



## Statistical system based on $p$ -adic numbers

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### ARTICLE INFO

#### Article history:

Received 5 May 2021

Received in revised form 25 May 2021

Accepted 26 May 2021

Available online 28 May 2021

Editor: N. Lambert

### ABSTRACT

We propose statistical systems based on  $p$ -adic numbers. In the systems, the Hamiltonian is a standard real number which is given by a map from the  $p$ -adic numbers. Therefore we can introduce the temperature as a real number and calculate the thermodynamical quantities like free energy, thermodynamical energy, entropy, specific heat, etc. Although we consider a very simple system, which corresponds to a free particle moving in one dimensional space, we find that there appear the behaviors like phase transition in the system. Usually in order that a phase transition occurs, we need a system with an infinite number of degrees of freedom but in the system where the dynamical variable is given by  $p$ -adic number, even if degree of freedom is unity, there might occur the phase transition.

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## 1. Introduction

Real numbers are obtained from rational numbers by the procedure of completion. For the completion, we need to define a distance between two numbers, which is the absolute value of the difference between two numbers. It is possible to define “absolute value” in a way different from the definition of the absolute value which we use when we define the real number. The  $p$ -adic numbers are obtained by the completion using the  $p$ -adic absolute value [1], where  $p$  is a prime number. For a review, see [2] and for recent developments, [3].

Let  $\mathbb{Q}_p$  be set of all two-sided sequences,  $\dots a_2 a_1 a_0. a_{-1} a_{-2} \dots$ , where “.” is a radix point and  $a_i \in \mathbb{F}_p \equiv \mathbb{Z}/p\mathbb{Z}$  for each  $i$ , that is,  $a_i \in \{0, 1, 2, \dots, p-1\}$ . An element of  $\mathbb{Q}_p$  is

$$x = \dots a_2 a_1 a_0. a_{-1} a_{-2} \dots = \dots a_2 p^2 + a_1 p + a_0 + a_{-1} p^{-1} + a_{-2} p^{-2} \dots, \quad (1)$$

where all but a finite set of digits with negative indices are zero. We define the *order*  $v_p(x)$  of  $x$  and the *absolute value (valuation)*  $|x|_p$  as follows,

$$v_p(x) \equiv \begin{cases} \infty & a_i = 0 \text{ for all } i, \\ \min\{s : a_s \neq 0\} & \text{otherwise; } \end{cases}, \quad |x|_p \equiv p^{-v_p(x)}. \quad (2)$$

For example, we find  $\dots, \left| \frac{1}{9} \right|_3 = 9, \left| \frac{1}{3} \right|_3 = 3, |1|_3 = 1, |3|_3 = \frac{1}{3}, |9|_3 = \frac{1}{9}, |27|_3 = \frac{1}{27}$ , etc. For the sequence of numbers  $\{p^n\}$ , we obtain  $|p^n|_p = p^{-n} \rightarrow 0$  when  $n \rightarrow +\infty$  and therefore the sequence converges to vanish. Then as an example, we find the following expansion by using  $|\cdot|_3$ ,

$$-\frac{1}{2} = \dots + 3^2 + 3 + 1 = \sum_{k=0}^{+\infty} 3^k, \quad (3)$$

which corresponds to the formal expansion  $-\frac{1}{2} = \frac{1}{1-3} = 1 + 3 + 3^2 + \dots$ .

