

Theory of Coulomb drag for massless Dirac fermions

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Abstract. Coulomb drag between two unhybridized graphene sheets separated by a dielectric spacer has recently attracted considerable theoretical interest. We first review, for the sake of completeness, the main analytical results which have been obtained by other authors. We then illustrate pedagogically the minimal theory of Coulomb drag between two spatially separated two-dimensional systems of massless Dirac fermions which are both away from the charge-neutrality point. This relies on second-order perturbation theory in the screened interlayer interaction and on Boltzmann-transport theory. In this theoretical framework and in the low-temperature limit, we demonstrate that, to leading (i.e. quadratic) order in temperature, the drag transresistivity is completely insensitive to the precise intralayer momentum-relaxation mechanism (i.e. to the functional dependence of the transport scattering time on energy). We also provide analytical results for the low-temperature drag transresistivity for both cases of ‘thick’ and ‘thin’ spacers and for arbitrary values of the dielectric constants of the media surrounding the two Dirac-fermion layers. Finally, we present numerical results for the low-temperature drag transresistivity for the case when one of the media surrounding the Dirac-fermion layers has a frequency-dependent dielectric constant. We conclude by suggesting an experiment that can potentially allow for the observation of departures from the canonical quadratic-in-temperature behavior of the transresistivity.

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1. Introduction

Electron–electron interactions are a source of coupling between closely spaced nano-electronic circuits. This coupling has commanded a great deal of attention during the past 30 years or so, since it constitutes a potential alternative to the inductive and capacitive couplings of conventional electronics. Early on it was realized [1, 2] that ‘Coulomb mutual scattering’ between spatially separated electronic systems provides a mechanism to relax momentum that tends to equalize drift velocities. This *intrinsic* friction due to electron–electron interactions is currently referred to as ‘Coulomb drag’ [3–11]. Early experimental work was carried out by Gramila *et al* [12] and Sivan *et al* [13] in semiconductor double quantum wells.

In these experiments, a constant current is imposed on the two-dimensional (2D) electron gas in one of the wells (the ‘active’ or ‘drive’ layer). If no current is allowed to flow in the other well (the ‘passive’ layer), an electric field develops whose associated force cancels the frictional drag force exerted by the electrons in the active layer on the electrons in the passive one. The transresistance ρ_D , defined as the ratio of the induced voltage in the passive layer to the applied current in the drive layer, directly measures the rate at which momentum is transferred from the current-carrying 2D electron gas to its neighbor. Coulomb drag is ultimately caused by *fluctuations* in the density of electrons in each layer since 2D layers with uniformly distributed charge will not exert any frictional forces upon each other [4].

The study of Coulomb-coupled 2D systems has now been revitalized by advances which have made it possible to prepare robust and ambipolar 2D electron systems (ESs), based on graphene [14] layers or on the surface states of topological insulators (TIs) [15], which are described by an ultrarelativistic wave equation instead of the nonrelativistic Schrödinger equation.

Single- and few-layer graphene systems can be produced, for example, by mechanical exfoliation of thin graphite [16] or by thermal decomposition of silicon carbide [17]. Isolated graphene layers host massless Dirac 2D electron systems (MD2DESs) with a fourfold (spin \times valley) flavor degeneracy, whereas topologically protected MD2DESs that have no additional spin or valley flavor labels appear automatically [15] at the top and bottom surfaces of a 3D TI thin film. The protected surface states of 3D TIs are associated with spin–orbit interaction-driven bulk band inversions. 3D TIs in a slab geometry offer two surface states that can be far enough apart to make single-electron tunneling negligible, but close enough for Coulomb interactions between surfaces to be important. Unhybridized MD2DES pairs can be realized in graphene by separating two layers by a dielectric [18] (such as Al_2O_3) or by a few layers of a one-atom-thick insulator such as BN [19–22]. In both cases interlayer hybridization is negligible and the nearby graphene layers are, from the point of view of single-particle physics, isolated. Isolated graphene layers can also be found on the surface of bulk graphite [23, 24] and in ‘folded graphene’ [25] (a natural byproduct of micromechanical exfoliation) or prepared by chemical vapor deposition [24]. We use the term *double-layer graphene* (DLG) to refer to a system with two graphene layers that are coupled only by Coulomb interactions, avoiding the term *bilayer graphene*, which typically refers to two adjacent graphene layers in the crystalline Bernal-stacking configuration [14].

DLG and TI thin films are both described at low energies by a Hamiltonian with two MD2DESs [14] coupled only by Coulomb interactions, in the absence of single-particle tunneling. Coulomb drag between two spatially separated MD2DESs has recently attracted a great deal of theoretical interest [26–31]. The calculations in [26–31] refer to the regime in which *both* layers are either electron- or hole-doped. The Coulomb drag transresistivity in this case is negative and vanishes like T^2 at low temperatures. Despite the considerable amount of work published on the subject recently [26–31], no clear consensus exists on the dependence of the drag transresistivity in the Fermi-liquid regime on carrier densities in the two layers, on the interlayer distance and on the dielectric constants of the media surrounding the two layers. The main analytical results obtained earlier by other authors will be summarized below in section 2. In passing, we note that the impact of plasmons on the drag transresistivity has recently been studied in [32].

The Coulomb drag transresistivity between two MD2DESs in the regime in which one layer is electron doped and the other is hole doped has recently been calculated by Mink *et al* [33]⁴. In this intriguing regime, the authors of [33] have found that ρ_D grows *logarithmically* upon lowering the temperature T towards the critical temperature T_c for exciton condensation [34] (condensation of electron–hole pairs in a dipolar condensate).

Coulomb drag between two graphene sheets has recently been measured by Kim *et al* [18]. This first experimental study represents an important milestone since the authors of this work have shown that the ‘strong-coupling’ regime, i.e. the regime in which the interlayer distance d is much smaller than the typical separation between two electrons in each layer, is easy to achieve experimentally with two independently contacted graphene sheets [18]. This study has indeed fueled recent theoretical investigations on Coulomb drag between the MD2DESs mentioned above.

⁴ For earlier theoretical works on Coulomb drag in electronic systems in the exciton-condensate phase or approaching an exciton-condensate instability, see Vignale and MacDonald [33] and Hu [33], respectively. For experimental anomalies of the Coulomb-drag transresistivity of electron–hole semiconductor bilayers (at zero magnetic field), see Croxall *et al* [33] and Seamons *et al* [33].

In this paper, we present in a pedagogical fashion the minimal theory of Coulomb drag between two spatially separated MD2DESs in the regime in which both layers are either electron or hole doped. We will only be concerned with the so-called ‘Fermi-liquid regime’ in which both layers are away from the charge neutrality point. Our theory relies on second-order perturbation theory in the screened interlayer interaction and on Boltzmann transport (BT) theory. In this theoretical framework and in the low-temperature limit, we demonstrate that, to leading (i.e. quadratic) order in temperature, the drag transresistivity is completely insensitive to the precise intralayer momentum-relaxation mechanism (i.e. to the functional dependence of the transport scattering time on energy). This is in disagreement with the findings reported in [29, 30]. We also provide new analytical results for the low-temperature drag transresistivity in both the cases of ‘thick’ and ‘thin’ spacers, correcting in the latter case a mistake contained in [28]. At odds with the literature, our results hold true for arbitrary values of the dielectric constants of the media surrounding the two Dirac-fermion layers. Finally, we present numerical results for the low-temperature drag transresistivity in the case when one of the media surrounding the two MD2DESs has a strongly frequency-dependent dielectric constant. We conclude by suggesting an experiment with a DLG deposited on SrTiO₃ [35]⁵ that can pave the way for the observation of departures from the canonical quadratic-in-temperature behavior of the transresistivity.

This paper is organized as follows. In section 2, we give a summary of the main analytical results obtained by other authors. In section 3, we present the model Hamiltonian and the most important basic definitions. In section 4, we present the Kubo formalism approach to the calculation of the drag conductivity, while in section 5 we present a series of simplifications that lead to the BT expression for the drag conductivity and resistivity. These two sections do not contain original results but make the paper completely self-contained. Experts can skip sections 4 and 5 and go directly to sections 6 and 7, which contain the most important results of this work and all our original results. In section 8, we summarize our main findings and draw our main conclusions.

2. Summary of the main analytical results obtained earlier by other authors

Despite the large body of theoretical work dedicated to Coulomb drag between two unhybridized MD2DESs [26–31], no consensus appears to exist among different authors. In what follows we summarize the main analytical results that can be found in the existing literature.

- (1) Tse *et al* [26] studied Coulomb drag between two unhybridized graphene sheets separated by a dielectric by employing BT theory. They neglected the spatial dependence of the dielectric constant in the \hat{z} -direction (see figure 1) and assumed a momentum-independent scattering time. In the weak-coupling limit, i.e. in the limit in which the interlayer distance d is much larger than the average distance between two electrons in each layer, Tse *et al* [26] demonstrated that the low-temperature drag resistivity is given by

$$\rho_D \rightarrow -\frac{h}{e^2} \frac{\pi \zeta(3)}{32} \frac{(k_B T)^2}{\varepsilon_{F,1} \varepsilon_{F,2}} \frac{1}{(q_{TF,1} d)(q_{TF,2} d)} \frac{1}{(k_{F,1} d)(k_{F,2} d)}. \quad (1)$$

In equation (1), $q_{TF,\ell}$ is the Thomas–Fermi screening wave vector, which is proportional to $k_{F,\ell}$, the Fermi wave number in each layer. In the symmetric $n_1 = n_2$ case the previous

⁵ For a recent experiment exploiting SrTiO₃ as a substrate for graphene transport experiments, see [35].

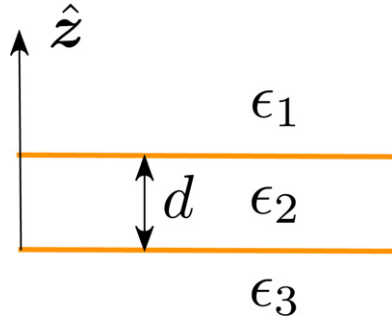


Figure 1. A side view of the double-layer system, which explicitly indicates the dielectric model used in these calculations. The two layers hosting massless Dirac fermions are located at $z = 0$ and $z = d$. In a Coulomb-drag transport setup a constant current flow is imposed in one layer (the bottom one, say). If no current is allowed to flow in the other layer, an electric field develops whose associated force cancels the frictional drag force exerted by the electrons in the bottom layer on the electrons in the top one.

equation yields

$$\rho_D \propto -\frac{h}{e^2} \frac{T^2}{n^3 d^4}. \quad (2)$$

- (2) We now turn to summarizing the main results of Peres *et al* [29]. The authors of this work used BT theory, took into account the momentum dependence of the scattering time and also the spatial dependence of the dielectric constant in the \hat{z} -direction. While the main approach followed by these authors is numerical, they also provide an analytical expression for the Coulomb-drag transresistivity in the weak-coupling regime. For an intralayer scattering time that depends *linearly* on momentum, Peres *et al* [29] found

$$\rho_D \propto -\frac{h}{e^2} \frac{T^2}{n^4 d^6}, \quad \text{for } k_F d \gg 1. \quad (3)$$

These authors have missed one power of momentum in the integral that yields the asymptotic behavior of the nonlinear susceptibility at low energy [36].

- (3) Katsnelson [28] used BT theory, took into account, at least partially, the spatial dependence of the dielectric constant in the \hat{z} -direction, but did not take into account the momentum dependence of the scattering time. The main results of [28] are

$$\begin{cases} \rho_D \propto -\frac{h}{e^2} T^2 \frac{|\ln(nd^2)|}{n}, & \text{for } k_F d \ll 1, \\ \rho_D \propto -\frac{h}{e^2} \frac{T^2}{n^3 d^4}, & \text{for } k_F d \gg 1. \end{cases} \quad (4)$$

Note that the weak-coupling result of Katsnelson is in agreement with that of Tse *et al* [26].

- (4) Hwang *et al* [30] used BT theory, took into account the momentum dependence of the scattering time, but neglected the spatial dependence of the dielectric constant in the

\hat{z} -direction. For a *momentum-independent* intralayer scattering time they find that

$$\begin{cases} \rho_D \propto -\frac{h}{e^2} \frac{T^2}{n^2 d^2}, & \text{for } k_F d \ll 1, \\ \rho_D \propto -\frac{h}{e^2} \frac{T^2}{n^4 d^6}, & \text{for } k_F d \gg 1. \end{cases} \quad (5)$$

Note that the weak-coupling result in equation (5) differs from the result of Tse *et al* [26]. For an intralayer scattering time that depends *linearly* on momentum, Hwang *et al* [30] find instead that

$$\begin{cases} \rho_D \propto -\frac{h}{e^2} T^2 \frac{|\ln(nd^2)|}{n}, & \text{for } k_F d \ll 1, \\ \rho_D \propto -\frac{h}{e^2} \frac{T^2}{n^3 d^4}, & \text{for } k_F d \gg 1. \end{cases} \quad (6)$$

By comparing equation (5) with equation (6), Hwang *et al* [30] concluded that the functional dependence of ρ_D on n and d is very sensitive to the functional dependence of the intralayer scattering time on momentum. According to Amorim and Peres [36], the calculations of Hwang *et al* [30] share the same problem as that of Peres *et al* [29].

- (5) Narozhny *et al* [31] have recently presented a systematic study of Coulomb drag between two MD2DESs which is based on perturbation theory in the dimensionless coupling constant $\alpha_{ee} = e^2/(\hbar v)$ with v being the Dirac velocity. The authors of [31] did not consider the spatial dependence of the dielectric constant along the \hat{z} -direction and discussed mostly the case of ‘low doping’ (i.e. the regime in which the chemical potential is comparable with or smaller than $k_B T$). Since the focus of our paper is on the opposite regime, i.e. the Fermi-liquid regime, we now provide a short summary of the results of Narozhny *et al* [31] in this regime only. The authors of [31] have demonstrated that the energy dependence of the relaxation time is completely irrelevant in relation to the low-temperature drag transresistivity. In the weak-coupling limit, Narozhny *et al* [31] found for the drag transresistivity the same doping- and interlayer separation-dependence as that in [26] and [28]. As the density decreases, Narozhny *et al* [31] found that the drag coefficient acquires logarithmic corrections—see figure 3 in [31]. In particular, in the limit $k_F d \ll 1$, equation (41) in [31] reads

$$\rho_D \propto -\frac{h}{e^2} \frac{T^2}{n} \ln \left(\frac{1}{N_F \alpha_{ee}} \right). \quad (7)$$

As we will see below, our results agree with those of Narozhny *et al* [31], and represent their generalization to the case of a finite Thomas–Fermi screening length ($\propto N_F \alpha_{ee}$) and spatially dependent dielectric constants along the \hat{z} -direction.

