

## Stochastic Navier–Stokes Equation with Colored Noise: Renormalization Group Analysis

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**Abstract.** In this work we study the fully developed turbulence described by the stochastic Navier–Stokes equation with finite correlation time of random force. Inertial-range asymptotic behavior is studied in one-loop approximation and by means of the field theoretic renormalization group. The inertial-range behavior of the model is described by limiting case of vanishing correlation time that corresponds to the nontrivial fixed point of the RG equation. Another fixed point is a saddle type point, i.e., it is infrared attractive only in one of two possible directions. The existence and stability of fixed points depends on the relation between the exponents in the energy spectrum  $\mathcal{E} \propto k^{1-y}$  and the dispersion law  $\omega \propto k^{2-\eta}$ .

### 1 Introduction and description of the model

One of the possible ways to consider the fully developed turbulence within the framework of some microscopic model is to study the stochastic Navier–Stokes equation with random external force [1]. It has the form

$$\partial_t v_i + (v_l \partial_l) v_i + \partial_i \varphi = \nu_0 \partial^2 v_i + \phi_i, \quad (1)$$

where  $v_i(x)$  is a transverse (owing to the incompressibility) velocity field,  $x \equiv \{t, \mathbf{x}\}$ ,  $\partial_t \equiv \partial/\partial t$ ,  $\partial_i \equiv \partial/\partial x_i$ ,  $\nu_0$  is molecular kinematic viscosity,  $\partial^2 = \partial_i \partial_i$  is the Laplace operator, and  $\varphi = -\partial^{-2}(\partial_i v_l)(\partial_l v_i)$  is pressure. The simplest way is to assume that random force  $\phi_i$  is decorrelated in space and time [2]. In this case, we preserve the Galilean symmetry of the system, therefore this approach is very interesting from the physical point of view. On the other hand, it is intriguing to consider such a model with colored noise. At the microscopical level such correlations of the random force  $\phi_i$  arise if the noise is itself the result of removing “faster” degrees of freedom. In Fourier space these correlations can be represented in the form [3]

$$\langle \phi_i(\omega, \mathbf{k}) \phi_j(\omega', \mathbf{k}') \rangle \propto \delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') P_{ij}(\mathbf{k}) D_\phi(\omega, k), \quad (2)$$

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where  $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$  is the transverse projector. We choose the function  $D_\phi(\omega, k)$  in the form [4]

$$D_\phi(\omega, k) = \tilde{D}_0 \frac{k^{8-d-y-2\eta}}{\omega^2 + \nu_0^2 u_0^2 k^{4-2\eta}}; \quad (3)$$

here  $k \equiv |\mathbf{k}|$  is the wave number,  $D_0(k) = \tilde{D}_0 k^{8-d-y-2\eta}$  is the zero-time (power-law) correlation function of the extracted degrees of freedom, a new parameter  $u_0$  is needed for the reason of dimensionality, and  $\tilde{D}_0 > 0$  is an amplitude factor. From (3) it follows that the energy spectrum of the velocity in the inertial range has the form  $\mathcal{E} \propto k^{1-y}$ , while the correlation time at the momentum  $k$  scales as  $k^{-2+\eta}$ .

This means that the random force  $\phi_i$  is simulated by the following statistical ensemble: it is assumed to be Gaussian, homogeneous, with the zero mean and the following correlation function:

$$\langle \phi_i(t, \mathbf{x}) \phi_j(t', \mathbf{x}') \rangle = \tilde{D}_0 \int \frac{d\omega}{2\pi} \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} P_{ij}(\mathbf{k}) \frac{k^{8-d-(y+2\eta)}}{\omega^2 + \nu_0^2 u_0^2 k^{4-2\eta}} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') - i\omega(t - t')}. \quad (4)$$

Here  $d$  is an arbitrary (for generality) dimension of  $\mathbf{x}$  space, and  $1/m$  is an integral turbulence scale, related to stirring. The function (4) involves two independent exponents  $y$  and  $\eta$ , which in the renormalization group (RG) approach play the role of two formal expansion parameters. Such ensemble was employed in some models, studied in [4–8]. It was shown that, depending on the values of the exponents  $y$  and  $\eta$ , the model reveals various types of inertial-range scaling regimes with nontrivial anomalous exponents, which were explicitly derived to the first [4] and second [5] orders of double expansion in  $y$  and  $\eta$ .