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The  $p$ -adic numbers attracted the attentions of the string physicist due to the  $p$ -adic like structure of the string amplitude [4–6]. After that, the quantum mechanics including the path integral formulation and statistical system have been studied [7–10]. In these formulations, the path-integrand or the Hamiltonian is also a  $p$ -adic numbers. In the standard statistical physics, the classical Hamiltonian is a standard real number or  $c$ -number and even for the quantum Hamiltonian, we consider the sum over the eigenvalues of the Hamiltonian. In this sense, the value of the Hamiltonian is a real number. The Hamiltonian of the Ising model can be regarded as a map from  $\mathbb{Z}_2$  to real numbers and the Hamiltonian for the fermionic fields can be a map from the anti-commuting Grassmann numbers to real numbers. This motivates us to consider the Hamiltonian which is given by a map from  $p$ -adic numbers to real numbers, that is, we consider a system where the dynamical variables are  $p$ -adic numbers but the Hamiltonian is given by real numbers. Then we can introduce the temperature  $T$  or several coupling constants as  $c$ -numbers and we can investigate the thermodynamical quantities like free energy, thermodynamical energy, entropy, specific heat, etc. A natural map from the  $p$ -adic numbers to real numbers is given by absolute value (valuation) in (2). Recently in [11], a model of the statistical system, where the Hamiltonian is given by the distance of two  $p$ -adic numbers, that is, the absolute value of the difference between the two  $p$ -adic numbers, has been proposed and well-studied. The model can be regarded as a  $p$ -adic analogue of the electrostatics. In this paper, we consider the simplest model corresponding to a single free particle moving in one dimensional space. Although the model is very simple but we show that the model shows rich thermodynamical structures and generates phenomena like phase transition in spite that we are considering only one degree of freedom.

In the next section, as a preparation to consider the model, we review on the measure of the  $p$ -adic number in order to define the integration which we use to calculate the partition function of the system. In Section 3, we propose the simplest model which corresponds to a free particle moving on one-dimensional space and calculate the thermodynamical quantities, whose structures are very rich and complicated. The calculations in Section 3 are mainly given numerically. In Section 4, we try to clarify the structure given in Section 3 analytically as possible as we can. The last section is devoted to the summary and discussion on the obtained results and we speculate some applications.

## 2. Invariant measure on the field $\mathbb{Q}_p$

In order to define the integration with respect to the  $p$ -adic numbers, we first consider the invariant measure on  $\mathbb{Q}_p$ . For details, see [2].

Let assume  $a, b \in \mathbb{Q}_p$ . Then there exists the Haar measure, which is positive and satisfies the conditions,

$$d(x+a) = dx, \quad d(xb) = |b|_p dx. \quad (4)$$

We normalize this measure so that

$$\int_{B_0} dx = 1. \quad (5)$$

Here  $B_0 = B_{\gamma=0}(a=0)$  is a region inside a circle on the  $p$ -adic number, which is defined by, for general  $\gamma$  and  $a$ ,

$$B_\gamma(a) = \{x : |x - a|_p \leq p^\gamma\}, \quad (6)$$

and we denote  $B_\gamma(a=0)$  by simply  $B_\gamma$ . We also define the circumference of the circle by

$$S_\gamma(a) = \{x : |x - a|_p = p^\gamma\}. \quad (7)$$

A function  $f \in L^1_{\text{loc}}$  is called *integrable* if there exists

$$\lim_{N \rightarrow \infty} \int_{B_N} f(x) dx = \lim_{N \rightarrow \infty} \sum_{-\infty < \gamma \leq N} \int_{S_\gamma} f(x) dx. \quad (8)$$

We also denote the integration by

$$\int_{\mathbb{Q}_p} f(x) dx = \sum_{-\infty < \gamma < \infty} \int_{S_\gamma} f(x) dx. \quad (9)$$

