergravité. Notre image suggère une possibilité plus conservatrice, inspirée par les relations entre la F-théorie, la supergravité 11-dimensionnelle et le type IIB. Nous ne devrions pas penser retrouver les supergravités quadridimensionnelles à partir d'une réduction de T^2 de la théorie exotique $(4,0)$, pas plus que nous ne nous attendons à ce qu'une réduction directe de la F-théorie sur un cercle donne la supergravité 11-dimensionnelle, ou que la M-théorie soit simplement réduite à IIB. Au lieu de cela, lorsque la M-théorie est placée sur un 2-tore, on peut prendre la limite dite de la F-théorie qui décompactifie à dix dimensions tout en conservant la $SL(2,\mathbb{Z})$, c'est-à-dire qu'elle donne la théorie de type IIB. La limite est également valable à partir de la M-théorie sur une variété elliptiquement fibrée, auquel cas la décompactification donne le type IIB sur la base de la fibration elliptique. L'idée est donc de considérer la supergravité maximale tridimensionnelle, c'est-à-dire la théorie $(4,0)$ sur un volume fixe T^3 dans les limites de décompactification. En dénotant les rayons des cercles dans T^3 par r_1, r_2, r_3 et en fixant le $\mathrm{Vol}(T^3) = 1$, jusqu'aux facteurs numériques on a $r_1 = 1/r_2 r_3$. On peut prendre $r_2, r_3 \to \infty$ et donc $r_1 \to 0$, c'est-à-dire décompactifier deux dimensions.
Le chemin

$$E_{8(8)} \supseteq SL(3,\mathbb{R}) \times E_{6(6)} \supseteq SL(2,\mathbb{R}) \times \mathbb{R}^+ \times E_{6(6)} \hookrightarrow GL(2,\mathbb{R}) \times E_{6(6)} \xrightarrow{r_2,r_3\to\infty} E_{6(6)} \quad \text{en D=5}$$

donne une théorie cinq-dimensionnelle avec une symétrie $E_{6(6)}$, c'est-à-dire la supergravité ordinaire en cinq dimensions. Une autre possibilité est $r_2, r_3 \to 0$ et donc $r_1 \to \infty$, c'est-à-dire décompactifier une seule dimension. Le chemin maintenemt est

$$E_{8(8)} \supseteq SL(3,\mathbb{R}) \times E_{6(6)} \supseteq SL(2,\mathbb{R}) \times \mathbb{R}^+ \times E_{6(6)} \longrightarrow SL(2,\mathbb{R}) \times E_{7(7)} \xrightarrow{r_1 \to \infty} E_{7(7)} \quad \text{en D=4.}$$

Cela explique l'apparition de $E_{6(6)}$ cinq-dimensionnels et de $E_{7(7)}$ quatre-dimensionnels dans les limites de décompactification de la supergravité maximale tridimensionnelle. Comme tout ce qui concerne le plongement de $SL(3,\mathbb{Z})$ dans le groupe de dualité tridimensionnel, ces chaînes continuent à tenir pour les théories à 16 et huit supercharges. En appelant le groupe de symétrie G, nous notons d'abord que $G^{\text{exotic}}_{D=6} = G_{D=5}$ et que $SL(3,\mathbb{R}) \times G^{\text{exotic}}_{D=6} \subseteq G_{D=3}$ ainsi que $SL(2,\mathbb{R}) \times G_{D=4} \subseteq G_{D=3}$. Les décompacités vers les supergravités ordinaires en quatre et cinq dimensions fonctionnent maintenant comme dans le cas maximalement supersymétrique.

Une autre observation qui suggère que nous ne considérons pas la théorie comme étant véritablement six-dimensionnelle provient de la considération des dualités de rang supérieur. En considérant les symétries de Kac-Moody conjecturées $E_{8+n(8+n)}$ pour $n = 1, 2, 3$, nous pourrions nous attendre à trouver que la commutante $SL(3,\mathbb{R})$ de $E_{6(6)}$ est étendue à $SL(3+n,\mathbb{R})$. Or, ce n'est pas le cas. En particulier, la $SL(3,\mathbb{R}) \times SL(3,\mathbb{R}) \times E_{6(6)}$ que nous considérons dans notre décomposition dimensionnelle (en trois dimensions externes, trois dimensions internes et une symétrie interne $E_{6(6)}$) ne s'étend pas à un sous-groupe $SL(6,\mathbb{R}) \times E_{6(6)}$ dans E_{11}.[13] Cependant, il existe une sous-algèbre $Spin(1,5)$ de KE_{11} correspondant à la décomposition de la représentation de spineur à 32 composants en 4 spineurs de la même chiralité en six dimensions, de sorte que E_{11} semble accommoder le multiplet au niveau de la superalgèbre. Le fait que le sous-groupe pertinent $SL(6,\mathbb{R})$ n'existe pas indique (sans surprise) qu'il n'y a pas de gravité six-dimensionnelle pour ce multiplet et potentiellement que la théorie n'est pas vraiment six-dimensionnelle.[14]

Couplages de Chern-Simons et anomalies

Pour étayer davantage cette image, nous incluons d'autres arguments suggérant que la réduction naïve de la théorie $(4,0)$ sur S^1 ou T^2 pourrait ne pas produire la dynamique de la supergravité en cinq ou quatre dimensions. Nous trouverons également des déclarations similaires pour la théorie $(3,1)$.

Tout d'abord, nous considérons la génération des interactions topologiques de Chern-Simons présentes dans la supergravité maximale en cinq dimensions [43]

$$S_{\text{CS}} = \int k_{\Lambda\Sigma\Delta}\ A^\Lambda \wedge F^\Sigma \wedge F^\Delta \tag{5}$$

où $k_{\Lambda\Sigma\Delta}$ est l'invariant cubique de $E_{6(6)}$ et les Λ, Σ, Δ sont des indices de $E_{6(6)}$ allant de 1 à 27. Cette interaction n'implique pas la métrique et n'admet pas de linéarisation. Par supersymétrie, l'incapacité à la générer indiquerait que les équations dérivées des champs tenseurs de rang trois et quatre ne concorderont pas avec celles de la gravité au-delà du niveau linéarisé. Des calculs similaires ont été effectués, notamment dans le contexte des théories à huit supercharges, où il a été montré comment les diagrammes triangulaires avec des modes KK massifs provenant des champs chiraux six-dimensionnels dans la boucle génèrent des termes de Chern-Simons en cinq dimensions [44–47]. Un point important ici est que, bien que les modes KK des champs six-dimensionnels soient impliqués, le calcul lui-même est effectué en cinq dimensions. Comme nous le montrons dans la section 3.2, sous des hypothèses raisonnables, seule la réduction de la supergravité six-dimensionnelle génère des (5) compatibles avec l'invariant cubique $E_{6(6)}$.

Comme les modes KK considérés ici proviennent de champs chiraux à six dimensions, le calcul ci-dessus est étroitement lié aux anomalies six-dimensionnelles et aux théorèmes d'indice. Puisque les multiplets exotiques comportent des champs chiraux, les questions relatives aux anomalies se posent naturellement. On peut objecter que celles-ci sont formulées dans l'espace plat, et que ce n'est qu'après réduction que la gravité cinq-dimensionnelle (linéarisée) et la symétrie du

[13] Nous remercions Guillaume Bossard de nous avoir expliqué ces caractéristiques de E_{11}.

[14] Une légère différence entre notre image et celle de la F-théorie est que, bien qu'il n'y ait pas de $SL(12,\mathbb{R})$ à l'intérieur de E_{11}, il n'y a pas non plus de groupe de spin douz-dimensionnel ou de charge d'impulsion.

difféomorphisme apparaissent. Les cinq degrés de liberté portés par le champ SDW doivent être considérés comme des excitations d'une métrique en cinq dimensions, de sorte que l'on ne s'attend pas à une symétrie de difféomorphisme six-dimensionnelle, mais plutôt à des symétries exotiques qui donnent lieu à des difféomorphismes en cinq dimensions.

En général, l'invariance des difféomorphismes est une propriété critique pour les théories de supergravité quantique. Elle correspond à la conservation du tenseur énergie-impulsion au niveau quantique et peut être vérifiée par des calculs à une boucle, les états externes étant des gravitons. En même temps, les anomalies correspondant à la rupture des difféomorphismes peuvent aussi être interprétées comme la transformation anormale de la mesure de l'intégrale de chemin des champs chiraux sous les transformations de difféomorphisme de l'espace-temps. Les anomalies de difféomorphisme sont équivalentes aux anomalies de symétrie locale de Lorentz jusqu'aux contre-termes locaux non polynomiaux (voir par exemple [48]). Ainsi, indépendamment des considérations sur la symétrie des difféomorphismes, il est logique de se demander si la théorie non gravitationnelle $(4,0)$ est invariante sous les transformations locales de Lorentz sur des variétés de fond arbitraires en six dimensions. On peut répondre à cette question en calculant les anomalies gravitationnelles au sens conventionnel.

Nous constatons que les champs exotiques de la théorie $(4,0)$ se trouvent à l'intérieur des domaines de certains opérateurs de Dirac, de la même manière que les p-formes auto-duelles se trouvent à l'intérieur du complexe de signature (voir par exemple [49]). Ce fait est intimement lié aux multiplets exotiques qui apparaissent comme des produits de multiplets de matière, et est très similaire au traitement des p-formes auto-duelles comme partie d'un champ bispineur. Comme nous le verrons, pour les champs exotiques, nous devons simplement prendre des puissances supérieures des représentations du spineur. Les calculs explicites se trouvent dans la section 3.1, avec plus de détails dans l'appendice B.2. La conclusion est que les multiplets $(4,0)$ et $(3,1)$ ont des anomalies non-vanentes. D'une certaine manière, la décomposition des multiplets maximalement supersymétriques mentionnée ci-dessus donne une explication heuristique à cela. Les multiplets ordinaires $(2,0)$ - gravité (GM), gravitino (GoM) et tenseur (TM) - bien que tous chiraux, ont des champs de chiralité différente qui apparaissent en eux, de sorte que une combinaison particulière d'entre eux devient même une théorie non chirale.[15] Au contraire, les multiplets exotiques ont des chiralités maximalement alignées de sorte qu'une annulation semble naïvement beaucoup moins probable, et ne se produit effectivement pas.

Quantification et anomalies des tenseurs-spineurs antisymétriques

Outre les champs exotiques bosoniques, des deux-formes fermioniques apparaissent dans les multiplets exotiques N= (4,0) et N= (3,1) maximalement supersymétriques en six dimensions [20].

L'étude des deux-formes fermioniques en six dimensions peut être ramenée à un ensemble plus général, à savoir les p-formes fermioniques en dimension arbitraire D. Considérons des champs fermioniques de la forme $\psi^\alpha_{\mu_1\mu_2\ldots\mu_p}$, où α est un indice de spineur et les μ_i sont des indices de l'espace-temps, qui sont totalement antisymétriques dans leurs indices de l'espace-temps :

$$\psi^\alpha_{\mu_1\mu_2\ldots\mu_p} = \psi^\alpha_{[\mu_1\mu_2\ldots\mu_p]}\,. \tag{6}$$

L'action libre pour un tel champ dans un espace-temps plat est une généralisation directe de l'action de Rarita-Schwinger pour une 1-forme fermionique ψ^α_μ et se lit [50, 51]

$$S_0[\psi] = -(-1)^{\frac{p(p-1)}{2}} \int d^Dx\; \bar{\psi}_{\mu_1\mu_2\ldots\mu_p}\, \gamma^{\mu_1\mu_2\ldots\mu_p\nu\rho_1\rho_2\ldots\rho_p}\, \partial_\nu \psi_{\rho_1\rho_2\ldots\rho_p}\,. \tag{7}$$

Cette action est invariante sous certaines symétries de jauge réductibles, c'est-à-dire avec des transformations "jauge pour jauge". Ces symétries sont

$$\delta\psi = d\Lambda^{(p-1)}\,, \quad \delta\Lambda^{(p-1)} = d\Lambda^{(p-2)}\,, \quad \ldots\,, \quad \delta\Lambda^{(1)} = d\Lambda^{(0)} \tag{8}$$

en notation de forme différentielle (avec un indice de spineur spectateur). Ici, chaque paramètre $\Lambda^{(k)}$ est un tenseur-spineur antisymétrique de rang k. Cette réductibilité introduit des subtilités

[15] En fait, les trois multiplets ont des polynômes d'anomalie proportionnels : $I_{\mathrm{TM}} = \frac{1}{4} I_{\mathrm{G}oM} = -\frac{1}{21} I_{\mathrm{G}M}$.

bien connues sur la quantification, que nous aborderons à l'aide du puissant formalisme champs-antichamps de Batalin-Vilkovisky (BV) [52, 53].

Les anomalies gravitationnelles pour ces multiplets exotiques sont calculées dans le chapitre 3 (les résultats ont été publiés dans [15]), mais certaines hypothèses étaient nécessaires puisque la structure fantôme précise pour la deux-forme fermionique était inconnue à l'époque. L'un des objectifs de cette partie est de combler cette lacune. Une autre motivation, plus lointaine, pour examiner ces types de champs provient de considérations sur la gravité duale [1, 3, 54, 55], où (dans le régime linéarisé) le graviton est dualisé à un tenseur de symétrie mixte de type $[D-3,1]$. Cependant, il n'existe toujours pas de modèle supersymétrique, manifestement covariant, dans lequel ce champ trouve un partenaire, et un champ fermionique de p-forme serait le candidat naturel (voir [56] pour une première tentative de dualisation des champs fermioniques, et [57, 58] pour des considérations connexes dans le formalisme du prépotentiel).

Dans la procédure de quantification des théories de jauge irréductibles (c'est-à-dire lorsqu'il n'y a pas de transformations "jauge pour jauge"), les jauges quadratiques contenant un opérateur différentiel donnent lieu à un troisième fantôme qui se propage. Ceci a été découvert pour la première fois en supergravité dans la quantification du champ de Rarita-Schwinger, et le troisième fantôme propageant est connu sous le nom de fantôme de Nielsen-Kallosh [59, 60]. Plus tard, le troisième fantôme a été dérivé à nouveau dans le formalisme BV [52] d'une manière manifestement locale par Batalin et Kallosh dans [61].

Le "troisième fantôme" pour les jauges quadratiques apparaît également dans le cas réductible, comme nous le prouvons dans le chapitre 5. Il faut souligner que cette affirmation est valable au-delà de l'action simple et des symétries de jauge pour la p-forme fermionique décrite ci-dessus : nous autorisons les algèbres de jauge non-abéliennes, la fermeture on-shell, etc. Ces subtilités sont toutes regroupées dans la forme explicite de l'action BV minimale pour le modèle en question, qui existe toujours et que nous gardons arbitraire.

La quantification des champs fermioniques libres de p-forme, à l'aide des résultats généraux mentionnés ci-dessus, (selon les termes de [53]) est "comme casser des noix avec un marteau de forgeron". Comme les champs fermioniques satisfont des équations de mouvement du premier ordre et que l'action (7) est déjà sous forme hamiltonienne, les méthodes de quantification hamiltonienne de [62–64] auraient été plus économiques. Le troisième fantôme a également été discuté dans ce formalisme dans la référence [65]. Cependant, l'approche que nous utilisons ici a l'avantage de préserver la covariance manifeste. Ceci est fait à la fois dans la fixation de jauge habituelle à fonction delta et dans la fixation de jauge Gaussienne où un fantôme de Nielsen-Kallosh généralisé apparaît; les propagateurs et les transformations BRST sont également discutés dans les deux schémas. Les détails explicites ne sont donnés que dans le cas à deux-forme, mais la généralisation au degré de forme supérieur ne pose aucune difficulté. Nous maintenons la localité manifeste et la covariance tout au long de l'étude.

Nous calculons l'anomalie gravitationnelle d'une p-forme chirale fermionique en dimensions $D = 4m+2$. Nous utilisons pour cela le spectre fantôme trouvé dans la quantification et appliquons le théorème de l'indice d'Atiyah-Singer [66, 67], en suivant les méthodes développées dans les articles classiques [68–71]. Nous décrivons en détail la procédure générale et affichons les résultats en dimensions $D = 2$, 6 et 10 dans les tableaux 6.3, 6.4 et 6.5. Un résultat intriguant est que dans les dimensions $D \geq 6$, l'anomalie d'une p-forme chirale fermionique correspond à celle d'une $(D-p-1)$-forme; il serait très intéressant d'utiliser ce fait pour tenter de construire de nouveaux modèles sans anomalies.

Nous devons mentionner une mise en garde importante liée au calcul de l'anomalie gravitationnelle : à notre connaissance, il n'existe actuellement aucun modèle qui couple de manière cohérente une p-forme fermionique à la gravité dynamique. On peut espérer que cette difficulté sera résolue à l'avenir, peut-être en incluant (un nombre infini) d'autres champs.[16] Cependant, puisque les calculs de l'anomalie de la section 6.4 sont uniquement basés sur le spectre fantôme et non sur la forme spécifique de l'action, nous sommes confiants que ces résultats survivront à de tels développements futurs.

[16] Quelque chose d'encore plus exotique devrait se produire dans les théories $D = 6$, $\mathcal{N} = (4,0)$ ou $(3,1)$, si elles existent, puisqu'elles ne contiennent aucune métrique. Comme discuté ci-dessus, on devrait probablement prendre la disparition de l'anomalie gravitationnelle comme un critère sélectionnant sur quelles variétés d'arrière-plan ces théories peuvent être formulées de manière cohérente dans certains régimes.

Structure

La thèse est divisée en trois parties et elle est organisée comme suit. La première partie est un chapitre introductif, nous passons en revue la construction et la structure des multiplets exotiques en six dimensions. Cette partie couvre également la discussion des champs tenseurs exotiques libres apparaissant dans ces multiplets exotiques.

La deuxième partie est basée sur l'article [15]. Dans le chapitre 2 nous discutons comment relier la superalgèbre $\mathcal{N} = (4,0)$ à celle de la supergravité onze-dimensionnelle et comment interpréter ses charges en termes d'objets $E_{8(8)}$, dans le cadre de la géométrie exceptionnelle. Le chapitre 3 section 3.1 contient les polynômes d'anomalie pour la symétrie de Lorentz locale des multiplets exotiques, qui s'avèrent non factorisables. Nous montrons également dans ce chapitre qu'il n'existe pas de mécanisme conventionnel pour générer les couplages de Chern-Simons de la supergravité maximale en cinq dimensions à partir de la compactification circulaire des champs $\mathcal{N} = (4,0)$. Dans le chapitre 4, nous présentons notre construction de "h-theories" sur des géométries fibrées T^3, dont les solutions correspondent aux équations de mouvement linéarisées du graviton exotique.

Enfin, dans la troisième partie, principalement basée sur [72], nous nous concentrons sur les tenseurs-spineurs antisymétriques. Dans le chapitre 5, nous donnons d'abord un bref aperçu du formalisme BV, où l'apparition du "troisième fantôme" est expliquée dans les théories irréductibles. Ensuite, nous montrons que le "troisième fantôme" apparaît également dans les théories réductibles. Le dernier chapitre est consacré à l'application de la quantification BV aux champs fermioniques libres de p-forme. Avec le spectre fantôme obtenu à partir du formalisme BV, le calcul complet des anomalies gravitationnelles est présenté.

Part I

Exotic supermultiplets in six dimensions

Chapter 1

The exotic multiplets and exotic tensor fields

In this chapter we provide some background discussion of the six-dimensional supermultiplets, whose highest spin field is a spin-2 boson which is not a graviton. The supermultiplets of extended Poincaré supersymmetry which correspond to possible local field theories were classified in [20]. Curiously, the list provided includes the multiplet which forms the basis for the $\mathcal{N} = (4,0)$ theory of [1], as well as a similar multiplet with $\mathcal{N} = (3,1)$ supersymmetry. However, similar multiplets with less supersymmetry were omitted, one can find a list of them in [73]. As these will form part of our discussion later, we will review the detailed construction of such multiplets with $\mathcal{N} = (1,0)$, $\mathcal{N} = (2,0)$ and $\mathcal{N} = (4,0)$ supersymmetry in the appendix A, and our conventions can also be found in there.

These multiplets are exotic in the sense that they are not based on a dynamical metric tensor, their field theories have not been constructed beyond the linear level. The free fields can be represented by tensor with mixed Young symmetry. For example, the two column Young diagrams $\yng(2,2)$ and $\yng(2,1)$ can be used to represent the bosonic exotic gravitons in these multiplets in six dimensions. The reduction of $\yng(2,2)$ to five dimensions give rise to fields corresponding to Young diagrams of type $\yng(2,2)$, $\yng(2,1)$, and $\yng(1,1)$; while the reduction of $\yng(2,1)$ yields $\yng(2,1)$ and $\yng(1,1)$. As we will see, these fields provide equivalent formulations of linearised gravity in five dimensions. The dual graviton $\yng(2,1)$ is first studied in [74] and then together with the double dual graviton $\yng(2,2)$ they are considered a while ago by Hull [1–3] in the context of gravitational dualities. More recently, there are various further investigations on the dynamics of these free tensor fields [14, 75–79]. We will briefly go through the gauge theories of these exotic free fields and mention their connections with the chiral exotic fields in the $\mathcal{N} = (4,0)$ and $\mathcal{N} = (3,1)$ multiplets.

1.1 The massless multiplets of the six dimensional maximal supersymmetry

In six dimensions the Lorentz group is $SO(5,1)$. It admits symplectic-Majorana-Weyl spinor representations, with such chiral spinors represented as pairs of four-component complex vectors ζ^A for $A = 1, \ldots, 2n$ satisfying the reality condition $(\zeta^A)^* = \Omega_{AB}\mathcal{B}\zeta^B$ (see appendix A for conventions). For the case of maximal supersymmetry, which will be our main focus here, one has 32 real supercharges Q which are made up of four such symplectic-Majorana-Weyl spinors. Up to interchange of chirality, the possible combinations of chiralities are $\mathcal{N} = (4,0)$, $(3,1)$ or $(2,2)$. The corresponding R-symmetry groups of these superalgebras are $G^R_{(p,q)} = Sp(2p) \times Sp(2q)$[1] for $\mathcal{N} = (p,q)$ supersymmetry.

[1]In this article, we denote by $Sp(2n)$ the compact symplectic group of rank n.

For instance, for the (4,0) chiral superalgebra, the anti-commutator of the supercharges is given by

$$\{Q^A_\alpha, Q^B_\beta\} = \Omega^{AB} P_\mu \gamma^\mu{}_{[\alpha\beta]} + \dot{Z}^{[AB]}_{[\alpha\beta]} + Z^{(AB)}_{(\alpha\beta)} \tag{1.1}$$

where P_μ is the momentum and the quantities denoted with a Z are central charges (with $\dot{Z}_{[\alpha\beta]A}{}^A = 0$).

To study the multiplets, we need to look at the representations of the superalgebras. The massless physical states form representations of the little group $G_{\text{little}} = SU(2) \times SU(2) \times G^R_{(p,q)}$, which is the subgroup of $Spin(5,1) \times G^R_{(p,q)}$ preserving a null-momentum vector. Representations of G_{little} will be denoted as e.g. $(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$, where we use a semicolon to separate the representations of the spacetime part and the R-symmetry part of the little group. The representations of these superalgebras with only states of helicity at most 2 were classified in [20], and are presented in Table 1.1.

$D=6,(p,q)=(4,0)$ $SU(2)\times SU(2)\times Sp(8)$ Q belongs to $(\mathbf{2},\mathbf{1};\mathbf{8})$	$\mathbf{2^8} = (\mathbf{5},\mathbf{1};\mathbf{1}) + (\mathbf{3},\mathbf{1};\mathbf{27}) + (\mathbf{1},\mathbf{1};\mathbf{42})$ $+(\mathbf{4},\mathbf{1};\mathbf{8}) + (\mathbf{2},\mathbf{1};\mathbf{48})$
$D=6,(p,q)=(3,1)$ $SU(2)\times SU(2)\times Sp(6)\times Sp(2)$ Q belongs to $(\mathbf{2},\mathbf{1};\mathbf{6},\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$	$\mathbf{2^8} = (\mathbf{4},\mathbf{2};\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2};\mathbf{14},\mathbf{1}) + (\mathbf{3},\mathbf{1};\mathbf{6},\mathbf{2})$ $+(\mathbf{1},\mathbf{1};\mathbf{14'},\mathbf{2}) + (\mathbf{4},\mathbf{1};\mathbf{1},\mathbf{2})$ $+(\mathbf{3},\mathbf{2};\mathbf{6},\mathbf{1})$ $+(\mathbf{2},\mathbf{1};\mathbf{14},\mathbf{2}) + (\mathbf{1},\mathbf{2};\mathbf{14'},\mathbf{1})$
$D=6,(p,q)=(2,2)$ $SU(2)\times SU(2)\times Sp(4)\times Sp(4)$ Q belongs to $(\mathbf{2},\mathbf{1};\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{4})$	$\mathbf{2^8} = (\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{5},\mathbf{1}) + (\mathbf{2},\mathbf{3};\mathbf{4},\mathbf{1})$ $+(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{5}) + (\mathbf{1},\mathbf{1};\mathbf{5},\mathbf{5}) + (\mathbf{2},\mathbf{1};\mathbf{4},\mathbf{5})$ $+(\mathbf{3},\mathbf{2};\mathbf{1},\mathbf{4}) + (\mathbf{1},\mathbf{2};\mathbf{5},\mathbf{4}) + (\mathbf{2},\mathbf{2};\mathbf{4},\mathbf{4})$ Graviton in the $(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$

Table 1.1: Six-dimensional multiplets with 32 supercharges

We can see that in dimension six, the chiral superalgebra $\mathcal{N} = (4,0)$ has only one massless multiplet

$$\mathbf{2^8} = (\mathbf{5},\mathbf{1};\mathbf{1}) + (\mathbf{3},\mathbf{1};\mathbf{27}) + (\mathbf{1},\mathbf{1};\mathbf{42}) + (\mathbf{4},\mathbf{1};\mathbf{8}) + (\mathbf{2},\mathbf{1};\mathbf{48})\,. \tag{1.2}$$

The representations $(\mathbf{3},\mathbf{1};\mathbf{27})$, $(\mathbf{1},\mathbf{1};\mathbf{42})$ and $(\mathbf{2},\mathbf{1};\mathbf{48})$ are immediately identified with self-dual 2-forms B^+_{ij}, scalars ϕ and chiral fermions λ.

The field in the $(\mathbf{5},\mathbf{1};\mathbf{1})$ representation of the little group $SU(2)\times SU(2)\times Sp(8)$ has been labeled the exotic graviton [1] and is represented as a four-index object C_{ijkl} with i,j,k,l being the little group $SO(4)$ vector indices. It has the following properties

$$C_{ijkl} = C_{klij} = C_{[ij]kl} = C_{ij[kl]}\,, \qquad C_{[ijk]l} = 0 \\ C^i{}_{jik} = 0 \tag{1.3}$$

$$C_{ijkl} = \frac{1}{2}\epsilon_{ijmn}C^{mn}{}_{kl} = \frac{1}{2}C_{ij}{}^{mn}\epsilon_{mnkl}\,. \tag{1.4}$$

We see that this little group representation corresponding to the exotic four-index field has the symmetries of a self-dual Weyl tensor in four-dimensional Euclidean space. For this reason, this field and the supermultiplets for which it is the top component are often described as "self-dual Weyl" (see e.g. [6]), and we will use this terminology interchangeably with the label "exotic". The covariant self-dual Weyl field is represented by $C_{\mu\nu\rho\sigma}$ with the same index symmetries as the Riemann tensor

$$C_{\mu\nu\rho\sigma} = C_{\rho\sigma\mu\nu} = C_{[\mu\nu]\rho\sigma} = C_{\mu\nu[\rho\sigma]}\,, \tag{1.5}$$

$$C_{[\mu\nu\rho]\sigma} = 0\,. \tag{1.6}$$

The field strength (in flat spacetime) is defined as[2]

$$G_{\mu\nu\rho\sigma\tau\kappa} = \partial_{[\mu}C_{\nu\rho][\sigma\tau,\kappa]} \tag{1.7}$$

[2] The comma in the subscript denotes a partial derivative.

so that

$$G_{\mu\nu\rho\sigma\tau\kappa} = G_{[\mu\nu\rho]\sigma\tau\kappa} = G_{\mu\nu\rho[\sigma\tau\kappa]} = G_{\sigma\tau\kappa\mu\nu\rho}\,, \tag{1.8}$$

and

$$G_{[\mu\nu\rho\sigma]\tau\kappa} = 0\,. \tag{1.9}$$

The field strength defined in such way is invariant under the transformation

$$\delta C_{\mu\nu\rho\sigma} = 2\partial_{[\mu}\eta_{\nu]\rho\sigma} + 2\partial_{[\rho}\eta_{\sigma]\mu\nu} - 4\partial_{[\mu}\eta_{\nu\rho\sigma]} \tag{1.10}$$

with the parameter $\eta_{\mu\nu\rho} = \eta_{\mu[\nu\rho]}$.

It satisfies the self-duality on both the first three and the last three indices $G = \star G = G\star$ where we use $\star$ to denote the Hodge-star operation and it is defined in the $6d$ Minkowski spacetime as

$$\begin{aligned} G_{\mu\nu\rho\sigma\tau\kappa} &= (\star G)_{\mu\nu\rho\sigma\tau\kappa} \equiv \frac{1}{3!}\epsilon_{\mu\nu\rho\alpha\beta\gamma}G^{\alpha\beta\gamma}{}_{\sigma\tau\kappa} \\ &= (G\star)_{\mu\nu\rho\sigma\tau\kappa} \equiv \frac{1}{3!}G_{\mu\nu\rho}{}^{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma\sigma\tau\kappa}\,. \end{aligned} \tag{1.11}$$

This it the equation of motion for the covariant field strength G. We see that in our convention with the $6d$ Minkowski metric, the Hodge-star $\star$ squares to identity $(\star)^2 = 1$.

The $(\mathbf{4}, \mathbf{1}; \mathbf{8})$ part of the multiplet corresponds to a covariant chiral fermionic 2-form-spinor field $\psi^A_{\mu\nu}$, which we refer to as the exotic gravitino. We omit the $Sp(8)$ index A. This spinor field is anti-symmetric and its field strength is self-dual and gauge invariant

$$\begin{aligned} &\psi_{\mu\nu} = -\psi_{\nu\mu} \\ \chi_{\mu\nu\rho} \equiv 3\partial_{[\mu}\psi_{\nu\rho]}\,, \qquad &\chi = \star\,\chi \qquad \text{invariant under} \qquad \delta\psi_{\mu\nu} = 2\partial_{[\mu}\epsilon_{\nu]} \end{aligned} \tag{1.12}$$

where ϵ_ν is an arbitrary vector-spinor. However, unlike the bosonic fields, these properties (1.12) are not strong enough to ensure that ψ_{ij} transforms in $(\mathbf{4}, \mathbf{1}; \mathbf{8})$ of the little group after some gauge fixing. An extra constraint must be included, and it is

$$\gamma^{dabc}\chi_{abc} = 0 \tag{1.13}$$

where a, b, c, d are spatial indices. In [11], it was proved that this constraint together with the self-duality in (1.12) is equivalent to the Rarita-Schwinger type equation

$$\gamma^{\alpha\beta\mu\nu\rho}\chi_{\mu\nu\rho} = 0\,, \tag{1.14}$$

which is the covariant field equation leading to $(\mathbf{4}, \mathbf{1}; \mathbf{8})$. Equivalently, it can be taken as

$$\gamma^\mu\chi_{\mu\nu\rho} = 0\,. \tag{1.15}$$

We will come back to this discussion in chapter 3.

As shown in [1–3], due to the double self-duality relations (1.11), the dimensional reduction of $C_{\mu\nu\rho\sigma}$ to five dimensions gives a single linearised graviton. This can happen because the various components of $C_{\mu\nu\rho\sigma}$ which appear in the reduction become the dual graviton and the double-dual graviton. We will review this process in next section.

This mechanism is essentially a "squared" version of the mechanism by which a self-dual two-form in six dimensions restricts to a single vector field in five. Similarly, the exotic gravitino reduces to a single gravitino in five dimensions, and in total the massless degrees of freedom of the $(4, 0)$ multiplet reduce to exactly the fields of five-dimensional $\mathcal{N} = 8$ supergravity. In addition, the Kaluza-Klein tower of massive modes arising from the massless $(4, 0)$ states on circle match perfectly the $\frac{1}{2}$-BPS-states of the five-dimensional maximal supergravity [1,3]. The scalars of the $(4, 0)$ multiplet transform in the correct $Sp(8)$ representation to form a non-linear sigma model based on the coset

$$E_{6(6)}/\,Sp(8) \tag{1.16}$$

which is the same as that parametrised by the scalars of five-dimensional maximal supergravity [43]. However, as discussed in the introduction, it is not clear that the $E_{6(6)}$ symmetry uplifts to the six-dimensional theory.

We also see that in addition to the $(4,0)$ and $(2,2)$ maximal SUSY multiplets, there is the $(3,1)$ multiplet [1,2,20]. The highest spin field corresponds to the $(\mathbf{4},\mathbf{2};\mathbf{1},\mathbf{1})$ representation of the little group $SU(2)\times SU(2)\times Sp(6)\times Sp(2)$ and is a three-index object whose covariant field is a three-index tensor $D_{\mu\nu\rho}$ which satisfies

$$D_{\mu\nu\rho}=D_{[\mu\nu]\rho}, \quad D_{[\mu\nu\rho]}=0\,. \tag{1.17}$$

Its field strength is defined as

$$S_{\mu\nu\rho\sigma\kappa}=\partial_{[\mu}D_{\nu\rho][\sigma,\kappa]} \tag{1.18}$$

which is invariant under

$$\delta D_{\mu\nu\rho}=2\partial_{[\mu}\alpha_{\nu]\rho}-2\partial_{[\mu}\alpha_{\nu\rho]}+\partial_{\rho}\beta_{\mu\nu}-\partial_{[\rho}\beta_{\mu\nu]} \tag{1.19}$$

with the parameters $\alpha_{\mu\nu}$, $\beta_{\mu\nu}=\beta_{[\mu\nu]}$. It is constrained to satisfy the one-side self-duality constraint

$$S_{\mu\nu\rho\sigma\kappa}=(\star S)_{\mu\nu\rho\sigma\kappa}=\frac{1}{3!}\epsilon_{\mu\nu\rho\alpha\beta\gamma}S^{\alpha\beta\gamma}{}_{\sigma\kappa}\,. \tag{1.20}$$

It can be shown that upon a circle reduction the $(3,1)$ multiplet also yields the linearised five-dimensional $\mathcal{N}=8$ supergravity multiplet. The scalars of this multiplet naively appear to have a coset structure [1]

$$\frac{F_4}{Sp(6)\times Sp(2)}\,. \tag{1.21}$$

but the vector and two-form fields appear only to transform in a representation of $F_{4(4)}$ when combined [12], making it unclear that this is a symmetry of the theory.

All three of these maximal six-dimensional supermultiplets can be thought of as products of smaller supermultiplets. The idea that maximal supergravity can be viewed as the square of maximal super Yang-Mills theory has proved to be extremely powerful for the computation of perturbative scattering amplitudes [13, 21–24]. However, this view is also useful for simply understanding the multiplet structures purely at the level of the representation theory. In fact, one can also obtain the supergravity multiplets with various amounts of supersymmetry by considering products of tensor multiplets with supercharges of opposite chirality [6, 10]

$$\begin{aligned}
[(2,0)_{\text{tensor}}]\otimes[(0,2)_{\text{tensor}}]&=[(2,2)_{\text{sugra}}]\\
[(2,0)_{\text{tensor}}]\otimes[(0,1)_{\text{tensor}}]&=[(2,1)_{\text{sugra}}]\\
[(1,0)_{\text{tensor}}]\otimes[(0,1)_{\text{tensor}}]&=[(1,1)_{\text{sugra}}]
\end{aligned} \tag{1.22}$$

By contrast, the exotic multiplets arise when the tensor multiplets in the product have supercharges of aligned chirality:

$$\begin{aligned}
[(2,0)_{\text{tensor}}]\otimes[(2,0)_{\text{tensor}}]&=[(4,0)_{\text{SD-Weyl}}]\\
[(2,0)_{\text{tensor}}]\otimes[(1,0)_{\text{tensor}}]&=[(3,0)_{\text{SD-Weyl}}]\\
[(1,0)_{\text{tensor}}]\otimes[(1,0)_{\text{tensor}}]&=[(2,0)_{\text{SD-Weyl}}]+[(2,0)_{\text{tensor}}]\,,
\end{aligned} \tag{1.23}$$

Note that there exists also a $[(1,0)_{\text{SD-Weyl}}]$ which can be constructed using the standard methods [20]. The $(2,0)_{\text{SD-Weyl}}$ case is similar to the squaring of the $(1,0)$ vector multiplet, for which the product gives $[(2,0)_{\text{sugra}}]+[(2,0)_{\text{tensor}}]$.

For the non-maximally supersymmetric case, notably $(2,0)$ and $(1,0)$ the SD-Weyl multiplets exist in parallel to the standard supergravity multiplets [73], and have the same numbers of degrees of freedom as the latter, but have fields living in the different representations of the symmetry groups, as summarised in the Table 1.2. Their field contents upon the circle reduction match, and correspond to the five-dimensional supergravity multiplets with 16 and 8 supercharges respectively.

A detailed construction and a complete list of $(1,0)$, $(2,0)$ and $(4,0)$ multiplets with low spins can be found in Appendix A.2.

Similar considerations apply to the last maximally supersymmetric multiplet, which receives much less attention in this thesis. The $(3,1)$ multiplet can be seen as a product of tensor and vector multiplets [6]

$$[(2,0)_{\text{tensor}}]\times[(1,1)_{\text{vector}}]=[(3,1)]_{\text{exotic}}\,. \tag{1.24}$$

	Exotic (or SD-Weyl)	Gravity
$D=6,(p,q)=(2,0)$ $SU(2)\times SU(2)\times Sp(4)$ $Q_{\frac{1}{2}}$ in $(\mathbf{2},\mathbf{1};\mathbf{4})$	$(\mathbf{3},\mathbf{1};\mathbf{1})\times\mathbf{2^4}$ $=(\mathbf{5},\mathbf{1};\mathbf{1})+(\mathbf{3},\mathbf{1};\mathbf{5})+(\mathbf{1},\mathbf{1};\mathbf{1})$ $+(\mathbf{4},\mathbf{1};\mathbf{4})+(\mathbf{2},\mathbf{1};\mathbf{4})+(\mathbf{3},\mathbf{1};\mathbf{1})$	$(\mathbf{1},\mathbf{3};\mathbf{1})\times\mathbf{2^4}$ $=(\mathbf{3},\mathbf{3};\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{5})+(\mathbf{2},\mathbf{3};\mathbf{4})$
$D=6,(p,q)=(1,0)$ $SU(2)\times SU(2)\times Sp(2)$ $Q_{\frac{1}{2}}$ in $(\mathbf{2},\mathbf{1};\mathbf{2})$	$(\mathbf{4},\mathbf{1};\mathbf{1})\times\mathbf{2^2}$ $=(\mathbf{5},\mathbf{1};\mathbf{1})+(\mathbf{3},\mathbf{1};\mathbf{1})+(\mathbf{4},\mathbf{1};\mathbf{2})$	$(\mathbf{2},\mathbf{3};\mathbf{1})\times\mathbf{2^2}$ $=(\mathbf{3},\mathbf{3};\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1})+(\mathbf{2},\mathbf{3};\mathbf{2})$

Table 1.2: Six-dimensional SD-Weyl vs. gravity multiplets

1.2 The exotic tensor fields and dualities

In the previous section, we considered the six dimensional chiral superalgebra and there are three exotic objects in their massless representations. They are realised by the covariant fields $C_{\mu\nu\rho\sigma}$, $D_{\mu\nu\rho}$ and $\psi_{\mu\nu}$ as representation of $SO(5,1)$, and they satisfy some symmetry properties and self-dualities. In the bosonic case, it also follows that their field equations are just the self-duality condition in each case. These self-dualities are very important to determine the little group irreducible representation, however one can also relax the self-duality constraints and have some "weaker" field equations for more general exotic Lorentz tensor fields in arbitrary dimensions. We introduce these exotic tensor fields and study their properties as well as their duality relations.

For simplicity, we only consider bosonic tensor fields. Our discussions are also restricted at the level of field equations. There are also intriguing results and analysis in terms of Lagrangian formalism and action principle for these fields, see e.g. [9, 11, 12, 74, 76–78, 80]. We start with d-dimensional Minkowskian spacetime and it is useful to describe the irreducible $GL(d,\mathbb{R})$-tensors in terms of Young diagrams and Young tableaux (see appendix A.3). For our interests, we will mainly deal with two-column Young diagrams. This is called the "Bi-form gauge theory" in [76] and we mostly follow the conventions there.

1.2.1 Some gauge theories

The linearised graviton (Pauli-Fierz field). The linearised graviton (or Pauli-Fierz field) $h_{\mu\nu}$ is represented by the Young tableaux of type $[1,1]=\begin{array}{|c|c|}\hline \; & \; \\ \hline\end{array}$. The linearised Riemann curvature tensor (field strength of h) is

$$R_{\mu\nu\rho\sigma}=\partial_{[\mu}h_{\nu][\rho,\sigma]} \tag{1.25}$$

and graphically we can represent it by the Young tableaux

$$R=\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline\end{array} \qquad \text{or} \qquad R[h]=\begin{array}{|c|c|}\hline \; & \; \\ \hline \partial & \partial \\ \hline\end{array}\,. \tag{1.26}$$

In the Young tableaux where we filled the boxes with the partial derivatives "∂" is to indicate that we take two derivatives of h and impose the the Young tableaux symmetry on the corresponding index. The gauge transformation is

$$\delta h_{\mu\nu}=2\partial_{[\mu}\xi_{\nu]}=\begin{array}{|c|c|}\hline \; & \partial \\ \hline\end{array} \qquad \text{with} \qquad \xi=\begin{array}{|c|}\hline \; \\ \hline\end{array} \tag{1.27}$$

where the parameter ξ_ν transforms in the fundamental representation of $GL(d,\mathbb{R})$. This leaves the field strength invariant by the Young tableaux symmetry and the commutativity of the partial derivatives:

$$\delta R=\begin{array}{|c|c|}\hline \; & \partial \\ \hline \partial & \partial \\ \hline\end{array}=0\,. \tag{1.28}$$

The symmetry properties of R is manifest in the representation of Young tableaux

$$R_{\mu\nu\rho\sigma}=R_{\rho\sigma\mu\nu}=R_{[\mu\nu]\rho\sigma}=R_{\mu\nu[\rho\sigma]}\,, \tag{1.29}$$

$$R_{[\mu\nu\rho]\sigma}=0\,. \tag{1.30}$$

The differential Bianchi identity is obvious if we look at the second graphic representation (1.26)

$$\partial_{[\lambda}R_{\mu\nu]\rho\sigma} = \begin{array}{|c|c|}\hline & \\ \hline \partial & \partial \\ \hline \partial \\ \cline{1-1}\end{array} = 0\,. \tag{1.31}$$

here we used again the commutative property of the partial derivatives. This also implies $R_{\mu\nu[\rho\sigma,\lambda]} = 0$ because of the symmetry of R (1.29). The field equation is exactly the linearised Einstein equation, which states that the single trace of the Riemann curvature tensor vanishes

$$R_{\nu\sigma} \equiv \eta^{\mu\rho}R_{\mu\nu\rho\sigma} = 0\,. \tag{1.32}$$

For $d \geqslant 4$, this field equation does not imply $R_{\mu\nu\rho\sigma} = 0$. Thus, it is non trivial, and on-shell the Pauli-Fierz field reduces to a traceless symmetry tensor h_{ij} of the little group $SO(d-2)$. But for $d = 3$, it is well-known that the vanishing of the Ricci tensor $R_{\mu\nu}$ implies the vanishing of the Riemann tensor, in this case, in order to get a non-topological theory, we should replace the single trace field equation by $R \equiv \eta^{\nu\sigma}R_{\nu\sigma} = 0$ and the theory propagates one physical degree of freedom in three dimensions.

The dual graviton. The Young tableaux for the dual graviton (the name "dual graviton" will be justified below) $D_{\mu_1\mu_2\ldots\mu_{d-3}\nu}$ is the "hook" of the type $[d-3,1]$

$$\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \\ \hline \\ \cline{1-1}\end{array}$$

with

$$D_{\mu_1\mu_2\ldots\mu_{d-3}\nu} = D_{[\mu_1\mu_2\ldots\mu_{d-3}]\nu} \qquad \text{and} \qquad D_{[\mu_1\mu_2\ldots\mu_{d-3}\nu]} = 0\,. \tag{1.33}$$

The field strength is defined similarly as in (1.18)

$$S_{\mu_1\mu_2\ldots\mu_{d-2}\nu_1\nu_2} = \partial_{[\mu_1}D_{\mu_2\ldots\mu_{d-2}][\nu_1,\nu_2]} \tag{1.34}$$

and represented by $[d-2,2]$

$$S[D] = \begin{array}{|c|c|}\hline & \\ \hline & \partial \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline \\ \cline{1-1} \partial \\ \cline{1-1}\end{array}\,. \tag{1.35}$$

One of the advantage to use Young tableaux is that we easily find that in this case there are two parameters needed to describe the gauge freedom of D, and they are α and β of the type $[d-4,1]$ and $[d-3]$. The gauge transformation is given by

$$\delta D_{\mu_1\mu_2\ldots\mu_{d-3}\nu} = \partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_{d-3}]\nu} + \partial_{[\mu_1}\beta_{\mu_2\ldots\mu_{d-3}]\nu} + (-1)^d\partial_\nu\beta_{\mu_1\ldots\mu_{d-3}} \tag{1.36}$$

and

$$\delta \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \\ \hline \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \\ \hline \partial \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline & \partial \\ \hline \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \\ \hline \\ \cline{1-1}\end{array}\,. \tag{1.37}$$

The symmetries are

$$S_{\mu_1\mu_2\ldots\mu_{d-2}\nu_1\nu_2} = S_{[\mu_1\mu_2\ldots\mu_{d-2}]\nu_1\nu_2} = S_{\mu_1\mu_2\ldots\mu_{d-2}[\nu_1\nu_2]} \tag{1.38}$$

the algebraic Bianchi identity

$$S_{[\mu_1\mu_2\ldots\mu_{d-2}\nu_1]\nu_2} = 0 \tag{1.39}$$

and the differential Bianchi identities

$$\partial_{[\rho}S_{\mu_1\mu_2\ldots\mu_{d-2}]\nu_1\nu_2} = 0\,, \qquad\qquad S_{\mu_1\mu_2\ldots\mu_{d-2}[\nu_1\nu_2,\rho]} = 0\,. \tag{1.40}$$

The field equation for D is again the traceless of its field strength

$$\eta^{\mu_1\nu_1}S_{\mu_1\mu_2\ldots\mu_{d-2}\nu_1\nu_2} = 0\,. \tag{1.41}$$

The double dual graviton. The double dual graviton (again, the name of this field will be explained later) $C_{\mu_1\mu_2\ldots\mu_{d-3}\nu_1\nu_2\ldots\nu_{d-3}}$ has the symmetry of a $[d-3, d-3]$ Young Tableaux

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \\ \vdots \;\; \vdots \\ \begin{array}{|c|c|}\hline & \\ \hline \end{array}$$

it means

$$C_{\mu_1\mu_2\ldots\mu_{d-3}\nu_1\nu_2\ldots\nu_{d-3}} = C_{[\mu_1\mu_2\ldots\mu_{d-3}]\nu_1\nu_2\ldots\nu_{d-3}} = C_{\mu_1\mu_2\ldots\mu_{d-3}[\nu_1\nu_2\ldots\nu_{d-3}]}\,, \tag{1.42}$$

$$C_{\mu_1\mu_2\ldots\mu_{d-3}\nu_1\nu_2\ldots\nu_{d-3}} = C_{\nu_1\nu_2\ldots\nu_{d-3}\mu_1\mu_2\ldots\mu_{d-3}}\,, \tag{1.43}$$

$$C_{[\mu_1\mu_2\ldots\mu_{d-3}\nu_1]\nu_2\ldots\nu_{d-3}} = 0\,. \tag{1.44}$$

The field strength is $[d-2, d-2]$

$$G_{\mu_1\mu_2\ldots\mu_{d-2}\nu_1\nu_2\ldots\nu_{d-2}} = \partial_{[\mu_1}C_{\mu_2\ldots\mu_{d-2}][\nu_1\nu_2\ldots\nu_{d-3},\nu_{d-2}]} \tag{1.45}$$

$$G[C] = \begin{array}{c} \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \\ \vdots \;\; \vdots \\ \begin{array}{|c|c|}\hline & \\ \hline \partial & \partial \\ \hline \end{array} \end{array}\,. \tag{1.46}$$

Now we need only one parameter with the symmetry $[d-3, d-4]$ for the gauge transformation

$$\delta C_{\mu_1\mu_2\ldots\mu_{d-3}\nu_1\nu_2\ldots\nu_{d-3}} = \eta_{\nu_1\nu_2\ldots\nu_{d-3}[\mu_1\mu_2\ldots\mu_{d-4},\mu_{d-3}]} + \eta_{\mu_1\mu_2\ldots\mu_{d-3}[\nu_1\nu_2\ldots\nu_{d-4},\nu_{d-3}]} \tag{1.47}$$

and

$$\delta \begin{array}{c} \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \\ \vdots \;\; \vdots \\ \begin{array}{|c|c|}\hline & \\ \hline \end{array} \end{array} = \begin{array}{c} \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \\ \vdots \;\; \vdots \\ \begin{array}{|c|c|}\hline & \partial \\ \hline \end{array} \end{array}\,. \tag{1.48}$$

Analogously, the field strength G satisfies the symmetries

$$G_{\mu_1\mu_2\ldots\mu_{d-2}\nu_1\nu_2\ldots\nu_{d-2}} = G_{[\mu_1\mu_2\ldots\mu_{d-2}]\nu_1\nu_2\ldots\nu_{d-2}} = G_{\mu_1\mu_2\ldots\mu_{d-2}[\nu_1\nu_2\ldots\nu_{d-2}]}\,, \tag{1.49}$$

$$G_{\mu_1\mu_2\ldots\mu_{d-2}\nu_1\nu_2\ldots\nu_{d-2}} = G_{\nu_1\nu_2\ldots\nu_{d-2}\mu_1\mu_2\ldots\mu_{d-2}}\,, \tag{1.50}$$

the algebraic Bianchi identities

$$G_{[\mu_1\mu_2\ldots\mu_{d-2}\nu_1]\nu_2\ldots\nu_{d-2}} = 0 = G_{\mu_1\mu_2\ldots[\mu_{d-2}\nu_1\nu_2\ldots\nu_{d-2}]} \tag{1.51}$$

and the differential Bianchi identities

$$\partial_{[\rho}G_{\mu_1\mu_2\ldots\mu_{d-2}]\nu_1\nu_2\ldots\nu_{d-2}} = 0 = G_{\mu_1\mu_2\ldots\mu_{d-2}[\nu_1\nu_2\ldots\nu_{d-2},\rho]}\,. \tag{1.52}$$

The new feature here that the single trace equation of motion

$$\eta^{\mu_1\nu_1}G_{\mu_1\mu_2\ldots\mu_{d-2}\nu_1\nu_2\ldots\nu_{d-2}} = 0 \tag{1.53}$$

is problematic, it may imply the vanishing of G in certain spacetime dimensions. Because in these dimensions the tensor G is completely determined by its trace part, see e.g. [14, 76, 81], and we should take the vanishing of higher trace of G as the equation of motion.

For example, in spacetime dimension five, the Levi-Civita symbol $\epsilon_{\mu_1\ldots\mu_5}$ has five indices and it is invariant under $SO(4,1)$. Then we have for $G[C] = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & \\ \hline \end{array}$ the following identity

$$G_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} = \left(\frac{1}{2!}\frac{1}{3!}\epsilon_{\mu_1\mu_2\mu_3\alpha\beta}\epsilon^{\rho_1\rho_2\rho_3\alpha\beta}\right)G_{\rho_1\rho_2\rho_3\sigma_1\sigma_2\sigma_3}\left(\frac{1}{2!}\frac{1}{3!}\epsilon_{\nu_1\nu_2\nu_3\kappa\tau}\epsilon^{\sigma_1\sigma_2\sigma_3\kappa\tau}\right)\,. \tag{1.54}$$

We see that this holds because of the ϵ identity (note that the minus sign is due to the Minkowski signature)

$$\epsilon_{\mu_1\mu_2\mu_3\alpha\beta}\epsilon^{\rho_1\rho_2\rho_3\alpha\beta} = -3!\,2!\,\delta^{[\rho_1}_{\mu_1}\delta^{\rho_2}_{\mu_2}\delta^{\rho_3]}_{\mu_3} \tag{1.55}$$

and alternatively, we can fuse one ϵ from the first bracket with another one of the second bracket

$$\epsilon^{\rho_1\rho_2\rho_3\alpha\beta}\epsilon^{\sigma_1\sigma_2\sigma_3\kappa\tau} = -5!\,\eta^{[\rho_1|\sigma_1|}\eta^{\rho_2|\sigma_2|}\dots\eta^{\beta]\tau} \tag{1.56}$$

where we only antisymmetrise the indices which are not separated by "| |". We see that because there are only 5 η's in the expression (1.56), when we contract it with $G_{\rho_1\rho_2\rho_3\sigma_1\sigma_2\sigma_3}$, only traces of G will appear. The single trace determines also higher traces, hence the traceless part of G vanishes and G is completely fixed by $G^{\mu}{}_{\nu\rho\mu\sigma\tau}$.

This generalises automatically for a tensor gauge field A with $[p,q]$ ($p \geqslant q$) Young tableaux symmetry and field strength $F[A]$ of the type $[p+1,q+1]$ in arbitrary dimensions: the single trace equation of motion

$$\eta^{\mu_1\nu_1}F_{\mu_1\mu_2\dots\mu_{p+1}\nu_1\nu_2\dots\nu_{q+1}} = 0 \tag{1.57}$$

is non-trivial in dimension $d \geqslant p+q+2$. Furthermore [76], the n-th trace of F vanishing is a non-trivial field equation in dimension $d = p+q+3-n$. If we apply this result for our $[d-3,d-3]$ double dual graviton in d dimensions then (1.53) is non-trivial only for $d=4$, and the vanishing of the $(d-3)$-th trace of G yields a natural and non-trivial field equation for $d \geqslant 4$

$$G_{\mu_1\mu_2\dots\mu_{d-3}\rho}{}^{\mu_1\mu_2\dots\mu_{d-3}}{}_{\sigma} = 0\,. \tag{1.58}$$

Specifically, the self-dual Weyl field written as $C_{\mu\nu\rho\sigma}$ is a $[p,q] = [2,2]$ gauge field in six dimensions. The self-duality constraint on its field strength G (1.11) and the algebraic Bianchi identity of G together imply that G satisfies the single trace field equation

$$\eta^{\mu\sigma}G_{\mu\nu\rho\sigma\tau\kappa} = 0\,. \tag{1.59}$$

Degrees of freedom. All the three two column gauge fields are represented by the Young tableaux, and allowing gauge transformations with appropriate parameters. One imposes the nature non-trivial field equations (1.32), (1.41) and (1.58). On-shell, for $d \geqslant p+q+2$, the linearised graviton and the dual graviton transform in the representations of the little group $SO(d-2)$ with the same Young tableaux and in addition, they are traceless. In the sense that if we pick one index from the left column and contract with another one from the right column via the $SO(d-2)$ invariant tensor δ_{ij}, the result vanishes.

The number of independent components of the $\widehat{[p,q]}$ little group Young tableaux is (we use the same notation as in [76], and " $\widehat{\ }$ " means traceless)

$$dim_{(d-2)}\widehat{[p,q]} = dim_{(d-2)}[p,q] - dim_{(d-2)}[p-1,q-1] \tag{1.60}$$

where $dim_d[p,q]$ denotes the number of independent components of the traceful $[p,q]$ Young tableaux in the Lorentz group $SO(d-1,1)$ (or in the orthogonal group $SO(d)$)

$$dim_d[p,q] = \binom{d}{p}\binom{d+1}{q}\left(1-\frac{q}{p+1}\right)\,. \tag{1.61}$$

We can then insert the three cases that we are interested in and read off their physical degrees of freedom

$$dim_{(d-2)}\widehat{[1,1]} = dim_{(d-2)}\widehat{[d-3,1]} = \frac{d(d-3)}{2} \tag{1.62}$$

for $d \geqslant 4$. The evaluation of $dim_{(d-2)}\widehat{[d-3,d-3]}$ through the equation (1.60) gives a negative number. Because the non-trivial field equation (1.58) for the $[d-2,d-2]$ field strength will lead to a $\widehat{[1,1]}$ Young tableaux for the $[d-3,d-3]$ double dual graviton in the little group. Thus, the double dual graviton propagates on-shell the same amount of degrees of freedom as the linearised graviton and the dual graviton.

Note that in general, the Young tableaux with traceless condition are not sufficient to determine irreducible representations of the Lorentz or orthogonal groups. For example, the free $[2,2]$ tensor C with single trace field equation have in six dimensions $dim_{(6-2)}\widehat{[2,2]} = 10$ degrees of freedom. The self-dual constraint (1.11) implies the $SO(4)$ little group self-dualities (1.4). Thus, it halves the degrees of freedom and finally gives a irreducible representation of $SO(4)$, and we see that imposing the self-duality is a necessary step to determine irreducible representations for the Lorentz (orthogonal) groups in some dimensions.

