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Symmetries of Multipartite Weyl Quantum Channels

Dariusz Chruściński [1,*], Bihalan Bhattacharya [1] and Saikat Patra [2]

[1] Institute of Physics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Grudziądzka 5/7, 87-100 Toruń, Poland; bihalan@umk.pl

[2] Department of Mathematical Sciences, Indian Institute of Science Education & Research (IISER) Berhampur, Transit Campus, Ganjam, Berhampur 760 010, Odisha, India; saikatp@iiserbpr.ac.in

* Correspondence: darch@fizyka.umk.pl

Abstract: Quantum channels define key objects in quantum information theory. They are represented by completely positive trace-preserving linear maps in matrix algebras. We analyze a family of quantum channels defined through the use of the Weyl operators. Such channels provide generalization of the celebrated qubit Pauli channels. Moreover, they are covariant with respective to the finite group generated by Weyl operators. In what follows, we study self-adjoint Weyl channels by providing a special Hermitian representation. For a prime dimension of the corresponding Hilbert space, the self-adjoint Weyl channels contain well-known generalized Pauli channels as a special case. We propose multipartite generalization of Weyl channels. In particular, we analyze the power of prime dimensions using finite fields and study the covariance properties of these objects.

Keywords: quantum channels; Weyl operators; Pauli maps

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1. Introduction

A quantum channel is represented by a linear map $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ that is completely positive and trace-preserving (CPTP) [1–3]. Recall that Φ is a positive map if for $X \geq 0$ one has $\Phi(X) \geq 0$. Complete positivity requires that the following extended map

$$\mathrm{id}_d \otimes \Phi : M_d(\mathbb{C}) \otimes M_d(\mathbb{C}) \to M_d(\mathbb{C}) \otimes M_d(\mathbb{C}), \tag{1}$$

defines a positive map but on the larger matrix algebra $M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$ (id_d denotes an identity map on the matrix algebra $M_d(\mathbb{C})$). Such maps define key objects of quantum information theory [4] since any legitimate quantum operation (like quantum measurement or quantum evolution) is represented by some completely positive map. Any such map can be represented by so-called Kraus representation [4]

$$\Phi(X) = \sum_i K_i X K_i^\dagger, \tag{2}$$

with a suitable choice of Kraus operators $\{K_i\}$. Now, Φ is trace-preserving if $\sum_i K_i^\dagger K_i = \mathbb{1}_d$ (identity operator in $M_d(\mathbb{C})$). Introducing the Hilbert–Schmidt inner product

$$(X, Y)_{\mathrm{HS}} := \mathrm{Tr}(X^\dagger Y), \tag{3}$$

one defines an adjoint (dual) map $\Phi^\ddagger$ via

$$(\Phi^\ddagger(X), Y)_{\mathrm{HS}} = (X, \Phi(Y))_{\mathrm{HS}}, \tag{4}$$

for all $X, Y \in M_d(\mathbb{C})$. A map Φ is self-adjoint (or self-dual) if $\Phi^\ddagger = \Phi$. Note that if Φ is trace-preserving, then $\Phi^\ddagger$ is unital, i.e., $\Phi^\ddagger(\mathbb{1}_d) = \mathbb{1}_d$. Indeed, if Φ is trace-preserving then $\mathrm{Tr}X = \mathrm{Tr}\Phi(X) = (\mathbb{1}_d, \Phi(X))_{\mathrm{HS}} = (\Phi^\ddagger(\mathbb{1}_d), X)_{\mathrm{HS}}$ which implies $\Phi^\ddagger(\mathbb{1}_d) = \mathbb{1}_d$. Any self-adjoint map is necessarily unital. A paradigmatic example of a self-adjoint completely positive trace-preserving (CPTP) map is a Pauli channel [4]

$$\Phi(X) = \sum_{\alpha=0}^{3} p_\alpha \sigma_\alpha X \sigma_\alpha, \tag{5}$$

where σ_α are Pauli matrices with $\sigma_0 = \mathbb{1}_2$, and p_α is a probability distribution. The characteristic feature of (5) is that Kraus operators σ_α are both unitary and Hermitian. Interestingly, σ_k for $k = 1,2,3$ are traceless, isospectral, and mutually orthogonal $(\sigma_k, \sigma_l)_{\mathrm{HS}} = 2\delta_{k\ell}$. Moreover, the elementary maps $\Delta_k(X) = \sigma_k X \sigma_k$ are mutually commuting, that is, $\Delta_k \circ \Delta_\ell = \Delta_\ell \circ \Delta_k$ for $k, \ell = 1,2,3$. Finally, Pauli maps are covariant with respect to the Pauli group G (a group generated by Pauli matrices), i.e., for any $U \in G$

$$U\Phi(X)U^\dagger = \Phi(UXU^\dagger). \tag{6}$$

Covariant maps were analyzed by several authors [5–12]. It is well known that unitary and at the same time Hermitian orthonormal basis exists only in $M_2(\mathbb{C})$. Hence, in $M_d(\mathbb{C})$ with $d > 2$ generalizing Pauli channel we have to relax either Hermiticity or unitarity. A natural generalization consists in replacing Pauli matrices by well known unitary Weyl operators $\{W_{k\ell}\}_{k,\ell=0}^{d-1}$ and define the corresponding quantum channel via

$$\Phi(X) = \sum_{k,\ell=0}^{d-1} w_{k\ell} W_{k\ell} X W_{k\ell}^\dagger, \tag{7}$$

with $w_{k\ell} \geq 0$. This map is no longer self-dual. However, it is still trace-preserving and unital. Moreover, the Kraus operators $W_{k\ell}$ (except $W_{00} = \mathbb{1}_d$) are traceless and isospectral and define an orthogonal basis in $M_d(\mathbb{C})$. Finally, (7) is covariant with respect to the group generated by Weyl operators [10]. Another generalization of (5) is based on the observation that eigenbases of three Pauli matrices define the maximal set of mutually unbiased bases (MUBs) in $\mathbb{C}^2$ [13] (see [14] for a review). It is well known that in $\mathbb{C}^d$ there exist at most $d+1$ MUBs [13,14], and if d is a power of prime, the explicit construction of the maximal set of $d+1$ MUBs is known. Assuming the existence of the maximal set of MUBs $|e_k^{(\alpha)}\rangle$ $(\alpha = 1, \ldots, d+1)$ such that for $\alpha \neq \beta$ one has $|\langle e_k^{(\alpha)}|e_l^{(\beta)}\rangle|^2 = \frac{1}{d}$, the generalized Pauli channels are defined by [15–17]

$$\Phi(X) = p_0 \frac{1}{d}\mathbb{1}_d \mathrm{Tr}X + \sum_{\alpha=1}^{d+1} p_\alpha \Delta_\alpha(X), \tag{8}$$

where $\Delta_\alpha(X) = \sum_{\alpha=0}^{d-1} P_k^{(\alpha)} X P_k^{(\alpha)}$ with $P_k^{(\alpha)} = |e_k^{(\alpha)}\rangle\langle e_k^{(\alpha)}|$.