Then we find the following formula,

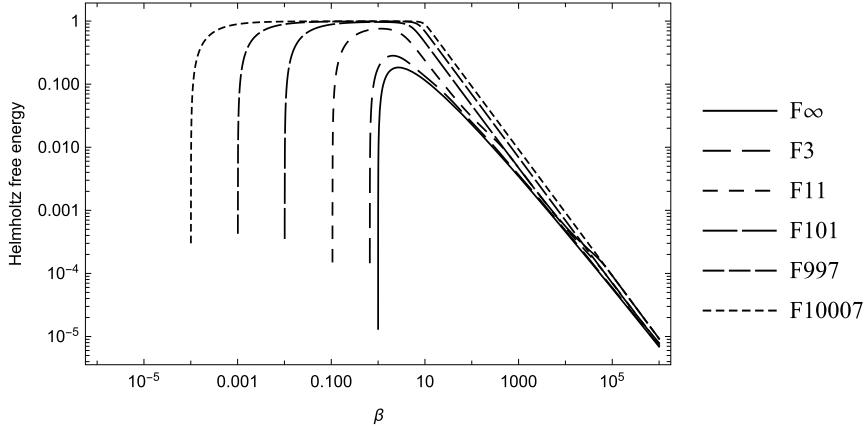
$$\int_{\mathbb{Q}_p} f(|x|_p) dx = \left(1 - \frac{1}{p}\right) \sum_{-\infty < \gamma < \infty} f(p^\gamma) p^\gamma. \quad (10)$$

We do not give any proof of the formula (10) but we use this formula to obtain the partition functions of the statistical system in the following section.

## 3. A model of a single particle in one dimensional space

We consider the system of a single  $p$ -adic particle, which corresponds to a single free particle in ideal gas in one dimension, and investigate the following partition function in the canonical ensemble,

$$Z = \int \frac{dq dx}{2\pi\hbar} e^{-\beta H}, \quad H = |q|^2, \quad (11)$$



**Fig. 1.** The figure expresses the behaviors of the Helmholtz free energies  $F_p(\beta)$  for  $p = 3$  (F3), 11 (F11), 101 (F101), 997 (F997), and 10007 (F10007) and  $F_\infty$  corresponds to  $F_\infty = -\beta^{-1} \ln Z_\infty(\beta)$ . The vertical axis corresponds to the values of the free energies and the horizontal axis to  $\beta$ .

where  $\beta$  is the inverse of the temperature  $T$  with the Boltzmann constant normalized to be unity and  $q$  can be identified with the momentum of the particle. Then by using the formula (10), we obtain

$$Z \propto Z_p(\beta) \equiv \int_{\mathbb{Q}_p} dq e^{-\beta|q|_p^2} = \left(1 - \frac{1}{p}\right) \sum_{-\infty < \gamma < \infty} p^\gamma e^{-\beta p^{2\gamma}}. \quad (12)$$

Just for the comparison, we may consider a function where the sum  $\sum_{-\infty < \gamma < \infty} \dots$  is replaced by the integration  $\int_{-\infty}^{\infty} dx \dots$ ,

$$Z_p^c(\beta) \equiv \left(1 - \frac{1}{p}\right) \int_{-\infty}^{\infty} dx p^x e^{-\beta p^{2x}} = \left(1 - \frac{1}{p}\right) \frac{1}{\ln p^2} Z_\infty(\beta), \quad Z_\infty(\beta) \equiv \sqrt{\frac{\pi}{\beta}}. \quad (13)$$

Here  $Z_\infty(\beta)$  is the partition function of the usual single free particle moving in one dimensional space. We should note that the factor  $\left(1 - \frac{1}{p}\right) \frac{1}{\ln p^2}$  does not depend on  $\beta$  and therefore the thermodynamical energy and the specific heat etc. corresponding to  $Z_p^c(\beta)$  do not depend on  $p$ . We should also note that the expression of  $Z_p(\beta)$  has a quasi-periodicity as follows,

$$Z_p(\beta p^2) = p^{-1} Z_p(\beta). \quad (14)$$

The Helmholtz free energy is defined by  $F_p(\beta) = -\beta^{-1} \ln Z_p(\beta)$ . In Fig. 1, the free energies for  $p = 3, 11, 101, 997$ , and 10007 are depicted as a function of  $\beta$ . The line for  $F_\infty$  corresponds to the free energy defined by  $F_\infty = -\beta^{-1} \ln Z_\infty(\beta)$ , which is nothing but the free energy of the standard (real number) free particle moving in one dimensional space. The free energies look smooth function of  $\beta$  and the difference from  $F_\infty$  becomes larger if  $p$  becomes larger.

