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Low-lying hadrons in the matrix model of two-color QCD at extreme strong coupling: quantum phases and the spin-puzzle

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2-color QCD (SU(2) gauge theory coupled to fundamental fermions) has several novel features: for instance, enhanced Pauli-Gursey symmetry yields degeneracies between mesons and di/tetra-quark states. The quantum mechanical matrix model provides a simplified platform to directly probe the properties of low-energy (spin-0 and spin-1) hadrons. Using variational calculation, we numerically obtain the energy eigenstates and eigenvalues of the matrix model at ultra-strong coupling. In chiral limit, the effects of non-perturbative axial anomaly are quantified. Interestingly, in chiral limit, gluons contribute significantly to spin of hadrons. These effects are suppressed in heavy quark limit. Further, at strong coupling, the system can undergo quantum phase transitions (in presence or absence of chemical potential). The ground state can be a spin-1 di-quark state which spontaneously breaks spatial rotational symmetry.

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1. Introduction

Non-Abelian gauge theories play a crucial role in describing the interactions between subatomic particles. It is well-known that computations in such a theory, especially in the strong coupling regime, is very challenging. A matrix model of $SU(N)$ gauge theory (proposed in [1]) captures many interesting features of the gauge theory [2, 3]. Being quantum mechanical, the matrix model provides a simplified computational platform and the low-energy spectrum of hadrons [4] and glueballs [5] obtained in the matrix model framework have remarkable accuracy.

Here, we consider the matrix model of 2-color 1-flavor QCD ($SU(2)$ gauge theory coupled to a single fundamental Dirac fermion, which we will refer to as matrix-QCD$_{2,1}$), which is interesting in its own right. The fundamental representation of $SU(2)$ is pseudo-real and as a consequence, the usual baryon number $U(1)_B$ symmetry is enhanced to $SU(2)_B$. This also gives rise to unusual bound states like physical di-quark states [6]. Further, the determinant of the Euclidean Dirac operator with two colors is real which facilitate its investigation with finite baryon density [7]. In this work, we estimate the low-energy spectrum of matrix-QCD$_{2,1}$ in the extreme strong coupling regime using a variational calculation. We demonstrate that the ground state can undergo level crossings, signifying quantum phase transitions. We study the properties of the phases and found that in some of these phases, the glue contribute significantly to the spin of the hadrons. Further, if we include a baryon number chemical potential, there is a possibility of a ground state with non-zero spin – a LOFF-like phase [8].

Matrix-QCD$_{2,1}$ - Hamiltonian and symmetries: Matrix-QCD$_{2,1}$ is quantum mechanical approximation of $SU(2)$ Yang-Mills theory on $\mathbb{R} \times \mathbb{S}^3$ coupled to a fundamental Dirac fermion [1, 4, 9]. In the matrix model, the $SU(2)$ gauge fields are $\mathcal{A}_i = M_{ia}T^a$, where M_{ia} with $i = 1, 2, 3$ and $a = 1, 2, 3$ are 3×3 real matrices (elements of $\mathcal{M}_3$ – the space of all M_{ia}) and T^a's are the generators of $SU(2)$ in the fundamental representation. Under spatial rotations and gauge transformations, the gauge fields transform as $\mathcal{A}_i \to \mathcal{R}_{ij}\mathcal{A}_j$ and $\mathcal{A}_i \to h^{-1}\mathcal{A}_i h$ where $\mathcal{R} \in SO(3)_{rot}$ and $h \in SU(2)$. Consequently, $\mathcal{M}_3$ is a principle $AdSU(2)$ bundle over the gauge configuration space $\mathcal{M}_3/AdSU(2)$. The bundle is twisted and does not admit any global section [2]. As a result, gauge fixing is impossible in the matrix model – the Gribov ambiguity which is a key feature of non-Abelian gauge theories. In the quantum mechanical model, the chromoelectric field is given by the conjugate momenta of the gauge field: $\Pi_i \equiv -iT^a \frac{\partial}{\partial M_{ia}}$ and the chromomagnetic field is obtained from the gauge field curvature: $\mathcal{B}_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = -\frac{1}{R}\mathcal{A}_i - \frac{i}{2}\epsilon_{ijk}\left[\mathcal{A}_j, \mathcal{A}_k\right]$ [9]. Here, R is the radius of S^3.