In this paper, we propose a different generalization which is valid for any qudit system. It turns out that (8) defines a special subclass of channels we construct. The paper is organized as follows: in Section 2 we provide the basic construction of Weyl channels for arbitrary $d \geq 2$ which for $d = 2$ reduces to Pauli maps. Section 3 discusses the Hermitian representation in terms of Hermitian $Q_{k\ell}$ operators introduced in [18]. We generalize our construction to multipartite scenario in Section 4, and in Section 5 we study a particular case when the dimension of the corresponding Hilbert space is a power of prime number. Final conclusions are collected in Section 6.

2. Weyl Channels

Let us recall the definition of Schwinger Heisenberg–Weyl operators in $M_d(\mathbb{C})$ [19–21]: one defines two unitary operators Z and X (clock and shift operators that generate displacements in discrete momentum and position, respectively) via

$$Z|k\rangle = \omega^k|k\rangle\ , \quad X|k\rangle = |k+1\rangle, \tag{9}$$

where $\omega = e^{2\pi i/d}$ and we add modulo d. Note that

$$Z^k X^l = \omega^{k\ell} X^\ell Z^k. \tag{10}$$

The unitary Weyl operators are defined by

$$W_{k\ell} = X^\ell Z^k = \sum_{m=0}^{d-1} \omega^{km}|\ell+m\rangle\langle m|, \tag{11}$$

for $k, \ell = 0, 1, \ldots, d-1$. They satisfy the following relations [22,23]

$$W_{k\ell}W_{rs} = \omega^{\ell r}W_{k+r,\ell+s}\ , \quad W^\dagger_{k\ell} = \omega^{k\ell}W_{-k,-\ell}, \tag{12}$$

Now, a Weyl channel is defined via (7) with $w_{k\ell} \geq 0$ and $\sum_{k,\ell} w_{k\ell} = 1$. Now, let G_d be a group generated by $W_{k\ell}$. It turns out [10] that $G_d = \{\omega^m W_{k\ell}\,|\,k,\ell,m = 0,1,\ldots,d-1\}$ and hence $|G_d| = d^3$. If $d = 2$ one recovers $G_2 = \{\pm\mathbb{1}_2, \pm\sigma_1, \pm i\sigma_2, \pm\sigma_3\}$ which is isomorphic to the quaternion group [9]. Now, the Weyl channel (7) is covariant with respect to G_d [10] (actually, it is irreducibly covariant since the standard unitary representation of G_d by Weyl operators is irreducible). In particular, the Pauli channel is (irreducibly) covariant with respect to G_2 [9].

Let us recall that if d is a prime number then the eigenbases of the following $d+1$ unitary operators $\{X, Z, XZ, XZ^2, \ldots, XZ^{d-1}\}$ are mutually unbiased [13,14]. In this case d^2-1, unitary Weyl operators $W_{k\ell}$ with $(k,\ell) \neq (0,0)$ can be grouped into $d+1$ classes of mutually commuting operators $W_{\alpha k,\alpha\ell}$ with $\alpha = 1,2,\ldots,d-1$. One defines the following class of Weyl channels called *generalized Pauli channels* [15–17]

$$\Phi(X) = \sum_{k,\ell=0}^{d-1} w_{k\ell}W_{k\ell}XW^\dagger_{k\ell}, \tag{13}$$

with $w_{\alpha k,\alpha\ell} = w_{k\ell}$. Defining

$$\pi_0 := w_{00},\ \pi_k := w_{1,k}\ (k = 1,\ldots,d-1),\ \pi_d := w_{01},\ \pi_{d+1} := w_{10}, \tag{14}$$

one finds the following representation

$$\Phi(X) = \pi_0 X + \frac{1}{d-1}\left(\sum_{k=1}^{d-1}\pi_k\sum_{\alpha=1}^{d-1}W_{\alpha,\alpha k}XW^\dagger_{\alpha,\alpha k} + \pi_d\sum_{\alpha=1}^{d-1}W_{0\alpha}XW^\dagger_{0\alpha} + \pi_{d+1}\sum_{\alpha=1}^{d-1}W_{\alpha 0}XW^\dagger_{\alpha 0}\right), \tag{15}$$

with $\sum_{j=0}^{d+1}\pi_j = 1$. Now,

$$\sum_{\alpha=1}^{d-1}W_{k,\alpha k}XW^\dagger_{k,\alpha k} + X = d\Delta_k(X), \quad k = 1,\ldots,d-1, \tag{16}$$

where $\Delta_k(X)$ is a decoherence map with respect to the eigenbasis of $W_{1,k}$ [15]. Similarly,

$$\sum_{\alpha=1}^{d-1}W_{0\alpha}XW^\dagger_{0\alpha} + X = d\Delta_d(X), \quad \sum_{\alpha=1}^{d-1}W_{\alpha 0}XW^\dagger_{\alpha 0} + X = d\Delta_{d+1}(X), \tag{17}$$

where $\Delta_d(X)$ and $\Delta_{d+1}(X)$ are decoherence map with respect to the eigenbases of W_{01} and W_{10}, respectively. Finally, using the following identity [15]

$$\sum_{\alpha=1}^{d+1} \Delta_\alpha(X) = X + \mathbb{1}_d \text{Tr} X, \tag{18}$$

one finds

$$\Phi(X) = p_0 \frac{1}{d} \mathbb{1}_d \text{Tr} X + \sum_{\alpha=1}^{d+1} p_\alpha \Delta_\alpha(X), \tag{19}$$

with

$$p_0 = \frac{d}{d-1}(1 - d\pi_0) \ , \quad p_k = \frac{1}{d-1}(d\pi_k + d\pi_0 - 1), \tag{20}$$

for $k = 1, \ldots, d+1$. Note, however, that in the above representation we do not require that $p_\alpha \geq 0$. For example, $p_0 < 0$ for $\pi_0 > \frac{1}{d}$. In particular an identity map $\Phi(X) = X$ corresponds to $p_0 = -d$ and $p_k = 1$ ($k = 1, \ldots, d+1$). Still, one has $\sum_{k=1}^{d+1} p_k = 1$.

In this paper, we analyze a class of Weyl channels which satisfy an additional symmetry

$$S_d \Phi(X) S_d = \Phi(S_d X S_d), \tag{21}$$

where S_d is a $d \times d$ permutation matrix defined by

$$S_d|k\rangle := |d-k\rangle \ , \quad k = 0, 1, \ldots, d-1 \tag{22}$$

Note that $S_d W_{k\ell} S_d = W_{-k,-\ell}$ and hence condition (21) is equivalent to the following constraint $w_{k\ell} = w_{-k,-\ell}$. Actually, S_d defines a parity operator with eigenvalues ± 1. We call such Weyl channels *mirrored symmetric*. It is evident that mirrored symmetric channels are self-dual. Note that for $d = 3$, mirrored symmetric channels coincide with generalized Pauli channels; however, it is no longer true for $d > 3$.

3. Hermitian Representation of Mirrored Symmetric Channels

Mirrored symmetric Weyl channels are self-dual and hence can be represented by a Hermitian Kraus representation $\Phi(X) = \sum_j a_j K_j X K_j$, with $a_j > 0$, and $K_j = K_j^\dagger$. In this section, we construct a family of self-dual completely positive maps by providing an appropriate Hermitian representation. These maps are completely positive but in general not trace-preserving. Interestingly, adding a mirror symmetry restores the trace-preservation property and hence gives rise to a family of mirrored symmetric Weyl quantum channels.