We may also investigate the thermodynamical energy  $E_p(\beta) = -\frac{\partial \ln Z_p(\beta)}{\partial \beta}$ , the entropy  $S_p(\beta) = \beta(E_p(\beta) - F_p(\beta))$ , and the specific heat  $C_p(\beta) = -\beta^2 \frac{\partial^2 E_p(\beta)}{\partial \beta^2}$  and compare them with the quantities corresponding to the free particle moving in one dimensional space, that is, thermodynamical energy  $E_\infty(\beta) = -\frac{\partial \ln Z_\infty(\beta)}{\partial \beta}$ , the entropy  $S_\infty(\beta) = \beta(E_\infty(\beta) - F_\infty(\beta))$ , and the specific heat  $C_\infty(\beta) = -\beta^2 \frac{\partial^2 E_\infty(\beta)}{\partial \beta^2}$ . In Fig. 2, Fig. 3, and Fig. 4, the behaviors of the thermodynamical energies, entropies, and specific heats are depicted, respectively, for  $p = 3, 11, 101, 997$ , and 10007 as a function of  $\beta$ .

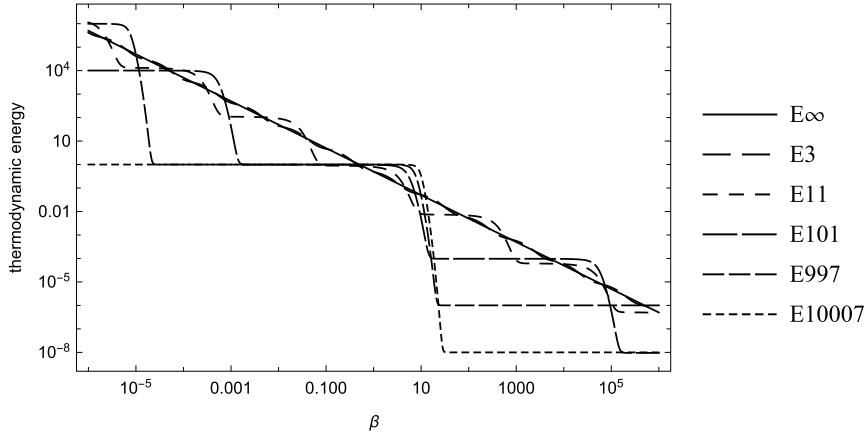
Although  $F_p(\beta)$  looks a smooth function but the thermodynamical energy  $E_p(\beta)$ , the entropy  $S_p(\beta)$ , and the specific heat  $C_p(\beta)$  look to show the oscillation. The oscillation could correspond to the quasi-periodicity in (14). An interesting point is that there seem to be jumps in the value of  $E_p(\beta)$ ,  $S_p(\beta)$ , and  $C_p(\beta)$ . Because the thermodynamical energy  $E_p(\beta)$  is the first derivative of the free energy  $F_p(\beta)$ , the jumps seem to correspond to the first order phase transition. Usually, in the system with a finite number of degrees of freedom, phase transitions cannot be generated.

#### 4. Analytical properties of model

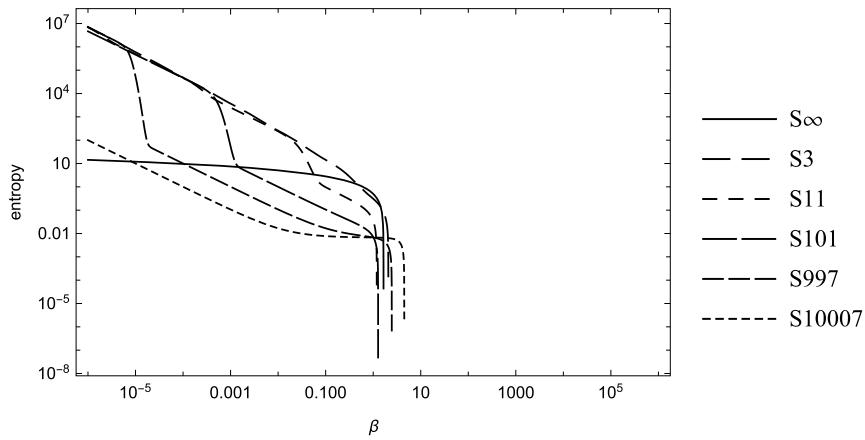
In the last section, we have found several specific structures for the thermodynamical quantities by the numerical calculations. In this section, we analyze the behaviors analytically as possible as we can.