In matrix-QCD$_{2,1}$, the quarks ψ are Grassmann-valued matrices which only depends on time [4]. In the quantum mechanical model we can represent ψ as $\psi = (b_{\alpha A}, d^\dagger_{\alpha A})^T$, where $\alpha = 1, 2$ and $A = 1, 2$ are the spin and color indices, respectively and $\{b^\dagger_{\alpha A}, b_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB} = \{d^\dagger_{\alpha A}, d_{\beta B}\}$. The quark transforms in the spin$-\frac{1}{2}$ representation of $SO(3)_{rot}$ and in the fundamental representation of the $SU(2)$ gauge group.

To study the dynamics of the quark and the glue in the extreme strong coupling regime, it is convenient to use the rescaled dimensionless variables: $\mathcal{A}'_i = Rg^{-1/3}\mathcal{A}_i$ and $\Pi'_i = R^{-1}g^{1/3}\Pi_i$, where g is the Yang-Mills coupling constant. To keep the notation simple we will drop the prime in the entire discussion. Using above, the Hamiltonian of matrix-QCD$_{2,1}$ in the Weyl basis is given by

$$H \equiv e_0 \left[\mathrm{Tr}\left(\Pi_i \Pi_i - \frac{1}{2}[\mathcal{A}_i, \mathcal{A}_j]^2 + g^{-\frac{4}{3}} \mathcal{A}_i \mathcal{A}_i + i g^{-\frac{2}{3}} \epsilon_{ijk} [\mathcal{A}_i, \mathcal{A}_j] \mathcal{A}_k \right) + c H_c + M H_m + H_{int} \right] \quad (1)$$

$$H_{int} = M_{ia} \left(b^\dagger_{\alpha A} \sigma^i_{\alpha\beta} T^a_{AB} b_{\beta B} - d_{\alpha A} \sigma^i_{\alpha\beta} T^a_{AB} d^\dagger_{\beta B} \right), \quad H_c = b^\dagger_{\alpha A} b_{\alpha A} - d_{\alpha A} d^\dagger_{\alpha A} \quad (2)$$

$$H_m = \left(e^{i\theta} b^\dagger_{\alpha A} d^\dagger_{\alpha A} + e^{-i\theta} d_{\alpha A} b_{\alpha A} \right) \quad (3)$$

where $e_0 \equiv g^{2/3} R^{-1}$ is the energy scale of the system, $M e_0$ is the fermion mass and the dimensionless constant c denotes the curvature coupling of the fermion on the S^3. In the double scaling limit $g, R \to \infty$, H has a meaningful spectra if e_0 is held finite.

The glue and quark transform in the spin-1 and spin-1/2 representation of $SO(3)_{rot}$ respectively which are generated by $L_i \equiv -2\epsilon_{ijk} \mathrm{Tr}\left(\Pi_j \mathcal{A}_k\right)$, and $S_i \equiv \frac{1}{2}(b^\dagger_{\alpha A} \sigma^i_{\alpha\beta} b_{\beta A} + d_{\alpha A} \sigma^i_{\alpha\beta} d^\dagger_{\beta A})$. It is easy to check that $[L_i, L_j] = i\epsilon_{ijk} L_k$, and $[S_i, S_j] = i\epsilon_{ijk} S_k$. The total spin $J_i = L_i + S_i$, commutes with H: $[H, J_i] = 0$ for $i = 1, 2, 3$ and generates the $SO(3)_{rot}$ symmetry in the system.

The $SU(2)$ gauge symmetry is generated by the Gauss law generators $G^a \equiv 2i\mathrm{Tr}\Big([\Pi_i, \mathcal{A}_i] T^a\Big) + (b^\dagger_{\alpha A} T^a_{AB} b_{\alpha B} + d_{\alpha A} T^a_{AB} d^\dagger_{\alpha B})$ which satisfies $[G_a, G_b] = i\epsilon^{abc} G^c$ and $[H, G^a] = 0$.