Following [18], let us introduce

$$D_{k\ell} := \omega^{-k\ell/2} Z^k X^\ell = \omega^{k\ell/2} W_{k\ell}. \tag{23}$$

Unitary operators $D_{k\ell}$ satisfy the following relations:

$$D_{k\ell}^\dagger = D_{-k,-\ell} \ , \quad D_{k\ell} D_{rs} = \omega^{\frac{ks - rl}{2}} D_{k+r,\ell+s}. \tag{24}$$

Finally, let us define a set of Hermitian operators [18]

$$Q_{k\ell} := (-1)^{k\ell} \left(\chi D_{k\ell} + \chi^* D_{k\ell}^\dagger \right), \tag{25}$$

where $\chi = \frac{1}{2}(1+i)$ (actually authors of [18] did not include the factor $(-1)^{k\ell}$). These so called Heisenberg-Weyl observables [18] were recently used in tomographic scenario [24–26]. Hermitian operators $Q_{k\ell}$ define an orthogonal basis in $M_d(\mathbb{C})$

$$\mathrm{Tr}(Q_{k\ell}^\dagger Q_{ij}) = d\, \delta_{ki}\delta_{\ell j}. \tag{26}$$

One has $Q_{00} = \mathbb{1}_d$ and the remaining operators $Q_{k\ell}$ are traceless.

Proposition 1. *Traceless operators $Q_{k\ell}$ are isospectral and their spectrum reads*

$$\sigma(Q_{k\ell}) = \{\cos(2\pi j/d) - \sin(2\pi j/d) \,|\, j = 0, 1, \ldots, d-1\}. \tag{27}$$

For a proof, see Appendix A.

Example 1. *Note that for $d = 2$ one recovers the spectrum of Pauli matrices $\{-1, 1\}$. For $d = 3$, the spectrum reads $\sigma(Q_{k\ell}) = \{\frac{1}{2}(-1-\sqrt{3}), \frac{1}{2}(-1+\sqrt{3}), 1\}$ and for $d = 4$ one has $\sigma(Q_{k\ell}) = \{-1, -1, 1, 1\}$.*

Remark 1. *There is another well-known orthogonal basis in $M_d(\mathbb{C})$ defined in terms of Hermitian generalized Gell–Mann matrices:*

$$\begin{aligned} \Lambda_{k\ell}^{(s)} &= E_{e\ell} + E_{\ell k}\,, \;\; 1 \le k < \ell \le d, \\ \Lambda_{k\ell}^{(a)} &= -i(E_{k\ell} - E_{\ell k})\,, \;\; 1 \le k < \ell \le d, \qquad (28) \\ \Lambda_{\ell}^{(d)} &= \sqrt{\frac{2}{\ell(\ell+1)}}\left(\sum_{j=1}^{\ell} E_{jj} - \ell E_{\ell+1,\ell+1}\right), \;\; \ell = 1, \ldots, d-1 \qquad (29) \end{aligned}$$

where $E_{ij} = |i\rangle\langle j|$. They define generators of $SU(d)$. Note, however, that contrary to $Q_{k\ell}$ generalized Gell–Mann matrices are not isospectral. Moreover, for the map defined via

$$\Phi(X) = \sum_{\ell=1}^{d-1} q_\ell \Lambda_\ell^{(d)} X \Lambda_\ell^{(d)} + \sum_{k<\ell}\left(q_{k\ell}^{(s)} \Lambda_{k\ell}^{(s)} X \Lambda_{k\ell}^{(s)} + q_{k\ell}^{(a)} \Lambda_{k\ell}^{(a)} X \Lambda_{k\ell}^{(a)}\right), \tag{30}$$

the trace-preservation condition is quite nontrivial. Finally, contrary to Weyl operators Gell–Mann matrices cannot be split into disjoint sets of mutually commuting operators.

Proposition 2. *Operators $Q_{k\ell}$ satisfy the following identity*

$$Q_{k\ell}^2 + Q_{-k,-\ell}^2 = 2\mathbb{1}_d. \tag{31}$$

Moreover, one has

$$Q_{k\ell} X Q_{k\ell} + Q_{-k,-\ell} X Q_{-k,-\ell} = W_{k\ell} X W_{k,\ell}^\dagger + W_{-k,-\ell} X W_{-k,-\ell}^\dagger. \tag{32}$$

Indeed, one easily finds

$$Q_{k\ell}^2 + Q_{-k,-\ell}^2 = 2\mathbb{1}_d + \frac{i}{2}\left(D_{k\ell}^2 + D_{-k,-\ell}^2 - D_{k\ell}^{\dagger 2} - D_{-k,-\ell}^{\dagger 2}\right), \tag{33}$$

and due to (24) the second term vanishes. Simple algebra proves (32).

Remark 2. *Interestingly, for $d = 4$ one has $Q_{k\ell}^2 = \mathbb{1}_4$.*

Consider now the following class of completely positive maps

$$\Phi(X) = \sum_{k,\ell=0}^{d-1} \kappa_{k\ell}\, Q_{k\ell} X Q_{k\ell}, \tag{34}$$

with $\kappa_{k\ell} \geq 0$. The above map is evidently self-dual. However, it is not trace-preserving (and hence also not unital).

Proposition 3. *If (34) is mirrored symmetric, i.e., $S_d\Phi(X)S_d = \Phi(S_dXS_d)$, where S_d is a permutation matrix defined in (22), and $\sum_{k,\ell}\kappa_{k\ell} = 1$, then (34) is trace-preserving.*

Proof. Indeed, observe that

$$S_dZ^kS_d = Z^{-k}\,, \quad S_dX^kS_d = X^{-k}, \tag{35}$$

and hence

$$S_dQ_{k\ell}S_d = (-1)^{k\ell}\omega^{-k\ell/2}\Big(\chi Z^{-k}X^{-\ell} + \chi^* Z^kX^\ell\Big) = Q_{-k,-\ell}. \tag{36}$$

It is, therefore, clear that $S_d\Phi(X)S_d = \Phi(S_dXS_d)$ is equivalent to the mirror symmetry $\kappa_{k\ell} = \kappa_{-k,-\ell}$ which, in turn, taking into account Proposition 2 implies that Φ is unital (and hence trace-preserving). □

Note that if d is odd, one has $\frac{d^2-1}{2}$ independent parameters $\kappa_{k\ell}$. If $d = 2n$ is even, then

$$Q_{0n}^2 = Q_{n0}^2 = Q_{nn}^2 = \mathbb{1}_d,$$

and one has $\frac{d^2-4}{2} + 3$ independent parameters $\kappa_{k\ell}$.

Proposition 4. *If n is even, then operators $\{Q_{0n}, Q_{n0}, Q_{nn}\}$ are mutually commuting. If $n = 2r+1$, then*

$$\Sigma_1 := \frac{1}{2}Q_{0n}\,, \ \ \Sigma_2 := \frac{1}{2}(-1)^rQ_{nn}\,, \ \ \Sigma_3 := \frac{1}{2}Q_{n0}, \tag{37}$$

satisfy commutation relation of the $su(2)$ Lie algebra, i.e.,

$$[\Sigma_k, \Sigma_\ell] = i\epsilon_{k\ell m}\Sigma_m, \tag{38}$$

where $\epsilon_{k\ell m}$ stands for the Levi–Civita symbol.