Depending on the parameter  $u_0$ , the function (4) demonstrates two interesting limiting cases: if  $u_0 \rightarrow 0$ , the situation corresponds to the independent of time (“frozen”) velocity field, the case  $u_0 \rightarrow \infty$  corresponds to the rapid-change (zero-time correlated) model. The relations

$$\tilde{D}_0 / \nu_0^5 u_0^2 = \tilde{g}_0 \equiv \Lambda^{y+2\eta} \quad (5)$$

define the coupling constant  $\tilde{g}_0$ , which plays the role of the expansion parameter in the ordinary perturbation theory, and the characteristic ultraviolet (UV) momentum scale  $\Lambda$ .

Such ensemble with arbitrary function  $D_\phi(\omega, k) \propto f(Wk^2/\omega^3)$  instead of (3) was used in [9] (here  $W$  is dissipation rate of energy). The first order term of the expansion of viscosity in the turbulent Reynolds number was obtained.

A very powerful method to study the various statistical models of turbulence is provided by the field theoretic renormalization group; see the monographs [10, 11] and references therein. In a number of papers this approach was applied to the case of passive vector (magnetic) fields advected by a turbulent flow with some prescribed properties: large-scale anisotropy, helicity, compressibility, finite correlation time, non-Gaussianity, a more general form of nonlinearity, etc.; see [12–16] and references therein.

## 2 Field theoretic formulation of the model

The stochastic problem (1) and (4) is equivalent to the field theoretic model of the set of two fields  $\Phi \equiv \{v', v\}$  with the De Dominicis–Janssen action functional

$$\mathcal{S}(\Phi) = \frac{1}{2} v'_i D_v v'_k + v'_k \left[ -\partial_t - (v_i \partial_i) + \nu_0 \partial^2 \right] v_k. \quad (6)$$

Here  $D_v$  is the correlator (4), the needed integrations over  $x = (t, \mathbf{x})$  and summations over the vector indices are implied [10, 11]. The field theoretic formulation means that statistical averages of random

quantities in the original stochastic problem coincide with functional averages with weight  $\exp S(\Phi)$  with the action (6).

The model (6) corresponds to a standard Feynman diagrammatic technique with the triple vertex  $-v'_k(v_i\partial_i)v_k$  and two bare propagators. In the frequency-momentum representation, the triple vertex corresponds to the expression

$$V_{ijk} = \frac{1}{2}i(\delta_{ik}k_j^{v'} + \delta_{ij}k_k^{v'}), \quad (7)$$

where  $k'$  is the momentum of the field  $v'$ ; in the diagrams it is represented by the point, in which three lines connect with each other. The two propagators are determined by the quadratic (free) part of the action functional and are represented in the diagrams as slashed straight (the slashed end corresponds to the field  $v'$ ) and straight (the end without a slash corresponds to the field  $v$ ) lines, respectively. In the frequency-momentum representation they have forms

$$\langle v_i v'_j \rangle_0 = \frac{P_{ij}(\mathbf{k})}{-i\omega + \nu_0 k^2}; \quad (8)$$

$$\langle v_i v_j \rangle_0 = \tilde{D}_0 \frac{P_{ij}(\mathbf{k})}{\omega^2 + \nu_0^2 k^4} \frac{k^{8-d-(y+2\eta)}}{\omega^2 + \nu_0^2 u_0^2 k^{4-2\eta}}. \quad (9)$$

From the analysis of canonical dimensions it follows that for any  $d > 2$  superficial divergences can be present only in the 1-irreducible functions of two types. The first example is the function  $\langle v'_\alpha v_\beta \rangle_{1\text{-ir}}$ , for which the formal index of divergence is  $d_\Gamma = 2$ . Another possibility is the function  $\langle v'_\alpha v_\beta v_\gamma \rangle_{1\text{-ir}}$  with  $d_\Gamma = 1$ .