1.2.2 The dual formulations of linearised gravity

The Maxwell theory is formulated by using an abelian vector field A_μ transforming in the fundamental representation $\square$ of $GL(d,\mathbb{R})$. The dual-formulation is based on the antisymmetric tensor gauge field $\tilde{A}_{\mu_1\mu_2\ldots\mu_{d-3}}$ represented by $\begin{array}{c}\square\\\square\\\vdots\\\square\end{array}$ with $d-3$ boxes in one single column. Their field strengths are related by the Hodge star operator $\tilde{F}[\tilde{A}] = \star F[A]$. The Bianchi identities and equation of motions get exchanged by this duality transformation. In the region where the electric source coupled to A_μ vanishes, one has the equation of motion

$$d \star F = 0 \qquad \Rightarrow \qquad d\tilde{F} = 0\,, \tag{1.63}$$

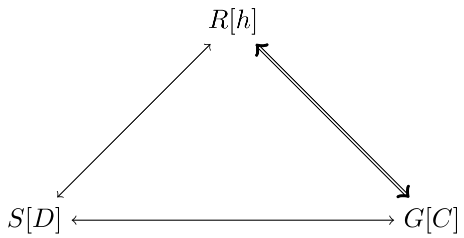
so $\tilde{F}$ can be solved by a gauge field $\tilde{A}$ up to gauge transformations. The relation between A and $\tilde{A}$ is non-local.

We had introduced the field strengths in terms of their gauge potentials in the previous paragraphs. Conversely, given the tensor fields with the Young Tableaux symmetries $[2,2]$, $[d-2,2]$ and $[d-2,d-2]$, if they also fulfill the differential Bianchi identities, there are the generalised Poincaré lemmata [75,82,83] ensure that they can be solved by the gauge potentials of type $[1,1]$, $[d-3,1]$ and $[d-3,d-3]$ up to gauge transformations. Hence, they admit similar duality relations to the electromagnetic duality.

We can start with the Pauli-Fierz field $h_{\mu\nu}$ with its field strength R, then define the $[d-2,2]$ tensor S and the $[d-2,d-2]$ tensor G by

$$\begin{aligned} S &= \star R \\ G &= \star R \star\,, \end{aligned} \tag{1.64}$$

the $\star$ acting from the left is to dualise the two indices on the first column of R (1.26) while the $\star$ coming from the right side is to take the dualisation on the second column. This following picture [84] shows the triality relations between these tensors. The double edged arrow between R and G means the double Hodge dualisation and note that we also have $G = S\star$.



(1.65)

The algebraic Bianchi identities of R (1.30) implies that S satisfies the field equation (1.41) and G satisfies the algebraic Bianchi identities (1.51). The field equation of R implies the algebraic Bianchi identity (1.39) of S and the non-trivial field equation (1.58) for G. Furthermore, it can be derived that both S and G satisfy the differential Bianchi identities (1.40) and (1.52). This is the requirements one needs to apply the generalised Poincaré lemma, so S and G can be solved with the gauge potential D and C of the type $[d-3,1]$ and $[d-3,d-3]$. Moreover, we can see from the discussion on counting degrees of freedom (1.62) that the $[1,1]$, $[d-3,1]$ and $[d-3,d-3]$ gauge potential describe the same physical degrees of freedom in the light-cone gauge. One can conclude that there are at least three equivalent formulations for linearised gravity via the gauge field with mix Young Tableaux symmetry. However, it was noticed in [55], only two independent sources which couple naturally to the three gauge potentials.

In the above duality transformations, the algebraic Bianchi identities and field equations get interchanged between R and S as well as between S and G. But between R and G Bianchi identities go to Bianchi identities and the field equation is mapped to the other field equation. Base on this fact, it is argued in [14] that the relation between the linearised graviton field h and its double-dual C is algebraic. It means one can introduce C by a algebraic combination of h together with some gauge transformations. There are recent studies on these exotic duals of differential forms and more general tensor fields, see e.g. [79, 84].

The "critical" dimension and the reduction of chiral exotic tensors. We want to apply the aforementioned duality to exotic gauge of type $[2,1]$ and $[2,2]$, so the spacetime dimension is $d = 2+3 = 5$. Five dimensions is also "critical" for the $[2,2]$ gauge field as we showed that the trace-free part of its field strength G vanishes identically, so G is completely determined by its single trace, and as we discussed, the non-trivial field equation in five dimensions for G is

$$G^{\mu\nu}{}_{\sigma\mu\nu\rho} = 0\,. \tag{1.66}$$

This equation describes 5 degrees of freedom in $5d$ and it belongs to one of the three equivalent formulations h, D, C for the linearised gravity in five dimensions. As discussed in [2], one would naively expect that if one identifies the three $5d$ fields with components of a free $[2,2]$ field C from six dimensions by circle reduction

$$h_{\mu\nu} = C_{\mu5\nu5} \qquad D_{\mu\nu\rho} = C_{\mu\nu\rho5} \qquad C^{5d}_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}\,, \tag{1.67}$$

then the single trace free field equation (1.59) in $6d$ could give three $5d$ nature non-trivial field equations. However, this is not the case, a straightforward counting tells us that a free $[2,2]$ field describe 10 degrees of freedoms in $6d$ and the three gauge fields in $5d$ have 15 degrees of freedom. The subtlety is explained in [2], upon reduction, the $6d$ field equation (1.59) fixes the trace of $G[C]$ in $5d$

$$G_{\mu\nu\rho\sigma\tau}{}^{\rho} \propto R_{\mu\nu\sigma\tau}\,, \tag{1.68}$$

and the linearised Riemann tensor determines G completely

$$G_{\mu\nu\rho}{}^{\alpha\beta\gamma} \propto R_{[\mu\nu}{}^{[\alpha\beta}\delta_{\rho]}{}^{\gamma]}\,. \tag{1.69}$$

In [14], in the example on the algebraic relation between h and C, a same formula to (1.69) in five dimensions is derived. If one starts with $R[h]$ and define G by (1.64) in five dimensions then one can fuse the two ϵ's and use $R_{\mu\nu} = 0$ to eliminate the traces of R

$$\begin{aligned} G_{\mu\nu\rho}{}^{\alpha\beta\gamma} &= \left(\frac{1}{2!}\right)^2 \epsilon_{\mu\nu\rho\sigma\tau} R^{\sigma\tau}{}_{\kappa\lambda}\epsilon^{\alpha\beta\gamma\kappa\lambda} \\ &= -\left(\frac{1}{2!}\right)^2 5!\, \delta^{[\alpha}_{[\mu}\,\delta^{\beta}_{\nu}\dots\delta^{\lambda]}_{\tau]} R^{\sigma\tau}{}_{\kappa\lambda} \\ &= -9R_{[\mu\nu}{}^{[\alpha\beta}\delta_{\rho]}{}^{\gamma]}\,. \end{aligned} \tag{1.70}$$

This equation holds just by using the definition of G, some ϵ algebra and the field equation $R_{\mu\nu} = 0$ without knowing the $6d$ reductions.

It is straightforward to substitute the gauge potentials back in (1.69) and by the generalised Poincaré lemma, one finds that C can be expressed in terms of algebraic combination of h up to gauge transformations [14]. In the context of the dimensional reduction (1.67), $C^{5d}_{\mu\nu\rho\sigma}$ and $h_{\mu\nu}$ are not independent. Thus, one gets back to the 10 degrees of freedom carried by the $6d$ free $[2,2]$ tensor field.

If we are starting with the self-dual Weyl tensor, then the equation (1.11) is even stronger and it relates $R[h]$, $S[D]$ and $G[C]$ exactly via the dualities (1.64), so the self-dual Weyl tensor gives one independent graviton in five dimensions. In a similar way to the SD-Weyl tensor, the three-index object in the $(3,1)$ multiplet gives a graviton and a vector field upon reduction to five dimensions.

Part II

On dynamics of the exotic multiplet

Chapter 2

The algebraic approach

The theory of eleven-dimensional supergravity can be formulated with eleven-dimensional Lorentz symmetry non-manifest, but broken to a subgroup $SO(1, 10-d) \times SO(d)$, as one would have in dimensional reductions of the theory. Remarkably, when this is done, one finds that this group can be enhanced [85] to a local symmetry $SO(1, 10-d) \times \tilde{H}_d$, where $\tilde{H}_d$ is the (double cover of) the maximal compact subgroup of the exceptional group $E_{d(d)}$ which would appear in the corresponding torus compactification [86]. As one increases d, this exceptional group becomes infinite dimensional, as does the corresponding $\tilde{H}_d$, and grand proposals as to how these infinite dimensional symmetries are realised in M theory have been put forward [54, 87].

Dimensions $= 11 - d$	$E_{d(d)}$	$\tilde{H}_d$
9	$E_{2(2)} \simeq SL(2,\mathbb{R}) \times \mathbb{R}$	$Spin(2)$
8	$E_{3(3)} \simeq SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$Spin(3) \times Spin(2)$
7	$E_{4(4)} \simeq SL(5,\mathbb{R})$	$Spin(5)$
6	$E_{5(5)} \simeq Spin(5,5)$	$Spin(5) \times Spin(5)$
5	$E_{6(6)}$	$Sp(8)$
4	$E_{7(7)}$	$SU(8)$
3	$E_{8(8)}$	$Spin(16)$

Table 2.1: SUGRA U-duality groups (split real forms) and double cover of their maximal compact subgroups in $(11-d)$ dimensions [86, 88].

Recently, work has been done constructing the exceptional field theories. For $d \leq 7$, these exceptional symmetries give rise to exceptional generalised geometries [88, 89] which can be used to describe the internal sector of the theory [33, 34]. The full theory can then be written with these symmetries manifest and the internal sector given by the generalised geometry formulation [36, 37]. Further, one finds that the formulation of exceptional geometry can describe also type IIA and IIB supergravity via the exact same equations. The only change is the choice of subgroup which corresponds to the action of spacetime diffeomorphisms on tensors (i.e. the choice of "gravity line" in the language of [90]). There are two inequivalent embeddings of $GL(d-1, \mathbb{R})$ into $E_{d(d)} \times \mathbb{R}^+$, giving different decompositions of the exceptional theory into ordinary tensor fields [33, 36, 39]. One of these embeddings gives type IIA and the other type IIB. In the language of [36], this is phrased as the choice of "section" of a higher dimensional space. Such sections are subspaces V of (the dual of) the generalised tangent space such that $V \otimes V$ is null in particular $E_{d(d)}$ covariant projections of the tensor product space. In generalised geometry discussions, the subspace V is simply the cotangent bundle of the underlying manifold.

In this chapter, we explore the possibility that a third choice of spacetime subgroup could give the $\mathcal{N} = (4, 0)$ theory of [1]. In the half maximal setting, it was established that both the ten-dimensional type I theory and the six-dimensional $\mathcal{N} = (2, 0)$ theory could be seen in this way [40]. However, the $\mathcal{N} = (4, 0)$ theory is not a standard type of gravitational theory, so we expect that the picture will be different. We will see here that some hints of its known features, at least at the linearised level, can be seen from this angle of investigation, but these will amount more to curiosities than conclusive evidence. An important realisation, though, is that there is no spacetime section inside the exceptional multiplet of charges, in the way that there is for standard supergravity, but only the embedding of the momentum charge, which does not solve the section condition and carries no natural action of a special linear group. We will also examine the corresponding pictures for exotic multiplets with $\mathcal{N} = (2,0)$ and $\mathcal{N} = (1,0)$ supersymmetry, finding the same pattern of behaviour.

We begin by studying the embedding of the spin groups into $\mathrm{Cliff}(10,1;\mathbb{R})$ and the relation of this to the higher dimensional enhanced symmetries $\tilde{H}_d$. We then comment on the interpretation of these embeddings in terms of charges and how this could correspond to different spacetime groups inside the duality group $E_{8(8)}$.

2.1 An almost universal construction of the maximal supersymmetry algebras

The maximal supersymmetry algebras can all be seen as subalgebras of a Lie superalgebra $\mathcal{A}$, which we briefly describe. The generators of $\mathcal{A}$ consist of 32 fermionic generators Q^α, transforming as the **32** representation of $SL(32,\mathbb{R})$.[1] The anti-commutators of these give 528 bosonic generators $X^{\alpha\beta}$

$$\begin{aligned} \{Q^\alpha, Q^\beta\} &= X^{\alpha\beta} = X^{(\alpha\beta)} \\ [X^{\alpha\beta}, Q^\gamma] &= 0 \end{aligned} \tag{2.1}$$

which have vanishing brackets with the Q's. Finally, we add the generators $\mathcal{M}_\alpha{}^\beta$ of $\mathfrak{sl}(32,\mathbb{R})$ which act on the Q's and X's via the adjoint action.

We can recover a maximal supersymmetry algebra from $\mathcal{A}$ by truncating the $\mathfrak{sl}(32,\mathbb{R})$ generators to a subalgebra of the form $\mathfrak{spin}(D-1,1) \oplus \mathfrak{k}$, where $\mathfrak{k}$ is the maximal compact commutant of $\mathfrak{spin}(D-1,1)$ inside $\mathfrak{sl}(32,\mathbb{R})$ ($\mathfrak{k}$ is the R-symmetry automorphism algebra). Decomposing Q^α and $X^{\alpha\beta}$ under $\mathfrak{spin}(D-1,1) \oplus \mathfrak{k}$, we recover the supersymmetry algebra. It is easy to see why this prescription works: the generators Q^α and $X^{\alpha\beta}$ of the algebra $\mathcal{A}$ are simply the supertranslational part, without specifying how they transform under the Lorentz symmetry and R symmetry. This is then fixed by choosing the subalgebra $\mathfrak{spin}(D-1,1) \oplus \mathfrak{k} \subset \mathfrak{sl}(32,\mathbb{R})$

$$Q \to Q^A_{\tilde{\alpha}} \tag{2.2}$$

with A being the R-symmetry index and $\tilde{\alpha}$ is the $\mathfrak{spin}(D-1,1)$ spinor index.

We now want to view the algebra $\mathfrak{sl}(32,\mathbb{R})$ as the irreducible matrix representation of the Clifford algebra $\mathrm{Cliff}(10,1;\mathbb{R})$, see appendix B.1 for more details. Choosing the natural $\mathfrak{spin}(10,1)$ subalgebra (which has no compact commutant in $\mathfrak{sl}(32,\mathbb{R})$), the **32** representation is irreducible, while the **528** decomposes into **11** + **55** + **462**, so that X becomes the momentum P_μ, a 2-form $Z_{\mu\nu}$ and a 5-form $Z_{\mu_1\ldots\mu_5}$. We thus recover the standard eleven-dimensional supersymmetry algebra. Furthermore, the maximal automorphism group for the eleven-dimensional superalgebra is $SL(32,\mathbb{R})$ and one can work out how the $SL(32,\mathbb{R})$ generators rotate the momentum and the central charges [92].

The standard (non-chiral) maximal supersymmetry algebras in lower dimensions are then obtained by taking $\mathfrak{spin}(D-1,1)$ subalgebras of this $\mathfrak{spin}(10,1)$ and then examining their compact commutants in $\mathfrak{sl}(32,\mathbb{R})$ to find the R-symmetry (though again there are exceptions to this rule – see footnote 1). We can decompose the eleven-dimensional Lorentz indices into indices $\mu,\nu = 0,1,\ldots D-1$ for the "external spacetime" $\mathfrak{spin}(D-1,1)$ Lorentz group and $m,n = 1,\ldots,d$ the orthogonal group indices for the "internal space".

We see that the parts of $X^{\alpha\beta}$ which form the momentum charge in D-dimensions are completely contained in the eleven-dimensional momentum charge P_μ, and that the d-dimensional Lorentz group is contained in the eleven-dimensional Lorentz group by construction. In the corresponding supergravity theories, this can be interpreted as saying that the lower-dimensional spacetime is a subspace of the higher dimensional spacetime.

[1] Finite dimensional unfaithful representations of $\tilde{K}(E_{11})$ exist, on which the ideal $\mathcal{I}$ acting trivially and for Dirac fermions the finite dimensional quotient is $\tilde{K}(E_{11})/\mathcal{I} \cong SL(32,\mathbb{R})$ [91].

However, in some dimensions D there are alternative embeddings of $\mathfrak{spin}(D-1,1)$ into $\mathfrak{sl}(32,\mathbb{R})$, such that the resulting supercharges Q have different chiralities to those in the simple embeddings above. For example, a different embedding of $\mathfrak{spin}(9,1)$ to that above gives the $\mathcal{N}=(2,0)$ supersymmetry algebra of type IIB supergravity in ten dimensions. A relatively clean way to see this is to construct the embedding explicitly in terms of the $\mathrm{Cliff}(10,1;\mathbb{R})$ gamma-matrices, so this is what we do next.

2.2 Spin embeddings into higher dimensional Clifford algebras

We start by giving a general picture of some different ways that one can embed the Lie algebra of $Spin(s+1,t)$ into $\mathrm{Cliff}(s+N,t)$. The construction is very explicit, using gamma matrices and a multitude of different indices. Readers who do not wish to indulge these details could skip straight to the examples.

2.2.1 Different embeddings of $Spin(s+1,t)$ into $\mathrm{Cliff}(s+N,t)$

Let i,j be indices for the vector representation of $SO(s,t)$ taking values in $\{-t,\dots,-1\}$ for the timelike directions and $\{1,\dots,s\}$ for the spacelike directions. Let Γ^M be the gamma matrices generating $\mathrm{Cliff}(s+N,t)$, with the index M similarly taking values in $\{-t,\dots,-1,1,\dots,s,s+1,\dots,s+N\}$. Introducing a further set of indices I,J taking values in $\{-t,\dots,-1,1,\dots,s,s+1\}$, consider the generators

$$\{\hat{\gamma}^{IJ}\} = \begin{cases} \Gamma^{ij}, & I=i, J=j \\ \Gamma^{i\; s+1\; s+2\;\dots\; s+n} & I=i, J=s+1 \end{cases} \tag{2.3}$$

in which $s+1,\dots,s+n$ label n spacelike directions in the space of signature $(s+N,t)$ which are invariant under $SO(s,t)$. One can check that these generate $Spin(s+1,t)$ or $Spin(s,t+1)$, where the signature of the extra direction is determined by the value of n as

$$\begin{array}{c|cccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \\ \hline \pm & - & + & + & - & - & + & + & - & - & \dots \end{array} \tag{2.4}$$

In what follows, we will take $n \in \{1,2,5,6,\dots\}$ so that the extra direction is spacelike ($+$ in the table).

If we have that $s+t+1$ is even, we can calculate the chirality matrix[2] $\hat{\gamma}^{(s+t+1)}$ for the embedded $\mathrm{Cliff}(s+1,t)^{\mathrm{even}}$. This tells us how the $(s+N,t)$ spinor decomposes into $(s+1,t)$ spinors. In particular, we note that if $n=N$ then this is

$$\hat{\gamma}^{(s+t+1)} = \Gamma^{-t\; -t+1}\dots\Gamma^{s-2\; s-1}\Gamma^{s\; s+1\;\dots\; s+N} = \Gamma^{-t}\dots\Gamma^{-1}\Gamma^{1}\dots\Gamma^{s+N} = \Gamma^{(s+t+N)} \tag{2.5}$$

which is the product of the gamma matrices in signature $(s+N,t)$ (i.e. $\pm\mathbb{1}$ or $\pm i\mathbb{1}$ if $s+t+N$ is odd, or the chirality matrix if $s+t+N$ is even). Thus, if in $\mathrm{Cliff}(s+N,t)$ we have $\Gamma^{(s+t+N)} = +\mathbb{1}$ then all spinors will decompose to have the same (positive) chirality. This will appear in our examples in the next section.

2.2.2 Examples

Example 1 : Type II into eleven dimensions

We start by looking at the nine-dimensional spin group $Spin(8,1)$, generated by Γ^{ij}, for $i,j = 0,1,\dots,8$, inside $\mathrm{Cliff}(10,1)$. We then consider how we could add generators to these to enhance the group to give a $Spin(9,1)$ inside $\mathrm{Cliff}(10,1)$. We see two inequivalent ways to do this, leading to decompositions of the eleven-dimensional spinor into two spinors of different chirality or into

[2] In our notation for this chapter, if a Clifford algebra is generated by gamma matrices γ^i, with the index i running over d values, then $\gamma^{(d)} = \prod_i \gamma^i$ is the product of the d distinct gamma matrices.

two spinors of the same chirality under the $Spin(9,1)$ subgroups. These correspond to type IIA (non-chiral) and type IIB (chiral) respectively.

For type IIA we simply add the spin generators corresponding to including one more direction of the eleven-dimensional space, so that our $Spin(9,1)$ group is generated by

$$\left\{\hat{\gamma}^{IJ}\right\} = \left\{\Gamma^{ij}, \Gamma^{i\,9}\right\} \tag{2.6}$$

which gives (recall that the Γ-matrices are the Cliff(10, 1) gamma matrices and we take $\Gamma^{(11)} = \Gamma^0\Gamma^1\Gamma^2\ldots\Gamma^{10} = +\mathbb{1}$)

$$\hat{\gamma}^{(10)} = \Gamma^{01}\Gamma^{23}\ldots\Gamma^{78}\Gamma^{89} = \Gamma^{(11)}\Gamma^{10} = \Gamma^{10} = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix} \quad \text{(in an appropriate basis)} \tag{2.7}$$

so we see that the eleven-dimensional spinor decomposes into one positive and one negative chirality ten-dimensional spinors.

The commutant of the type IIA $\mathfrak{spin}(9,1)$ subalgebra inside $\mathfrak{sl}(32,\mathbb{R})$ is generated by $\{\Gamma^{10}\}$. This generates an $\mathbb{R}^+$ subgroup of $SL(32,\mathbb{R})$, and so there is no non-trivial compact commutant. This matches the R-symmetry of type IIA.

For type IIB, we instead take

$$\left\{\hat{\gamma}^{IJ}\right\} = \left\{\Gamma^{ij}, \Gamma^{i\,9\,10}\right\} \tag{2.8}$$

leading to

$$\hat{\gamma}^{(10)} = \Gamma^{01}\Gamma^{23}\ldots\Gamma^{8\,9\,10} = \Gamma^{(11)} = \mathbb{1} \tag{2.9}$$

Thus, the 32 component spinor decomposes into only positive chirality spinors for this $Spin(9,1)$ subgroup, as all spinors have eigenvalue +1 under $\hat{\gamma}^{(10)}$.

The commutant of the type IIB $\mathfrak{spin}(9,1)$ subalgebra inside $\mathfrak{sl}(32,\mathbb{R})$ is generated by $\{\Gamma^{9\,10}\}$. This generates an $SO(2)$ subgroup, which matches the R-symmetry of type IIB.

Example 2 : Six-dimensional $\mathcal{N} = (4,0)$ into eleven dimensions

We start with the $Spin(4,1)$ generators Γ^{ij}, for $i,j = 0,1,\ldots,4$, inside Cliff(10, 1) and look to extend this to an embedding of $Spin(5,1)$. Taking the additional generators Γ^{i5} would result in the $\mathfrak{spin}(5,1)$ subalgebra for standard $\mathcal{N} = (2,2)$ supergravity in six-dimensions. If instead we take

$$\left\{\hat{\gamma}^{IJ}\right\} = \left\{\Gamma^{ij}, \Gamma^{i56789\,10}\right\} \tag{2.10}$$

then, similarly to the situation for type IIB above, we obtain

$$\hat{\gamma}^{(6)} = \Gamma^{01}\Gamma^{23}\ldots\Gamma^{456789\,10} = \Gamma^{(11)} = +\mathbb{1} \tag{2.11}$$

so that again the 32 component spinor decomposes into only positive chirality spinors for this $Spin(5,1)$ subgroup.

The commutant of this $\mathfrak{spin}(5,1)$ subalgebra inside $\mathfrak{sl}(32,\mathbb{R})$ is generated by $\{\Gamma^m, \Gamma^{m_1m_2},\ldots, \Gamma^{m_1\ldots m_6}\}$ for $m,n = 5,6,\ldots,10$. Of these, only the generators $\{\Gamma^{m_1m_2}, \Gamma^{m_1m_2m_3}, \Gamma^{m_1\ldots m_6}\}$ square to $-\mathbb{1}$ and hence are compact. The compact commutant group these generate is $Sp(8)$, which matches the R-symmetry of the $\mathcal{N} = (4,0)$ multiplet.

2.2.3 Irreducible decomposition of charges

In the examples of section 2.2.2 we gave the embedding of two inequivalent $Spin(9,1)$ groups and two inequivalent $Spin(5,1)$ groups into Cliff(10, 1; ℝ). In terms of $Spin(10,1)$ objects the charges ($X_{\alpha\beta}$ above) can be written as an eleven-dimensional vector, two-form and five-form via

$$\{Q_\alpha, Q_\beta\} = P_M(\tilde{C}\Gamma^M)_{\alpha\beta} + \tfrac{1}{2}Z_{MN}(\tilde{C}\Gamma^{MN})_{\alpha\beta} + \tfrac{1}{5!}Z_{M_1\ldots M_5}(\tilde{C}\Gamma^{M_1\ldots M_5})_{\alpha\beta} \tag{2.12}$$

where we have explicitly included the transpose intertwiner $\tilde{C}$ as defined in appendix B.1. We can then calculate explicitly the action of our other $Spin$ groups on the charges $(P, Z_{(2)}, Z_{(5)})$, written in terms of a decomposition under the common subgroup with $Spin(10,1)$. We provide a sketch of these calculations here, noting that our $Spin$ groups are acting as subgroups of $GL(32,\mathbb{R})$. This means that the action of a matrix M is given by

$$\begin{aligned} M\cdot(\tilde{C}\Gamma^{\cdots}) &= -M^T(\tilde{C}\Gamma^{\cdots}) - (\tilde{C}\Gamma^{\cdots})M \\ &= -\tilde{C}\Big((\tilde{C}^{-1}M^T\tilde{C})\Gamma^{\cdots} + \Gamma^{\cdots}M\Big). \end{aligned} \tag{2.13}$$

Example 1 : Type II into eleven dimensions

For type IIA the generators of the relevant $Spin(9,1)$ were found above to be $M^{\mu\nu} = \Gamma^{\mu\nu}$ and $M^{\mu 9} = \Gamma^{\mu 9}$, for $\mu,\nu = 0,1,\ldots,8$ the vector indices of $Spin(8,1)$. Clearly, these simply generate a $Spin(9,1)$ subgroup of $Spin(10,1)$ preserving the tenth spatial direction. As such it is clear that the $Spin(9,1)$ irreducible combinations of charges will be

$$\begin{array}{lll} (P^\mu, P^9) & (Z_{\mu\nu}, Z_{\mu 9}) & (Z_{\mu_1\ldots\mu_5}, Z_{\mu_1\ldots\mu_4 9}) \\ (P^{10}) & (Z_{\mu 10}, Z_{9\ 10}) & (Z_{\mu_1\ldots\mu_4 10}, Z_{\mu_1\ldots\mu_3 9\ 10}) \end{array} \tag{2.14}$$

We can check this explicitly, noting that

$$M^{\mu 9}\cdot(\tilde{C}\Gamma^{\cdots}) = \tilde{C}[M^{\mu 9},\Gamma^{\cdots}] \tag{2.15}$$

From this, we can see that as e.g. $[M^{\mu 9},\Gamma^{10}] = 0$ we have that P^{10} is invariant under our $Spin(9,1)$. Similarly, we see that $[M^{\mu 9},\Gamma_{\nu 10}] = 2\delta^\mu{}_\nu\Gamma^9{}_{10}$ and $[M^{\mu 9},\Gamma_{9\ 10}] = -2\Gamma^\mu{}_{10}$ so that $(Z_{\mu 10}, Z_{9\ 10})$ forms a vector of $Spin(9,1)$.

For type IIB, the situation is more complicated as the generators of the relevant $Spin(9,1)$ are now $M^{\mu\nu} = \Gamma^{\mu\nu}$ and $M^{\mu 9} = \Gamma^{\mu 9\ 10}$. We then have

$$M^{\mu 9}\cdot(\tilde{C}\Gamma^{\cdots}) = -\tilde{C}\{M^{\mu 9},\Gamma^{\cdots}\} \tag{2.16}$$

We must then calculate the anti-commutators to see which charges are rotated into each other by $M^{\mu 9}$. For example, $\{M^{\mu 9},\Gamma^\nu\} = 2g^{\mu\nu}\Gamma^{9\ 10}$ and $\{M^{\mu 9},\Gamma^{9\ 10}\} = -2\Gamma^\mu$, so that $(P^\mu, Z_{9\ 10})$ now forms a vector of this $Spin(9,1)$. Continuing in this way, one finds that the $Spin(9,1)$ irreducible combinations are

$$\begin{array}{lll} (P^\mu, Z_{9\ 10}) & (Z_{\mu\nu}, Z_{\mu\nu\lambda 9\ 10}) & (Z_{\mu_1\ldots\mu_5}) \\ & (Z_{\mu i}, P^i) & (Z_{\mu_1\ldots\mu_4 i}) \end{array} \tag{2.17}$$

where $i = 9,10$. In ten dimensions, these are a vector, a three-form, a self-dual five-form and doublets of vectors and self-dual five forms, which are precisely the charges appearing on the right hand side of the supersymmetry algebra for type IIB.

Example 2 : Six-dimensional $\mathcal{N} = (4,0)$ into eleven dimensions

Let us now perform the same calculations for the $\mathcal{N} = (4,0)$ embedding of $Spin(5,1)$ in (2.10). Letting $\mu,\nu = 0,1,\ldots,4$, we have the generators $M^{\mu\nu} = \Gamma^{\mu\nu}$ and $M^{\mu 5} = \Gamma^{\mu 56789\ 10}$, leading to

$$M^{\mu 5}\cdot(\tilde{C}\Gamma^{\cdots}) = -\tilde{C}\{M^{\mu 5},\Gamma^{\cdots}\} \tag{2.18}$$

Calculating the relevant anti-commutators, using indices $m,n = 5,6,\ldots,10$, organises the charges into $1+6+6+15$ vectors of $Spin(5,1)$

$$\begin{array}{llll} (P^\mu, Z_{\mu_1\ldots\mu_5}) & (Z_{\mu_1\ldots\mu_4 m}, P^m) & (Z_{\mu m}, Z_{m_1\ldots m_5}) & (Z_{\mu p_1\ldots p_4}, Z_{mn}) \end{array} \tag{2.19}$$

together with $1+15+20$ self-dual three-forms

$$\begin{array}{lll} (Z_{\mu\nu}) & (Z_{\mu_1\mu_2\mu_3 mn}) & (Z_{\mu\nu m_1 m_2 m_3}) \end{array} \tag{2.20}$$

Of course, these charges precisely agree with the representations expected on the right hand side of the supersymmetry algebra (1.1), and one can check that they combine into representations of $Sp(8)$ as generated by $\{\Gamma^{mn},\Gamma^{mnp},\Gamma^{m_1\ldots m_6}\}$.

2.3 Dimensional splits, hidden symmetries and the 6d space

Consider the formulation of eleven-dimensional supergravity on a product space, as considered in [33, 34, 36, 85]. Letting $\mu, \nu = 0, 1, \dots 10 - d$ be spacetime indices for the external space, and $m, n = 1, \dots, d$ be those for the internal space, we have that the hidden symmetry group $\tilde{H}_d$ can be realised inside Cliff$(10, 1; \mathbb{R})$ with the generators

$$\mathfrak{h}_d \sim \{\Gamma^{m_1 m_2}, \Gamma^{m_1 m_2 m_3}, \Gamma^{m_1 \dots m_6}, \Gamma^m \Gamma^{m_1 \dots m_8}\} \tag{2.21}$$

for $d \leq 8$, where for $d < 8$ we truncate the generators which are automatically zero by antisymmetry. The first generator $\Gamma^{m_1 m_2}$ is simply the generator of $Spin(d)$, while the remaining terms correspond to the fields of the theory: the three-form A_3, its magnetic dual $\tilde{A}_6$ and the conjectured dual graviton [54, 93] $\tilde{h}_{1,8}$.

To relate the spin embeddings of the previous section to this formalism, we need to look at the parts of the spin group which are in common in the two descriptions. For example, consider a dimensional split with seven external dimensions. The (continuous) U-duality group is $E_{4(4)} \simeq SL(5, \mathbb{R})$ and we write our theory in terms of objects transforming under $GL(7, \mathbb{R}) \times SL(5, \mathbb{R}) \times \mathbb{R}^+$. To describe eleven-dimensional supergravity in the relevant generalised geometry formalism, the generalised tangent space on the internal four-dimensional part of the space is (for the expressions of the generalised tangent space of $E_{d(d)} \times \mathbb{R}^+$ and its decomposition under the group $GL(d, \mathbb{R})$ see e.g. table 2 in [88])

$$E \simeq T_4 \oplus \Lambda^2 T_4^* \tag{2.22}$$

where T_4 transforms under the natural $GL(4, \mathbb{R})$ group of the frame bundle in four dimensions. E itself transforms as a ten-dimensional representation of $SL(5, \mathbb{R}) \times \mathbb{R}^+$. We view this simply as the vector space of charges of the objects living only in these four dimensions, here the four-dimensional momentum and the M2-branes wrapping directions in the four-dimensional space. The analogue of the spin group then becomes $Spin(6, 1) \times Spin(5)$, which is generated by the eleven-dimensional Γ-matrices ($\mu, \nu = 0, 1, \dots, 6$ and $m, n = 7, 8, 9, 10$)

$$\{\Gamma^{\mu\nu}, \Gamma^{m_1 m_2}, \Gamma^{m_1 m_2 m_3}\} \tag{2.23}$$

The first two sets of generators in the list generate part of the usual spacetime spin group $Spin(6, 1) \times Spin(4) \subset Spin(10, 1)$, while the $\Gamma^{m_1 m_2 m_3}$ enhance the $Spin(4)$ factor to the $Spin(5)$ hidden symmetries which are not manifest in the standard formulation with manifest eleven-dimensional covariance. The intersection of the $Spin(9, 1)$ groups relevant to type IIA and type IIB with this are then each isomorphic to $Spin(6, 1) \times Spin(3)$.

With this dimensional split in place, the above discussion of extending the $Spin(8, 1)$ in nine dimensions to $Spin(9, 1)$ for type IIA or type IIB becomes a discussion of how to extend the $Spin(6, 1) \times Spin(2)$ generated by ($\mu, \nu = 0, 1, \dots, 6$ and $\underline{m}, \underline{n} = 7, 8$)

$$\{\Gamma^{\mu\nu}, \Gamma^{\underline{m}_1 \underline{m}_2}\} \tag{2.24}$$

to $Spin(6, 1) \times Spin(3)$.

In type IIA, the relevant $Spin(6, 1) \times Spin(3)$ is generated by

$$\left\{\Gamma^{\mu\nu}, \Gamma^{\underline{m}_1 \underline{m}_2}, \Gamma^{\underline{m} 9}\right\} \tag{2.25}$$

and this simply corresponds to including one more of the spatial directions rotated into each other by the eleven-dimensional spin group. To see this more explicitly, we decompose the generalised tangent space (2.22) under the $GL(2, \mathbb{R})$ containing the $Spin(2)$ factor in our $Spin(6, 1) \times Spin(2)$, giving

$$E \simeq (T_2 \oplus \mathbb{R}_9 \oplus \mathbb{R}_{10}) \oplus (\Lambda^2 T_2^* \oplus T_2^* \oplus T_2^* \oplus \mathbb{R}_{9,10}) \tag{2.26}$$

We then consider which parts of this are combined into irreducible representations of the $Spin(3)$ factor in (2.25), which is the compact subgroup of an $SL(3, \mathbb{R})$ with generators $(T_2 \otimes T_2^*) \oplus (T_2 \otimes \mathbb{R}_9^*) \oplus (\mathbb{R}_9 \otimes T_2^*)$. We see that this $Spin(3)$ rotates T_2 into $\mathbb{R}_9$, forming $T_3 = T_2 \oplus \mathbb{R}_9$. This $SL(3, \mathbb{R})$ can be extended to a $GL(3, \mathbb{R})$ inside $SL(5, \mathbb{R}) \times \mathbb{R}^+$ containing our $Spin(3)$ and T_3 becomes its vector representation. We then have

$$E \simeq (T_3 \oplus \mathbb{R}_{10}) \oplus (\Lambda^2 T_3^* \oplus T_3^*) \tag{2.27}$$

with the internal momentum charges spanning the T_3 factor, as this is the vector representation of the corresponding general linear group. Thus, our ten-dimensional spacetime for type IIA then has directions corresponding to the seven external dimensions and the three directions in T_3. These are simply ten of the original eleven directions we started with in the first place. The passage from (2.26) to (2.27) exactly mirrors the discussion of the charges in the supersymmetry algebra (2.14), which when restricted to the singlets of $Spin(6,1)$, reduces to the combinations

$$(P^{\underline{m}}, P^9) \qquad (P^{10}) \qquad (Z_{\underline{mn}}, Z_{\underline{m}9}) \qquad (Z_{\underline{m}10}, Z_{9\,10}) \tag{2.28}$$

For our type IIB embedding of $Spin(9,1)$ in Cliff$(10,1)$, the intersection with $Spin(6,1) \times Spin(5)$ is instead the $Spin(6,1) \times Spin(3)$ generated by

$$\left\{\Gamma^{\mu\nu}, \Gamma^{\underline{m}_1\underline{m}_2}, \Gamma^{\underline{m}9\,10}\right\} \tag{2.29}$$

We again look at which directions in (2.26) are rotated into each other by this $Spin(3)$ group. In this case, the $Spin(3)$ is contained in an $SL(3,\mathbb{R})$ with generators $(T_2 \otimes T_2^*) \oplus (T_2 \otimes \mathbb{R}_9 \otimes \mathbb{R}_{10}) \oplus (T_2^* \otimes \mathbb{R}_9^* \otimes \mathbb{R}_{10}^*)$ which rotates T_2 into $\mathbb{R}_{9,10}$ and these are combined into T_3'. This is again the fundamental representation of a $GL(3,\mathbb{R}) \subset SL(5,\mathbb{R}) \times \mathbb{R}^+$ containing our $Spin(3)$ and the full generalised tangent space then becomes

$$E \simeq T_3' \oplus T_3'^* \oplus T_3'^* \oplus \Lambda^3 T_3'^* \tag{2.30}$$

In the type IIB case, the momentum direction we have added to T_2 corresponds to the charge of the $M2$-brane wrapping the 9 and 10 directions in the eleven-dimensional picture, as in the well-known duality between type IIB on S^1 and M theory on T^2 [94–96]. Again, the combinations of charges which become representations of $GL(3,\mathbb{R})$ perfectly match those found in (2.17) restricted to the singlets of $Spin(6,1)$:

$$(P^{\underline{m}}, Z_{9\,10}) \qquad (Z_{\underline{m}i}, P^i) \qquad (Z_{\underline{mn}}) \tag{2.31}$$

This discussion of type IIA and type IIB is usually presented in the exceptional geometry literature in terms of these inequivalent embeddings of the general linear groups into the exceptional groups [33, 35, 39] (different "gravity lines") or different solutions to a section condition [36, 97]. However, we wanted to start instead from the details of the corresponding spin groups and central charges, as in our main case of interest in this article that is the most accessible information.

Let us now consider the embedding of $Spin(5,1)$ into Cliff$(10,1)$ given in (2.10). By naive comparison with (2.29) and its interpretation, one could expect that the sixth direction in this case could correspond to the charge of some six-brane in the eleven-dimensional picture. However, M-theory does not contain such an object (see [93] for a full discussion of this point). We will see that in fact, the new generator can be embedded into the last generator listed in (2.21), corresponding to the dual graviton. This exists only for dimensional splits with three external dimensions or fewer. As the only case with a finite-dimensional duality group is that of three external dimensions, for convenience we choose to examine the situation in that framework.

Thus we consider a $(3+8)$-dimensional split of eleven-dimensional supergravity. The corresponding generalised geometry description would feature objects transforming under $GL(3,\mathbb{R}) \times E_{8(8)} \times \mathbb{R}^+$ and the analogue of the spin group inside this would be $Spin(2,1) \times SO(16)$. In fact, for our purposes it will suffice to truncate $E_{8(8)} \times \mathbb{R}^+$ to the $SL(9,\mathbb{R}) \times \mathbb{R}^+$ sector which contains only the graviton and dual-graviton fields [35]. In this subsector, the charges on the eight-dimensional part of the space transform in the rank two antisymmetric bivector representation of $SL(9,\mathbb{R})$, which has the $GL(8,\mathbb{R})$ decomposition

$$E \simeq T_8 \oplus (T_8^* \otimes \Lambda^7 T_8^*) \oplus (\Lambda^8 T_8^* \otimes \Lambda^8 T_8^* \otimes T_8^*) \tag{2.32}$$

while the decomposition of the adjoint of $SL(9,\mathbb{R})$ is

$$\mathrm{ad}\, SL(9,\mathbb{R}) \simeq (T_8 \otimes T_8^*) \oplus (\Lambda^8 T_8 \otimes T_8) \oplus (\Lambda^8 T_8^* \otimes T_8^*) \tag{2.33}$$

The corresponding spin group is $Spin(2,1) \times Spin(9)$ generated by ($\mu,\nu = 0,1,2$ and $\hat{m},\hat{n} = 3,4,\ldots,9,10$)

$$\left\{\Gamma^{\mu\nu}, \Gamma^{\hat{m}\hat{n}}, \Gamma^{\hat{m}}\Gamma^{(8)}\right\} \qquad \text{where} \qquad \Gamma^{(8)} = \Gamma^3\Gamma^4 \ldots \Gamma^9\Gamma^{10} \tag{2.34}$$

The intersection of the $Spin(4,1)$ group from section 2.2.2 with the $Spin(2,1)\times Spin(9)$ considered here is then $Spin(2,1)\times Spin(2)$, which is generated by

$$\{\Gamma^{\mu\nu},\Gamma^{ab}\} \tag{2.35}$$

Here we define the index ranges $\mu,\nu=0,1,2$ and $a,b=3,4$, while $m,n=5,6,7,8,9,10$ and $\underline{m},\underline{n}=6,7,8,9,10$ so that $\hat{m}=(a,m)=(a,5,\underline{m})$. We seek to enhance this to the $Spin(2,1)\times Spin(3)$ groups which are the intersections of the $Spin(5,1)$ groups described in section 2.2.2 with $Spin(2,1)\times Spin(9)$. The $Spin(2,1)\times Spin(3)$ of standard $\mathcal{N}=(2,2)$ supergravity in six dimensions is generated by

$$\{\Gamma^{\mu\nu},\Gamma^{ab},\Gamma^{a5}\} \tag{2.36}$$

which corresponds simply to including one more of the standard eleven-dimensional momenta to give a total of six spacetime momenta out of the eleven.

However, the $Spin(5,1)$ group which corresponds to the $\mathcal{N}=(4,0)$ decomposition gives rise to a $Spin(2,1)\times Spin(3)$ group generated by

$$\left\{\Gamma^{\mu\nu},\Gamma^{ab},\Gamma^{a}\Gamma^{(8)}\right\} \tag{2.37}$$

which are clearly contained in the generators of $Spin(2,1)\times Spin(9)$ in (2.34).

To see how to interpret this in terms of charges, we note that this $Spin(9)$ is contained inside the $SL(9,\mathbb{R})$ group generated by (2.33). Decomposing

$$T_8=A_3\oplus B_5=C_2\oplus\mathbb{R}_5\oplus B_5 \tag{2.38}$$

(according to $\hat{m}=(a,m)=(a,5,\underline{m})$) we see that the $Spin(9)$ generators featuring in (2.37) are inside the $SL(3,\mathbb{R})$ subgroup generated by

$$(C_2\otimes C_2^*)\oplus(\Lambda^2C_2\otimes\Lambda^5B_5\otimes C_2)\oplus(\Lambda^2C_2^*\otimes\Lambda^5B_5^*\otimes C_2^*)\subset \operatorname{ad}SL(9,\mathbb{R}) \tag{2.39}$$

The five-dimensional dual graviton field (for the five-dimensional spacetime consisting of the external directions together with the momenta in C_2) corresponds to the term $C^*\otimes\Lambda^2C^*$, and we see that this is the term appearing in (2.39). We then look at the decomposition of the charges (2.32)

$$\begin{aligned}E\simeq\ & C\oplus\mathbb{R}\oplus B\\ &\oplus(C^*\otimes C^*\otimes\Lambda^5B^*)\oplus(C^*\otimes\Lambda^2C^*\otimes\Lambda^5B^*)\oplus(C^*\otimes\Lambda^5B^*)\oplus(\Lambda^2C^*\otimes\Lambda^5B^*)\\ &\oplus(B^*\otimes C^*\otimes\Lambda^5B^*)\oplus(B^*\otimes\Lambda^2C^*\otimes\Lambda^5B^*)\\ &\oplus(C^*\otimes\Lambda^2C^*\otimes\Lambda^4B^*)\oplus(\Lambda^2C^*\otimes\Lambda^4B^*)\oplus(\Lambda^2C^*\otimes B^*\otimes\Lambda^4B^*)\\ &\oplus\left[(\Lambda^2C^*\otimes\Lambda^5B^*)^2\right]\oplus\left[(\Lambda^2C^*\otimes\Lambda^5B^*)^2\otimes C^*\right]\oplus\left[(\Lambda^2C^*\otimes\Lambda^5B^*)^2\otimes B^*\right]\end{aligned} \tag{2.40}$$

and see which parts are combined into representations of this $SL(3,\mathbb{R})$. Here we find a very different result to the $\mathcal{N}=(2,2)$ case. The terms which combine with C to form an $SL(3,\mathbb{R})$ representation make up not a triplet but an octuplet of $SL(3,\mathbb{R})$:

$$C\oplus\left(C^*\otimes C^*\otimes\Lambda^5B^*\right)\oplus\left[(\Lambda^2C^*\otimes\Lambda^5B^*)^2\otimes C^*\right] \tag{2.41}$$

This subspace does not satisfy the section condition of $E_{8(8)}$ exceptional field theory[3], and thus it seems difficult to interpret it as the coordinate directions of a higher-dimensional spacetime. Clearly, it also does not match the naive expectation of (2.19), which would suggest that the two five-dimensional momenta P^a in C would simply be joined by one additional charge $Z_{\mu_1\mu_2\mu_3ab}$ to form a triplet. We will examine this further in section 2.4. The decompositions (2.41) and (2.39) are essentially the same as (2.32) and (2.33) and are the charges and adjoint relevant for five-dimensional pure gravity reduced to three dimensions, with the $SL(3,\mathbb{R})$ simply interpreted as the

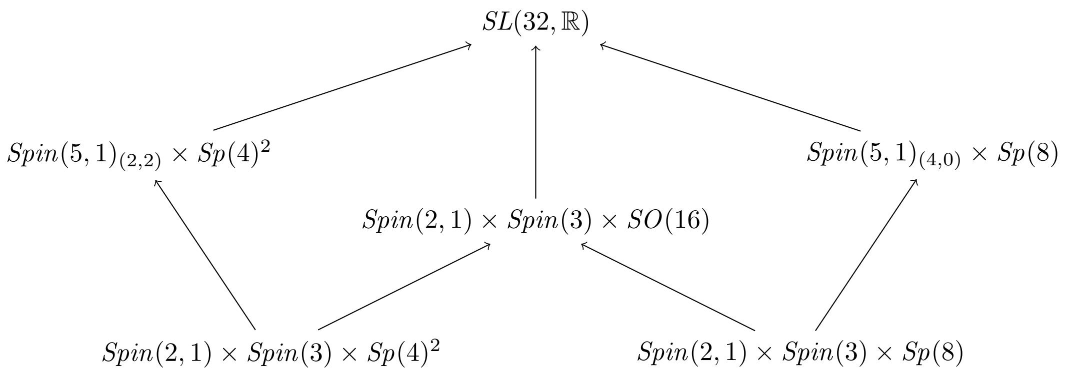
[3]The $E_{8(8)}$ section condition determines whether a subspace $V\subset E$ has $V\otimes V$ null in the projection $\mathbf{248}\times\mathbf{248}\to\mathbf{1}+\mathbf{248}+\mathbf{3875}$. This tensor product contains terms contracting T_8 into the $\Lambda^7T_8^*$ factor of $T_8^*\otimes\Lambda^7T_8^*$ and into both factors of $T_8^*\otimes\Lambda^7T_8^*$. It is the non-vanishing of these contractions which demonstrate that several subspaces we consider in this article do not satisfy this condition.

Ehlers symmetry.
We note also that the $SL(3,\mathbb{R})$ subgroup (2.39) is conjugate to the standard one by an $SL(9,\mathbb{R}) \subset E_{8(8)}$ transformation. To see this explicitly, it is convenient to think about the action of our two $SL(3,\mathbb{R})$ subgroups instead on the vector representation of $SL(9,\mathbb{R}) \times \mathbb{R}^+$

$$V \simeq T_8 \oplus \Lambda^8 T_8^* \simeq C_2 \oplus \mathbb{R}_5 \oplus B_5 \oplus (\Lambda^2 C^* \otimes \Lambda^5 B^*) \tag{2.42}$$

The $\mathcal{N} = (2,2)$ $SL(3,\mathbb{R})$ subgroup has $C_2 \oplus \mathbb{R}_5$ as the triplet part of the decomposition of V, while the $\mathcal{N} = (4,0)$ $SL(3,\mathbb{R})$ has $C_2 \oplus (\Lambda^2 C^* \otimes \Lambda^5 B^*)$. The difference is simply the interchange of the $\mathbb{R}_5$ and $(\Lambda^2 C^* \otimes \Lambda^5 B^*)$ directions in V, i.e. interchange of the $\Lambda^8 T_8^*$ direction in (2.42) with one of the directions in T_8, which can be implemented via a rotation operation inside $SO(9)$. Thus, these two $SL(3,\mathbb{R})$ subgroups are conjugate via this rotation inside $SL(9,\mathbb{R})$. It follows that the decompositions of the charges E are also related by this swapping of directions. As such, any triplet of this $SL(3,\mathbb{R})$ that we could have found would be equivalent to the standard triplet of momenta for standard $\mathcal{N} = (2,2)$ supergravity by a U-duality.

At this point, let us also make some brief remarks about the commutant groups of our $Spin(2,1) \times Spin(3)$ groups inside $Spin(2,1) \times SO(16)$, as this reveals some subtle points for consideration. The chains of embeddings of the spin groups we have considered so far can be summarised in the following diagram:



(2.43)

The group at the bottom right of this diagram has the generators[4]

$$\left\{\Gamma^{\mu\nu}, \Gamma^{ab}, \Gamma^{a}\Gamma^{(8)}, \Gamma^{m_1 m_2}, \Gamma^{m_1 m_2 m_3}, \Gamma^{m_1 \dots m_6}\right\} \tag{2.44}$$

while the group at the bottom left has the generators

$$\left\{\Gamma^{\mu\nu}, \Gamma^{ab}, \Gamma^{a5}, \Gamma^{\underline{m}_1 \underline{m}_2}, \Gamma^{\underline{m}_1 \underline{m}_2 \underline{m}_3}\right\} \tag{2.45}$$

The first three terms of each generate their respective $Spin(2,1) \times Spin(3)$ factors, and are related by exchanging Γ^5 and $\Gamma^{(8)}$ as one would expect from the discussion of the $SL(9,\mathbb{R})$ rotation operation above. However, one can perform this exchange on the remaining generators in (2.44) to obtain generators for a $Spin(2,1) \times Spin(3) \times Sp(8)$ group containing (2.45):

$$\left\{\Gamma^{\mu\nu}, \Gamma^{ab}, \Gamma^{a5}, \Gamma^{\underline{m}_1 \underline{m}_2}, \Gamma^{\underline{m}_1 \underline{m}_2 \underline{m}_3}, \Gamma^{\underline{m}}\Gamma^{(8)}, \Gamma^{\underline{m}_1 \underline{m}_2}\Gamma^{(8)}, \Gamma^{\underline{m}_1 \cdots \underline{m}_5}\Gamma^{(8)}\right\} \tag{2.46}$$

Very naively, one might then wonder why the group $Spin(5,1)_{(2,2)} \times Sp(4)^2$ in (2.43) is not $Spin(5,1)_{(2,2)} \times Sp(8)$. The reason is because the generators added to those in (2.45) do not commute with the generators Γ^{i5} which are present in $Spin(5,1)_{(2,2)}$, but which are not part of its $Spin(2,1) \times Spin(3)$ subgroup.

This shows that one should be careful about making conclusions when imposing dimensional splits in the way that we have done in this section. Indeed, there is an apparent paradox in our work here. The embeddings of $Spin(5,1)$ into $SL(32,\mathbb{R})$ really are inequivalent as they give

[4] Recall that we defined the index ranges $\mu,\nu = 0,1,2$ and $a,b = 3,4$, while $m,n = 5,6,7,8,9,10$ and $\underline{m},\underline{n} = 6,7,8,9,10$ so that $\hat{m} = (a,m) = (a,5,\underline{m})$.

different decompositions of the **32** representation into irreducible parts. However, on imposing the dimensional split that we have done, the corresponding $Spin(2,1) \times Spin(3)$ subgroups have been found to be conjugate by an $SO(9)$ transformation. Thus, this inequivalence is not apparent from the point of view of our dimensional split. Similarly, the corresponding $SL(3,\mathbb{R})$ subgroups inside $SL(9,\mathbb{R}) \subset E_{8(8)}$ also appear to be equivalent, unlike in the case of the type IIA vs type IIB embeddings. From our analysis it thus remains unclear exactly how the inequivalent decompositions of the spinor can be seen within the framework of exceptional groups. To learn more, one would need to include the full external $Spin(5,1)$ group as well as the dual graviton charges, which would be contained only in a full E_{11} analysis. The details go beyond the scope of our current investigation, though the resolution appears to be that there simply does not exist an $\mathfrak{sl}(6,\mathbb{R})$ subalgebra containing our $\mathfrak{spin}(5,1)_{(4,0)}$ whose possible equivalence one can ask about [98].

Let us now turn to a comparison of what we have found with the construction of [1]. In that picture, one examines the five-dimensional maximal supersymmetry algebra

$$\{Q_{\alpha A}, Q_{\beta B}\} = \Omega_{AB} P_\mu {\gamma^\mu}_{[\alpha\beta]} + K\Omega_{AB} C_{\alpha\beta} + \dot{Z}_{[AB]} C_{\alpha\beta} + \dot{Z}_{\mu[AB]} {\gamma^\mu}_{[\alpha\beta]} + Z_{[\mu\nu](AB)} \gamma^{\mu\nu}_{(\alpha\beta)} \tag{2.47}$$

The central charge K is singled out as it is a singlet of the bosonic subalgebra $\mathfrak{spin}(4,1) \times \mathfrak{sp}(8)$, and it is remarked that it is not the charge of any of the five-dimensional vector fields, but becomes the magnetic charge of the gravi-photon on reduction to four dimensions. To identify the higher-dimensional physical object carrying the charge K, it is useful to consider that, in terms of the eleven-dimensional charges, it is the five-form charge $Z_{(5)}$ carried by the M5-brane but with all indices in the five-dimensional external space. (This was shown to be paired with the five-dimensional momentum to form a vector of $Spin(5,1)_{(4,0)}$ in (2.19).) Possibly the simplest picture of this arises from the type IIA decomposition. We think of the fifth direction of the five-dimensional external space as the M theory circle and note that the charge K can then be seen as a D6-brane with legs along the six internal directions.

In terms of the decomposition (2.40), the D6-brane is part of the M-theory dual graviton, but to see this, we need to decompose further. Thus we go back to (2.38), and this time give explicit labels to three one dimensional subspaces spanning A_3

$$A_3 = \mathbb{R}_3 \oplus \mathbb{R}_4 \oplus \mathbb{R}_5 \tag{2.48}$$

where our previous $C_2 = \mathbb{R}_3 \oplus \mathbb{R}_4$. We then imagine $\mathbb{R}_4$ to correspond to the M theory circle direction. In terms of these labels, the internal D6-charge corresponds to the dual graviton charge $\mathbb{R}_4^* \otimes (\mathbb{R}_4^* \otimes \mathbb{R}_5^* \otimes \Lambda^5 B^*) \subset T^* \otimes \Lambda^7 T^*$. The momentum charge around the M theory circle becomes the D0-brane charge in the IIA picture and corresponds to $\mathbb{R}_4 \subset T_8$. Thus, naively it appears[5] that the charges

$$\mathbb{R}_3 \oplus \mathbb{R}_4 \oplus \left[\mathbb{R}_4^* \otimes (\mathbb{R}_4^* \otimes \mathbb{R}_5^* \otimes \Lambda^5 B^*)\right] \subset E \tag{2.49}$$

are thought of as the three momenta which, in conjunction with the three momenta in the external space, make up the momenta in the six-dimensional spacetime of [1].

While the smaller subspaces $\mathbb{R}_3 \oplus \mathbb{R}_4$ or $\mathbb{R}_3 \oplus \left[\mathbb{R}_4^* \otimes (\mathbb{R}_4^* \otimes \mathbb{R}_5^* \otimes \Lambda^5 B^*)\right]$ solve the section constraint of $E_{8(8)}$ exceptional field theory, the three charges (2.49) together do not. This is because the charge $\mathbb{R}_4$ has a non-zero contraction with the charge $\mathbb{R}_4^* \otimes (\mathbb{R}_4^* \otimes \mathbb{R}_5^* \otimes \Lambda^5 B^*)$ in the relevant tensor product. Thus, these charges fail to satisfy the usual requirements to be a spacetime section.

Further, in [2], the conjectured six-dimensional theory is compactified on T^2 to give a maximally supersymmetric four-dimensional theory with an $SL(2,\mathbb{R})$ internal symmetry. It was noted there that this $SL(2,\mathbb{R})$ symmetry must be outside of the usual $E_{7(7)}$ symmetry of four-dimensional maximal supergravity.[6] However, if we view the two momenta on T^2 as the D0 and D6 charges $\mathbb{R}_4 \oplus \left[\mathbb{R}_4^* \otimes (\mathbb{R}_4^* \otimes \mathbb{R}_5^* \otimes \Lambda^5 B^*)\right]$, then we see that in fact there is also no $SL(2,\mathbb{R})$ subgroup of $E_{8(8)}$ which rotates these charges into each other, as this would have to contain a generator $\mathbb{R}_4^* \otimes \mathbb{R}_4^* \otimes (\mathbb{R}_4^* \otimes \mathbb{R}_5^* \otimes \Lambda^5 B^*)$. Thus, the $SL(2,\mathbb{R})$ symmetry of [2] also appears to lie outside of the $E_{8(8)}$ duality group.