3. The model Hamiltonian and basic definitions

We consider two unhybridized layers of massless Dirac fermions, each one described at the noninteracting level by the following single-channel Hamiltonian ($\hbar = 1$):

$$\hat{\mathcal{H}}_\ell = v \sum_{\mathbf{k}, \alpha, \beta} \hat{\psi}_{\mathbf{k}, \alpha, \ell}^\dagger (\boldsymbol{\sigma}_{\alpha\beta} \cdot \mathbf{k}) \hat{\psi}_{\mathbf{k}, \beta, \ell}. \quad (8)$$

Here v is the bare electron velocity, \mathbf{k} is the $\mathbf{k} \cdot \mathbf{p}$ momentum, α, β are (sublattice) pseudospin labels and $\boldsymbol{\sigma}_{\alpha\beta} = (\sigma_{\alpha\beta}^x, \sigma_{\alpha\beta}^y)$ is a vector of Pauli matrices which act on the sublattice pseudospin degree-of-freedom. The field operator $\hat{\psi}_{\mathbf{k},\alpha,\ell}^\dagger$ ($\hat{\psi}_{\mathbf{k},\alpha,\ell}$) creates (destroys) an electron with momentum \mathbf{k} , pseudospin α and layer index $\ell = 1, 2$.

The Hamiltonian (8) can be easily diagonalized in each layer by the matrix

$$\mathcal{U}(\mathbf{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi_{\mathbf{k}}/2} & e^{-i\varphi_{\mathbf{k}}/2} \\ e^{i\varphi_{\mathbf{k}}/2} & -e^{i\varphi_{\mathbf{k}}/2} \end{pmatrix}, \quad (9)$$

where $\varphi_{\mathbf{k}}$ is the polar angle of the vector \mathbf{k} .

The i th Cartesian component of the current-density operator ($i = x, y$) in the ℓ th layer is given by

$$\hat{J}_{\mathbf{q},\ell}^i = \sum_{\mathbf{k},\alpha,\beta} \hat{\psi}_{\mathbf{k}-\mathbf{q},\alpha,\ell}^\dagger (v\sigma_{\alpha\beta}^i) \hat{\psi}_{\mathbf{k},\beta,\ell}. \quad (10)$$

The Fermi wave number in the ℓ th layer is defined by

$$k_{\text{F},\ell} \equiv \sqrt{\frac{4\pi n_\ell}{N_{\text{f},\ell}}}, \quad (11)$$

where $n_\ell > 0$ is the excess electron density in the ℓ th layer⁶ and $N_{\text{f},\ell}$ is a degeneracy factor ($N_{\text{f},\ell} = 4$ for a graphene layer, accounting for spin and valley degeneracies, while $N_{\text{f},\ell} = 1$ for a TI surface state).

In the absence of disorder, the Green's function corresponding to the Hamiltonian in equation (8) in the imaginary frequency axis is given by the following 2×2 matrix:

$$G_\ell(\mathbf{k}, i\omega) = \frac{(\mu_\ell + i\omega)\mathbb{I}_\sigma + v\mathbf{k} \cdot \boldsymbol{\sigma}}{(\mu_\ell + i\omega)^2 - v^2\mathbf{k}^2}, \quad (12)$$

where \mathbb{I}_σ is the 2×2 identity matrix in the sublattice-pseudospin representation and μ_ℓ is the chemical potential in the ℓ th layer. In the zero-temperature limit, $\mu_\ell \rightarrow \varepsilon_{\text{F},\ell} = vk_{\text{F},\ell}$, where $\varepsilon_{\text{F},\ell}$ is the Fermi energy in the ℓ th layer.

The disorder-free Green's function in the eigenstate representation reads

$$\mathcal{G}_{\ell,\lambda}(\mathbf{k}, i\omega) = \frac{1}{i\omega + \mu_\ell - \varepsilon_{\mathbf{k},\lambda}}, \quad (13)$$

where $\varepsilon_{\mathbf{k},\lambda} = \lambda v|\mathbf{k}|$ are Dirac-band energies.

In what follows we will need the following matrix elements:

$$\rho_{\mathbf{k}\lambda,\mathbf{k}'\lambda'} \equiv [\mathcal{U}^\dagger(\mathbf{k})\mathcal{U}(\mathbf{k}')]_{\lambda\lambda'} = \frac{e^{i(\varphi_{\mathbf{k}}-\varphi_{\mathbf{k}'})/2} + \lambda\lambda' e^{-i(\varphi_{\mathbf{k}}-\varphi_{\mathbf{k}'})/2}}{2}, \quad (14)$$

and

$$\sigma_{\mathbf{k}\lambda,\mathbf{k}'\lambda'}^x \equiv [\mathcal{U}^\dagger(\mathbf{k})\sigma^x\mathcal{U}(\mathbf{k}')]_{\lambda\lambda'} = \frac{\lambda e^{-i(\varphi_{\mathbf{k}}+\varphi_{\mathbf{k}'})/2} + \lambda' e^{i(\varphi_{\mathbf{k}}+\varphi_{\mathbf{k}'})/2}}{2}. \quad (15)$$

⁶ We discuss electron doping for the sake of definiteness. The Coulomb-drag transresistivity is a particle-hole symmetric property, i.e. $\rho_{\text{D}}(-n_1, -n_2) = \rho_{\text{D}}(n_1, n_2)$.

The two MD2DESs described by equation (8) are coupled electrostatically by long-range Coulomb interactions, which are influenced by the layered dielectric environment (see figure 1). The coupling Hamiltonian reads

$$\hat{\mathcal{H}}_{\text{ee}} = \frac{1}{2S} \sum_{\mathbf{q}, \ell \neq \ell'} V_{\ell\ell'}(\mathbf{q}) \hat{\rho}_{\mathbf{q},\ell} \hat{\rho}_{-\mathbf{q},\ell'}, \quad (16)$$

where

$$\hat{\rho}_{\mathbf{q},\ell} = \sum_{\mathbf{k}, \alpha} \hat{\psi}_{\mathbf{k}-\mathbf{q},\alpha,\ell}^\dagger \hat{\psi}_{\mathbf{k},\alpha,\ell} \quad (17)$$

is the density operator for the ℓ th layer and $V_{\ell\ell'}(\mathbf{q})$ (with $\ell \neq \ell'$) is the 2D Fourier transform of the interlayer Coulomb interaction

$$V_{12}(\mathbf{q}) = V_{21}(\mathbf{q}) = \frac{8\pi e^2}{qD(\mathbf{q})} \epsilon_2. \quad (18)$$

Here

$$D(\mathbf{q}) = [(\epsilon_1 + \epsilon_2)(\epsilon_2 + \epsilon_3)e^{qd} + (\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)e^{-qd}]. \quad (19)$$

For future purposes, we introduce the dynamically screened interlayer interaction $U_{12}(\mathbf{q}, \omega)$, which, at the random phase approximation (RPA) level, is given by [37, 38]

$$U_{12}(\mathbf{q}, \omega) = \frac{V_{12}(\mathbf{q})}{\varepsilon(\mathbf{q}, \omega)}, \quad (20)$$

where

$$\varepsilon(\mathbf{q}, \omega) = [1 - V_{11}(\mathbf{q})\chi_1^{(0)}(\mathbf{q}, \omega)][1 - V_{22}(\mathbf{q})\chi_2^{(0)}(\mathbf{q}, \omega)] - V_{12}^2(\mathbf{q})\chi_1^{(0)}(\mathbf{q}, \omega)\chi_2^{(0)}(\mathbf{q}, \omega) \quad (21)$$

is the RPA dynamical dielectric function.

In equation (21), $\chi_\ell^{(0)}(\mathbf{q}, \omega)$ is the well-known [39–41] density–density (Lindhard) response function of a noninteracting MD2DES at arbitrary doping n_ℓ . The Coulomb interaction in the $\ell = 1$ (top) layer is given by

$$V_{11}(\mathbf{q}) = \frac{4\pi e^2}{qD(\mathbf{q})} [(\epsilon_2 + \epsilon_3)e^{qd} + (\epsilon_2 - \epsilon_3)e^{-qd}], \quad (22)$$

while the Coulomb interaction in the bottom layer, $V_{22}(\mathbf{q})$, can be simply obtained from $V_{11}(\mathbf{q})$ by interchanging $\epsilon_3 \leftrightarrow \epsilon_1$.

Equations (18), (19) and (22) first appeared in [42] and their explicit derivation has been reported in [28]. Note that in the ‘uniform’ $\epsilon_1 = \epsilon_2 = \epsilon_3 \equiv \epsilon$ limit we recover the familiar expressions $V_{11}(\mathbf{q}) = V_{22}(\mathbf{q}) \rightarrow 2\pi e^2/(\epsilon q)$ and $V_{12}(\mathbf{q}) = V_{21}(\mathbf{q}) \rightarrow V_{11}(\mathbf{q}) \exp(-qd)$. Most of the previous works on Coulomb drag in DLG have assumed this limit, which rarely applies experimentally.

The aim of this paper is to present a theory of Coulomb drag, which is valid up to *second order* in the dynamically screened interaction $U_{12}(\mathbf{q}, \omega)$, for the system described by the Hamiltonian

$$\hat{\mathcal{H}} = \sum_{\ell} \hat{\mathcal{H}}_{\ell} + \hat{\mathcal{H}}_{\text{ee}}. \quad (23)$$

Note that we are not including in equation (23) any term describing *intralayer* electron–electron interactions. These can be treated in an approximate fashion by invoking

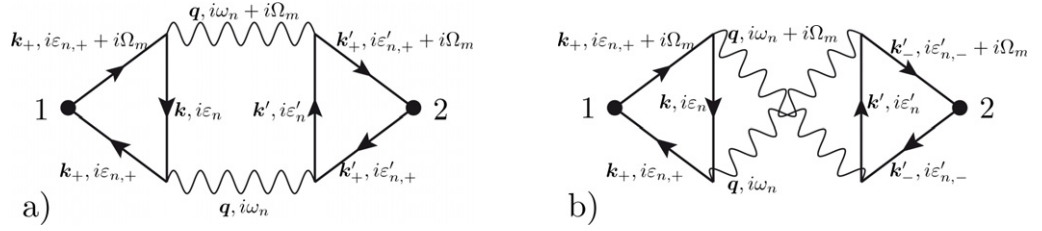


Figure 2. Second-order Aslamazov–Larkin-type diagrams contributing to the Coulomb-drag conductivity (finite-temperature Matsubara formalism). Solid lines denote the single-particle Green’s function in the presence of disorder. Wavy lines denote the *screened* interlayer interaction U_{12} in equation (20). Black dots in the triangular portions of the diagrams denote vertices of current operators in the two layers, which are both proportional to the Pauli matrix σ^x , say, if one is interested in the longitudinal drag conductivity. Finally, $\mathbf{k}_{\pm} = \mathbf{k} \pm \mathbf{q}$, $k'_{\pm} = k' \pm q$, $i\varepsilon_{n,\pm} = i\varepsilon_n \pm i\omega_n$, and $i\varepsilon'_{n,\pm} = i\varepsilon'_n \pm i\omega_n$.

Landau’s theory of normal Fermi liquids [38], i.e. by renormalizing the microscopic parameters of the intralayer Hamiltonian $\hat{\mathcal{H}}_{\ell}$. For example, in equation (8) one can use phenomenologically the renormalized quasiparticle velocity [43], v_{ℓ}^* , instead of the bare Dirac velocity v . Anyway, a treatment of the impact of intralayer interactions on the Coulomb drag transresistivity is well beyond the scope of this paper.

4. The Kubo-formula approach to calculating the drag conductivity

The Coulomb drag starts at second order in a perturbative expansion for the ‘drag conductivity’ σ_D in powers of the interlayer Coulomb interaction. To avoid infrared pathologies (stemming from the long-range nature of the Coulomb interaction) of certain integrals that appear in the theory, Coulomb interactions must be screened: from now on, we will work with the dynamically screened interaction U_{12} introduced in equation (20) rather than with the bare potential V_{12} in equation (18).

The two Aslamazov–Larkin-type diagrams [44] that contribute to σ_D up to second order in U_{12} are depicted in figure 2. In this figure, solid lines represent disordered propagators in the sublattice-pseudospin representation, while wavy lines represent the screened Coulomb interaction U_{12} . Finally, the vertices (black dots in the triangular diagrams) are current operators, one in each layer, evaluated at $\mathbf{q} = \mathbf{0}$ from the very beginning since we are interested in the drag conductivity in the uniform limit. Both current operators can be taken along the \hat{x} -direction since we are interested in the current response of the ‘passive’ layer in the \hat{x} -direction to an electric field applied in the same direction in the ‘active’ layer, i.e. we are interested in the *longitudinal* drag conductivity.

Straightforward algebraic manipulations lead to the following compact expression for the sum of the two diagrams in figure 2:

$$\begin{aligned} \sigma_D(i\Omega_m) &= \frac{e^2}{2\Omega_m} \int \frac{d^2\mathbf{q}}{(2\pi)^2} \frac{1}{\beta} \sum_{\omega_n} U_{12}(\mathbf{q}, i\omega_n) U_{12}(\mathbf{q}, i\omega_n + i\Omega_m) \\ &\quad \times \Gamma_1(\mathbf{q}, i\omega_n + i\Omega_m, i\omega_n) \Gamma_2(\mathbf{q}, i\omega_n, i\omega_n + i\Omega_m), \end{aligned} \quad (24)$$

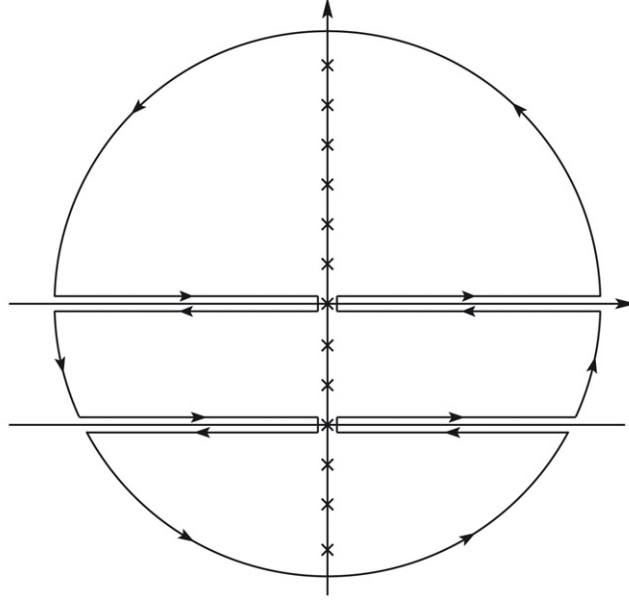


Figure 3. The contour of integration \mathcal{C} chosen to carry out the Matsubara sum in equation (27). Note the branch cuts on the real axis and in the lower half of the complex plane at $\Im m(z) = -i\Omega_m$. The crosses denote the poles of the Bose–Einstein thermal factor $n_B(z)$.

where $\beta = (k_B T)^{-1}$ is the usual thermal factor, Ω_m and ω_n are the *bosonic* Matsubara frequencies and

$$\Gamma_\ell(\mathbf{q}, i\omega_1, i\omega_2) = \frac{vN_{f,\ell}}{\beta} \sum_{\varepsilon_n} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \text{Tr} \left[G_\ell(\mathbf{k}, i\varepsilon_n) G_\ell(\mathbf{k} + \mathbf{q}, i\varepsilon_n + i\omega_2) \sigma^x G_\ell(\mathbf{k} + \mathbf{q}, i\varepsilon_n + i\omega_1) \right. \\ \left. + G_\ell(\mathbf{k}, i\varepsilon_n) G_\ell(\mathbf{k} - \mathbf{q}, i\varepsilon_n - i\omega_1) \sigma^x G_\ell(\mathbf{k} - \mathbf{q}, i\varepsilon_n - i\omega_2) \right] \quad (25)$$

is the so-called ‘non-linear susceptibility’. Here ε_n is a *fermionic* Matsubara frequency, the index $\ell = 1, 2$ refers to the layer degree of freedom and the symbol $\text{Tr}[\dots]$ denotes a trace over sublattice-pseudospin indices.