### 3 Feynman diagrammatic technique

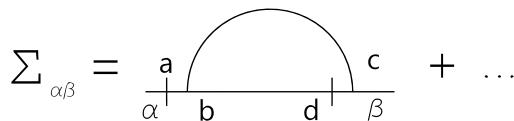
Consider the generating functional of the 1-irreducible Green's functions:

$$\Gamma(\Phi) = S(\Phi) + \tilde{\Gamma}(\Phi), \quad (10)$$

where for the functional arguments we have used the same symbols  $\Phi = \{v, v'\}$  as for the corresponding random fields;  $S(\Phi)$  is the action functional (6) and  $\tilde{\Gamma}(\Phi)$  is the sum of all the 1-irreducible diagrams with loops [11]. This means that

$$\Gamma_2^{ab} = [i\omega - \nu_0 p^2] P_{ab}(\mathbf{p}) + \Sigma_{ab}, \quad (11)$$

where  $\mathbf{p}$  is an external momentum,  $P_{ab}(\mathbf{p})$  is the transverse projector and  $\Sigma_{ab}$  is the “self-energy operator,” diagrammatic representation of which is given in Fig. 1.



**Figure 1.** The one-loop approximation of the 1-irreducible response function  $\langle v'_\alpha v_\beta \rangle_{1\text{-ir}}$

Scalarization of the expression (11) produces

$$\Gamma_2 = i\omega - \nu_0 \mathbf{p}^2 + \Sigma, \quad (12)$$

and after calculations for the one-loop approximation of the scalar quantity  $\Sigma$  one obtains

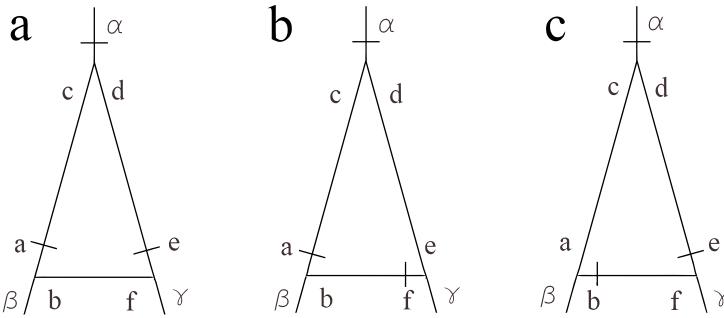
$$\Sigma = -\frac{1}{16}p^2 \times \tilde{g}_0 \nu_0 \frac{u_0^3 d(d-1) + 3u_0^2 d(d-1) + 2u_0(d^2 - d + 2)}{d(d+2)(u_0 + 1)^3} C_d \frac{m^{-y}}{y}, \quad (13)$$

where  $C_d \equiv S_d/(2\pi)^d$ . Moreover, we use the minimal subtraction (MS) renormalization scheme, in which all the anomalous dimensions  $\gamma$  are independent of the regularizers such as  $y$  and  $\eta$ , and we may choose them arbitrary with the only restriction – our diagrams have to remain UV finite; see [5] for a detailed discussion. The most convenient way which we adopt in what follows is to put  $\eta = 0$ .

The one-loop expansion of the second divergent function  $\langle v'_\alpha v_\beta v_\gamma \rangle_{1-\text{ir}}$  has the form

$$\langle v'_\alpha v_\beta v_\gamma \rangle_{1-\text{ir}} = V_{\alpha\beta\gamma} + (\Delta_1 + \Delta_2 + \Delta_3), \quad (14)$$

where  $V_{\alpha\beta\gamma}$  is the vertex (7), and diagrams  $\Delta_1$ ,  $\Delta_2$ , and  $\Delta_3$  are presented in Figs. 2a–2c.



**Figure 2.** The one-loop approximation of the 1-irreducible function  $\langle v'_\alpha v_\beta v_\gamma \rangle_{1-\text{ir}}$

Integrations over the momenta in the diagrams depicted in Figs. 1–2 are performed via averaging over the angles:

$$\int d\mathbf{k} f(\mathbf{k}) = S_d \int_m^\infty dk k^{d-1} \langle f(\mathbf{k}) \rangle, \quad (15)$$

where  $\langle \dots \rangle$  is the averaging over the unit sphere in the  $d$ -dimensional space,  $S_d$  is its surface area, and  $k = |\mathbf{k}|$ .