A strongly related fact is that there is also no $SL(3,\mathbb{R})$ subgroup of the $E_{8(8)}$ duality group for which the charges (2.49) form a triplet representation. As we found above, these can only be

[5]See section 2.4 for a more complete discussion.

[6]The lack of this $SL(2,\mathbb{R})$ is related to the absence [99,100] of uplifts of the deformed $SO(8)$ gauged supergravities of [101]. It is also related to the missing $U(1)$ factor of footnote 1.

combined into an octuplet of $SL(3,\mathbb{R})$. The D0 and D6 charges then sit inside this octuplet in such a way that there is no $SL(2,\mathbb{R})$ subgroup under which they form a doublet.

One then wonders if there is a different triplet of charges for our $SL(3,\mathbb{R})$ group (2.39), which could form the six-dimensional space of the $\mathcal{N} = (4,0)$ theory. One quickly see that there is precisely such a set: writing $C = \mathbb{R}_3 \oplus \mathbb{R}_4$ as before, we have the triplet

$$\mathbb{R}_5 \oplus \left(C^* \otimes \Lambda^2 C^* \otimes \Lambda^5 B^*\right) \tag{2.50}$$

comprising one of the spatial momenta in M theory together with the six-dimensional dual gravitons with no leg along that direction. This set of charges thus solves the section condition of $E_{8(8)}$ exceptional geometry. However, as noted above, the same $SO(9)$ transformation which related the $SL(3,\mathbb{R})$ subgroup (2.39) to the standard one relates this section to the standard one spanned by $\mathbb{R}_3 \oplus \mathbb{R}_4 \oplus \mathbb{R}_5$. As such, the charges (2.50) are simply U-dual to the three momentum charges along $\mathbb{R}_3$, $\mathbb{R}_4$ and $\mathbb{R}_5$. This would indicate that something has gone wrong, as the corresponding theories are supposed to be very different, as are the relevant spinor decompositions. Further still, by considering the orbits of the charges in the supersymmetry algebra under $Spin(5,1)_{(4,0)}$ and how these are mapped into the **248** representation of $E_{8(8)}$ we can see that (2.50) does not match the momenta of the six-dimensional space. We will do this explicitly in the next section.

2.4 Charges in $E_{8(8)}$ and the triplet of $SO(3)$

In this section we will see that our identification of charges in (2.49) is not quite right. Unlike the lower rank exceptional groups, in $E_{8(8)}$ the internal charges appearing in the anti-commutator of supersymmetries do not map onto the **248** representation. Rather, they span only the subspace forming the **120** representation of the maximal compact subgroup $SO(16)$. As such, the momentum charge $P^{\hat{m}}$ of eleven-dimensional supergravity in the eight internal directions, embeds into not just the obvious vector T_8 in (2.32), but it also has a component along $T_8^* \otimes (\Lambda^8 T_8^*)^2$. The interpretation of this is that the supersymmetry algebra closes not just onto local translations, but a combination of these with higher gauge transformations of the dual gravitons. We also note that the subspace of the charges into which the momentum directly embeds does not solve the section condition.

For standard supergravities, one could identify the spacetime section from the momentum charge coming from the supersymmetry algebra in the following way. The embedded momentum charge in fact lives in a subspace of the sum of two isomorphic vector representations of the orthogonal group inside E. For the momentum $P_{\hat{m}}$ above, these two become the T_8 and $T_8^* \otimes (\Lambda^8 T_8^*)^2$ representations of the $GL(8,\mathbb{R})$ subgroup of $E_{8(8)}$ containing $SO(8)$. One can project onto these two subspaces in a $GL(8,\mathbb{R})$ covariant way. More generally, there are $SO(8)$ covariant projectors onto any linear combination of them. The property that picks out the subspace T_8 (or $T_8^* \otimes (\Lambda^8 T_8^*)^2$ which is the same up to an automorphism of $SL(9,\mathbb{R})$) is that it solves the section condition (while any linear combination does not). Thus, even though the momentum charge does not directly live in the directions T_8 of the spacetime section, it is fairly simple to identify the spacetime section and project onto it. Indeed, the generalised Lie derivative of exceptional geometry effectively implements such a projection, as it receives no contribution from the $T_8^* \otimes (\Lambda^8 T_8^*)^2$ piece.

Let us contrast this with the situation for the momentum charge of the $\mathcal{N} = (4,0)$ theory. There, the result (2.19) tells us that two of the five-dimensional momenta are combined with the charge labelled K above into a triplet, which makes up the three internal momenta of the six-dimensional spacetime. This triplet is invariant under the $Sp(8)$ R-symmetry, which uniquely identifies it inside the **248** of $E_{8(8)}$ as the generators of $SO(3)_{(4,0)}$ (see (2.60) later). In terms of the charges in (2.40) this triplet consists of $\Lambda^2 C^* \otimes \Lambda^5 B^*$ together with a two-dimensional subspace of $C \oplus (C^* \otimes \Lambda^2 C^* \otimes \Lambda^5 B^*)$. We would then like to project this onto a triplet of an $SL(3,\mathbb{R})$ group containing $SO(3)_{(4,0)}$, as we did for the standard supergravity case. Naively it would even seem reasonable that the projected subspace could be similar to the charges (2.49). However, here there is no such projection. The $SL(3,\mathbb{R})$ group containing $SO(3)_{(4,0)}$ makes the triplet of $SO(3)_{(4,0)}$ into an octuplet. It is not a subspace of the sum of two triplets.

What we have learned here is that there is no spacetime section for the $\mathcal{N} = (4,0)$ theory in the standard sense. Rather, the momentum charge is the triplet of $SO(3)_{(4,0)}$ which is invariant under $Sp(8)$, and like the embedded momentum charge in other cases, this does not solve the section condition. Moreover, the identification of this subspace appears to require the decomposition under $SO(3)_{(4,0)} \times Sp(8)$, which requires knowledge of the physical fields. Thus, very differently to the case of standard supergravity, it appears that the momentum charge, or even a relevant subspace of the correct dimension, can only be identified once a field configuration is specified. This picture also resonates with the earlier mentioned observation that the $Spin(5,1)_{(4,0)} \times Sp(8)$ group is present inside KE_{11}, but there appears to be no $SL(6,\mathbb{R}) \times E_{6(6)}$ subgroup which contains it, suggesting that a description of the $\mathcal{N} = (4,0)$ theory in the E_{11} formalism must make explicit use of the Lorentz symmetry.

2.5 Interpretation of $SL(3,\mathbb{R}) \times E_{6(6)}$ inside $E_{8(8)}$

In the previous section, we argued that the role of $SO(3)_{(4,0)} \subset SL(3,\mathbb{R})$ is very different for the $\mathcal{N} = (4,0)$ theory compared with the role of the Lorentz and general linear groups in standard supergravity. In particular, there is no three-dimensional spacetime section satisfying the section condition, but only the analogue of the embedding of the momentum charge in the $\mathbf{248}$ of $E_{8(8)}$. Noting that any $SL(3,\mathbb{R})$ subgroup of $E_{8(8)}$ with commutant $E_{6(6)}$ will be conjugate as $SL(3,\mathbb{R}) \times E_{6(6)} \subset E_{8(8)}$ is a maximal subgroup, we now examine the decompositions of the generalised tangent space and the adjoint of $E_{8(8)}$ under $SL(3,\mathbb{R}) \times E_{6(6)}$. Remarkably, despite all that has been said in the previous sections, some aspects of the $\mathcal{N} = (4,0)$ theory do fit into this picture as we now discuss.

We start from the $GL(8,\mathbb{R})$ decomposition of the $E_{8(8)} \times \mathbb{R}^+$ multiplet of charges related to eleven-dimensional supergravity on an eight-dimensional internal space [35]

$$\begin{aligned} E \simeq \mathbf{248}_{+1} \simeq T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus (T^* \otimes \Lambda^7 T^*) \\ \oplus (\Lambda^8 T^* \otimes \Lambda^3 T^*) \oplus (\Lambda^8 T^* \otimes \Lambda^6 T^*) \oplus ((\Lambda^8 T^*)^2 \otimes T^*) \end{aligned} \tag{2.51}$$

This corresponds to the decomposition of the adjoint representation of $E_{8(8)}$

$$\mathbf{248}_0 \simeq (T \otimes T^*) \oplus \Lambda^3 T \oplus \Lambda^3 T^* \oplus \Lambda^6 T \oplus \Lambda^6 T^* \oplus (\Lambda^8 T \otimes T) \oplus (\Lambda^8 T^* \otimes T^*) \tag{2.52}$$

together with the embedding of $GL(8,\mathbb{R})$ into $E_{8(8)} \times \mathbb{R}^+$ such that $\mathbf{1}_{+1} = (\Lambda^8 T^*)$. These expressions do not provide a generalised geometry in the usual way due to problems with diffeomorphism covariance associated to the dual graviton field (see [35] for a discussion) but one can argue that using additional section conditions to constrain certain compensator fields in the tensor hierarchy it is possible to write an exceptional field theory construction based on them [38].

We now wish to study further splits of the dimensions. In particular, we choose three of the eight dimensions to join the three external dimensions, leaving 5 remaining internal dimensions (in the eleven-dimensional picture). This mirrors our study of the spin groups in section 2.3.

As such, let us decompose under $GL(3,\mathbb{R}) \times GL(5,\mathbb{R}) \subset GL(8,\mathbb{R})$ so that

$$T_8 = A_3 \oplus B_5. \tag{2.53}$$

as before. We reiterate that the straightforward $SL(3,\mathbb{R})$ subgroup of the $GL(3,\mathbb{R})$ factor is appropriate for our purposes here, as the choice which seems most naturally related to the six-dimensional $\mathcal{N} = (4,0)$ theory is equivalent to this one (as shown explicitly in section 2.3). Indeed, whichever $SL(3,\mathbb{R})$ subgroup we chose, we would wish to write our eventual decompositions in terms of its triplet representation and tensor products thereof. As $SL(3,\mathbb{R}) \times E_{6(6)}$ is a maximal subgroup, the result of doing this will be the same whichever $SL(3,\mathbb{R})$ we chose initially.

The $GL(5,\mathbb{R})$ factor can be seen to be a subgroup of a $Spin(5,5) \times \mathbb{R}^+$ group inside $E_{8(8)} \times \mathbb{R}^+$ which commutes with our $GL(3,\mathbb{R})$. Identifying the $Spin(5,5) \times \mathbb{R}^+$ representations as is familiar from five-dimensional exceptional generalised geometry via

$$\begin{aligned} (B \otimes B^*) \oplus \Lambda^3 B \oplus \Lambda^3 B^* &\simeq \mathfrak{spin}(5,5) \\ \Lambda^5 B^* &\simeq \mathbf{1}_{+4} \\ B \oplus \Lambda^2 B^* \oplus \Lambda^5 B^* &\simeq \mathbf{16}_{+1} \\ B^* \oplus \Lambda^4 B^* &\simeq \mathbf{10}_{+2} \end{aligned} \tag{2.54}$$

we find the $GL(3,\mathbb{R})\times Spin(5,5)\times\mathbb{R}^+$ decompositions

$$\begin{aligned}\mathbf{248}_0 \simeq\ & \mathfrak{gl}(3,\mathbb{R})\oplus\mathfrak{spin}(5,5)\\ &\oplus(A\otimes\mathbf{16}^-_{\mathbf{-1}})\oplus(A^*\otimes\mathbf{16}^+_{\mathbf{+1}})\\ &\oplus(\Lambda^2A\otimes\mathbf{10}_{\mathbf{-2}})\oplus(\Lambda^2A^*\otimes\mathbf{10}_{\mathbf{+2}})\\ &\oplus(\Lambda^3A\otimes\mathbf{16}^+_{\mathbf{-3}})\oplus(\Lambda^3A^*\otimes\mathbf{16}^-_{\mathbf{+3}})\\ &\oplus(\Lambda^3A\otimes A\otimes\mathbf{1}_{\mathbf{-4}})\oplus(\Lambda^3A^*\otimes A^*\otimes\mathbf{1}_{\mathbf{+4}})\end{aligned}\tag{2.55}$$

and

$$\begin{aligned}E\simeq\mathbf{248}_{+1}\simeq\ & A\oplus\mathbf{16}^+_{\mathbf{+1}}\oplus(A^*\otimes\mathbf{10}_{\mathbf{+2}})\\ &\oplus(\Lambda^2A^*\otimes\mathbf{16}^-_{\mathbf{+3}})\oplus(\Lambda^3A^*\otimes\mathbf{45}_{\mathbf{+4}})\oplus(\Lambda^2A^*\otimes A^*\otimes\mathbf{1}_{\mathbf{+4}})\\ &\oplus(\Lambda^3A^*\otimes A^*\otimes\mathbf{16}^+_{\mathbf{+5}})\\ &\oplus(\Lambda^3A^*\otimes\Lambda^2A^*\otimes\mathbf{10}_{\mathbf{+6}})\\ &\oplus((\Lambda^3A^*)^2\otimes\mathbf{16}^-_{\mathbf{+7}})\\ &\oplus((\Lambda^3A^*)^2\otimes A^*\otimes\mathbf{1}_{\mathbf{+8}})\end{aligned}\tag{2.56}$$

From this, we see explicitly that the commutant of $GL(3,\mathbb{R})$ inside $E_{8(8)}\times\mathbb{R}^+$ cannot be enhanced further than $Spin(5,5)\times\mathbb{R}^+$, as (2.55) contains no trivial $GL(3,\mathbb{R})$ singlets beyond the $\mathfrak{spin}(5,5)$ summand. This agrees with the standard picture in supergravity, where we expect six-dimensional $\mathcal{N}=(2,2)$ supergravity to have global symmetry $Spin(5,5)$.

However, we expect the six-dimensional $\mathcal{N}=(4,0)$ theory to have global symmetry $E_{6(6)}$, and thus it would be desirable if we could see a way to make $E_{6(6)}$ the commutant of our spacetime subgroup inside $E_{8(8)}$. To match this to the above, we decompose the above under $SL(3,\mathbb{R})\times Spin(5,5)\times\mathbb{R}^+\subset GL(3,\mathbb{R})\times Spin(5,5)\times\mathbb{R}^+$. Under $SL(3,\mathbb{R})$ we have additional identifications $\Lambda^3A\simeq\Lambda^3A^*\simeq\mathbf{1}$ and $\Lambda^2A\simeq A^*$ and thus we have the decompositions

$$\begin{aligned}\mathbf{248}_0\simeq\ &\mathfrak{sl}(3,\mathbb{R})\oplus\Big(\mathbb{R}\oplus\mathfrak{spin}(5,5)\oplus\mathbf{16}^+_{\mathbf{-3}}\oplus\mathbf{16}^-_{\mathbf{+3}}\Big)\\ &\oplus\Lambda^2A^*\otimes\Big(\mathbf{1}_{\mathbf{-4}}\oplus\mathbf{10}_{\mathbf{+2}}\oplus\mathbf{16}^-_{\mathbf{-1}}\Big)\oplus\Lambda^2A\otimes\Big(\mathbf{1}_{\mathbf{+4}}\oplus\mathbf{10}_{\mathbf{-2}}\oplus\mathbf{16}^+_{\mathbf{+1}}\Big)\end{aligned}\tag{2.57}$$

$$\begin{aligned}E\simeq\mathbf{248}_{+1}\simeq\mathbf{1}_{\mathbf{+4}}\otimes\Big[&\Big(\mathbb{R}\oplus\mathfrak{spin}(5,5)\oplus\mathbf{16}^+_{\mathbf{-3}}\oplus\mathbf{16}^-_{\mathbf{+3}}\Big)\oplus A^*\otimes\Big(\mathbf{1}_{\mathbf{+4}}\oplus\mathbf{10}_{\mathbf{-2}}\oplus\mathbf{16}^+_{\mathbf{+1}}\Big)\\ &\oplus\Lambda^2A^*\otimes\Big(\mathbf{1}_{\mathbf{-4}}\oplus\mathbf{10}_{\mathbf{+2}}\oplus\mathbf{16}^-_{\mathbf{-1}}\Big)\oplus(\Lambda^2A^*\otimes A^*)_0\Big]\end{aligned}\tag{2.58}$$

where $(\Lambda^2A^*\otimes A^*)_0$ denotes the irreducible part of $(\Lambda^2A^*\otimes A^*)$ whose totally anti-symmetric part is zero. The summands $\mathbb{R}\oplus\mathfrak{spin}(5,5)\oplus\mathbf{16}^+_{\mathbf{-3}}\oplus\mathbf{16}^-_{\mathbf{+3}}$ form an $\mathfrak{e}_{6(6)}$ subalgebra of $\mathfrak{e}_{8(8)}$ and we recognise the decompositions

$$\begin{aligned}\mathfrak{e}_{6(6)}&\to\mathbb{R}\oplus\mathfrak{spin}(5,5)\oplus\mathbf{16}^+_{\mathbf{-3}}\oplus\mathbf{16}^-_{\mathbf{+3}}\\ \mathbf{27}&\to\mathbf{1}_{-\mathbf{4}}\oplus\mathbf{10}_{+\mathbf{2}}\oplus\mathbf{16}^-_{-\mathbf{1}}\\ \mathbf{27}'&\to\mathbf{1}_{+\mathbf{4}}\oplus\mathbf{10}_{-\mathbf{2}}\oplus\mathbf{16}^+_{+\mathbf{1}}\end{aligned}\tag{2.59}$$

Ignoring the overall $\mathbb{R}^+$ weight (as there is no non-trivial homomorphism $SL(3,\mathbb{R})\times E_{6(6)}\to\mathbb{R}^+$) and choosing to use the isomorphisms $\Lambda^3A\simeq\Lambda^3A^*\simeq\mathbb{R}$ and $\Lambda^2A\simeq A^*$ to write the result in a suggestive way, we find the standard decompositions

$$\mathbf{248}_0\to\mathfrak{sl}(3,\mathbb{R})\oplus\mathfrak{e}_{6(6)}\oplus(\Lambda^2A^*\otimes\mathbf{27})\oplus(\Lambda^2A\otimes\mathbf{27}')\tag{2.60}$$

$$E\simeq\mathbf{248}_{+1}\to(A^*\otimes\mathbf{27}')\oplus(\Lambda^2A^*\otimes A^*)_0\oplus(\Lambda^2A^*\otimes\Lambda^3A^*\otimes\mathbf{27})\oplus(\Lambda^3A^*\otimes\mathbf{78})\tag{2.61}$$

We could have written these down at the outset. The reason for presenting this chain of decompositions and recombinations at this level of detail is to keep track of all of how the different charges combine into the $E_{6(6)}$ representations, and to show very explicitly that all that is needed to realise $E_{6(6)}$ is to break $GL(3,\mathbb{R})$ to $SL(3,\mathbb{R})$.

Naively applying the usual assignment of forms in the adjoint to physical fields and scalars to a sigma model, one would suspect that the six-dimensional parent theory would have two-forms in the **27** of $E_{6(6)}$ and scalars in the coset $E_{6(6)}/\,Sp(8)$, exactly as one would hope for the $\mathcal{N}=(4,0)$ theory.

However, this is also problematic, as one would also like to interpret the forms in the generalised vector as their charges. The one-forms in E are in the wrong $E_{6(6)}$ representation to be the charges of the two-forms in the adjoint. This is because in the adjoint the $\Lambda^2 A$ and $\Lambda^2 A^*$ terms also live in different representations. In the usual Kac-Moody prescription we would want to interpret the corresponding charges in E as being dual in some higher sense. However, a possible resolution is that under the maximal compact subgroup $Sp(8)$, these become equal. This suggests that really the symmetry of any theory underlying these observations is $Sp(8)$ rather than $E_{6(6)}$ (c.f. the situation for $F_{4(4)}$ in the $\mathcal{N}=(3,1)$ multiplet as discussed in the introduction). An alternative resolution would be to decompose under $SO(3)\subset SL(3,\mathbb{R})$, which allows the identification of vectors and two-forms, so that the third term in (2.61) could be viewed as the charges of the two-forms.

Further signs in this direction come from comparison of (2.61) with the charges in the superalgebra (1.1). We expect to find vector charges in the $\mathbf{1}\oplus\mathbf{27}$ of $Sp(8)$ together with (anti-self-dual) three forms in the **36**. These objects are present inside (2.61), but to see them we must decompose under $SO(3)\times Sp(8)$, as we noted in the previous section. In order to see $E_{6(6)}$ we have to combine the magnetic charges of the scalars with the three-form central charges, while the singlet vector momentum charge becomes part of a non-vector representation of $SL(3,\mathbb{R})$. This again shows that moving from Lorentz to special linear group is be problematic in this context, and that to identify a subspace for the momentum of the correct dimension we must decompose under $SO(3)$.

However, there are also encouraging signs in this, in that the non-vector representation of $SL(3,\mathbb{R})$ which absorbs the singlet vector central charge has the correct index structure to be a charge for the exotic graviton $C_{\mu\nu\rho\sigma}$ from section 1, as a charge $\Lambda_{m[np]}$ can give a gauge transformation $\delta C_{mnpq}\sim\partial_{[m}\Lambda_{n][pq]}+\partial_{[p}\Lambda_{q][mn]}-2\partial_{[m}\Lambda_{npq]}$, where the last term vanishes identically in a three-dimensional restriction.

Indeed, one can see that this does in fact appear in the following way. If we consider $\mathbb{R}^3$ with standard Euclidean metric (and now take $m,n=1,2,3$) and define

$$\partial^m{}_n=\epsilon^m{}_n{}^p\partial_p \qquad \Lambda^m{}_n=\tfrac{1}{2}\epsilon^{mpq}\Lambda_{n[pq]} \tag{2.62}$$

we can then compute the part of the projection of $\partial\Lambda$ into the $\mathfrak{sl}(3,\mathbb{R})$ part of the adjoint in (2.60):

$$[\partial,\Lambda]^m{}_n=\partial_q\Lambda_n{}^{qm}-3\delta^m{}_{[n}\partial^p\Lambda^q{}_{pq]} \tag{2.63}$$

If we then define a dualised variable

$$\tilde{\Lambda}_{m,pq}=\tfrac{1}{2}\epsilon_m{}^{rs}\epsilon_{pq}{}^t\Lambda_{t[rs]} \tag{2.64}$$

and restrict to considering $\tilde{\Lambda}$ in the **5** representation of $SO(3)$ (so that the **8** of $SL(3,\mathbb{R})$ splits into the momentum charge and the gauge parameter) then we find

$$[\partial,\Lambda]^m{}_n=-\epsilon^{mpq}\epsilon_n{}^{rs}\Big(\partial_{[p}\tilde{\Lambda}_{q]rs}+\partial_{[r}\tilde{\Lambda}_{s]pq}\Big) \tag{2.65}$$

Considering a variation of the exotic graviton $C_{[mn][pq]}$ to transform in the adjoint of $SL(3,\mathbb{R})$ via defining

$$\delta C^m{}_n=\epsilon^{mpq}\epsilon_n{}^{rs}\delta C_{[mn][pq]} \tag{2.66}$$

we find

$$\delta C_{[mn][pq]}=-\Big(\partial_{[p}\tilde{\Lambda}_{q]rs}+\partial_{[r}\tilde{\Lambda}_{s]pq}\Big) \tag{2.67}$$

The projection of $\partial\Lambda$ we have calculated would naively become part of the action of the generalised Lie derivative or exceptional Dorfman derivative as introduced in [33]. Recall that this object has the general form[7]

$$L_V = \partial_V - (\partial \times_{\text{ad}} V)\cdot \tag{2.68}$$

where $V \in E$ is a generalised vector. The first term is a straightforward derivative, while the second term term gives the action of the appropriate derivatives of the gauge parameter. What we have discovered here is that, with the definitions made above, we seem to be able to recover the gauge transformation of the exotic graviton as part of this object. In particular, the derivative (2.65) which would be the only place where $\tilde{\Lambda}$ would appear in (2.68), appears to give the correct gauge transformation (2.67). This gives us some confidence in our interpretation of the momentum charge and that our assertion of the necessity of working under the Lorentz group $SO(3)$ is justified.

Overall, it seems that there is some hope of identifying the terms in (2.60) and (2.61) in the usual way. In (2.60), the $\mathfrak{sl}(3,\mathbb{R})$, $\mathfrak{e}_{6(6)}$ and $\Lambda^2 A^*$ terms correspond to the exotic graviton, scalar sigma model and two-forms respectively, while in (2.61) the terms match the charges of the two-forms, the exotic graviton, higher duals of the two-forms, the three-form charges in (A.10) and the magnetic duals of the scalars in that order. However, as discussed, it is really only under $SO(3) \subset SL(3,\mathbb{R})$ that we can identify the triplet A with spacetime, which makes these apparent matches at least slightly surprising.

All of these comments should be taken as suggestive but in no way conclusive. However, they are in harmony with other proposals made in this thesis concerning the importance of a fixed volume T^3 fibred manifold, leaving only an action of $SL(3,\mathbb{R}) \subset GL(3,\mathbb{R})$ and the absence of a six-dimensional "section". The observation that one needs to work under $SO(3)$ to identify the six-dimensional momentum charge is also curious, as it suggests that knowledge of the exotic graviton field configuration is needed to identify the six-dimensional space. They also fit a pattern of behaviour shared by multiplets with less supersymmetry, as we explore next.

2.6 Exotic gravity with less supersymmetry

In this section, we examine the versions of the decompositions (2.60) and (2.61) relevant to the cases of theories with less than maximal supersymmetry. In all cases we see that a special role is played by the five-dimensional Ehlers symmetry $\mathfrak{sl}(3,\mathbb{R})$, which becomes the terms relevant to the exotic graviton in our decompositions. In a sense, the decompositions for these theories are built by adding additional terms to this $\mathfrak{sl}(3,\mathbb{R})$ base in a similar sense to the way that conventional generalised geometries are built as extensions of ordinary geometry with frame bundle group $GL(d,\mathbb{R})$.

2.6.1 $\mathcal{N} = (2,0)$ supersymmetry and $SO(8,8+n)$

If, instead of looking at eleven-dimensional supergravity, we look at type I supergravity (which has half-maximal supersymmetry in ten-dimensions) the analogous group to $E_{8(8)}$ appearing in reductions to three dimensions (with Abelian gauge symmetry) is $SO(8,8+n)$, where n is the number of vector multiplets in ten dimensions.

We can then ask if the same procedure outlined above for the charges and adjoint representation of $E_{8(8)}$ will go through to match the field content of half-maximal exotic gravity. In this section we will show that it does.

Rather than examining first the decompositions under a standard spacetime $GL(7,\mathbb{R})$ group (corresponding to the spatial directions on the seven-torus in a type I compactification), let us assume that exotic gravity will correspond to an $SL(3,\mathbb{R})$ subgroup as in the previous section and simply decompose under the product of $SL(3,\mathbb{R})$ with a suitable commutant inside $SO(8,8+n) \times \mathbb{R}^+$. As such, consider the maximal subgroup $SO(3,3) \times SO(5,5+n) \times \mathbb{R}^+$, noting that $Spin(3,3) \simeq SL(4,\mathbb{R})$. We then decompose the adjoint under the $SL(3,\mathbb{R}) \times SO(5,5+n) \times \mathbb{R}^+$

[7] In fact, for $E_{8(8)}$ it has been argued that one must add additional terms to this formula, including a second constrained gauge parameter, in order to correctly account for the tensor hierarchy and address issues with closure of the gauge algebra and covariance [38]. Here we consider only a local patch of flat space and ignore these issues, as we are merely looking for signs of agreement in the core part of the object.

subgroup and give the two presentations of the result corresponding to (2.60) and (2.60)

$$\begin{aligned}\mathfrak{spin}(8,8+n) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R} \oplus \mathfrak{spin}(5,5+n)\\ \oplus \Big[\Lambda^2 A^* \otimes \big(\mathbf{1_{+2}} \oplus \square_{\mathbf{-1}}\big)\Big] \oplus \Big[\Lambda^2 A \otimes \big(\mathbf{1_{-2}} \oplus \square_{\mathbf{+1}}\big)\Big]\end{aligned} \tag{2.69}$$

$$\begin{aligned}E \simeq \Big[A^* \otimes \big(\mathbf{1_{-2}} \oplus \square_{\mathbf{+1}}\big)\Big] \oplus (\Lambda^2 A^* \otimes A^*)_0\\ \oplus \Big[\big(\mathbb{R} \oplus \boxminus\big) \otimes \Lambda^3 A^*\Big] \oplus \Big[\Lambda^2 A^* \otimes \Lambda^3 A^* \otimes \big(\mathbf{1_{+2}} \oplus \square_{\mathbf{-1}}\big)\Big]\end{aligned} \tag{2.70}$$

This would correspond to having two-forms transforming in the $\big(\mathbf{1_{+2}} \oplus \square_{\mathbf{-1}}\big)$ representation of $SO(5,5+n)$ together with scalars in the coset $SO(5,5+n) \times \mathbb{R}^+ / SO(5) \times SO(5+n)$. Together with the exotic graviton, this would precisely match the bosonic field content of one $\mathcal{N} = (2,0)$ exotic graviton multiplet together with $(5+n)$ $\mathcal{N} = (2,0)$ tensor multiplets. However, again we see that the representation of the A^* charges in (2.70) does not quite match that of the fields $\Lambda^2 A^*$ in (2.69) as the $\mathbb{R}^+$ weights do not match. Thus again we see a sign that the full $SO(5,5+n) \times \mathbb{R}^+$ may not be a symmetry of any corresponding theory, or that we may not be able to move from $SO(3)$ to $SL(3,\mathbb{R})$ in the usual way.

2.6.2 $\mathcal{N} = (1,0)$ supersymmetry

We can also consider what happens for various theories with eight supercharges which (on reduction to three dimensions) have scalars living in symmetric spaces as for the maximal and half-maximal theories considered above. A list of such theories and their corresponding coset manifolds can be found in [102].

For example, let us first consider pure five-dimensional supergravity. On reduction to three dimensions, we obtain scalars living in the coset space $G_{2(2)}/ SU(2) \times SU(2)$, thus the analogue of the group $E_{8(8)}$ from the maximal case here is $G_{2(2)}$. This has an $SL(3,\mathbb{R})$ subgroup, under which the decomposition of the adjoint representation is

$$\mathfrak{g}_{2(2)} \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \Lambda^2 A^* \oplus \Lambda^2 A \tag{2.71}$$

which would match a theory in six-dimensions with an exotic graviton and a single self-dual two-form. Thus, as expected, this matches the field content of the $\mathcal{N} = (1,0)$ exotic graviton multiplet.

Next, consider pure $\mathcal{N} = (1,0)$ supergravity in six-dimensions, which upon reduction to three-dimensions has scalar manifold $SO(4,3)/ SO(4) \times SO(3)$. The group $SO(4,3)$ again has an $SL(3,\mathbb{R})$ decomposition of the relevant type:

$$\begin{aligned}\mathfrak{so}(4,3) \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}\\ \oplus \Big[\Lambda^2 A^* \otimes (\mathbf{1_{+2}} \oplus \mathbf{1_{-1}})\Big] \oplus \Big[\Lambda^2 A \otimes (\mathbf{1_{-2}} \oplus \mathbf{1_{+1}})\Big]\end{aligned} \tag{2.72}$$

This matches a theory with an exotic graviton, two self-dual two-forms and one scalar, which is the bosonic field content of an exotic graviton multiplet together with one tensor multiplet.

This pattern continues for the other theories outlined in [102]. A more involved example is six-dimensional minimal supergravity coupled to two vector multiplets and two tensor multiplets. On reduction to three dimensions, one obtains the scalar manifold $F_{4(4)}/ Sp(6) \times Sp(2)$. One then looks at the decomposition

$$\begin{aligned}\mathfrak{f}_{4(4)} \simeq \mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})\\ \oplus \Big[\Lambda^2 A^* \otimes \mathbf{6}\Big] \oplus \Big[\Lambda^2 A \otimes \mathbf{6'}\Big]\end{aligned} \tag{2.73}$$

Thus we hypothesise an exotic graviton, self-dual two-forms in the $\mathbf{6}$ representation of $SL(3,\mathbb{R})$ and five scalars in the coset manifold $SL(3,\mathbb{R})/ SO(3)$. This field content matches an exotic graviton multiplet together with five tensor multiplets, and we expect a global symmetry group $SL(3,\mathbb{R})$, modulo the same problems with charges and fields living in different representations.

Table 2.2 summarises the corresponding results for this collection of theories. In all cases, the $SL(3,\mathbb{R})$ subgroup gives a decomposition which exactly matches a combination of an exotic graviton multiplet and some number of tensor multiplets, identifying the conjectured global symmetry group as its commutant. This global symmetry and its coset are precisely those of the corresponding six-dimensional conventional supergravity theory on S^1. If one assumes that the reduction of these theories on S^1 should give the same five-dimensional theory as reducing the standard $\mathcal{N} = (1, 0)$ supergravity then this is inevitable, since the five-dimensional scalars must come only from the six-dimensional scalars of the exotic theory. Below we explain why other features of this table inevitably must work out.

We also note that in all cases but the first row, the charges of the two-forms do not match the representation for the two-forms, as we found in the cases considered in sections 2.5 and 2.6.1. Thus, we again see that the numerator group of the scalar coset may not be a true symmetry of the corresponding theory, or that really one must work under $SO(3)$ to make these match.

Finally, we explain why the decomposition of the the duality group in 3d inevitably has the form

$$\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{k} \oplus (\mathbf{3} \otimes \mathbf{r}) \oplus (\mathbf{3}' \otimes \mathbf{r}') \tag{2.74}$$

if the three-dimensional theory can be written as a torus reduction of a five-dimensional supergravity theory. The existence of the $\mathfrak{sl}(3, \mathbb{R})$ is the usual Ehlers symmetry appearing in the reduction of 5d gravity to three dimensions. Under $GL(2, \mathbb{R})$ this has the form

$$\mathfrak{sl}(3, \mathbb{R}) = (C \otimes C^*) \oplus (\Lambda^2 C^* \otimes C^*) \oplus (\Lambda^2 C \otimes C) \tag{2.75}$$

If the three-dimensional theory comes from the reduction of a five-dimensional supergravity theory, then the only other degrees of freedom are standard scalars and p-form fields. Thus the adjoint can only contain $GL(2, \mathbb{R})$ representations of the form $\Lambda^p C^* \oplus \Lambda^p C^*$ together with scalars and $\mathfrak{sl}(3, \mathbb{R})$ as above. The only options for p are $p = 0, 1, 2, 3$. Any $SL(3, \mathbb{R})$ representation in (2.74) other than $\mathbf{1}$, $\mathbf{3}$ or $\mathbf{3}'$ would give other types of $GL(2, \mathbb{R})$ representations and thus is not allowed. Thus the decomposition (2.74) is universal. Further, once it is known that the degrees of freedom of pure five-dimensional supergravity lift to $\mathcal{N} = (1, 0)$ exotic gravity and both vector and tensor multiplets lift to $\mathcal{N} = (1, 0)$ tensor multiplets, it is clear that this decomposition will match the decomposition of an $\mathcal{N} = (1, 0)$ exotic gravity. Thus the matching of the degrees of freedom between the $SL(3, \mathbb{R})$ decompositions and the exotic gravity theories is inevitable once one assumes that they reduce to those of standard gravitational theories in three dimensions.

3d coset	6d Supergravity			6d Exotic gravity		
	n_V	n_T	6d coset	n_T	$B^-_{\mu\nu}$ rep	6d coset
$\frac{G_{2(2)}}{SU(2)\times SU(2)}$	5d sugra		-	0	$\mathbf{1}$	-
$\frac{SO(4,3)}{SO(4)\times SO(3)}$	0	0	-	1	$\mathbf{1_{+2}}\oplus\mathbf{1_{-1}}$	$\mathbb{R}^+$
$\frac{SO(4,4+n)}{SO(4)\times SO(4+n)}$	n	1	$\mathbb{R}^+$	n+2	$\mathbf{1}_{+2}\oplus\mathbf{n{+}2}_{-1}$	$\frac{SO(1,n+1)}{SO(n+1)}\times\mathbb{R}^+$
$\frac{F_{4(4)}}{Sp(6)\times Sp(2)}$	2	2	$\frac{SO(2,1)}{SO(2)}$	5	$\mathbf{6}$	$\frac{SL(3,\mathbb{R})}{SO(3)}$
$\frac{E_{6(2)}}{SU(6)\times SU(2)}$	4	3	$\frac{SO(3,1)}{SO(3)}$	8	$[\mathbf{3}\otimes\bar{\mathbf{3}}]_{\mathbb{R}}$	$\frac{SL(3,\mathbb{C})}{SU(3)}$
$\frac{E_{7(-5)}}{SO(12)\times SU(2)}$	8	5	$\frac{SO(5,1)}{SO(5)}$	14	$\mathbf{15}$	$\frac{SU^*(6)}{Sp(6)}$
$\frac{E_{8(-24)}}{E_7\times SU(2)}$	16	9	$\frac{SO(9,1)}{SO(9)}$	26	$\mathbf{27}$	$\frac{E_{6(-26)}}{F_4}$

Table 2.2: $\mathcal{N}=(1,0)$ supergravity and exotic supergravity theories and their duality groups

Chapter 3

Chern-Simons couplings and anomalies

3.1 Anomalies of exotic multiplets

Since the exotic multiplets contain chiral fields, they may suffer from anomalies. This section is aimed at extending the results known for chiral spin $\frac{1}{2}$, spin $\frac{3}{2}$ and self-dual fields to the SD Weyl field and the exotic gravitino.

We start by a brief review of the classical results on anomalies. Then we consider a Dirac operator coupled to a vector bundle V, and briefly go through the relation between the index theory and anomalies. Some choices of V are well-understood and relate to the standard anomalies for fields that appear in supergravity multiplets [68–71]. These cases, i.e. the chiral spin 1/2 and 3/2 fermions, and self-dual tensor fields, will be reviewed in subsection 3.1.2, mostly following the conventions of a recent review [48]. As we shall show, the curvatures of all relevant exotic fields, i.e. the SD Weyl field of $(4,0)$ multiplet, its counterpart in the $(3,1)$ multiplet as well as the exotic gravitino can be found in the domain of the Dirac operators for appropriate choices of V. The index calculation for the fields in the $(4,0)$ multiplet will be presented in subsection 3.1.4. The anomaly polynomials for other six-dimensional exotic multiplets with different number of supercharges will be given in subsection 3.1.5.

3.1.1 Consistent anomaly and descent equations

In spacetime dimension $D = 2n$, we denote by Φ the collection of quantum fields sensitive to anomalies and A is the "external" gauge field to which the fields in Φ are coupled. In the case of gravitational anomalies, A can be taken as the spin connection for local Lorentz symmetry or the Christoffel connection for diffeomorphisms. The partition function $Z[A]$ is a functional of A defined as the functional integral

$$Z[A] = e^{-\Gamma[A]} = \int \mathcal{D}\Phi \, e^{-S[\Phi,A]} \tag{3.1}$$

where $S[\Phi, A]$ is action and we only integrate over the fields in Φ which explains the notion "external" for A. Note that, in the above integral we are already in the Euclidean signature.[1] The action is then assumed to be invariant under some gauge transformation with parameter $\epsilon = \epsilon(x)$

$$\begin{aligned} \Phi \to \Phi' &= \Phi + \delta_\epsilon \Phi \\ A \to A' &= A + \delta_\epsilon A \\ S[\Phi', A'] &= S[\Phi, A] \,. \end{aligned} \tag{3.2}$$

In general, the path integral $\mathcal{D}\Phi$ measure is not necessarily invariant under such transformations and we write

$$\mathcal{D}\Phi \to \mathcal{D}\Phi' = \mathcal{D}\Phi \, [\det(J)]^c = \mathcal{D}\Phi e^{\int \epsilon(x)\cdot\mathcal{A}(x)\, d^D x} \tag{3.3}$$

[1] See appendix B.2 for the definitions and conventions.

where J is the Jacobian for the transformation and $c = 1$ for the transformation of a commuting field while $c = -1$ for a pair of anti-commuting fields.

For A' we have

$$\begin{aligned} e^{-\Gamma[A']} &= \int \mathcal{D}\Phi \, e^{-S[\Phi,A']} \\ &= \int \mathcal{D}\Phi' \, e^{-S[\Phi',A']} \\ &= \int \mathcal{D}\Phi \, e^{-S[\Phi,A]+\int \epsilon(x)\mathcal{A}(x)\, d^D x} \\ &= e^{-\Gamma[A]+\int \epsilon(x)\cdot\mathcal{A}(x)\, d^D x} \end{aligned} \tag{3.4}$$

and it follows

$$\delta_\epsilon \Gamma[A] \equiv \Gamma[A'] - \Gamma[A] = \int -\epsilon(x)\cdot \mathcal{A}(x)\, d^D x \,. \tag{3.5}$$

Here, we only used the gauge invariance of the action and renamed the dummy index of integration in the second step of (3.4). The integrand in (3.5) is a local function and it is the anomaly[2] which we are looking for. We will write it as $I^1_{2n}(\epsilon, A)$ or just simply as I^1_{2n} omitting the variables

$$\delta\Gamma = \int_{M_{2n}} I^1_{2n} \tag{3.6}$$

where the superscript means the ghost number when formulated in the BRST scheme, and it is 1 for $I^1_{2n}(\epsilon, A)$ because we replace the gauge parameter ϵ by a ghost gauge function v [69]. Clearly, the anomaly I^1_{2n} is defined in this way up to a exact term dG_{2n-1} (and also up to an "s"-exact piece that we are going to explain below). This local function $I^1_{2n}(\epsilon, A)$ is called the consistent anomaly, due to the fact that it is a solution of the Wess-Zumino consistency condition (see e.g. [48] for a proof)

$$\delta_{\epsilon_1}\delta_{\epsilon_2}\Gamma - \delta_{\epsilon_2}\delta_{\epsilon_1}\Gamma = \delta_{[\epsilon_1,\epsilon_2]}\Gamma \,, \tag{3.7}$$

when writing with $I^1_D(\epsilon, A)$ this condition is

$$\delta_{\epsilon_1}\int I^1_{2n}(\epsilon_2, A) - \delta_{\epsilon_2}\int I^1_{2n}(\epsilon_1, A) = \int I^1_{2n}([\epsilon_1,\epsilon_2], A)\,. \tag{3.8}$$

This consistency condition indicates that the gauge variation of the effective action Γ reflects the gauge algebra structure of the theory. If one starts from the equation (3.8) and wants to compute the anomalies, it is to look for a non-trivial solution (not a gauge variation of a local functional in the gauge fields, in that case one can introduce a local counterterm in the Lagrangian to remove the anomaly) of the Wess-Zumino consistency condition.

We will not give much details about how to solve the equation and there is a very comprehensive approach in [69] and most of our conventions also follow this paper. The key ingredient of the solution to (3.8) is the descent equations (formulated in the BRST language)

$$\begin{aligned} I_{2n+2} &= dI_{2n+1} \\ s\, I_{2n+1} &= dI^1_{2n} \\ sI^1_{2n} &= dI^2_{2n-1} \\ &\vdots \\ sI^{2n+1}_0 &= 0 \end{aligned}$$

where s is the BRST-operator and the result is that $\int I^1_{2n}(v, A)$ is a representative of the BRST cohomology at ghost number one and the whole integral is defined up to a BRST-exact term. The integrand $I^1_{2n}(v, A)$ is not only determined up to a d-exact term but also an s-exact term

$$I^1_{2n}(v, A) \simeq I^1_{2n}(v, A) + dG^1_{2n-1} + sF_{2n}\,. \tag{3.9}$$

[2]In the literature, also the whole integral which is a functional is sometimes called the anomaly.

In practice, it is much easier to manipulate and compare the top term I_{2n+2}, it is also what we refer to as anomaly polynomial. Fortunately, this term is related to the index density $\mathrm{Ind}(\mathcal{D})$ of a Dirac operator D in $2n+2$ dimensions [69, 70]

$$I_{2n+2} = [\mathrm{Ind}(\mathcal{D})]_{2n+2} \tag{3.10}$$

where the index density $\mathrm{Ind}(\mathcal{D})$ is expressed in characteristic classes and it is a polyform of different degrees. The subscript "$2n+2$" means to restrict to the $(2n+2)$-form part.

3.1.2 Anomalies in standard supergravity fields

In last section we discussed the consistent anomalies and its relation to the index of some Dirac operators. Now we move to a concrete scenario, i.e. supergravities and talk about gravitational anomalies.

Pure gravitational anomalies only arise in spacetime dimension $D = 2n = 4k+2$ [68], and the anomalies in $4k+2$ dimensional theories are encoded by characteristic classes in $4k+4$ dimensions, which can be computed using the index theorems for the Dirac operators.

Suppose our space-time manifold with Euclidean signature has a spin structure and let S be the spinor bundle. Then the Dirac operator on the smooth section of the spinor bundle $\mathcal{C}^\infty(S)$ is defined as the composition

$$\mathcal{D} = cl \circ \nabla^S : \mathcal{C}^\infty(S) \xrightarrow{\nabla^S} \mathcal{C}^\infty(T^*M \otimes S) \xrightarrow{cl} \mathcal{C}^\infty(S), \tag{3.11}$$

where ∇^S is the spin connection and cl is the Clifford multiplication. In local coordinates, this is the Dirac trace of the covariant derivative in some representations of the gamma matrices[3]

$$\mathcal{D} = cl(e^\mu)\nabla^S_{e_\mu} = \gamma^\mu \nabla^S_\mu \,. \tag{3.12}$$

In space-time dimension $4k+2$, the spinor bundle decomposes into subbundles of definite chiralities with respect to the Euclidean chirality operator $\Gamma = i^{2k+1}\gamma^0\gamma^1\cdots\gamma^{4k+1}$, i.e. $S = S^+ \oplus S^-$. Consequently, $\mathcal{D}$ takes an off-diagonal form:

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix} \tag{3.13}$$

and the relevant positively projected Dirac operator flips the chirality of the spinor field

$$\mathcal{D}^+ : \mathcal{C}^\infty(S^+) \longrightarrow \mathcal{C}^\infty(S^-). \tag{3.14}$$

Since the full Dirac operator $\mathcal{D}$ is self-adjoint, it has always vanishing index. It is $\mathcal{D}^+$, whose adjoint is $\mathcal{D}^- : \mathcal{C}^\infty(S^-) \longrightarrow \mathcal{C}^\infty(S^+)$, that has a non-trivial index. For the rest of the paper we shall omit the superscript $+$ and use $\mathcal{D}$ to denote the appropriate Dirac operator.

The Dirac operator can be twisted by some vector bundle V (i.e. act on spinors coupled to some vector gauge field) .

$$\mathcal{D} : \mathcal{C}^\infty(S^+ \otimes V) \longrightarrow \mathcal{C}^\infty(S^- \otimes V) \tag{3.15}$$

Applying the index theorem [103], its index density is given by[4]

$$\mathrm{Ind}(\mathcal{D}) = \hat{A}(M)\mathrm{ch}(V)\,, \tag{3.16}$$

where $\hat{A}(M)$ is the roof genus and $\mathrm{ch}(V)$ is the Chern character, see appendix B.2 for their definitions.

Furthermore, one can also generalise the definition of the Dirac operator to the Clifford module E, a vector bundle whose fiber admits a Clifford action. In the definition (3.11) we just replace

[3] In section 3.1 we use indices μ, ν for the $4k+2$ dimensional spacetime.

[4] In the literature, this formula is also written as $\mathrm{Ind}(D) = \hat{A}(M)\,\mathrm{ch}(R)\,\mathrm{ch}(F)$, the first two factor refer to pure gravitational anomalies and R stands for the curvature 2-form in some tensor bundle of the Lorentz group. The third factor is responsible for gauge anomalies and F is the curvature of the gauge bundle, and it is absent if the fermions do not couple to any gauge fields.

the Clifford multiplication cl by the Clifford action and replace the spin connection ∇^S by the connection ∇^E on E.

To talk about the index theory of the generalised Dirac operator we would like to put it in the twisted form (3.15). If our even-dimensional base manifold is spin and oriented, then every E Clifford module has a product structure $E = S \otimes V$, where V is a vector bundle determined by E, S and the Clifford action on E [104]. By making use of the chiral decomposition of S we define $E^\pm := S^\pm \otimes V$ and thus

$$\mathcal{D}_E : \mathcal{C}^\infty(E^+) \longrightarrow \mathcal{C}^\infty(E^-)\,. \tag{3.17}$$

A pertinent example of Clifford module is given by the bundle of differential forms $\Lambda^\bullet T^*M$, which is a tensor product of spinor bundles $\Lambda^\bullet T^*M = S \otimes S$. The sections of S are spinors transforming in the spinor representation of $SO(4k+2)$. One could further restrict to the chiral S^+ and anti-chiral S^- subrepresentations and obtain

$$S^\pm \otimes S^\pm = \Lambda^1 T^*M \oplus \Lambda^3 T^*M \oplus \ldots \oplus \Lambda^{2k+1}_\pm T^*M \tag{3.18}$$

and

$$S^+ \otimes S^- = \Lambda^0 T^*M \oplus \Lambda^2 T^*M \oplus \ldots \oplus \Lambda^{2k} T^*M \tag{3.19}$$

where $\Lambda^{2k+1}_\pm T^*M$ are the self-dual (anti-self-dual) forms. In Euclidean signature a n-form $F_{\mu_1\ldots\mu_n}$ is self-dual if it obeys $F_{\mu_1\ldots\mu_n} = \frac{i}{n!}\epsilon_{\mu_1\ldots\mu_{2n}} F^{\mu_{n+1}\ldots\mu_{2n}}$.

The Hirzebruch signature operator is given by

$$\tau : \mathcal{C}^\infty(S^+ \otimes (S^+ \oplus S^-)) \longrightarrow \mathcal{C}^\infty(S^- \otimes (S^+ \oplus S^-)) \tag{3.20}$$

with $V = S^+ \otimes S^-$ (cf (3.16)), and its index is given by the Hirzebruch L-polynomial. From other side, the complexifications of self-dual even forms and anti-self-dual odd forms are given by $S^+ \otimes S^-$ and $S^- \otimes S^-$ respectively, and we are interested in the index of

$$\mathcal{D}_A : \mathcal{C}^\infty(S^+ \otimes S^-) \longrightarrow \mathcal{C}^\infty(S^- \otimes S^-) \tag{3.21}$$

with $V = S^-$. It can be shown that the result for the index is equal to half of the Hirzebruch L-polynomial with an additional $-$ sign due to Bose rather than Fermi statistics, and is given by

$$I^A_{2n+2} = \left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)[\hat{A}(M)\,\mathrm{ch}(\tilde{R})]_{2n+2} = \left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)[L(M)]_{2n+2} \tag{3.22}$$

where $\tilde{R} = \frac{1}{2}R_{\mu\nu}\gamma^{\mu\nu}$ with R being the Riemann tensor of M and $\gamma^{\mu\nu}$ the generator in the spinor representation. The pre-factor factorizes $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$, where the first $\frac{1}{2}$ due is to the chirality projector of the second spinor and the second $\frac{1}{2}$ comes from the constraint that we consider F as a real field when analytically continuing to Lorentzian signature.

For the gravitino field, the relevant Rarita-Schwinger complex is given by

$$\mathcal{C}^\infty(S^+ \otimes T^*M) \longrightarrow \mathcal{C}^\infty(S^- \otimes T^*M) \tag{3.23}$$

The gravitino anomalies are actually given by the map:

$$\mathcal{D} : \mathcal{C}^\infty(S^+ \otimes (T^*M - 1)) \longrightarrow \mathcal{C}^\infty(S^- \otimes (T^*M - 1))\,. \tag{3.24}$$

The origin of this formal shift is explained in [68] and we shall come back to it in the next subsection. The tensor product $S^+ \otimes T^*M$ contains an anti-chiral spinor S^- that needs to be projected out. In addition a vector potential in D dimensions has $D-2$ physical degrees of freedom. These together lead to the $-\hat{A}(M)$ in the expression for the index:

$$I^{\mathrm{spin}\frac{3}{2}}_{2n+2} = [\hat{A}(M)\,\mathrm{ch}(R) - 2\hat{A}(M) + \hat{A}(M)]_{2n+2} = [\hat{A}(M)(\mathrm{ch}(R) - 1)]_{2n+2}\,, \tag{3.25}$$

where R is the curvature two-form in the vector representation of $SO(4k+2)$.

3.1.3 Anomalies for product multiplets

Many supergravity theories can be seen as products of Yang-Mills multiplets with less supersymmetry. In cases when the resulting supergravity is chiral, the anomalous part of the spectrum can be analysed like in the previous subsection. All the fields are in the domain of a Dirac operator with choices of V being given by the tangent bundle or (products of) spin bundles. As a result all standard supermultiplets have anomalies of very constrained form.

Type IIB supergravity is a prime example of such a product theory, and can be obtained as a double copy two $(1,0)$ Yang-Mills multiplets. The anomalous part (a couple of left gravitini, two right dilatini and a tensor field with a self-dual five-form field strength) is given by

$$\lambda^L \circ A_\mu + A_\mu \circ \lambda^L + \lambda^L \circ \lambda^L$$

Hereafter we shall use $\circ$ to denote the products of fields. As already mentioned $\lambda^L \circ A_\mu$ projects into the left gravitino and a right spin 1/2 field. Note that both are in the IIB spectrum, and one only needs to worry about the subtraction of 2 vectorial degrees of freedom. The whole IIB anomalous complex can the be thought of as

$$\mathcal{C}^\infty(S^+ \otimes (2 \times (T^*M - 2) \oplus S^+)) \longrightarrow \mathcal{C}^\infty(S^- \otimes (2 \times (T^*M - 2) \oplus S^+)) \,. \tag{3.26}$$

with the resulting anomaly given by the 12-form

$$I^{\text{IIB}} = -\Big[\hat{A}(M)\Big(\text{ch}(R) - 2 - \frac{1}{8}\,\text{ch}(\tilde{R})\Big)\Big]_{12} \tag{3.27}$$

that vanishes [68].

The reduction of IIB on a $K3$ surface yields a six-dimensional $(2,0)$ theory that contains a supergravity multiplet and 21 tensor multiplets and is also anomaly free. One can also see that the non-chiral and obviously non-anomalous maximal $(2,2)$ supergravity can be decomposed into $(2,0)$ multiplets and contains a $(2,0)$ gravity multiplet, together with four gravitino multiplets and five tensor multiplets. Hence the three standard $(2,0)$ multiplets have anomaly polynomials that are proportional

$$-\frac{1}{21} I_{\text{gravity}} = \frac{1}{4} I_{\text{gravitino}} = I_{\text{tensor}} := X_8 = \frac{1}{48}\left(\frac{p_1^2}{4} - p_2\right). \tag{3.28}$$

Because of the M5-brane anomalies and inflow, the X_8 polynomial appears in the M-theory action via gravitational Chern-Simons couplings. The contraction structure in X_8 is given by the t_8 tensor that appears naturally in the string amplitudes.