We now need to perform the analytic continuation $i\Omega_m \rightarrow \Omega + i0^+$ in equation (24) and then take the dc $\Omega \rightarrow 0$ limit. It is very well known that the analytic continuation must be carried out *after* performing the sum over the Matsubara frequencies ω_n . Let us assume that $U_{12}(q, z)$, seen as a function of the complex frequency z , is *analytic*, i.e. that it has *no* branch cuts. Under this assumption, the function $\Gamma_\ell(\mathbf{q}, z, z')$ has *two* branch cuts when z or z' are on the real axis, but is analytic elsewhere, including the points $z = 0$ and $z' = 0$. We then introduce the function

$$f(\mathbf{q}, z, z') = U_{12}(q, z) U_{12}(q, z') \Gamma_1(\mathbf{q}, z', z) \Gamma_2(\mathbf{q}, z, z'), \quad (26)$$

and perform the Matsubara sum in equation (24) using a standard procedure:

$$\mathcal{J}(i\Omega_m) \equiv \frac{1}{\beta} \sum_{\omega_n} f(\mathbf{q}, i\omega_n, i\Omega_m + i\omega_n) = \oint_{\mathcal{C}} \frac{dz}{2\pi i} n_B(z) f(\mathbf{q}, z, i\Omega_m + z) \quad (27)$$

where $n_B(z) = (e^{\beta z} - 1)^{-1}$ is the Bose–Einstein occupation factor on the complex plane and \mathcal{C} is an appropriate contour which excludes the branch cuts of the integrand (see figure 3), located

at $\Im m(z) = 0, -i\Omega_m$. Note that the contour \mathcal{C} as defined in figure 3 includes the points $z = 0$ and $z = -i\Omega_m$, where the integrand is analytic. We obtain

$$\begin{aligned} \mathcal{J}(i\Omega_m) = \mathcal{P} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} n_B(\omega) [f(\mathbf{q}, \omega^+, i\Omega_m + \omega) - f(i\Omega_m, \omega^-, i\Omega_m + \omega) \\ + f(i\Omega_m, \omega - i\Omega_m, \omega^+) - f(i\Omega_m, \omega - i\Omega_m, \omega^-)], \end{aligned} \quad (28)$$

since only the integrals around the branch cuts contribute to $\mathcal{J}(i\Omega_m)$, while the integral over the circle vanishes when its radius is sent to infinity. Here $\omega^\pm = \omega \pm i0^+$ and \mathcal{P} stands for the Cauchy principal value, i.e. the point $\omega = 0$ is excluded from the integration.

We are now ready to perform the analytical continuation to real frequencies. After some algebra we find that

$$\begin{aligned} \mathcal{J}(\Omega + i0^+) = \mathcal{P} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \left\{ [n_B(\Omega + \omega) - n_B(\omega)] f(\mathbf{q}, \omega^-, \Omega + \omega^+) \right. \\ \left. + n_B(\omega) f(\mathbf{q}, \omega^+, \Omega + \omega^+) - n_B(\Omega + \omega) f(\mathbf{q}, \omega^-, \Omega + \omega^-) \right\}. \end{aligned} \quad (29)$$

We finally take the dc limit $\Omega \rightarrow 0$ in equation (29) and obtain

$$\lim_{\Omega \rightarrow 0} \mathcal{J}(\Omega + i0^+) \rightarrow \Omega \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{\partial n_B(\omega)}{\partial \omega} f(\mathbf{q}, \omega^-, \omega^+), \quad (30)$$

since the terms in the last line of equation (29) vanish at least like Ω^2 . Thus the drag conductivity in the dc limit reads [6]

$$\sigma_D = \frac{\beta e^2}{16\pi} \int \frac{d^2\mathbf{q}}{(2\pi)^2} \int_{-\infty}^{+\infty} d\omega \frac{|U_{12}(q, \omega)|^2}{\sinh^2(\beta\omega/2)} \Gamma_1(\mathbf{q}, \omega^+, \omega^-) \Gamma_2(\mathbf{q}, \omega^-, \omega^+). \quad (31)$$

This is the central result of this section and was first obtained in the context of Coulomb drag between ordinary 2D parabolic-band electron gases—see, e.g., [6].

4.1. The nonlinear susceptibility

Let us go back to the definition of the nonlinear susceptibility Γ_ℓ given in equation (25). We introduce the following definition:

$$g_\ell(\mathbf{q}, i\varepsilon, i\varepsilon', i\varepsilon'') = v N_{f,\ell} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \text{Tr} [G_\ell(\mathbf{k}, i\varepsilon) G_\ell(\mathbf{k} + \mathbf{q}, i\varepsilon') \sigma^x G_\ell(\mathbf{k} + \mathbf{q}, i\varepsilon'')], \quad (32)$$

so that equation (25) reads

$$\begin{aligned} \Gamma_\ell(\mathbf{q}, i\omega_1, i\omega_2) = \frac{1}{\beta} \sum_{\varepsilon_n} [g_\ell(\mathbf{q}, i\varepsilon_n, i\varepsilon_n + i\omega_2, i\varepsilon_n + i\omega_1) + g_\ell(\mathbf{q}, i\varepsilon_n, i\varepsilon_n - i\omega_1, i\varepsilon_n - i\omega_2)] \\ \equiv \mathcal{J}_g(\mathbf{q}, i\omega_1, i\omega_2) + \mathcal{J}_g(-\mathbf{q}, -i\omega_2, -i\omega_1). \end{aligned} \quad (33)$$

We here remind the reader that ω_1 and ω_2 (ε_n) are bosonic (fermionic) Matsubara frequencies. Let us now concentrate on the first term on the rhs of equation (33), i.e. on $\mathcal{J}_g(\mathbf{q}, i\omega_1, i\omega_2)$. For the sake of simplicity, in what follows we omit to indicate explicitly the \mathbf{q} -dependence of the functions $g_\ell(\mathbf{q}, i\varepsilon, i\varepsilon', i\varepsilon'')$ and $\mathcal{J}_g(\mathbf{q}, i\omega_1, i\omega_2)$ every time we write equations that involve them. The complete notation will be restored only at the end of the mathematical manipulations that we perform on equation (33).

To begin with, we perform the fermionic Matsubara sum in the first term on the rhs of equation (33) by the standard procedure:

$$\begin{aligned}\mathcal{J}_g(i\omega_1, i\omega_2) &= \frac{1}{\beta} \sum_{\varepsilon_n} g_\ell(i\varepsilon_n, i\varepsilon_n + i\omega_2, i\varepsilon_n + i\omega_1) \\ &= \oint_{\mathcal{C}} \frac{dz}{2\pi i} n_F(z) g_\ell(z, z + i\omega_2, z + i\omega_1),\end{aligned}\quad (34)$$

where $n_F(z) = (e^{\beta z} + 1)^{-1}$ is the Fermi–Dirac occupation factor on the complex plane. The function $g(z, z + i\omega_2, z + i\omega_1)$ has three branch cuts in the complex plane located at $\Im m(z) = 0, -i\omega_1, -i\omega_2$. Choosing a suitable contour of integration \mathcal{C} , which encircles only the poles of $n_F(z)$ and leaves outside the branch cuts (in complete analogy with the contour drawn in figure 3), we find that

$$\begin{aligned}\mathcal{J}_g(i\omega_1, i\omega_2) &= \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi i} n_F(\varepsilon) [g_\ell(\varepsilon^+, \varepsilon + i\omega_2, \varepsilon + i\omega_1) - g_\ell(\varepsilon^-, \varepsilon + i\omega_2, \varepsilon + i\omega_1) \\ &\quad + g_\ell(\varepsilon - i\omega_1, \varepsilon^+ + i\omega_2 - i\omega_1, \varepsilon^+) - g_\ell(\varepsilon - i\omega_1, \varepsilon^- + i\omega_2 - i\omega_1, \varepsilon^-) \\ &\quad + g_\ell(\varepsilon - i\omega_2, \varepsilon^+, \varepsilon^+ + i\omega_1 - i\omega_2) - g_\ell(\varepsilon - i\omega_2, \varepsilon^-, \varepsilon^- + i\omega_1 - i\omega_2)].\end{aligned}\quad (35)$$

Here $\varepsilon^\pm = \varepsilon \pm i0^+$ and we used that $n_F(\varepsilon + i\omega_m) = n_F(\varepsilon)$ if ω_m is a bosonic Matsubara frequency. Note that we are allowed to remove the index \pm from ε^\pm when it is summed to $i\omega_1$ or $i\omega_2$ (since they will be analytically continued to ω^\pm) but not when it is summed to $i\omega_1 - i\omega_2$. Analytically continuing $i\omega_1 \rightarrow \omega_1^+$ and $i\omega_2 \rightarrow \omega_2^-$ (to obtain, e.g., $\Gamma_1(\mathbf{q}, \omega^+, \omega^-)$ in the integrand on the rhs of equation (31)), we finally obtain

$$\begin{aligned}\mathcal{J}_g(\omega_1^+, \omega_2^-) &= \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi i} n_F(\varepsilon) [g_\ell(\varepsilon^+, \varepsilon^- + \omega_2, \varepsilon^+ + \omega_1) - g_\ell(\varepsilon^-, \varepsilon^- + \omega_2, \varepsilon^+ + \omega_1) \\ &\quad + g_\ell(\varepsilon^- - \omega_1, \varepsilon^- + \omega_2 - \omega_1, \varepsilon^+) - g_\ell(\varepsilon^- - \omega_1, \varepsilon^- + \omega_2 - \omega_1, \varepsilon^-) \\ &\quad + g_\ell(\varepsilon^+ - \omega_2, \varepsilon^+, \varepsilon^+ + \omega_1 - \omega_2) - g_\ell(\varepsilon^+ - \omega_2, \varepsilon^-, \varepsilon^- + \omega_1 - \omega_2)].\end{aligned}\quad (36)$$

We observe that the second term on the third line (containing only ε^-) and the first term on the fourth line (containing only ε^+) of equation (36) are exactly zero since they involve products of three Green's functions with poles on the same half of the complex plane. After some algebraic manipulations, we obtain

$$\begin{aligned}\mathcal{J}_g(\omega_1^+, \omega_2^-) &= \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi i} \{ [n_F(\varepsilon) - n_F(\varepsilon + \omega_2)] g_\ell(\varepsilon^+, \varepsilon^- + \omega_2, \varepsilon^+ + \omega_1) \\ &\quad + [n_F(\varepsilon + \omega_1) - n_F(\varepsilon)] g_\ell(\varepsilon^-, \varepsilon^- + \omega_2, \varepsilon^+ + \omega_1) \}.\end{aligned}\quad (37)$$

We now take the limit $\omega_1, \omega_2 \rightarrow \omega$ and recast equation (34) in the following form:

$$\begin{aligned}\mathcal{J}_g(\omega^+, \omega^-) &= v N_{f,\ell} \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi i} [n_F(\varepsilon + \omega) - n_F(\varepsilon)] \\ &\quad \times \int \frac{d^2\mathbf{k}}{(2\pi)^2} \text{Tr}\{ [G_\ell^A(\mathbf{k}, \varepsilon) - G_\ell^R(\mathbf{k}, \varepsilon)] G_\ell^A(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) \sigma^x G_\ell^R(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) \},\end{aligned}\quad (38)$$

where we have introduced the retarded (advanced) Green's function $G_\ell^{R(A)}(\mathbf{k}, \varepsilon)$.

The second term on the rhs of the last line of equation (33) can be treated in an analogous manner. The final expression for the nonlinear susceptibility reads

$$\begin{aligned} \Gamma_\ell(\mathbf{q}, \omega^+, \omega^-) &= vN_{f,\ell} \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi i} [n_F(\varepsilon + \omega) - n_F(\varepsilon)] \\ &\times \int \frac{d^2\mathbf{k}}{(2\pi)^2} \text{Tr}\{[G_\ell^A(\mathbf{k}, \varepsilon) - G_\ell^R(\mathbf{k}, \varepsilon)]G_\ell^A(\mathbf{k} + \mathbf{q}, \varepsilon + \omega)\sigma^x \\ &\times G_\ell^R(\mathbf{k} + \mathbf{q}, \varepsilon + \omega)\} + \{(\mathbf{q}, \omega) \rightarrow (-\mathbf{q}, -\omega)\}. \end{aligned} \quad (39)$$

Equation (39) together with equation (31) represent the most important results of this section and are the starting point for the calculation of the drag transresistivity ρ_D .

5. The Boltzmann transport limit

Equations (31) and (39) need to be simplified for practical purposes and for a quantitative estimate of the Coulomb-drag transresistivity.

We first switch from the non-diagonal sublattice-pseudospin representation of the Green's functions to the diagonal representation. In the latter representation equation (39) reads

$$\begin{aligned} \Gamma_\ell(\mathbf{q}, \omega^+, \omega^-) &= vN_{f,\ell} \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi i} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \sum_{\lambda, \lambda', \lambda''} \left\{ [n_F(\varepsilon + \omega) - n_F(\varepsilon)] \right. \\ &\times [\mathcal{G}_{\ell, \lambda}^A(\mathbf{k}, \varepsilon) - \mathcal{G}_{\ell, \lambda}^R(\mathbf{k}, \varepsilon)] \mathcal{G}_{\ell, \lambda'}^A(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) \mathcal{G}_{\ell, \lambda''}^R(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) \\ &\times \rho_{k\lambda, k+q\lambda'} \sigma_{k+q\lambda', k+q\lambda''}^x \rho_{k+q\lambda'', k\lambda} \left. \right\} + \{(\mathbf{q}, \omega) \rightarrow (-\mathbf{q}, -\omega)\}, \end{aligned} \quad (40)$$

where we used the definitions in equations (14) and (15) for the density and current vertices.