For the proof cf. Appendix B.

Define the following family of unital quantum channels

$$\Delta_{k\ell}(X) = \frac{1}{2}(Q_{k\ell}XQ_{k\ell} + Q_{-k,-\ell}XQ_{-k,-\ell}), \tag{39}$$

which are S_d-covariant, i.e., $S_d\Delta_{k\ell}(X)S_d = \Delta_{k\ell}(S_dXS_d)$. One easily proves that

$$\Delta_{k\ell} \circ \Delta_{ij} = \Delta_{ij} \circ \Delta_{k\ell}. \tag{40}$$

Assuming mirror symmetry $\kappa_{k\ell} = \kappa_{-k,-\ell}$, the map (34) can be represented as follows

$$\Phi = \sum_{k,l=0}^{d-1} \kappa_{k\ell}\,\Delta_{k\ell}. \tag{41}$$

Due to the commutativity property (40), the spectral properties of Φ are fully controlled by the spectral properties of $\Delta_{k\ell}$. Simple analysis leads to

$$\Delta_{k\ell}(Q_{ij}) = \frac{1}{2}\Big(\omega^{kj-\ell i} + \omega^{\ell i-kj}\Big)Q_{ij} = \mathrm{Re}\,\omega^{kj-\ell i}\,Q_{ij}, \tag{42}$$

and hence

$$\Phi(Q_{ij}) = \lambda_{ij} Q_{ij}, \tag{43}$$

with real eigenvalues

$$\lambda_{ij} = \sum_{k,l=0}^{d-1} \kappa_{k\ell} \, \mathrm{Re} \, \omega^{kj-\ell i} = \sum_{k,l=0}^{d-1} \kappa_{k\ell} \, \cos(2\pi(kj-\ell i)/d). \tag{44}$$

Note that $\lambda_{ij} = \lambda_{-i,-j}$ and $\lambda_{00} = 1$.

Recall that any matrix $X \in M_d(\mathbb{C})$ can be mapped to a vector $|X\rangle\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ via

$$|X\rangle\rangle = \sum_i |i\rangle \otimes X|i\rangle = \sum_{i,j} X_{ij} |i \otimes j\rangle, \tag{45}$$

where X_{ij} are matrix elements of X. It simply means that one defines $|x\rangle\rangle$ as a column vector in $\mathbb{C}^d \otimes \mathbb{C}^d$ by stacking the rows of the matrix [27,28]. Using this operation (so-called vectorization) one may assign to any linear map $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ a linear super-operator $\widehat{\Phi} : \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$ as follows [27,28]

$$\widehat{\Phi}|X\rangle\rangle := |\Phi(X)\rangle\rangle. \tag{46}$$

Vectorization enjoys the following property

$$|AXB^\dagger\rangle\rangle = A \otimes B^* |X\rangle\rangle, \tag{47}$$

and hence one finds the following super-operator corresponding to (34)

$$\widehat{\Phi} = \sum_{k,l=0}^{d-1} \kappa_{k\ell} \, Q_{k\ell} \otimes Q^*_{k\ell}. \tag{48}$$

The spectral representation of $\widehat{\Phi}$ reads

$$\widehat{\Phi} = \frac{1}{d} \sum_{k,l=0}^{d-1} \lambda_{k\ell} \, |Q_{k\ell}\rangle\rangle\langle\langle Q_{k\ell}| \, , \quad \langle\langle Q_{k\ell}|Q_{ij}\rangle\rangle = d\delta_{ki}\delta_{\ell j}. \tag{49}$$

Finally, using the following identity

$$\sum_{i,j=0}^{d-1} |i\rangle\langle j| \otimes |i\rangle\langle j| = \frac{1}{d} \sum_{k,l=0}^{d-1} Q_{k\ell} \otimes Q^*_{k\ell}, \tag{50}$$

one finds the Choi matrix of Φ

$$C_\Phi = \frac{1}{d} \sum_{k,l=0}^{d-1} \lambda_{k\ell} \, Q_{k\ell} \otimes Q^*_{k\ell}, \tag{51}$$

with the corresponding spectral decomposition

$$C_\Phi = \sum_{k,l=0}^{d-1} \kappa_{k\ell} \, |Q_{k\ell}\rangle\rangle\langle\langle Q_{k\ell}|, \tag{52}$$

which clearly shows that Φ is completely positive if and only if $C_\Phi \geq 0$, i.e., $\kappa_{k\ell} \geq 0$.

Representing a density operator via the following Bloch tensor $\boldsymbol{x} = (x_{ij})$

$$\rho = \frac{1}{d} \sum_{i,j=0}^{d-1} x_{ij} Q_{ij} \, , \ \ x_{00} = 1, \ x_{ij} \in \mathbb{R}, \tag{53}$$

one has $|\rho\rangle\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} x_{ij} |Q_{ij}\rangle\rangle$, and hence

$$\widehat{\Phi}|\rho\rangle\rangle = \frac{1}{d}\sum_{i,j=0}^{d-1}(\boldsymbol{\lambda}\circ\boldsymbol{x})_{ij}|Q_{ij}\rangle\rangle, \tag{54}$$

where $(\boldsymbol{\lambda}\circ\boldsymbol{x})_{ij} = \lambda_{ij}x_{ij}$. Hence, on the level of a Bloch tensor $\boldsymbol{x}$, the map simply operates via the Hadamard product with the matrix $\boldsymbol{\lambda} = (\lambda_{ij})$ of eigenvalues of Φ. This provides a natural generalization of the Bloch representation of the Pauli channel

$$\Phi(\rho) = \frac{1}{2}\sum_{\alpha=0}^{3}\lambda_\alpha\sigma_\alpha\mathrm{Tr}(\sigma_\alpha\rho)\ ,\ \ \lambda_0 = 1, \tag{55}$$

which maps the Bloch vector $\mathbf{x} = (x_1, x_2, x_3)$ of ρ to $\mathbf{x}' = (\lambda_1x_1, \lambda_2x_2, \lambda_3x_3)$.

Let us observe that using $Q_{k\ell}$ operators one may easily restore a set of unitary operators

$$U_{k\ell} := Q_{k\ell} + iQ_{-k,-\ell}. \tag{56}$$

Indeed, one finds

$$U_{kl} = \xi_{k\ell}W_{\ell k}, \tag{57}$$

with $\xi_{k\ell} = (-1)^{k\ell}\omega^{k\ell/2}\chi$. Note that $|\xi_{k\ell}| = 1$ and hence $U_{k\ell}$ defines a collection of unitary operators. It shows, therefore, that any mirrored symmetric map (34) satisfies

$$U_{k\ell}\Phi(X)U_{k\ell}^\dagger = \Phi(U_{k\ell}XU_{k\ell}^\dagger), \tag{58}$$

that is, we restored the unitary covariance provided the self-adjoint map is mirrored symmetric.