Working directly with six-dimensional multiplets, we note that the product of two $\mathcal{N} = (1,0)$ vector multiplets is a sum of $\mathcal{N} = (2,0)$ gravity and tensor multiplets:

Table 3.1: 6d $\mathcal{N} = (1,0)$ Yang-Mills squared

$\mathcal{N} = (1,0)$	A_μ vector $(\mathbf{2},\mathbf{2};\mathbf{1})$	λ^L chiral fermion $(\mathbf{1},\mathbf{2};\mathbf{2})$
A_μ vector $(\mathbf{2},\mathbf{2};\mathbf{1})$	$g_{\mu\nu}$ $(\mathbf{3},\mathbf{3};\mathbf{1})$ ${B_{\mu\nu}}^-$ $(\mathbf{3},\mathbf{1};\mathbf{1})$ ${B_{\mu\nu}}^+$ $(\mathbf{1},\mathbf{3};\mathbf{1})$ ϕ $(\mathbf{1},\mathbf{1};\mathbf{1})$	ψ_μ^L $(\mathbf{2},\mathbf{3};\mathbf{2})$ λ^R $(\mathbf{2},\mathbf{1};\mathbf{2})$
λ^L chiral fermion $(\mathbf{1},\mathbf{2};\mathbf{2})$	ψ_μ^L $(\mathbf{2},\mathbf{3};\mathbf{2})$ λ^R $(\mathbf{2},\mathbf{1};\mathbf{2})$	ϕ $(\mathbf{1},\mathbf{1};\mathbf{4})$ ${B_{\mu\nu}}^+$ $(\mathbf{1},\mathbf{3};\mathbf{4})$

The anomalous part of the product is given by

$$(A_\mu + 2 \times \lambda^L) \circ (A_\mu + 2 \times \lambda^L) \Rightarrow 2\lambda^L \circ A_\mu + 2A_\mu \circ \lambda^L + 4\lambda^L \circ \lambda^L, \tag{3.29}$$

and like in the ten-dimensional case, $\lambda^L \circ A_\mu$ and $A_\mu \circ \lambda^L$ contains the left-moving gravitini and the right-moving tensorini, which are in the spectrum with a net contribution to the anomaly given by

$-\frac{1}{2}\hat{A}(TM)[\mathrm{ch}(R)-2]$. The product of the two chiral spinors λ^L results in a self-dual 2-form. The total anomaly of is

$$\begin{aligned} &-2\cdot 2\cdot\frac{1}{2}\left[\hat{A}(TM)[\mathrm{ch}(R)-2]\right]_8 - 4I^A \\ =&-\frac{1}{5760}\left(536{p_1}^2-1952p_2\right)-4\cdot\frac{1}{5760}\left(16{p_1}^2-112p_2\right) = -20X_8 \end{aligned} \tag{3.30}$$

Another six-dimensional example dimensions is the studied in [8], where the tensor product of super Yang-Mills multiplets with $\mathcal{N}=(1,0)$ and $\mathcal{N}=(1,1)$ supersymmetries is shown to yield the gravity multiplet in $\mathcal{N}=(2,1)$. The details of the tensor product are summarised in the following table:

Table 3.2: 6d $\mathcal{N}=(1,0)$ Yang-Mills tensor with $\mathcal{N}=(1,1)$ Yang-Mills

	$\mathcal{N}=(1,1)$ A_μ $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$	λ^R $(\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{2})$	λ^L $(\mathbf{1},\mathbf{2};\mathbf{2},\mathbf{1})$	ϕ $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$
$\mathcal{N}=(1,0)$ A_μ $(\mathbf{2},\mathbf{2};\mathbf{1})$	$g_{\mu\nu}$ $(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$ ${B_{\mu\nu}}^-$ $(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})$ ${B_{\mu\nu}}^+$ $(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$ ϕ $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	ψ^R_μ $(\mathbf{3},\mathbf{2};\mathbf{1},\mathbf{2})$ λ^L $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$	ψ^L_μ $(\mathbf{2},\mathbf{3};\mathbf{2},\mathbf{1})$ λ^R $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$	A_μ $(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$
λ^L $(\mathbf{1},\mathbf{2};\mathbf{1})$	ψ^L_μ $(\mathbf{2},\mathbf{3};\mathbf{2},\mathbf{1})$ λ^R $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$	A_μ $(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	ϕ $(\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{1})$ ${B_{\mu\nu}}^+$ $(\mathbf{1},\mathbf{3};\mathbf{4},\mathbf{1})$	λ^L $(\mathbf{1},\mathbf{2};\mathbf{4},\mathbf{2})$

The resulting $\mathcal{N}=(2,1)$ supergravity multiplet [20] contains one graviton $g_{\mu\nu}$, 4 left-handed gravitini ψ^L_μ, 2 right-handed gravitini ψ^R_μ, 8 vectors A_μ, one anti-self-dual 2-form ${B_{\mu\nu}}^-$, 5 self-dual 2-form ${B_{\mu\nu}}^+$, 4 right-handed fermions λ^R, 10 left-handed fermions λ^L and 5 scalars ϕ. We may once more consider the anomaly as the sum of the anomalies from individual terms in the product

$$\begin{aligned} & (A_\mu + 2\times\lambda^L)\circ(A_\mu + 2\times\lambda^R + 2\times\lambda^L + 4\times\phi) \\ \Rightarrow\ & 2A_\mu\circ\lambda^R + 2A_\mu\circ\lambda^L + 2\lambda^L\circ A_\mu + 4\lambda^L\circ\lambda^R + 4\lambda^L\circ\lambda^L + 8\lambda^L\circ\phi \\ =\ & 2\lambda^L\circ A_\mu + 4\lambda^L\circ\lambda^L + 8\lambda^L\circ\phi\,, \end{aligned} \tag{3.31}$$

where only the anomalous terms are kept. The total anomaly is given by

$$\begin{aligned} I^{\text{product}} &= -2\cdot\frac{1}{2}\left[\hat{A}(TM)[\mathrm{ch}(R)-2]\right]_8 - 4I^A - 8\cdot\frac{1}{2}I^{\text{spin}\frac{1}{2}} \\ &= -\frac{1}{5760}\left(268{p_1}^2-976p_2\right)-4\cdot\frac{1}{5760}\left(16{p_1}^2-112p_2\right)-4\cdot\frac{1}{5760}\left(7{p_1}^2-4p_2\right) \\ &= -\frac{360}{5760}(p_1^2-4p_2) \end{aligned} \tag{3.32}$$

and agrees with the direct calculation

$$\begin{aligned} I^{\text{gravity}}_{\mathcal{N}=(2,1)} &= \frac{1}{2}\cdot(-4+2)I^{\text{spin}\frac{3}{2}} + (1-5)I^A + \frac{1}{2}(4-10)I^{\text{spin}\frac{1}{2}} \\ &= -I^{\text{spin}\frac{3}{2}} - 4I^A - 3I^{\text{spin}\frac{1}{2}} = -12X_8\,. \end{aligned} \tag{3.33}$$

3.1.4 Index densities of exotic Dirac operators

The indices of the exotic fields (and multiplets) can be computed using (3.16). The only essential difference from the calculations reviewed above is that V is now given by a product of bundles. In the $\mathcal{N}=(4,0)$ multiplet (1.2) there are two exotic anomalous objects, namely the exotic gravitino $\psi_{\mu\nu}$ in $(\mathbf{4},\mathbf{1};\mathbf{8})$ and the exotic graviton $C_{\mu\nu\rho\sigma}$ in $(\mathbf{5},\mathbf{1};\mathbf{1})$. We treat each in turn.

3.1.4.1 Exotic gravitino

We start with the fermion $\psi_{\mu\nu}$. The field strength χ is anti-self-dual with respect to $SO(6)$:[5]

$$\chi = -i \star_{\mathrm{E}} \chi \Longleftrightarrow \chi_{\mu\nu\rho} = -\frac{i}{3!}\epsilon_{\mu\nu\rho\alpha\beta\gamma}\chi^{\alpha\beta\gamma} \tag{3.34}$$

where the Hodge-star $\star_{\mathrm{E}}$ is taken in the Euclidean convention.

The advantage of working directly with χ is that we do not need to worry about the ghost contribution, and the calculation follows the treatment of the self-dual forms [68] (see also [71]). See chapter 6 section 6.2 for a clear overlook about the ghost spectrum and the computation for anomalies in that context. In [68], a generic potential A and its field strength F are viewed as independent variables in the path-integral formalism, and one integrates over both of them by using a first order action. In the path-integral measure, the self-dual F^+ part and anti-self-dual part F^- both appear, the contribution to the anomalies from them cancel each other. But one can extract anomaly for the (anti-)self-dual part alone as the Jacobian generated by it under transformations of the Lorentz group in the corresponding (anti-)self-dual representation. Since there is no gauge freedom in F^+ or F^-, there is no need to subtract ghost contributions. Back to our case with ψ and χ, a first order Lagrangian formalism in which χ is algebraic is yet unknown, but we assume its existence and compute in the spirit of [68].

To compute the anomaly via the field strength $\chi_{\mu\nu\rho}$, we need to determine in which representation of the Lorentz group (orthogonal group) it transforms.

Recall that the Dynkin label of the negative chiral spinor representation of $\mathfrak{su}^*(4)$ is $[0,0,1]$.[6] The field strength $\chi_{\mu\nu\rho}$ is fermionic and it transforms as a part in the tensor product of the anti-self-dual 3 form $[0,0,2]$ with the anti-chiral spinor $[0,0,1]$:

$$[0,0,2] \otimes [0,0,1] = [0,0,3] \oplus [0,1,1]\,, \tag{3.35}$$

and the duality constraint (3.34) alone does not imply that $\chi_{\mu\nu\rho}$ transforms as the irreducible piece $[0,0,3]$. We mentioned this fact in chapter 1 section 1.1 (see [11] for more details), the duality condition is weaker than the field equation (1.14)

$$\gamma^{\alpha\beta\mu\nu\rho}\chi_{\mu\nu\rho} = 0 \tag{3.36}$$

which is equivalent to (see appendix B.3)

$$\gamma^{\mu}\chi_{\mu\nu\rho} = 0\,. \tag{3.37}$$

The condition (3.37) together with the chirality imply that $\chi_{\mu\nu\rho}$ transforms in the irreducible representation $[0,0,3]$ of $\mathfrak{su}^*(4)$. This can be seen as follows, if we decompose the field strength as

$$\chi_{\mu\nu\rho} = \hat{\chi}_{\mu\nu\rho} + \gamma_{[\mu}\sigma_{\nu\rho]} + \gamma_{[\mu\nu}\epsilon_{\rho]} + \gamma_{\mu\nu\rho}\eta \tag{3.38}$$

where $\gamma^{\mu}\hat{\chi}_{\mu\nu\rho} = 0 = \gamma^{\nu}\sigma_{\nu\rho} = \gamma^{\rho}\epsilon_{\rho}$. The equation (3.37) would set the variables $\{\sigma_{\nu\rho}, \epsilon_{\rho}, \eta\}$ to zero, it is the gamma-traceless part $\hat{\chi}_{\mu\nu\rho}$ that corresponds to $[0,0,3]$. Since the field equation (1.14) leads to the little group representation $(\mathbf{4},\mathbf{1};\mathbf{8})$, we will refer to the anomaly computation for $[0,0,3]$ as the "on-shell" anomaly computation, i.e. we used the field equation to navigate to the correct representation.

On the other hand, the duality constraint (3.34) alone will kill the components $\sigma_{\nu\rho}$ and η (the computational details can be find in appendix B.3), so

$$\chi_{\mu\nu\rho} = \hat{\chi}_{\mu\nu\rho} + \gamma_{[\mu\nu}\epsilon_{\rho]}\,. \tag{3.39}$$

This just match the decomposition of the tensor product (3.35) and $\epsilon_{\rho} \simeq [0,1,1]$, it is a gamma-traceless 1-form spinor. We will refer to the anomaly computation for $\chi_{\mu\nu\rho} = \hat{\chi}_{\mu\nu\rho} + \gamma_{[\mu\nu}\epsilon_{\rho]}$ as the "off-shell" anomaly computation, because in this case the we introduce $\chi_{\mu\nu\rho}$ as an independent

[5]Note that in the Euclidean space-time it is anti-self-dual and it is self-dual in the Minkowskian cases. Similarly, left-handed spinors have negative chirality in the Euclidean space-time, while they are right-handed with positive chirality in Minkowskian as explained in appendix B.2.

[6]Our conventions for the Dynkin labels are outlined in appendix B.2.

variable in the path integral and only require $\chi_{\mu\nu\rho}$ to be anti-self-dual. We are integrating over both $\hat{\chi}_{\mu\nu\rho}$ and ϵ_ρ. The difference between the on- and off-shell computations is just the index density contribution from $[0,1,1]$.

From now on we focus on the index theory anomaly computation for the irreducible piece $[0,0,3]$. It is convenient to have it in the tri-spinor product $(S^-)^{\otimes 3}$.

$$[0,0,1]\otimes[0,0,1]\otimes[0,0,1]=[0,0,3]\oplus[0,1,1]\oplus[0,1,1]\oplus[1,0,0]$$

We can recast the result for representations of $\mathfrak{su}^*(4)$ in terms of the sections of the corresponding bundles:

$$\begin{aligned}\mathcal{D}_\chi &: \mathcal{C}^\infty(S^+\otimes S^-\otimes S^-)-\mathcal{C}^\infty(S^+\otimes T^*M)-\mathcal{C}^\infty(S^+\otimes T^*M)+\mathcal{C}^\infty(S^-)\\ &\longrightarrow \mathcal{C}^\infty(S^-\otimes S^-\otimes S^-)-\mathcal{C}^\infty(S^-\otimes T^*M)-\mathcal{C}^\infty(S^-\otimes T^*M)+\mathcal{C}^\infty(S^+)\end{aligned}$$

leading to the definition of the complex for the exotic gravitino

$$\Rightarrow \mathcal{D}_\chi : \mathcal{C}^\infty(S^+\otimes[S^-\otimes S^- - T^*M^{\oplus 2}-1])\longrightarrow \mathcal{C}^\infty(S^-\otimes[S^-\otimes S^- - T^*M^{\oplus 2}-1]). \quad (3.40)$$

The formal manipulation above is allowed in K-theory [103], and effectively we have the index theorem for the index density of $\mathcal{D}_\chi$

$$\begin{aligned}\mathrm{Ind}(\mathcal{D}_\chi) &= \hat{A}(M)[\mathrm{ch}((S^-)^{\otimes 2})-\mathrm{ch}(T^*M^{\oplus 2})-1]\\ &= \hat{A}(M)[\mathrm{ch}(S^-)^2-2\mathrm{ch}(T^*M)-1]\end{aligned} \quad (3.41)$$

According to the famous results [103], we have

$$\mathrm{ch}(S^+\oplus S^-)=\prod_{j=1}^n 2\cosh\frac{x_j}{2} \quad \text{and} \quad \mathrm{ch}(S^+-S^-)=\prod_{j=1}^n 2\sinh\frac{x_j}{2} \quad (3.42)$$

for the space-time manifold in $2n$ dimensions. It follows that

$$\mathrm{ch}(S^+)=\frac{1}{2}\left(\prod_{j=1}^n 2\cosh\frac{x_j}{2}+\prod_{j=1}^n 2\sinh\frac{x_j}{2}\right) \quad (3.43)$$

$$\mathrm{ch}(S^-)=\frac{1}{2}\left(\prod_{j=1}^n 2\cosh\frac{x_j}{2}-\prod_{j=1}^n 2\sinh\frac{x_j}{2}\right) \quad (3.44)$$

Inserting this into (3.41) and using the relation (B.25), we arrive at

$$[\mathrm{Ind}(\mathcal{D}_\chi)]_8=\frac{1}{5760}(501p_1^2+3828p_2)\,. \quad (3.45)$$

The contribution to the gravitational anomaly from χ is obtained from the above result by multiplying it by $(-1)^2\frac{1}{2}$. The first -1 comes from the fact that χ is fermionic and the second -1 is because the map in (3.40) is actually in the opposite direction [71]. The division by 2 is due to the fact that self-dual tensor in Lorentz signature satisfies the reality condition.

$$I_\chi=(-1)^2\frac{1}{2}\left[\mathrm{Ind}(D_\chi)\right]_8=\frac{1}{5760}(\frac{501}{2}p_1^2+1914p_2). \quad (3.46)$$

3.1.4.2 SD Weyl field

We now turn to the index density of the field strength of the exotic graviton defined in (1.7), $G_{\mu\nu\rho\sigma\tau\kappa}=\partial_{[\mu}C_{\nu\rho][\sigma\tau,\kappa]}$. The computation is also under the assumption of the existence of an action with G as an independent variable. A remarkable fact for the bosonic case is that the self-duality condition (1.11) is stronger than the single trace field equation (1.59), it halves the degrees of freedom determined by the single trace field equation. Importantly, the symmetry, single traceless condition and the self-duality condition together, imply that $G_{\mu\nu\rho\sigma\tau\kappa}$ transforms in an irreducible representation of $\mathfrak{su}^*(4)$ [81].

We conclude that G transforms in the $[0,0,4]$ of $\mathfrak{su}^*(4)$, and in order to obtain it from a tensor product, one can take a pair of the field strengths F_3^- of self-dual 2-forms:

$$[0,0,2] \otimes [0,0,2] = [0,0,4] \oplus [0,1,2] \oplus [0,2,0]. \tag{3.47}$$

For the $[0,1,2]$ part,

$$[0,1,0] \otimes [0,0,2] = [0,1,2] \oplus [1,0,1]. \tag{3.48}$$

The representations $[0,2,0]$ and $[1,0,1]$ are immediately recognised as the metric $g_{(\mu\nu)}$ and the two-form $B_{[\mu\nu]}$ respectively. The individual $[0,0,2]$ appears also as an irreducible part in the tensor product of 2 negative chirality spinors:

$$[0,0,1] \otimes [0,0,1] = [0,1,0] \oplus [0,0,2]. \tag{3.49}$$

We can consider a product of four chiral spinors $[0,0,1]$ and, applying the tensor product decomposition, obtain

$$\begin{aligned} [0,0,1]^{\otimes 4} &= ([0,1,0] \oplus [0,0,2]) \otimes ([0,1,0] \oplus [0,0,2]) \\ &= ([0,1,0] \otimes [0,1,0]) \oplus ([0,1,0] \otimes [0,0,2]) \oplus ([0,0,2] \otimes [0,1,0]) \\ &\quad \oplus [0,0,4] \oplus [0,1,2] \oplus [0,2,0], \end{aligned} \tag{3.50}$$

where (3.47) is used to get the last three terms.

The $[0,0,4]$ can now be extracted, and the result can be recast in terms of sections of corresponding bundles. The details of this calculation can be found in Appendix B.4. The resulting complex for $\mathcal{D}_G$ operator is given by

$$\begin{aligned} \mathcal{D}_G : \mathcal{C}^\infty &\left(S^+ \otimes [S^- \otimes S^- \otimes S^- - (S^- \otimes T^*M)^{\oplus 3} + (S^+)^{\oplus 2}]\right) + B + g \\ &\longrightarrow \mathcal{C}^\infty \left(S^- \otimes [S^- \otimes S^- \otimes S^- - (S^- \otimes T^*M)^{\oplus 3} + (S^+)^{\oplus 2}]\right) + B + g. \end{aligned} \tag{3.51}$$

At this stage, we can state that the sections to which B and g belong do not contribute to the index density. Simply said, the metric and a generic two-form field are anomaly free. It follows the relevant complex is

$$\begin{aligned} \mathcal{D}_G : \mathcal{C}^\infty &\left(S^+ \otimes [S^- \otimes S^- \otimes S^- - (S^- \otimes T^*M)^{\oplus 3} + (S^+)^{\oplus 2}]\right) \\ &\longrightarrow \mathcal{C}^\infty \left(S^- \otimes [S^- \otimes S^- \otimes S^- - (S^- \otimes T^*M)^{\oplus 3} + (S^+)^{\oplus 2}]\right), \end{aligned} \tag{3.52}$$

and is again in the form (3.15). The index density for $\mathcal{D}_G$ is then

$$\mathrm{Ind}(\mathcal{D}_G) = \hat{A}(M)\left(\mathrm{ch}((S^-)^3) - 3\mathrm{ch}(S^-)\mathrm{ch}(T^*M) + 2\mathrm{ch}(S^+)\right) \tag{3.53}$$

Every individual factor is known and one can show that

$$[\mathrm{Ind}(\mathcal{D}_G)]_8 = \frac{1}{3}(2p_1^2 + 10p_2) = \frac{1}{5760}(3840p_1^2 + 19200p_2). \tag{3.54}$$

Since G is bosonic and the reality condition is imposed on it in order to move to the Minkowski signature, the anomaly for the field strength is

$$I_G = (-1)(\frac{1}{2})\mathrm{Ind}(\mathcal{D}_G) = \frac{1}{5760}(-1920p_1^2 - 9600p_2)\,. \tag{3.55}$$

3.1.4.3 Exotic graviton in the $(3,1)$ multiplet

The field strength of the three-index exotic graviton D in the $(3,1)$ multiplet $S_{\mu\nu\rho\sigma\kappa} = \partial_{[\mu} D_{\nu\rho][\sigma,\kappa]}$ is also subject to self-duality condition, and hence the field is expected to have a non-vanishing index. The discussion follows closely the previous section and we focus on the field strength S which is in the $[1,0,3]$ representation. Due to the absence of residual gauge symmetry, one can avoid the discussion of ghosts and quantisation.

The relevant Dirac operator for S is given by (details of the computation can be found in the Appendix B.4):

$$\begin{aligned} \mathcal{D}_S : \mathcal{C}^\infty &\left(S^+ \otimes [S^- \otimes S^- \otimes S^+ - (S^+ \otimes T^*M)^{\oplus 2} - (S^-)^{\oplus 2}]\right) \\ &\longrightarrow \mathcal{C}^\infty \left(S^- \otimes [S^- \otimes S^- \otimes S^+ - (S^+ \otimes T^*M)^{\oplus 2} - (S^-)^{\oplus 2}]\right). \end{aligned} \tag{3.56}$$

It follows that

$$\mathrm{Ind}(\mathcal{D}_S) = \hat{A}(M)\left((\mathrm{ch}(S^-))^2 \mathrm{ch}(S^+) - 2\mathrm{ch}(S^+)\mathrm{ch}(T^*M) - 2\mathrm{ch}(S^-)\right), \tag{3.57}$$

and the anomaly polynomial can be computed as

$$I_S = (-1)(\frac{1}{2})\,[\mathrm{Ind}(\mathcal{D}_S)]_8 = \frac{1}{5760}(-3808p_1^2 - 7904p_2). \tag{3.58}$$

3.1.5 Anomalies of the exotic multiplets with different supersymmetries

Now we are able to collect everything together and present the anomaly formulae for different multiplets.

The anomalous objects among the 6d $\mathcal{N}=(4,0)$ multiplet are the exotic graviton $(\mathbf{5},\mathbf{1};\mathbf{1})$, the self-dual 2-forms $(\mathbf{3},\mathbf{1};\mathbf{27})$, the exotic gravitini $(\mathbf{4},\mathbf{1};\mathbf{8})$ and the chiral fermions $(\mathbf{2},\mathbf{1};\mathbf{48})$. Taking into account signs due to chirality the total anomaly is given by

$$I_{(4,0)} = I_G + 27I_A + 8I_\chi + \frac{1}{2}\times 48 I_{\frac{1}{2}} = \frac{1}{5760}(684p_1^2 + 2592p_2) \neq 0 \tag{3.59}$$

Since this multiplet is a product [6, 10], we could obtain the same result by following the section 3.1.3. A concrete product construction is described in Table 12 of [6], here we just give the construction of exotic graviton in the light-cone

$$(\mathbf{3},\mathbf{1}) \otimes (\mathbf{3},\mathbf{1}) = (\mathbf{5},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}). \tag{3.60}$$

It follows that if we view the exotic graviton field strength as product of field strengths of a pair of chiral 2 forms and apply the same product construction to other fields, we end up with the same equation (3.59) for the total anomaly.

The anomaly contributions in the $(3,1)$ multiplet are given by

$$I_{(3,1)} = I_S + 12I_A + 2I_\chi + \frac{1}{2}\times 6I_{\frac{3}{2}} + \frac{1}{2}\times(28-14)I_{\frac{1}{2}} = \frac{1}{5760}\left(-2241p_1^2 - 8388p_2\right). \tag{3.61}$$

The $(2,0)$ SD Weyl multiplet consists of $C_{\mu\nu\rho\sigma}; 6B^-_{\mu\nu}; \phi; 4\psi^R_{\mu\nu}; 4\psi^R$. Its anomaly is given by

$$I_{(2,0)\ \text{exotic}} = I_G + 6I_A + 4I_\chi + \frac{1}{2}\times 4I_{\frac{1}{2}} = \frac{1}{5760}(-808p_1^2 - 2624p_2) \tag{3.62}$$

Finally, the $(1,0)$ SD Weyl multiplet comprises $C_{\mu\nu\rho\sigma}; B^-_{\mu\nu}; \phi; 2\psi^R_{\mu\nu}$ and has an anomaly polynomial

$$I_{(1,0)\ \text{exotic}} = I_G + I_A + 2I_\chi = \frac{1}{5760}(-1403p_1^2 - 5884p_2) \tag{3.63}$$

3.1.6 The Lagrangian for the exotic gravitino anomaly computation

So far we have performed the index-theoretical computation of the exotic field anomalies. While a universal feature is that all fields that appear in these supermultiplets are in the domain of a Dirac operator for some choice of vector bundle V, for the exotic fields V is given by a product of spin bundles. Hence, much like for the self-dual tensor fields (and unlike the gravitino) the computation involves the field strengths rather than potentials. This comes with a certain advantage - since there is no gauge freedom of the field strength, there is no need to manually add any ghost field contributions to the anomaly as one would do for the gravitino.

One can ask the question about computing anomalies with the potential itself. An example of such computation is the ordinary gravitino anomaly [68, 69, 71]. According to [68], this type of computation generalizes to the cases where a chiral gravitino ψ_A transforms in a tensor representation of $SO(6)$ (or of $SO(5,1)$) with the tensor index A. Contributions from unwanted parts that appear in the tensor product and the ghost contributions are yet to be subtracted.

For our interests, we take the chiral fermions in the rank 2 antisymmetric representation. We will give a complete quantisation scheme for the free classical field $\psi_{\mu\nu}$ in the next part of the thesis. Before moving to it, we can still carry out a study on the classical Lagrangian of the antisymmetric rank 2 tensor-spinor and the (anti-)self-duality constraint.

- **The (anti-)self-duality**
  A generic tensor field by itself has no contribution to the gravitational anomaly. It is the self-dual or anti-self-dual part that are individually anomalous, and their anomalies cancel when they are combined to an unconstrained tensor field. At first glance, it seems to be necessary for us to impose the (anti-)self-dual constraint.
  There is a generalized Rarita-Schwinger action of the fermionic two-form proposed in [50, 51]

$$S = \int d^6x\, \bar{\psi}_{\mu\nu}\Gamma^{\mu\nu\rho\sigma\tau}\partial_\rho\psi_{\sigma\tau} \tag{3.64}$$

  we will also use this action in the quantisation later. The equation of motion derived from this action is

$$\Gamma^{\alpha\beta\mu\nu\rho}\partial_\mu\psi_{\nu\rho} = 0\,, \tag{3.65}$$

  which is, as we mentioned, shown [11] to be equivalent to the anti-self-dual condition (1.12) and a constraint (1.13).
  This way the (anti-)self-duality constraint on the field strength of $\psi_{\mu\nu}$ is automatically satisfied, provided it is on-shell. One can thus use this Lagrangian for the anomaly computation for the exotic gravitino.

- **Representation** The above consideration becomes clearer when one looks a the representations. An antisymmetric two-form of $SO(5,1)$ corresponds to the $\mathfrak{su}^*(4)$ highest weight $[1,0,1]$. We take the product of it with a chiral spinor

$$[1,0,1]\otimes[1,0,0] = [2,0,1]\oplus[0,1,1]\oplus[1,0,0]. \tag{3.66}$$

  The exotic gravitino is in the $[2,0,1]$ and $[0,1,1]$ is describing an ordinary gravitino (to be unambiguous, it is a gamma-traceless spinorial one-form) with opposite chirality.
  We can also check this using the little group $SO(4)\equiv SU(2)\times SU(2)$. The physical degrees of freedom of a two-form $B_{\mu\nu}$ are given by B_{ij} with $i,j = 1,2,3,4$ running over the $SO(4)$ indices. Then, if we take the self-dual and anti-self-dual part together

$$[(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})]\otimes(\mathbf{2},\mathbf{1}) = (\mathbf{4},\mathbf{1})\oplus(\mathbf{2},\mathbf{3})\oplus(\mathbf{2},\mathbf{1}), \tag{3.67}$$

  which is nothing but the decomposition (3.66) translated in the little group. For the anti-self-dual part of B alone

$$(\mathbf{3},\mathbf{1})\otimes(\mathbf{2},\mathbf{1}) = (\mathbf{4},\mathbf{1})\oplus(\mathbf{2},\mathbf{1})\,. \tag{3.68}$$

3.2 Five-dimensional Chern-Simons interactions

The non-triviality of the index bundle discussed without any obvious anomaly cancellation mechanism in view (at least for the maximally supersymmetric $(4,0)$ and $(3,1)$ cases) might be just one of the signs of trouble with the multiplets involving the SD Weyl field C or its three-index counterpart D. Given the lack of general covariance, this might appear to be neither too surprising nor lethal if mechanisms for reproducing the non-linear dynamics of lower dimensional gravitational theories can be established.

As prior mentioned, when compactified on a circle the degrees of freedom of these multiplets can be arranged into the fields of the five-dimensional supergravity [1]. The SD Weyl field C can in five dimensions be represented in terms of a symmetric field $h_{\mu\nu}$, while D reduces to $h_{\mu\nu}$ plus a vector.[7] The six-dimensional (linearised) equations of motion are consistent with the interpretation of h as the linearised excitation around the flat metric. A direct study of the dynamics of C or D fields beyond linearisation, and hence the comparison with the non-linear five dimensional gravity, is very difficult and this is the key problem in establishing whether interacting $(4,0)$ and $(3,1)$ theories exist.

From other side, the maximal five-dimensional supergravity is unique, and contains interactions that do not involve the metric [43]. As we mentioned in the introduction there is the topological Chern-Simons term (1)

$$S_{\rm CS} = \int k_{\Lambda\Sigma\Delta}\ A^\Lambda \wedge F^\Sigma \wedge F^\Delta\,, \tag{3.69}$$

and it does not admit linearisation. Hence probing its origin could be the first step towards understanding the interaction in six-dimensional $(4,0)$ and $(3,1)$ theories, while avoiding the complications associated with the C and D fields (the D field yields a vector in five dimensions, we will see that its test is slightly different).

All vectors of the five-dimensional maximal supergravity are in the **27** representation of $E_{6(6)}$. The interaction (3.69) is possible due to the fact that there is a $E_{6(6)}$ singlet in the cubic tensor product of the fundamentals $\mathbf{27}\otimes\mathbf{27}\otimes\mathbf{27} = \mathbf{1}\oplus\cdots$. There is a more refined structure: under $E_{6(6)} \longrightarrow SL(6,\mathbb{R})\times SL(2,\mathbb{R})$, we have $\mathbf{27}\to(\mathbf{15},\mathbf{1})+(\mathbf{6},\mathbf{2})$ and the only allowed trilinear couplings involve either three fields in **15** of $SL(6,\mathbb{R})$ that are $SL(2,\mathbb{R})$ singlets or a single vector field in **15** and a doublet of $SL(2,\mathbb{R})$ in **6** of $SL(6,\mathbb{R})$. This structure is perfectly consistent with eleven-dimensional origin of the Chern-Simons interactions, and arises in the reduction of the six-dimensional $(2,2)$ supergravity on a circle. The $\mathbf{15}-\mathbf{15}-\mathbf{15}$ interaction can be seen directly from the T^6 reduction of eleven-dimensional Chern-Simons terms. The doublet of **6** corresponds to the metric and the three-form field having one leg along the torus. Note that even if the Chern-Simons interactions do not involve five-dimensional gravitons, 6 of the 27 vector fields have eleven-dimensional gravitational origin.

In theories with 16 and 8 supercharges, the intimate connections between the six-dimensional anomalies and five-dimensional Chern-Simons couplings has been studied, and it is expected that only the anomaly-free theories yield gauge invariant Chern-Simons interactions upon circle reduction [105, 106]. In the maximally supersymmetric case, the refined structure of the Chern-Simons couplings makes their compatibility with a non-vanishing gravitational index in the $(4,0)$ or $(3,1)$ multiplets very unlikely.

It is instructive to review the five-dimensional Chern-Simons terms in theories with 8 supercharges [44, 45, 47, 107] and their six-dimensional $\mathcal{N}=(1,0)$ origin (the case with 16 supercharges and six-dimensional $\mathcal{N}=(2,0)$ is very similar). There are two ways of generating these upon the circle reduction. The first involves either simple dimensional reduction of existing six-dimensional Chern-Simons terms, or field redefinitions involving the gravi-photon field A^0 coming form the six-dimensional metric

$$ds_6^2 = ds_5^2 + g_{55}(dx^5 + A^0_\mu dx^\mu)^2 \tag{3.70}$$

where $\mu = 0,1,...,4$. In the reduction of the eleven-dimensional supergravity to five dimensions, the entire (3.69) can be generated in this fashion. The second mechanism involves integrating out

[7] In this section we will use five-dimensional indices μ,ν and six-dimensional indices M,N. The gamma matrix in 5d is written as γ^μ while the 6d gammas are Γ^M.

at one loop the massive spin 1/2, 3/2 and two-form, i.e. potentially anomalous, fields coupled to A^0 or six-dimensional vector fields.

A generic six-dimensional $(1,0)$ theory has n_T tensor multiplets with an anti-self-dual three-form in each, and a self-dual three-form in the gravity multiplet, leading to an $O(1,n_T)$ symmetry, and gauge multiplets with a gauge group of dimension n_V. The six-dimensional interactions lead via reduction to the following triple interactions

$$A^0 \wedge F^\alpha \wedge F^\beta \eta_{\alpha\beta} + k_{\alpha ij} A^\alpha \wedge F^i \wedge F^j \tag{3.71}$$

with α, β being $O(1,n_T)$ index and i running over the Cartan subalgebra of the six-dimensional gauge group.

The $O(1,n_T)$ symmetry does not allow generation of any terms cubic in A^α [107], but couplings

$$k_0 A^0 \wedge F^0 \wedge F^0 + k_{0ij} A^0 \wedge F^i \wedge F^j + k_{ijk} A^i \wedge F^j \wedge F^k \tag{3.72}$$

are allowed, and are in fact a part of the five-dimensional low energy effective action arising after integrating out the massive fields. For example, the first term in (3.72), can be traced to a triangle diagram with three external legs being gravi-photons with some massive fields running in the loop.

By taking the ansatz (3.70), all six-dimensional fields that are coupled minimally to graviton will provide massive fields in five dimensions that couple minimally to the gravi-photon with charges given by the corresponding Kaluza-Klein level. We list the minimal five-dimensional coupling between the $U(1)$ vector fields and massive spin 1/2 and spin 3/2 fermions and complex two-forms:

$$\begin{aligned} & iq\bar\psi\gamma^\mu A_\mu \psi \\ & iq\bar\psi_\rho \gamma^{\rho\mu\nu} A_\mu \psi_\nu \\ & \pm \frac{1}{4} iq\epsilon^{\mu\nu\rho\sigma\tau} \bar B_{\mu\nu} A_\rho B_{\sigma\tau}, \end{aligned} \tag{3.73}$$

where the sign in the last line is correlated with the six-dimensional chirality of the B-field. We have followed the conventions of [44, 45]. A lengthy one-loop computation indeed leads to the appearance of the cubic interactions of the form (3.72).

The five-dimensional theory also has non-minimal couplings

$$\begin{aligned} & \frac{1}{2} i\tilde q_{1/2} F^{\mu\nu} \bar\psi \gamma^{\mu\nu} \psi \\ & \frac{1}{2} i\tilde q_{3/2} F^{\mu\nu} \bar\psi_\rho \gamma^{\mu\nu\rho\sigma} \psi_\sigma + \frac{1}{2} i\tilde q'_{3/2} F^{\mu\nu} \bar\psi_\mu \psi_\nu \\ & \tilde q_B \bar B_{\mu\nu} F^{\nu\rho} B_\rho{}^\mu + \tilde q'_B \bar B_{\mu\nu} F^{\nu\rho} B_{\rho\sigma} F^{\sigma\mu}. \end{aligned} \tag{3.74}$$

However, as shown in [44] these can be used to cancel divergences in relevant diagrams and do not affect the Chern-Simons couplings.

The five-dimensional Chern-Simons interactions (3.71) and (3.72) do not contain any scalars and are gauge invariant by virtue of six-dimensional anomaly cancellation [105, 106]. We shall not establish any direct relation between the non-vanishing index for the $(4,0)$ and $(3,1)$ multiplets and the impossibility of recovering the gauge invariant Chern-Simons couplings of the maximal five-dimensional supergravity. Instead we shall show that there are no diffeomorphism invariant couplings compatible with the structure of these multiplets that can be reduced on the circle or give rise to interactions like (3.73) that are needed in order to generate the five-dimensional Chern-Simons terms.

3.2.1 Testing the (4,0) multiplet

We should recall that the six-dimensional multiplets do not contain gravity, and while the five-dimensional Planck length is given by the radius of the compactification circle, the reduction procedure is by no means the conventional Kaluza-Klein. The most notable difference is the absence of the “gravi-photon”, i.e. the KK vector that usually arises from the reduction of the metric. Further more, we still assume the six-dimensional Lorentz invariance and the locality of interactions.

The 27 chiral two forms B^{α}_{MN} in the $(4,0)$ multiplet are in the **27** of $Sp(8) \subset E_{6(6)}$ [1]. Due to self-duality each six-dimensional B_{MN} yields a five-dimensional vector A_μ and there are no KK vectors arising in the reduction of other fields in the $(4,0)$ multiplet. In other words the 27 five-dimensional vectors A^{Λ}_μ in five-dimensional $\mathcal{N} = 8$ supergravity all originate form the six-dimensional tensor fields.

In order to explore the possibility of the coupling (3.69) governed by the $E_{6(6)}$ cubic invariant being generated via loop integration of the massive states with three external five-dimensional vector fields, the $E_{6(6)}$ invariant three-vertices involving six-dimensional B_{MN}-fields have to be examined.

The first immediate observation is that these tests do not involve the SD Weyl field. Indeed, in order to get a contribution from the exotic graviton C_{MNPQ} running in the loop, $\mathbf{1} \otimes \mathbf{27} \otimes \mathbf{1}$ needs to contain an $E_{6(6)}$ (or $Sp(8)$) singlet, which is clearly not possible.

Turning to the fermions we start from the chiral spin 1/2 fields in the **48** of $Sp(8)$. The minimal five-dimensional coupling is of the form

$$iqc_{\Lambda ij}\bar{\psi}^i\gamma^\mu A^{\Lambda}_\mu\psi^j, \tag{3.75}$$

where Λ is in **27**, i, j are **48** indices and $c_{\Lambda ij}$ is a constant. Such a tri-vertex is allowed since there is a singlet contained in $\mathbf{48} \otimes \mathbf{27} \otimes \mathbf{48}$. However, in order to lift this coupling to six dimensions we must complete the term

$$iqc_{\Lambda ij}\bar{\psi}^i\gamma^\mu B^{\Lambda}_{\mu 5}\psi^j \tag{3.76}$$

to a Lorentz scalar. The easiest way is to put a derivative on B and thus yielding[8]

$$iqc_{\Lambda ij}\bar{\psi}^i\Gamma^M\partial^N B^{\Lambda}_{MN}\psi^j. \tag{3.77}$$

However $\partial^N B^{\Lambda}_{MN} = 0$ serves like the Lorenz gauge just as in the case for Abelian vector field, and (3.77) vanishes. Another option is to increase the rank of the gamma matrix sandwiched by the fermions

$$iqc_{\alpha ij}\bar{\psi}^i\Gamma^{MN}B^{\Lambda}_{MN}\psi^j. \tag{3.78}$$

This could give rise to the wanted minimal coupling when the index $N = 5$, but a chiral fermion bilinear in six dimensions with two fermions of the same chirality does not contain any two forms. Hence, the above expression is identically zero. Further possible six-dimensional couplings dimensions lead to non-minimal couplings in five dimensions, which as already explained do not give quantum contributions to the one-loop Chern-Simons terms.

The exotic gravitino ψ^a_{MN} is in the **8** of $Sp(8)$, and the trilinear coupling with the vector $\mathbf{8} \otimes \mathbf{27} \otimes \mathbf{8}$ contains a singlet. The gravitino-vector coupling as listed in (3.73)

$$iq\tilde{k}'_{\Lambda ab}\bar{\psi}^a_\rho\gamma^{\rho\mu\nu}A^{\Lambda}_\mu\psi^b_\nu \tag{3.79}$$

is lifted to

$$iq\tilde{k}'_{\Lambda ab}\bar{\psi}^a_{\rho 5}\Gamma^{\rho\mu\nu}B^{\Lambda}_{\mu 5}\psi^b_{\nu 5} \tag{3.80}$$

There are three six-dimensional candidates that do not have any derivatives acting on B_{MN} or ψ_{MN}

$$\begin{aligned} &iq\tilde{k}'_{\Lambda ab}\bar{\psi}^a_{MQ}\Gamma^{MNPQRS}\psi^b_{NR}B^{\Lambda}_{PS}, \quad iq\tilde{k}'_{\Lambda ab}\bar{\psi}^a_{MR}\Gamma^{MNPQ}\psi^{b\ \ R}_{N}B^{\Lambda}_{PQ}, \\ &iq\tilde{k}'_{\Lambda ab}\bar{\psi}^a_{MR}\Gamma^{MNPQ}\psi^b_{NP}B^{\Lambda\, R}_{Q}. \end{aligned} \tag{3.81}$$

The first one is allowed by $SO(5,1)$ representation, but to achieve (3.79) we would have to set $Q = R = S = 5$ so the gamma matrix vanishes by antisymmetry. The other two vanish due to the tensor product decomposition of the exotic gravitini.

The trilinear coupling of vectors with the massive two-forms B also need to be considered. Such couplings for the reduction of $(1,0)$ theory in (3.73) contain the gravi-photon and originate from

[8] Note that we raise and lower six-dimensional indices with the flat metric η_{MN} in this section, as it is assumed to be non-dynamical in the six-dimensional theory.

self-duality of the six-dimensional tensor fields [46]. This is no longer the case, and one should be looking for a six dimensional cubic invariant built solely from the bare potentials B^{α}_{MN}

$$k_{\Lambda\Delta\Gamma}\ B^{\Lambda}\wedge B^{\Delta}\wedge B^{\Gamma}\Longrightarrow k_{\Lambda\Delta\Gamma}\epsilon^{MNPQRS}\ B^{\Lambda}_{MN}B^{\Delta}_{PQ}B^{\Gamma}_{RS}\,, \tag{3.82}$$

the reduction of which would contain a minimal term proportional to

$$+iqk_{\Lambda\Delta\Gamma}\epsilon^{\mu\nu\rho\sigma\tau}\bar{B}^{\Lambda}_{\mu\nu}A^{\Delta}_{\rho}B^{\Gamma}_{\sigma\tau}. \tag{3.83}$$

This product contains an $E_{6(6)}$ singlet and hence is allowed. As discussed, under $E_{6(6)}\longrightarrow SL(6,\mathbb{R})\times SL(2,\mathbb{R})$, $\mathbf{27}\to(\mathbf{15},\mathbf{1})+(\mathbf{6},\mathbf{2})$ and the trilinear couplings are either between three **15** or between one **15** and two different **6**. This means that any possible contribution to (3.69) from (3.83) should have a massive two form in **15** in the loop. From other side as shown in [108], the only two-forms allowed to enter the five-dimensional action are in one of the **6** representations. Hence contributions from the massive two-forms to (3.69) seem to be ruled out by supersymmetry. At any rate it would be very hard to imagine a gauge invariant completion of (3.82).[9]

This seems to exhaust the possibilities for generating the five-dimensional Chern-Simons couplings using the five-dimensional massive modes coupled to the fields of the five-dimensional maximal supergravity in a way that can be lifted to six-dimensional Lorentz and gauge invariant interactions.

Finally, one may entertain the possibility of a coupling like

$$S_{\rm CSE}=\int k_{\Lambda\Delta\Gamma}\ B^{\Lambda}_{MN}\wedge H^{\Delta}_{PQV}\wedge H^{\Gamma}_{RSW}\eta^{VW}\epsilon^{MNPQRS}\,. \tag{3.84}$$

and its direct reduction to five-dimensions. Clearly this coupling is not gauge invariant in six dimensions. But that is not the only problem - upon reduction only the part involving η^{55} gives a sensible and gauge invariant five dimensional coupling. On the other hand, five-dimensional interactions cannot contain $\eta^{\mu\nu}$. Hence constraints need to be imposed on (3.84) in order to eliminate the unwanted parts. This would come at the expense of the Lorentz invariance, and we do not consider the possibility of breaking this here (even in a "specific and limited way"). Running slightly ahead, we remark that the possibility of even such - however dubious - cures is not available for the $(3,1)$ multiplet.

3.2.2 Testing the (3,1) multiplet

The (3,1) multiplet, written in the representation of $\mathfrak{su}(2)\times\mathfrak{su}(2)\times\mathfrak{sp}(6)\times\mathfrak{sp}(2)$ is

$$\begin{array}{c}(\mathbf{4},\mathbf{2};\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{2};\mathbf{14},\mathbf{1})+(\mathbf{3},\mathbf{1};\mathbf{6},\mathbf{2})+(\mathbf{1},\mathbf{1};\mathbf{14'},\mathbf{2})\\ +(\mathbf{4},\mathbf{1};\mathbf{1},\mathbf{2})+(\mathbf{3},\mathbf{2};\mathbf{6},\mathbf{1})+(\mathbf{2},\mathbf{1};\mathbf{14},\mathbf{2})+(\mathbf{1},\mathbf{2};\mathbf{14'},\mathbf{1}).\end{array} \tag{3.85}$$

It follows [1] that, the $(\mathbf{4},\mathbf{2};\mathbf{1},\mathbf{1})$ field D_{MNP} gives a linearised metric $h_{\mu\nu}$ and a vector $A^0_{\mu}=D_{\mu55}$, which we denote with a superscript 0 to distinguish from other vectors.

There are also five-dimensional vectors A^i_{μ} given by $(\mathbf{2},\mathbf{2};\mathbf{14},\mathbf{1})$ and those A^{α}_{μ} from the chiral two-form $(\mathbf{3},\mathbf{1};\mathbf{6},\mathbf{2})$. With respect to the chain of groups

$$Sp(6)\times Sp(2)\subset F_{4(4)}\subset E_{6(6)}, \tag{3.86}$$

the **27** of $E_{6(6)}$ has a decomposition under $Sp(6)\times Sp(2)$

$$\mathbf{27}=(\mathbf{1},\mathbf{1})+(\mathbf{14},\mathbf{1})+(\mathbf{6},\mathbf{2}). \tag{3.87}$$

To build a six-dimensional vertex with $(3,1)$ field content, a $Sp(6)\times Sp(2)$ singlet needs to be constructed. Recalling the structure of the $E_{6(6)}$ cubic invariant (3.69), it is not hard to see that

$$\begin{array}{llll}(\alpha,\beta,i) & (\alpha,\beta,0) & & \\ (0,0,0) & (0,0,i) & (0,i,j) & (i,j,k)\end{array} \tag{3.88}$$

[9]Note that in [44, 45] the five-dimensional couplings generated using the vertex with a massive two-form in the loop and an external vector field involve the gravi-photon. As mentioned, in the reduction of the $(4,0)$ theory this field does not even arise.

trilinear couplings need to be generated. Note that this structure is rather different from that of the trilinear couplings arising in the reduction of $(1,0)$ theory which has e.g. (α, i, j) couplings obtainable by direct reduction, and does not have (α, β, i) couplings. It can be shown, using arguments from the previous subsection, that it is not possible either to directly lift this structure to Lorentz and gauge invariant couplings of $(3,1)$ multiplet, or to generate them by integrating out massive modes in the loop.

Chapter 4

Evidence for h-theories

In this chapter we shall discuss how trying to solve the equations of motion for the SD Weyl field C_{MNPQ} may suggest an alternative way of thinking about some of the six-dimensional exotic multiplets. The basic construction works for theories with 32, 16 or 8 supercharges, but for the latter two a number of $(2,0)$ tensor and $(1,0)$ vector multiplets respectively need to be added. Bellow, we shall mostly discuss the maximally supersymmetric case of $(4,0)$ exotic multiplet.

The SD Weyl field has 5 physical degrees of freedom. While this is the same number of degrees of freedom as that of five-dimensional metric, we argued (however indirectly) that the dynamics of this field when reduced on a circle is unlikely to be the same as that of gravity. From other side, this number also matches the number of the parameters of the $SL(3,\mathbb{R})/SO(3)$ coset, i.e. a three-torus of of fixed volume. Moreover, as discussed in section 2.5 this coset is closely related to the SD Weyl field, both in terms of the degrees of freedom of the field itself and its gauge transformation parameters. So one may wonder if the system of five scalars parametrising the coset coupled to thee-dimensional gravity, which carries no dynamical degrees of freedom, may be related to the solutions of the equations of motion for the SD Weyl field. This system is familiar, and it has been shown in [42] that its solutions can be summarised by a Ricci-flatness condition of a semi-classical metric on a six-dimensional space X obtained as a T^3 fibration over the three-dimensional base.[1]

As we shall review shortly, the Ricci-flatness condition is equivalent to a real two-form k on X, constructed from the coset element of $SL(3,\mathbb{R})/SO(3)$, being covariantly constant. One can think of k as the Kähler form on X, but in the context of supersymmetric theories, one cannot establish a duality between any six-dimensional supergravity (with 32, 16 or 8 supercharges) on X and solutions of the above three-dimensional system, preserving a quarter of supersymmetry. On the contrary, the latter are consistent with the $(4,0)$ and exotic $(2,0)$ and $(1,0)$ supersymmetry respectively, and k can be squared to a SD Weyl field satisfying its flatness condition. It can be shown that at the linearised level, the differential conditions on k reduce to the equations of motion for the SD Weyl field.

In order to see this we just need to examine the duality groups of these theories. Let us start from the 32 supercharge case. The three-dimensional theory has $E_{8(8)}$ symmetry and it's scalar manifold is the coset space $E_{8(8)}/SO(16)$. All but five of these scalars are set to zero in the solution, leaving the $E_{6(6)}$ symmetry, which is stabilised by $SL(3,\mathbb{R})$ inside $E_{8(8)}$ intact. So when geometrising the $SL(3,\mathbb{R})$ symmetry and thinking of the solutions of the three-dimensional system of gravity and five scalars in terms of solutions of some six-dimensional theory on X, one expects the latter to have manifest $E_{6(6)}$ symmetry. This is not the case for the maximal six-dimensional supergravity, but it is for the $(4,0)$ SD Weyl multiplet. As it is clear from the details of the construction in subsection 4.2, on three-dimensional bases one can construct at most T^3 and hence geometrise only $SL(3,\mathbb{R})$ this way. In many ways the construction is reminiscent of geometrisation of $SL(2,\mathbb{R})$ in type IIB and F-theory. Moreover since the construction involves two groups of intersecting co-dimension two defects, the $SL(3,\mathbb{R})$ arises as the group generated by two $SL(2,\mathbb{R})$ subgroups. This is reminiscent of the two pairs of charges in (2.49) which have accompanying $SL(2,\mathbb{R})$ actions, even though the full triplet (2.49) has no $SL(3,\mathbb{R})$, as discussed in sub-section 2.3. However, to make closer comparison with that discussion, it may be better to

[1] We shall work with a Euclideanised version of the three-dimensional theory.

consider the points made in section 2.4 and look at the two $SO(2)$ groups generating an $SO(3)$ under which the momenta transform as a triplet.

Similarly in the case with 16 supercharges, embedding $SL(3,\mathbb{R})$ in the duality group $SO(8,8+n)$ leaves invariant a $SO(5,5+n)\times\mathbb{R}^+$ suggesting that this is the symmetry on the resulting six-dimensional theory. While all $(2,0)$ multiplets have the same R-symmetry, this symmetry is a bit bigger than that admitted by the $(2,0)$ gravity plus $n+5$ tensor multiplets, and the $\mathbb{R}^+$ factor accounts for the extra scalar in the $(2,0)$ SD Weyl multiplet.

Theories with 8 supercharges are a bit harder to analyse, notably because there are many options for the scalar manifolds available. Yet the most "typical" quaternionic coset is given by $SO(4,4+n)/SO(4)\otimes SO(4+n)$. Under $SL(3,\mathbb{R})$ embedding one gets a coset space $SO(1,1+n)/SO(1+n)$ which can describe the moduli space of $n_T = 1+n$ six-dimensional $(1,0)$ tensor multiplets coupled either to $(1,0)$ gravity multiplet or to $(1,0)$ SD Weyl multiplet. Note however that the latter case has fewer degrees of freedom (a $(1,0)$ gravity multiplet is "worth" a $(1,0)$ SD Weyl multiplet + a tensor multiplet).

In this section, we will discuss the equations of motion of the SD Weyl field and their T^3 reduction, looking to capture the solutions of three-dimensional gravity with varying scalars in terms of a six-dimensional geometric construction involving the SD Weyl field. In particular, we will construct a T^3 fibered manifold together with a tensor field $\mathcal{C}_{MNPQ}$ using five scalar fields with dependence only on the three-dimensional base. On imposing that the five scalars solve the supersymmetry conditions of three-dimensional supergravity, this field will solve an equation

$$\mathcal{G}^{Q}_{\ NPQRS} = 0. \qquad \mathcal{G}_{MNPQRS} = \nabla_{[M}\mathcal{C}_{NP][QR;S]} \tag{4.1}$$

However, the field $\mathcal{C}_{MN,PQ}$ cannot directly be interpreted as the SD Weyl field on the curved space. To identify the SD Weyl field of [1] we should linearise the system by thinking of the three-dimensional scalar fields underlying the construction as small fluctuations. The corresponding geometry will then be seen as a small fluctuation of a flat manifold $\mathbb{R}^3\times T^3$, with the five scalars determining the metric on the fibres. As we shall see shortly, in the expansion of $\mathcal{C}$ in the powers of the scalar fields

$$\mathcal{C}_{MNPQ} = \mathcal{C}^{(0)}_{MNPQ} + C_{MNPQ} + \dots \tag{4.2}$$

the linear fluctuations of the SD Weyl field of [1] will be identified with the the first order term, denoted C_{MNPQ}. To the first order in fluctuations, (4.1) reduces to the standard equation for the SD Weyl field of [1]

$$G^{Q}{}_{NPQRS} = 0. \qquad G_{MNPQRS} = \partial_{[M}C_{NP][QR,S]} \tag{4.3}$$

provided that C_{MNPQ} is taken to have no dependence on the T^3 directions of the geometry.

Note that geometric fluctuations around the flat geometry $\mathbb{R}^3\times T^3$ would only affect the non-linear parts in the expansion of (4.1) without spoiling the agreement of it with the linearised equation (4.3). Thus one could view equation (4.1) as a non-linear extension of the SD Weyl field equation of motion. Here we only check that the two agree at the linearised level. Another feature of this construction which mirrors comments made in section 2.5 is that the T^3 fibered geometry uses the physical degrees of freedom in its definition, and thus the six-dimensional space requires the physical fields for its definition.

This reformulation of three-dimensional theories in terms of the (diffeomorphism non-invariant) six-dimensional one on (non-compact) manifolds with certain geometric properties, suggests the interpretation of the latter as lower-dimensional cousins of F-theory.

4.1 SD Weyl field on $\mathbb{R}^3\times T^3$

Since to the linear order in scalar fields, the equation of motion for the SD Weyl field C_{MNPQ} is not sensitive to the metric fluctuations, in this section we will examine the reduction of the equations on a flat $\mathbb{R}^3\times T^3$.

Following [1, 2], we define the SD Weyl field strength in flat space as

$$G_{MNPQRS} = \partial_{[M}C_{NP][QR,S]} \tag{4.4}$$

subject to self-duality constraint (we are working in Euclidean signature, hence the factor of i):

$$G_{MNPQRS} = \frac{i}{3!}\epsilon_{MNPTUV}G^{TUV}{}_{QRS} = \frac{i}{3!}\epsilon_{QRSTUV}G_{MNP}{}^{TUV} \tag{4.5}$$

with $M, N, P... = 1, ..., 6$. The equation of motion for C in 6 dimensions is then given by

$$G^{Q}{}_{NPQRS} = 0. \tag{4.6}$$

We shall now assume that the SD Weyl field C depends only on three of the coordinates. We can separate the the coordinates into the $\mathbb{R}^3$ part x^α with $\alpha = 1, 2, 3$ and T^3 part ξ^i with $i = 1, 2, 3$, and allow the dependence only on the x^α-coordinates of $\mathbb{R}^3$, i.e. take

$$\partial_i C_{ABCD} = 0, \tag{4.7}$$

This, together with the self-dual condition of the field strength of C eliminates the some of the components

$$\begin{aligned} G_{ijkMNP} &= \partial_{[i}C_{jk][MN;P]} = 0 \\ G_{\alpha\beta\gamma MNP} &= \epsilon_{\alpha\beta\gamma ijk}G^{ijk}{}_{MNP} = 0 \end{aligned} \tag{4.8}$$

The non-vanishing components field strength of C in the product ansatz are of the type

$$G_{\alpha ijMNP} \quad \text{or} \quad G_{\alpha\beta iMNP}, \tag{4.9}$$

where because of the exchanging symmetry $C_{MNPQ} = C_{PQMN}$ we only need to focus on the first 3 indices of the field strength G. Since these two types of components are related again by the self-duality of G

$$G_{\alpha ijMNP} = \frac{i}{3!}\epsilon_{\alpha ij\beta\gamma k}G_{\beta\gamma kMNP} \quad \text{or} \quad G_{\beta\gamma kMNP} = \frac{i}{3!}\epsilon_{\beta\gamma k\alpha ij}G_{\alpha ijMNP}. \tag{4.10}$$

it is sufficient to consider the components $G_{\alpha\beta i\gamma\delta j}$ arising from the potentials $C_{\mu i\nu j}$. In each pair of indices on C there is one $\mathbb{R}^3$ coordinate index and one T^3 coordinate index. The equation of motion for C in 6 dimensions (4.6) reduces to

$$\delta^{\gamma\delta}\partial_{[\gamma}C_{\alpha i][\beta j,\delta]} = 0 \tag{4.11}$$

This is a set of (linear) three-dimensional equations for five degrees of freedom contained in the SD Weyl field. As already mentioned the $SL(3,\mathbb{R})/SO(3)$ coset has the same number of degrees of freedom. In what follows, we will construct a T^3 fibred geometry together with a tensor field $\mathcal{C}_{MNPQ}$ satisfying similar equations to those for C_{MNPQ} above (but on the curved geometry), such that the linearisation of the total system reduces to (4.11). The fact that the geometry is given by $\mathbb{R}^3 \times T^3$ only at zeroth order in fluctuations does not affect the linearised equation (4.11).

4.2 The $SL(3,\mathbb{R})/SO(3)$ sigma-model and the SD Weyl field

The symmetric space $SL(3,\mathbb{R})/\,SO(3)$ has dimension five. Thus, we need 5 real scalars to parametrise the non-linear sigma-model with target $SL(3,\mathbb{R})/SO(3)$. Its vielbein V_{ai} in Borel gauge can be written as follows

$$V = e^{\Phi_1/\sqrt{3}} \begin{pmatrix} 1 & a & b \\ 0 & e^{-(\sqrt{3}\Phi_1 - \Phi_2)/2} & ce^{-(\sqrt{3}\Phi_1 - \Phi_2)/2} \\ 0 & 0 & e^{-(\sqrt{3}\Phi_1 + \Phi_2)/2} \end{pmatrix}, \tag{4.12}$$

where a is a $SO(3)$ index, while i is a $SL(3,\mathbb{R})$ index. The two dilatonic scalars Φ_1 and Φ_2 correspond to the two Cartan generators of $\mathfrak{sl}(3,\mathbb{R})$ and the three other scalars a, b and c are nilpotent generators which complete the coset.