We now show that the terms with $\lambda' \neq \lambda''$ do *not* contribute to $\Gamma_\ell(\mathbf{q}, \omega^+, \omega^-)$. Let us indeed consider all the terms in the triple sum in equation (40) in which $\lambda'' = -\lambda' \equiv \bar{\lambda}'$. We first recall that $\sigma_{k+q\lambda', k+q\bar{\lambda}'}^x = i\lambda' \sin(\varphi_{k+q})$ and

$$\rho_{k\lambda, k+q\lambda'} \rho_{k+q\bar{\lambda}', k\lambda} = \frac{i\lambda\lambda'}{2} \sin(\varphi_k - \varphi_{k+q}). \quad (41)$$

We then consider the last term on the third line of equation (40) and perform the following changes of variables: $\mathbf{k} \rightarrow -\mathbf{k}$ and $\varepsilon \rightarrow -\varepsilon$. Using the fact that $\sin(\varphi_{-k-q}) = -\sin(\varphi_{k+q})$ and $\sin(\varphi_{-k} - \varphi_{-k-q}) = \sin(\varphi_k - \varphi_{k+q})$, we can write the off-diagonal (OD) contribution ($\lambda'' = -\lambda'$) to $\Gamma_\ell(\mathbf{q}, \omega^+, \omega^-)$ as

$$\begin{aligned} \Gamma_\ell^{\text{OD}}(\mathbf{q}, \omega^+, \omega^-) &= vN_{f,\ell} \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi i} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \sum_{\lambda, \lambda'} \left(-\frac{\lambda}{2} \right) \left\{ [n_F(\varepsilon + \omega) + n_F(-\varepsilon - \omega) - n_F(\varepsilon) \right. \\ &- n_F(-\varepsilon)] [\mathcal{G}_{\ell, \lambda}^A(\mathbf{k}, \varepsilon) - \mathcal{G}_{\ell, \lambda}^R(\mathbf{k}, \varepsilon)] \mathcal{G}_{\ell, \lambda'}^A(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) \mathcal{G}_{\ell, \bar{\lambda}'}^R(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) \\ &\times \sin(\varphi_k - \varphi_{k+q}) \sin(\varphi_{k+q}) \left. \right\}. \end{aligned} \quad (42)$$

Since $n_F(z) + n_F(-z) = 1$, the term in square brackets on the second line of the previous equation is identically zero. The OD contribution to $\Gamma_\ell(\mathbf{q}, \omega^+, \omega^-)$ thus vanishes. From now on, we can set $\lambda'' = \lambda'$ in equation (40) and sum only over λ and λ' .

We can further simplify equation (40) by assuming that, in the presence of weak disorder, the Green's function in the ℓ th layer can be well approximated by the expression

$$\mathcal{G}_{\ell,\lambda}^{\text{R(A)}}(\mathbf{k}, \omega) \approx \left[\omega - \xi_{\mathbf{k},\lambda}^{(\ell)} \pm \frac{i}{2\tau_{\ell}(\mathbf{k})} \right]^{-1}, \quad (43)$$

where $\tau_{\ell}(\mathbf{k})$ represents a momentum-dependent scattering time in the ℓ th layer and $\xi_{\mathbf{k},\lambda}^{(\ell)} \equiv \lambda v|\mathbf{k}| - \mu_{\ell}$ are Dirac-band energies measured from the chemical potential μ_{ℓ} of the ℓ th layer.

Note that here $\tau_{\ell}(\mathbf{k})$ should not be interpreted as a one-particle scattering time, but rather as the *transport scattering time* [7]. This statement can be justified within the Kubo formalism by including disorder-related vertex corrections or, in a much more transparent way, by using the BT equation, see [appendix](#).

Below, we will assume that the transport scattering time $\tau_{\ell}(\mathbf{k})$ is isotropic, i.e. that it depends only on $k = |\mathbf{k}|$, but we allow different scattering times in the two layers. The specific functional dependence of τ_{ℓ} on k depends on a particular model of intralayer impurity scattering. In the simple case of a momentum-independent scattering time, we recover the usual ‘relaxation time approximation’. A scattering time which depends linearly on momentum, $\tau_{\ell}(\mathbf{k}) \propto k$ is a very popular model in the graphene literature. Early on, it was understood [45] that such a functional dependence of τ_{ℓ} on k is needed to explain the linear-in-carrier-density dc conductivities that are experimentally measured in samples on dielectric substrates such as SiO₂ (and is typically attributed to charged impurities located close to the graphene sheet). Below, we will not assume any specific functional dependence of τ_{ℓ} on k . Our low-temperature analytical results for the Coulomb-drag transresistivity do *not* depend on the particular scattering model that one chooses.

At this point, it is useful to introduce the one-particle spectral function corresponding to equation (43),

$$\begin{aligned} \mathcal{A}_{\ell,\lambda}(\mathbf{k}, \omega) &= -\frac{1}{\pi} \Im m \mathcal{G}_{\ell,\lambda}^{\text{R}}(\mathbf{k}, \omega) \\ &= i \left[\mathcal{G}_{\ell,\lambda}^{\text{A}}(\mathbf{k}, \omega) - \mathcal{G}_{\ell,\lambda}^{\text{R}}(\mathbf{k}, \omega) \right], \end{aligned} \quad (44)$$

and the identity

$$\mathcal{G}_{\ell,\lambda}^{\text{R}}(\mathbf{k}, \omega) \mathcal{G}_{\ell,\lambda}^{\text{A}}(\mathbf{k}, \omega) = \tau_{\ell}(\mathbf{k}) \mathcal{A}_{\ell,\lambda}(\mathbf{k}, \omega). \quad (45)$$

Using these definitions in equation (40) we obtain the following approximate expression for the nonlinear susceptibility:

$$\begin{aligned} \Gamma_{\ell}(\mathbf{q}, \omega^+, \omega^-) &\approx v N_{\text{f},\ell} \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi i} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \sum_{\lambda,\lambda'} \left\{ [n_{\text{F}}(\varepsilon + \omega) - n_{\text{F}}(\varepsilon)] i \tau_{\ell}(\mathbf{k} + \mathbf{q}) \right. \\ &\quad \times \mathcal{A}_{\ell,\lambda}(\mathbf{k}, \varepsilon) \mathcal{A}_{\ell,\lambda'}(\mathbf{k} + \mathbf{q}, \varepsilon + \omega) \frac{1 + \lambda\lambda' \cos(\varphi_{\mathbf{k}} - \varphi_{\mathbf{k}+\mathbf{q}})}{2} \\ &\quad \left. \times \sigma_{\mathbf{k}+\mathbf{q}\lambda', \mathbf{k}+\mathbf{q}, \lambda'}^x \right\} + \left\{ (\mathbf{q}, \omega) \rightarrow (-\mathbf{q}, -\omega) \right\}. \end{aligned} \quad (46)$$

From now on, we will introduce the simplified notation $J_{\mathbf{k},\lambda}^x \equiv v \sigma_{\mathbf{k}\lambda, \mathbf{k}\lambda}^x = v \lambda \cos(\varphi_{\mathbf{k}})$, where in the last equality we have used equation (15). The definition of $J_{\mathbf{k},\lambda}^x$ should not be confused with the definition of the current-density operator $\hat{J}_{\mathbf{q},\ell}^i$ in second quantization given in equation (10).

In this paper, we are interested in the limit in which intralayer scattering is weak, which is, for example, the most relevant regime for high-quality DLG samples. In this limit, we can approximate the spectral functions in equation (46) with δ functions,

$$\mathcal{A}_{\ell,\lambda}(\mathbf{k}, \varepsilon) \approx \delta(\varepsilon - \xi_{\mathbf{k},\lambda}^{(\ell)}). \quad (47)$$

Straightforward algebraic manipulations of equation (46) with the use of equation (47) yield the following BT expression for the nonlinear susceptibility:

$$\begin{aligned} \Gamma_{\ell}^{\text{BT}}(\mathbf{q}, \omega^+, \omega^-) \equiv \Gamma_{\ell}^{\text{BT}}(\mathbf{q}, \omega) &= N_{\text{f},\ell} \sum_{\lambda,\lambda'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} [\tau_{\ell}(\mathbf{k} + \mathbf{q}) J_{\mathbf{k}+\mathbf{q},\lambda'}^x - \tau_{\ell}(\mathbf{k}) J_{\mathbf{k},\lambda}^x] \\ &\times \Im m \left\{ \frac{n_{\text{F}}(\xi_{\mathbf{k},\lambda}^{(\ell)}) - n_{\text{F}}(\xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)})}{\omega + \xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)} + i0^+} [1 + \lambda\lambda' \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}})] \right\}. \end{aligned} \quad (48)$$

We have dubbed equation (48) as the ‘BT expression’ for the nonlinear susceptibility since after inserting it into equation (31) one obtains precisely an expression for the Coulomb-drag conductivity that can be derived within a Boltzmann-equation approach. This will be shown in the [appendix](#).

Note that, in the usual ‘relaxation-time approximation’ (τ_{ℓ} independent of k), the BT expression for the nonlinear susceptibility is simply proportional to the intralayer transport scattering time, $\Gamma_{\ell}^{\text{BT}}(\mathbf{q}, \omega) \propto \tau_{\ell}$.

5.1. Coulomb drag transresistivity in the Boltzmann transport limit

We now focus our attention on the quantity that is actually measured in experiments, i.e. the drag transresistivity ρ_{D} , which is precisely the ratio between the voltage drop in the passive layer and the current in the active layer. This quantity can be easily found by inverting the 2×2 conductivity matrix,

$$\rho_{\text{D}} = -\sigma_{\text{D}} \frac{1}{\det \begin{pmatrix} \sigma_1 & \sigma_{\text{D}} \\ \sigma_{\text{D}} & \sigma_2 \end{pmatrix}} \approx -\frac{\sigma_{\text{D}}}{\sigma_1 \sigma_2}, \quad (49)$$

where σ_{ℓ} is the intralayer conductivity and the last approximation in equation (49) holds true only if $\sigma_{\text{D}} \ll \sigma_{\ell}$. Within BT theory the intralayer conductivity at finite temperature is given by [46]

$$\sigma_{\ell} = \frac{e^2}{2} v^2 N_{\text{f},\ell} \sum_{\lambda} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \tau_{\ell}(\mathbf{k}) \left[-\frac{\partial n_{\text{F}}(\xi_{\mathbf{k},\lambda}^{(\ell)})}{\partial \xi_{\mathbf{k},\lambda}^{(\ell)}} \right], \quad (50)$$

while the drag conductivity σ_{D} is given by equation (31) with the expression (48) for the nonlinear susceptibility. The final expression for the BT drag resistivity reads

$$\rho_{\text{D}}^{\text{BT}} = -\frac{\beta e^2}{16\pi \sigma_1 \sigma_2} \int \frac{d^2\mathbf{q}}{(2\pi)^2} \int_{-\infty}^{+\infty} d\omega \frac{|U_{12}(\mathbf{q}, \omega)|^2}{\sinh^2(\beta\omega/2)} \Gamma_1^{\text{BT}}(\mathbf{q}, \omega) \Gamma_2^{\text{BT}}(\mathbf{q}, \omega). \quad (51)$$

Note that in the relaxation-time approximation the BT drag transresistivity does not depend on the transport scattering times in the two layers (even if these are different), since, as noted earlier in section 5, $\Gamma_{\ell}^{\text{BT}}$ is proportional to τ_{ℓ} in this approximation and so is σ_{ℓ} .

The situation turns out to be much more complicated in the case when τ_{ℓ} is an arbitrary function of $k = |\mathbf{k}|$. A full numerical treatment of the 3D integral in equation (51) at any

finite temperature is beyond the scope of this work and will be the subject of a forthcoming publication.

In the next section, however, we demonstrate that a dramatic simplification occurs in the low-temperature limit. We will indeed prove that when $k_B T$ is much smaller than the Fermi energies in the two layers, $k_B T \ll \min_\ell(\varepsilon_{F,\ell})$, ρ_D^{BT} is *insensitive* to the precise functional dependence $\tau_\ell = \tau_\ell(k)$ of the intralayer scattering time. Since for typical values of doping the Fermi energy in a graphene sheet is very large (≈ 1400 K for a carrier density of the order of 10^{12} cm^{-2}), analytical results in the low-temperature limit are very useful when both layers are sufficiently away from the charge neutrality point.

In the intriguing regime in which one layer (both layers) lies (lie) at the charge neutrality point, BT theory is inapplicable. Coulomb drag in this regime is dominated by unavoidable electron–hole puddles [47] and is certainly very interesting but well beyond the scope of this paper. For reasons of symmetry, lowest-order BT theory applied to the regime in which one of the two layers is at the neutrality point gives $\rho_D^{\text{BT}} = 0$ [26, 27].

6. The low-temperature limit

In this section, we use equation (51) to calculate ρ_D^{BT} *analytically* in the low-temperature limit, for arbitrary values of the dielectric constants ε_i in figure 1, and for a generic scattering time $\tau_\ell(k)$.

As already anticipated above, the low-temperature regime is readily identified by the inequality $k_B T \ll \min_\ell(\varepsilon_{F,\ell})$. In this limit we can

- (i) use the low-temperature expression for the intralayer conductivity [45, 46]

$$\lim_{T \rightarrow 0} \sigma_\ell = \frac{e^2}{4\pi} N_{f,\ell} \varepsilon_{F,\ell} \tau_\ell(k_{F,\ell}); \quad (52)$$

- (ii) evaluate the nonlinear BT susceptibilities $\Gamma_1^{\text{BT}}(\mathbf{q}, \omega)$ and $\Gamma_2^{\text{BT}}(\mathbf{q}, \omega)$ at zero temperature and only to lowest order in ω in the low-frequency $\omega/[2 \min_\ell(\varepsilon_{F,\ell})] \rightarrow 0$ limit⁷; and
 (iii) replace the dynamically screened interlayer interaction $U_{12}(q, \omega)$ with the much simpler statically screened interaction $U_{12}(q, 0)$.

Indeed, the thermal factor $\sinh^{-2}(\beta\omega/2)$ in equation (51) represents an effective cutoff on the values of ω in the integral in equation (51): it must be $\omega \ll 2/\beta \ll 2 \min_\ell(\varepsilon_{F,\ell})$.

To find the asymptotic behavior of $\Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega)$ in the limit $\omega \rightarrow 0$, we first re-write equation (48) in the form

$$\begin{aligned} \Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega) = & -\pi N_{f,\ell} \sum_{\lambda,\lambda'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} [\tau_\ell(\mathbf{k} + \mathbf{q}) J_{\mathbf{k}+\mathbf{q},\lambda'}^x - \tau_\ell(\mathbf{k}) J_{\mathbf{k},\lambda}^x] [n_F(\xi_{\mathbf{k},\lambda}^{(\ell)}) - n_F(\xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)})] \\ & \times \delta(\omega + \xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)}) [1 + \lambda\lambda' \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}})]. \end{aligned} \quad (53)$$

⁷ In principle, the nonlinear BT susceptibility for an intralayer scattering time that depends on k can diverge at $\omega = vq$. This behavior originates from collinear scattering (\mathbf{k} parallel or antiparallel to $\mathbf{k} + \mathbf{q}$) and has been studied by several authors in other contexts [48–51]. Dynamical screening in equation (51) completely cures these divergencies [51] however, yielding a finite *subleading* contribution to the drag transresistivity in the low-temperature limit.