Naively, the limit  $p \rightarrow \infty$  is expected to correspond to the standard real number but the results obtained in this paper seem to conflict with this naive speculation. In fact, in any thermodynamical quantity  $F_p(\beta)$ ,  $E_p(\beta)$ ,  $S_p(\beta)$ , or  $C_p(\beta)$  which we have calculated, the difference of the quantity from that in the system of a real free particle,  $F_\infty(\beta)$ ,  $E_\infty(\beta)$ ,  $S_\infty(\beta)$ , or  $C_\infty(\beta)$  becomes larger when  $p$  becomes larger. The breakdown of this naive speculation could come from the definition of the absolute value (valuation) in (2), if we fix a value of  $q$ , the definition (2) tells  $|q|_p \rightarrow 1$  if  $v_p(q) > 0$ , which tells that the region of  $q$  which gives a non-trivial contribution to the partition function is rather restricted. In order to find what happens, we rewrite the r.h.s. in (12) as below,

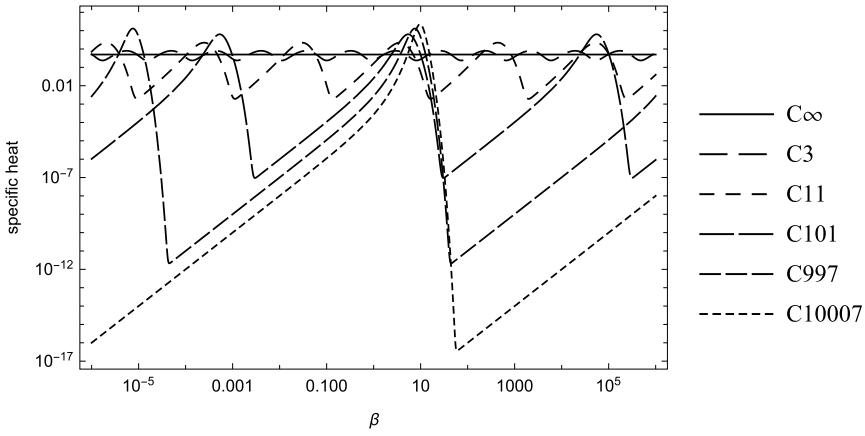
$$\left(1 - \frac{1}{p}\right) \sum_{-\infty < \gamma < \infty} p^\gamma e^{-\beta p^{2\gamma}} = \left(1 - \frac{1}{p}\right) \sum_{-\infty < \gamma < \infty} e^{-\beta e^{2s} + s}, \quad s \equiv \gamma \ln p. \quad (15)$$



**Fig. 2.** The behaviors of the thermodynamical energies  $E_p(\beta)$  for  $p = 3$  (E3), 11 (E11), 101 (E101), 997 (E997), and 10007 (E10007) and  $E_\infty$  ( $E_\infty$ ) are depicted. The vertical axis corresponds to the values of the thermodynamical energies and the horizontal axis to  $\beta$ .



**Fig. 3.** The behaviors of the entropies  $S_p(\beta)$  for  $p = 3$  (S3), 11 (S11), 101 (S101), 997 (S997), and 10007 (S10007) and  $S_\infty$  ( $S_\infty$ ) are depicted. The vertical axis corresponds to the values of the entropies and the horizontal axis to  $\beta$ .



**Fig. 4.** The behaviors of the specific heat  $C_p(\beta)$  for  $p = 3$  (C3), 11 (C11), 101 (C101), 997 (C997), and 10007 (C10007) and  $C_\infty$  ( $C_\infty$ ) are depicted. The vertical axis corresponds to the values of the specific heats and the horizontal axis to  $\beta$ .