The Mauer-Cartan form dVV^{-1} can be split into the part symmetric in $SO(3)$ indices, $P^{ab} = P^{(ab)}$, and the anti-symmetric part $Q^{ab} = Q^{[ab]}$

$$(\partial_\alpha VV^{-1})^{ab} = P_\alpha^{ab} + Q_\alpha^{ab}. \tag{4.13}$$

Here the partial derivative ∂_α is taken with respect to the 3 dimensional space on which we put the $SL(3,\mathbb{R})/SO(3)$ sigma-model. (We think of this as the base space in what follows.) The involution σ under which the $\mathfrak{so}(3)$ subalgebra is invariant corresponds to taking minus the matrix transpose, and thus (4.13) splits $\partial_\alpha VV^{-1}$ into its σ-eigenvector parts. The symmetric part P_α transforms covariantly under the action of base-coordinates dependent $SO(3)$ elements while Q_α transforms like a connection.

The action of this sigma-model coupled to three-dimensional gravity is given by

$$S = \int d^3x\, \frac{1}{2\kappa^2}\sqrt{-g}(R - g^{\alpha\beta}\,\mathrm{Tr}\,P_\alpha P_\beta), \tag{4.14}$$

with the field equations

$$\begin{aligned} \mathcal{D}_\alpha P^\alpha &= g^{\alpha\beta}(\nabla_\alpha P_\beta + [Q_\alpha, P_\beta]) \\ R_{\alpha\beta} &= \mathrm{Tr}\,P_\alpha P_\beta. \end{aligned} \tag{4.15}$$

In terms of the scalar fields (4.12), the Lagrangian can be brought into a simple form

$$\begin{aligned} \mathcal{L} = \frac{1}{2\kappa^2}\sqrt{-g}(R &- \frac{1}{2}(\partial\Phi_1)^2 - \frac{1}{2}(\partial\Phi_2)^2 - \\ &- \frac{1}{2}e^{\sqrt{3}\Phi_1-\Phi_2}(\partial a)^2 - \frac{1}{2}e^{2\Phi_2}(\partial c)^2 - \frac{1}{2}e^{\sqrt{3}\Phi_1+\Phi_2}(\partial b - c\partial a)^2). \end{aligned} \tag{4.16}$$

The Lagrangian (4.16) can be embedded into three-dimensional supersymmetric theories with the varying amounts of the supersymmetry. The relevant part of the supersymmetry transformations for the spin 3/2 and 1/2 fermions[2] is given by

$$\begin{aligned} \delta\psi_\alpha &= \mathcal{D}_\alpha\epsilon = (\nabla_\alpha + \frac{1}{4}Q_\alpha^{ab}T^{ab})\epsilon \\ \delta\chi^a &= -\frac{1}{2}P_\alpha^{ab}T^b\epsilon \end{aligned} \tag{4.17}$$

where T^{ab} and T^a are the $SO(3)$ generators in the adjoint and spin representation respectively.

Solutions with varying moduli consistent with $SL(3,\mathbb{Z})$ were constructed in [42]. The solution takes the form of overlapping codimension two objects. One can start by solving for each such object, which will be picking a specific $SO(2)$ inside the $SO(3)$ automorphism group. A $\frac{1}{2}$-BPS projector for a brane with transverse $x^{\bar{a}_1} - x^{\bar{a}_2}$ plane can be written as[3]

$$P = \frac{1}{2}\left(1 + \gamma^{\bar{a}_1\bar{a}_2}\Lambda_{a_1a}\Lambda_{a_2b}T^{ab}\right) \tag{4.18}$$

where $\Lambda_{ab}(x)$ is an $SO(3)$ rotation matrix. Solving the BPS conditions for a single codimension-two object yields a solution very much like the standard seven-branes in ten dimensions. On a three-dimensional base there is room for two groups of intersecting objects with a net quarter of supersymmetry preserved.[4] Two groups of such overlapping objects will now fill out the entire $SO(3)$, and the solution geometrically realises a T^3 fibration over the three-dimensional base space.

Using the $SL(3,\mathbb{R})$-invariant form of metric on T^3 one can summarise the solution using a six-dimensional metric of the form

$$ds_6^2 = ds_{\text{base}}^2 + (V^TV)_{ij}(x)d\xi^i d\xi^j \tag{4.19}$$

[2]Note that the spin 1/2 fields transform under the maximal compact subgroup of the symmetry group. Here we restricted to the relevant $SO(3)$ subgroup.

[3]The barred indices $\bar{a}_i = 1, 2, 3$ refer to tangent space

[4]It is not hard to verify that there are only two independent projectors of the type (4.18) on a three-dimensional space.

with the metric on the three-dimensional base space taken as

$$ds^2_{\text{base}} = e^{2\phi_1(x)}dx_1^2 + e^{2\phi_2(x)}dx_2^2 + e^{2\phi_3(x)}dx_3^2. \tag{4.20}$$

As shown in [42], solution of the killing spinor equations is equivalent to the following two-form on the six-dimensional space

$$k_{MN} = \begin{pmatrix} 0 & -e^{-\phi_\alpha}\delta^\alpha_a V_{ai} \\ e^{\phi_\beta}V^T_{ib}\delta^\beta_b & 0 \end{pmatrix} \tag{4.21}$$

being covariantly constant

$$\nabla_M k_{NP} = 0. \tag{4.22}$$

One could think of k as the fundamental form on the resulting six-dimensional manifold. However, as mentioned the solution to (4.16) cannot be lifted to a solution of six-dimensional supergravity on any six-manifold since its symmetry group is $SO(5,5)$ and not the $E_{6(6)}$ group that is stabilised by $SL(3,\mathbb{R})$ inside $E_{8(8)}$ for the case of maximal supersymmetry. The group stabilised by $SL(3,\mathbb{R})$ inside the three-dimensional duality group is compatible with the six-dimensional $(4,0)$ theory or less supersymmetric exotic theories.

In order to describe the six-dimensional lift in terms of the exotic graviton, one can build a four-index object with the properties of Riemann tensor:

$$\mathcal{C}_{MNPQ} = k_{MN}k_{PQ} - k_{[MN}k_{PQ]}, \tag{4.23}$$

which has the non-trivial components $\mathcal{C}_{\alpha i\beta j}$. The algebraic symmetries of $\mathcal{C}$ are manifestly the same as in (1.5). By virtue of (4.22) $\mathcal{C}$ satisfies

$$\nabla_\gamma \mathcal{C}_{\alpha i\beta j;\gamma} = 0. \tag{4.24}$$

Notice that the three-dimensional covariant derivatives are used here. Consistently (4.24) and with the self-duality properties the field strength of $\mathcal{C}$, $\mathcal{C}_{ijkl}$ components can be taken to zero.

In order to compare with (4.11), we need to consider the linearisation of (4.24). Using

$$k_{MN} = k^{(0)}_{MN} + k^{(1)}_{MN} + ... = \begin{pmatrix} 0 & -\delta^\alpha_i \\ \delta^\alpha_i & 0 \end{pmatrix} + k^{(1)}_{MN} + ... \tag{4.25}$$

one can expand $\mathcal{C}$ in a similar fashion, with $\mathcal{C}^{(0)}_{MNPQ} = k^{(0)}_{MN}k^{(0)}_{PQ} - k^{(0)}_{[MN}k^{(0)}_{PQ]}$ having only constant components. It is the linear term in the expansion of $\mathcal{C}$ that is taken to be equal to the SD Weyl field C_{MNPQ}

$$\mathcal{C}_{MNPQ} = \mathcal{C}^{(0)}_{MNPQ} + C_{MNPQ} + ... \tag{4.26}$$

It can be checked then, that the linearised equations of motion for the five three-dimensional scalar fields (4.16) imply $\partial_{[\alpha}C_{\beta i][\gamma j,\alpha]} = 0$.

In other words the three-dimensional gravity coupled to scalars in $SL(3,\mathbb{R})/SO(3)$ coset is solved at linearised level by $(4,0)$ SD Weyl supersymmetry on a T^3 fibered Ricci-flat manifold M. Note that on M, the covariantly constant tensor $\mathcal{C}$ is globally defined. This is not the case for the SD Weyl field C which is obtained by picking the part linear in scalar fields in the expansion of $\mathcal{C}$.

From other side the conspiracy between $SL(3,\mathbb{Z})$ and the duality groups in three and six dimensions makes this construction unique. One could construct a T^2 fibered four-manifold in a similar fashion, but the group $E_{7(7)}$ group that is stabilised by $SL(2,\mathbb{R})$ inside $E_{8(8)}$ is too big for any five-dimensional theory. This has a well-known realisation in terms of codimension-two objects with a deficit angle. The T^3 fibered construction corresponds to two sets of intersecting codimension-two objects, each realising an $SO(2)$ within $SO(3)$. As mentioned, on a three-dimensional base there are only two independent such groups each preserving half supersymmetry (any other half-supersymmetric projector can be built out of the above two). In agreement with this, no other $SL(n,\mathbb{R})$ group (for $n > 3$) inside $E_{8(8)}$ stabilises any known duality group for an $(n+3)$-dimensional theory (since the stabiliser is $E_{9-n(9-n)}$ as can easily be seen from the extended Dynkin diagram).

Here we have concentrated on the maximally supersymmetric theory in three dimensions and its lift to the six-dimensional $(4,0)$. From other side, there is very little dependence on the details of the multiplet or amount of supersymmetry, and as discussed above similar relation exists between three-dimensional theories with 16 and 8 supercharges, and $(2,0)$ and $(1,0)$ SD-Weyl multiplets completed by matter multiplets.

Part III

On quantisations and anomalies of fermionic p-forms

Chapter 5

Introduction to the Batalin-Vilkovisky (BV) formalism

In this chapter, we introduce the Batalin–Vilkovisky (BV) field-antifield formalism [52, 53]. This formalism provides a systematic way of quantising gauge systems which is in general very involved. The antibracket, antifields and the master equation are introduced, they play important role in analysing gauge transformations and the associated gauge structure. Due to our interests and purpose, we will focus on procedures of gauge-fixing, and the notation and conventions mostly follow the comprehensive reviews [109, 110].

5.1 Gauge transformations

We start from a classical action $S_0[\varphi^i]$ depending on fields φ^i with $i = 1, 2, \ldots, n$. The action is assumed to be invariant under some gauge transformations $\delta\varphi^i = R^i_\alpha \Lambda^\alpha$ with generators R^i_α and parameters Λ^α. We use DeWitt's condensed notation, where a contracted index includes spacetime integration. For example, in the gauge transformations

$$\delta\varphi^i(y) = \sum_\alpha \int R^i_\alpha(x, y)\Lambda^\alpha(y)\,. \tag{5.1}$$

The range of the index α indicates the total number of gauge transformations. The classical equation of motion is

$$\frac{\delta^R S_0}{\delta\varphi^i(x)} = 0\,, \tag{5.2}$$

where the superscript "R" means that the functional derivative $\frac{\delta^R}{\delta\varphi^i}$ is acting from the right.[1] The vanishing of gauge variations of $S_0[\varphi^i]$ yields the *Noether identities*

$$0 = \delta_\Lambda S_0 = \frac{\delta^R S_0}{\delta\varphi^i}\delta\varphi^i = \frac{\delta^R S_0}{\delta\varphi^i} R^i_\alpha \Lambda^\alpha$$

$$\Longrightarrow \quad \frac{\delta^R S_0}{\delta\varphi^i} R^i_\alpha = 0\,. \tag{5.3}$$

The fields and gauge parameters are allowed to be bosonic or fermionic, their Grassmann parities are written as $\epsilon(\varphi^i) \equiv \epsilon_i$ and $\epsilon(\Lambda^\alpha) \equiv \epsilon_\alpha$. We see that in this convention, the indices of an expression reveal the Grassmann parity, e.g. $\epsilon(R^i_\alpha) = \epsilon_i + \epsilon_\alpha$ mod 2. The parity gives a $\mathbb{Z}_2$ grading to the set of fields, we will omit the "mod 2" for convenience.

[1] We also define the left derivative $\frac{\delta^L}{\delta\varphi^i}$ to acting from the left. For any function or functional X of the field ϕ we have the variation $\delta X(\phi) = \delta\phi \frac{\delta^L X}{\delta\phi} = \frac{\delta^R X}{\delta\phi}\delta\phi$.

The complete set of gauge transformations. We assume that any solution $\delta\varphi^i$ to the Noether identities (5.3) can be expressed as

$$\delta\varphi^i = R^i_\alpha \Lambda^\alpha + \frac{\delta^R S_0}{\delta\varphi^j} M^{ij} \,, \tag{5.4}$$

where M^{ij} is some arbitrary function with the graded index symmetry $M^{ji} = -(-1)^{\epsilon_i\epsilon_j} M^{ij}$. The second term $\frac{\delta^R S_0}{\delta\varphi^j} M^{ij}$ alone is a gauge transformation due to the symmetry of M^{ij} and it vanishes on-shell. Gauge transformations of this type are called *trivial* gauge transformations and the commutator of two trivial gauge transformations is another trivial gauge transformation. So the set of all trivial gauge transformations form an ideal of the gauge algebra and we can quotient out them in practice. On-shell, the set $\{R^i_\alpha\}$ gives a complete description of the gauge freedoms of the equation of motion. Consider the commutator of two gauge transformations from the complete set and use the assumption (5.4) again, we find

$$\frac{\delta^R R^i_\alpha}{\delta\varphi^j} R^j_\beta - (-1)^{\epsilon_\alpha\epsilon_\beta} \frac{\delta^R R^i_\beta}{\delta\varphi^j} R^j_\alpha = R^i_\gamma T^\gamma_{\alpha\beta} - \frac{\delta^R S_0}{\delta\varphi^j} M^{ji}_{\alpha\beta} \,, \tag{5.5}$$

where we omitted the gauge parameters and the new variables $T^\gamma_{\alpha\beta}$, $M^{ij}_{\alpha\beta}$ have the graded symmetry

$$\begin{aligned} T^\gamma_{\alpha\beta} &= -(-1)^{\epsilon_\alpha\epsilon_\beta} T^{\beta\alpha} \\ M^{ij}_{\alpha\beta} &= -(-1)^{\epsilon_\alpha\epsilon_\beta} M^{ij}_{\beta\alpha} = -(-1)^{\epsilon_i\epsilon_j} M^{ji}_{\alpha\beta} \,. \end{aligned} \tag{5.6}$$

If $M^{ji}_{\alpha\beta} = 0$, the gauge algebra is called a *"closed"* algebra. It is a Lie algebra if the structure constants $T^\gamma_{\alpha\beta}$ are field independent and the commutators of elements from the complete set also satisfy the Jacobi identity. The gauge algebra is said to be *"open"* if $M^{ji}_{\alpha\beta} \neq 0$, and in this case it is also referred to as the gauge algebra *"close only on-shell"*, since it is multiplied by the equation of motion $\frac{\delta^R S_0}{\delta\varphi^j}$ in the relation defining this gauge algebra. The situation for open gauge algebra occurs for instance, in supergravity theories and it is the case that Faddeev-Popov quantisation can not be implemented.

It is convenient to introduce the *ghost fields* to describe the gauge algebra. We simply replace the parameters Λ^α by the ghosts C^α with opposite Grassmann parity $\epsilon(C^\alpha) = \epsilon^\alpha + 1$. Writing the equations (5.3) and (5.5) with the ghosts

$$\frac{\delta^R S_0}{\delta\varphi^i} R^i_\alpha C^\alpha = 0 \,, \tag{5.7}$$

$$\left(2\frac{\delta^R R^i_\alpha}{\delta\varphi^j} R^j_\beta - R^i_\gamma T^\gamma_{\alpha\beta} + \frac{\delta^R S_0}{\delta\varphi^j} M^{ji}_{\alpha\beta} \right) (-1)^{\epsilon_\alpha} C^\beta C^\alpha = 0 \,, \tag{5.8}$$

we get the very first two equations that characterise the gauge structure of the theory.[2] Obviously, with only these two conditions one does not conclude that the gauge algebra is a "Lie" algebra. A further step is to study the Jacobi identity

$$[\delta_1, [\delta_2, \delta_3]] + [\delta_2, [\delta_3, \delta_1]] + [\delta_3, [\delta_1, \delta_2]] = 0 \,, \tag{5.9}$$

and it leads to extra constraints (involving cubic terms of the ghosts) on the generators and constants, see e.g. [110] for details.

Reducibility of gauge transformations. In the last paragraph, we only assumed that the set $\{R^i_\alpha\}$ exhausts the solutions to Noether identity. In general, we get rid of the trivial gauge transformations by restricting to the solution space $\mathcal{S}$ of the equation of motion, and on-shell, these gauge generators of R^i_α can still be linearly dependent. Gauge theories with linearly dependent generators are phrased as *reducible theories*, otherwise they are called *irreducible*.

Following [52,111], we adopt the notations for generators and ghosts with extra numbers: $R^i_{\alpha_0}$ and $C^{\alpha_0}_0$ with $\alpha_0 = 1, 2, \ldots, m_0$.

[2] One needs to pay attention to the extra factor $(-1)^{\epsilon_\alpha}$ in (5.8), this is necessary because of the graded symmetry for the ghosts $C^\alpha C^\beta = (-1)^{(\epsilon_\alpha+1)(\epsilon_\beta+1)} C^\beta C^\alpha$.

- **Irreducible gauge theories**

  By definition, the gauge generators are all independent

$$\text{rank } R^i_{\alpha_0}\big|_{\mathcal{S}} = m_0\,, \tag{5.10}$$

  and we have the completeness assumption for the Hessian matrix

$$\text{rank } \frac{\delta^L\delta^R S_0}{\delta\varphi^i\varphi^j}\bigg|_{\mathcal{S}} = n - m_0\,. \tag{5.11}$$

  This completeness assumption is seminal, it states that the non-existence of the propagator or degeneracy of the Hessian is completely due to the gauge invariance.

- **First-stage reducible theories**

  When the gauge generators are linearly dependent, the rank condition (5.10) no longer holds. The gauge theory is then reducible. Let us start with the simplest case. Suppose there are $m_0 - m_1$ independent generators

$$\text{rank } R^i_{\alpha_0}\big|_{\mathcal{S}} = m_0 - m_1\,, \tag{5.12}$$

  then on-shell there are m_1 null vectors $Z^{\alpha_0}_{1\alpha_1}$ of $R^i_{\alpha_0}$ and they are labelled by α_1

$$R^i_{\alpha_0} Z^{\alpha_0}_{1\alpha_1}\big|_{\mathcal{S}} = 0\,, \tag{5.13}$$

$$\text{rank } Z^{\alpha_0}_{1\alpha_1}\big|_{\mathcal{S}} = m_1\,, \tag{5.14}$$

  with the parity $\epsilon(Z^{\alpha_0}_{1\alpha_1}) = \epsilon_{\alpha_0} + \epsilon_{\alpha_1}$. The condition (5.14) is the linear independence of the null vectors. There are "gauge transformations" for the ghosts $C^{\alpha_0}_0$

$$\delta C^{\alpha_0}_0 = Z^{\alpha_0}_{1\alpha_1} C^{\alpha_1}_1\,, \tag{5.15}$$

  these transformations are parametrised by the next generator ghost fields $C^{\alpha_1}_1$, a.k.a. ghosts for ghosts. For them, the parities are $\epsilon(C^{\alpha_1}_1) = (\epsilon_{\alpha_1} + 1) + 1$.

  The assumption on the Hessian matrix changes to

$$\text{rank } \frac{\delta^L\delta^R S_0}{\delta\varphi^i\varphi^j}\bigg|_{\mathcal{S}} = n - m_0 + m_1\,. \tag{5.16}$$

- ***L*-th-stage reducible**

  Similarly, we can postulate the existence of m_2 null vectors of $Z^{\alpha_0}_{1\alpha_1}$ and modify the rank conditions for all gauge generators and the Hessian matrix accordingly. This process can be extended to any finite stage of reducibility. For the L-th-stage the extra generators (gauge transformation for ghosts, for ghosts for ghosts and etc.) are

$$Z^{\alpha_{s-1}}_{s\alpha_s}\,, \quad \alpha_s = 1, \ldots, m_s\,, \quad s = 1, \ldots, L\,, \tag{5.17}$$

$$\epsilon(Z^{\alpha_{s-1}}_{s\alpha_s}) = \epsilon_{\alpha_{s-1}} + \epsilon_{\alpha_s}\,, \tag{5.18}$$

$$Z^{\alpha_{s-2}}_{s-1\alpha_{s-1}} Z^{\alpha_{s-1}}_{s\alpha_s}\Big|_{\mathcal{S}} = 0\,, \quad s = 2, \ldots, L\,. \tag{5.19}$$

  Just like in the first two cases, we also introduce here the ghosts $C^{\alpha_s}_s$ corresponding to further gauge transformations. The parity formula for them is

$$\epsilon(C^{\alpha_s}_s) = (\epsilon_{\alpha_s} + s) + 1\,, \quad s = 0, \ldots, L\,. \tag{5.20}$$

  We will have ghosts from different stages, to distinguish between them, we introduce the *ghost numbers* denoted as $\text{gh}(X)$ for a field X. For our classical fields, $\text{gh}(\varphi^i) = 0$ whereas for the ghosts they are just the stage number plus one: $\text{gh}(C^{\alpha_s}_s) = s + 1$.

The rank conditions read

$$\text{rank } R^i_{\alpha_0}\big|_{\mathcal{S}} = \sum_{s=0}^{L}(-1)^s m_s\,, \tag{5.21}$$

$$\text{rank } Z^{\alpha_{s-1}}_{s\alpha_s}\big|_{\mathcal{S}} = \sum_{t=s}^{L}(-1)^{t-s} m_s\,, \quad s = 1,\ldots,L\,, \tag{5.22}$$

$$\text{rank } \frac{\delta^L\delta^R S_0}{\delta\varphi^i\varphi^j}\bigg|_{\mathcal{S}} = n - \sum_{s=0}^{L}(-1)^s m_s\,. \tag{5.23}$$

Note that for reducible gauge theories the gauge algebra structure can be much more complicated than those of irreducible theories. However, the Noether identities (5.7) and the commutation relations of the original gauge generator (5.8) remain the same.

5.2 The Field-Antifield formalism

5.2.1 The Antifields and the Antibracket

To describe the gauge structure, we introduced the ghosts as gauge transformations parameters so far. In the field-antifield formalism of Batalin and Vilkovisky [52, 53, 111], the space of variables is extended to include the *antifields* φ^*_i and $C^*_{s\alpha_s}$. We will denote the set of all fields collectively by Φ^I, and the antifields by Φ^*_I, where $I = 1, 2, \ldots, N$. The ghost number assignments and Grassmann parities of the antifields are related to their ordinary counterparts

$$\text{gh}(\Phi^*_I) = -\,\text{gh}(\Phi^I) - 1\,, \quad \epsilon(\Phi^*_I) = \epsilon(\Phi^I) + 1\,. \tag{5.24}$$

The set $\{\varphi_i, {C_s}^{\alpha_s}\}$ and $\{\varphi^*_i, C^*_{s\alpha_s}\}$ are called the *minimal* sets of fields and antifields.

The action $S_0[\varphi^i]$ is then extended to a functional of all the fields and antifields $S[\Phi,\Phi^*]$. At the moment, we will only require that it reduces to the classical action when all the antifields are set to zero:

$$S_0[\varphi^i] = S[\Phi, \Phi^* = 0]\,. \tag{5.25}$$

To get a concrete expression of $S[\Phi,\Phi^*]$ in terms of the fields and antifields we need to introduce the following device.

The Antibracket. For functions (or functionals) X and Y defined on the phase space spanned by $\{\Phi, \Phi^*\}$, we define an operation $(\,\cdot\,,\,\cdot\,)$ which is called the *antibracket*:

$$(X,Y) = \frac{\delta^R X}{\delta\Phi^I}\frac{\delta^L Y}{\delta\Phi^*_I} - \frac{\delta^R X}{\delta\Phi^*_I}\frac{\delta^L Y}{\delta\Phi^I}\,. \tag{5.26}$$

This bracket is very similar to the Poisson bracket in classical mechanics and we see immediately that the antifields Φ^* can be thought of as the conjugate momenta[3] to the normal fields Φ

$$(\Phi^I, \Phi^*_J) = \delta^I_J\,. \tag{5.27}$$

The antibracket carries ghost number +1 and it has odd Grassmann parity

$$\text{gh}[(X,Y)] = \text{gh}[X] + \text{gh}[Y] + 1\,, \epsilon[(X,Y)] = \epsilon_X + \epsilon_Y + 1\,. \tag{5.28}$$

The symmetric of the antibracket is

$$(X,Y) = -(-1)^{(\epsilon_X+1)(\epsilon_Y+1)}(Y,X)\,, \tag{5.29}$$

and it also satisfies a graded Jacobi identity (see [110] for a proof)

$$((X,Y),Z) + (-1)^{(\epsilon_X+1)(\epsilon_Y+\epsilon_Z)}((Y,Z),X) + (-1)^{(\epsilon_Z+1)(\epsilon_X+\epsilon_Y)}((Z,X),Y) = 0\,. \tag{5.30}$$

[3] There are canonical transformations which map $\{\Phi^I, \Phi^*_I\}$ to new pairs of variables that are functions of the old ones and the antibracket is preserved at the same time [109, 110], we will see examples of these transformations in our applications.

5.2.2 The Proper Solution

Let S be a ghost number zero, even functional of the collective variable $z^a = \{\Phi, \Phi^*\}$, $a = 1, 2, \ldots, 2N$. We call the equation

$$(S, S) = 0 \tag{5.31}$$

the *classical master equation.*
By introducing the matrix

$$\zeta^{ab} = \begin{pmatrix} 0 & \delta^I_J \\ -\delta^I_J & 0 \end{pmatrix}, \tag{5.32}$$

the antibracket can be brought into the form

$$(X, Y) = \frac{\delta^R X}{\delta z^a} \zeta^{ab} \frac{\delta^L Y}{\delta z^b}. \tag{5.33}$$

The master equation for S becomes

$$(S, S) = \frac{\delta^R S}{\delta z^a} \zeta^{ab} \frac{\delta^L S}{\delta z^b} = 0. \tag{5.34}$$

We compute the second derivative of S and define the matrix

$$\mathcal{R}^a_c \equiv \zeta^{ab} \frac{\delta^L \delta^R S}{\delta z^b \delta z^c}. \tag{5.35}$$

Then, for any solutions to the master equation (5.34) the following condition holds by taking a further derivative

$$\frac{\delta^R S}{\delta z^a} \mathcal{R}^a_b = 0, \tag{5.36}$$

which is an extended version of the Noether identities. It follows that the solution S admits gauge transformations generated by $\mathcal{R}^a_b$.

At the stationary point z_0 of S (where the "extended equation of motion" holds: $\left.\frac{\delta^R S}{\delta z}\right|_{z=z_0} = 0$), we have the equations

$$\left.\mathcal{R}^a_b \mathcal{R}^b_c\right|_{z_0} = 0. \tag{5.37}$$

Thus, the matrix $\mathcal{R}$ is a nilpotent matrix at the stationary points and have maximally rank N. The Hessian matrix of S is obtained by $\zeta^{-1}\mathcal{R}$, where ζ^{-1} is the inverse of the invertible matrix ζ and it preserves the rank. Thus,

$$\text{rank} \left.\frac{\delta^L \delta^R S}{\delta z^b \delta z^c}\right|_{z_0} \leq N, \tag{5.38}$$

and when the rank of the Hessian equals N, we call this solution S a *proper* solution of the master equation. One of the reasons to have the proper solution is that we need N independent gauge conditions, it is an indication that under this circumstance the N antifields can be removed.

From now on, we only focus on the proper solutions of the master equation. In order to make contact with the classical action S_0, we required the boundary condition (5.25). Being a proper solution with the the boundary condition (5.25) imply that S satisfies further conditions

$$\begin{aligned} \left.\frac{\delta^L \delta^R S}{\delta \varphi^*_i \delta C^{\alpha_0}_0}\right|_{\Phi^*=0} &= R^i_{\alpha_0}, \\ \left.\frac{\delta^L \delta^R S}{\delta C^*_{s-1\alpha_{s-1}} \delta C^{\alpha_s}_s}\right|_{\Phi^*=0} &= Z^{\alpha_{s-1}}_{s\alpha_s}, \quad s = 1, \ldots, L, \end{aligned} \tag{5.39}$$

where these conditions ensure that S contains information of the gauge algebra given by S_0. The existence of these conditions is shown in [112]. The proper solution satisfying (5.25) and based only on the minimal sector is called the minimal proper solution S^M. Its general form reads

$$S^{\mathrm{M}}[\Phi, \Phi^*] = S_0[\varphi] + \varphi^*_i R^i_{\alpha_0} C^{\alpha_0}_0 + \sum_{s=1}^{L} C^*_{s-1\alpha_{s-1}} Z^{\alpha_{s-1}}_{s\alpha_s} C^{\alpha_s}_s + \ldots, \tag{5.40}$$

where the omitted terms contain cubic interactions, higher power of the antifields and etc. For example, we can consider the Yang-Mills theory and it has a closed irreducible gauge algebra. The enlarged set of fields contains the gauge field $A^{a\mu}$, the ghost C^a for gauge transformation parameters their antifields, i.e. $\Phi = \{A^{a\mu}, C^a\}$ and $\Phi^* = \{A^*_{a\mu}, C^*_a\}$. The proper solution is

$$S = \int d^dx \left(-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + A^*_{a\mu} D^{\mu a}{}_b C^b + \frac{1}{2} C^*_c f_{ab}{}^c C^b C^a \right) , \tag{5.41}$$

where $D^{\mu a}{}_b$ is the covariant derivative and $f_{ab}{}^c$ is the properly normalised structure constant for the gauge group. One can easily check that $f_{ab}{}^c$ is to be identified with the gauge structure constant $T^\gamma_{\alpha\beta}$ in (5.8). In Maxwell theory the gauge group is abelian and $f_{ab}{}^c$ (or $T^\gamma_{\alpha\beta}$) vanishes, so there are only the first two terms of (5.41) appearing in the minimal proper solution. We see in this example that the cubic term carries the explicit information about the gauge algebra. More structures of the gauge symmetry (such as the on-shell closure) are also encoded in the proper solution depending on the case that one examines.

The BRST-Symmetry of the proper solution. The *BRST-symmetry* appears in the quantisation of gauge theories under the Faddeev-Popov method. In that procedure, one fixes the gauge and breaks gauge invariance of the action, but a graded, nilpotent symmetry with ghost number +1 arises for the gauge-fixed action and it is the BRST-symmetry. The original fields and ghosts transform both under the BRST-symmetry.

This symmetry comes very naturally out of the field-antifield formalism; it appears already in the proper solution before we proceed to the gauge-fixing procedure. We define the classical BRST-transformation s for any functional X as its antibracket with the proper solution

$$sX \equiv (X, S) , \tag{5.42}$$

as consequence of the classical master equation this is a symmetry of the proper solution $sS = (S, S) = 0$. The graded property reads

$$s(XY) = XsY + (-1)^{\epsilon_Y}(sX)Y , \tag{5.43}$$

note that on the product s acts from the right due to our definition. We write the Jacobi identity (5.30)

$$((X, S), S) + ((S, S), X) + (-1)^{\epsilon_X}((S, X), S) = 0 , \tag{5.44}$$

where we used $\epsilon[S]$=0. Because of $(S, S) = 0$, the Jacobi identity becomes

$$\begin{aligned} &((X, S), S) + (-1)^{\epsilon_X}((S, X), S) = 0 \\ \Longrightarrow\ &[1 + (-1)^{\epsilon_X}(-1)(-1)^{\epsilon_X+1}]((X, S), S) = 2((X, S), S) = 0 \\ \Longrightarrow\ &s^2 X = ((X, S), S) = 0 , \end{aligned} \tag{5.45}$$

where we used the graded symmetry (5.29) of the antibracket and s is a nilpotent symmetry. The BRST transformations on the fields and antifields is easy to compute, by definition of s they are given by the functional derivatives of the action S with respect to the corresponding antifields and fields

$$s\Phi^I = \frac{\delta^L S}{\delta \Phi^*_I} , \quad s\Phi^*_I = (-1)^{\epsilon_I+1} \frac{\delta^R S}{\delta \Phi^*_I} . \tag{5.46}$$

For instance, we have

$$s\Phi^I = R^i_{\alpha_0} C^{\alpha_0} + \dots , \tag{5.47}$$

and by looking at (5.40), one comes to the idea that antifields can be interpreted as the sources for the BRST-transformations, although we did not write the terms in the ... and this is not completely recognisable for the moment. This interpretation will be clear after we gauge-fix the action and introduce the gauge-fixed BRST-symmetry.

5.3 The gauge-fixing fermion

In this section we turn to the question of gauge-fixing and we reach the key point of covariant path integral quantisations. To eliminate the antifields, we consider the surfaces in the phase space given by the following condition[4]

$$\Sigma = \left\{ \{\Phi, \Phi^*\} | \Phi^*_A = \frac{\delta\Psi(\Phi)}{\delta\Phi^A} \right\}, \tag{5.48}$$

where Ψ is a Grassmann odd functional with ghost number -1, i.e.

$$\epsilon(\Psi) = 1\,, \quad \text{gh}(\Psi) = -1\,. \tag{5.49}$$

When a functional X restricted on the surface Σ, we mean to replace the dependence of X on the antifields Φ^*_A simply by $\frac{\delta\Psi(\Phi)}{\delta\Phi^A}$.

The independence of gauge-fixing. Passing to the quantisation, we need to have a well-defined path integral and some quantum actions which appear in the integrand. Let us start with introducing the operator

$$\Delta \equiv (-1)^{\epsilon_A+1} \frac{\delta^R}{\delta\Phi^A} \frac{\delta^R}{\delta\Phi^*_A}\,. \tag{5.50}$$

Let bosonic functional $W = W[\Phi, \Phi^*]$ be assumed to satisfy the equation

$$\Delta \exp\left(\frac{i}{\hbar} W\right) = 0\,, \tag{5.51}$$

which is equivalent to

$$\frac{1}{2}(W, W) = i\hbar \Delta W\,. \tag{5.52}$$

The equation (5.52) is called the *quantum master equation* and $\hbar$ is the reduced Planck's constant. The functional W plays the role of a extended quantum action. We want the path integral

$$\begin{aligned} I_\Psi[\Phi, \Phi^*] &= \int \mathcal{D}[\Phi]\mathcal{D}[\Phi^*]\delta\left(\Phi^*_A - \frac{\delta\Psi}{\delta\Phi^A}\right) \exp\left(\frac{i}{\hbar} W[\Phi, \Phi^*]\right) \\ &= \int \mathcal{D}[\Phi] \exp\left(\frac{i}{\hbar} W[\Phi, \Phi^* = \frac{\delta\Psi}{\delta\Phi}]\right)\,. \end{aligned} \tag{5.53}$$

to be well-defined, meaning, at stationary points propagators exist. The form of the path integral also indicates that the elimination of antifields is realised by the delta-function. If the quantum master equation (5.52) is satisfied, we have a path integral which is independent of the choice of the gauge-fixing fermions. Let the gauge-fixing fermion have a infinitesimal change $\Psi \to \Psi + \delta\Psi$, the change of the path integral is[5]

$$\begin{aligned} I_{\Psi+\delta\Psi} - I_\Psi &= \int \mathcal{D}[\Phi] \left(\exp\left(\frac{i}{\hbar} W[\Phi, \frac{\delta(\Psi+\delta\Psi)}{\delta\Phi}]\right) - \exp\left(\frac{i}{\hbar} W[\Phi, \frac{\delta\Psi}{\delta\Phi}]\right) \right) \\ &= \int \mathcal{D}[\Phi] \left. \frac{\delta^R\left(\exp\left(\frac{i}{\hbar} W[\Phi, \Phi^*]\right)\right)}{\delta\Phi^*_A} \right|_\Sigma \frac{\delta^L(\delta\Psi)}{\delta\Phi^A} + O\left((\delta\Psi)^2\right) \\ &= \int \mathcal{D}[\Phi] \Delta \exp\left(\frac{i}{\hbar} W\right) \delta\Psi + O\left((\delta\Psi)^2\right) \\ &= O\left((\delta\Psi)^2\right) \end{aligned} \tag{5.54}$$

[4]There is no need to distinguish between left or right derivatives of Ψ with respect to the fields $\frac{\delta\Psi(\Phi)}{\delta\Phi^A} \equiv \frac{\delta^L\Psi(\Phi)}{\delta\Phi^A} = (-1)^{\epsilon_A(\epsilon(\psi)+1)} \frac{\delta^R\Psi(\Phi)}{\delta\Phi^A} = \frac{\delta^R\Psi(\Phi)}{\delta\Phi^A}$.

[5]We used the integration by part in the second last step: $\int \mathcal{D}\phi \frac{\delta^R X}{\delta\phi} Y = (-1)^{\epsilon(\phi)+1} \int \mathcal{D}\phi X \frac{\delta^L Y}{\delta\phi}$.

If the solution to the quantum master equation (5.52) is expanded in power series of $\hbar$ we will write

$$W = S + \sum_{p=1}^{\infty} \hbar^p M_p \,, \tag{5.55}$$

and the equation (5.52) becomes

$$\begin{aligned} &p = 0 : (S, S) = 0 \,, \\ &p = 1 : (M_1, S) = i\Delta S \,, \\ &p \geq 2 : (M_p, S) = i\Delta M_{p-1} - \frac{1}{2}\sum_{q=1}^{p-1}(M_p, M_{p-q}) \,. \end{aligned} \tag{5.56}$$

In the classical limit $\hbar \longrightarrow 0$, W reduces to S and we identify it with the classical action. The requirement that W satisfies the quantum master equation implies S is a solution of the classical master equation.

In the above discussions, the assumption of the well-definedness of the path integral (5.53) is crucial, and it is the only restriction that one needs when constructing an appropriate gauge-fixing fermion. The reducibility of gauge structure plays a central role in determining this restriction and for reducible theories of different stages the restriction conditions on the gauge-fixing fermion need to be formulated similarly but in a tedious way. The gauge-fixing fermion appears in our applications will be given explicitly and they are the admissible ones that we are going to explain. For general cases, we refer to [109, 110] for pedagogical reviews.

Trivial pairs and auxiliary fields. We want to use the condition $\Phi^*_A = \frac{\delta\Psi(\Phi)}{\delta\Phi^A}$ to eliminate all the antifields. But the gauge-fixing fermion has ghost number -1 and the variables required for the minimal proper solution have all non-negative ghost numbers. Thus, by ghost number conservation, we need to introduce new fields and their corresponding antifields to make the elimination possible. This can be done by adding so called *"trivial pairs"*. For example, we add to our minimal sector the a pair of fields and their antifields $\{X, Y,\ X^*, Y^*\}$, such that

$$\mathrm{gh}[Y] = \mathrm{gh}[X] + 1 \,, \epsilon(Y) = \epsilon(X) + 1 \,. \tag{5.57}$$

Then we have $\mathrm{gh}[X^*] = -\,\mathrm{gh}[X] - 1 = -\,\mathrm{gh}[X]$ and $\epsilon(X^*) = \epsilon(X) + 1 = \epsilon(Y)$, so we can add

$$S_{\text{trivial}} = \int X^* Y \tag{5.58}$$

to the action S^M to get the *non-minimal solution* S^{MN} to the master equation.[6]

5.4 Gauge-fixing for irreducible theories and the Nielsen-Kallosh ghost

We begin with the irreducible case and start from an action $S_0[\varphi^i]$ depending on fields φ^i, invariant under some gauge invariances $\delta\varphi^i = R^i_\alpha \Lambda^\alpha$. To avoid cumbersome notations, we just write c^α for the ghost corresponding to the gauge parameter Λ^α, and the minimal sector includes the antifields φ^*_i and c^*_α. The gauge-fixing condition will be written $\chi^\alpha(\varphi) = 0$; the function $\chi^\alpha(\varphi)$ has the same index structure and Grassmann parity as the gauge parameter Λ^α. Ghost number assignments and Grassmann parities are collected in table 5.1.

The minimal solution to the classical master equation is

$$S^{\mathrm{M}}[\varphi^i, c^\alpha; \varphi^*_i, c^*_\alpha] = S_0[\varphi] + \varphi^*_i R^i_\alpha c^\alpha + \dots \,, \tag{5.59}$$

again we omitted the terms in higher power of the antifields (cf. (5.40)).

[6] This is based on the fact that the proper solution of the classical master equation is unique up to addition of trivial pairs and canonical transformations [110].

	φ_i	c^α	c'^α	b^α	S_0, S^{M}, S^{NM}	Ψ	χ^α
gh	0	1	-1	0	0	-1	0
ϵ	ϵ_i	$\epsilon_\alpha+1$	$\epsilon_\alpha+1$	ϵ_α	0	1	ϵ_α

Table 5.1: Ghost numbers and Grassmann parities of the various objects appearing in the irreducible case.

To gauge-fix the theory, one further extends the space of fields by adding a trivial pair (c'^α, b^α) of ghost numbers -1 and 0 respectively, along with their antifields.[7] The minimal action S^{M} is then extended to the non-minimal

$$S^{\mathrm{NM}} = S^{\mathrm{M}}[\varphi^i, c^\alpha; \varphi^*_i, c^*_\alpha] + c'^*_\alpha b^\alpha\,, \tag{5.60}$$

which still satisfies the master equation. As planned, we eliminated the antifields according to the formula

$$\Phi^*_I = \frac{\delta\Psi}{\delta\Phi^I}\,. \tag{5.61}$$

If the gauge-fixing fermion Ψ is well-chosen, the resulting action is properly gauge-fixed and possesses well-defined propagators.

Delta-function gauge-fixing. The simplest example is the *delta-function gauge-fixing*: here, one can simply take the gauge-fixing fermion as

$$\Psi_\delta = c'_\alpha \chi^\alpha(\varphi)\,. \tag{5.62}$$

It has the requisite ghost number -1 and statistical parity.

This gives the gauge-fixed action

$$S_\delta[\varphi^i, c^\alpha, c'_\alpha, b_\alpha] \equiv S^{\mathrm{NM}}\left[\Phi^I, \Phi^*_I = \frac{\delta\Psi_\delta}{\delta\Phi^I}\right] \tag{5.63}$$

$$= S^{\mathrm{M}}\left[\varphi^i, c^\alpha; \varphi^*_i = c'_\alpha\frac{\delta^R\chi^\alpha}{\delta\varphi^i}, c^*_\alpha = 0\right] + \chi^\alpha(\varphi)b_\alpha\,. \tag{5.64}$$

The field b^α is auxiliary and enforces the gauge-fixing constraint $\chi^\alpha(\varphi) = 0$. Integration over the field b^α leads to the insertion of $\delta\left(\chi^\alpha(\varphi) = 0\right)$. The fields c^α and c'_α are the usual Faddeev-Popov ghost-antighost pairs. Note however that formula (5.64) is also correct for e.g. theories with open algebras where the usual Faddeev-Popov procedure cannot be applied; these subtleties only appear in the explicit form of S^{M}.

Gaussian gauge-fixing. It can also be convenient to use the *Gaussian gauge-fixing*, which will bring a gauge-breaking term in the action of the form $\chi_\alpha(\varphi)M^{\alpha\beta}\chi_\beta(\varphi)$ with some non-degenerate matrix M. This can be obtained by including terms linear in the auxiliary fields in Ψ_{G}:

$$\Psi_{\mathrm{G}} = c'_\alpha\chi^\alpha(\varphi) + \frac{1}{2}c'_\alpha(M^{-1})^{\alpha\beta}b_\beta\,, \tag{5.65}$$

giving

$$S[\varphi^i, c^\alpha, c'_\alpha, b_\alpha] = S^M\left[\varphi^i, c^\alpha; \varphi^*_i = c'_\alpha\frac{\delta^R\chi^\alpha}{\delta\varphi^i}, c^*_\alpha = 0\right] + \chi^\alpha(\varphi)b_\alpha + \frac{1}{2}(M^{-1})^{\alpha\beta}b_\beta b_\alpha\,. \tag{5.66}$$

Here, b^α is a simple auxiliary field appearing quadratically in the action. Eliminating it using its own equation of motion yields the looked-after gauge-breaking term $\chi_\alpha M^{\alpha\beta}\chi_\beta$. However, in some applications one would like this term to contain a differential operator, $M = \mathcal{D}$. Then, the above procedure is problematic since the non-local object $\mathcal{D}^{-1}$ appears in the gauge-fixing fermion (5.65) and the action (5.66).

[7] In some cases, it can be more convenient to take fields with down indices instead, and we will sometimes do this in the following. Note also that c' is often written $\bar{c}$; however, since we will be dealing with fermionic theories in the applications, this could be confused with the Dirac conjugate and we will stick with the prime notation in this part of the thesis.

The Nielsen-Kallosh ghost. It is well-known that gauge-breaking terms of the above mentioned form lead to a *third* propagating ghost, the *Nielsen-Kallosh ghost* (the first two ghosts being the usual Faddeev-Popov ghosts c^α and antighosts c'^α) [59, 60]. The third ghost is nothing but the b^α field, which stops being auxiliary and propagates with kinetic operator $\mathcal{D}$. This was first described within this formalism, while maintaining manifest locality throughout, by Batalin and Kallosh in reference [61] as we review shortly now.

The trick is to use the freedom to do a canonical transformation, which preserves the antibracket and maps solutions of the master equation to solutions, and only after that replace the antifields using a gauge-fixing fermion. In the simple case where the gauge condition χ^α only depends on the original fields φ^i, the canonical transformation reads

$$b^\alpha \to \tilde{b}^\alpha = b^\alpha - \chi^\alpha(\varphi) \tag{5.67}$$

$$\varphi^*_i \to \tilde{\varphi}^*_i = \varphi^*_i + b^*_\alpha \frac{\delta^R \chi^\alpha}{\delta \varphi^i} \,. \tag{5.68}$$

(the gauge condition χ^α is also allowed to depend on the ghost fields c^α, c'^α or b^α, in which case the canonical transformation is more complicated; see [61]), with other variables unchanged.

To check that this transformation is canonical, compute

$$\tilde{\varphi}^*_i d\tilde{\varphi}^i + \tilde{b}^*_\alpha d\tilde{b}^\alpha = \left(\varphi^*_i + b^*_\alpha \frac{\delta^R \chi^\alpha}{\delta\varphi^i}\right) d\varphi^i + b^*_\alpha d(b^\alpha - \chi^\alpha(\varphi)) = \varphi^*_i d\varphi^i + b^*_\alpha db^\alpha \,,$$

which is the field-antifield analogue of the condition $p'_i dq'^i = p_i dq^i$ in classical mechanics. Another way is to notice that this transformation is generated by $F = b^*_\alpha \chi^\alpha(\varphi)$ via the antibracket, i.e. takes the form

$$\Phi^I \to \Phi^I + (F, \Phi^I) \,, \quad \Phi^*_I \to \Phi^*_I + (F, \Phi^*_I) \,, \quad F = b^*_\alpha \chi^\alpha(\varphi) \,.$$

This maps the non-minimal action (5.60) to

$$\widetilde{S}^{\rm NM} = S^{\rm M}\left[\varphi^i, c^\alpha, \varphi^*_i + b^*_\alpha \frac{\delta^R \chi^\alpha}{\delta \varphi^i}, c^*_\alpha\right] + c'^*_\alpha (b^\alpha - \chi^\alpha(\varphi)) \,, \tag{5.69}$$

which still satisfies the master equation since the transformation is canonical. Using the gauge-fixing fermion

$$\Psi_{\rm G} = \frac{1}{2} c'^\alpha \mathcal{D}_{\alpha\beta}(\varphi) \left(\chi^\beta(\varphi) + b^\beta\right) \,. \tag{5.70}$$

now gives

$$S_{\rm G}[\varphi^i, c^\alpha, c'_\alpha, b_\alpha] \equiv \widetilde{S}^{\rm NM}\left[\Phi^I, \Phi^*_I = \frac{\delta \Psi_{\rm G}}{\delta \Phi^I}\right] \tag{5.71}$$

$$= S^{\rm M}\left[\varphi^i, c^\alpha; c'^\alpha \mathcal{D}_{\alpha\beta} \frac{\delta^R \chi^\beta}{\delta\varphi^i} + \frac{1}{2} c'^\alpha \frac{\delta^R \mathcal{D}_{\alpha\beta}}{\delta \varphi^i} \left(\chi^\beta + b^\beta\right)(-1)^{\epsilon_i \epsilon_\beta}, 0\right]$$

$$- \frac{1}{2} \mathcal{D}_{\alpha\beta}\, \chi^\beta(\varphi) \chi^\alpha(\varphi) + \frac{1}{2} \mathcal{D}_{\alpha\beta} b^\beta b^\alpha \,. \tag{5.72}$$

This action contains the desired gauge-breaking term $\mathcal{D}_{\alpha\beta}\, \chi^\beta(\varphi)\chi^\alpha(\varphi)$, along with a quadratic term in b^α. This construction is most relevant when the operator $\mathcal{D}_{\alpha\beta}$ is field dependent: then, b^α is coupled to the other fields (including the ghosts c^α and c'^α) and cannot be ignored in Feynman diagram computations.

5.5 Gauge-fixing for first-stage reducible theories and the "third ghost"

In the rest part of this chapter, we will show that the "third ghost" will appear in the Gaussian gauge-fixing procedure in reducible theories. We consider now a first-stage reducible theory with

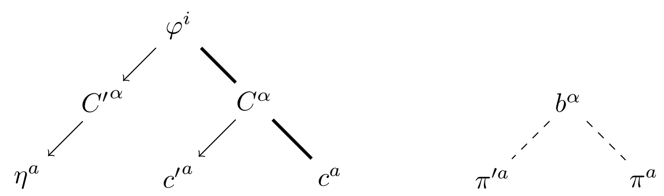


Figure 5.1: The pyramid of ghosts fields in the first-stage reducible case [111]. The fields linked by a thick line constitute the minimal BV sector; an arrow $a \to b$ indicates that the field b (along with its partner in a trivial pair) is introduced to fix the gauge freedom of a. The second pyramid shows the partners of the non-minimal fields of the first pyramid. Ghost numbers of the fields have alternating increasing and decreasing patterns for both pyramids, see figure 5.2 for an explicit assignment.

	φ^i	C^α	C'^α	c^a	c'^a	η^a	b^α	π^a	π'^a
gh	0	1	-1	2	-2	0	0	-1	1
ϵ	ϵ_i	$\epsilon_\alpha + 1$	$\epsilon_\alpha + 1$	ϵ_a	ϵ_a	ϵ_a	ϵ_α	$\epsilon_a + 1$	$\epsilon_a + 1$

Table 5.2: Ghost numbers and Grassmann parities of the various fields in the first-stage reducible case.

an action $S_0[\varphi^i]$ invariant under m gauge transformations $\delta\varphi^i = R^i_\alpha \Lambda^\alpha$, which themselves are invariant under n reducibility ("gauge-for-gauge") transformations $\delta\Lambda^\alpha = Z^\alpha_a \lambda^a$,

$$\frac{\delta^R S_0}{\delta\varphi^i} R^i_\alpha = 0\,, \quad R^i_\alpha Z^\alpha_a = 0\,. \tag{5.73}$$

We assume that there are no further reducibilities. The number of independent gauge redundancies in the fields φ^i is therefore $m - n$. Accordingly, the gauge-fixing condition $\chi^\alpha(\varphi) = 0$ must only contain $m - n$ independent conditions. We will take it to satisfy n constraints:

$$X_{a\alpha}\,\chi^\alpha(\varphi) = 0 \tag{5.74}$$

with $X_{a\alpha}$ of maximal rank.

In the minimal BV sector, we therefore have the original fields φ^i, the ghost C^α corresponding to the gauge parameter Λ^α, and the ghost-for-ghost c^a corresponding to the reducibility parameter λ^a, along with their antifields. The proper solution S^{M} to the master equation starts as

$$S^{\mathrm{M}}[\varphi^i, C^\alpha, c^a; \varphi^*_i, C^*_\alpha, c^*_a] = S_0[\varphi] + \varphi^*_i R^i_\alpha C^\alpha + C^*_\alpha Z^\alpha_a c^a + \dots\,. \tag{5.75}$$

To build the non-minimal action, one introduces *three* extra trivial pairs: $\{C'^\alpha, b^\alpha\}$ to fix the gauge freedom of φ^i, but also two more, $\{\eta^a, \pi'^a\}$ and $\{c'^a, \pi^a\}$, to fix the gauge freedom of the ghosts C'^α and C^α, as in figure 5.1. Their ghost numbers and Grassmann parities can be found in table 5.2.

The non-minimal action is then

$$S^{\mathrm{NM}} = S^{\mathrm{M}} + C'^*_\alpha b^\alpha + c'^*_a \pi^a + \eta^*_a \pi'^a\,. \tag{5.76}$$

Delta function gauge-fixing. This case is well-known [53]: simply take the gauge-fixing fermion

$$\Psi_\delta = C'_\alpha \chi^\alpha(\varphi) + c'^a \omega_{a\alpha} C^\alpha + \eta^a \sigma^\alpha_a C'_\alpha\,, \tag{5.77}$$

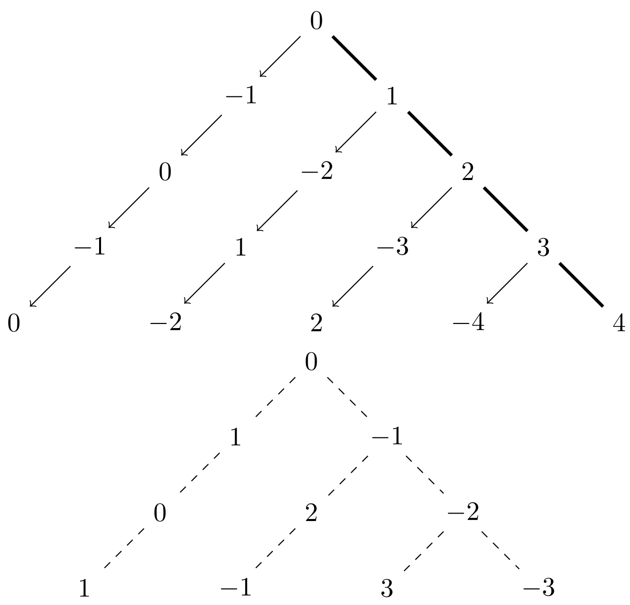


Figure 5.2: The example of ghost numbers assignments in the third-stage reducible case, this generalises to higher reducibility cases.

where ω and σ are of maximal rank and we take the pair (C'_α, b_α) to have indices down for this paragraph only. The gauge-fixed action then reads

$$S_\delta \equiv S^{\rm NM}\left[\Phi^I, \Phi^*_I = \frac{\delta\Psi_\delta}{\delta\Phi^I}\right] \tag{5.78}$$

$$= S^{\rm M}\left[\varphi^i, C^\alpha, c^a; \varphi^*_i = C'_\alpha \frac{\delta^R\chi^\alpha}{\delta\varphi^i}, C^*_\alpha = c'^a\omega_{a\alpha}, c^*_a = 0\right]$$
$$+ \left(\chi^\alpha(\varphi) + \eta^a\sigma^\alpha_a\right) b_\alpha + \omega_{a\alpha}C^\alpha\pi^a + \sigma^\alpha_a C'_\alpha \pi'^a\,. \tag{5.79}$$

In this action, the ghosts C^α and C'_α are both gauge fields. Their gauge invariances are fixed by the $2m$ gauge conditions $\omega_{a\alpha}C^\alpha = 0$ and $\sigma^\alpha_a C'^\alpha = 0$ imposed by the auxiliary fields π^a and π'^a. The field b^α is also auxiliary and imposes the equation

$$\chi^\alpha(\varphi) + \eta^a\sigma^\alpha_a = 0\,. \tag{5.80}$$

Among these m conditions, $m-n$ fix the gauge invariance of the original fields φ^i, and the remaining n set the extra ghost η to zero.

Gaussian gauge-fixing. Now, we would like to achieve the gauge-fixing term $\mathcal{D}_{\alpha\beta}\chi^\beta\chi^\alpha$. As before, the field b^α will become propagating if $\mathcal{D}$ is a differential operator. However, an important difference with the irreducible case is that b^α is a constrained field, satisfying the same constraint (5.74) as the gauge condition.

One starts with the same canonical transformation as in the irreducible case:

$$b^\alpha \to \tilde b^\alpha = b^\alpha - \chi^\alpha(\varphi) \tag{5.81}$$

$$\varphi^*_i \to \tilde\varphi^*_i = \varphi^*_i + b^*_\alpha \frac{\delta^R\chi^\alpha}{\delta\varphi^i} \tag{5.82}$$

with other fields unchanged. We take the gauge-fixing fermion

$$\Psi_{\rm G} = \frac{1}{2}C'^\alpha\mathcal{D}_{\alpha\beta}(\varphi)\left(\chi^\beta(\varphi) + b^\beta\right) + c'^a\omega_{a\alpha}C^\alpha + \eta^a\sigma_{a\alpha}C'^\alpha\,, \tag{5.83}$$

which is of the same form as Ψ_δ, with only the first term modified. Eliminating the antifields using Ψ_G then gives

$$S_G \equiv \widetilde{S}^{NM}\left[\Phi^I, \Phi^*_I = \frac{\delta\Psi_G}{\delta\Phi^I}\right] \tag{5.84}$$

$$\begin{aligned} &= S^M\left[\varphi^i, C^\alpha, c^a; C'^\alpha\mathcal{D}_{\alpha\beta}\frac{\delta^R\chi^\beta}{\delta\varphi^i} + \frac{1}{2}C'^\alpha\frac{\delta^R\mathcal{D}_{\alpha\beta}}{\delta\varphi^i}\left(\chi^\beta + b^\beta\right)(-1)^{\epsilon_i\epsilon_\beta}, c'^a\omega_{\alpha a}, 0\right] \\ &\quad - \frac{1}{2}\mathcal{D}_{\alpha\beta}\chi^\beta\chi^\alpha + \frac{1}{2}\mathcal{D}_{\alpha\beta}b^\beta b^\alpha \\ &\quad + \eta^a\sigma_{a\alpha}(b^\alpha - \chi^\alpha) + \pi^a\omega_{a\alpha}C^\alpha + \pi'^a\sigma_{a\alpha}C'^\alpha . \end{aligned} \tag{5.85}$$

Because of the constraint (5.74) satisfied by $\chi^\alpha(\varphi)$, there is a privileged choice for the matrix $\sigma_{a\alpha}$: simply take $\sigma = X$. This gets rid of the unwanted term $\eta^a\sigma_{a\alpha}\chi^\alpha$ in the last line, and one remains with

$$\begin{aligned} S_G = S^M&\left[\varphi^i, C^\alpha, c^a; C'^\alpha\mathcal{D}_{\alpha\beta}\frac{\delta^R\chi^\beta}{\delta\varphi^i} + \frac{1}{2}C'^\alpha\frac{\delta^R\mathcal{D}_{\alpha\beta}}{\delta\varphi^i}\left(\chi^\beta + b^\beta\right)(-1)^{\epsilon_i\epsilon_\beta}, c'^a\omega_{\alpha a}, 0\right] \\ &- \frac{1}{2}\mathcal{D}_{\alpha\beta}\chi^\beta\chi^\alpha + \frac{1}{2}\mathcal{D}_{\alpha\beta}b^\beta b^\alpha + \pi^a\omega_{a\alpha}C^\alpha + \pi'^a X_{a\alpha}C'^\alpha + \eta^a X_{a\alpha}b^\alpha , \end{aligned} \tag{5.86}$$

featuring the desired gauge-breaking term $\mathcal{D}_{\alpha\beta}\chi^\beta\chi^\alpha$.