4. Multipartite Channels

Consider now a multipartite system living in $\mathcal{H} = \mathbb{C}^{d_1}\otimes\ldots\otimes\mathbb{C}^{d_n}$. Multipartite Weyl channels were recently analyzed in [29]. Let us define

$$W_{\boldsymbol{k\ell}} := W_{k_1\ell_1}\otimes\ldots\otimes W_{k_n\ell_n}, \tag{59}$$

were $\boldsymbol{k} = (k_1,\ldots,k_n)$, $\boldsymbol{\ell} = (\ell_1,\ldots,\ell_n)$, and $W_{k_i\ell_i}$ are Weyl operators in $\mathbb{C}^{d_i}$. The multipartite Weyl channel is defined as follows [29]

$$\Phi(X) = \sum_{\boldsymbol{k},\boldsymbol{\ell}} p_{\boldsymbol{k\ell}}W_{\boldsymbol{k\ell}}XW_{\boldsymbol{k\ell}}^\dagger, \tag{60}$$

where $p_{\boldsymbol{k\ell}}$ is a probability distribution. Now, let S_{d_i} be a $d_i\times d_i$ permutation matrix such that $S_{d_i}|k_i\rangle = |-k_i\rangle$ and let

$$\mathbf{S}_d := S_{d_1}\otimes\ldots\otimes S_{d_n}. \tag{61}$$

Proposition 5. *A multipartite Weyl channel (60) is self-adjoint if*

$$\mathbf{S}_d\Phi(X)\mathbf{S}_d = \Phi(\mathbf{S}_dX\mathbf{S}_d), \tag{62}$$

for all $X\in M_d(\mathbb{C})$.

It is clear that self-adjoint Weyl channel satisfies the following mirror symmetry $p_{\boldsymbol{k\ell}} = p_{-\boldsymbol{k},-\boldsymbol{\ell}}$. In a similar way, we define multipartite operators $Q_{\boldsymbol{k\ell}}$

$$Q_{\boldsymbol{k\ell}} := Q_{k_1\ell_1}\otimes\ldots\otimes Q_{k_n\ell_n}, \tag{63}$$

were $Q_{k_i \ell_i}$ are operators in $\mathbb{C}^{d_i}$.

Proposition 6. *The operators* $Q_{\boldsymbol{k\ell}}$ *are isospectral and satisfy*

$$Q^2_{\boldsymbol{k\ell}} + Q^2_{-\boldsymbol{k},-\boldsymbol{\ell}} = 2\mathbb{1}_{d_1} \otimes \ldots \otimes \mathbb{1}_{d_n}, \tag{64}$$

together with

$$Q_{\boldsymbol{k\ell}} X Q_{\boldsymbol{k\ell}} + Q_{-\boldsymbol{k},-\boldsymbol{\ell}} X Q_{-\boldsymbol{k},-\boldsymbol{\ell}} = W_{\boldsymbol{k\ell}} X W^\dagger_{\boldsymbol{k\ell}} + W_{-\boldsymbol{k},-\boldsymbol{\ell}} X W^\dagger_{-\boldsymbol{k},-\boldsymbol{\ell}}. \tag{65}$$

Using operators $Q_{\boldsymbol{k\ell}}$, we define the following completely positive map

$$\Phi(X) = \sum_{\boldsymbol{k},\boldsymbol{\ell}} \kappa_{\boldsymbol{k\ell}} Q_{\boldsymbol{k\ell}} X Q_{\boldsymbol{k\ell}}, \tag{66}$$

with $\kappa_{\boldsymbol{k\ell}} \geq 0$. Again, being completely positive, it is generally not trace-preserving.

Proposition 7. *If (60) is mirrored symmetric, i.e.,* $\mathbf{S}_d \Phi(X) \mathbf{S}_d = \Phi(\mathbf{S}_d X \mathbf{S}_d)$, *where* $\mathbf{S}_d$ *is a permutation matrix defined in (61), and* $\sum_{\boldsymbol{k\ell}} \kappa_{\boldsymbol{k\ell}} = 1$, *then (60) is trace-preserving.*

Proof. Observe that defining

$$Z^{\boldsymbol{k}} := Z^{k_1} \otimes \ldots \otimes Z^{k_n}\ , \quad X^{\boldsymbol{\ell}} := X^{\ell_1} \otimes \ldots \otimes X^{\ell_n}, \tag{67}$$

one finds

$$\mathbf{S}_d Z^{\boldsymbol{k}} \mathbf{S}_d = Z^{-\boldsymbol{k}}\ , \quad \mathbf{S}_d X^{\boldsymbol{\ell}} \mathbf{S}_d = X^{-\boldsymbol{\ell}}, \tag{68}$$

and hence $\mathbf{S}_d Q_{\boldsymbol{k\ell}} \mathbf{S}_d = Q_{-\boldsymbol{k},-\boldsymbol{\ell}}$. Finally, observe that if each $d_i = 2m_i$, then

$$Q^2_{\boldsymbol{0m}} = Q^2_{\boldsymbol{m0}} = Q^2_{\boldsymbol{mm}} = \mathbb{1}_{d_1} \otimes \ldots \otimes \mathbb{1}_{d_n}, \tag{69}$$

where $\boldsymbol{m} = (m_1, \ldots, m_n)$ and $\mathbf{0} = (0, \ldots, 0)$. □

Define the following family of unital quantum channels

$$\Delta_{\boldsymbol{k\ell}}(X) = \frac{1}{2}(Q_{\boldsymbol{k\ell}} X Q_{\boldsymbol{k\ell}} + Q_{-\boldsymbol{k},-\boldsymbol{\ell}} X Q_{-\boldsymbol{k},-\boldsymbol{\ell}}). \tag{70}$$

One has $\mathbf{S}_d \Delta_{\boldsymbol{k\ell}}(X) \mathbf{S}_d = \Delta_{\boldsymbol{k\ell}}(\mathbf{S}_d X \mathbf{S}_d)$. Let us observe that

$$\Delta_{\boldsymbol{k\ell}} = \Delta_{k_1 \ell_1} \otimes \ldots \otimes \Delta_{k_n \ell_n}, \tag{71}$$

and hence

$$\Delta_{\boldsymbol{k\ell}} \circ \Delta_{\boldsymbol{ij}} = \Delta_{\boldsymbol{ij}} \circ \Delta_{\boldsymbol{k\ell}}. \tag{72}$$

Assuming the mirror symmetry $\kappa_{\boldsymbol{k\ell}} = \kappa_{-\boldsymbol{k}k,-\boldsymbol{\ell}}$ the map (60) can be represented as follows

$$\Phi = \sum_{\boldsymbol{k\ell}} \kappa_{\boldsymbol{k\ell}}\, \Delta_{\boldsymbol{k\ell}}. \tag{73}$$

Due to the commutativity property (72), the spectral properties of Φ are fully controlled by the spectral properties of $\Delta_{\boldsymbol{k\ell}}$. One obviously has

$$\Delta_{\boldsymbol{k\ell}}(Q_{\boldsymbol{ij}}) = \prod_{r=1}^{n} \mathrm{Re}\, \omega_{d_r}^{k_r j_r - \ell_r i_r}\, Q_{\boldsymbol{ij}}, \tag{74}$$

where $\omega_d = e^{2\pi i/d}$, and hence

$$\Phi(Q_{\boldsymbol{ij}}) = \lambda_{\boldsymbol{ij}} Q_{\boldsymbol{ij}}, \tag{75}$$

with real eigenvalues

$$\lambda_{\boldsymbol{ij}} = \sum_{\boldsymbol{k},\boldsymbol{\ell}} \kappa_{\boldsymbol{k\ell}} \prod_{r=1}^{n} \cos(2\pi(k_r j_r - \ell_r i_r)/d_r). \tag{76}$$

Note that $\lambda_{\boldsymbol{ij}} = \lambda_{-\boldsymbol{i},-\boldsymbol{j}}$ and $\lambda_{\boldsymbol{00}} = 1$.