If we like to consider the integration corresponding to the free particle in one dimensional space as in (13), we need to consider the limit where  $ds = d\gamma \ln p$  vanishes for a finite  $d\gamma$ . The limit is not the limit of  $p \rightarrow \infty$  but  $\ln p \rightarrow 0$ , that is, the limit of  $p \rightarrow 1$ . In the limit, we can replace  $\sum_{-\infty < \gamma < \infty} \dots$  by  $\frac{1}{\ln p} \int_{-\infty}^{\infty} ds \dots$  and we obtain the result in (13).

Now we consider why the jumps observed in this paper could occur. First we estimate which  $\gamma$  contributes to the thermodynamical quantities by investigating the saddle point in the expression of (15). The saddle point  $s = s_0$  is given by

$$0 = \frac{d}{ds} \left( -\beta e^{2s} + s \right) \Big|_{s=s_0} = -2\beta e^{2s_0} + 1, \quad (16)$$

that is,

$$s_0 = -\frac{1}{2} \ln(2\beta). \quad (17)$$

Then we find that  $s_0$  is monotonically decreasing function of  $\beta$ . We should note, however, that  $\gamma_0 \equiv \frac{s_0}{\ln p}$  is not always an integer. Therefore especially for large  $p$ , only one of the integer value  $\gamma = \gamma_0$  which satisfies  $|\gamma_0 - \frac{s_0}{\ln p}| < 1$  dominates and we can use the following approximation,

$$\sum_{-\infty < \gamma < \infty} e^{-\beta e^{2s} + s} \sim e^{-\beta e^{2\gamma_0 \ln p} + \gamma_0 \ln p}. \quad (18)$$

If the value of  $\beta$  increases, that is, the temperature decreases, and goes beyond a critical value, the contribution coming from  $\gamma = \gamma_0 - 1$  becomes larger than that coming from  $\gamma = \gamma_0$ . Therefore there occurs a jump in the dominant contribution, which also generates the jumps in the thermodynamical quantities.

For example, when  $\beta = \frac{1}{2}$ , we find  $s_0 = 0$  and therefore  $\gamma = 0$ .

$$p^\gamma e^{-\beta p^{2\gamma}} \Big|_{\beta=\frac{1}{2}, \gamma=0} = 1, \quad p^\gamma e^{-\beta p^{2\gamma}} \Big|_{\beta=\frac{1}{2}, \gamma=-1} = \frac{e^{-\frac{1}{2p^2}}}{p}, \quad p^\gamma e^{-\beta p^{2\gamma}} \Big|_{\beta=\frac{1}{2}, \gamma=1} = p e^{-\frac{p^2}{2}}. \quad (19)$$

In the limit of  $p \rightarrow \infty$ , we find  $p^\gamma e^{-\beta p^{2\gamma}} \Big|_{\beta=\frac{1}{2}, \gamma=-1}, p^\gamma e^{-\beta p^{2\gamma}} \Big|_{\beta=\frac{1}{2}, \gamma=1} \rightarrow 0$  and therefore only  $p^\gamma e^{-\beta p^{2\gamma}} \Big|_{\beta=\frac{1}{2}, \gamma=0}$  contribute. This tells that when  $\beta \sim \frac{1}{2}$ , only the term with  $\gamma = 0$  dominates when  $p$  is large. On the other hand, when  $\beta = \frac{p^2}{2} > \frac{1}{2}$ , the term with  $\gamma = -1$  dominates and we find

$$p^\gamma e^{-\beta p^{2\gamma}} \Big|_{\beta=\frac{p^2}{2}, \gamma=-1} = \frac{e^{-\frac{1}{2}}}{p}, \quad p^\gamma e^{-\beta p^{2\gamma}} \Big|_{\beta=\frac{p^2}{2}, \gamma=-2} = \frac{e^{-\frac{1}{2p^2}}}{p^2}, \quad p^\gamma e^{-\beta p^{2\gamma}} \Big|_{\beta=\frac{p^2}{2}, \gamma=0} = e^{-\frac{p^2}{2}}. \quad (20)$$