The fermionic Hilbert space $\mathcal{H}_F$ is finite dimensional: maximum occupancy of fermions in a state is 8. On the other hand, the Hilbert space of the glue $\mathcal{H}_G$ is infinite dimensional: the space of square-integrable functions of M_{ia} with inner product measure $dM_{11} dM_{12} \ldots dM_{33}$. The physical Hilbert space is $\mathcal{H}_{phys} \subset \mathcal{H}_G \otimes \mathcal{H}_F$ such that any state $|\cdot\rangle \in \mathcal{H}_{phys}$ satisfies $G^a |\cdot\rangle = 0$. Hence, $\mathcal{H}_{phys}$ is the space of color-singlet (gauge invariant) states.

Using the $b_{\alpha A}$ and $d_{\alpha A}$, we can define

$$\widehat{B}_+ = \epsilon_{\alpha\beta} \epsilon_{AB} b^\dagger_{\alpha A} d_{\beta B}, \qquad \widehat{B}_- = \epsilon_{\alpha\beta} \epsilon_{AB} d^\dagger_{\alpha A} b_{\beta B}, \qquad \widehat{B}_3 = \frac{1}{2}(b^\dagger_{\alpha A} b_{\alpha A} - d^\dagger_{\alpha A} d_{\alpha A}) \quad (4)$$

which satisfies $[\widehat{B}_+, \widehat{B}_-] = 2\widehat{B}_3$ and $[\widehat{B}_\pm, \widehat{B}_3] = \mp \widehat{B}_\pm$. Further, it is easy to check that $[H, \widehat{B}_\pm] = 0 = [H, \widehat{B}_3]$. Thus the system has a $SU(2)_B$ global symmetry generated by the $\{\widehat{B}_+, \widehat{B}_-, \widehat{B}_3\}$. This is the Pauli-Gürsey symmetry.

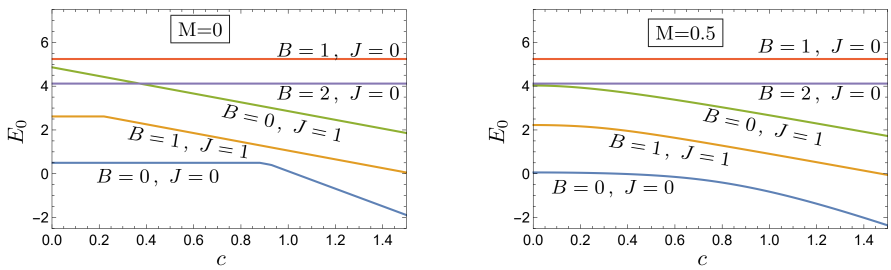

Figure 1: Energies of the ground states in the different (B, J) sectors as function of c for $M = 0$ (left) and for $M = 0.5$ (right). Here, the baryon number chemical potential is zero.

If the fermions are massless (i.e. $M = 0$), the Hamiltonian (1) also commutes with the chiral charge $Q_0 = \frac{1}{2}(b^\dagger_{\alpha A} b_{\alpha A} - d_{\alpha A} d^\dagger_{\alpha A})$. However, the global $U(1)_A$ symmetry generated by Q_0 in the theory with massless fermions is broken to $\mathbb{Z}_2$ by quantum anomalies [3]. When $M \neq 0$, $U(1)_A$ is explicitly broken to $\mathbb{Z}_2$. So the global symmetry of matrix-QCD$_{2,1}$ is $SU(2)_B \times \mathbb{Z}_2$.

In the Hamiltonian, $H_c = 2Q_0$ and the parameter c can be thought of as the "chiral chemical potential". Because the symmetry generated by Q_0 is anomalous, c is not a chemical potential in the thermodynamic sense. Nonetheless, it is interesting to study the effect c on the energy spectrum and expectation values of other interesting observables.

The physical Hilbert space $\mathcal{H}_{phys}$ can be spanned by the colorless eigenstates of the Hamiltonian. Such color-singlet states must have even number of fermions and can be simultaneous eigenstates of J_iJ_i, J_3, B_iB_i and B_3 with the eigenvalues $J(J+1)$,J_3,$B(B+1)$, and B_3. Here, J can take integer values $J = 0, 1, 2 \ldots$ with $J_3 = -J, -J+1 \ldots J-1, J$. On the other hand B can values $B = 0, 1, 2$ with $B_3 = -B, -B+1, \ldots B-1, B$. The states with $B_3 = 0$ has equal number of quarks and anti-quarks and are mesons. The states with $B_3 = \pm 1$ are di-quarks and $B_3 = \pm 2$ are tetra-quarks.