If $\kappa_{\boldsymbol{k\ell}} = \kappa^{(1)}_{k_1\ell_1} \dots \kappa^{(n)}_{k_n\ell_n}$, then Φ is a separable quantum channel

$$\Phi = \Phi^{(1)} \otimes \dots \otimes \Phi^{(n)}, \tag{77}$$

with $\Phi^{(r)}(X) = \sum_{k_r,\ell_r=0}^{d_r-1} \kappa^{(r)}_{k_r\ell_r} Q_{k_r\ell_r} X Q_{k_r\ell_r}$, where $Q_{k_r\ell_r}$ are Q-operators in $\mathbb{C}^{d_r}$. In general, however, $\kappa_{\boldsymbol{k\ell}}$ does not factorize, and the map Φ cannot be represented as a tensor product of single-partite maps which implies that Φ acting on a separable state in $\mathbb{C}^{d_1} \otimes \dots \otimes \mathbb{C}^{d_n}$ can create an entangled state.

5. Power of Prime Dimension: A Case Study

In this section, we analyze a particular scenario when $d_1 = \dots = d_n = p$ is a prime number, that is, $d := \dim\mathcal{H} = p^n$ is a power of prime. It is well known that in this case there exist the maximal set of $d+1 = p^n + 1$ mutually unbiased bases in $\mathcal{H} = \mathbb{C}^d$. Since d is power of prime denote by $\mathbb{F}_d$ a finite field with d elements [30,31]. Let us introduce a computational basis $|a\rangle$ in $\mathcal{H}$, with $a \in \mathbb{F}_d$ and define

$$\mathbb{X}_a := \sum_{x\in\mathbb{F}_d} |a+x\rangle\langle x| \ , \ \ \mathbb{Z}_a := \sum_{x\in\mathbb{F}_d} \chi(ax)|x\rangle\langle x|, \tag{78}$$

where the operations '$x+a$' and 'ax' are defined within $\mathbb{F}_d$, and

$$\chi(x) := \exp\left(\frac{2\pi i}{p}\mathrm{tr}(x)\right) = \omega_p^{\mathrm{tr}(x)}, \tag{79}$$

where the trace operation $\mathrm{tr} : \mathbb{F}_d \to \mathbb{F}_p$ is defined as follows

$$\mathrm{tr}(x) = x + x^2 + \dots + x^{p^{n-1}}. \tag{80}$$

Note that $\mathbb{F}_p$ is a finite subfield of $\mathbb{F}_d$ and hence $\mathbb{F}_d = \{0, 1, \dots, p-1, a_1, \dots, a_{d-p}\}$. The character $\chi : \mathbb{F}_d \to \mathbb{C}$ satisfies $\chi(a+b) = \chi(a)\chi(b)$. Note that if $n = 1$, i.e., $d = p$, then $\mathbb{F}_d = \{0, 1, \dots, d-1\}$ and (78) recovers the original definition of X^k and Z^ℓ. Finally, let us define the following family of Weyl operators

$$\mathbb{W}_{a,b} := \mathbb{X}_a \mathbb{Z}_b. \tag{81}$$

One easily proves

$$\mathbb{Z}_a \mathbb{X}_b = \chi(ab) \mathbb{X}_b \mathbb{Z}_a, \tag{82}$$

and hence

$$\mathbb{W}_{a,b}\mathbb{W}_{c,s} = \chi(bc)\mathbb{W}_{a+c,b+s} \ , \ \ \mathbb{W}^\dagger_{a,b} = \chi(ab)\mathbb{W}_{-a,-b}. \tag{83}$$

Now, let us define a Weyl channel via

$$\Phi(X) = \sum_{a,b\in\mathbb{F}_d} p_{a,b} \mathbb{W}_{a,b} X \mathbb{W}^\dagger_{a,b}, \tag{84}$$

where $p_{a,b}$ is a probability distribution on $\mathbb{F}_d \times \mathbb{F}_d$. Φ is self-adjoint if $p_{a,b} = p_{-a,-b}$. Introducing the following permutation matrix $\mathbb{S}_d|a\rangle = |-a\rangle$ one arrives at the following

Proposition 8. *The Weyl channel is self-adjoint if*

$$\mathbb{S}_d \Phi(X) \mathbb{S}_d = \Phi(\mathbb{S}_d X \mathbb{S}_d), \tag{85}$$

for all $X \in M_d(\mathbb{C})$.

The key property of the family $\{\mathbb{W}_{a,b}\}$ is that eigenbases of the following $d+1$ operators

$$\mathbb{Z}_1, \mathbb{X}_1 \mathbb{Z}_a \ (a \in \mathbb{F}_d), \tag{86}$$

defines the maximal set of mutually unbiased bases in $\mathcal{H} = \mathbb{C}^d$. Defining

$$\pi_0 := p_{0,0}, \ \pi_a := p_{1,a} \ (a \in \mathbb{F}_d^*), \ \pi_d := p_{0,1}, \ \pi_{d+1} := p_{1,0}, \tag{87}$$

with $\mathbb{F}_d^* = \mathbb{F}_d - \{0\}$, one finds the following representation of the generalized Pauli channel

$$\Phi(X) = \pi_0 X + \frac{1}{d-1} \left(\sum_{a \in \mathbb{F}_d^*} \pi_a \sum_{\alpha \in \mathbb{F}_d^*} \mathbb{W}_{\alpha,\alpha a} X \mathbb{W}_{\alpha,\alpha a}^\dagger + \pi_d \sum_{\alpha \in \mathbb{F}_d^*} \mathbb{W}_{0\alpha} X \mathbb{W}_{0\alpha}^\dagger + \pi_{d+1} \sum_{\alpha \in \mathbb{F}_d^*} \mathbb{W}_{\alpha 0} X \mathbb{W}_{\alpha 0}^\dagger \right), \tag{88}$$

with $\sum_{a \in \mathbb{F}_d} \pi_a = 1$. Defining $\mathbb{D}_{a,b} := \chi(ab)^{-1/2} \mathbb{Z}_a \mathbb{X}_b$, let us introduce

$$\mathbb{Q}_{a,b} = (-1)^{\mathrm{tr}(ab)} \left(\chi \mathbb{D}_{a,b} + \chi^* \mathbb{D}_{a,b}^\dagger \right), \tag{89}$$

with $\chi = \frac{1}{2}(1+i)$. One has an analog of Proposition 1.

Proposition 9. *Hermitian operators* $\mathbb{Q}_{x,y}$ *are isospectral, and they define an orthogonal basis in* $M_d(\mathbb{C})$*, i.e.,* $\mathrm{Tr}(\mathbb{Q}_{x,y} \mathbb{Q}_{x',y'}) = d\delta_{x,x'}\delta_{y,y'}$.