The extra propagating field. Just as in the irreducible case, the field b^α is propagating whenever $\mathcal{D}$ contains derivatives, and couples to the other fields and ghosts if $\mathcal{D}$ is field-dependent. This generalises a result of [61] about the "third ghost" to the reducible case.

In the action (5.86), the auxiliary fields π^a and π'^a impose the gauge conditions

$$\omega_{a\alpha}C^\alpha = 0\,, \quad X_{a\alpha}C'^\alpha = 0 \tag{5.87}$$

on the ghost fields C^α and C'^α, as in the delta-function gauge-fixing case. On the other hand, η^a plays here a very different role as it did in (5.79): it is now a Lagrange multiplier for the constraint

$$X_{a\alpha}\, b^\alpha = 0 \tag{5.88}$$

on the field b^α. Notice how both C'^α and b^α satisfy the same constraint as $\chi^\alpha(\varphi)$ in this gauge-fixing scheme.

Strictly speaking, this Gaussian gauge-fixing procedure is only "partially" Gaussian in the sense that there is only the gauge-breaking term for φ^i but not for the gauge fields C^α, C'^α and b^α. They are delta-function gauge-fixed.

5.6 Higher stage reducibility

This procedure generalises straightforwardly to theories with higher degree of reducibility. For concreteness, we write out the second-stage reducible case here. So, we consider an action $S_0[\varphi^i]$ with second-stage reducible gauge symmetries:

$$\delta\varphi^i = R^i{}_{\alpha_0}\Lambda^{\alpha_0}\,, \quad \delta\Lambda^{\alpha_0} = Z^{\alpha_0}{}_{\alpha_1}\lambda^{\alpha_1}\,, \quad \delta\lambda^{\alpha_1} = z^{\alpha_1}{}_{\alpha_2}\epsilon^{\alpha_2} \tag{5.89}$$

with $\alpha_0 = 1, \ldots, m$, $\alpha_1 = 1, \ldots, n$ and $\alpha_2 = 1, \ldots, r$. Invariance under these transformations is equivalent to the relations

$$\frac{\delta^R S_0}{\delta\varphi^i}R^i{}_{\alpha_0} = 0\,, \quad R^i{}_{\alpha_0}Z^{\alpha_0}{}_{\alpha_1} = 0\,, \quad Z^{\alpha_0}{}_{\alpha_1}z^{\alpha_1}{}_{\alpha_2} = 0\,, \tag{5.90}$$

and we assume that there are no further reducibilities. The gauge condition $\chi^{\alpha_0}(\varphi) = 0$ must fix the $m - n + r$ independent gauge transformations: we take it to satisfy constraints $X_{\alpha_1\alpha_0}\chi^{\alpha_0} = 0$ as in the previous case, but here with a degenerate matrix X of rank $n - r$.

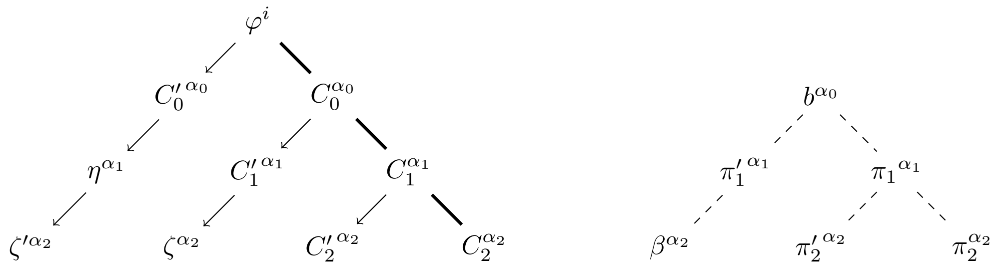


Figure 5.3: The pyramids of ghost fields in the second-stage reducible case [61]. This should be read in the same way as figure 5.1.

In the minimal BV sector, there are now three generations of ghosts: $C_0^{\alpha_0}$, $C_1^{\alpha_1}$ and $C_2^{\alpha_2}$. The first few terms in the minimal action are simply

$$\begin{aligned} &S^M[\varphi^i, C_0^{\alpha_0}, C_1^{\alpha_1}, C_2^{\alpha_2};\, \varphi_i^*, C^*_{0\,\alpha_0}, C^*_{1\,\alpha_1}, C^*_{2\,\alpha_2}] \\ &\quad = S_0[\varphi^i] + \varphi_i^* R^i{}_{\alpha_0} C_0^{\alpha_0} + C^*_{0\,\alpha_0} Z^{\alpha_0}{}_{\alpha_1} C_1^{\alpha_1} + C^*_{1\,\alpha_1} z^{\alpha_1}{}_{\alpha_2} C_2^{\alpha_2} + \dots \, . \end{aligned} \tag{5.91}$$

For gauge fixing, we add the usual extra pairs, as in figure 5.3.

The Non-minimal action reads

$$\begin{aligned} S^{NM} = {} & S^M[\varphi^i, C_0^{\alpha_0}, C_1^{\alpha_1}, C_2^{\alpha_2};\, \varphi_i^*, C^*_{0\,\alpha_0}, C^*_{1\,\alpha_1}, C^*_{2\,\alpha_2}] \\ & + C'^*_{0\alpha_0} b^{\alpha_0} \\ & + C'^*_{1\alpha_1} \pi_1^{\alpha_1} + \eta^*_{\alpha_1} \pi_1'^{\alpha_1} \\ & + C'^*_{2\alpha_2} \pi_2{}^{\alpha_2} + \zeta^*_{\alpha_2} \pi_2'^{\alpha_2} + \zeta'^*{}_{\alpha_2} \beta^{\alpha_2} \, . \end{aligned} \tag{5.92}$$

Delta-function gauge-fixing. We use the following gauge-fixing fermion

$$\Psi_\delta = C_0'^{\alpha_0} \chi_{\alpha_0}(\varphi) + C_1'^{\alpha_1} (\omega_1)_{\alpha_1\alpha_0} C_0{}^{\alpha_0} + C_2'^{\alpha_2} (\omega_2)_{\alpha_2\alpha_1} C_1{}^{\alpha_1} \tag{5.93}$$

$$\quad + \eta^{\alpha_1} (\sigma_1)_{\alpha_1\alpha_0} C_0'^{\alpha_0} + \zeta^{\alpha_2} (\sigma_2)_{\alpha_2\alpha_1} C_1'^{\alpha_1} + \zeta'^{\alpha_2} (\sigma_2')_{\alpha_2\alpha_1} \eta^{\alpha_1} \, . \tag{5.94}$$

This gives the gauge-fixed action

$$\begin{aligned} S_\delta = S^M \Big[& \varphi^i, C_0^{\alpha_0}, C_1^{\alpha_1}, C_2^{\alpha_2};\, C_0'^{\alpha_0} \frac{\delta^R \chi_{\alpha_0}(\varphi)}{\delta \varphi^i}, C_1'^{\alpha_1} (\omega_1)_{\alpha_1\alpha_0}, C_2'^{\alpha_2} (\omega_2)_{\alpha_2\alpha_1}, 0 \Big] \\ & + (\chi_{\alpha_0}(\varphi) + \eta^{\alpha_1} (\sigma_1)_{\alpha_1\alpha_0}) b^{\alpha_0} \\ & + ((\omega_1)_{\alpha_1\alpha_0} C_0{}^{\alpha_0} + \zeta^{\alpha_2} (\sigma_2)_{\alpha_2\alpha_1}) \pi_1^{\alpha_1} + ((\sigma_1)_{\alpha_1\alpha_0} C_0'^{\alpha_0} + \zeta'^{\alpha_2} (\sigma_2')_{\alpha_2\alpha_1}) \pi_1'^{\alpha_1} \\ & + (\omega_2)_{\alpha_2\alpha_1} C_1{}^{\alpha_1} \pi_2^{\alpha_2} + (\sigma_2)_{\alpha_2\alpha_1} C_1'^{\alpha_1} \pi_2'^{\alpha_2} + (\sigma_2')_{\alpha_2\alpha_1} \eta^{\alpha_1} \beta^{\alpha_2} \, . \end{aligned} \tag{5.95}$$

The propagating fields are φ^i and the Faddeev-Popov ghost-antighost pairs $\{C_0^{\alpha_0}, C_0'^{\alpha_0}\}$, $\{C_1^{\alpha_1}, C_1'^{\alpha_1}\}$ and $\{C_2^{\alpha_2}, C_2'^{\alpha_2}\}$.

Gaussian gauge-fixing. The non-minimal action, after the canonical transformation (5.81), then reads

$$\begin{aligned} \widetilde{S}^{\rm NM} = S^{\rm M} \Big[& \varphi^i, C_0^{\alpha_0}, C_1^{\alpha_1}, C_2^{\alpha_2};\, \varphi_i^* + b^*_{\alpha_0} \frac{\delta^R \chi^{\alpha_0}}{\delta \varphi^i}, C^*_{0\,\alpha_0}, C^*_{1\,\alpha_1}, C^*_{2\,\alpha_2} \Big] \\ & + C'^*_{0\alpha_0} (b^{\alpha_0} - \chi^{\alpha_0}(\varphi)) + C'^*_{1\alpha_1} \pi_1^{\alpha_1} + \eta^*_{\alpha_1} \pi_1'^{\alpha_1} \\ & + C'^*_{2\alpha_2} \pi_2{}^{\alpha_2} + \zeta^*_{\alpha_2} \pi_2'^{\alpha_2} + \zeta'^*{}_{\alpha_2} \beta^{\alpha_2} \, . \end{aligned} \tag{5.96}$$

We use the gauge-fixing fermion

$$\begin{aligned} \Psi_{\rm G} = {} & \frac{1}{2} C_0'^{\alpha_0} \mathcal{D}_{\alpha_0\beta_0}(\varphi) (\chi^{\beta_0}(\varphi) + b^{\beta_0}) + C_1'^{\alpha_1} (\omega_1)_{\alpha_1\alpha_0} C_0{}^{\alpha_0} + C_2'^{\alpha_2} (\omega_2)_{\alpha_2\alpha_1} C_1{}^{\alpha_1} \\ & + \eta^{\alpha_1} X_{\alpha_1\alpha_0} C_0'^{\alpha_0} + \zeta^{\alpha_2} (\sigma_2)_{\alpha_2\alpha_1} C_1'^{\alpha_1} + \zeta'^{\alpha_2} (\sigma_2')_{\alpha_2\alpha_1} \eta^{\alpha_1} \, , \end{aligned} \tag{5.97}$$

where ω_1 and X are of rank $n-r$ and ω_2, σ_2 and σ_2' are of maximal rank r. The gauge-fixed action is then

$$\begin{aligned}
S_{\mathrm{G}} = S^{\mathrm{M}}\Big[&\varphi^i, C_0^{\alpha_0}, C_1^{\alpha_1}, C_2^{\alpha_2};\\
&C_0'^{\,\alpha_0}\mathcal{D}_{\alpha_0\beta_0}\frac{\delta^R\chi^{\beta_0}}{\delta\varphi^i} + \frac{1}{2}C_0'^{\,\alpha_0}\frac{\delta^R\mathcal{D}_{\alpha_0\beta_0}}{\delta\varphi^i}\left(\chi^{\beta_0}+b^{\beta_0}\right)(-1)^{\epsilon_i\epsilon_{\beta_0}},\\
&C_1'^{\,\alpha_1}(\omega_1)_{\alpha_1\alpha_0}, C_2'^{\,\alpha_2}(\omega_2)_{\alpha_2\alpha_1}, 0\Big]\\
&-\frac{1}{2}\mathcal{D}_{\alpha_0\beta_0}\chi^{\beta_0}\chi^{\alpha_0} + \frac{1}{2}\mathcal{D}_{\alpha_0\beta_0}b^{\beta_0}b^{\alpha_0} + \eta^{\alpha_1}(X_{\alpha_1\alpha_0}b^{\alpha_0} + (\sigma_2')_{\alpha_2\alpha_1}\beta^{\alpha_2})\\
&+ ((\omega_1)_{\alpha_1\alpha_0}C_0{}^{\alpha_0} + \zeta^{\alpha_2}(\sigma_2)_{\alpha_2\alpha_1})\pi_1^{\alpha_1} + (X_{\alpha_1\alpha_0}C_0'^{\,\alpha_0} + \zeta'^{\alpha_2}(\sigma_2')_{\alpha_2\alpha_1})\pi_1'^{\,\alpha_1}\\
&+ (\omega_2)_{\alpha_2\alpha_1}C_1{}^{\alpha_1}\pi_2^{\alpha_2} + (\sigma_2)_{\alpha_2\alpha_1}C_1'^{\,\alpha_1}\pi_2'^{\,\alpha_2}\,,
\end{aligned}\tag{5.98}$$

with the same structure as (5.86) and where we already used the constraint $X_{\alpha_1\alpha_0}\chi^{\alpha_0} = 0$. The auxiliary fields η^{α_1}, $\pi_1^{\alpha_1}$ and $\pi_1'^{\,\alpha_1}$ impose the constraints

$$X_{\alpha_1\alpha_0}b^{\alpha_0} + (\sigma_2')_{\alpha_2\alpha_1}\beta^{\alpha_2} = 0 \tag{5.99}$$

$$(\omega_1)_{\alpha_1\alpha_0}C_0{}^{\alpha_0} + (\sigma_2)_{\alpha_2\alpha_1}\zeta^{\alpha_2} = 0 \tag{5.100}$$

$$X_{\alpha_1\alpha_0}C_0'^{\,\alpha_0} + (\sigma_2')_{\alpha_2\alpha_1}\zeta'^{\alpha_2} = 0 \tag{5.101}$$

which give $n-r$ constraints on b^{α_0}, $C_0{}^{\alpha_0}$, $C_0'^{\,\alpha_0}$ and imply the vanishing of the extra ghosts, $\beta^{\alpha_2} = \zeta^{\alpha_2} = \zeta'^{\alpha_2} = 0$. The fields $\pi_2^{\alpha_2}$ and $\pi_2'^{\,\alpha_2}$ impose the gauge-fixing conditions

$$(\omega_2)_{\alpha_2\alpha_1}C_1{}^{\alpha_1} = 0\,, \quad (\sigma_2)_{\alpha_2\alpha_1}C_1'^{\,\alpha_1} = 0\,. \tag{5.102}$$

This Gaussian gauge-fixing procedure for the second-stage theories, is analogous to the one we had for the first-stage: we perform the canonical transformation and produce a term proportional to $\mathcal{D}_{\alpha_0\beta_0}\chi^{\beta_0}\chi^{\alpha_0}$ in the gauge-fixed action with help of the gauge-fixing fermion. As a consequence, the fields φ^i have non-degenerate kinetic term in (5.98), and the top sitting auxiliary fields b^{α_0} becomes propagating if $\mathcal{D}_{\alpha_0\beta_0}$ contains derivatives; whereas the zero-stage ghost pair $C_0{}^{\alpha_0}$ and $C_0'^{\,\alpha_0}$ can have higher derivative kinetic terms depending on the exact form of the operator $\mathcal{D}_{\alpha_0\beta_0}$ (this we will see explicitly in the applications in the next chapter).

For theories that have higher levels of reducibility, both the delta-function gauge-fixing and the Gaussian gauge-fixing obviously work. However, it is a non-trivial task to also produce non-degenerate kinetic terms for all the ghost pairs by some "well-chosen" canonical transformations and modifications of gauge-fixing fermions. In general, there are cross terms between different fields and the diagonalisation of these cross terms can be very involved. Since this is neither necessary for the well-definedness of the path integral nor important for our interests on quantisation of the antisymmetric tensor-spinors, we will not move to any further discussions on this issue.

Chapter 6

Free fermionic p-form fields

This chapter turns to the quantisation of free fermionic p-form fields, using the general results of the previous chapter. After the quantisation, we move to the gravitational anomaly of a chiral fermionic p-form in dimensions $D = 4m + 2$ in section 6.4.

6.1 Review: the BV quantisation of the Rarita-Schwinger Lagrangian

As an example, we apply in this section the field-antifield method described in chapter 5 to the quantisation of the free spin 3/2 field ψ_μ^α (μ is a spacetime index and α a spinor index). We will use Dirac spinors to avoid dimension-dependent discussions of chirality and/or reality conditions, but these can be included without difficulty. Our spinor conventions are as in the textbook [113].

The Rarita-Schwinger action and gauge invariances are

$$S_0[\psi] = -\int d^D x\, \bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho\,, \quad \delta\psi_\mu^\alpha = \partial_\mu \Lambda^\alpha\,, \tag{6.1}$$

where the bar denotes the usual Dirac conjugate, $\bar{\psi}_\mu \equiv i(\psi_\mu)^\dagger \gamma^0$. We will impose the gauge condition

$$\chi(\psi) \equiv \gamma^\mu \psi_\mu = 0\,. \tag{6.2}$$

In the minimal sector, there are the field ψ_μ^α, the ghost c^α, and their antifields $\psi_\alpha^{*\mu}$, c_α^*. Notice that the antifields carry naturally an index down, so they transform as conjugate spinors under Lorentz transformations. The minimal BV action is

$$S^{\mathrm{M}} = \int d^D x \left(-\frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho + \psi_\alpha^{*\mu} \partial_\mu c^\alpha + \text{c.c.} \right) \tag{6.3}$$

$$= \int d^D x \left(-\bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho + \psi^{*\mu} \partial_\mu c + (\partial_\mu \bar{c}) \bar{\psi}^{*\mu} \right)\,, \tag{6.4}$$

where in the second line we suppressed the spinor indices and introduced the notation

$$\bar{\psi}^{*\mu} \equiv i\gamma^0 (\psi^{*\mu})^\dagger \tag{6.5}$$

for the "Dirac conjugate" of a conjugate (index-down) spinor. With this notation, we have $\bar{\bar{\chi}} = +\chi$ for any spinor or conjugate spinor χ, and the property $(ab)^\dagger = +\bar{b}\bar{a}$ for any conjugate spinor a and spinor b.

For the non-minimal sector, one adds one pair of spinors $\{c'^\alpha, b^\alpha\}$ and their antifields. The ghost numbers are given in table 6.1. Grassmann parity is ghost number *plus one* modulo two, since we have a fermionic theory and take the convention where degrees add up when determining signs. In particular, c and c' are bosonic (commuting) spinors, while b has the correct spin-statistics. The

	ψ^α_μ	c^α	c'^α	b^α	$\psi^{*\mu}_\alpha$	c^*_α	c'^*_α	b^*_α
gh	0	1	-1	0	-1	-2	0	-1
Grassmann parity	1	0	0	1	0	1	1	0

Table 6.1: The ghost numbers and parities of the fields and antifields appearing in the quantisation of the Rarita-Schwinger Lagrangian.

non-minimal action, adding the trivial pair, is simply

$$S^{\rm NM} = S^{\rm M} + \int d^Dx\ (c'^*_\alpha b^\alpha + \text{c.c.}) \tag{6.6}$$

$$= \int d^Dx\ \left(-\bar\psi_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + \psi^{*\mu}\partial_\mu c + (\partial_\mu\bar c)\bar\psi^{*\mu} + c'^* b + \bar b \bar c'^*\right)\ . \tag{6.7}$$

Delta-function gauge-fixing. For δ-function gauge-fixing, one takes the gauge-fixing fermion

$$\Psi_\delta = \int d^Dx\ (\bar c'\chi(\psi) + \text{c.c.}) = \int d^Dx\ \left(\bar c'\gamma^\mu\psi_\mu - \bar\psi_\mu\gamma^\mu c'\right)\ , \tag{6.8}$$

which gives the gauge-fixed action

$$S_\delta[\Phi^I] = S^{\rm NM}\left[\Phi^I, \Phi^*_I = \frac{\delta\Psi_\delta}{\delta\Phi^I}\right] \tag{6.9}$$

$$= \int d^Dx\ \left(-\frac{1}{2}\bar\psi_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + \bar c'\,\gamma^\mu\partial_\mu c + \bar b\gamma^\mu\psi_\mu + \text{c.c.}\right) \tag{6.10}$$

$$= \int d^Dx\ \left(-\bar\psi_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + \bar c'\,\gamma^\mu\partial_\mu c + \bar c\,\gamma^\mu\partial_\mu c' + \bar b\gamma^\mu\psi_\mu - \bar\psi_\mu\gamma^\mu b\right)\ . \tag{6.11}$$

The auxiliary field b enforces the gauge-fixing condition $\gamma^\mu\psi_\mu = 0$. Using this condition, the kinetic term for ψ_μ reduces to $-\bar\psi^\mu\not\partial\psi_\mu$.

Gaussian gauge-fixing. We now want to produce the Gaussian gauge-breaking term

$$\xi\,\bar\chi(\psi)\not\partial\,\chi(\psi) = -\xi\,\bar\psi_\mu\gamma^\mu\gamma^\nu\gamma^\rho\partial_\nu\psi_\rho \tag{6.12}$$

with an arbitrary parameter $\xi \neq 0$. As indicated above, this is done by the canonical transformation

$$b \to b - \chi(\psi)\,, \quad \psi^{*\mu} \to \psi^{*\mu} + b^*\frac{\delta\chi}{\delta\psi_\mu} = \psi^*_\mu + b^*\gamma_\mu \tag{6.13}$$

(and similarly for the Dirac conjugates), which gives the non-minimal action

$$\widetilde{S}^{\rm NM} = \int d^Dx\ \left(-\frac{1}{2}\bar\psi_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + (\psi^{*\mu} + b^*\gamma^\mu)\partial_\mu c + c'^*(b-\chi) + \text{c.c.}\right)\ . \tag{6.14}$$

Eliminating the antifields by means of the gauge-fixing fermion

$$\Psi_{\rm G} = -\frac{\xi}{2}\int d^Dx\ \bar c'\not\partial\,(\chi(\psi) + b) + \text{c.c.} \tag{6.15}$$

then produces

$$S_{\rm G} = \int d^Dx\ \left(-\bar\psi_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + \xi\,\bar\chi\not\partial\chi - \xi\,(\bar c'\,\Box c + \bar c\,\Box c') - \xi\,\bar b\not\partial b\right)\ . \tag{6.16}$$

The field b is now a propagating spin 1/2 field. Note that the ghosts c, c' count for four, since they come with the second-order $\Box = \not\partial\not\partial$ as kinetic operator.

This can be "undoubled" with a well-known trick (cf. for example the textbook [114], exercise VIA4.2). Introduce a Lagrange multiplier λ, of ghost number -1, to impose the equation $\not\partial c = f$

with f a new spinor of ghost number 1. This gives the equivalence $\bar{c}' \Box c \sim \bar{c}' \not{\partial} f + \bar{\lambda}(\not{\partial} c - f)$. This is diagonalised by the non-local triangular change of variables $c \to c + \not{\partial}^{-1} f$, with final result $\bar{c}' \Box c \sim \bar{c}' \not{\partial} f + \bar{\lambda} \not{\partial} c$ featuring four fields with a first-order Lagrangian instead of two with a second-order one.

The field b, being of ghost number zero, has the correct spin-statistics. This gives indeed the requisite number of ghosts: $3 = 4 - 1$.

In the action (6.16), the Nielsen-Kallosh ghost b is decoupled. However, in supergravity the operator $\not{\partial}$ appearing in the gauge-breaking term is covariantised and contains the vielbein and the spin-connection. Then, the field b couples to the other fields and ghosts of the theory [59–61].

6.2 Quantisation of the fermionic 2-form

In this section, we carry out the quantisation of the fermionic 2-form explicitly along the lines explained in section 5.6, both in delta-function gauge-fixing and in the Gaussian gauge-fixing where an extra Nielsen-Kallosh ghost appears.

The action, gauge transformations and reducibilities read

$$S_0[\psi] = \int d^D x \, \bar{\psi}_{\mu\nu} \gamma^{\mu\nu\rho\sigma\tau} \partial_\rho \psi_{\sigma\tau} \,, \quad \delta\psi^\alpha_{\mu\nu} = 2\, \partial_{[\mu} \Lambda^\alpha_{\nu]} \,, \quad \delta\Lambda^\alpha_\mu = \partial_\mu \lambda^\alpha \,. \tag{6.17}$$

The gauge parameter Λ has $n = Ds$ components and λ has $m = s$ components, for a total of $n - m = s(D-1)$ independent gauge transformations, where s is the dimension of the spinor representation at hand. (In this section as in section 6.1, we consider Dirac spinors so $s = 2^{[D/2]}$, but this counting is of course also valid with when reality and/or chirality conditions are imposed on the fields.) This action was used in the papers [11,12] as part of a complete free action principle for the exotic $\mathcal{N} = (4,0)$ and $\mathcal{N} = (3,1)$ multiplets in $D = 6$.

Accordingly, the minimal spectrum in the Batalin-Vilkovisky formalism consists of the fields and antifields

$$\{\psi^\alpha_{\mu\nu} \,, C^\alpha_\mu \,, c^\alpha \,, \psi^{*\mu\nu}_\alpha \,, C^{*\mu}_\alpha \,, c^*_\alpha\} \,, \tag{6.18}$$

where C^α_μ is the ghost associated to the Λ^α_μ gauge parameter and c^α is the ghost-for-ghost associated to the reducibility parameter λ^α. The minimal BV master action reads

$$S^{\mathrm{M}} = \int d^D x \left(\frac{1}{2} \bar{\psi}_{\mu\nu} \gamma^{\mu\nu\rho\sigma\tau} \partial_\rho \psi_{\sigma\tau} + 2\, \psi^{*\mu\nu} \partial_\mu C_\nu + C^{*\mu} \partial_\mu c + \text{c.c.} \right) . \tag{6.19}$$

Note that antifields transform naturally as a conjugate spinor. The non-minimal action, with the usual trivial pairs, is

$$S^{\mathrm{NM}} = S^{\mathrm{M}} + \int d^D x \, \left(C'^{*\mu} b_\mu + c'^* \pi + \eta^* \pi' + \text{c.c.} \right) , \tag{6.20}$$

where $\mathrm{gh}[C'^\mu, b_\mu] = [-1, 0]$, $\mathrm{gh}[c', \pi] = [-2, -1]$ and $\mathrm{gh}[\eta, \pi'] = [0, 1]$.

Both the minimal and non-minimal action are ghost number zero functionals satisfying the master equation

$$(S^{\mathrm{M}}, S^{\mathrm{M}}) = 0 = (S^{\mathrm{NM}}, S^{\mathrm{NM}}) \,. \tag{6.21}$$

We will use the redundant gauge condition

$$\begin{aligned} \chi_\mu(\psi) &\equiv \gamma^\nu \psi_{\mu\nu} - \frac{1}{D-2} \gamma_{\mu\nu\rho} \psi^{\nu\rho} \\ &= 0 \,, \end{aligned} \tag{6.22}$$

which satisfies the constraint

$$\gamma^\mu \chi_\mu(\psi) = 0 \tag{6.23}$$

identically and hence gives the correct number $n - m = s(D-1)$ of gauge conditions to fix the independent gauge transformations. To understand this gauge condition better, it is useful to write the different trace components of $\psi_{\mu\nu}$ explicitly,

$$\psi_{\mu\nu} = \hat{\psi}_{\mu\nu} + (\gamma_\mu \sigma_\nu - \gamma_\nu \sigma_\mu) + \gamma_{\mu\nu} \rho \tag{6.24}$$

where $\hat{\psi}_{\mu\nu}$ and σ_μ are gamma-traceless, $\gamma^\nu\hat{\psi}_{\mu\nu} = 0 = \gamma^\mu\sigma_\mu$. A short computation then shows that the condition $\chi_\mu(\psi) = 0$ is equivalent to $\sigma_\mu = 0$, i.e. setting the spin 3/2 component σ_μ to zero but *not* the spin 1/2 part ρ, indeed removing $s(D-1)$ components of $\psi_{\mu\nu}$.

Delta-function gauge-fixing. The gauge-fixing fermion is taken as

$$\Psi_\delta = \int d^Dx \,\left(\bar{C}'_\mu\,\chi^\mu(\psi) + \bar{c}'\gamma^\mu C_\mu + \bar{\eta}\gamma^\mu C'_\mu + \text{c.c.}\right)\,, \tag{6.25}$$

leading to

$$\begin{aligned} S_\delta = \int d^Dx \Big(&\frac{1}{2}\bar{\psi}_{\mu\nu}\gamma^{\mu\nu\rho\sigma\tau}\partial_\rho\psi_{\sigma\tau} + 2\,\bar{C}'^\sigma\frac{\delta\chi_\sigma}{\delta\psi_{\mu\nu}}\partial_\mu C_\nu + \bar{c}'\partial\!\!\!/ c \\ &+ \bar{b}_\mu\left(\chi^\mu(\psi) - \gamma^\mu\eta\right) + \bar{\pi}\gamma^\mu C_\mu + \bar{\pi}'\gamma^\mu C'_\mu + \text{c.c.}\Big) \end{aligned} \tag{6.26}$$

where

$$\frac{\delta\chi_\sigma}{\delta\psi_{\mu\nu}} = \delta^{[\mu}_\sigma\gamma^{\nu]} - \frac{1}{D-2}\gamma_\sigma{}^{\mu\nu}\,. \tag{6.27}$$

The auxiliary fields enforce the gauge conditions

$$\chi^\mu(\psi) - \gamma^\mu\eta = 0\,,\quad \gamma^\mu C_\mu = 0\,,\quad \gamma^\mu C'_\mu = 0\,. \tag{6.28}$$

Contracting the first condition with γ_μ gives $\eta = 0$ owing to the constraint satisfied by $\chi^\mu(\psi)$, and then it also implies the gauge condition $\chi^\mu(\psi) = 0$.

Using this gauge condition, the kinetic term for ψ can then be simplified using the decomposition (6.24) with $\sigma_\mu = 0$, which gives

$$\frac{1}{2}\bar{\psi}_{\mu\nu}\gamma^{\mu\nu\rho\sigma\tau}\partial_\rho\psi_{\sigma\tau} = -\bar{\hat{\psi}}^{\mu\nu}\partial\!\!\!/\hat{\psi}_{\mu\nu} - \frac{1}{2}(D-1)(D-2)(D-3)(D-4)\,\bar{\rho}\partial\!\!\!/\rho\,. \tag{6.29}$$

The gamma-tracelessness conditions on C_μ and C'_μ can also be used to reduce their kinetic term to

$$2\,\bar{C}'^\sigma\frac{\delta\chi_\sigma}{\delta\psi_{\mu\nu}}\partial_\mu C_\nu = -\frac{2D}{D-2}\bar{C}'^\mu\partial\!\!\!/ C_\mu\,. \tag{6.30}$$

Rescaling the fields and using an auxiliary field d^μ to impose the gamma-tracelessness of $\hat{\psi}_{\mu\nu}$, the final result can then be written as

$$\begin{aligned} S_\delta = \int d^Dx \Big(&-\bar{\hat{\psi}}^{\mu\nu}\partial\!\!\!/\hat{\psi}_{\mu\nu} - \bar{\rho}\partial\!\!\!/\rho + \bar{C}'^\mu\partial\!\!\!/ C_\mu + \bar{c}'\partial\!\!\!/ c \\ &+ \bar{d}^\mu\gamma^\nu\hat{\psi}_{\mu\nu} + \bar{\pi}\gamma^\mu C_\mu + \bar{\pi}'\gamma^\mu C'_\mu + \text{c.c.}\Big)\,. \end{aligned} \tag{6.31}$$

Notice that, although, as we expect, there is no extra auxiliary field propagating, we shall not ignore the fact that the double gamma trace ρ of $\psi^{\mu\nu}$ survives and have a kinetic term in the gauge-fixed action. This would be crucial to the computation of anomalies.

Gaussian gauge-fixing. We now would like to achieve a gauge-breaking term of the form $\bar{\chi}_\mu\mathcal{D}^{\mu\nu}\chi_\nu$, where $\mathcal{D}^{\mu\nu}$ is some first-order differential operator. Using only gamma matrices, the flat metric $\eta^{\mu\nu}$ and one spacetime derivative, we find using the gamma-tracelessness of χ_μ that the only independent possibility for $\mathcal{D}^{\mu\nu}$ is the very simple

$$\mathcal{D}^{\mu\nu} = \eta^{\mu\nu}\partial\!\!\!/\,. \tag{6.32}$$

As indicated in section 5.5, we start with the canonical transformation

$$b_\mu \to b_\mu - \chi_\mu(\psi)\,,\quad \psi^{*\mu\nu} \to \psi^{*\mu\nu} + b^{*\sigma}\frac{\delta\chi_\sigma}{\delta\psi_{\mu\nu}}\,, \tag{6.33}$$

which gives the new non-minimal action

$$\widetilde{S}^{\text{NM}} = \int d^Dx \Big(\frac{1}{2} \bar{\psi}_{\mu\nu} \gamma^{\mu\nu\rho\sigma\tau} \partial_\rho \psi_{\sigma\tau} + 2 \Big(\psi^{*\mu\nu} + b^{*\sigma} \frac{\delta \chi_\sigma}{\delta \psi_{\mu\nu}} \Big) \partial_\mu C_\nu + C^{*\mu} \partial_\mu c + C'^{*\mu}(b_\mu - \chi_\mu) + c'^{*} \pi + \eta^* \pi' + \text{c.c.} \Big) , \tag{6.34}$$

and we use the gauge-fixing fermion

$$\Psi_{\text{G}} = \int d^Dx \left(-\frac{\xi}{2} \bar{C}'^{\sigma} \not{\partial} (b_\sigma + \chi_\sigma) + \bar{c}' \gamma^\mu C_\mu + \bar{\eta} \gamma^\mu C'_\mu + \text{c.c.} \right) , \tag{6.35}$$

where $\xi \neq 0$ is an arbitrary parameter. This gives the gauge-fixed action

$$S_{\text{G}} = \int d^Dx \Big(\frac{1}{2} \bar{\psi}_{\mu\nu} \gamma^{\mu\nu\rho\sigma\tau} \partial_\rho \psi_{\sigma\tau} - 2\xi \, \bar{C}'^{\sigma} \not{\partial} \frac{\delta \chi_\sigma}{\delta \psi_{\mu\nu}} \partial_\mu C_\nu + \bar{c}' \not{\partial} c - \frac{\xi}{2} \bar{b}_\mu \not{\partial} b^\mu + \frac{\xi}{2} \bar{\chi}_\mu \not{\partial} \chi^\mu + \bar{\eta} \gamma^\mu (b_\mu + \chi_\mu) + \bar{\pi} \gamma^\mu C_\mu + \bar{\pi}' \gamma^\mu C'_\mu + \text{c.c.} \Big) . \tag{6.36}$$

The term $\bar{\eta}\gamma^\mu \chi_\mu(\psi)$ in this action identically vanishes thanks to the constraint satisfied by χ_μ. The auxiliary fields η, π and π' impose the gamma-tracelessness of C_μ, C'_μ and b_μ.

Using the gamma-tracelessness conditions on the ghosts C_μ and C'_μ, their kinetic term can be simplified as

$$-2\xi \, \bar{C}'^{\sigma} \not{\partial} \frac{\delta \chi_\sigma}{\delta \psi_{\mu\nu}} \partial_\mu C_\nu = \frac{2\xi D}{D-2} \bar{C}'_\mu \Box C^\mu \tag{6.37}$$

The final result is then, after rescaling some of the fields,

$$S_{\text{G}} = \int d^Dx \Big[\bar{\psi}_{\mu\nu} \gamma^{\mu\nu\rho\sigma\tau} \partial_\rho \psi_{\sigma\tau} + \xi \, \bar{\chi}_\mu(\psi) \not{\partial} \chi^\mu(\psi) + (\bar{C}'_\mu \Box C^\mu + \bar{C}_\mu \Box C'^\mu) - \bar{b}_\mu \not{\partial} b^\mu + (\bar{c}' \not{\partial} c + \bar{c} \not{\partial} c') + (\bar{\eta} \gamma^\mu b_\mu - \bar{b}_\mu \gamma^\mu \eta) + (\bar{\pi} \gamma^\mu C_\mu - \bar{C}_\mu \gamma^\mu \pi) + (\bar{\pi}' \gamma^\mu C'_\mu - \bar{C}'_\mu \gamma^\mu \pi') \Big] \tag{6.38}$$

with the desired gauge-breaking term. The field b_μ has a kinetic term and is a propagating spin 3/2 field: it is the Nielsen-Kallosh ghost for the fermionic two-form. As in the Rarita-Schwinger case, the ghosts C_μ and C'_μ have a second-order kinetic term and hence count for four. The field b_μ has the correct spin-statistics and there are effectively three spin 3/2 ghosts.

BRST transformations. The gauge-fixed actions above are invariant under a nilpotent BRST transformation $\bar{s}$ of ghost number +1, and the extra terms in the action (gauge-breaking terms and ghosts terms) are BRST-exact. As we had already seen in chapter 5 section 5.2.2, even before gauge-fixing, this comes very naturally out of the field-antifield formalism. Now, the BRST symmetry we will discuss, is present after the process of gauge-fixing, they are thus called the gauge-fixed classical BRST symmetry. Our interests here restrict to the fermionic two-form case, see [109, 110] for a general discussion.

In the delta-function gauge-fixing case, the action of the BRST differential $\bar{s}$ on a functional A depending on the fields Φ^I of the non-minimal sector (but not on the antifields Φ^*_I) is given by

$$\bar{s}A = (A, S^{\text{NM}})\Big|_{\Phi^* = \frac{\delta \Psi_\delta}{\delta \Phi}} = \frac{\delta^R A}{\delta \Phi^I} \frac{\delta^L S^{\text{NM}}}{\delta \Phi^*_I}\Big|_{\Phi^* = \frac{\delta \Psi_\delta}{\delta \Phi}} , \tag{6.39}$$

where the non-minimal action is in eq. (6.20). Notice, however, that (6.20) is linear in the antifields[1]: therefore, $\frac{\delta^L S^{NM}}{\delta \Phi^*_I}$ is antifield-independent and the definition of $\bar{s}$ is in fact independent of the gauge-fixing fermion. On the fields, $\bar{s}$ explicitly reads

$$\bar{s}\psi_{\mu\nu} = 2\partial_{[\mu} C_{\nu]} , \quad \bar{s}C_\mu = \partial_\mu c , \quad \bar{s}C'_\mu = b_\mu , \quad \bar{s}c' = \pi , \quad \bar{s}\eta = \pi' , \quad \bar{s}(\text{other}) = 0 . \tag{6.40}$$

[1]We discussed in the last chapter that terms of higher order in antifields would be expected in a putative interacting theory with a more involved gauge structure, e.g. if the gauge algebra were open.

The nilpotency

$$\bar{s}^{\,2} = 0 \tag{6.41}$$

is immediate and holds off-shell. On $\psi_{\mu\nu}$ (and C_μ due to the reducibility), $\bar{s}$ takes of course the familiar form 'gauge transformations with parameter replaced by ghost'. The gauge-fixed action (6.26) can then be written as

$$S_\delta = S_0 + \bar{s}\,\Psi_\delta\,, \tag{6.42}$$

with S_0 the original action (6.17) and Ψ_δ the gauge-fixing fermion (6.25). Indeed, since $S^{\rm NM}$ is linear in antifields, we have $S^{\rm NM} = S_0 + \Phi^*_I \frac{\delta^L S^{\rm NM}}{\delta \Phi^*_I}$. Therefore,

$$S_\delta = S^{\rm NM}\left[\Phi^I, \Phi^*_I = \frac{\delta \Psi_\delta}{\delta \Phi^I}\right] = S_0 + \frac{\delta \Psi_\delta}{\delta \Phi^I}\frac{\delta^L S^{\rm NM}}{\delta \Phi^*_I} = S_0 + \bar{s}\,\Psi_\delta\,. \tag{6.43}$$

(This can also be checked explicitly using formulas (6.40).) BRST invariance

$$\bar{s}\,S_\delta = 0 \tag{6.44}$$

of the gauge-fixed action then follows from the gauge-invariance of S_0 (indeed, $\bar{s}S_0 = 0$ is equivalent to its gauge invariance since it only depends on $\psi_{\mu\nu}$) and $\bar{s}^2 = 0$.

We now do the same for the Gaussian gauge-fixing case. Even though (6.39) doesn't depend on the choice of gauge-fixing fermion, in the Gaussian case the non-minimal action from which we started is different. The BRST transformation in this case is then defined as

$$\tilde{s}A = (A, \widetilde{S}^{\rm NM})\Big|_{\Phi^* = \frac{\delta \Psi_{\rm G}}{\delta \Phi}} = \frac{\delta^R A}{\delta \Phi^I}\frac{\delta^L \widetilde{S}^{\rm NM}}{\delta \Phi^*_I}\Bigg|_{\Phi^* = \frac{\delta \Psi_{\rm G}}{\delta \Phi}}\,, \tag{6.45}$$

with $\widetilde{S}^{\rm NM}$ given in (6.34) (which is also linear in antifields, so no antifields appear on the right-hand-side of (6.45) and this definition is again independent of the choice of gauge-fixing fermion). It takes the explicit form

$$\begin{aligned} &\tilde{s}\,\psi_{\mu\nu} = 2\,\partial_{[\mu}C_{\nu]}\,, \quad \tilde{s}\,C_\mu = \partial_\mu c\,, \quad \tilde{s}\,C'_\mu = b_\mu - \chi_\mu \\ &\tilde{s}\,b_\mu = 2\,\frac{\delta \chi_\mu}{\delta \psi_{\rho\sigma}}\partial_\rho C_\sigma\,, \quad \tilde{s}\,c' = \pi\,, \quad \tilde{s}\,\eta = \pi'\,, \quad \tilde{s}\,(\text{other}) = 0\,. \end{aligned} \tag{6.46}$$

Notice that $\tilde{s}\,b_\mu = \tilde{s}\,\chi_\mu$. The properties

$$\tilde{s}^{\,2} = 0\,, \quad S_{\rm G} = S_0 + \tilde{s}\,\Psi_{\rm G}\,, \quad \tilde{s}\,S_{\rm G} = 0 \tag{6.47}$$

then follow straightforwardly. By comparing the gauge-fixed BRST transformations (6.40) and (6.46) with the non-miminal actions (6.20) and (6.34) we confirmed that the antifields can be thought as sources for the gauge transformation of their corresponding fields (cf. (5.47)).

Propagators. We finish this section by exhibiting the propagators for $\psi_{\mu\nu}$ in both gauge-fixing schemes. The propagator $S^{\mu\nu}{}_{\sigma\tau}(p)$ is obtained by solving

$$K^{\rho\kappa}{}_{\mu\nu}(p) S^{\mu\nu}{}_{\sigma\tau}(p) = \delta^\rho_{[\sigma}\delta^\kappa_{\tau]} \tag{6.48}$$

where $K^{\rho\kappa}{}_{\mu\nu}$ is the kinetic operator of $\psi_{\mu\nu}$ in momentum space.

In the delta-function gauge-fixing case, the kinetic part of the action for the gamma-traceless component $\hat{\psi}_{\mu\nu}$ of $\psi_{\mu\nu}$ is simply $-\not{\partial}$. The propagator for that component is therefore simply given by (including Feynman's $i\varepsilon$ prescription)

$$S^{\mu\nu}{}_{\sigma\tau}(p) = -\mathbb{P}^{\mu\nu}{}_{\kappa\lambda}\,\frac{\not{p}}{p^2 - i\varepsilon}\,\mathbb{P}^{\kappa\lambda}{}_{\sigma\tau}\,, \tag{6.49}$$

where $\mathbb{P}$ is the projector onto the gamma-traceless subspace

$$\mathbb{P}^{\mu\nu}{}_{\rho\sigma} = \delta^{\mu\nu}_{\rho\sigma} + \frac{2}{D-2}\gamma^{[\mu}\delta^{\nu]}_{[\rho}\gamma_{\sigma]} - \frac{1}{(D-1)(D-2)}\gamma^{\mu\nu}\gamma_{\rho\sigma}\,. \tag{6.50}$$

It is antisymmetric in both pairs of indices, satisfies $\gamma_\mu \mathbb{P}^{\mu\nu}{}_{\rho\sigma} = 0 = \mathbb{P}^{\mu\nu}{}_{\rho\sigma}\gamma^\rho$ and $\mathbb{P}^{\mu\nu}{}_{\kappa\lambda}\mathbb{P}^{\kappa\lambda}{}_{\rho\sigma} = \mathbb{P}^{\mu\nu}{}_{\rho\sigma}$. The other component of $\psi_{\mu\nu}$ is ρ, which is simply a Dirac field with the usual propagator $-\not{p}/(p^2 - i\varepsilon)$.

In the Gaussian gauge-fixing case, the kinetic operator appearing in (6.38) reads, in momentum space,

$$K^{\rho\kappa}{}_{\mu\nu}(p) = \gamma^{\rho\kappa\lambda}{}_{\mu\nu}p_\lambda - \xi\left(\delta^{[\rho}_\lambda \gamma^{\kappa]} + \frac{1}{D-2}\gamma_\lambda{}^{\rho\kappa}\right)\not{p}\left(\delta^\lambda_{[\mu}\gamma_{\nu]} - \frac{1}{D-2}\gamma^\lambda{}_{\mu\nu}\right). \tag{6.51}$$

We find the result

$$\begin{aligned} S^{\mu\nu}{}_{\sigma\tau}(p) = \frac{1}{p^2 - i\varepsilon}\frac{1}{(D-4)}\Bigg[&-\frac{1}{2}(D-4)\delta^{\mu\nu}_{\sigma\tau}\not{p} - 2\left(p^{[\mu}\delta^{\nu]}_{[\sigma}\gamma_{\tau]} + \gamma^{[\mu}\delta^{\nu]}_{[\sigma}p_{\tau]}\right) + \gamma^{[\mu}\delta^{\nu]}_{[\sigma}\not{p}\gamma_{\tau]} \\ &+ \frac{2}{D-2}\left(p^{[\mu}\gamma^{\nu]}\gamma_{\sigma\tau} + \gamma^{\mu\nu}\gamma_{[\sigma}p_{\tau]}\right) + \frac{1}{2(D-3)}\gamma^{\mu\nu}\not{p}\gamma_{\sigma\tau} \\ &+ \frac{4}{(D-2)}\left(1 + \frac{(D-4)[(D-2)^2+4]}{D^2\xi}\right)p^{[\mu}\gamma^{\nu]}\frac{\not{p}}{p^2}\gamma_{[\sigma}p_{\tau]} \\ &+ 4\left(1 + \frac{(D-4)(D-2)^2}{D^2\xi}\right)p^{[\mu}\delta^{\nu]}_{[\sigma}p_{\tau]}\frac{\not{p}}{p^2}\Bigg]. \end{aligned} \tag{6.52}$$

6.3 Quantisation of fermionic p-forms (a general case)

Action and gauge symmetries. The action for a fermionic p-form field, that is, a tensor-spinor $\psi^\alpha_{\mu_1\mu_2\dots\mu_p}$ totally antisymmetric in its spacetime indices, is a direct generalisation of the Rarita-Schwinger action for a fermionic one-form ψ^α_μ and reads [50, 51]

$$S_0[\psi] = -(-1)^{\frac{p(p-1)}{2}}\int d^Dx\, \bar{\psi}_{\mu_1\mu_2\dots\mu_p}\,\gamma^{\mu_1\mu_2\dots\mu_p\nu\rho_1\rho_2\dots\rho_p}\,\partial_\mu\psi_{\rho_1\rho_2\dots\rho_p}. \tag{6.53}$$

Due to the rank $2p+1$ antisymmetric gamma matrix, it is manifestly invariant under the gauge transformations

$$\delta\psi^\alpha_{\mu_1\mu_2\dots\mu_p} = p\,\partial_{[\mu_1}\Lambda^{(p-1)\alpha}{}_{\mu_2\dots\mu_p]}, \tag{6.54}$$

where the gauge parameter $\Lambda^{(p-1)}$ is an arbitrary antisymmetric tensor-spinor of rank $p-1$. This system is $(p-1)$-stage reducible: (6.54) comes with the chain of gauge-for-gauge transformations

$$\delta\Lambda^{(p-1)\alpha}{}_{\mu_2\dots\mu_p} = (p-1)\,\partial_{[\mu_2}\Lambda^{(p-2)\alpha}{}_{\mu_3\dots\mu_p]} \tag{6.55}$$

$$\delta\Lambda^{(p-2)\alpha}{}_{\mu_3\dots\mu_p} = (p-2)\,\partial_{[\mu_3}\Lambda^{(p-3)\alpha}{}_{\mu_4\dots\mu_p]} \tag{6.56}$$

$$\vdots$$

$$\delta\Lambda^{(1)\alpha}_\mu = \partial_\mu\Lambda^{(0)\alpha}, \tag{6.57}$$

where each parameter $\Lambda^{(k)}$ is a rank-k antisymmetric tensor-spinor. In form notation (with a spectator spinor index), this is

$$\delta\psi = d\Lambda^{(p-1)}, \quad \delta\Lambda^{(p-1)} = d\Lambda^{(p-2)}, \quad \dots, \quad \delta\Lambda^{(1)} = d\Lambda^{(0)}. \tag{6.58}$$

The equations of motion coming from the action (6.53) read

$$\gamma^{\mu_1\dots\mu_p\nu_1\dots\nu_{p+1}}H_{\nu_1\dots\nu_{p+1}} = 0, \quad H_{\mu_1\dots\mu_{p+1}} \equiv (p+1)\partial_{[\mu_1}\psi_{\mu_2\dots\mu_{p+1}]}, \tag{6.59}$$

where $H = d\psi$ is the gauge-invariant field strength of the field ψ (we denote by H instead of χ to avoid confusions with the gauge-fixing conditions). Equivalently, they can be written as the single-gamma-trace equation

$$\gamma^{\mu_1}H_{\mu_1\mu_2\dots\mu_{p+1}} = 0. \tag{6.60}$$

These equations propagate the correct representation of the massless little group: the rank p antisymmetric tensor-spinor of $SO(D-2)$ satisfying a gamma-traceless condition.

Such a tensor-spinor identically vanishes for $p \geq D/2$. This is consistent with the covariant action (6.53): it identically vanishes when $2p+1 > D$ because of the antisymmetric gamma matrix and, for $2p+1 = D$, the equations of motion are equivalent to $H[\psi] = 0$ which implies that ψ is pure gauge. So, the theory described by (6.53) has propagating degrees of freedom only for $2p < D$, and we will see it explicitly in the counting (6.61) below. We will assume this inequality for the remainder of this section.

Let q_p denote the number of components of the rank p antisymmetric tensor-spinor in the little group $SO(D-2)$, while $\hat{q}_p$ equals the number of components of the little group gamma-traceless rank p antisymmetric tensor-spinor. The physical degrees of freedom of a fermionic p-form given by the action (6.53) is $\hat{q}_p$. We also need the number of components of the Dirac spinor of the little group, denoted by s, and $s = 2^{[\frac{D-2}{2}]}$, where $[x]$ is the largest integer less than or equal to x. We have the identity for $p < \frac{D-1}{2}$

$$\begin{aligned}\hat{q}_p = q_p - q_{p-1} &= s\left(\frac{(D-2)!}{p!(D-2-p)!} - \frac{(D-2)!}{(p-1)!(D-1-p)!}\right) \\ &= \frac{(D-2)!(D-2p-1)}{p!(D-p-1)!}\, 2^{[\frac{D-2}{2}]}\,.\end{aligned} \tag{6.61}$$

One can check, for example, for $p = 0$ we get $\hat{q}_0 = 2^{[\frac{D-2}{2}]}$ and for $p = 1$, $\hat{q}_1 = (D-3)2^{[\frac{D-2}{2}]}$ agree with the physical degrees of freedom for Dirac fermion and gravitino. When $p = 2$ and $D = 6$ we have $\hat{q}_2 = 8$, if we take the spinor to be chiral this number reduces to 4. This is the known result for the exotic gravitino (cf. (1.14)).

Gauge conditions. We now turn to the gauge condition that we will impose on the field $\psi^\alpha_{\mu_1\mu_2\dots\mu_p}$. It is given by an equation of the form

$$\chi^\alpha_{\mu_1\dots\mu_{p-1}}(\psi) = 0\,, \tag{6.62}$$

with the same index structure as the gauge parameter, that must only contain as many independent conditions as there are independent gauge transformations. That number is

$$Q_{p-1} - Q_{p-2} + Q_{p-3} - Q_{p-4} + \cdots \pm Q_0 \tag{6.63}$$

where Q_k is the number of components of an antisymmetric tensor-spinor of rank k in the spacetime Lorentz group $SO(D-1,1)$ (we also denote by $\hat{Q}_k$ the number of components of such a tensor-spinor which is in addition gamma-traceless) and where the final sign depends on the parity of p. One way to realise this is to take a gauge condition $\chi^\alpha_{\mu_1\dots\mu_{p-1}}(\psi)$ that satisfies $Q_{p-2} - Q_{p-3} + Q_{p-4} - \cdots$ independent constraints; this can work if the operator X in the constraint equations $X^\alpha_{\mu_1\dots\mu_{p-2}}(\chi) = 0$ (cf. section 5.6) itself satisfies $Q_{p-3} - Q_{p-4} + \cdots$ independent constraints, etc. This reasoning shows that it is sufficient to define operators $T^{(k)}$ mapping fermionic k-forms to $(k-1)$-forms such that the nilpotency condition

$$T^{(k)} \circ T^{(k+1)} = 0 \tag{6.64}$$

holds and exhausts the constraints satisfied by $T^{(k+1)}$ (extra constraints would of course upset the counting above). Then, the gauge condition

$$\chi(\psi) \equiv T^{(p)}(\psi) = 0 \tag{6.65}$$

satisfies $T^{(p-1)}(\chi) = 0$ and gives the correct number of independent conditions. From the analysis of the Rarita-Schwinger case ($p = 1$), eq. (6.2), we take

$$T^{(1)}(\psi) = \gamma^\mu \psi_\mu \tag{6.66}$$

as a suitable starting point. The next operators can then be determined recursively using the nilpotency condition: the first few read explicitly

$$T^{(2)}(\psi)_\mu = \gamma^\nu \psi_{\mu\nu} - \frac{1}{D}\gamma_\mu \gamma^{\nu\rho}\psi_{\nu\rho} \tag{6.67}$$

$$T^{(3)}(\psi)_{\mu\nu} = \gamma^\rho \psi_{\mu\nu\rho} + \frac{2}{D-2}\gamma_{[\mu}\gamma^{\rho\sigma}\psi_{\nu]\rho\sigma} \tag{6.68}$$

$$T^{(4)}(\psi)_{\mu\nu\rho} = \gamma^\sigma \psi_{\mu\nu\rho\sigma} - \frac{3}{D-4}\gamma_{[\mu}\gamma^{\sigma\tau}\psi_{\nu\rho]\sigma\tau} - \frac{2}{D(D-2)(D-4)}\gamma_{\mu\nu\rho}\gamma^{\sigma\tau\kappa\lambda}\psi_{\sigma\tau\kappa\lambda} \tag{6.69}$$

$$T^{(5)}(\psi)_{\mu\nu\rho\sigma} = \gamma^\tau \psi_{\mu\nu\rho\sigma\tau} + \frac{4}{D-6}\gamma_{[\mu}\gamma^{\tau\kappa}\psi_{\nu\rho\sigma]\tau\kappa} + \frac{8}{(D-2)(D-4)(D-6)}\gamma_{[\mu\nu\rho}\gamma^{\tau\kappa\lambda\zeta}\psi_{\sigma]\tau\kappa\lambda\zeta} \tag{6.70}$$

$$\ldots .$$

We point out that $T^{(2)}(\psi)_\mu = \frac{D-2}{D}\chi_\mu(\psi)$ where $\chi_\mu(\psi)$ is the redundant gauge condition (6.22) for the fermionic two-form.

Gauge-fixing. The generalisation of the gauge-fixing processes described in the last two sections to higher form degrees is direct: the non-minimal sector in both gauge-fixing procedures for the p-forms $\psi_{\mu_1\mu_2\ldots\mu_p}$ is represented by two pyramids as in figure 5.3, where we extend the first pyramid to have $p+1$ levels. At level k, the ghosts are $(k-1)$-form tensor-spinors. As for the auxiliary fields pyramid, it has p levels and starts from a $(p-1)$-form tensor-spinor from the top. The arrows that indicating the gauge-fixing directions, the ghost number assignments and the Grassmann parities are just the same as usual.