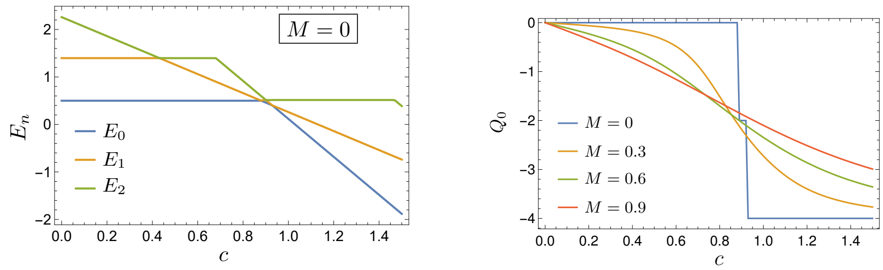

Figure 2: Triple crossing and QPT in $(B, J) = (0, 0)$ sector. Left: First three energy levels vs c at $M = 0$. Right: Q_0 vs c with different M

2. Results

We employed the Rayleigh-Ritz variational method to construct the color-singlet eigenstates of the Hamiltonian (for details of the numerical strategy see [9]). The energy of the lightest colorless eigenstates of H belonging to different (B, J)-sectors at $g \to \infty$ limit are shown in Fig.1. We find that the ground state of the system in the extreme strong coupling limit is a unique meson state belonging to $(B, J) = (0, 0)$ sector for any value of c and M.

Quantum phase transitions: In the chiral limit ($M = 0$), the ground state energies of the different (B, J)-sectors undergoes level crossing if we tune c. Such level-crossings in the ground state are signatures of quantum phase transitions (QPTs) in the system. In particular, the QPT in the $(B, J) = (0, 0)$ sector is quite special as it is a triple crossing as shown in Fig.2 (left). In this sector,

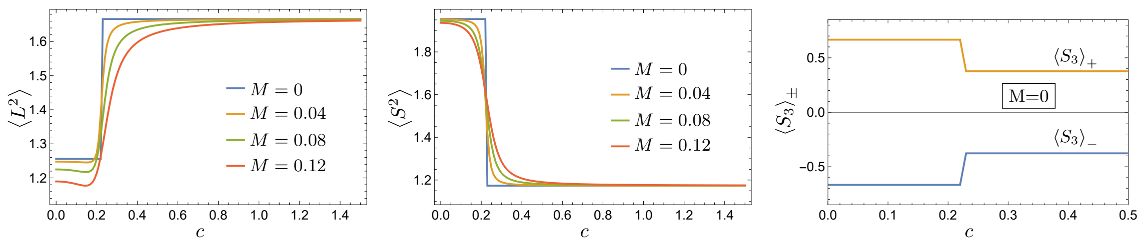

Figure 3: $\langle L^2 \rangle$, $\langle S^2 \rangle$ and $\langle S_3 \rangle_\pm$ in the ground state for $(B, J) = (1, 1)$ sector as a function of c.

we find that the phase transition at the chiral limit happens at $c = c_0^* \approx 0.928$. The QPT is captured as discontinuities in several observables. As shown in Fig.2 (right), the expectation value of the chiral charge Q_0 in the ground state of $(B, J) = (0, 0)$ jumps discontinuously at c_0^* when $M = 0$. As $Q_0 = (\partial E_0/\partial c)$, this discontinuity demonstrates that the QPT at c_0^* is first order in nature.

When M is non-zero, the abrupt discontinuous jumps in the observables at c_0^* get smoothed, and the first order transition is replaced by continuous crossover.

Quark contribution to the spin of hadrons: In QCD, the contribution of quarks to spin of hadrons (especially proton) has been a topic of immense interest [10]. In the matrix model, we can directly estimate the contribution of the quark and the glue to the spin of any energy eigenstate. Here, in matrix-QCD$_{2,1}$, we can compute the expectation values of $\langle L^2\rangle$, $\langle S^2\rangle$ and $\langle S_3\rangle$ in any energy eigenstate and estimate the distribution of its spin among the quark and the glue. We find that in the chiral limit, the contribution of the quark to the spin of a state can be very small and in the heavy quark limit, the entire spin of the state can come from the quark.