Now, consider a self-adjoint completely positive map

$$\Phi(X) = \sum_{x,y \in \mathbb{F}_d} \kappa_{x,y} \mathbb{Q}_{x,y} X \mathbb{Q}_{x,y}. \tag{90}$$

If Φ is mirrored symmetric, i.e., $\mathbb{S}_d \Phi(X) \mathbb{S}_d = \Phi(\mathbb{S}_d X \mathbb{S}_d)$, then it is trace-preserving, i.e., it defines a quantum channel.

Example 2. *As an example, consider a two-qubit scenario corresponding to* $p = n = 2$*, i.e.,* $d = 4$*. In Appendix C, we provide a list of* $Q_{k\ell}$ *with* $k, \ell = 0, 1, 2, 3$ *and* $\mathbb{Q}_{x,y}$ *with* $x, y \in \mathbb{F}_4 = \{0, 1, a, b = a+1\}$*. Interestingly, one finds the following five sets of mutually commuting operators*

$$\begin{array}{lll}
\mathbb{Q}_{0,1} = \sigma_0 \otimes \sigma_1 & \mathbb{Q}_{0,a} = \sigma_1 \otimes \sigma_0 & \mathbb{Q}_{0,b} = \sigma_1 \otimes \sigma_1 \\
\mathbb{Q}_{1,0} = \sigma_3 \otimes \sigma_0 & \mathbb{Q}_{a,0} = \sigma_3 \otimes \sigma_3 & \mathbb{Q}_{b,0} = \sigma_0 \otimes \sigma_3 \\
\mathbb{Q}_{1,1} = \sigma_3 \otimes \sigma_1 & \mathbb{Q}_{a,a} = \sigma_2 \otimes \sigma_3 & \mathbb{Q}_{b,b} = \sigma_1 \otimes \sigma_2 \\
\mathbb{Q}_{1,a} = \sigma_2 \otimes \sigma_0 & \mathbb{Q}_{a,b} = \sigma_2 \otimes \sigma_2 & \mathbb{Q}_{b,1} = \sigma_0 \otimes \sigma_2 \\
\mathbb{Q}_{1,b} = \sigma_2 \otimes \sigma_1 & \mathbb{Q}_{a,1} = \sigma_3 \otimes \sigma_2 & \mathbb{Q}_{b,a} = \sigma_1 \otimes \sigma_3.
\end{array}$$

It should be stressed that operators $Q_{k\ell}$ cannot be divided into five disjoint sets of mutually commuting operators. Note that $\mathbb{Q}^2_{x,y} = \mathbb{1}_4$. Moreover, note that $\mathbb{S}_4 = \mathbb{1}_4$; hence, $\mathbb{S}_4$-covariance trivially holds in this case. The map Φ has the following form

$$\Phi(X) = \sum_{\mu,\nu=0}^{3} q_{\mu\nu}\sigma_\mu \otimes \sigma_\nu X \sigma_\mu \otimes \sigma_\nu, \tag{91}$$

with $\sum_{\mu,\nu=0}^{3} q_{\mu\nu} = 1$. Actually, one easily proves

Proposition 10. *If $d = 2^n$, then $\mathbb{Q}^2_{x,y} = \mathbb{1}_d$ and $\mathbb{S}_d = \mathbb{1}_d$. The n-partite quantum channel Φ reads*

$$\Phi(X) = \sum_{\boldsymbol{\mu}} q_{\boldsymbol{\mu}}\, \sigma_{\boldsymbol{\mu}} X \sigma_{\boldsymbol{\mu}}, \tag{92}$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$, $\sigma_{\boldsymbol{\mu}} = \sigma_{\mu_1} \otimes \ldots \otimes \sigma_{\mu_n}$, and $\sum_{\boldsymbol{\mu}} q_{\boldsymbol{\mu}} = 1$.

6. Conclusions

The multipartite Weyl channels provide an important class of maps used in quantum information theory. These maps define a direct generalization of Pauli qubit channels. It is well known that Weyl channels are covariant with respect to the finite group generated by Weyl operators [10]. In this paper we analyze Weyl channels which are self-adjoint with respect to the standard Hilbert–Schmidt inner product. It is shown that self-adjoint channels are additionally covariant with respect to a particular permutation S_d (a parity operator). Interestingly, self-adjoint Weyl channels allow for a Hermitian Kraus representation in terms of Hermitian $Q_{k\ell}$ operators (introduced in [18]). Q-operators enjoy several interesting properties: they are isospectral and define an orthonormal basis in $M_d(\mathbb{C})$. Interestingly, for a map $\Phi(X) = \sum_{k,\ell} \kappa_{k\ell} Q_{k\ell} X Q_{k\ell}$ covariance with respect to S_d implies that Φ is trace-preserving. We call such maps *mirrored symmetric* due to the following property $\kappa_{k\ell} = \kappa_{-k,-\ell}$. This analysis is then generalized for multipartite scenario. In particular, we studied the structure of self-adjoint multipartite Weyl channels in power of prime dimensions. If $d = p^r$ with p a prime number, then there exists a maximal set of '$d+1$' mutually unbiased bases which enables one to construct generalized Pauli channels. Our analysis is illustrated for the simplest scenario $d = 2^2$. In this case, we found a set of $\mathbb{Q}_{x,y}$ operators with $x, y \in \mathbb{F}_4$. It turns out that $\mathbb{Q}_{x,y}$ are simply tensor product of Pauli matrices and can be grouped into five subsets of mutually commuting operators (it is not the case for $Q_{k\ell}$ operators in $M_4(\mathbb{C})$).

It would be interesting to apply these class of maps to study the quantum evolution of open systems. In particular, in connection to quantum non-Markovianity (see [32–34]). Moreover, presented formalism can be generalized to continuous variables (CV) systems living in the infinite dimensional Hilbert spaces. One defines standard unitary displacement operators

$$\mathcal{D}(x,p) := e^{ip\hat{x}} e^{-ix\hat{p}} e^{-ixp/2}, \tag{93}$$

where $(x,p) \in \mathbb{R}^2$ and $\hat{x}$ and $\hat{p}$ are position and momentum operators satisfying $[\hat{x},\hat{p}] = i$ (we put $\hbar = 1$). Defining annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \;,\;\; \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}),$$

one finds $\mathcal{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$, with a complex parameter $\alpha = x + ip$. Now, orthogonality relations for $\mathcal{Q}(\alpha)$ operators read $\mathrm{Tr}(\mathcal{Q}(\alpha)\mathcal{Q}(\alpha')) = \pi\delta^2(\alpha - \alpha')$ [18]. In particular,

$\text{Tr}\mathcal{Q}(\alpha) = \pi\delta^2(\alpha)$. This formalism, therefore, enables one to consider a family of Gaussian CV channels [35,36]

$$\Phi(\rho) = \frac{1}{\pi}\int d^2\alpha\, q(\alpha)\mathcal{Q}(\alpha)\rho\mathcal{Q}(\alpha), \tag{94}$$

with $q(\alpha) \geq 0$. Finally, the permutation matrix S_d is replaced by a parity operator $\Pi := e^{i\pi\hat{n}}$, with $\hat{n} = \hat{a}^\dagger\hat{a}$. One finds $\Pi\mathcal{D}(x,p)\Pi = \mathcal{D}(-x,-p)$ and hence

$$\Pi\mathcal{Q}(\alpha)\Pi = \mathcal{Q}(-\alpha).$$

Hence, the Gaussian channel (94) is mirrored symmetric or rather parity covariant if

$$\Pi\Phi(\rho)\Pi = \Phi(\Pi\rho\Pi), \tag{95}$$

which is equivalent to $q(\alpha) = q(-\alpha)$. It would be interesting to study further properties of such covariant channels. We plan to address these problems in the future work.