For the divergent part of diagram 2a, which is proportional to an external momentum  $\mathbf{p}$ , we have

$$\Delta_1 = \frac{i}{4} \tilde{g}_0 \frac{u_0(u_0 + 2)}{4(u_0 + 1)^2} \frac{(d+1)p_\alpha \delta_{\beta\gamma} - p_\beta \delta_{\alpha\gamma} - p_\gamma \delta_{\alpha\beta}}{d(d+2)} C_d \frac{m^{-y}}{y}. \quad (16)$$

Two remaining diagrams give

$$\Delta_2 = \Delta_3 = -\frac{i}{4} \tilde{g}_0 \frac{u_0(u_0^2 + 3u_0 + 4)}{8(u_0 + 1)^3} \frac{(d+1)p_\alpha \delta_{\beta\gamma} - p_\beta \delta_{\alpha\gamma} - p_\gamma \delta_{\alpha\beta}}{d(d+2)} C_d \frac{m^{-y}}{y}. \quad (17)$$

Using the transversality condition  $\partial_i v_i = 0$  together with expression (7) and moving the derivative in the vertex from the field  $\mathbf{v}'$  to the field  $\mathbf{v}$  (using integration by parts) one can conclude that the term proportional to  $p_\alpha$  gives no contribution, therefore the sum of the three triangle diagrams gives

$$\Delta_1 + \Delta_2 + \Delta_3 = \frac{i}{8} \tilde{g}_0 \frac{u_0}{(u_0 + 1)^3} \frac{p_\beta \delta_{\alpha\gamma} + p_\gamma \delta_{\alpha\beta}}{d(d+2)} C_d \frac{m^{-y}}{y}. \quad (18)$$

## 4 Renormalization of the model and fixed points

The model (6) is multiplicatively renormalizable with two independent renormalization constants  $Z_1$  and  $Z_2$ ; the renormalized action functional has the form

$$\mathcal{S}_R(\Phi) = \frac{1}{2}v'_i D_v v'_k + v'_k \left[ -\partial_t - Z_1(v_i \partial_i) + Z_2 v_0 \partial^2 \right] v_k. \quad (19)$$

These renormalization constants can be found from the requirement of UV finiteness of the expressions (12) and (14). Taking into account expressions (13) and (18) and introducing new coupling  $g = \tilde{g}C_d/4d(d+2)$  for the anomalous dimensions  $\gamma_1$  and  $\gamma_2$  one obtains

$$\gamma_1 = g \frac{u}{(u+1)^3}; \quad (20)$$

$$\gamma_2 = g \frac{u^3 d(d-1) + 3u^2 d(d-1) + 2u(d^2 - d + 2)}{4(u+1)^3}. \quad (21)$$

Dimensionality considerations together with expressions (9) and (19) give

$$g_0 = g\mu^\eta Z_g, \quad u_0 = u\mu^\eta Z_u; \quad \gamma_g = 2\gamma_1 - 3\gamma_2, \quad \gamma_u = -\gamma_2, \quad (22)$$

where  $\mu$  is “reference mass” (additional free parameter of the renormalized theory) in the MS renormalization scheme. One of the basic RG statements is that the asymptotic behavior of the model is governed by the fixed points  $\{g^*, u^*\}$ , defined by the relations  $\beta_g = 0, \beta_u = 0$ . From the expressions (22) it follows that

$$\beta_g = g(-y - 2\gamma_1 + 3\gamma_2); \quad \beta_u = u(-\eta + \gamma_2). \quad (23)$$

The type of a fixed point (IR/UV attractive or a saddle point), i.e., the character of the RG flow in the vicinity of the point, is determined by the matrix  $\Omega_{ik} = \partial\beta_i/\partial g_k$ , where  $\beta_i$  is a full set of  $\beta$ -functions and  $g_k$  is a full set of couplings. For an IR attractive fixed point, the matrix  $\Omega$  has to be positive, i.e., the real parts of all its eigenvalues are positive.

From the analysis of  $\beta$  functions it follows that if  $\frac{1}{3} < \alpha < \frac{1}{3} + \frac{4/3}{3d(d-1)+2}$  the system possesses fixed point  $\{g^*, u^*\}$  with coordinates

$$u^* = \frac{-3 + \sqrt{1 - \frac{16(\alpha-1)}{d(d-1)(3\alpha-1)}}}{2}; \quad (24)$$

$$g^* = \frac{3\alpha-1}{2}y \frac{(u^*+1)^3}{u^*}, \quad (25)$$

where  $\alpha = \eta/y$ ; see Fig. 3.