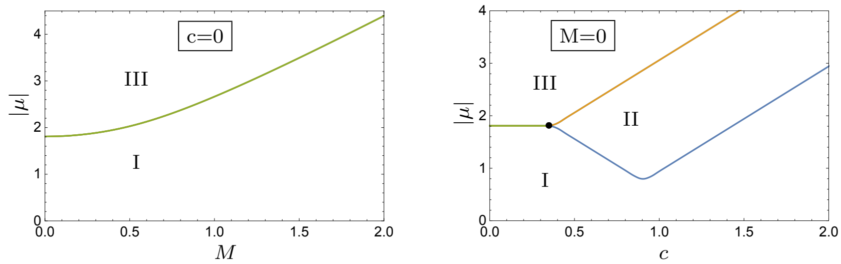

Figure 4: Phase diagram in the presence of Baryon chemical potential for $c = 0$ (left) and $M = 0$ (right). Blue, orange and green curves are the coexistence line between the phases and the black dot is the triple point where all three phases can coexists.

To demonstrate that, we consider the ground state of the $(B, J) = (1, 1)$ sector in the chiral limit: this state undergoes a level crossing at $c = c_1^* \approx 0.22$. The QPT is again captured by the discontinuity in observables (see Fig.3). In both phases, the ground state has $J = 1$ and we study the contribution of quark spin to the states with $J_3 = \pm 1$ by estimating $\langle S_3\rangle_\pm$. As shown in Fig.3, in the phase with $c < c_1^*$, the quark contributes significantly to the chiral ground state with $\langle S_3\rangle_\pm \approx 0.67$. On the other hand, in the phase with $c > c_1^*$, the spin of the ground state gets dominant contribution from glue and the quark account for only 33% of the spin with $\langle S_3\rangle_\pm \approx 0.33$.

For small non-zero M, the disconituities in Fig.3 smoothens and the QPT is replaced by a continuous crossover. For very large M, we get $\langle S^2\rangle_\pm \approx 2$ and $\langle S_3\rangle_\pm \approx 1$ irrespective of c. This indicates that the quark and glue are effectively decoupled in the heavy quark limit.

Phases with finite baryon chemical potential: We can study matrix-QCD$_{2,1}$ with finite baryon chemical potential term by adding $H_\mu = e_0\mu\widehat{B}_3$ to the Hamiltonian (1). As $[H_\mu, \widehat{B}_\pm] \neq 0$, the global $SU(2)_B$ is explicitly broken and only a $U(1)_B$ generated by $\widehat{B}_3$ is preserved. Thus the global symmetry of the Hamiltonian with non-zero μ is $U(1)_B \times \mathbb{Z}_2$. When $\mu \neq 0$, the energy of the states are given by $E(\mu) = E(\mu = 0) + \mu B_3$. When $\mu = 0$, each state in $B > 0$ sectors are $(2B + 1)-$fold degenerate with energy $E(\mu = 0)$. Turning on μ, the degeneracy between the states with different B_3 is lifted: the mesons, di-quarks and the tetra-quarks of a sector do not have the same energy. As a result it might happen that when $|\mu|$ is sufficiently large (but finite), the lighest meson state from

$(B, J) = (0, 0)$ sector is heavier than the di-quark and/or tetera-quark states from $(B, J) = (1, 1)$ and/or $(2, 0)$-sectors.

Consequently, the system can have three distinct phases: Phase I - ground state is a spin-0 meson with $(B, J) = (0, 0)$, Phase II - ground state is a spin-1 di-quark with $(B, J) = (1, 1)$ and Phase III - ground state is a spin-0 tetra-quark with $(B, J) = (2, 0)$. The choice of the parameters c, M and μ determines the phase of the system (see Fig. 4).

Interestingly, in phase II, the ground state is a spin-1 di-quark. This spin-triplet state is not rotationally invariant: if we add a small perturbation ϵJ_3, the degeneracy of the triplet is lifted and the ground state is an unique linear combination of the triplet. In the limit of $\epsilon \to 0$, the degeneracy is restored but the ground state remains frozen as a linear combination of the triplet. Thus, $SO(3)_{rot}$ is spontaneously broken in phase II, which is a reminiscent of the LOFF phase of 2-color QCD [8]. The possiblity of such spin-1 di-quark ground state also agrees with the findings in [6].

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