An arrow $a_p \to b_{p-1}$ indicates that the variable b_{p-1} along with its trivial pair partner fix the gauge freedom of a_p by imposing the the gauge condition $T^{(p)}$. This means we have such terms

$$\Psi = \int d^D x \left(\ldots + \bar{b}^{\mu_1\mu_2\ldots\mu_{p-1}} T^{(p)}(a)_{\mu_1\mu_2\ldots\mu_{p-1}} + \text{c.c.} + \ldots\right) \tag{6.71}$$

in the gauge-fixing fermion. For the delta-function gauge-fixing, Ψ_δ consists only this kind of terms; and for the Gaussian gauge-fixing, the trick of canonical transformations (5.81) is still valid, one can reach a gauge-breaking term of the form

$$\overline{T^{(p)}(\psi)}^{\mu_1\mu_2\ldots\mu_p} \not{\partial}\, T^{(p)}(\psi)_{\mu_1\mu_2\ldots\mu_p} \,. \tag{6.72}$$

The price we payed for achieving this term is that the top sitting $(p-1)$-form-spinor auxiliary field becomes a propagating field in the dynamics, this is the "extra ghost" in the general case.

The same process we presented in the last section for the two-form-spinor can thus be extended naturally to p-form-spinors, we are not going to give the explicit gauge-fixed action, this can be worked out on a case-by-case basis.

Counting degrees of freedom. A good cross-check to the quantisation is to count the degrees of freedom. In the delta-function gauge-fixing case, the dynamical variables are the p-form-spinor ψ_p (p stands for form degree in the following discussion) and pairs of Faddeev-Popov ghost-antighost $\{C'_k, C_k\}$ for $k = 0, 1, \ldots, p-1$. The aforementioned gauge condition $T^{(k)}(C) = 0$ will eliminate the $(k-1)$-th gamma-trace, the $(k-3)$-th gamma-trace... and etc. With all the $k-(2l+1)$-th gamma-trace removed, the resulting components for such a C_k is (same for C'_k)

$$\hat{Q}_k + \hat{Q}_{k-2} + \hat{Q}_{k-4} + \cdots = \sum_{l=0}^{[\frac{k}{2}]} \hat{Q}_{k-2l} \,, \tag{6.73}$$

and we also take into account that $\{C'_k, C_k\}$ comes with the Grassmann parity $\epsilon_k = 1 + (-1)^{p-k}$. Put these ingredients together, we have for the resulting net components of fields

$$\begin{aligned} N_{\text{components}} = \quad & \hat{Q}_p + \hat{Q}_{p-2} + \hat{Q}_{p-4} + \hat{Q}_{p-6} + \hat{Q}_{p-8} + \dots \\ & - 2\left(\hat{Q}_{p-1} + \hat{Q}_{p-3} + \hat{Q}_{p-5} + \hat{Q}_{p-7} + \dots\right) \\ & + 2\left(\hat{Q}_{p-2} + \hat{Q}_{p-4} + \hat{Q}_{p-6} + \hat{Q}_{p-8} + \dots\right) \\ & \quad \dots \\ & = \hat{Q}_p - 2\hat{Q}_{p-1} + 3\hat{Q}_{p-2} - 4\hat{Q}_{p-3} + 5\hat{Q}_{p-4} - \dots \\ & = (Q_p - Q_{p-1}) - 2(Q_{p-1} - Q_{p-2}) + 3(Q_{p-2} - Q_{p-3}) - 4(Q_{p-3} - Q_{p-4}) + \dots \\ & = Q_p - 3Q_{p-1} + 5Q_{p-2} - 7Q_{p-3} + 9Q_{p-4} - \dots \\ & = \sum_{k=0}^{p} (-1)^k (2k+1)\, Q_{p-k} \\ & = 2^{[\frac{D}{2}]} \sum_{k=0}^{p} (-1)^k (2k+1) \frac{D!}{(p-k!)(D-p+k)!} \\ & = \frac{(D-2)!(D-2p-1)}{p!(D-p-1)!}\, 2^{[\frac{D}{2}]}\,, \end{aligned} \tag{6.74}$$

where we used $\hat{Q}_k = Q_k - Q_{k-1}$. Because we have a fermionic theory, the physical degrees of freedom is a half of the net components [109, 113]

$$N_{\text{physical}} = \frac{1}{2} N_{\text{components}} = \frac{(D-2)!(D-2p-1)}{p!(D-p-1)!}\, 2^{[\frac{D-1}{2}]} = \hat{q}_p\,. \tag{6.75}$$

This matches our previous counting in the little group (6.61). In the computation, we also see that the pattern $Q_p - 3Q_{p-1} + 5Q_{p-2} - 7Q_{p-3} + 9Q_{p-4} - \dots$ appears. This is somehow a evidence of the statement made in [115] that in fermionic theory one has successive fields in numbers of $1, 3, 5, 7\dots$ as ghost structures in the quantisation, whereas in the bosonic case the known result is $1, 2, 3, 4\dots$.

Now we turn to the Gaussian gauge-fixing process. Here, to the p-form spinor kinetic term, a gauge breaking term (6.72) is added and when we count, all the gamma-traces of ψ_p contribute. However, the $\{C'_{p-1}, C_{p-1}\}$ get a second order kinetic term and their components are doubled. At the same time, because of the canonical transformation, the auxiliary field b_{p-1} have a kinetic term in the action. This field is the trivial pair partner of C'_{p-1} and have the opposite Grassmann parity. Effectively, at the $(p-1)$-form level we have 3 ghosts with even Grassmann parity (cf. (6.37) and the discussion below it). The treatment to the all other ghosts remain the same.
We compute again the number of components

$$\begin{aligned} \tilde{N}_{\text{components}} = \quad & Q_p + \\ & - 3\left(\hat{Q}_{p-1} + \hat{Q}_{p-3} + \hat{Q}_{p-5} + \hat{Q}_{p-7} + \dots\right) \\ & + 2\left(\hat{Q}_{p-2} + \hat{Q}_{p-4} + \hat{Q}_{p-6} + \hat{Q}_{p-8} + \dots\right) \\ & \quad \dots \\ & = Q_p - 3\hat{Q}_{p-1} + 2\hat{Q}_{p-2} - 5\hat{Q}_{p-3} + 4\hat{Q}_{p-4} - 7\hat{Q}_{p-5} + 6\hat{Q}_{p-6} \dots \\ & = (Q_p - Q_{p-1}) - 2(Q_{p-1} - Q_{p-2}) + 3(Q_{p-2} - Q_{p-3}) - 4(Q_{p-3} - Q_{p-4}) + \dots \\ & = Q_p - 3Q_{p-1} + 5Q_{p-2} - 7Q_{p-3} + 9Q_{p-4} - \dots \\ & = \sum_{k=0}^{p} (-1)^k (2k+1)\, Q_{p-k} \\ & = N_{\text{components}} = 2\hat{q}_p\,, \end{aligned} \tag{6.76}$$

where the same pattern $1, 3, 5, 7\dots$ of gamma-traceful fields appears, and we arrive at the same result.

	$\psi_{\mu\nu}$	$\hat{\psi}_{\mu\nu}$	ρ	C_μ	C'_μ	c	c'	b_μ	η	d_μ	π	π'
S_δ chirality	+	+	+	+	+	+	+	+	+	+	+	+
S_{G} chirality	+			+	−	+	+	−	−		+	−
Grassmann parity	1	1	1	0	0	1	1	1	1	1	0	0

Table 6.2: The chirality of the various fields appearing in the gauge-fixed actions (6.31) and (6.38) for the chiral fermionic two-form. Notice that the chirality can depend on the gauge-fixing scheme. Grassmann parity is also included; even fields have abnormal spin-statistics.

6.4 Gravitational anomalies of antisymmetric tensor-spinors

In this section, we compute the gravitational anomalies for chiral fermionic p-forms in $D = 4m+2$ dimensions using the Atiyah-Singer index theorem [66–71]. The computations of this section only rely on the spectrum of ghosts and they therefore apply to any theory with the same structure (6.58) of gauge transformations and reducibilities, whether or not it has a kinetic term of the form (6.53). For this reason, we will also consider in this section fermionic p-forms with $2p \geq D$, which carry no degrees of freedom. For such fields, the action (6.53) vanishes identically, but one could nevertheless imagine the existence of topological models in which they are coupled to other fields while still having the same structure of gauge transformations and reducibilities; our computations would then be applicable to such models. The prime example of this case is the gravitino in $D = 2$, which doesn't have a Rarita-Schwinger kinetic term and carries no degree of freedom, but for which we nevertheless reproduce the classic result of [68].
As the first step, we would like to exam the chirality structure of the quantisation of the chiral fermionic two-form $\psi_{\mu\nu}$ in $D = 4k+2$ dimensions. The chirality condition imposed is

$$\gamma_* \psi_{\mu\nu} = +\psi_{\mu\nu} \,, \tag{6.77}$$

where γ_* is the usual chirality matrix.

The chiralities of the ghosts depend on the gauge-fixing procedure, for example, for the 2-form-spinor we have the expression $\bar{C}'_\mu \, \chi^\mu(\psi)$ in the delta-function gauge-fixing fermion (6.25), while the corresponding term is $\bar{C}'_\mu \, \partial\!\!\!/ \, \chi^\mu(\psi)$ in the Gaussian gauge-fixing case (6.35). The ghost field C' have thus opposite chiralities in the two gauge-fixing schemes. Notice that the Grassmann parity of the variables plays an important role, it determines the signs of contributions from each field (cf. (3.3)). Some formal manipulations and careful handling of signs are required to reach the standard form (3.15). Let us show how they work explicitly in the chiral two-form-spinor case and the chiralities and spin-statistics of each fields are listed in the table 6.2. The computation is slightly different in the two gauge-fixing schemes of this thesis, but the result is of course the same.

The gravitational anomalies for the chiral 2-form-spinor field.

- **Computation in the delta-function gauge-fixing**

  In this case, the dynamic variables are $\{\hat{\psi}_{\mu\nu},\, \rho,\, \hat{C'}_\mu,\, \hat{C}_\mu,\, c',\, c\}$ (cf. (6.31)). The relevant path integral measure is

$$\int \left[\mathcal{D}\hat{\psi}_{\mu\nu}\mathcal{D}\overline{\hat{\psi}}_{\mu\nu}\right] \left[\mathcal{D}\hat{C'}_\mu\mathcal{D}\overline{\hat{C'}}_\mu\right] \left[\mathcal{D}\hat{C}_\mu\mathcal{D}\overline{\hat{C}}_\mu\right] \left[\mathcal{D}c'\mathcal{D}\bar{c'}\right] \left[\mathcal{D}c\mathcal{D}\bar{c}\right] \left[\mathcal{D}\rho\mathcal{D}\bar{\rho}\right] \,, \tag{6.78}$$

  where a 'hat' denotes a gamma-traceless field.

  Since the gamma-trace has the opposite chirality as the field itself, the field $\hat{\psi}_{\mu\nu}$ for example is an element of the formal difference

$$\mathcal{C}^\infty(S^+ \otimes \Lambda^2 T^*M - S^- \otimes T^*M) \,, \tag{6.79}$$

  i.e. a positive chirality fermionic two-form without the negative chirality one-form component. This is not in the standard form (3.15) for $\mathcal{D}_2$ (the subscript 2 stands for the fermionic 2-form, and later for fermionic p-forms we use $\mathcal{D}_p$) to act upon; however, a fermion of negative chirality gives the opposite contribution to the index density as a fermion of positive chirality. We can then replace S^- by S^+ in (6.79) and change the sign: $\hat{\psi}_{\mu\nu}$ therefore contributes as

$$\mathcal{C}^\infty(S^+ \otimes [\Lambda^2 T^*M + T^*M])\,, \tag{6.80}$$

which is now in the form (3.15). Another rule is that fields with the wrong spin-statistics, in our case $\hat{C}_\mu$ and $\hat{C}'_\mu$, also contribute with a minus sign. Combining these two rules, the (wrong spin-statistics, positive chirality, gamma-traceless) field $\hat{C}_\mu$ for example contributes as

$$-\mathcal{C}^\infty(S^+ \otimes T^*M - S^-) = \mathcal{C}^\infty(S^+ \otimes [-T^*M - 1])\,. \tag{6.81}$$

One must sum the contributions of all fields appearing in the measure (6.78), using these two rules and the chirality and spin-statistics of table 6.2. The complex $\mathcal{C}^\infty(S^+ \otimes V_\delta)$ on which the Dirac operator acts in this case is then

$$\begin{aligned}\mathcal{C}^\infty(S^+ \otimes V_\delta) =& \mathcal{C}^\infty(S^+ \otimes \Lambda^2 T^*M - S^- \otimes T^*M) - \mathcal{C}^\infty(S^+ \otimes T^*M - S^-)\\ &- \mathcal{C}^\infty(S^+ \otimes T^*M - S^-) + \mathcal{C}^\infty(S^+) + \mathcal{C}^\infty(S^+) + \mathcal{C}^\infty(S^+)\\ =& \mathcal{C}^\infty(S^+ \otimes [\Lambda^2 T^*M - T^*M + 1])\end{aligned} \tag{6.82}$$

and we have

$$\begin{aligned}V_\delta =& \Lambda^2 T^*M - T^*M + 1\\ \mathcal{D}_2 :& \mathcal{C}^\infty(S^+ \otimes V_\delta) \longrightarrow \mathcal{C}^\infty(S^- \otimes V_\delta)\,.\end{aligned} \tag{6.83}$$

- **Computation in the Gaussian gauge-fixing**

  After the Gaussian gauge-fixing, the generalised Nielsen-Kallosh ghost b_μ enters the dynamics and we have the measure

$$\int [\mathcal{D}\psi_{\mu\nu}\mathcal{D}\overline{\psi}_{\mu\nu}] \left[\mathcal{D}\hat{C'}_\mu\mathcal{D}\overline{\hat{C'}}_\mu\right] \left[\mathcal{D}\hat{C}_\mu\mathcal{D}\overline{\hat{C}}_\mu\right] \left[\mathcal{D}\hat{b}_\mu\mathcal{D}\bar{\hat{b}}_\mu\right] [\mathcal{D}c'\mathcal{D}\bar{c}'] [\mathcal{D}c\mathcal{D}\bar{c}] \tag{6.84}$$

  after integrating out the auxiliary fields. Here as before, a hat indicates a gamma-traceless field. Notice that in this case we integrate over unconstrained $\psi_{\mu\nu}$. The ghosts $\hat{C}_\mu$ and $\hat{C}'_\mu$ have opposite chiralities but otherwise identical properties and their contributions to the index density cancel out. The total complex on which $\mathcal{D}_2$ acts in this case is then

$$\begin{aligned}\mathcal{C}^\infty(S^+ \otimes V_\mathrm{G}) =& \mathcal{C}^\infty(S^+ \otimes \Lambda^2 T^*M) - \mathcal{C}^\infty(S^- \otimes T^*M - S^+)\\ &- \mathcal{C}^\infty(S^+ \otimes T^*M - S^-) + \mathcal{C}^\infty(S^- \otimes T^*M - S^+)\\ &+ \mathcal{C}^\infty(S^+) + \mathcal{C}^\infty(S^+)\\ =& \mathcal{C}^\infty(S^+ \otimes [\Lambda^2 T^*M - T^*M + 1])\,.\end{aligned} \tag{6.85}$$

  Therefore,

$$V_\delta = V_\mathrm{G} \equiv V_2 \tag{6.86}$$

  as expected; both gauge-fixing procedures will give the same result for the anomaly.

Anomaly for fermionic p-forms. For $p \geq 3$, the ghost spectrum extended similarly and we prefer to perform the above discussion only in the δ-function gauge-fixing scheme to avoid the dynamics of the Nielsen-Kallosh ghosts. The p-form gauge theory is $(p-1)$-stage reducible. The chirality of ψ_p is $+1$, so all the Faddeev-Popov ghost-antighost pairs $\{C'_k, C_k\}$ for $k = 0, 1, \ldots, p-1$, have the same chirality $+1$. The vital point is that when we remove the gamma traces, different chiralities appear. We use the superscript $+$ or $-$ to indicate the positive or negative chiralities. For example, consider a positive chiral fermionic 3-form ghost C_3^+ in the gamma trace decomposition

$$C_3^+ = \hat{C}_3^+ + \gamma^{(1)}\hat{J}_2^- + \gamma^{(2)}\hat{E}_1^+ + \gamma^{(3)}\hat{F}_0^-\,, \tag{6.87}$$

where $\hat{C}_3^+, \hat{J}_2^-, \hat{E}_1^+, \hat{F}_0^-$ are all gamma-traceless, and they come with alternating chiralities. We denote by $\hat{\mathcal{I}}_k^+$ the index density contribution from a positive chiral gamma-traceless k-form-spinor and denote by $\mathcal{I}_k^+$ the index density contribution from the gamma-traceful ones. Clearly, for the negative chirality we have

$$\hat{\mathcal{I}}_k^- = -\hat{\mathcal{I}}_k^+ , \quad \mathcal{I}_k^- = -\mathcal{I}_k^+ , \tag{6.88}$$

and similar to the number of components we have the identity

$$\hat{\mathcal{I}}_k^+ = \mathcal{I}_k^+ - \mathcal{I}_{k-1}^- \tag{6.89}$$

for the index density contribution but with the chiralities get flipped. The total contribution is

$$\begin{aligned}
\mathcal{I}_{p-\text{form}}^+ = \ & \hat{\mathcal{I}}_p^+ + \hat{\mathcal{I}}_{p-2}^+ + \hat{\mathcal{I}}_{p-4}^+ + \hat{\mathcal{I}}_{p-6}^+ + \hat{\mathcal{I}}_{p-8}^+ + \dots \\
& - 2\left(\hat{\mathcal{I}}_{p-1}^+ + \hat{\mathcal{I}}_{p-3}^+ + \hat{\mathcal{I}}_{p-5}^+ + \hat{\mathcal{I}}_{p-7}^+ + \dots\right) \\
& + 2\left(\hat{\mathcal{I}}_{p-2}^+ + \hat{\mathcal{I}}_{p-4}^+ + \hat{\mathcal{I}}_{p-6}^+ + \hat{\mathcal{I}}_{p-8}^+ + \dots\right) \\
& \dots \\
= & \hat{\mathcal{I}}_p^+ - 2\hat{\mathcal{I}}_{p-1}^+ + 3\hat{\mathcal{I}}_{p-2}^+ - 4\hat{\mathcal{I}}_{p-3}^+ + 5\hat{\mathcal{I}}_{p-4}^+ - \dots \\
= & (\mathcal{I}_p^+ - \mathcal{I}_{p-1}^-) - 2(\mathcal{I}_{p-1}^+ - \mathcal{I}_{p-2}^-) + 3(\mathcal{I}_{p-2}^+ - \mathcal{I}_{p-3}^-) - 4(\mathcal{I}_{p-3}^+ - \mathcal{I}_{p-4}^-) + \dots \\
= & \mathcal{I}_p^+ - \mathcal{I}_{p-1}^+ + \mathcal{I}_{p-2}^+ - \mathcal{I}_{p-3}^+ + \mathcal{I}_{p-4}^+ - \dots ,
\end{aligned} \tag{6.90}$$

where the third equality is the deciding step. Without the chirality flipping between gamma traces, the pattern would be identical as in the degrees of freedom counting (6.74).

From this index density contribution we can find the complex $\mathcal{C}^\infty(S^+ \otimes V_p)$, on which the Dirac operator $\mathcal{D}_p$ acts and we get

$$\begin{aligned}
V_p = \ & \Lambda^p T^*M - \Lambda^{p-1} T^*M + \Lambda^{p-2} T^*M - \Lambda^{p-3} T^*M + \dots \pm 1 , \\
= & \sum_{k=0}^{p} (-1)^k \, \Lambda^{p-k} \, T^*M .
\end{aligned} \tag{6.91}$$

$$\mathcal{D}_p : \mathcal{C}^\infty(S^+ \otimes V_p) \longrightarrow \mathcal{C}^\infty(S^- \otimes V_p) . \tag{6.92}$$

For $p = 0$, $\mathcal{D}_0$ acts on $\mathcal{C}^\infty(S^+)$, whereas for $p = 1$, $\mathcal{D}_1$ acts on $\mathcal{C}^\infty(S^+ \otimes [T^*M - 1])$ and we recovered the known results for chiral fermion and chiral gravitino.

The next step is to apply the index theorem to get the index density and it will give the anomaly polynomial $I_{D+2}^{(p)}$, we need to pick out the $(D+2)$-form part

$$I_{D+2}^{(p)} = [\mathrm{Ind}(\mathcal{D}_p)]_{D+2} = \left[\hat{A}(M)\left(\sum_{k=0}^{p}(-1)^k \, \mathrm{ch}\,(\Lambda^{p-k} T^*M)\right)\right]_{D+2} . \tag{6.93}$$

Chern characters and traces. In the expression (6.93), Chern characters for higher exterior powers of the cotangent bundle T^*M arise. The generator in the fundamental representation of $SO(D)$ is $(t^{ab})_{cd} = \delta_c^a \delta_d^b - \delta_d^a \delta_c^b$, then the generator of the rank k antisymmetric tensor representation of $SO(D)$ can be expressed as

$$(T_{[k]}^{ab})^{i_1 \dots i_k}{}_{j_1 \dots j_k} = k! \sum_{l=1}^{k} \delta_{j_1}^{[i_1} \delta_{j_2}^{i_2} \cdots (t^{ab})^{i_l}{}_{j_l} \cdots \delta_{j_k}^{i_k]} . \tag{6.94}$$

Now let $R_{[k]}$ be the curvature 2-form in these tensor bundles, it is given as

$$R_{[k]} \equiv \frac{1}{2} R_{ab} T_{[k]}^{ab} , \tag{6.95}$$

where R_{ab} is the curvature 2-form on the tangent bundle. A non-trivial problem is to write the traces of powers of $R_{[k]}$ in terms of traces of powers of R in the fundamental representation. This can be done using the explicit formula (6.94).

For example, for $k = 2$

$$(T^{ab}_{[2]})^{cd}{}_{ef} = 2\left((t^{ab})^c{}_{[e}\delta^d{}_{f]} + (t^{ab})^d{}_{[f}\delta^c{}_{e]}\right) , \tag{6.96}$$

$$\left(R_{[2]}\right)_{cd,ef} = (R_{ce}\delta_{df} + R_{df}\delta_{ce} - R_{cf}\delta_{de} - R_{de}\delta_{cf}) . \tag{6.97}$$

Since $R_{[2]}$ is an antisymmetric matrix, itself and its odd powers are traceless. The first non-vanishing contributions to the Chern character are

$$\operatorname{tr} R^2_{[2]} = \frac{1}{2}\sum_{a,b}(R_{[2]})^2_{ab,ab} = (D-2)\operatorname{tr} R^2, \tag{6.98}$$

where the factor of $\frac{1}{2}$ accounts for the fact that we are summing over independent pairs of indices a, b instead of taking them as anti-symmetric double indices, and

$$\operatorname{tr} R^4_{[2]} = (D-8)\operatorname{tr} R^4 + 3(\operatorname{tr} R^2)^2. \tag{6.99}$$

Details of the computation can be found in Appendix B.6. We see that this direct computation becomes very cumbersome for higher rank of the tensors and for higher powers.

Fortunately, there is the generating formula [116]:

$$\sum_{k=0}^{\infty} x^k \operatorname{ch}(R_{[k]}) = \det\left(1 + x\, e^{\frac{iR}{2\pi}}\right) = \exp\operatorname{tr}\log\left(1 + x\, e^{\frac{iR}{2\pi}}\right) \tag{6.100}$$

with x a formal variable. So, $\operatorname{ch}(R_{[k]})$ can be found by expanding the right-hand-side and selecting the k-th power of x. For example, for the second-rank antisymmetric one finds

$$\operatorname{ch}(R_{[2]}) = \frac{1}{2}\left(\operatorname{tr} e^{\frac{iR}{2\pi}}\right)^2 - \frac{1}{2}\operatorname{tr} e^{\frac{i2R}{2\pi}} . \tag{6.101}$$

This formula contains all $\operatorname{tr}(R^n_{[2]})$ in terms of the fundamental traces: for example, four-form component of this equation gives (6.98), the eight-form component recovers (6.99), and so on. Likewise, all traces of the form $\operatorname{tr}(R^n_{[k]})$ can be found by expanding equation (6.100) to order x^k and to form degree $2n$.

Putting everything together, the anomaly polynomial for the chiral fermionic p-forms is given by

$$I^{(p)}_{D+2} = \left[\hat{A}(M)\sum_{k=0}^{p}(-1)^{p-k}\operatorname{ch}(R_{[k]})\right]_{D+2} . \tag{6.102}$$

This can be computed for any desired D and p using the ingredients detailed above. We display explicitly the results in terms of Pontryagin classes in dimensions $D = 2$, 6 and 10 in tables 6.3, 6.4 and 6.5. Of course, for spin 1/2 and 3/2 fields ($p = 0$ and $p = 1$ respectively), these tables reproduce the classic results of [68]. The anomaly polynomial for the chiral bosons (i.e. the self-dual scalar, 2-form and 4-form) in those dimensions are also listed for convenience [68].

Interestingly, in dimensions $D \geq 6$ we find[2] that the anomaly of a chiral fermionic p-form matches that of a $(D-p-1)$-form,

$$I^{(p)}_{D+2} = I^{(D-p-1)}_{D+2} . \tag{6.103}$$

For example, the anomaly of a chiral fermionic 2-form in $D = 6$ could be cancelled by a 3-form of the opposite chirality. Similarly, one could imagine canceling the anomaly of a bosonic, self-dual 4-form in $D = 10$ (such as the one appearing in type IIB supergravity) using topological fermionic 8- and 9-forms of opposite chirality. This is of course subject to the caveats mentioned in the introduction, namely, the current lack of explicit Lagrangians coupling fermionic p-forms to dynamical gravity. Nevertheless, it would be very interesting to see whether these possibilities can be realised in physically relevant models.

[2]To be more precise: this is apparent in $D = 6$ and 10 from tables 6.4 and 6.5, and has been checked explicitly in $D = 14$ and 18; however, we have no general proof for arbitrary D.

p	$I_4^{(p)}$
0	$-\frac{1}{24}\,p_1$
1	$\frac{23}{24}\,p_1$
2	$-p_1$
I_4^A	$-\frac{1}{24}\,p_1$

Table 6.3: The anomaly polynomials for chiral fermionic p-forms in $D = 2$.

p	$I_8^{(p)}$
0	$\frac{1}{5\,760}\left(7\,p_1^2 - 4\,p_2\right)$
1	$\frac{1}{5\,760}\left(275\,p_1^2 - 980\,p_2\right)$
2	$\frac{1}{5\,760}\left(790\,p_1^2 + 2\,840\,p_2\right)$
3	$\frac{1}{5\,760}\left(790\,p_1^2 + 2\,840\,p_2\right)$
4	$\frac{1}{5\,760}\left(275\,p_1^2 - 980\,p_2\right)$
5	$\frac{1}{5\,760}\left(7\,p_1^2 - 4\,p_2\right)$
6	0
I_8^A	$\frac{1}{5\,760}\left(16\,p_1^2 - 112\,p_2\right)$

Table 6.4: The anomaly polynomials for chiral fermionic p-forms in $D = 6$.

p	$I_{12}^{(p)}$
0	$\frac{1}{967\,680}\left(-31\,p_1^3 + 44\,p_1p_2 - 16\,p_3\right)$
1	$\frac{1}{967\,680}\left(225\,p_1^3 - 1\,620\,p_1p_2 + 7\,920\,p_3\right)$
2	$\frac{1}{967\,680}\left(2\,412\,p_1^3 + 27\,792\,p_1p_2 - 186\,048\,p_3\right)$
3	$\frac{1}{967\,680}\left(7\,980\,p_1^3 + 162\,960\,p_1p_2 - 73\,920\,p_3\right)$
4	$\frac{1}{967\,680}\left(13\,734\,p_1^3 + 338\,184\,p_1p_2 + 764\,064\,p_3\right)$
5	$\frac{1}{967\,680}\left(13\,734\,p_1^3 + 338\,184\,p_1p_2 + 764\,064\,p_3\right)$
6	$\frac{1}{967\,680}\left(7\,980\,p_1^3 + 162\,960\,p_1p_2 - 73\,920\,p_3\right)$
7	$\frac{1}{967\,680}\left(2\,412\,p_1^3 + 27\,792\,p_1p_2 - 186\,048\,p_3\right)$
8	$\frac{1}{967\,680}\left(225\,p_1^3 - 1\,620\,p_1p_2 + 7\,920\,p_3\right)$
9	$\frac{1}{967\,680}\left(-31\,p_1^3 + 44\,p_1p_2 - 16\,p_3\right)$
10	0
I_{12}^A	$\frac{1}{967\,680}\left(-256\,p_1^3 + 1\,664\,p_1p_2 - 7\,936\,p_3\right)$

Table 6.5: The anomaly polynomials for chiral fermionic p-forms in $D = 10$.

The 6d exotic gravitino anomaly revisited. We are at the stage to compare this result with the one we computed via field strength of exotic gravitino in chapter 3, section 3.1.4. Recall that in six dimensions, we use the A_3 Dynkin label, and now consider $[1,0,0]$ as the Weyl spinor with positive chirality, then we have the decomposition for self-dual $\chi_{\mu\nu\rho}$ (now we use $\chi_{\mu\nu\rho}$ for the field strength as in chapter 3)

$$[2,0,0]\otimes[1,0,0]=[3,0,0]\oplus[1,1,0]\,. \tag{6.104}$$

With respect to this tensor product decomposition, we have two interpretations: the "on-shell" computation is associated to a chiral three-form spinor $\chi_{\mu\nu\rho}$ satisfying $\gamma^{\alpha\beta\mu\nu\rho}\chi_{\mu\nu\rho}=0$ and it picks out only the $[3,0,0]$ piece; while the "off-shell" computation is for self-dual $\chi_{\mu\nu\rho}$ which contains both $[3,0,0]$ and $[1,1,0]$, as we explained in the section 3.1.4, the self-duality constraint is weaker.

In $D=6$, the index density contribution of the "on-shell" chiral three-form spinor is given by (3.45) as

$$[\mathrm{Ind}(\mathcal{D}_\chi)]_8=\frac{1}{5760}(501p_1^2+3828p_2)\,. \tag{6.105}$$

From table 6.4 we read

$$I_8^{(2)}=[\mathrm{Ind}(\mathcal{D}_2)]_8=\frac{1}{5760}(790p_1^2+2840p_2)\,. \tag{6.106}$$

$$[\mathrm{Ind}(\mathcal{D}_2)]_8-[\mathrm{Ind}(\mathcal{D}_\chi)]_8=\frac{1}{5760}(289p_1^2-988p_2)=I_{\frac{3}{2}}+2I_{\frac{1}{2}}\,. \tag{6.107}$$

The difference is exactly the same as the index density contribution from a gamma-traceless chiral one-form spinor:

$$I_{\frac{3}{2}}+2I_{\frac{1}{2}}\simeq[1,1,0]\,. \tag{6.108}$$

Indeed,

$$\begin{aligned}[1,1,0]&\simeq\mathcal{C}^\infty(S^+\otimes T^*M-S^-)\\&=\mathcal{C}^\infty(S^+\otimes[T^*M-1])+2\mathcal{C}^\infty(S^+)\,.\end{aligned} \tag{6.109}$$

We see that it is the "off-shell" computation via the field strength that matches the result through BV quantisation.

Conclusion

We conclude by briefly mentioning some of the many aspects of the exotic supersymmetric multiplets that we have not addressed.

The algebraic structure of the exotic six-dimensional multiplets and the embedding into the exceptional geometry framework appears to be an interesting story, which we have only scratched the surface of here. It is clear that the six-dimensional momenta and spin group can be described in the algebraic framework, such that they agree with the supersymmetry algebra, but there is no spacetime section in the usual sense. This should not be a great surprise as these are not standard gravitational multiplets. However, the wider interpretation of the matching of momentum charges and section condition is subtle issue for the higher-rank exceptional groups which perhaps deserves further study in its own right. One could wonder whether the presence of the additional **248** constrained fields needed to accommodate the gauge algebra and tensor hierarchy in [38] could play a role in this. Naively, one would expect some modification to the usual generalised Lie derivative picture would be needed in order for the gauge algebra to close in the absence of a spacetime solving the section condition.

One could also wonder whether there is a similar story for the $D_{[\mu\nu]\lambda}$ exotic graviton of (1.17). In the case of the $\mathcal{N}=(3,1)$ theory, the decomposition of the adjoint of $E_{8(8)}$ under $SL(3,\mathbb{R})\times F_{4(4)}\subset G_{2(2)}\times F_{4(4)}$ is

$$\mathbf{248}\to\Big(\mathfrak{sl}(3,\mathbb{R})\oplus(\mathbf{3},\mathbf{1})\oplus(\mathbf{3}',\mathbf{1})\Big)\oplus\mathfrak{f}_{4(4)}\oplus(\mathbf{3},\mathbf{26})\oplus(\mathbf{3}',\mathbf{26})\oplus(\mathbf{1},\mathbf{26}) \tag{6.110}$$

where the three terms in the bracket make up $\mathfrak{g}_{2(2)}$. Decomposing under $SO(3)\times Sp(6)\times Sp(2)$ one can see that the non-compact generators of $\mathfrak{g}_{2(2)}$ match the $\mathbf{5}\oplus\mathbf{3}$ of $SO(3)$ for the relevant exotic graviton, while the $\mathfrak{f}_{4(4)}$ term corresponds to the scalar coset. There is also a $(\mathbf{3}',\mathbf{14},\mathbf{1})$ for the vectors and $(\mathbf{3},\mathbf{6},\mathbf{2})$ for the self-dual two-forms. The final term is slightly harder to interpret, but the 14 non-compact generators could be matched to the three-form magnetic duals of the vectors.

More generally, it appears that the special role played by $SL(3,\mathbb{R})$ for the exotic graviton for the $\mathcal{N}=(4,0)$ multiplet could become $G_{2(2)}$ for the exotic graviton of the $\mathcal{N}=(3,1)$ theory. For example, there is an $\mathcal{N}=(1,0)$ supermultiplet (with $V=(\mathbf{3},\mathbf{2},\mathbf{1})$ in the notation of appendix A.2) with field content

$$\begin{array}{ccccc}(\mathbf{2},\mathbf{2},\mathbf{1}) & \oplus & (\mathbf{3},\mathbf{2},\mathbf{2}) & \oplus & (\mathbf{4},\mathbf{2},\mathbf{1})\\ A_\mu & & \psi^R_\mu & & D_{[\mu\nu]\lambda}\end{array} \tag{6.111}$$

This multiplet appears to match the decomposition of the group $SO(4,3)$ by

$$\mathfrak{so}(4,3)\to\mathfrak{g}_{2(2)}\oplus\mathbf{7}\to\Big(\mathfrak{sl}(3,\mathbb{R})\oplus(\mathbf{3},\mathbf{1})\oplus(\mathbf{3}',\mathbf{1})\Big)\oplus\mathbf{3}\oplus\mathbf{3}'\oplus\mathbf{1} \tag{6.112}$$

Again, the three terms in the bracket correspond to the field $D_{[\mu\nu]\lambda}$ while the $\mathbf{3}\oplus\mathbf{3}'$ of $SL(3,\mathbb{R})$ correspond to a vector field. Finally, the remaining non-compact singlet generator is the magnetic dual three-form to this vector. This pattern is repeated across other examples, with $\mathfrak{g}_{2(2)}$ playing the role for $D_{[\mu\nu]\lambda}$. It could thus be worth considering how the rest of our analysis would work out for these cases.[3]

The existence of the exotic six-dimensional multiplets could have been dismissed as a mere curiosity if not for the possible far reaching implications for gaining insights into strongly coupled

[3] Many of the considerations of this paper could also be applied to exotic two-dimensional theories with fields built as product involving chiral bosons. We have not studied two-dimensional theories in this thesis.

gravitational theories [1–3]. Just like the gravity multiplet can be thought of the product of two YM multiplets, the $(4, 0)$ multiplet is a product of two $(2, 0)$ tensor multiplets [4, 10]. From other side while the circle reduction of a $(2, 0)$ theory yields five-dimensional YM, it appears to be impossible to reconcile the nonlinear couplings of the five-dimensional maximal supergravity with the symmetries of the six dimensional $(4, 0)$ and $(3, 1)$ multiplet consistently with six-dimensional gauge or Lorentz invariance. We have not studied the possibility of couplings which manifestly break these properties in any detail.

Our results on anomalies in exotic six-dimensional multiplets are perhaps not too surprising – after all they do not display full covariance. One should have in mind formal properties of elliptic operators on a six-manifold M, rather that anomalous box diagrams. While we express the result in terms of local curvatures on M, there can be no cancellation mechanism short of full automatic cancellation. Such cancellations are not happening for any of the exotic multiplets. In the $(2, 0)$ case the SD Weyl multiplet is the only multiplet that does not have an anomaly polynomial proportional to X_8 (3.28). Both for $(2, 0)$ and for $(1, 0)$ SD Weyl multiplets, trying to find a combination of matter that would lead to cancellation of the irreducible part of the anomaly is not useful, in spite of abundance of 2-form tensor fields. Due to absence of gravitons, one could not possibly compute counterterms that could lead to anomaly cancellation. On the other hand, for the $(2, 0)$ and $(1, 0)$ SD Weyl multiplets one could contemplate coupling to respectively $(2, 0)$ and $(1, 0)$ gravity multiplets, together with appropriate matter, in order to cancel the irreducible part of the anomaly. We have neither studied if this can be done supersymmetrically or thought about any other aspects of such "exotic bi-gravity" theories.

Finally, we discussed the quantisation of the exotic spinor fields in detail. Our starting point is the rank two antisymmetric tensor-spinor and its quantisation in the BV formalism generalises to fermionic p-forms provided that they possess non-trivial Rarita-Schwinger type Lagrangians. An extra propagating ghost appear in every case when the quadratic gauges contain differential operators in Gaussian gauge-fixings. The spectrum of dynamical fields effectively corresponds to 1, 3, 5, 7, ... gamma-traceful fields. With the apparent ghost spectrum, the gravitational anomalies are computed by using the Atiyah-Singer index theorem. We recover the classic results [68] for spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ fermions and a new matching (6.103) between anomalies given by chiral fermionic p- and $(D - p - 1)$-forms is discovered.

Appendix A

Conventions for the exotic multiplets and tensor fields

A.1 $6d$ spinors and gamma matrices

In this appendix, we give our conventions for the six-dimensional gamma matrices. For more details one can find, for example in [11, 113, 117].

The flat metric $\eta_{\mu\nu}$ is with the "mostly plus" signature, i.e. $\eta = \text{diag}(-+++++)$.

By Cliff$(p,q,\mathbb{R})$, $Spin(p,q)$ and $SO(p,q)$, as well as their associated Lie-algebras, we mean to use the metric $\eta = \text{diag}(-\cdots-++\cdots+)$ with p positive entries, and q negative entries. One should pay attention that we have exactly the reverse of the conventional notation, unless stated otherwise.

The Clifford algebra Cliff$(5,1)$ is generated by the gamma matrices defined by

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \tag{A.1}$$

and the higher rank gamma matrices are defined by the total antisymmetrisation of the product of single gammas

$$\gamma^{\mu_1\mu_2\cdots\mu_r} = \gamma^{[\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_r]}. \tag{A.2}$$

The chirality matrix is given by $\gamma_7 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5$, it anti-commutes with all the gamma matrices $\{\gamma_7,\gamma^\mu\} = 0$. We define the Dirac conjugate by $\bar{\psi} = i\psi^\dagger\gamma^0$.

The chirality projector is defined by

$$P_\pm = \frac{1}{2}\left(1 \pm \gamma_7\right). \tag{A.3}$$

With our convention, all the spatial gamma matrices are hermitian and ${\gamma^0}^\dagger = -\gamma^0$. The important symmetry property of the gamma matrix is given by the unitary transpose intertwiner C (charge conjugation matrix)

$$(\gamma^\mu)^T = -C\gamma^\mu C^{-1} \qquad C^T = C. \tag{A.4}$$

We use the transpose intertwiner to lower and raise the spinor index

$$\psi^\alpha = \mathcal{C}^{\alpha\beta}\psi_\beta \qquad (\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\rho\,\mathcal{C}_{\rho\beta} = (\gamma_\mu C^{-1})_{\alpha\beta}, \tag{A.5}$$

here $\mathcal{C}^{\alpha\beta}$ are the components of C^T while $\mathcal{C}_{\alpha\beta}$ are the components of C^{-1}.

The symmetry property and the (anti-)hermitian property together also fix the complex conjugation property of the gammas. To see this, we define the unitary matrix

$$\mathcal{B} = -iC\gamma^0, \tag{A.6}$$

which gives

$$(\gamma^\mu)^* = \mathcal{B}\gamma^\mu\mathcal{B}^{-1} \qquad \text{and} \qquad \mathcal{B}^*\mathcal{B} = -I. \tag{A.7}$$

This definition ensures that for a generic spinor ψ, the expression $\mathcal{B}\psi$ transforms in the same way as ψ^* under the Lorentz transformation. However, because of the minus sign in the second equation of (A.7) we can not identify the complex conjutage spinor ψ^* with $\mathcal{B}\psi$.

Instead, for pairs of spinors ζ^A for $A = 1, \dots, 2n$, we can impose the symplectic-realilty condition

$$(\zeta^A)^* = \Omega_{AB}\mathcal{B}\zeta^B, \tag{A.8}$$

where the Ω_{AB} is the $Sp(2n)$ symplectic form and its inverse is denoted by Ω^{AB}. They are numerically the same, but we can use them to lower and raise the $Sp(2n)$ index, if we denote $(\zeta^A)^*$ by putting down the $Sp(2n)$ index: ζ^*_A.

Spinors satisfy the condition (A.8) are called symplectic-Majorana spinors. Sometimes, the symplectic-realilty condition is written with the Dirac conjugate

$$\bar{\zeta}^A = \Omega_{AB}(\zeta^B)^T C \qquad \text{or} \qquad \bar{\zeta}^A_\alpha = \Omega_{AB}\mathcal{C}^{\alpha\beta}\zeta^B_\beta \tag{A.9}$$

Furthermore, we can impose the Weyl condtion $\gamma_7\zeta^A = \pm\zeta^A$ on the spinors and this is compatible with the reality condition (A.8) because of $(\gamma_7)^* = \mathcal{B}\gamma_7\mathcal{B}^{-1}$. In this way, we define the symplectic-Majorana-Weyl spinor in six dimensions.

A.2 Constructions of the chiral multiplets

We briefly review the construction of massless multiplets of the chiral supersymmetry algebras in six dimensions. For the construction of these multiplets, let us first look at the chiral supersymmetric algebra $\mathcal{N} = (N, 0)$ without central charges

$$\{Q^A_\alpha, Q^B_\beta\} = \Omega^{AB}\left(\mathrm{P}_+\gamma^\mu C^{-1}\right)_{[\alpha\beta]} P_\mu \tag{A.10}$$

where the C is the transpose intertwiner (charge conjugation matrix) and P_+ is the positive chiral projector. The index $\alpha = 1, \dots, 4$ is the $SU^*(4) \simeq Spin(1,5)$ spinor index and the R-symmetry $Sp(2N)$ index A runs from 1 to $2N$. The supercharges $Q_{\alpha A}$ thus live in the $(\mathbf{4}, \mathbf{2N})$ representation of $Spin(1,5) \times Sp(2N)$. where Ω_{AB} is the $Sp(2N)$ symplectic form, P_μ is the momentum.

As usual, to analyse the spin content of massless multiplets, we decompose under $Spin(1,1) \times Spin(4) \subset Spin(1,5)$, writing the $\mathrm{Cliff}(1,5;\mathbb{R})$ gamma matrices as the tensor products

$$\gamma^0 = \mathrm{i}\sigma^2 \otimes \mathbb{1} \qquad \gamma^1 = \sigma^1 \otimes \mathbb{1} \qquad \gamma^m = \sigma^3 \otimes \gamma^m \tag{A.11}$$

where σ^i are the Pauli matrices and γ^m are the generators of $\mathrm{Cliff}(4;\mathbb{R})$. The transpose intertwiners $C_{1,5}$ for $\mathrm{Cliff}(1,5)$ and C_4 for $\mathrm{Cliff}(4)$, which we use to raise and lower spinor indices, are then related by $C_{1,5} = \sigma^1 \otimes C_4$. Taking zero central charges and momentum $(P^\mu) = (k, k, 0, \dots, 0)$ for a massless representation, we see that

$$\left[(\mathrm{P}_+\gamma^\mu C^{-1})_{[\alpha\beta]}P_\mu\right] = 2k\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \otimes C_4 \tag{A.12}$$

The supercharges with non-trivial algebra are thus those with positive chirality under the $Spin(1,1)$.[1] As $(\mathbf{4})_{Spin(5,1)} \to ((\mathbf{2},\mathbf{1})_+ + (\mathbf{1},\mathbf{2})_-)_{SU(2)_1\times SU(2)_2\times Spin(1,1)}$ we have that these transform in the $(\mathbf{2},\mathbf{1},\mathbf{2N})$ representation of $SU(2)_1 \times SU(2)_2 \times Sp(2N)$. Decomposing under $U(1)_1 \subset SU(2)_1$ we have $\mathbf{2} \to \mathbf{1}_+ + \mathbf{1}_-$. Denote now by $Q_\pm$ the supercharges with $U(1)_1$ charge ± 1. In an appropriate complex basis we have that these satisfy the usual Clifford algebra of raising and lowering operators

$$\{Q_{+A}, Q_{+B}\} = 0 \qquad \{Q_{+A}, Q^{-B}\} = \delta_A{}^B \qquad \{Q^{-A}, Q^{-B}\} = 0 \tag{A.13}$$

We can then build a multiplet by acting on a vacuum state $|0\rangle$ with the raising operators Q_{+A}. The basic multiplet thus has the form

$$|0\rangle \quad Q_{+A}|0\rangle \quad Q_{+A}Q_{+B}|0\rangle \quad Q_{+A}Q_{+B}Q_{+C}|0\rangle \quad \dots \tag{A.14}$$

With each term having one unit more $U(1)_1$ charge than the previous. These various terms can then be combined into $SU(2)_1$ representations.

[1]The negative chirality supercharges are nilpotent and generate physically irrelevant zero-norm states, so we discard them at this point.

A.2.1 $\mathcal{N} = (1, 0)$ multiplets

We list massless multiplets of the chiral supersymmetry algebras in the non-maximal cases six dimensions. Here, the R-symmetry is $Sp(2)$ and the basic multiplet, in which the vacuum has only a $U(1)_1$ charge of -1, has the structure

$$\begin{array}{cccc} & |0\rangle & Q_{+A}|0\rangle & Q_{+A}Q_{+B}|0\rangle \\ U(1)_1 \text{ charge} & -1 & 0 & +1 \\ Sp(2) \text{ irreps} & \mathbf{1} & \mathbf{2} & \mathbf{1} \end{array} \tag{A.15}$$

This is the hyper-multiplet and by combining the $U(1)_1$ charges into $SU(2)_1$ representations we can read-off its field content as

$$\begin{array}{ccc} \text{Spin} & 0 & \frac{1}{2} \\ SU(2)_1 \times SU(2)_2 \times Sp(2) \text{ rep} & (\mathbf{1}, \mathbf{1}, \mathbf{2}) & (\mathbf{2}, \mathbf{1}, \mathbf{1}) \end{array} \tag{A.16}$$

The other multiplets are then formed by taking tensor products of this multiplet with some representation V of $G_{\text{little}} = SU(2)_1 \times SU(2)_2 \times Sp(2)$. We have:

$$\begin{array}{cccc} V = (\mathbf{1}, \mathbf{1}, \mathbf{1}) & \text{(Hyper)} & & \\ \text{Field} & \phi & \lambda^R & \\ G_{\text{little}} \text{ rep} & (\mathbf{1}, \mathbf{1}, \mathbf{2}) & (\mathbf{2}, \mathbf{1}, \mathbf{1}) & \\ & & & \\ V = (\mathbf{1}, \mathbf{2}, \mathbf{1}) & \text{(Vector)} & & \\ \text{Field} & & \lambda^L & A_\mu \\ G_{\text{little}} \text{ rep} & & (\mathbf{1}, \mathbf{2}, \mathbf{2}) & (\mathbf{2}, \mathbf{2}, \mathbf{1}) \\ & & & \\ V = (\mathbf{2}, \mathbf{1}, \mathbf{1}) & \text{(Tensor)} & & \\ \text{Field} & \phi & \lambda^R & B^-_{\mu\nu} \\ G_{\text{little}} \text{ rep} & (\mathbf{1}, \mathbf{1}, \mathbf{1}) & (\mathbf{2}, \mathbf{1}, \mathbf{2}) & (\mathbf{3}, \mathbf{1}, \mathbf{1}) \end{array} \tag{A.17}$$

$$\begin{array}{cccccc} V = (\mathbf{1}, \mathbf{3}, \mathbf{1}) & (\text{Gravitino}^L) & & & & \\ \text{Field} & & & B^+_{\mu\nu} & \psi^L_\mu & \\ G_{\text{little}} \text{ rep} & & & (\mathbf{1}, \mathbf{3}, \mathbf{2}) & (\mathbf{2}, \mathbf{3}, \mathbf{1}) & \\ & & & & & \\ V = (\mathbf{2}, \mathbf{2}, \mathbf{1}) & (\text{Gravitino}^R) & & & & \\ \text{Field} & & \lambda^L & A_\mu & \psi^R_\mu & \\ G_{\text{little}} \text{ rep} & & (\mathbf{1}, \mathbf{2}, \mathbf{1}) & (\mathbf{2}, \mathbf{2}, \mathbf{2}) & (\mathbf{3}, \mathbf{2}, \mathbf{1}) & \\ & & & & & \\ V = (\mathbf{2}, \mathbf{3}, \mathbf{1}) & \text{(Gravity)} & & & & \\ \text{Field} & & & B^+_{\mu\nu} & \psi^L_\mu & g_{\mu\nu} \\ G_{\text{little}} \text{ rep} & & & (\mathbf{1}, \mathbf{3}, \mathbf{1}) & (\mathbf{2}, \mathbf{3}, \mathbf{2}) & (\mathbf{3}, \mathbf{3}, \mathbf{1}) \\ & & & & & \\ V = (\mathbf{3}, \mathbf{1}, \mathbf{1}) & \text{(Exotic Gravitino)} & & & & \\ \text{Field} & & \lambda^R & B^-_{\mu\nu} & \psi^R_{\mu\nu} & \\ G_{\text{little}} \text{ rep} & & (\mathbf{2}, \mathbf{1}, \mathbf{1}) & (\mathbf{3}, \mathbf{1}, \mathbf{2}) & (\mathbf{4}, \mathbf{1}, \mathbf{1}) & \\ & & & & & \\ V = (\mathbf{4}, \mathbf{1}, \mathbf{1}) & \text{(Exotic Gravity)} & & & & \\ \text{Field} & & & B^-_{\mu\nu} & \psi^R_{\mu\nu} & C_{[\mu\nu][\lambda\kappa]} \\ G_{\text{little}} \text{ rep} & & & (\mathbf{3}, \mathbf{1}, \mathbf{1}) & (\mathbf{4}, \mathbf{1}, \mathbf{2}) & (\mathbf{5}, \mathbf{1}, \mathbf{1}) \end{array} \tag{A.18}$$

A.2.2 $\mathcal{N} = (2,0)$ multiplets

Here, the R-symmetry is $Sp(4)$ and the basic multiplet, in which the vacuum has only a $U(1)_1$ charge of -2, has the structure

$$\begin{array}{cccccc} & |0\rangle & Q_{+A}|0\rangle & Q_{+A}Q_{+B}|0\rangle & Q_{+A}Q_{+B}Q_{+C}|0\rangle & Q_{+A}Q_{+B}Q_{+C}Q_{+D}|0\rangle \\ U(1)_1 \text{ charge} & -2 & -1 & 0 & +1 & +2 \\ Sp(4) \text{ irreps} & \mathbf{1} & \mathbf{4} & \mathbf{1}+\mathbf{5} & \mathbf{4} & \mathbf{1} \end{array} \tag{A.19}$$

This gives the tensor multiplet, whose field content is

$$\begin{array}{cccc} \text{Field} & \phi & \lambda^R & B^-_{\mu\nu} \\ G_{\text{little}} \text{ rep} & (\mathbf{1},\mathbf{1},\mathbf{5}) & (\mathbf{2},\mathbf{1},\mathbf{4}) & (\mathbf{3},\mathbf{1},\mathbf{1}) \end{array} \tag{A.20}$$

The other multiplets are then formed by taking tensor products of this multiplet with some representation V of $G_{\text{little}} = SU(2)_1 \times SU(2)_2 \times Sp(4)$. We have:

$$\begin{array}{cccccc}
V = (\mathbf{1},\mathbf{1},\mathbf{1}) & \text{(Tensor)} & & & & \\
\text{Field} & \phi & \lambda^+ & B^-_{\mu\nu} & & \\
G_{\text{little}} \text{ rep} & (\mathbf{1},\mathbf{1},\mathbf{5}) & (\mathbf{2},\mathbf{1},\mathbf{4}) & (\mathbf{3},\mathbf{1},\mathbf{1}) & & \\
 & & & & & \\
V = (\mathbf{1},\mathbf{2},\mathbf{1}) & \text{(Gravitino}^+\text{)} & & & & \\
\text{Field} & & \lambda^- & A_\mu & \psi^+_\mu & \\
G_{\text{little}} \text{ rep} & & (\mathbf{1},\mathbf{2},\mathbf{5}) & (\mathbf{2},\mathbf{2},\mathbf{4}) & (\mathbf{3},\mathbf{2},\mathbf{1}) & \\
 & & & & & \\
V = (\mathbf{1},\mathbf{3},\mathbf{1}) & \text{(Gravity)} & & & & \\
\text{Field} & & & B^+_{\mu\nu} & \psi^-_\mu & g_{\mu\nu} \\
G_{\text{little}} \text{ rep} & & & (\mathbf{1},\mathbf{3},\mathbf{5}) & (\mathbf{2},\mathbf{3},\mathbf{4}) & (\mathbf{3},\mathbf{3},\mathbf{1}) \\
 & & & & & \\
V = (\mathbf{2},\mathbf{1},\mathbf{1}) & \text{(Exotic Gravitino)} & & & & \\
\text{Field} & \phi & \lambda^+ & B^-_{\mu\nu} & \psi^+_{\mu\nu} & \\
G_{\text{little}} \text{ rep} & (\mathbf{1},\mathbf{1},\mathbf{4}) & (\mathbf{2},\mathbf{1},\mathbf{5}+\mathbf{1}) & (\mathbf{3},\mathbf{1},\mathbf{4}) & (\mathbf{4},\mathbf{1},\mathbf{1}) & \\
 & & & & & \\
V = (\mathbf{3},\mathbf{1},\mathbf{1}) & \text{(Exotic Gravity)} & & & & \\
\text{Field} & \phi & \lambda^+ & B^-_{\mu\nu} & \psi^+_{\mu\nu} & C_{[\mu\nu][\lambda\kappa]} \\
G_{\text{little}} \text{ rep} & (\mathbf{1},\mathbf{1},\mathbf{1}) & (\mathbf{2},\mathbf{1},\mathbf{4}) & (\mathbf{3},\mathbf{1},\mathbf{5}+\mathbf{1}) & (\mathbf{4},\mathbf{1},\mathbf{4}) & (\mathbf{5},\mathbf{1},\mathbf{1})
\end{array} \tag{A.21}$$

Note that there is no Gravitino^L multiplet. This is consistent with the absence of a gravity mulitplet when $\mathcal{N} = (3,0)$ or $\mathcal{N} = (4,0)$.

We can also decompose these multiplets into multiplets of the $\mathcal{N} = (1,0)$ algebra. The resulting $\mathcal{N} = (2,0) \to \mathcal{N} = (1,0)$ decompositions are given below.

$$\begin{array}{rcl}
\text{Tensor} & \to & \text{Tensor} + 2 \times \text{Hyper} \\
\text{Gravitino}^R & \to & \text{Gravitino}^R + 2 \times \text{Vector} \\
\text{Gravity} & \to & \text{Gravity} + 2 \times \text{Gravitino}^L \\
\text{Exotic Gravitino} & \to & \text{Exotic Gravitino} + 2 \times \text{Tensor} + 2 \times \text{Hyper} \\
\text{Exotic Gravity} & \to & \text{Exotic Gravity} + 2 \times \text{Exotic Gravitino} + \text{Tensor}
\end{array} \tag{A.22}$$

A.2.3 $\mathcal{N} = (4, 0)$ multiplet

Here, the R-symmetry is $Sp(8)$ and the basic multiplet, in which the vacuum has only a $U(1)_1$ charge of -4, is the exotic gravity multiplet and has the structure

$U(1)_1$ charge	-4	-3	-2	-1			
$Sp(4)$ irreps	$\mathbf{1}$	$\mathbf{8}$	$\mathbf{1}+\mathbf{27}$	$\mathbf{8}+\mathbf{48}$			
				0			
				$\mathbf{1}+\mathbf{27}+\mathbf{42}$			
				$+1$	$+2$	$+3$	$+4$
				$\mathbf{8}+\mathbf{48}$	$\mathbf{1}+\mathbf{27}$	$\mathbf{8}$	$\mathbf{1}$

(A.23)

Thus the field content is

Field	ϕ	λ^R	$B^-_{\mu\nu}$	$\psi^R_{\mu\nu}$	$C_{[\mu\nu][\lambda\kappa]}$
G_{little} rep	$(\mathbf{1},\mathbf{1},\mathbf{42})$	$(\mathbf{2},\mathbf{1},\mathbf{48})$	$(\mathbf{3},\mathbf{1},\mathbf{27})$	$(\mathbf{4},\mathbf{1},\mathbf{8})$	$(\mathbf{5},\mathbf{1},\mathbf{1})$

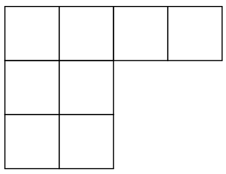
(A.24)

and we have the $\mathcal{N} = (4, 0) \to \mathcal{N} = (2, 0)$ decomposition:

$$\text{Exotic Gravity} \quad \to \quad \text{Exotic Gravity} + 4 \times \text{Exotic Gravitino} + 5 \times \text{Tensor} \tag{A.25}$$

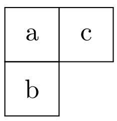
A.3 Young diagrams and Young tableaux

A Young diagram consists of n boxes set in k rows of non-increasing length: $l_1 \geq l_2 \geq \cdots \geq l_k, \sum_i l_i = n$. In the following example $n = 8$, with $l_1 = 4$, $l_2 = 2$, $l_3 = 2$

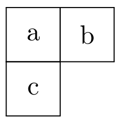


and we write a Young diagram either with $(l_1, l_2, \ldots, l_k)$ by the number of boxes of each row or with $[f_1, f_2, \ldots, f_n]$ indicating the number of boxes of each column. The above example is then $(4, 2, 2)$ or $[3, 3, 1, 1]$.