Clearly, $\Gamma_\ell^{\text{BT}}(\mathbf{q}, 0) = 0$. We thus need to calculate the derivative of $\Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega)$ with respect to ω at $\omega = 0$ and in the zero-temperature limit. We find that

$$\begin{aligned} \left. \frac{\partial \Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega)}{\partial \omega} \right|_{\omega=0} &= -\pi N_{\text{f},\ell} \sum_{\lambda, \lambda'} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \left. \frac{\partial [n_{\text{F}}(\omega + \xi_{\mathbf{k},\lambda}^{(\ell)})]}{\partial \omega} \right|_{\omega=0} \delta(\xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)}) \\ &\quad \times [\tau_\ell(\mathbf{k} + \mathbf{q}) J_{\mathbf{k}+\mathbf{q},\lambda'}^x - \tau_\ell(\mathbf{k}) J_{\mathbf{k},\lambda}^x] [1 + \lambda \lambda' \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}})] \\ &\stackrel{T \rightarrow 0}{=} -\pi N_{\text{f},\ell} \sum_{\lambda, \lambda'} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} [\tau_\ell(\mathbf{k} + \mathbf{q}) J_{\mathbf{k}+\mathbf{q},\lambda'}^x - \tau_\ell(\mathbf{k}) J_{\mathbf{k},\lambda}^x] \\ &\quad \times \delta(\xi_{\mathbf{k},\lambda}^{(\ell)}) \delta(\xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)}) [1 + \lambda \lambda' \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}})], \end{aligned} \quad (54)$$

where the last equality is valid only in the zero-temperature limit. More explicitly, we find that

$$\begin{aligned} \lim_{T \rightarrow 0} \Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega \rightarrow 0) &= -\pi \omega N_{\text{f},\ell} \sum_{\lambda, \lambda'} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} [\tau_\ell(\mathbf{k} + \mathbf{q}) J_{\mathbf{k}+\mathbf{q},\lambda'}^x - \tau_\ell(\mathbf{k}) J_{\mathbf{k},\lambda}^x] \\ &\quad \times \delta(\xi_{\mathbf{k},\lambda}^{(\ell)}) \delta(\xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)}) [1 + \lambda \lambda' \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}})]. \end{aligned} \quad (55)$$

Equation (55) is one of the most important results of this section. An explicit expression for $\lim_{T \rightarrow 0} \Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega \rightarrow 0)$ will be given below in equation (58). Note that the two δ -functions in the above expression impose that only intraband excitations ($\lambda = \lambda' = +1$) contribute to $\Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega \rightarrow 0)$ in the low-temperature limit. Moreover, the two δ -functions together pin the absolute values of \mathbf{k} and $\mathbf{k} + \mathbf{q}$ to be equal to $k_{\text{F},\ell}$. This implies that in the low-temperature limit the transport scattering times, $\tau_\ell(\mathbf{k} + \mathbf{q})$ and $\tau_\ell(\mathbf{k})$ inside the square brackets in equation (55) must both be evaluated on the Fermi surface of the ℓ th layer, i.e. $|\mathbf{k} + \mathbf{q}| = |\mathbf{k}| = k_{\text{F},\ell}$. Since $\tau_\ell(\mathbf{k})$ depends *only* on the absolute value of its argument (and not on the polar angle of \mathbf{k}), *both $\tau_\ell(\mathbf{k} + \mathbf{q})$ and $\tau_\ell(\mathbf{k})$ can be factorized out of the integral in equation (55)*. This is precisely the reason why at the end of section 5.1 we claimed that in the low-temperature limit $\rho_{\text{D}}^{\text{BT}}$ is completely insensitive to the precise intralayer scattering mechanism.

More mathematically, we have

$$\begin{aligned} \lim_{T \rightarrow 0} \Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega \rightarrow 0) &= -\pi \omega N_{\text{f},\ell} \tau_\ell(k_{\text{F},\ell}) \int_0^{+\infty} \frac{k dk}{2\pi} \delta(vk - \varepsilon_{\text{F},\ell}) \int_0^{2\pi} \frac{d\varphi_{\mathbf{k}}}{2\pi} [J_{\mathbf{k}+\mathbf{q},+}^x - J_{\mathbf{k},+}^x] \\ &\quad \times \delta(vk - v|\mathbf{k} + \mathbf{q}|) [1 + \cos(\varphi_{\mathbf{k}+\mathbf{q}} - \varphi_{\mathbf{k}})]. \end{aligned} \quad (56)$$

We now employ the following identity,

$$J_{\mathbf{k}+\mathbf{q},+}^x - J_{\mathbf{k},+}^x = \frac{q}{k_{\text{F},\ell}} \cos(\varphi_{\mathbf{q}}), \quad (57)$$

which can be easily derived by using the aforementioned condition $|\mathbf{k} + \mathbf{q}| = |\mathbf{k}| = k_{\text{F},\ell}$. The two integrals in equation (56) can be easily carried out analytically: we find that

$$\lim_{T \rightarrow 0} \Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega \rightarrow 0) = -N_{\text{F},\ell} \frac{\omega q_x \tau_\ell(k_{\text{F},\ell})}{2\pi v q} \Theta(2k_{\text{F},\ell} - q) \sqrt{1 - \frac{q^2}{4k_{\text{F},\ell}^2}}, \quad (58)$$

where we have used that $q_x = q \cos(\varphi_{\mathbf{q}})$. Equation (58) is one of the most important results of this section. We stress again that the applicability of this equation is not limited to the case of a momentum-independent scattering time. Rather, equation (58) applies to a generic intralayer scattering time $\tau_\ell = \tau_\ell(k)$. The physical meaning of this result is that in the low-temperature

limit the nonlinear susceptibility is determined by what happens close to the Fermi surface. What matters thus is only the magnitude of $\tau_\ell(k)$ evaluated at the Fermi momentum $k_{F,\ell}$ of the ℓ th layer. This conclusion is in agreement with Narozhny *et al* [31].

Using equation (58) and the following integral,

$$\int_{-\infty}^{+\infty} dx \frac{x^2}{\sinh^2(yx/2)} = \frac{8\pi^2}{3y^3}, \quad (59)$$

we finally arrive at the desired result for the low-temperature BT drag transresistivity:

$$\lim_{T \rightarrow 0} \rho_D^{\text{BT}} = -\frac{(k_B T)^2}{6e^2 \varepsilon_{F,1} \varepsilon_{F,2} v^2} \int_0^{q_{\max}} dq q |U_{12}(q, 0)|^2 \sqrt{1 - \frac{q^2}{4k_{F,1}^2}} \sqrt{1 - \frac{q^2}{4k_{F,2}^2}}, \quad (60)$$

where $q_{\max} \equiv \min(2k_{F,1}, 2k_{F,2})$. Once again, we stress that the result in equation (60) does *not* depend on the precise functional form of $\tau_\ell(k)$.

6.1. Dimensionless variables

It is now useful to introduce dimensionless variables. We scale the wave number q in equation (60) with $\sqrt{k_{F,1}k_{F,2}}$ by introducing $x = q/\sqrt{k_{F,1}k_{F,2}}$ and the effective interaction U_{12} with $e^2/\sqrt{k_{F,1}k_{F,2}}$ by introducing $\bar{U}_{12} = U_{12}\sqrt{k_{F,1}k_{F,2}}/e^2$.

In these reduced units, equation (60) reads (for physical reasons we restore Planck's constant from now on)

$$\lim_{T \rightarrow 0} \rho_D^{\text{BT}} = -\frac{h}{e^2} \frac{\alpha_{ee}^2 (k_B T)^2}{12\pi \varepsilon_{F,1} \varepsilon_{F,2}} \int_0^{x_{\max}} dx x |\bar{U}_{12}(x, 0)|^2 \sqrt{1 - \frac{x^2 k_{F,1}}{4 k_{F,2}}} \sqrt{1 - \frac{x^2 k_{F,2}}{4 k_{F,1}}}, \quad (61)$$

where $x_{\max} = \min(2\sqrt{k_{F,1}/k_{F,2}}, 2\sqrt{k_{F,2}/k_{F,1}})$. In equation (61), we have introduced the dimensionless coupling constant $\alpha_{ee} = e^2/(\hbar v)$, which has a value ≈ 2.2 in DLG and ≈ 4.4 in Bi_2Te_3 TIs if we use the respective Dirac velocities $v_G \approx 10^6 \text{ m s}^{-1}$ and $v_{\text{TI}} \approx 5 \times 10^5 \text{ m s}^{-1}$. Equation (61) is the most important result of this section.

To proceed further, we write in an explicit manner the statically screened dimensionless interlayer interaction $\bar{U}_{12}(x, 0)$:

$$\bar{U}_{12}(x, 0) = \frac{8\pi x f_{12}(x\xi)}{\mathcal{D}(x, \xi)}, \quad (62)$$

where

$$\begin{aligned} \mathcal{D}(x, \xi) \equiv & [x + 2N_{f,1}\alpha_{ee}\sqrt{k_{F,1}/k_{F,2}} f_{11}(x\xi)][x + 2N_{f,2}\alpha_{ee}\sqrt{k_{F,2}/k_{F,1}} f_{22}(x\xi)] \\ & - 16N_{f,1}N_{f,2}\alpha_{ee}^2 f_{12}^2(x\xi). \end{aligned} \quad (63)$$

In equations (62) and (63), we have introduced the crucially important dimensionless parameter

$$\xi \equiv d\sqrt{k_{F,1}k_{F,2}} \quad (64)$$

and the dimensionless form factors

$$f_{11}(x) = \frac{(\epsilon_2 + \epsilon_3) e^x + (\epsilon_2 - \epsilon_3) e^{-x}}{g(x)}, \quad (65)$$

$$f_{22}(x) = \frac{(\epsilon_2 + \epsilon_1) e^x + (\epsilon_2 - \epsilon_1) e^{-x}}{g(x)}, \quad (66)$$

$$f_{12}(x) = \frac{\epsilon_2}{g(x)} \quad (67)$$

with

$$g(x) = (\epsilon_1 + \epsilon_2)(\epsilon_2 + \epsilon_3)e^x + (\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)e^{-x}. \quad (68)$$

The only ingredient we have used in writing equation (62) is the well-known behavior of the static Lindhard function of a doped MD2DES for wave numbers q smaller than twice the Fermi momentum $k_{F,\ell}$:

$$\chi_\ell^{(0)}(q \leq 2k_{F,\ell}, 0) = -\frac{N_{f,\ell} k_{F,\ell}}{2\pi \hbar v}. \quad (69)$$

Two asymptotic limits can now be easily inferred from the general low-temperature theory in equations (61)–(68): (i) the weak-coupling (‘large interlayer distance’ and/or ‘high density’) limit, i.e. the limit in which $\xi \gg 1$, and (ii) the strong-coupling (‘small interlayer distance’ and/or ‘low density’) limit, i.e. the limit in which $\xi \ll 1$.

As discussed briefly above, the applicability of BT theory in the low-density regime is questionable: for the BT theory to remain valid, one should think of entering the strong-coupling regime by reducing the interlayer separation d while keeping fixed the factor $\sqrt{k_{F,1}k_{F,2}}$. Note also that, in the strong-coupling regime, diagrams of higher order with respect to those presented in figure 2 might become relevant. The applicability of RPA—equation (21)—is also questionable in this regime. Only a comparison with experimental results can tell us about the importance of these subtle issues of many-body theory.

6.2. The weak-coupling limit

In the limit $\xi \gg 1$, straightforward algebraic manipulations of equations (61)–(68) yield the following result for the low-temperature BT drag transresistivity:

$$\lim_{\xi \rightarrow \infty} \lim_{T \rightarrow 0} \rho_D^{\text{BT}} = -\frac{h}{e^2} \frac{\pi}{3} \frac{(k_B T)^2}{\epsilon_{F,1} \epsilon_{F,2} \xi^4 N_{f,1}^2 N_{f,2}^2 \alpha_{ee}^2} \int_0^{+\infty} dy \frac{y^3 f_{12}^2(y)}{[f_{11}(y) f_{22}(y) - 4f_{12}^2(y)]^2}, \quad (70)$$

where we have introduced a new reduced variable, $y = x\xi$. The quadrature with respect to the variable y can be carried out analytically yielding the following result:

$$\int_0^{+\infty} dy \frac{y^3 f_{12}^2(y)}{[f_{11}(y) f_{22}(y) - 4f_{12}^2(y)]^2} = \frac{1}{4} \epsilon_2^2 \int_0^{+\infty} dy \frac{y^3}{\sinh^2(y)} = \frac{3}{8} \zeta(3) \epsilon_2^2, \quad (71)$$

where $\zeta(x)$ is the Riemann zeta function and $\zeta(3) \approx 1.2$. Note that this quadrature does not depend either on ϵ_1 or on ϵ_3 .

In summary, we find that

$$\begin{aligned} \lim_{\xi \rightarrow \infty} \lim_{T \rightarrow 0} \rho_D^{\text{BT}} &= -\frac{h}{e^2} \frac{\pi \zeta(3)}{8} \frac{\epsilon_2^2}{d^4 \alpha_{ee}^2 N_{f,1}^2 N_{f,2}^2 k_{F,1}^2 k_{F,2}^2 \epsilon_{F,1} \epsilon_{F,2}} (k_B T)^2 \\ &\propto -\frac{h}{e^2} \epsilon_2^2 \frac{(k_B T)^2}{n_1^{3/2} n_2^{3/2} d^4}. \end{aligned} \quad (72)$$

The dependence of ρ_D^{BT} on the temperature, densities and interlayer distance displayed by equation (72) is in agreement with the results of Tse *et al* [26], Katsnelson [28] and Narozhny *et al* [31], even though the numerical prefactor we find is four times smaller (we remind the

reader that for DLG, $N_{f,1} = N_{f,2} = 4$). It is important to observe that the weak-coupling and low-temperature BT Coulomb-drag transresistivity is sensitive only to the dielectric constant ϵ_2 between the two MD2DESs (see figure 1).