Therefore in the limit of  $p \rightarrow \infty$ , we find the term  $p^\gamma e^{-\beta p^{2\gamma}} \Big|_{\beta=\frac{p^2}{2}, \gamma=-1}$  dominates. If the value of  $\beta$  changes from  $\beta = \frac{1}{2}$  to  $\beta = \frac{p^2}{2}$ , there should occur a transition where the dominant contribution changes from the term with  $\gamma = 0$  to that with  $\gamma = -1$ . The critical value  $\beta_c$ ,  $\frac{1}{2} < \beta_c < \frac{p^2}{2}$ , is given by solving the equation

$$p^\gamma e^{-\beta_c p^{2\gamma}} \Big|_{\gamma=0} = p^\gamma e^{-\beta_c p^{2\gamma}} \Big|_{\gamma=-1}, \quad (21)$$

that is,

$$e^{-\beta_c} = \frac{e^{-\frac{\beta_c}{p^2}}}{p}, \quad (22)$$

whose solution is given by

$$\beta_c = \frac{\ln p}{1 - \frac{1}{p^2}}. \quad (23)$$

Therefore we obtain  $\beta_c \sim 5$  for  $p = 997$  and  $\beta_c \sim 10$  for  $p = 10007$ , which may correspond to the behaviors around  $\beta \sim 10$  in Figs. 2 and 4. The generalization of the critical value  $\beta_c$  corresponding to the transition between  $\gamma = \gamma_0$  and  $\gamma = \gamma_0 - 1$  can be obtained by solving the equation

$$p^{\gamma_0} e^{-\beta_c \gamma_0 p^{2\gamma_0}} = p^{\gamma_0-1} e^{-\beta_c \gamma_0 p^{2(\gamma_0-1)}}, \quad (24)$$

as

$$\beta_{c\gamma_0} = \frac{\ln p}{p^{2\gamma_0} \left(1 - \frac{1}{p^2}\right)}. \quad (25)$$

The transition from  $\gamma = \gamma_0$  to  $\gamma = \gamma_0 - 1$  is very similar to the standard first order phase transition and therefore the expectation value of  $\gamma$  could be the order parameter specifying the phases.

## 5. Summary and discussions

In this paper, we have investigated the thermodynamics of the simplest model given in (11), where the dynamical variable  $q$  is a  $p$ -adic number but the Hamiltonian is given by a real number. Although the degree of freedom is unity, the system shows the behaviors like phase transition and we have found that the system has rich structures. Anyway at present, the physical meaning of the jump in the thermodynamical energy is still not clear although we have given some analytical arguments. Maybe we need to clarify it in future works for further understanding of the models.

Similar to the situation that the fermion fields are described by the Grassmann number, there could be a situation that some fields are described by the  $p$ -adic numbers. Such theories might be realized by considering a lattice instead of the continuous space-time as in

the lattice field theories, and putting the  $p$ -adic dynamical degrees of freedom on the sites of the lattice. If there exists a model which generates the second order phase transition corresponding to the continuum limit, we may obtain the  $p$ -adic field theory.

The behavior of the thermodynamical energy might be interesting if we consider the cosmology. For large  $p$ , the energy is almost constant in the large range of  $\beta$  and when  $\beta$  becomes large enough, that is, the temperature becomes low enough, there appears a jump in the value of the energy and the value becomes much smaller. The constant energy might play the role of the cosmological constant. Then the large constant value of the thermodynamical energy for high temperature (small  $\beta$ ) might generate the inflation in the early universe and the small constant energy for the low temperature (large  $\beta$ ) might correspond to the dark energy in the present universe.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Acknowledgements

This work is partially supported by the JSPS Grant-in-Aid for Scientific Research (C) No. 18K03615 (S.N.).

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