We can fill the Young diagram with different integers ranging between 1 and n, thus making a Young tableau. A standard way to do this is to keep integers increasing in each row from left to right, and in each column from top to bottom. Young tableaux obtained in such sense is called standard Young tableaux. For example we can fill the $[2, 1]$ Young diagram as



or



(A.26)

the first Young tableau can represent a $GL(d, \mathbb{R})$ tensor $T_{abc} = T_{[ab]c}$ while the tensor $\tilde{T}_{abc}$ represented by the second one is antisymmetric in the index a and c. We can see that there is a nature symmetric group S_m action on the $GL(d, \mathbb{R})$ indices via permutations. According to the representation theory of the symmetric group S_m there is a bijection between irreducible representations and Young diagrams with n boxes. The dimension of that representation is given by the number of standard tableaux. In our conventions, a tensor is said to be of (symmetry) type $[f_1, f_2, \ldots, f_n]$ if it transforms by S_m under that representation. The symmetry is explicitly given as: the tensors are totally antisymmetric in the indices corresponding to a column of the Young tableau; and any antisymmetrization over all the indices of a column, plus one index belonging to another column to its right, vanishes. There are further discussions involving e.g. computing the numbers of the standard Young tableaux, Young projectors, hook length and etc. we refer to [81, 117].

Appendix B

Conventions and useful formulae

B.1 More on Cliff$(10,1;\mathbb{R})$

We follow the conventions as in [34,118], consider the Clifford algebra Cliff$(10,1;\mathbb{R})$ and define the highest gamma matrix as

$$\Gamma^{(11)} = \Gamma^0\Gamma^1 \dots \Gamma^9\Gamma^{10} \tag{B.1}$$

and we have for the signature $(10,1)$ we have $\left(\Gamma^{(11)}\right)^2 = \mathbb{1}$. We choose $\Gamma^{(11)} = \Gamma^0\Gamma^1 \dots \Gamma^9\Gamma^{10} = -\mathbb{1} = -\Gamma_0\Gamma_1 \dots \Gamma_9\Gamma_{10}$ and define the eleven dimensional Levi-Civita symbol as $\epsilon_{01\dots10} = 1$. With this choice we have the duality relation between the rank p and rank $(11-p)$ gamma matrices

$$\Gamma^{\mu_1\mu_2\dots\mu_p} = (-1)^{\frac{(p+1)(p-2)}{2}} \frac{1}{(11-p)!} \epsilon^{\mu_1\mu_2\dots\mu_p}{}_{\mu_{p+1}\mu_{p+2}\dots\mu_1 1}\Gamma^{\mu_{p+1}\mu_{p+2}\dots\mu_1 1}\,. \tag{B.2}$$

It follows that $\{\mathbb{1}, \Gamma^{\mu_1}, \Gamma^{\mu_1\mu_2}, \Gamma^{\mu_1\mu_2\mu_3}, \Gamma^{\mu_1\mu_2\mu_3\mu_4}, \Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}\}$ is a basis of Cliff$(10,1;\mathbb{R}) \simeq \mathfrak{gl}(32,\mathbb{R})$. If we take out the generator $\mathbb{1}$, we would realise an $\mathfrak{sl}(32,\mathbb{R})$ algebra.

The above representation of Cliff$(10,1;\mathbb{R})$ is Majorana and it acts on the eleven dimensional spinor Q via the standard Majorana spinor representation.

We define the intertwiner as $\tilde{C}\Gamma^\mu\tilde{C}^{-1} = -(\Gamma^\mu)^T$ and $\tilde{C}^T = -\tilde{C}$, and $\tilde{C}\Gamma^{\mu_1\dots\mu_r}$ is symmetric for $r = 1,2$ and antisymmetric for $r = 0,3$.

B.2 Conventions for anomalies

Dynkin labels. We start with a brief account of our conventions for the representations of the space-time Lorentz group $SO(5,1)$ and the orthogonal group $SO(6)$. Their Lie algebras are different real forms of the complex Lie algebras of type $A_3 \sim D_3$ in the Cartan classification, with $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ and $\mathfrak{so}(5,1) \cong \mathfrak{su}^*(4)$. There are then two common conventions for the ordering of the Dynkin labels, and we use both in places. In the "D-type" conventions, the vector representation is $[1,0,0]$, the spinor with positive chirality is $[0,1,0]$ while the spinor with negative chirality is represented by $[0,0,1]$. We then have, for instance,

$$[1,0,0] \otimes [0,1,0] = [1,1,0] \oplus [0,0,1], \tag{B.3}$$

which recovers the discussion below (3.24). In the "A-type" conventions, we write the vector representation as $[0,1,0]$ and the spinor with positive chirality as $[1,0,0]$, while the spinor with negative chirality is represented by $[0,0,1]$. We use "A-type" conventions whenever referring to the Lie algebra as $\mathfrak{su}(4)$ or $\mathfrak{su}^*(4)$.

Minkowskian and Euclidean signatures. Here there is a subtlety on the signature of the metric, we need to clarify it in order to apply the family's index theorem to obtain the anomaly polynomials. Following [119], in spacetime dimension $D = 2n$ with the metric have a Minkowskian signature $(- + \cdots + +)$ we define

$$\epsilon_{01\dots D-1} = \sqrt{-g} \Longleftrightarrow \epsilon^{01\dots D-1} = \frac{-1}{\sqrt{-g}} \tag{B.4}$$

where g is the determinant of the metric. The canonical volume form is

$$\begin{aligned} \mathrm{d}V &= \frac{1}{D!}\epsilon_{\mu_1\mu_2\ldots\mu_D}\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\cdots\wedge\mathrm{d}x^{\mu_D} \\ &= \epsilon_{01\ldots D-1}\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^{D-1} \\ &= \sqrt{-g}\,\mathrm{d}^Dx\,. \end{aligned} \tag{B.5}$$

A differential p-form ω is given by

$$\omega = \frac{1}{D!}\,\omega_{\mu_1\mu_2\ldots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\,. \tag{B.6}$$

The Hodge dual of ω is a $(D-p)$-form

$$\star\omega = \frac{1}{p!(D-p)!}\omega_{\mu_1\mu_2\ldots\mu_p}\epsilon^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\ldots\nu_{(D-p)}}\,\mathrm{d}x^{\nu_1}\wedge\cdots\wedge\mathrm{d}x^{\nu_{(D-p)}} \tag{B.7}$$

and it follows $\star(\star\omega) = (-1)^{p(D-p)+1}$. Moreover, we define the components of the exterior derivative of ω as

$$(\mathrm{d}\omega)_{\mu_1\mu_2\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1}\omega_{\mu_2\mu_3\ldots\mu_{p+1}]} \tag{B.8}$$

and there is also the identity

$$\star\omega\wedge\omega = \frac{1}{p!}\omega_{\mu_1\mu_2\ldots\mu_p}\omega^{\mu_1\mu_2\ldots\mu_p}\sqrt{-g}\,\mathrm{d}^Dx\,. \tag{B.9}$$

The integration of a D-form η is defined as

$$\int\eta = \int\frac{1}{D!}\,\eta_{\mu_1\mu_2\ldots\mu_D}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_D} = \int\eta_{01\ldots(D-1)}\mathrm{d}x^0\mathrm{d}x^1\ldots\mathrm{d}x^{D-1}\,. \tag{B.10}$$

In the functional integral written in the Minkowskian signature, one has the factor e^{iS} with the action S. The continuation to the Euclidean case gives factor e^{-S_E}, where we need $S = iS_E$ and $x^0 = -ix^0_E$. We still count the Euclidean coordinates x^μ from 0 to $D-1$ in order to have the correct orientation [119] for writing down integrals. The continuation have a direct change on the definition of gamma matrices. Now we have in the Euclidean signature $\gamma^0_E = i\gamma^0$, where γ^μ is the Minkowskian gamma matrices and we use the subscript "E" to emphasize the difference. A convenient definition of the chirality matrix for computing anomalies is given in [69] as

$$\gamma_* = \xi\prod_{\mu=0}^{D-1}\gamma^\mu \tag{B.11}$$

where ξ is i^n for Euclidean and i^{n-1} for Minkowskian signature. One can compute

$$\gamma^E_* = i^n\prod_{\mu=0}^{D-1}\gamma^\mu_E = i^n\,i\prod_{\mu=0}^{D-1}\gamma^\mu = -i^{n-1}\prod_{\mu=0}^{D-1}\gamma^\mu = -\gamma_*\,. \tag{B.12}$$

Thus, the Weyl fermions of positive chirality in Minkowskian signature are negative chiral in the Euclidean signature and vice versa. To talk about Hodge dual, we now need to define the Euclidean ϵ_E and it is

$$\epsilon^E_{01\ldots\;D-1} = \sqrt{|g|} \Longleftrightarrow \epsilon_E^{01\ldots\;D-1} = \frac{1}{\sqrt{|g|}}\,. \tag{B.13}$$

We have the Hodge $\star_E$ defined similarly as in (B.7), we just replace ϵ by ϵ_E and

$$(\star_E)^2 = (-1)^{p(D-p)} \tag{B.14}$$

on p-forms. In the case that are relevant for gravitational anomalies $D = 2n = 4k+2$, a n-form $F_{\mu_1\mu_2\ldots\mu_n}$ is said to satisfy the self-dual constraint in the Minkowskian case if

$$F = \star F \Longleftrightarrow F_{\mu_1\mu_2\ldots\mu_n} = \frac{1}{n!}\,\epsilon_{\mu_1\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n}F^{\nu_1\nu_2\ldots\nu_n} \tag{B.15}$$

while in the Euclidean signature we need to insert a i to describe the self-duality

$$F = i \star_E F \Longleftrightarrow F_{\mu_1\mu_2\ldots\mu_n} = i\,\frac{1}{n!}\epsilon^E_{\mu_1\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} F^{\nu_1\nu_2\ldots\nu_n} \tag{B.16}$$

this is consistent because of (B.14). Moreover, we can obtain the Euclidean self-dual $2k+1$-forms F^+_E by spinor bilinears with positive Euclidean chirality

$$F^+_E \in \mathcal{C}^\infty(S^+_E \otimes S^+_E) \tag{B.17}$$

where S^+_E is the bundle for Euclidean positive chiral spinors and $\mathcal{C}^\infty$ means the section of bundles. As discussed, this means that the spinor bundle S^+_E corresponds to S^-, the bundle of negative Minkowskian chiral spinors. If we take tensor product of those we find the anti-self dual $2k+1$-forms in the Minkowskian signature

$$F^- \in \mathcal{C}^\infty(S^- \otimes S^-)\,. \tag{B.18}$$

We arrive at the conclusion that the when a $2k+1$ form is self-dual in the Euclidean spacetime, then it is anti-self-dual in the Minkowskian cases and vice versa.

Useful characteristic classes. The roof-genus and the Chern character are defined as [68]:

$$\hat{A}(M_{2n}) = 1 - \frac{1}{24}p_1(TM) + \frac{1}{5760}(7p_1^2(TM) - 4p_2(TM)) + \ldots\,, \tag{B.19}$$

$$\mathrm{ch}(V) \equiv \mathrm{tr}\left(\exp\left(\frac{i}{2\pi}F\right)\right) = \mathrm{rk}(V) + \frac{i}{2\pi}\,\mathrm{tr}_V\,F + \ldots + \frac{i^k}{k!(2\pi)^k}\,\mathrm{tr}_V\,F^k + \ldots\,, \tag{B.20}$$

where F is the curvature two-form of a connection on the vector bundle. Sometimes we just write $\mathrm{ch}(F)$ instead of $\mathrm{ch}(V)$ to stress with which curvature that we are computing the Chern character explicitly.

The expressions $p_i(TM)$ are the Pontryagin classes of the tangent bundle which in conventions we use are given in terms of the curvature two-form as:

$$\det\left(1 - \frac{R}{2\pi}\right) = 1 + p_1 + p_2 + p_3 + p_4 + \ldots \tag{B.21}$$

The first three Pontryagin classes are sufficient for our purposes

$$\begin{aligned} p_1 &= \frac{1}{(2\pi)^2}\left(-\frac{1}{2}\,\mathrm{tr}\,R^2\right) \\ p_2 &= \frac{1}{(2\pi)^4}\left(-\frac{1}{4}\,\mathrm{tr}\,R^4 + \frac{1}{8}(\mathrm{tr}\,R^2)^2\right) \\ p_3 &= \frac{1}{(2\pi)^6}\left(-\frac{1}{6}\,\mathrm{tr}\,R^6 + \frac{1}{8}\,\mathrm{tr}\,R^2\,\mathrm{tr}\,R^4 - \frac{1}{48}(\mathrm{tr}\,R^2)^3\right)\,. \end{aligned} \tag{B.22}$$

The spin 3/2 fermion anomaly is computed using

$$\begin{aligned} \hat{A}(M_{2n})\,(\mathrm{ch}(R) - 1) &= \hat{A}(M_{2n})\left(\mathrm{tr}(e^{\frac{i}{2\pi}R}) - 1\right) \\ &= \hat{A}(M_{2n})\left(\mathrm{tr}(e^{\frac{i}{2\pi}R} - \mathbb{1}) + \dim(T) - 1\right) \end{aligned} \tag{B.23}$$

where $\dim(T)$ is the dimension of the tensor representation of $SO(2n)$ and R is the curvature 2-form R_{ab} with the orthogonal frame indices a, b contracted with the generator T^{ab} of $SO(2n)$. Since R_{ab} is anti-symmetric in a and b, the matrix $\frac{1}{2\pi}R$ can be brought in the skew-symmetric form

$$\begin{pmatrix} & x_1 & & & & & & \\ -x_1 & & & & & & & \\ & & & x_2 & & & & \\ & & -x_2 & & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & & x_n \\ & & & & & & -x_n & \end{pmatrix} \tag{B.24}$$

where each x_j is a 2-form and the first three Pontryagin classes can also be expressed in power of x_j's

$$\begin{aligned} p_1 &= \sum_{j=1}^{n} x_j^2 \\ p_2 &= \sum_{i<j}^{n} x_i^2 x_j^2 \\ p_3 &= \sum_{i<j<k}^{n} x_i^2 x_j^2 x_k^2 \,. \end{aligned} \tag{B.25}$$

We also make use of the representation independent quantity

$$\hat{A}(M_{2n})\,\mathrm{tr}(e^{\frac{i}{2\pi}R} - \mathbb{1}) = \frac{1}{2^2}(4p_1) + \frac{1}{2^4}(\frac{2}{3}p_1^2 - \frac{8}{3}p_2) + \dots \tag{B.26}$$

and the Hirzebruch L-polynomial, expressed in terms of Pontryagin classes as

$$L(M_{2n}) = 1 + \frac{1}{3}p_1 + (-\frac{1}{45}p_1^2 + \frac{7}{45}p_2) + \dots \,. \tag{B.27}$$

As an example, the anomaly formulas for six-dimensional fields are given by [68]

$$\begin{aligned} I^{\mathrm{spin}\frac{1}{2}} &= \frac{1}{5760}\left(7{p_1}^2 - 4p_2\right) \\ I^{\mathrm{spin}\frac{3}{2}} &= \frac{1}{5760}\left(275{p_1}^2 - 980p_2\right) \\ I^{A} &= \frac{1}{5760}\left(16{p_1}^2 - 112p_2\right). \end{aligned} \tag{B.28}$$

The invariant polynomials in (B.28) correspond to anomalies for local Lorentz transformations.

B.3 Decomposition of the (anti-)self-dual field strength

For simplicity, the computations are perform in Minkowskian signature. We have the $6d$ gamma matrix duality

$$\gamma_{\mu_1\mu_2\ldots\mu_p}\gamma_7 = -\frac{1}{(6-p)!}\epsilon_{\mu_p\mu_{p-1}\ldots\mu_1\nu_1\nu_2\ldots\nu_{6-p}}\gamma^{\nu_1\nu_2\ldots\nu_{6-p}} \tag{B.29}$$

and in particular, for $p = 6$ it is

$$\gamma_{\mu\nu\rho\alpha\beta\gamma}\gamma_7 = \epsilon_{\mu\nu\rho\alpha\beta\gamma}\,. \tag{B.30}$$

The following identity allows us to split the higher rank gamma matrices into lower rank ones

$$\gamma^{\mu_1\mu_2\ldots\mu_p\alpha} = \gamma^{\mu_1\mu_2\ldots\mu_p}\gamma^{\alpha} - p\,\gamma^{[\mu_1\mu_2\ldots\mu_p-1}\eta^{\mu_p]\alpha}\,. \tag{B.31}$$

For a generic chiral 3-form spinor $\chi_{\mu\nu\rho}$, we can always do the decomposition

$$\chi_{\mu\nu\rho} = \hat{\chi}_{\mu\nu\rho} + \gamma_{[\mu}\sigma_{\nu\rho]} + \gamma_{[\mu\nu}\epsilon_{\rho]} + \gamma_{\mu\nu\rho}\eta \tag{B.32}$$

where $\gamma^{\mu}\hat{\chi}_{\mu\nu\rho} = 0 = \gamma^{\nu}\sigma_{\nu\rho} = \gamma^{\rho}\epsilon_{\rho}$. The gamma traces are defined as

$$\begin{aligned} \chi'_{\nu\rho} &\equiv \gamma^{\mu}\chi_{\mu\nu\rho} = \frac{2}{3}\sigma_{\nu\rho} + 2\gamma_{[\nu}\epsilon_{\rho]} + 4\gamma_{\nu\rho}\eta \\ \chi''_{\rho} &\equiv \gamma^{\nu}\gamma^{\mu}\chi_{\mu\nu\rho} = 4\epsilon_{\rho} + 20\gamma_{\rho}\eta \\ \chi''' &\equiv \gamma^{\rho}\gamma^{\nu}\gamma^{\mu}\chi_{\mu\nu\rho} = 120\eta\,. \end{aligned} \tag{B.33}$$

- First we show the equivalence

$$\gamma^{\alpha\beta\mu\nu\rho}\chi_{\mu\nu\rho} = 0 \quad \Longleftrightarrow \quad \gamma^{\mu}\chi_{\mu\nu\rho} = 0\,. \tag{B.34}$$

One computes

$$0 = \gamma_{\alpha\beta\mu\nu\rho}\chi^{\mu\nu\rho} = -\gamma_{\alpha\beta}\chi''' + 6\gamma_{[\alpha}\chi''_{\beta]} - 6\chi'_{\alpha\beta}\,, \tag{B.35}$$

and then uses the traces (B.33) to arrive

$$4\sigma_{\alpha\beta} - 12\gamma_{[\alpha}\epsilon_{\beta]} + 24\gamma_{\alpha\beta}\eta = 0\,. \tag{B.36}$$

The contraction with γ^α yields

$$\epsilon_\beta = 5\gamma_\beta\eta \tag{B.37}$$

and a further contraction with γ^β kills η. This also eliminates $\sigma_{\alpha\beta}$ by looking back at the relation (B.36). The field equation $\gamma^{\alpha\beta\mu\nu\rho}\chi_{\mu\nu\rho} = 0$ just tells us that $\chi_{\mu\nu\rho}$ is completely gamma-traceless which is equivalent to $\gamma^\mu\chi_{\mu\nu\rho} = 0$.

- Recall that the self-duality reads

$$\chi_{\mu\nu\rho} = \frac{1}{3!}\epsilon_{\mu\nu\rho\alpha\beta\gamma}\chi^{\alpha\beta\gamma} \tag{B.38}$$

and $\gamma_7\chi_{\mu\nu\rho} = +\chi_{\mu\nu\rho}$.

Insert (B.30) back in (B.38) and use the chirality condition for $\chi_{\mu\nu\rho}$

$$\chi_{\mu\nu\rho} = -\frac{1}{3!}\gamma_{\mu\nu\rho\gamma\beta\alpha}\chi^{\alpha\beta\gamma} \tag{B.39}$$

We also use the definition of traces (B.33), then (B.39) becomes

$$\gamma_{\mu\nu\rho}\chi''' - 9\gamma_{[\mu\nu}\chi''_{\rho]} + 18\gamma_{[\mu}\chi'_{\nu\rho]} = 0\,. \tag{B.40}$$

Now we put back the decomposition (B.32) we find a simpler equation

$$\gamma_{\mu\nu\rho}\eta = \gamma_{[\mu}\sigma_{\nu\rho]}\,. \tag{B.41}$$

Contract both sides with $\gamma^\nu\gamma^\mu$, we finally arrive at

$$20\gamma_\rho\eta = \frac{2}{3}\gamma^\nu\sigma_{\nu\rho} = 0 \implies \eta = 0 \implies \sigma_{\nu\rho} = 0\,. \tag{B.42}$$

B.4 Computational details for section 3.1

The Dirac operator for SD Weyl field:
In order to compute the relevant Dirac operator for the SD Weyl field one needs to extract the $[0,0,4]$ piece of the $\mathfrak{su}^*(4)$ representation in (3.47):

$$\begin{aligned}
R \in\ & \mathcal{C}^\infty(S^-\otimes S^-\otimes S^-\otimes S^-) - \mathcal{C}^\infty(T^*M\otimes T^*M) - \mathcal{C}^\infty(T^*M\otimes F_3^-) - \mathcal{C}^\infty(F_3^-\otimes T^*M)\\
&- \big(\mathcal{C}^\infty(T^*M\otimes F_3^-) - B\big) - g\\
=\ & \mathcal{C}^\infty(S^-\otimes S^-\otimes S^-\otimes S^-) - \mathcal{C}^\infty(T^*M\otimes T^*M) - \mathcal{C}^\infty(T^*M\otimes(S^-\otimes S^- - T^*M))\\
&- \mathcal{C}^\infty((S^-\otimes S^- - T^*M)\otimes T^*M) - \mathcal{C}^\infty(T^*M\otimes(S^-\otimes S^- - T^*M)) + B - g\\
=\ & \mathcal{C}^\infty(S^-\otimes S^-\otimes S^-\otimes S^-) - \mathcal{C}^\infty(T^*M\otimes T^*M) - \mathcal{C}^\infty(T^*M\otimes S^-\otimes S^-)\\
&+ \mathcal{C}^\infty(T^*M\otimes T^*M) - \mathcal{C}^\infty(S^-\otimes S^-\otimes T^*M) + \mathcal{C}^\infty(T^*M\otimes T^*M))\\
&- \mathcal{C}^\infty(T^*M\otimes S^-\otimes S^-) + \mathcal{C}^\infty(T^*M\otimes T^*M)) + B - g\\
=\ & \mathcal{C}^\infty(S^-\otimes S^-\otimes S^-\otimes S^-) - \mathcal{C}^\infty(T^*M\otimes S^-\otimes S^-)\\
&- \mathcal{C}^\infty(S^-\otimes S^-\otimes T^*M) - \mathcal{C}^\infty(T^*M\otimes S^-\otimes S^-)\\
&+ \mathcal{C}^\infty(T^*M\otimes T^*M)) + \mathcal{C}^\infty(T^*M\otimes T^*M) + B - g
\end{aligned} \tag{B.43}$$

$$
\begin{aligned}
&= \mathcal{C}^\infty(S^- \otimes S^- \otimes S^- \otimes S^-) - \mathcal{C}^\infty(T^*M \otimes S^- \otimes S^-) \\
&\quad - \mathcal{C}^\infty(S^- \otimes S^- \otimes T^*M) - \mathcal{C}^\infty(T^*M \otimes S^- \otimes S^-) \\
&\quad + \mathcal{C}^\infty(T^*M \otimes T^*M) + \mathcal{C}^\infty(T^*M \otimes T^*M) + B - g \\
&= \mathcal{C}^\infty(S^- \otimes S^- \otimes S^- \otimes S^-) - \mathcal{C}^\infty(T^*M \otimes S^- \otimes S^-) \\
&\quad - \mathcal{C}^\infty(S^- \otimes S^- \otimes T^*M) - \mathcal{C}^\infty(T^*M \otimes S^- \otimes S^-) \\
&\quad + \mathcal{C}^\infty(S^- \otimes S^+ + g) + \mathcal{C}^\infty(S^- \otimes S^+ + g) + B - g \\
&= \mathcal{C}^\infty(S^- \otimes S^- \otimes S^- \otimes S^-) - \mathcal{C}^\infty(T^*M \otimes S^- \otimes S^-) \\
&\quad - \mathcal{C}^\infty(S^- \otimes S^- \otimes T^*M) - \mathcal{C}^\infty(T^*M \otimes S^- \otimes S^-) \\
&\quad + \mathcal{C}^\infty(S^- \otimes S^+) + \mathcal{C}^\infty(S^- \otimes S^+) + B + g \\
&= \mathcal{C}^\infty\left(S^- \otimes [S^- \otimes S^- \otimes S^- - (S^- \otimes T^*M)^{\oplus 3} + (S^+)^{\oplus 2}]\right) + B + g
\end{aligned}
$$

The Dirac operator for exotic graviton in $(3,1)$ multiplet:
For the D field in the $(3,1)$ multiplet, we focus on its field strength S in the $[1,0,3]$ of $\mathfrak{su}^*(4)$. From

$$[1,0,1] \otimes [0,0,2] = [1,0,3] \oplus [1,1,1] \oplus [0,0,2] \oplus [0,1,0] \tag{B.44}$$

and

$$[1,0,1] \otimes [0,1,0] = [1,1,1] \oplus [0,0,2] \oplus [0,1,0] \oplus [2,0,0] \tag{B.45}$$

we get

$$[1,0,3] = [1,0,1] \otimes [0,0,2] \ominus ([1,0,1] \otimes [0,1,0] \ominus [2,0,0]) \tag{B.46}$$

Thus

$$
\begin{aligned}
S \in\ & \mathcal{C}^\infty(B \otimes F_3^-) - \mathcal{C}^\infty(B \otimes T^*M) + \mathcal{C}^\infty(F_3^+) \\
&= \mathcal{C}^\infty\left([S^+ \otimes S^- - \phi] \otimes [S^- \otimes S^- - \phi]\right) - \mathcal{C}^\infty\left([S^+ \otimes S^- - \phi] \otimes T^*M\right) + \mathcal{C}^\infty(S^+ \otimes S^+ - \phi) \\
&= \mathcal{C}^\infty\left(S^+ \otimes S^- \otimes S^- \otimes S^-\right) - \mathcal{C}^\infty\left(S^- \otimes S^+ \otimes T^*M\right) - \mathcal{C}^\infty\left(S^- \otimes S^- \otimes \phi\right) + \mathcal{C}^\infty\left(T^*M \otimes \phi\right) \\
&\quad - \mathcal{C}^\infty\left(S^- \otimes S^+ \otimes T^*M\right) + \mathcal{C}^\infty\left(T^*M \otimes \phi\right) + \mathcal{C}^\infty\left(S^+ \otimes S^+\right) - \mathcal{C}^\infty\left(T^*M\right) \\
&= \mathcal{C}^\infty\left(S^- \otimes [S^- \otimes S^- \otimes S^+ - (S^+ \otimes T^*M)^{\oplus 2} - (S^-)^{\oplus 2}]\right).
\end{aligned}
\tag{B.47}
$$

B.5 Independent components of SD Weyl field strength

Deducing which components of the field strength $G_{MNP,QRS}$ of the SD Weyl field on T^3 are independent is a cumbersome task due to the double self-duality of the field strength. Here we present a brief group-theoretical account which enables us to be sure that we have not missed parts of the equations of motion in equation (4.11).

The components of the SD Weyl field form a representation of $SO(6)$, whose Lie algebra coincides with that of $SU(4)$. Using Dynkin label conventions in which the six-dimensional vector representation is $[1,0,0]$, while the positive chirality spinor representation is $[0,1,0]$, the SD Weyl field strength $G_{MNP,QRS}$ transforms in the reducible representation $[0,4,0]+[2,0,0]$. Under the relevant $SO(3)\times SO(3)$ subgroup we have the decompositions

$$
\begin{aligned}
[1,0,0] &\longrightarrow [2,0]+[0,2] \\
[0,1,0] &\longrightarrow [1,1] \\
[0,4,0] &\longrightarrow [0,0]+[2,2]+[4,4] \\
[2,0,0] &\longrightarrow [0,0]+[2,2]+[4,0]+[0,4]
\end{aligned}
\tag{B.48}
$$

Splitting the index $M=(\alpha,i)$ as in section 4.1, the corresponding parts of the field G can be identified as follows (the symbol $\sim$ here is taken to mean "represents the same independent components of G”):

$$\begin{aligned} G_{ijk,i'j'k'} \sim G_{ijk,\alpha\beta\gamma} \sim G_{\alpha\beta\gamma,ijk} \sim G_{\alpha\beta\gamma,\alpha'\beta'\gamma'} &\sim [0,0] \\ G^{ij\alpha}{}_{ij\alpha} \sim G^{i\alpha\beta}{}_{i\alpha\beta} &\sim [0,0] \\ G^{i}{}_{\alpha\beta,ijk} \sim G^{\alpha\beta}{}_{i,\alpha\beta\gamma} &\sim [2,2] \\ G^{ij}{}_{\alpha,ij\beta} \sim G^{\alpha}{}_{ij,\alpha\beta\gamma} &\sim [2,2] \\ \alpha\beta\text{-traceless part of } G^{ij}{}_{\alpha,ij\beta} &\sim [4,0] \\ ij\text{-traceless part of } G^{\alpha\beta}{}_{i,\alpha\beta j} &\sim [0,4] \\ ij\text{- and } \alpha\beta\text{-traceless parts of } G^{k}{}_{\alpha i,k\beta j} \sim G^{\gamma}{}_{\alpha i,\gamma\beta j} &\sim [4,4] \end{aligned} \tag{B.49}$$

Imposing that $\partial_i C_{MNPQ} = 0$ as in section 4.1, we see that the first $[0,0]$ parts and the first [2,2] parts in this list vanish. All of the remaining components are then related to $G^{\gamma}{}_{\alpha i,\gamma\beta j}$ and its traces. Thus we conclude that the equation of motion $G^{M}{}_{NP,MRS} = 0$ indeed reduces to $G^{\gamma}{}_{\alpha i,\gamma\beta j} = 0$.

B.6 A explicit example for the computation of the trace formulas

The $SO(6)$ generator in the rank 2-tensor representation:
In order to find the index of the exotic gravitino (see section 6.4),

$$\operatorname{tr} e^{\frac{i}{2\pi} R_{[2]}} = \frac{D(D-1)}{2} - \frac{1}{2(2\pi)^2} \operatorname{tr} R_{[2]}^2 + \frac{1}{4!(2\pi)^4} \operatorname{tr} R_{[2]}^4 + \dots \tag{B.50}$$

for the $\frac{D(D-1)}{2} \times \frac{D(D-1)}{2}$ matrix $R_{[2]}$ given by

$$\left(R_{[2]}\right)_{cd,ef} = \left(R_{ce}\delta_{df} + R_{df}\delta_{ce} - R_{cf}\delta_{de} - R_{de}\delta_{cf}\right) \tag{B.51}$$

has to be computed. $\operatorname{tr} R_{[2]}^2$ and $\operatorname{tr} R_{[2]}^4$ are evaluated as follows.

$$\begin{aligned} \left(R_{[2]}^2\right)_{ab,ef} &= \frac{1}{2}\left(R_{[2]}\right)_{ab,cd}\left(R_{[2]}\right)_{cd,ef} \\ &= \frac{1}{2}\left(R_{ac}\delta_{bd} + R_{bd}\delta_{ac} - R_{ad}\delta_{bc} - R_{bc}\delta_{ad}\right)\left(R_{ce}\delta_{df} + R_{df}\delta_{ce} - R_{cf}\delta_{de} - R_{de}\delta_{cf}\right) \\ &= R_{ae}^2\delta_{bf} + R_{bf}^2\delta_{ae} - R_{af}^2\delta_{be} - R_{be}^2\delta_{af} + 2R_{ae}R_{bf} - 2R_{af}R_{be} \end{aligned}$$

$$\begin{aligned} \Rightarrow \operatorname{tr}\left(R_{[2]}\right)^2 &= \frac{1}{2}\sum_{a,b}\left(\left(R_{[2]}\right)^2\right)_{ab,ab} \\ &= \frac{1}{2}\sum_{a,b}\left(R_{aa}^2\delta_{bb} + R_{bb}^2\delta_{aa} - R_{ab}^2\delta_{ba} - R_{ba}^2\delta_{ab} + 2R_{aa}R_{bb} - 2R_{ab}R_{ba}\right) \\ &= \frac{1}{2}\left(D\operatorname{tr} R^2 + D\operatorname{tr} R^2 - \operatorname{tr} R^2 - \operatorname{tr} R^2 + 0 - 2\operatorname{tr} R^2\right) \\ &= (D-2)\operatorname{tr} R^2 \end{aligned} \tag{B.52}$$

$$\begin{aligned} \left(\left(R_{[2]}\right)^4\right)_{ab,ef} &= \frac{1}{2}\left(\left(R_{[2]}\right)^2\right)_{ab,cd}\left(\left(R_{[2]}\right)^2\right)_{cd,ef} \\ &= R_{ae}^4\delta_{bf} + 6R_{ae}^2R_{bf}^2 - 6R_{be}^2R_{af}^2 - R_{be}^4\delta_{af} + 4R_{ae}^3R_{bf} - 4R_{be}^3R_{af} \\ &\quad + R_{bf}^4\delta_{ae} - R_{af}^4\delta_{be} + 4R_{bf}^3R_{ae} - 4R_{af}^3R_{be} \\ \Rightarrow \operatorname{tr}\left(R_{[2]}\right)^4 &= \frac{1}{2}\sum_{a,b}\left(\left(R_{[2]}\right)^4\right)_{ab,ab} \\ &= \frac{1}{2}\sum_{a,b}\left(R_{aa}^4\delta_{bb} + 6R_{aa}^2R_{bb}^2 - 6R_{ba}^2R_{ab}^2 - R_{ba}^4\delta_{ab} + 4R_{aa}^3R_{bb}\right. \\ &\quad \left. - 4R_{ba}^3R_{ab} + R_{bb}^4\delta_{aa} - R_{ab}^4\delta_{ba} + 4R_{bb}^3R_{aa} - 4R_{ab}^3R_{ba}\right) \\ &= (D-8)\operatorname{tr} R^4 + 3(\operatorname{tr} R^2)^2 \end{aligned} \tag{B.53}$$

Bibliography

[1] C. M. Hull, "Strongly coupled gravity and duality," *Nucl. Phys. B* **583** (2000) 237–259, arXiv:hep-th/0004195.

[2] C. M. Hull, "Symmetries and compactifications of (4,0) conformal gravity," *JHEP* **12** (2000) 007, arXiv:hep-th/0011215.

[3] C. M. Hull, "BPS supermultiplets in five-dimensions," *JHEP* **06** (2000) 019, arXiv:hep-th/0004086.

[4] M. Chiodaroli, M. Gunaydin, and R. Roiban, "Superconformal symmetry and maximal supergravity in various dimensions," *JHEP* **03** (2012) 093, arXiv:1108.3085 [hep-th].

[5] L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, "Magic Square from Yang-Mills Squared," *Phys. Rev. Lett.* **112** no. 13, (2014) 131601, arXiv:1301.4176 [hep-th].

[6] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, "A magic pyramid of supergravities," *JHEP* **04** (2014) 178, arXiv:1312.6523 [hep-th].

[7] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, "Yang-Mills origin of gravitational symmetries," *Phys. Rev. Lett.* **113** no. 23, (2014) 231606, arXiv:1408.4434 [hep-th].

[8] A. Anastasiou, L. Borsten, M. J. Hughes, and S. Nagy, "Global symmetries of Yang-Mills squared in various dimensions," *JHEP* **01** (2016) 148, arXiv:1502.05359 [hep-th].

[9] M. Henneaux, V. Lekeu, and A. Leonard, "Chiral Tensors of Mixed Young Symmetry," *Phys. Rev. D* **95** no. 8, (2017) 084040, arXiv:1612.02772 [hep-th].

[10] L. Borsten, "$D = 6$, $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (4, 0)$ theories," *Phys. Rev. D* **97** no. 6, (2018) 066014, arXiv:1708.02573 [hep-th].

[11] M. Henneaux, V. Lekeu, and A. Leonard, "The action of the (free) (4, 0)-theory," *JHEP* **01** (2018) 114, arXiv:1711.07448 [hep-th]. [Erratum: JHEP 05, 105 (2018)].

[12] M. Henneaux, V. Lekeu, J. Matulich, and S. Prohazka, "The Action of the (Free) $\mathcal{N} = (3, 1)$ Theory in Six Spacetime Dimensions," *JHEP* **06** (2018) 057, arXiv:1804.10125 [hep-th].

[13] F. Cachazo, A. Guevara, M. Heydeman, S. Mizera, J. H. Schwarz, and C. Wen, "The S Matrix of 6D Super Yang-Mills and Maximal Supergravity from Rational Maps," *JHEP* **09** (2018) 125, arXiv:1805.11111 [hep-th].

[14] M. Henneaux, V. Lekeu, and A. Leonard, "A note on the double dual graviton," *J. Phys. A* **53** no. 1, (2020) 014002, arXiv:1909.12706 [hep-th].

[15] R. Minasian, C. Strickland-Constable, and Y. Zhang, "On symmetries and dynamics of exotic supermultiplets," *JHEP* **01** (2021) 174, arXiv:2007.08888 [hep-th].

[16] Y. Bertrand, S. Hohenegger, O. Hohm, and H. Samtleben, "Toward exotic 6D supergravities," *Phys. Rev. D* **103** no. 4, (2021) 046002, arXiv:2007.11644 [hep-th].

[17] M. Gunaydin, "Unified non-metric (1, 0) tensor-Einstein supergravity theories and (4, 0) supergravity in six dimensions," *JHEP* **06** (2021) 081, `arXiv:2009.01374 [hep-th]`.

[18] G. Galati and F. Riccioni, "On Exotic Six-Dimensional Supergravity Theories," *Phys. Part. Nucl. Lett.* **17** no. 5, (2020) 650–653.

[19] M. Cederwall, "Superspace formulation of exotic supergravities in six dimensions," `arXiv:2012.02719 [hep-th]`.

[20] J. A. Strathdee, "EXTENDED POINCARE SUPERSYMMETRY," *Int. J. Mod. Phys. A* **2** (1987) 273.

[21] Z. Bern, J. J. M. Carrasco, and H. Johansson, "New Relations for Gauge-Theory Amplitudes," *Phys. Rev. D* **78** (2008) 085011, `arXiv:0805.3993 [hep-ph]`.

[22] Z. Bern, J. J. M. Carrasco, and H. Johansson, "Perturbative Quantum Gravity as a Double Copy of Gauge Theory," *Phys. Rev. Lett.* **105** (2010) 061602, `arXiv:1004.0476 [hep-th]`.

[23] Z. Bern, T. Dennen, Y.-t. Huang, and M. Kiermaier, "Gravity as the Square of Gauge Theory," *Phys. Rev. D* **82** (2010) 065003, `arXiv:1004.0693 [hep-th]`.

[24] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson, and R. Roiban, "The Duality Between Color and Kinematics and its Applications," `arXiv:1909.01358 [hep-th]`.

[25] S. Nagy, "Chiral Squaring," *JHEP* **07** (2016) 142, `arXiv:1412.4750 [hep-th]`.

[26] L. Borsten and M. J. Duff, "Gravity as the square of Yang–Mills?," *Phys. Scripta* **90** (2015) 108012, `arXiv:1602.08267 [hep-th]`.

[27] A. Anastasiou, L. Borsten, M. J. Duff, A. Marrani, S. Nagy, and M. Zoccali, "Are all supergravity theories Yang–Mills squared?," *Nucl. Phys. B* **934** (2018) 606–633, `arXiv:1707.03234 [hep-th]`.

[28] R. Monteiro, D. O'Connell, and C. D. White, "Black holes and the double copy," *JHEP* **12** (2014) 056, `arXiv:1410.0239 [hep-th]`.

[29] A. Luna, R. Monteiro, D. O'Connell, and C. D. White, "The classical double copy for Taub–NUT spacetime," *Phys. Lett. B* **750** (2015) 272–277, `arXiv:1507.01869 [hep-th]`.

[30] G. L. Cardoso, S. Nagy, and S. Nampuri, "A double copy for $\mathcal{N} = 2$ supergravity: a linearised tale told on-shell," *JHEP* **10** (2016) 127, `arXiv:1609.05022 [hep-th]`.

[31] G. Cardoso, S. Nagy, and S. Nampuri, "Multi-centered $\mathcal{N} = 2$ BPS black holes: a double copy description," *JHEP* **04** (2017) 037, `arXiv:1611.04409 [hep-th]`.

[32] D. S. Berman, E. Chacón, A. Luna, and C. D. White, "The self-dual classical double copy, and the Eguchi-Hanson instanton," *JHEP* **01** (2019) 107, `arXiv:1809.04063 [hep-th]`.

[33] A. Coimbra, C. Strickland-Constable, and D. Waldram, "$E_{d(d)} \times \mathbb{R}^+$ generalised geometry, connections and M theory," *JHEP* **02** (2014) 054, `arXiv:1112.3989 [hep-th]`.

[34] A. Coimbra, C. Strickland-Constable, and D. Waldram, "Supergravity as Generalised Geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M theory," *JHEP* **03** (2014) 019, `arXiv:1212.1586 [hep-th]`.

[35] C. Strickland-Constable, "Subsectors, Dynkin Diagrams and New Generalised Geometries," *JHEP* **08** (2017) 144, `arXiv:1310.4196 [hep-th]`.

[36] O. Hohm and H. Samtleben, "Exceptional Form of D=11 Supergravity," *Phys. Rev. Lett.* **111** (2013) 231601, `arXiv:1308.1673 [hep-th]`.

[37] H. Godazgar, M. Godazgar, O. Hohm, H. Nicolai, and H. Samtleben, "Supersymmetric $E_{7(7)}$ Exceptional Field Theory," *JHEP* **09** (2014) 044, `arXiv:1406.3235 [hep-th]`.

[38] O. Hohm and H. Samtleben, "Exceptional field theory. III. $E_{8(8)}$," *Phys. Rev. D* **90** (2014) 066002, `arXiv:1406.3348 [hep-th]`.

[39] I. Schnakenburg and P. C. West, "Kac-Moody symmetries of 2B supergravity," *Phys. Lett. B* **517** (2001) 421–428, `arXiv:hep-th/0107181`.

[40] F. Ciceri, G. Dibitetto, J. J. Fernandez-Melgarejo, A. Guarino, and G. Inverso, "Double Field Theory at SL(2) angles," *JHEP* **05** (2017) 028, `arXiv:1612.05230 [hep-th]`.

[41] C. Vafa, "Evidence for F theory," *Nucl. Phys. B* **469** (1996) 403–418, `arXiv:hep-th/9602022`.

[42] J. T. Liu and R. Minasian, "U-branes and T**3 fibrations," *Nucl. Phys. B* **510** (1998) 538–554, `arXiv:hep-th/9707125`.

[43] E. Cremmer, "Supergravities in 5 Dimensions,".

[44] F. Bonetti, T. W. Grimm, and S. Hohenegger, "One-loop Chern-Simons terms in five dimensions," *JHEP* **07** (2013) 043, `arXiv:1302.2918 [hep-th]`.

[45] F. Bonetti, T. W. Grimm, and S. Hohenegger, "Exploring 6D origins of 5D supergravities with Chern-Simons terms," *JHEP* **05** (2013) 124, `arXiv:1303.2661 [hep-th]`.

[46] F. Bonetti, T. W. Grimm, and S. Hohenegger, "A Kaluza-Klein inspired action for chiral p-forms and their anomalies," *Phys. Lett. B* **720** (2013) 424–427, `arXiv:1206.1600 [hep-th]`.

[47] K. Ohmori, H. Shimizu, Y. Tachikawa, and K. Yonekura, "Anomaly polynomial of general 6d SCFTs," *PTEP* **2014** no. 10, (2014) 103B07, `arXiv:1408.5572 [hep-th]`.

[48] A. Bilal, "Lectures on Anomalies," `arXiv:0802.0634 [hep-th]`.

[49] T. Eguchi, P. B. Gilkey, and A. J. Hanson, "Gravitation, Gauge Theories and Differential Geometry," *Phys. Rept.* **66** (1980) 213.

[50] Y. M. Zinoviev, "Note on antisymmetric spin-tensors," *JHEP* **04** (2009) 035, `arXiv:0903.0262 [hep-th]`.

[51] A. Campoleoni, D. Francia, J. Mourad, and A. Sagnotti, "Unconstrained Higher Spins of Mixed Symmetry. II. Fermi Fields," *Nucl. Phys. B* **828** (2010) 405–514, `arXiv:0904.4447 [hep-th]`.

[52] I. A. Batalin and G. A. Vilkovisky, "Gauge Algebra and Quantization," *Phys. Lett. B* **102** (1981) 27–31.

[53] I. Batalin and G. Vilkovisky, "Quantization of Gauge Theories with Linearly Dependent Generators," *Phys. Rev. D* **28** (1983) 2567–2582. [Erratum: Phys.Rev.D 30, 508 (1984)].

[54] P. C. West, "E(11) and M theory," *Class. Quant. Grav.* **18** (2001) 4443–4460, `arXiv:hep-th/0104081`.

[55] C. M. Hull, "Duality in gravity and higher spin gauge fields," *JHEP* **09** (2001) 027, `arXiv:hep-th/0107149`.

[56] P. K. Townsend, "Gauge Invariance for Spin 1/2," *Phys. Lett. B* **90** (1980) 275–276.

[57] C. Bunster and M. Henneaux, "Supersymmetric electric-magnetic duality as a manifest symmetry of the action for super-Maxwell theory and linearized supergravity," *Phys. Rev. D* **86** (2012) 065018, `arXiv:1207.1761 [hep-th]`.

[58] V. Lekeu and A. Leonard, "Prepotentials for linearized supergravity," *Class. Quant. Grav.* **36** (2019) 045012, `arXiv:1804.06729 [hep-th]`.

[59] N. Nielsen, "Ghost Counting in Supergravity," *Nucl. Phys. B* **140** (1978) 499–509.

[60] R. E. Kallosh, "Modified Feynman Rules in Supergravity," *Nucl. Phys. B* **141** (1978) 141–152.

[61] I. Batalin and R. Kallosh, "Quantization of gauge theories with open algebra in the representation with the third ghost," *Nucl. Phys. B* **222** (1983) 139–151.

[62] E. S. Fradkin and G. A. Vilkovisky, "QUANTIZATION OF RELATIVISTIC SYSTEMS WITH CONSTRAINTS," *Phys. Lett. B* **55** (1975) 224–226.

[63] I. A. Batalin and G. A. Vilkovisky, "Relativistic S Matrix of Dynamical Systems with Boson and Fermion Constraints," *Phys. Lett. B* **69** (1977) 309–312.

[64] E. S. Fradkin and T. E. Fradkina, "Quantization of Relativistic Systems with Boson and Fermion First and Second Class Constraints," *Phys. Lett. B* **72** (1978) 343–348.

[65] M. Henneaux, "QUANTIZATION OF GAUGE FIELDS IN GAUGES INVOLVING EXTRA GHOSTS," *Phys. Rev. D* **27** (1983) 3040–3041.

[66] M. F. Atiyah and I. M. Singer, "The Index of elliptic operators. 4," *Annals Math.* **93** (1971) 119–138.

[67] M. F. Atiyah and I. M. Singer, "The Index of elliptic operators. 5.," *Annals Math.* **93** (1971) 139–149.

[68] L. Alvarez-Gaume and E. Witten, "Gravitational Anomalies," *Nucl. Phys. B* **234** (1984) 269.

[69] L. Alvarez-Gaume and P. H. Ginsparg, "The Structure of Gauge and Gravitational Anomalies," *Annals Phys.* **161** (1985) 423. [Erratum: Annals Phys. 171, 233 (1986)].

[70] L. Alvarez-Gaume and P. H. Ginsparg, "The Topological Meaning of Nonabelian Anomalies," *Nucl. Phys. B* **243** (1984) 449–474.

[71] O. Alvarez, I. M. Singer, and B. Zumino, "Gravitational Anomalies and the Family's Index Theorem," *Commun. Math. Phys.* **96** (1984) 409.

[72] V. Lekeu and Y. Zhang, "On the quantisation and anomalies of antisymmetric tensor-spinors," *JHEP* **11** (2021) 078, `arXiv:2109.03963 [hep-th]`.

[73] B. de Wit, "Supergravity," in *Les Houches Summer School: Session 76: Euro Summer School on Unity of Fundamental Physics: Gravity, Gauge Theory and Strings*. 12, 2002. `arXiv:hep-th/0212245`.

[74] T. Curtright, "GENERALIZED GAUGE FIELDS," *Phys. Lett. B* **165** (1985) 304–308.

[75] X. Bekaert and N. Boulanger, "Tensor gauge fields in arbitrary representations of GL(D,R): Duality and Poincare lemma," *Commun. Math. Phys.* **245** (2004) 27–67, `arXiv:hep-th/0208058`.

[76] P. de Medeiros and C. Hull, "Exotic tensor gauge theory and duality," *Commun. Math. Phys.* **235** (2003) 255–273, `arXiv:hep-th/0208155`.

[77] P. de Medeiros and C. Hull, "Geometric second order field equations for general tensor gauge fields," *JHEP* **05** (2003) 019, `arXiv:hep-th/0303036`.

[78] X. Bekaert and N. Boulanger, "Tensor gauge fields in arbitrary representations of GL(D,R). II. Quadratic actions," *Commun. Math. Phys.* **271** (2007) 723–773, `arXiv:hep-th/0606198`.

[79] N. Boulanger and V. Lekeu, "Higher spins from exotic dualisations," *JHEP* **03** (2021) 171, `arXiv:2012.11356 [hep-th]`.

[80] X. Bekaert, N. Boulanger, and M. Henneaux, "Consistent deformations of dual formulations of linearized gravity: A No go result," *Phys. Rev. D* **67** (2003) 044010, arXiv:hep-th/0210278.

[81] X. Bekaert and N. Boulanger, "The unitary representations of the Poincar\'e group in any spacetime dimension," *SciPost Phys. Lect. Notes* **30** (2021) 1, arXiv:hep-th/0611263.

[82] M. Dubois-Violette and M. Henneaux, "Generalized cohomology for irreducible tensor fields of mixed Young symmetry type," *Lett. Math. Phys.* **49** (1999) 245–252, arXiv:math/9907135.

[83] M. Dubois-Violette and M. Henneaux, "Tensor fields of mixed Young symmetry type and N complexes," *Commun. Math. Phys.* **226** (2002) 393–418, arXiv:math/0110088.

[84] A. Chatzistavrakidis and G. Karagiannis, "Relation between standard and exotic duals of differential forms," *Phys. Rev. D* **100** no. 12, (2019) 121902, arXiv:1911.00419 [hep-th].

[85] B. de Wit and H. Nicolai, "$d = 11$ Supergravity With Local SU(8) Invariance," *Nucl. Phys. B* **274** (1986) 363–400.

[86] E. Cremmer and B. Julia, "The SO(8) Supergravity," *Nucl. Phys. B* **159** (1979) 141–212.

[87] T. Damour, M. Henneaux, and H. Nicolai, "E(10) and a 'small tension expansion' of M theory," *Phys. Rev. Lett.* **89** (2002) 221601, arXiv:hep-th/0207267.

[88] C. M. Hull, "Generalised Geometry for M-Theory," *JHEP* **07** (2007) 079, arXiv:hep-th/0701203.

[89] P. Pires Pacheco and D. Waldram, "M-theory, exceptional generalised geometry and superpotentials," *JHEP* **09** (2008) 123, arXiv:0804.1362 [hep-th].

[90] A. Kleinschmidt, I. Schnakenburg, and P. C. West, "Very extended Kac-Moody algebras and their interpretation at low levels," *Class. Quant. Grav.* **21** (2004) 2493–2525, arXiv:hep-th/0309198.

[91] G. Bossard, A. Kleinschmidt, and E. Sezgin, "On supersymmetric E_{11} exceptional field theory," *JHEP* **10** (2019) 165, arXiv:1907.02080 [hep-th].

[92] O. Barwald and P. C. West, "Brane rotating symmetries and the five-brane equations of motion," *Phys. Lett. B* **476** (2000) 157–164, arXiv:hep-th/9912226.

[93] C. M. Hull, "Gravitational duality, branes and charges," *Nucl. Phys. B* **509** (1998) 216–251, arXiv:hep-th/9705162.

[94] J. H. Schwarz, "An SL(2,Z) multiplet of type IIB superstrings," *Phys. Lett. B* **360** (1995) 13–18, arXiv:hep-th/9508143. [Erratum: Phys.Lett.B 364, 252 (1995)].

[95] P. S. Aspinwall, "Some relationships between dualities in string theory," *Nucl. Phys. B Proc. Suppl.* **46** (1996) 30–38, arXiv:hep-th/9508154.

[96] J. H. Schwarz, "The power of M theory," *Phys. Lett. B* **367** (1996) 97–103, arXiv:hep-th/9510086.

[97] C. D. A. Blair, E. Malek, and J.-H. Park, "M-theory and Type IIB from a Duality Manifest Action," *JHEP* **01** (2014) 172, arXiv:1311.5109 [hep-th].

[98] G. Bossard, Private communication.

[99] H. Godazgar, M. Godazgar, and H. Nicolai, "Nonlinear Kaluza-Klein theory for dual fields," *Phys. Rev. D* **88** no. 12, (2013) 125002, arXiv:1309.0266 [hep-th].

[100] K. Lee, C. Strickland-Constable, and D. Waldram, "New Gaugings and Non-Geometry," *Fortsch. Phys.* **65** no. 10-11, (2017) 1700049, arXiv:1506.03457 [hep-th].

[101] G. Dall'Agata, G. Inverso, and M. Trigiante, "Evidence for a family of SO(8) gauged supergravity theories," *Phys. Rev. Lett.* **109** (2012) 201301, arXiv:1209.0760 [hep-th].

[102] A. Van Proeyen, "Special geometries, from real to quaternionic," in *Workshop on Special Geometric Structures in String Theory*. 10, 2001. arXiv:hep-th/0110263.

[103] M. F. Atiyah and I. M. Singer, "The Index of elliptic operators. 3.," *Annals Math.* **87** (1968) 546–604.

[104] N. Berline, E. Getzler, and M. Vergne, *Heat Kernels and Dirac Operators*. Grundlehren Text Editions. Springer Berlin Heidelberg, 1992.

[105] P. Corvilain, T. W. Grimm, and D. Regalado, "Chiral anomalies on a circle and their cancellation in F-theory," *JHEP* **04** (2018) 020, arXiv:1710.07626 [hep-th].

[106] P. Corvilain, "6d $\mathcal{N} = (1, 0)$ anomalies on S^1 and F-theory implications," *JHEP* **08** (2020) 133, arXiv:2005.12935 [hep-th].

[107] S. Ferrara, R. Minasian, and A. Sagnotti, "Low-energy analysis of M and F theories on Calabi-Yau threefolds," *Nucl. Phys. B* **474** (1996) 323–342, arXiv:hep-th/9604097.

[108] A. Baguet, O. Hohm, and H. Samtleben, "Consistent Type IIB Reductions to Maximal 5D Supergravity," *Phys. Rev. D* **92** no. 6, (2015) 065004, arXiv:1506.01385 [hep-th].

[109] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*. Princeton University Press, 1992.

[110] J. Gomis, J. Paris, and S. Samuel, "Antibracket, antifields and gauge theory quantization," *Phys. Rept.* **259** (1995) 1–145, arXiv:hep-th/9412228.

[111] I. A. Batalin and G. A. Vilkovisky, "Quantization of Gauge Theories with Linearly Dependent Generators," *Phys. Rev. D* **28** (1983) 2567–2582. [Erratum: Phys.Rev.D 30, 508 (1984)].

[112] I. A. Batalin and G. A. Vilkovisky, "Existence Theorem for Gauge Algebra," *J. Math. Phys.* **26** (1985) 172–184.

[113] D. Z. Freedman and A. Van Proeyen, *Supergravity*. Cambridge Univ. Press, Cambridge, UK, 5, 2012.

[114] W. Siegel, "Fields," arXiv:hep-th/9912205.

[115] W. Siegel, "Hidden Ghosts," *Phys. Lett. B* **93** (1980) 170–172.

[116] A. N. Schellekens and N. P. Warner, "Anomalies, Characters and Strings," *Nucl. Phys. B* **287** (1987) 317.

[117] V. Lekeu, *Aspects of electric-magnetic dualities in maximal supergravity*. PhD thesis, Brussels U., 2018. arXiv:1807.01077 [hep-th].

[118] J. P. Gauntlett and S. Pakis, "The Geometry of D = 11 killing spinors," *JHEP* **04** (2003) 039, arXiv:hep-th/0212008.

[119] A. Bilal and S. Metzger, "Anomaly cancellation in M theory: A Critical review," *Nucl. Phys. B* **675** (2003) 416–446, arXiv:hep-th/0307152.