6.3. The strong-coupling limit

In the limit $\xi \rightarrow 0$, we can expand all the functions $f_{ij}(x\xi)$ that appear in the effective interaction (62) in powers of their argument since x is bounded from above, $0 \leq x \leq x_{\max}$. Following this procedure we find the following asymptotic expression for $\bar{U}_{12}(x, 0)$ in the limit $\xi \rightarrow 0$:

$$\lim_{\xi \rightarrow 0} \bar{U}_{12}(x, 0) = \frac{4\pi}{(\epsilon_1 + \epsilon_3)x + 2 \frac{q_{\text{TF},1} + q_{\text{TF},2}}{\sqrt{k_{\text{F},1}k_{\text{F},2}}}}, \quad (73)$$

where we have introduced the Thomas–Fermi screening wave number $q_{\text{TF},\ell} = N_{f,\ell} \alpha_{\text{ee}} k_{\text{F},\ell}$. Using this result in equation (61), we find the final expression for the low-temperature BT drag transresistivity in the strong-coupling limit:

$$\begin{aligned} \lim_{\xi \rightarrow 0} \lim_{T \rightarrow 0} \rho_{\text{D}}^{\text{BT}} = & -\frac{h}{e^2} \frac{4\pi}{3} \alpha_{\text{ee}}^2 \frac{(k_{\text{B}}T)^2}{\epsilon_{\text{F},1}\epsilon_{\text{F},2}} \int_0^{x_{\max}} dx \sqrt{1 - \frac{x^2 k_{\text{F},1}}{4 k_{\text{F},2}}} \sqrt{1 - \frac{x^2 k_{\text{F},2}}{4 k_{\text{F},1}}} \\ & \times \frac{x}{\left[(\epsilon_1 + \epsilon_3)x + 2 \frac{q_{\text{TF},1} + q_{\text{TF},2}}{\sqrt{k_{\text{F},1}k_{\text{F},2}}} \right]^2}. \end{aligned} \quad (74)$$

Equation (74) simplifies considerably if we assume equal densities and degeneracies in the two layers: $n_1 = n_2 \equiv n$, $k_{\text{F},1} = k_{\text{F},2} \equiv k_{\text{F}}$, $\epsilon_{\text{F},1} = \epsilon_{\text{F},2} \equiv \epsilon_{\text{F}}$ and $N_{f,1} = N_{f,2} \equiv N_{\text{f}}$. In this case, equation (74) reduces to

$$\lim_{\xi \rightarrow 0} \lim_{T \rightarrow 0} \rho_{\text{D}}^{\text{BT}} = -\frac{h}{e^2} \frac{4\pi}{3} \alpha_{\text{ee}}^2 \left(\frac{k_{\text{B}}T}{\epsilon_{\text{F}}} \right)^2 \mathcal{F}(\epsilon_1 + \epsilon_3, N_{\text{f}}\alpha_{\text{ee}}), \quad (75)$$

where

$$\mathcal{F}(\epsilon_1 + \epsilon_3, N_{\text{f}}\alpha_{\text{ee}}) \equiv \int_0^2 dx \frac{x(1 - x^2/4)}{[(\epsilon_1 + \epsilon_3)x + 4N_{\text{f}}\alpha_{\text{ee}}]^2}. \quad (76)$$

Note that in the strong-coupling limit, $\lim_{T \rightarrow 0} \rho_{\text{D}}^{\text{BT}}$ does not depend on the interlayer distance d and scales like $1/n$. As far as the dielectric constants ϵ_i are concerned, the strong-coupling and low-temperature BT Coulomb-drag transresistivity depends only on $\epsilon_1 + \epsilon_3$ but not on ϵ_2 .

The quadrature in equation (76) can be easily carried out analytically. The result is

$$\begin{aligned} \mathcal{F}(\epsilon_1 + \epsilon_3, N_{\text{f}}\alpha_{\text{ee}}) = & \frac{1}{2(\epsilon_1 + \epsilon_3)^4} \left\{ 12N_{\text{f}}\alpha_{\text{ee}}(\epsilon_1 + \epsilon_3) - (\epsilon_1 + \epsilon_3)^2 [3 + 2 \ln(2)] \right. \\ & \left. + 24N_{\text{f}}^2\alpha_{\text{ee}}^2 \ln(2) + 2[(\epsilon_1 + \epsilon_3)^2 - 12N_{\text{f}}^2\alpha_{\text{ee}}^2] \ln\left(2 + \frac{\epsilon_1 + \epsilon_3}{N_{\text{f}}\alpha_{\text{ee}}} \right) \right\}. \end{aligned} \quad (77)$$

In the ‘uniform’ limit ($\epsilon_1 = \epsilon_2 = \epsilon_3$), equations (75) and (76) reduce to equations (44a)–(44c) of [31], even though the numerical prefactor we find is four times smaller. Finally, $\mathcal{F}(\epsilon_1 + \epsilon_3, N_{\text{f}}\alpha_{\text{ee}})$ diverges logarithmically in the weak-screening $N_{\text{f}}\alpha_{\text{ee}} \rightarrow 0$ limit, in agreement with equation (41) of [31].

7. Deviations from the low-temperature quadratic behavior due to a frequency-dependent substrate dielectric constant

In this section, we illustrate how deviations from the quadratic-in-temperature dependence of the Coulomb-drag transresistivity can occur in the situation when the dielectric constant ϵ_i of one of the media surrounding the two Dirac-fermion layers has a strong frequency dependence.

Consider, for example, a DLG system on a substrate such as SrTiO₃. Following Han *et al* [52], we model the dielectric constant of this substrate by a frequency-dependent function of the form

$$\epsilon_3(\omega) = \epsilon_\infty + (\epsilon_0 - \epsilon_\infty) \frac{\omega_0^2}{\omega_0^2 - \omega^2 + i\gamma\omega}. \quad (78)$$

At room temperature $\epsilon_\infty = 5.2$, $\epsilon_0 = 310$, $\omega_0/(2\pi) = 2.7$ THz and $\gamma/(2\pi) = 1.3$ THz.

We would like to understand what is the qualitative role played by the change $\epsilon_3 \rightarrow \epsilon_3(\omega)$ in the Coulomb-drag transresistivity. For simplicity, we assume that the two graphene sheets constituting the double layer have the same density $n_1 = n_2 = n$ ($k_{F,1} = k_{F,2} \equiv k_F$). We also assume that the Fermi energy ϵ_F corresponding to n is the largest energy scale in the problem: under this assumption, we are still allowed to expand the BT nonlinear susceptibility $\Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega)$ to lowest order in the small parameter $\hbar\omega/\epsilon_F$.

Despite $\hbar\omega \ll \epsilon_F$, we can have two distinct regimes since $\epsilon_3(\omega)$ brings in a new frequency scale, i.e. ω_0 : (i) $\omega \ll \omega_0 \ll \epsilon_F/\hbar$ and (ii) $\omega_0 \ll \omega \ll \epsilon_F/\hbar$. In the first regime, what matters is obviously $\epsilon_3(\omega = 0) = \epsilon_0$. In the second regime, instead, we have to retain the full frequency dependence of $\epsilon_3(\omega)$. The existence of this second regime ensures the possibility of observing deviations from $\rho_D^{\text{BT}} \propto T^2$ above a certain temperature scale.

The low-temperature Coulomb-drag transresistivity for the situation described above is given by equation (51) with $\Gamma_\ell^{\text{BT}}(\mathbf{q}, \omega)$ given by the expression in equation (58) and σ_ℓ given by the expression in equation (52):

$$\lim_{T \rightarrow 0} \rho_D^{\text{BT}} = -\frac{h}{e^2} \frac{\hbar\beta}{16\pi^3 v^2 \epsilon_F^2} \int_0^{2k_F} dq q \left(1 - \frac{q^2}{4k_F^2}\right) \int_0^{+\infty} d\omega \frac{\omega^2 |U_{12}(q, 0)|_{\epsilon_3 \rightarrow \epsilon_3(\omega)}|^2}{\sinh^2(\beta\omega/2)}. \quad (79)$$

Here the notation $U_{12}(q, 0)|_{\epsilon_3 \rightarrow \epsilon_3(\omega)}$ means that from the point of view of the electronic system we still have to use the statically screened interlayer interaction, while from the point of view of the substrate we have to take into account the frequency dependence of ϵ_3 through the use of equation (78).

The double integral in equation (79) can be easily performed numerically and some illustrative results are summarized in figure 4. For temperatures $T \lesssim \hbar\omega_0/k_B$ we find the usual quadratic-in-temperature behavior, $-\rho_D^{\text{BT}}/\rho_0 = aT^2$, where $\rho_0 = h/e^2$ and a is a numerical coefficient whose actual value depends on the carrier density n and on the interlayer distance d . However, we clearly see from figure 4 that, for temperatures $T \gtrsim \hbar\omega_0/k_B$, the low-temperature Coulomb-drag transresistivity deviates from the canonical quadratic-in-temperature behavior.

Further deviations from the quadratic-in-temperature dependence are induced by the explicit (and strong) temperature dependence of the dielectric constant of SrTiO₃—see, for example, [35]—which here has been neglected for simplicity.

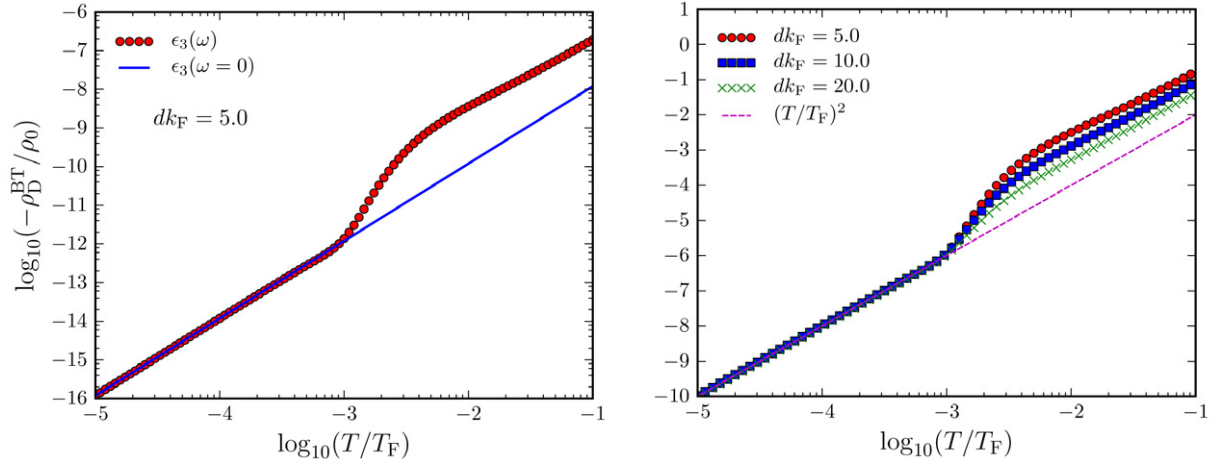


Figure 4. Left panel: the Coulomb-drag transresistivity ρ_D^{BT} (in units of $-\rho_0 = -h/e^2$) is plotted as a function of temperature T (in units of the Fermi temperature $T_F = \varepsilon_F/k_B$). Note that both axes are in the logarithmic scale. Data in this panel refer to: a carrier density $n = 3.1 \times 10^{14} \text{ cm}^{-2}$, an interlayer distance $d \approx 2 \text{ nm}$ ($dk_F = 5.0$), $\varepsilon_1 = 1$ and $\varepsilon_2 = 7.8$. Filled circles label the data obtained by taking into account the frequency dependence $\varepsilon_3 = \varepsilon_3(\omega)$ reported in equation (78). The solid line labels the results obtained by setting $\varepsilon_3 = \varepsilon_3(\omega=0) = \varepsilon_0$ in equation (79). Right panel: the Coulomb-drag transresistivity ρ_D^{BT} (in units of $-\rho_0 a$, where the coefficient a changes with changing dk_F —see text) is plotted as a function of T/T_F . Note that also in this panel both the axes are in the logarithmic scale. Curves labeled by different symbols correspond to different values of the interlayer distance d (density is fixed at the same value as that used in the top panel, i.e. $n = 3.1 \times 10^{14} \text{ cm}^{-2}$).

8. Summary of our main results and conclusions

In summary, we have presented in as complete and pedagogical a manner as possible the minimal theory of Coulomb drag between two spatially separated 2D systems of massless Dirac fermions, which are both away from the charge-neutrality point. Our theory relies on second-order perturbation theory in the screened interlayer interaction and on BT theory.

In this well-established theoretical framework, which has also been adopted by other authors in earlier works [26–31], we have clearly demonstrated that the precise functional dependence of the intralayer transport scattering time on momentum plays absolutely no role in determining the low-temperature Coulomb-drag transresistivity in the Fermi-liquid regime. This is in disagreement with the findings reported in [29, 30], while it agrees with the conclusions reached by Narozhny *et al* [31].

For two layers with identical degeneracies ($N_{f,1} = N_{f,2} \equiv N_f$) and densities ($n_1 = n_2 \equiv n$), we find the following results for the low-temperature Coulomb-drag transresistivity:

- (1) In the weak-coupling limit—equation (72)—we find that

$$\lim_{T \rightarrow 0} \rho_D^{\text{BT}} = -\frac{h}{e^2} \frac{\pi \zeta(3)}{8} \frac{\varepsilon_2^2}{d^4 \alpha_{ee}^2 N_f^4 k_F^6} \frac{k_B^2 T^2}{\hbar^2 v^2}. \quad (80)$$

The dependence of this result on layer separation d and doping n agrees with that found in [26, 28, 31].

(2) In the strong-coupling limit—equation (75)—we find that

$$\lim_{T \rightarrow 0} \rho_D^{\text{BT}} = -\frac{h}{e^2} \frac{4\pi}{3} \frac{\alpha_{\text{ee}}^2}{k_{\text{F}}^2} \frac{k_{\text{B}}^2 T^2}{\hbar^2 v^2} \mathcal{F}(\epsilon_1 + \epsilon_3, N_{\text{f}}\alpha_{\text{ee}}), \quad (81)$$

where the explicit expression for the function $\mathcal{F}(\epsilon_1 + \epsilon_3, N_{\text{f}}\alpha_{\text{ee}})$ is reported in equation (77). The independence of this result from layer separation d and the dependence on doping n ($\propto 1/n$) agree with the findings of [31], even though the authors of this work captured only the weak-screening $N_{\text{f}}\alpha_{\text{ee}} \rightarrow 0$ asymptotic behavior of the function $\mathcal{F}(\epsilon_1 + \epsilon_3, N_{\text{f}}\alpha_{\text{ee}})$ and did not take into account the spatial dependence of the dielectric constant along the \hat{z} -direction.

General results for $n_1 \neq n_2$ and $N_{\text{f},1} \neq N_{\text{f},2}$ can be found in equations (72) and (75).

Finally, we have shown how deviations from the canonical quadratic-in-temperature dependence can occur when one of the dielectric constants of the media surrounding the Dirac-fermion layers depends strongly on the frequency and temperature. DLG systems deposited, for example, on SrTiO₃, a very-well-known insulator close to a ferroelectric instability [35, 52], can be used as a test bed for this idea.

In future, we plan to present a systematic numerical study of equation (51) at arbitrary temperatures and investigate more deeply the strong-coupling limit by studying (i) the impact of diagrams of order higher than two in the screened interlayer interaction and (ii) beyond-RPA corrections.

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Appendix. The Boltzmann transport approach to drag conductivity

The BT equation for the distribution function $f_\ell(\mathbf{k}, \lambda)$ of the ℓ th layer reads

$$\partial_t f_\ell(\mathbf{k}, \lambda) + \nabla_{\mathbf{k}} f_\ell(\mathbf{k}, \lambda) \cdot \partial_t \mathbf{k} = I_\ell(\mathbf{k}, \lambda) + \gamma_{\ell, \bar{\ell}}(\mathbf{k}, \lambda), \quad (\text{A.1})$$

where $I_\ell(\mathbf{k}, \lambda)$ is the so-called collision integral, which describes intralayer scattering events, while the term $\gamma_{\ell, \bar{\ell}}(\mathbf{k}, \lambda)$ describes momentum transfer between the two Dirac-fermion layers ($\bar{\ell} = 2$ if $\ell = 1$ and $\bar{\ell} = 1$ if $\ell = 2$) and is obviously absolutely crucial in the Coulomb-drag problem. We are interested in the steady-state regime in which $\partial_t f_\ell(\mathbf{k}, \lambda) = 0$ and employ a ‘generalized relaxation time approximation’ for the intralayer collision integral, i.e.

$$I_\ell(\mathbf{k}, \lambda) = -\frac{f_\ell(\mathbf{k}, \lambda) - f_\ell^{(0)}(\mathbf{k}, \lambda)}{\tau_\ell(\mathbf{k})}, \quad (\text{A.2})$$

where $f_\ell^{(0)}(\mathbf{k}, \lambda) = n_F(\xi_{\mathbf{k},\lambda}^{(\ell)})$ is the equilibrium distribution function. Below we will assume that the transport scattering time $\tau_\ell(\mathbf{k})$ is isotropic.