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Appendix A. Proof of Proposition 1

Let us start with the following simple

Lemma A1. *If d is even*

$$(Z^kX^\ell)^d = (-1)^{k\ell}\mathbb{1}, \tag{A1}$$

and if d is odd

$$(Z^kX^\ell)^d = \mathbb{1}, \tag{A2}$$

Proof. It immediately follows from commutation relations $ZX = \omega XZ$.

It is therefore clear that a spectrum of Z^kX^ℓ has the following structure: if d is odd or d is even and $k\ell$ is even

$$\sigma(Z^kX^\ell) = \{1, \omega, \ldots, \omega^{d-1}\}\ , \tag{A3}$$

and if d is even and $k\ell$ is odd

$$\sigma(Z^kX^\ell) = \{\omega^{\frac{1}{2}}, \omega^{\frac{3}{2}}, \ldots, \omega^{\frac{2d-1}{2}}\}\ . \tag{A4}$$

Both spectra are invariant under multiplication by ω^m, i.e., if λ belongs to the spectrum $\omega^m\lambda$. Now, $D_{k\ell} = \omega^{-\frac{k\ell}{2}}Z^kX^\ell$. One finds for any d

$$\sigma(D_{k\ell}) = \begin{cases} \{1, \omega, \ldots, \omega^{d-1}\}\ , & k\ell \text{ even} \\ \{-1, -\omega, \ldots, -\omega^{d-1}\}\ , & k\ell \text{ odd} \end{cases} \tag{A5}$$

□

Corollary A1. *One has therefore*

$$\sigma((-1)^{k\ell} D_{k\ell}) = \{1, \omega, \ldots, \omega^{d-1}\} \,, \tag{A6}$$

that is, $(-1)^{k\ell} D_{k\ell}$ are isospectral.

Corollary A2. *Since $(-1)^{k\ell} D_{k\ell}$ is normal (being unitary), one concludes that $Q_{k\ell}$ are isospectral.*

Appendix B. Proof of Proposition 4

One finds for $d = 2n$:

$$Q_{n0} = Z^n \,, \;\; Q_{0n} = X^n \,, \;\; Q_{nn} = i^n Z^n X^n \,, \tag{A7}$$

and hence

$$[Q_{n0}, Q_{0n}] = (1 - \omega^{-nn/2}) Z^n X^n, \tag{A8}$$

that is, $[Q_{n0}, Q_{0n}] = 0$ if n is even, and

$$[Q_{n0}, Q_{0n}] = 2Z^n X^n = 2(-i)^n Q_{nn} = (-1)^r i Q_{nn}, \tag{A9}$$

for $n = 2r + 1$. Hence, commutation relations (38) follows.

Appendix C. $Q_{k\ell}$ and $\mathbb{Q}_{x,y}$ Operators for $d = 2^2$

For the reader's convenience, we present both Weyl operators $Q_{k\ell}$ and $\mathbb{Q}_{k\ell}$ operators for $d = 4$. Recall that $Q_{00} = \mathbb{Q}_{00} = \mathbb{1}_4$. One constructs the following set of Hermitian $Q_{k\ell}$ operators [18]

$$Q_{01} = \begin{pmatrix} 0 & \chi^* & 0 & \chi \\ \chi & 0 & \chi^* & 0 \\ 0 & \chi & 0 & \chi^* \\ \chi^* & 0 & \chi & 0 \end{pmatrix} \quad Q_{02} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad Q_{03} = \begin{pmatrix} 0 & \chi & 0 & \chi^* \\ \chi^* & 0 & \chi & 0 \\ 0 & \chi^* & 0 & \chi \\ \chi & 0 & \chi^* & 0 \end{pmatrix}$$

$$Q_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad Q_{11} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 & -1 \\ -i & 0 & 1 & 0 \\ 0 & 1 & 0 & -i \\ -1 & 0 & i & 0 \end{pmatrix} \quad Q_{12} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$Q_{13} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 & 1 \\ -i & 0 & -1 & 0 \\ 0 & -1 & 0 & -i \\ 1 & 0 & i & 0 \end{pmatrix} \quad Q_{20} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad Q_{21} = \begin{pmatrix} 0 & -\chi & 0 & \chi^* \\ -\chi^* & 0 & \chi & 0 \\ 0 & \chi^* & 0 & -\chi \\ \chi & 0 & -\chi^* & 0 \end{pmatrix}$$

$$Q_{22} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad Q_{23} = \begin{pmatrix} 0 & -\chi^* & 0 & \chi \\ -\chi & 0 & \chi^* & 0 \\ 0 & \chi & 0 & -\chi^* \\ \chi^* & 0 & -\chi & 0 \end{pmatrix} \quad Q_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$Q_{31} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & -i & 0 & -1 \\ -i & 0 & -1 & 0 \end{pmatrix} \quad Q_{32} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad Q_{33} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & 1 & 0 \end{pmatrix}$$

To construct $\mathbb{W}_{x,y}$, let us consider $\mathbb{F}_4 = \{0, 1, a, b = 1 + a\}$ with the following rules of addition and multiplication [30,31]

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

and

$\times$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

.

One finds the following five sets of mutually commuting Weyl operators $\mathbb{W}_{x,y}$ with $x, y \in \mathbb{F}_4$:

$$\mathbb{W}_{0,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathbb{W}_{0,a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbb{W}_{0,b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\mathbb{W}_{1,0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathbb{W}_{a,0} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathbb{W}_{b,0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbb{W}_{1,1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \mathbb{W}_{a,a} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \mathbb{W}_{b,b} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbb{W}_{1,a} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \mathbb{W}_{a,b} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \mathbb{W}_{b,1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbb{W}_{1,b} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathbb{W}_{a,1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathbb{W}_{b,a} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now, the corresponding $\mathbb{Q}_{x,y}$ operators are defined as follows:

$$\mathbb{Q}_{x,y} = \frac{1}{2}(-1)^{\mathrm{tr}(xy)}\Big([1+i]\chi(xy)\mathbb{W}_{y,x} + [1-i]\chi(xy)^*\mathbb{W}_{y,x}^\dagger\Big) = \frac{1}{2}\Big([1+i]\mathbb{W}_{y,x} + [1-i]\mathbb{W}_{y,x}^\dagger\Big),$$

due to $\chi(xy) = (-1)^{\mathrm{tr}(xy)}$. One finds

$$\mathbb{Q}_{0,1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathbb{Q}_{0,a} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathbb{Q}_{0,b} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbb{Q}_{1,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \mathbb{Q}_{a,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbb{Q}_{b,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\mathbb{Q}_{1,1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \mathbb{Q}_{a,a} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \mathbb{Q}_{b,b} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbb{Q}_{1,a} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \mathbb{Q}_{a,b} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \mathbb{Q}_{b,1} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\mathbb{Q}_{1,b} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \mathbb{Q}_{a,1} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \mathbb{Q}_{b,a} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

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