Moreover, it turns out that one of the two eigenvalues of the matrix  $\Omega$  for this point is less than zero, thus, this fixed point is saddle one, i.e., it is IR attractive only in one of the two possible directions.

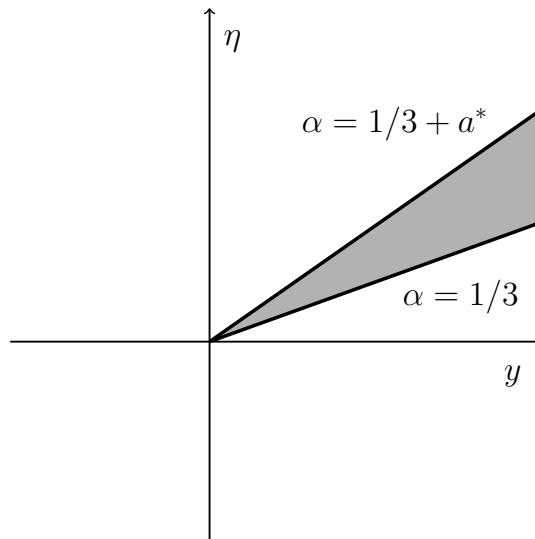
Another interesting case to be considered is  $u^* \rightarrow \infty$ . From (4) it follows that this case corresponds to the rapid-change model; this means that in this situation one should obtain the well-known fixed point of the model with zero-time correlations [2]. This is indeed so: taking into account that

$$\gamma_2 = \tilde{g} \frac{d(d-1)}{4} \quad \text{at} \quad u^* \rightarrow \infty, \quad (26)$$

one can obtain that in this case

$$\hat{g}^* = y \frac{4(d+2)}{3(d-1)}, \quad (27)$$

where  $\hat{g} = \tilde{g}C_d/4$ . This fixed point is IR attractive if  $y > 0, \eta > y/3$ .



**Figure 3.** Domain of the existence of the fixed point in the model (6);  $a^* = \frac{4/3}{3d(d-1)+2}$

## 5 Critical dimensions

In the leading order of IR asymptotic behavior, Green's functions satisfy the RG equation with the substitution  $g \rightarrow g^*$  and  $u \rightarrow u^*$ . This feature, together with canonical scale invariance, gives us critical dimensions of the fields in the model, which, in fact, govern the asymptotic behaviour of arbitrary correlation functions.

If  $u^* \rightarrow \infty$  one obtains

$$\Delta_v = 1 - y/3, \quad \Delta_{v'} = d - 1 + y/3, \quad (28)$$

which is in agreement with [2].

The saddle fixed point (24)–(25) gives

$$\Delta_v = 1 + \frac{\eta - y}{2}, \quad \Delta_{v'} = d - 1 + \frac{\eta - y}{2}. \quad (29)$$

## 6 Conclusion

We applied the field theoretic renormalization group to the analysis of the Navier–Stokes equation with colored random force, i.e., to the model with arbitrary *finite* correlation time of the velocity field. The critical dimensions of the fields are calculated. As it should be for the case of infinite fixed point, they coincide with the zero-time rapid-change model considered earlier in [2].

One loop approximation provides that, depending on the two exponents  $y$  and  $\eta$  that describe the energy spectrum  $\mathcal{E} \propto k^{1-y}$  and the dispersion law  $\omega \sim k^{2-\eta}$  of the velocity field, the possible *nontrivial* types of the IR behavior appear to reduce to the only one limiting case: the rapid-change type behavior, realized for  $y > 0, \eta > y/3$ . Another fixed point, realized for  $\frac{1}{3} < \alpha < \frac{1}{3} + \frac{4/3}{3d(d-1)+2}$ , where  $\alpha = y/\eta$ , is a saddle type point. The fact that the only IR attractive fixed point corresponds to vanishing correlation time means, in particular, that the Galilean symmetry, violated by the colored random force, is automatically restored in the IR limit. This gives a quantitative illustration of the

general concept that the symmetries of the Navier–Stokes equation, broken spontaneously and by initial or boundary conditions, are restored in the statistical sense for the fully developed turbulence; see the discussion in [1].

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