The interlayer collision integral reads

$$\begin{aligned} \gamma_{\ell,\bar{\ell}}(\mathbf{k}, \lambda) = & 2\pi N_{f,\bar{\ell}} \sum_{\lambda',\lambda'',\lambda'''} \int \frac{d^2\mathbf{q}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} |U_{12}(q, \xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)})|^2 \\ & \times \delta(\xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)} + \xi_{\mathbf{k}',\lambda''}^{(\bar{\ell})} - \xi_{\mathbf{k}'-\mathbf{q},\lambda'''}^{(\bar{\ell})}) \left\{ f_\ell(\mathbf{k}, \lambda) [1 - f_\ell(\mathbf{k} + \mathbf{q}, \lambda')] f_{\bar{\ell}}(\mathbf{k}', \lambda'') \right. \\ & \times [1 - f_{\bar{\ell}}(\mathbf{k}' - \mathbf{q}, \lambda''')] - [1 - f_\ell(\mathbf{k}, \lambda)] f_\ell(\mathbf{k} + \mathbf{q}, \lambda') [1 - f_{\bar{\ell}}(\mathbf{k}', \lambda'')] f_{\bar{\ell}}(\mathbf{k}' - \mathbf{q}, \lambda''') \left. \right\} \\ & \times \frac{1 + \lambda\lambda' \cos(\varphi_{\mathbf{k}} - \varphi_{\mathbf{k}+\mathbf{q}})}{2} \frac{1 + \lambda''\lambda''' \cos(\varphi_{\mathbf{k}'} - \varphi_{\mathbf{k}'-\mathbf{q}})}{2}. \end{aligned} \quad (\text{A.3})$$

The degeneracy factor $N_{f,\bar{\ell}}$ can be understood from a careful analysis of the conservation of spin and valley degrees of freedom during the scattering process.

In the semiclassical limit, $\partial_t \mathbf{k} = -e \mathbf{E}_\ell$, where \mathbf{E}_ℓ is the electric field in the ℓ th layer. In the regime of small electric fields we can linearize the BT equation by writing

$$f_\ell(\mathbf{k}, \lambda) = f_\ell^{(0)}(\mathbf{k}, \lambda) + f_\ell^{(1)}(\mathbf{k}, \lambda). \quad (\text{A.4})$$

The linearized Boltzmann equation in the steady-state regime reads

$$-e \nabla_{\mathbf{k}} f_\ell^{(0)}(\mathbf{k}, \lambda) \cdot \mathbf{E}_\ell = -\frac{f_\ell^{(1)}(\mathbf{k}, \lambda)}{\tau_\ell(\mathbf{k})} + \gamma_{\ell,\bar{\ell}}^{(\text{linear})}(\mathbf{k}, \lambda), \quad (\text{A.5})$$

where the linearized interlayer collision integral reads

$$\begin{aligned} \gamma_{\ell,\bar{\ell}}^{(\text{linear})}(\mathbf{k}, \lambda) = & 2\pi N_{f,\bar{\ell}} \sum_{\lambda',\lambda'',\lambda'''} \int \frac{d^2\mathbf{q}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} |U_{12}(q, \xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)})|^2 \\ & \times \delta(\xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)} + \xi_{\mathbf{k}',\lambda''}^{(\bar{\ell})} - \xi_{\mathbf{k}'-\mathbf{q},\lambda'''}^{(\bar{\ell})}) \\ & \times \left\{ f_\ell^{(1)}(\mathbf{k}, \lambda) f_{\bar{\ell}}^{(0)}(\mathbf{k}', \lambda'') [1 - f_\ell^{(0)}(\mathbf{k} + \mathbf{q}, \lambda')] [1 - f_{\bar{\ell}}^{(0)}(\mathbf{k}' - \mathbf{q}, \lambda''')] \right. \\ & - f_\ell^{(1)}(\mathbf{k} + \mathbf{q}, \lambda') f_\ell^{(0)}(\mathbf{k}, \lambda) f_{\bar{\ell}}^{(0)}(\mathbf{k}', \lambda'') [1 - f_{\bar{\ell}}^{(0)}(\mathbf{k}' - \mathbf{q}, \lambda''')] \\ & + f_{\bar{\ell}}^{(1)}(\mathbf{k}', \lambda'') f_\ell^{(0)}(\mathbf{k}, \lambda) [1 - f_\ell^{(0)}(\mathbf{k} + \mathbf{q}, \lambda')] [1 - f_{\bar{\ell}}^{(0)}(\mathbf{k}' - \mathbf{q}, \lambda''')] \\ & - f_{\bar{\ell}}^{(1)}(\mathbf{k}' - \mathbf{q}, \lambda''') f_\ell^{(0)}(\mathbf{k}, \lambda) f_{\bar{\ell}}^{(0)}(\mathbf{k}', \lambda'') [1 - f_\ell^{(0)}(\mathbf{k} + \mathbf{q}, \lambda')] \\ & - f_\ell^{(1)}(\mathbf{k}, \lambda) f_\ell^{(0)}(\mathbf{k} + \mathbf{q}, \lambda') f_{\bar{\ell}}^{(0)}(\mathbf{k}' - \mathbf{q}, \lambda''') [1 - f_{\bar{\ell}}^{(0)}(\mathbf{k}', \lambda'')] \\ & + f_\ell^{(1)}(\mathbf{k} + \mathbf{q}, \lambda') f_{\bar{\ell}}^{(0)}(\mathbf{k}' - \mathbf{q}, \lambda''') [1 - f_\ell^{(0)}(\mathbf{k}, \lambda)] [1 - f_{\bar{\ell}}^{(0)}(\mathbf{k}', \lambda'')] \\ & - f_{\bar{\ell}}^{(1)}(\mathbf{k}', \lambda'') f_\ell^{(0)}(\mathbf{k} + \mathbf{q}, \lambda') f_{\bar{\ell}}^{(0)}(\mathbf{k}' - \mathbf{q}, \lambda''') [1 - f_\ell^{(0)}(\mathbf{k}, \lambda)] \\ & \left. + f_{\bar{\ell}}^{(1)}(\mathbf{k}' - \mathbf{q}, \lambda''') f_\ell^{(0)}(\mathbf{k} + \mathbf{q}, \lambda') [1 - f_\ell^{(0)}(\mathbf{k}, \lambda)] [1 - f_{\bar{\ell}}^{(0)}(\mathbf{k}', \lambda'')] \right\} \\ & \times \frac{1 + \lambda\lambda' \cos(\varphi_{\mathbf{k}} - \varphi_{\mathbf{k}+\mathbf{q}})}{2} \frac{1 + \lambda''\lambda''' \cos(\varphi_{\mathbf{k}'} - \varphi_{\mathbf{k}'-\mathbf{q}})}{2}. \end{aligned} \quad (\text{A.6})$$

We now seek the solutions of equations (A.5)–(A.6) of the form

$$f_\ell^{(1)}(\mathbf{k}, \lambda) = \psi_\ell^{(\text{intra})}(\mathbf{k}, \lambda) \hat{\mathbf{k}} \cdot \mathbf{E}_\ell + \psi_\ell^{(\text{inter})}(\mathbf{k}, \lambda) \hat{\mathbf{k}} \cdot \mathbf{E}_{\bar{\ell}}. \quad (\text{A.7})$$

In the following, we will assume that $|\psi_\ell^{(\text{intra})}(k, \lambda)| \gg |\psi_\ell^{(\text{inter})}(k, \lambda)|$. The coefficients $\psi_\ell^{(\text{intra})}(k, \lambda)$ and $\psi_\ell^{(\text{inter})}(k, \lambda)$ can be found by calculating the functional derivative of the linearized BT equation with respect to \mathbf{E}_ℓ and $\mathbf{E}_{\bar{\ell}}$, respectively, i.e.

$$-e\nabla_{\mathbf{k}} f_\ell^{(0)}(\mathbf{k}, \lambda) \simeq -\hat{\mathbf{k}} \frac{\psi_\ell^{(\text{intra})}(k, \lambda)}{\tau_\ell(\mathbf{k})} \quad (\text{A.8})$$

and

$$0 = -\hat{\mathbf{k}} \frac{\psi_\ell^{(\text{inter})}(k, \lambda)}{\tau_\ell(\mathbf{k})} + \frac{\delta\gamma_{\ell, \bar{\ell}}^{(\text{linear})}(\mathbf{k}, \lambda)}{\delta\mathbf{E}_{\bar{\ell}}}. \quad (\text{A.9})$$

In writing equation (A.8), we have neglected the small term $\delta\gamma_{\ell, \bar{\ell}}^{(\text{linear})}(\mathbf{k}, \lambda)/\delta\mathbf{E}_\ell$. The solution to equation (A.8) reads

$$\psi_\ell^{(\text{intra})}(k, \lambda) = e\hat{\mathbf{k}} \cdot \nabla_{\mathbf{k}} \xi_{\mathbf{k}, \lambda}^{(\ell)} \frac{\partial f_\ell^{(0)}(\mathbf{k}, \lambda)}{\partial \xi_{\mathbf{k}, \lambda}^{(\ell)}} \tau_\ell(\mathbf{k}). \quad (\text{A.10})$$

The second term on the rhs of equation (A.9) can be calculated from equation (A.6). Solving equation (A.9) we find that

$$\begin{aligned} \psi_\ell^{(\text{inter})}(k, \lambda) = & -2\pi e\beta\tau_\ell(\mathbf{k})N_{\mathbf{f}, \bar{\ell}} \sum_{\lambda', \lambda'', \lambda'''} \int \frac{d^2\mathbf{q}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} |U_{12}(q, \xi_{\mathbf{k}, \lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q}, \lambda'}^{(\ell)})|^2 \\ & \times \delta(\xi_{\mathbf{k}, \lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q}, \lambda'}^{(\ell)} + \xi_{\mathbf{k}', \lambda''}^{(\bar{\ell})} - \xi_{\mathbf{k}'-\mathbf{q}, \lambda'''}^{(\bar{\ell})}) \left\{ f_\ell^{(0)}(\mathbf{k}, \lambda) f_{\bar{\ell}}^{(0)}(\mathbf{k}', \lambda'') [1 - f_\ell^{(0)}(\mathbf{k} + \mathbf{q}, \lambda')] \right. \\ & \times [1 - f_{\bar{\ell}}^{(0)}(\mathbf{k}' - \mathbf{q}, \lambda''')] \left. \right\} \left[\tau_{\bar{\ell}}(\mathbf{k}') \nabla_{\mathbf{k}'} \xi_{\mathbf{k}', \lambda''}^{(\bar{\ell})} - \tau_{\bar{\ell}}(\mathbf{k}' - \mathbf{q}) \nabla_{\mathbf{k}'} \xi_{\mathbf{k}'-\mathbf{q}, \lambda'''}^{(\bar{\ell})} \right] \cdot \hat{\mathbf{k}} \\ & \times \frac{1 + \lambda\lambda' \cos(\varphi_{\mathbf{k}} - \varphi_{\mathbf{k}+\mathbf{q}})}{2} \frac{1 + \lambda''\lambda''' \cos(\varphi_{\mathbf{k}'} - \varphi_{\mathbf{k}'-\mathbf{q}})}{2}. \end{aligned} \quad (\text{A.11})$$

To obtain equation (A.11), we used the identities

$$\frac{\partial f_\ell^{(0)}(\mathbf{k}, \lambda)}{\partial \xi_{\mathbf{k}, \lambda}^{(\ell)}} = \beta f_\ell^{(0)}(\mathbf{k}, \lambda) [1 - f_\ell^{(0)}(\mathbf{k}, \lambda)] \quad (\text{A.12})$$

and

$$f_\ell^{(0)}(\mathbf{k}, \lambda) [1 - f_\ell^{(0)}(\mathbf{k}', \lambda')] = \frac{f_\ell^{(0)}(\mathbf{k}, \lambda) - f_\ell^{(0)}(\mathbf{k}', \lambda')}{1 - \exp\left[\beta\left(\xi_{\mathbf{k}, \lambda}^{(\ell)} - \xi_{\mathbf{k}', \lambda'}^{(\ell)}\right)\right]}. \quad (\text{A.13})$$

The current density in the ℓ th layer reads

$$\begin{aligned} \mathbf{J}_\ell = & -eN_{\mathbf{f}, \ell} \sum_{\lambda} \int \frac{d^2\mathbf{k}}{(2\pi)^2} f_\ell^{(1)}(\mathbf{k}, \lambda) \nabla_{\mathbf{k}} \xi_{\mathbf{k}, \lambda}^{(\ell)} \\ & \equiv \mathbf{J}_\ell^{(\text{intra})} + \mathbf{J}_\ell^{(\text{inter})}. \end{aligned} \quad (\text{A.14})$$

Using equation (A.10) we find $\mathbf{J}_\ell^{(\text{intra})}$:

$$\begin{aligned} \mathbf{J}_\ell^{(\text{intra})} = & e^2 N_{\mathbf{f}, \ell} \sum_{\lambda} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \tau_\ell(\mathbf{k}) [\hat{\mathbf{k}} \cdot \nabla_{\mathbf{k}} \xi_{\mathbf{k}, \lambda}^{(\ell)}] (\hat{\mathbf{k}} \cdot \mathbf{E}_\ell) \left[-\frac{\partial f_\ell^{(0)}(\mathbf{k}, \lambda)}{\partial \xi_{\mathbf{k}, \lambda}^{(\ell)}} \right] \nabla_{\mathbf{k}} \xi_{\mathbf{k}, \lambda}^{(\ell)} \\ = & \mathbf{E}_\ell \frac{e^2 N_{\mathbf{f}, \ell}}{2} \sum_{\lambda} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \tau_\ell(\mathbf{k}) \left(\frac{\partial \xi_{\mathbf{k}, \lambda}^{(\ell)}}{\partial \mathbf{k}} \right)^2 \left[-\frac{\partial f_\ell^{(0)}(\mathbf{k}, \lambda)}{\partial \xi_{\mathbf{k}, \lambda}^{(\ell)}} \right]. \end{aligned} \quad (\text{A.15})$$

In the last equality of equation (A.15), we used the fact that

$$\nabla_{\mathbf{k}} \xi_{\mathbf{k},\lambda}^{(\ell)} = \hat{\mathbf{k}} \frac{\partial \xi_{\mathbf{k},\lambda}^{(\ell)}}{\partial k} \quad (\text{A.16})$$

(since $\xi_{\mathbf{k},\lambda}^{(\ell)}$ depends only on $|\mathbf{k}|$) and that the average over the angle of $(\hat{\mathbf{k}} \cdot \mathbf{v})\hat{\mathbf{k}} = \mathbf{v}/2$ for any constant vector \mathbf{v} (in $D = 2$ spatial dimensions). From equation (A.15), we find immediately the longitudinal intralayer conductivity

$$\begin{aligned} \sigma_{\ell} &\equiv \frac{\delta J_{\ell}^{(i)}}{\delta E_{\ell}^{(i)}} = \frac{e^2 N_{f,\ell}}{2} \sum_{\lambda} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \tau_{\ell}(\mathbf{k}) \left(\frac{\partial \xi_{\mathbf{k},\lambda}^{(\ell)}}{\partial k} \right)^2 \left[- \frac{\partial f_{\ell}^{(0)}(\mathbf{k}, \lambda)}{\partial \xi_{\mathbf{k},\lambda}^{(\ell)}} \right] \\ &= \frac{e^2 v^2 N_{f,\ell}}{2} \sum_{\lambda} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \tau_{\ell}(\mathbf{k}) \left[- \frac{\partial f_{\ell}^{(0)}(\mathbf{k}, \lambda)}{\partial \xi_{\mathbf{k},\lambda}^{(\ell)}} \right], \end{aligned} \quad (\text{A.17})$$

where i is a Cartesian index and in the last equality we used the fact that $\xi_{\mathbf{k},\lambda}^{(\ell)} = \lambda v |\mathbf{k}| - \mu_{\ell}$. Note that equation (A.17) coincides with equation (50) in the main text.

The interlayer current density reads

$$\mathbf{J}_{\ell}^{(\text{inter})} = -e N_{f,\ell} \sum_{\lambda} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \psi^{(\text{inter})}(k, \lambda) (\mathbf{k} \cdot \mathbf{E}_{\bar{\ell}}) \nabla_{\mathbf{k}} \xi_{\mathbf{k},\lambda}^{(\ell)}, \quad (\text{A.18})$$

where $\psi^{(\text{inter})}(k, \lambda)$ is given in equation (A.11). After some straightforward algebra we obtain

$$\begin{aligned} \mathbf{J}_{\ell}^{(\text{inter})} &= \mathbf{E}_{\bar{\ell}} \frac{2\pi N_{f,\ell} N_{f,\bar{\ell}} \beta e^2}{2} \sum_{\lambda, \lambda', \lambda'', \lambda'''} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \int \frac{d^2 \mathbf{k}'}{(2\pi)^2} |U_{12}(\mathbf{q}, \xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)})|^2 \\ &\quad \times \delta(\xi_{\mathbf{k},\lambda}^{(\ell)} - \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)} + \xi_{\mathbf{k}',\lambda''}^{(\bar{\ell})} - \xi_{\mathbf{k}'-\mathbf{q},\lambda'''}^{(\bar{\ell})}) \left\{ f_{\ell}^{(0)}(\mathbf{k}, \lambda) f_{\bar{\ell}}^{(0)}(\mathbf{k}', \lambda'') [1 - f_{\ell}^{(0)}(\mathbf{k} + \mathbf{q}, \lambda')] \right. \\ &\quad \times [1 - f_{\bar{\ell}}^{(0)}(\mathbf{k}' - \mathbf{q}, \lambda''')] \left. \right\} \left[\tau_{\ell}(\mathbf{k}) \nabla_{\mathbf{k}} \xi_{\mathbf{k},\lambda}^{(\ell)} - \tau_{\ell}(\mathbf{k} + \mathbf{q}) \nabla_{\mathbf{k}} \xi_{\mathbf{k}+\mathbf{q},\lambda'}^{(\ell)} \right] \\ &\quad \times \left[\tau_{\bar{\ell}}(\mathbf{k}') \nabla_{\mathbf{k}'} \xi_{\mathbf{k}',\lambda''}^{(\bar{\ell})} - \tau_{\bar{\ell}}(\mathbf{k}' - \mathbf{q}) \nabla_{\mathbf{k}'} \xi_{\mathbf{k}'-\mathbf{q},\lambda'''}^{(\bar{\ell})} \right] \\ &\quad \times \frac{1 + \lambda \lambda' \cos(\varphi_{\mathbf{k}} - \varphi_{\mathbf{k}+\mathbf{q}})}{2} \frac{1 + \lambda'' \lambda''' \cos(\varphi_{\mathbf{k}'} - \varphi_{\mathbf{k}'-\mathbf{q}})}{2}. \end{aligned} \quad (\text{A.19})$$

Note that equation (A.19) has been recast in a symmetric form by means of a simultaneous exchange of dummy variables ($\mathbf{k} \leftrightarrow \mathbf{k} + \mathbf{q}$, $\mathbf{k}' \leftrightarrow \mathbf{k}' - \mathbf{q}$, $\lambda \leftrightarrow \lambda'$ and $\lambda'' \leftrightarrow \lambda'''$). Consequently, the integral has been multiplied by a factor of 1/2. From equation (A.19), we can derive the longitudinal drag conductivity σ_{D} , which reads as follows:

$$\sigma_{\text{D}} \equiv \frac{\delta J_{\ell}^{(i)}}{\delta E_{\bar{\ell}}^{(i)}} = \frac{\beta e^2}{16\pi} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \int_{-\infty}^{+\infty} d\omega \frac{|U_{12}(\mathbf{q}, \omega)|^2}{\sinh^2(\beta\omega/2)} \Gamma_1^{\text{BT}}(\mathbf{q}, \omega) \Gamma_2^{\text{BT}}(\mathbf{q}, \omega), \quad (\text{A.20})$$

where the function $\Gamma_{\ell}^{\text{BT}}(\mathbf{q}, \omega)$ coincides with the one defined in equation (48) of the main text. In writing equation (A.20), we have introduced an auxiliary variable ω to disentangle \mathbf{k} from \mathbf{k}' :

$$\delta(\xi - \xi') = \int_{-\infty}^{+\infty} d\omega \delta(\xi - \omega) \delta(\xi' - \omega). \quad (\text{A.21})$$

References

- [1] Pogrebinskii M B 1977 *Sov. Phys.—Semicond.* **11** 372
- [2] Price P J 1983 *Physica B* **117** 750
- [3] Tso H C, Vasilopoulos P and Peeters F M 1992 *Phys. Rev. Lett.* **68** 2516
Tso H C, Vasilopoulos P and Peeters F M 1993 *Phys. Rev. Lett.* **70** 2146
Tso H C, Vasilopoulos P and Peeters F M 1993 *Phys. Rev. Lett.* **71** 1960
- [4] Zheng L and MacDonald A H 1993 *Phys. Rev. B* **48** 8203
- [5] Jauho A P and Smith H 1993 *Phys. Rev. B* **47** 4420
- [6] Kamenev A and Oreg Y 1995 *Phys. Rev. B* **52** 7516
- [7] Flensberg K, Hu B Y H, Jauho A P and Kinaret J M 1995 *Phys. Rev. B* **52** 14761
- [8] Lunde A M, Flensberg K and Jauho A P 1995 *Phys. Rev. B* **71** 125408
- [9] Badalyan S M, Kim C S, Vignale G and Senatore G 2007 *Phys. Rev. B* **75** 125321
- [10] Asgari R, Tanatar B and Davoudi B 2008 *Phys. Rev. B* **77** 115301
- [11] Rojo A G 1999 *J. Phys.: Condens. Matter* **11** R31
- [12] Gramila T J, Eisenstein J P, MacDonald A H, Pfeiffer L N and West K W 1991 *Phys. Rev. Lett.* **66** 1216
- [13] Sivan U, Solomon P M and Shtrikman H 1992 *Phys. Rev. Lett.* **68** 1196
- [14] Geim A K and Novoselov K S 2007 *Nature Mater.* **6** 183
Castro Neto A H, Guinea F, Peres N M R, Novoselov K S and Geim A K 2009 *Rev. Mod. Phys.* **81** 109
Geim A K 2009 *Science* **324** 1530
- [15] Moore J E 2009 *Nature Phys.* **5** 378
Moore J E *Nature* **464** 194
Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
Qi X L and Zhang S C 2011 *Rev. Mod. Phys.* **83** 1057
- [16] Novoselov K S, Geim A K, Morozov S V, Jiang D, Zhang Y, Dubonos S V, Grigorieva I V and Firsov A A 2004 *Science* **306** 666
Novoselov K S, Jiang D, Schedin F, Booth T J, Khotkevich V V, Morozov S V and Geim A K 2005 *Proc. Natl Acad. Sci. USA* **102** 10451
- [17] Rollings E, Gweon G H, Zhou S Y, Mun B S, McChesney J L, Hussain B S, Fedorov A V, First P N, de Heer W A and Lanzara A 2006 *J. Phys. Chem. Solids* **67** 2172
de Heer W A *et al* 2007 *Solid State Commun.* **143** 92
First P N, de Heer W A, Seyller T, Berger C, Stroschio J A and Moon J S 2010 *MRS Bull.* **35** 296
- [18] Kim S, Jo I, Nah J, Yao Z, Banerjee S K and Tutuc E 2011 *Phys. Rev. B* **83** 161401
- [19] Dean C R *et al* 2010 *Nature Nanotechnol.* **5** 722
- [20] Ponomarenko L A *et al* 2011 *Nature Phys.* **7** 958
- [21] Britnell L *et al* 2012 *Science* **335** 947
- [22] Britnell L *et al* 2012 *Nano Lett.* **12** 1707
- [23] Li G, Luican A and Andrei E Y 2009 *Phys. Rev. Lett.* **102** 176804
Neugebauer P, Orlita M, Faugeras C, Barra A L and Potemski M 2009 *Phys. Rev. Lett.* **103** 136403
- [24] Li G, Luican A, Lopes dos Santos J M B, Castro Neto A H, Reina A, Kong J and Andrei E Y 2009 *Nature Phys.* **6** 109
- [25] Schmidt H, Lüdtke T, Barthold P, McCann E, Fal'ko V I and Haug R J 2008 *Appl. Phys. Lett.* **93** 172108
Schmidt H, Lüdtke T, Barthold P and Haug R J 2010 *Phys. Rev. B* **81** 121403
- [26] Tse W K, Hu B Y K and Das Sarma S 2007 *Phys. Rev. B* **76** 081401
- [27] Narozhny B N 2007 *Phys. Rev. B* **76** 153409
- [28] Katsnelson M I 2011 *Phys. Rev. B* **84** 041407
- [29] Peres N M R, Lopes dos Santos J M B and Castro Neto A H 2011 *Europhys. Lett.* **95** 18001
- [30] Hwang E H, Sensarma R and Das Sarma S 2011 *Phys. Rev. B* **84** 245441
- [31] Narozhny B N, Titov M, Gornyi I V and Ostrovsky P M 2011 *Phys. Rev. B* **85** 195421

- [32] Badalyan S M and Peeters F M 2012 arXiv:1204.4598
- [33] Mink M P, Stoof H T C, Duine R A, Polini M and Vignale G 2012 *Phys. Rev. Lett.* **108** 186402
Vignale G and MacDonald A H 1996 *Phys. Rev. Lett.* **76** 2786
Hu B Y K 2000 *Phys. Rev. Lett.* **85** 820
Croxall A F, Das Gupta K, Nicoll C A, Thangaraj M, Beere H E, Farrer I, Ritchie D A and Pepper M 2008 *Phys. Rev. Lett.* **101** 246801
Seamons J A, Morath C P, Reno J L and Lilly M P 2009 *Phys. Rev. Lett.* **102** 026804
- [34] Min H, Bistrizter R, Su J J and MacDonald A H 2008 *Phys. Rev. B* **78** 121401
Lozovik Y E and Sokolik A A 2008 *JETP Lett.* **87** 55
Zhang C H and Joglekar Y N 2008 *Phys. Rev. B* **77** 233405
Kharitonov M Y and Efetov K B 2008 *Phys. Rev. B* **78** 241401
Bistrizter R, Min H, Su J J and MacDonald A H 2008 arXiv:0810.0331
Mink M P, Stoof H T C, Duine R A and MacDonald A H 2011 *Phys. Rev. B* **84** 155409
- [35] Couto N J G, Sacépé B and Morpurgo A F 2011 *Phys. Rev. Lett.* **107** 225501
- [36] Amorim B and Peres N M R 2012 arXiv:1203.6777
- [37] Zheng L and MacDonald A H 1994 *Phys. Rev. B* **49** 5522
- [38] Giuliani G F and Vignale G 2005 *Quantum Theory of the Electron Liquid* (Cambridge: Cambridge University Press)
- [39] Wunsch B, Stauber T, Sols F and Guinea F 2006 *New J. Phys.* **8** 318
- [40] Hwang E H and Das Sarma S 2007 *Phys. Rev. B* **75** 205418
- [41] Barlas Y, Pereg-Barnea T, Polini M, Asgari R and MacDonald A H 2007 *Phys. Rev. Lett.* **98** 236601
- [42] Profumo R E V, Polini M, Asgari R, Fazio R and MacDonald A H 2010 *Phys. Rev. B* **82** 085443
- [43] González J, Guinea F and Vozmediano M A H 1994 *Nucl. Phys. B* **424** 595
González J, Guinea F and Vozmediano M A H 1999 *Phys. Rev. B* **59** R2474
Luican A, Li G and Andrei E Y 2011 *Phys. Rev. B* **83** 041405
Elias D C *et al* 2011 *Nature Phys.* **7** 701
- [44] Larkin A and Varlamov A 2004 *Theory of Fluctuations in Superconductors* (Oxford: Clarendon)
- [45] Ando T 2006 *J. Phys. Soc. Japan* **75** 074716
Nomura K and MacDonald A H 2006 *Phys. Rev. Lett.* **96** 256602
Cheianov V V and Fal'ko V I 2006 *Phys. Rev. Lett.* **97** 226801
- [46] Das Sarma S, Adam S, Hwang E H and Rossi E 2011 *Rev. Mod. Phys.* **83** 407
- [47] Martin J, Akerman N, Ulbricht G, Lohmann T, Smet J H, von Klitzing K and Yacoby A 2008 *Nature Phys.* **4** 144
Zhang Y, Brar V W, Girit C, Zettl A and Crommie M F 2009 *Nature Phys.* **5** 722
Deshpande A, Bao W, Miao F, Lau C N and LeRoy B Y 2009 *Phys. Rev. B* **79** 205411
- [48] Sachdev S 1998 *Phys. Rev. B* **57** 7157
- [49] Kashuba A B 2008 *Phys. Rev. B* **78** 085415
- [50] Fritz L, Schmalian J, Müller M and Sachdev S 2008 *Phys. Rev. B* **78** 085416
- [51] Gangadharaiah S, Farid A M and Mishchenko E G 2008 *Phys. Rev. Lett.* **100** 166802
- [52] Han J, Wan F, Zhu Z and Zhang W 2007 *Appl. Phys. Lett.* **90** 031104