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Information theoretic inequalities as bounds in superconformal field theory

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In this paper, an information theoretic approach to bounds in superconformal field theories is proposed. It is proved that the supersymmetric Rényi entropy $\bar{S}_\alpha$ is a monotonically decreasing function of α and $(\alpha-1)\bar{S}_\alpha$ is a concave function of α. Under the assumption that the thermal entropy associated with the "replica trick" time circle is bounded from below by the charge at $\alpha \to \infty$, it is further proved that both $\frac{\alpha-1}{\alpha}\bar{S}_\alpha$ and $(\alpha-1)\bar{S}_\alpha$ monotonically increase as functions of α. Because $\bar{S}_\alpha$ enjoys universal relations with the Weyl anomaly coefficients in even-dimensional superconformal field theories, one therefore obtains a set of bounds on these coefficients by imposing the inequalities of $\bar{S}_\alpha$. Some of the bounds coincide with Hofman–Maldacena bounds and the others are new. We also check the inequalities for examples in odd-dimensions.

Keywords: Renyi divergence; superconformal bounds.

1. Introduction

Quantum information theoretic ideas, such as quantum entanglement, have recently played significant roles in condensed matter physics,[1–3] particle physics[4–7] and string theory.[8] To characterize the entanglement in states of a quantum mechanical system, one often bipartitions the system and computes the entanglement entropy, S_{EE}. Another interesting measure is the Rényi entropy, S_α, which is a one-parameter generalization of entanglement entropy and provides additional information about the entanglement structure for the same bipartition and returns to S_{EE} in the limit $\alpha \to 1$. α is called its order. In quantum field theory (QFT), one defines the entanglement entropy associated with a global state and a geometric region A by tracing over the field variables outside A, creating a reduced density matrix ρ_A and then evaluating $S_{\rm EE}$.[9] While $S_{\rm EE}$ (or S_α) generally includes UV divergences in QFT, its universal part contains important physical information, such as central charges characterizing degrees of freedom.[10–14] In many aspects, these universal terms are the counterparts of quantum-mechanical entropies, which satisfy a set of inequalities inspired from information theory. One natural question is: What are the QFT counterparts of these entropy inequalities and what are their roles? One inequality of $S_{\rm EE}$ called strong sub-additivity plays significant roles in constructing monotonically decreasing c-functions along RG flows, such as the two-dimensional entropic c-function[4,5] and the three-dimensional F-function.[6,14] Other applications of information theoretic inequalities include refining Bekenstein bound,[4] deriving the integrated null energy condition[15] and deriving gravitational positive energy conditions.[16]

In this paper, we concern the Rényi entropy inequalities related to its order α, which were proven in information theory[17] and still hold in quantum mechanics.[a] One therefore expects that these inequalities also play significant roles in QFT.[18] However, the exact results of Rényi entropy are very rare in QFT (except for $2d$ conformal field theories).[19–25] We therefore focus on a subset of field theories, supersymmetric ones with a conserved R-symmetry. By twisting the ordinary Rényi entropy to be supersymmetric,[26] $S_\alpha \to \bar{S}_\alpha$, we are able to obtain exact results at any coupling. For even-dimensional superconformal field theories (SCFTs), the supersymmetric Rényi entropy $\bar{S}_\alpha$ enjoys universal relations with the Weyl anomaly coefficients. These relations are independent of the specific theory and therefore can be used to bound the space of SCFTs. That is, imposing $\bar{S}_\alpha$'s inequalities to these relations gives a set of bounds on the Weyl anomaly coefficients. The key step in this derivation is to find the inequalities satisfied by $\bar{S}_\alpha$, which is the main topic of this paper. The idea is that, $\bar{S}_\alpha$ can be expressed as the Rényi divergence of the energy distribution from the R-charge distribution. By studying the α-dependence of the Rényi divergence, one can get the inequalities satisfied by $\bar{S}_\alpha$. It is proved along this way that $\bar{S}_\alpha$ monotonically decreases as a function of α and $(\alpha-1)\bar{S}_\alpha$ is a concave function of α. On the other hand, $\bar{S}_\alpha$ of CFTs associated with a spherical entangling surface is related to other physical quantities such as thermal entropy S, energy E and charge Q defined on the hyperbolic space $\mathbb{S}^1_\alpha \times \mathbb{H}^{d-1}$.[27] Under the assumption that the thermal entropy is bounded from below by the charge at $\alpha \to \infty$, it is further proved that both $\frac{\alpha-1}{\alpha}\bar{S}_\alpha$ and $(\alpha-1)\bar{S}_\alpha$ monotonically increase as functions of α.

We will start by introducing Rényi divergence in information theory and studying its behavior as a function of α, which will be used for the later proof of the

[a]To generalize the proof of classical information theoretic inequalities to quantum mechanical ones, one simply diagonalizes density matrices ρ, σ with unitary matrices, which does not change the Rényi entropy (or Rényi divergence).

supersymmetric Rényi entropy inequalities. Then the applications of these inequalities in even dimensions will be discussed and the validity of them will be checked for some odd-dimensional examples. A holographic derivation of the bound $S \geq 2\pi Q$ will be given in Appendix A.

2. Rényi Divergence

In information theory, Rényi divergence is related to Rényi entropy much like Kullback–Leibler divergence (relative entropy) is related to Shannon entropy. For a probability distribution $P = (p_1, \ldots, p_n)$, which satisfies $p_i \geq 0$ and $\sum_{i=1}^n p_i = 1$, the Shannon entropy is given by

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i, \tag{1}$$

and the Rényi entropy is given by ($\alpha > 0$)

$$H_\alpha(P) = \frac{1}{1-\alpha} \log \sum_{i=1}^{n} p_i^\alpha, \tag{2}$$

which reduces to the Shannon entropy (1) in the limit $\alpha \to 1$ and can be considered as the α-extension of the Shannon entropy. Let Q be another probability distribution, $Q = (q_1, \ldots, q_n)$. The relative entropy between P and Q is given by

$$D(P\|Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}, \tag{3}$$

which can be proven to be non-negative for two normalized distributions P and Q. Note that the relative entropy is regular only if $q_i = 0$ implies $p_i = 0$ for all i, in another word P is absolutely continuous with respect to Q, $P \ll Q$. In our later set up in QFT, P and Q will be identified as energy distribution ($\propto e^{-K}$) and charge distribution ($\propto e^{-\hat{Q}'}$), respectively. Therefore, $P \ll Q$ is guaranteed by the Bogomol'nyi–Prasad–Sommerfield bound. The α-extension of the relative entropy (3) is the Rényi divergence

$$D_\alpha(P\|Q) = \frac{1}{\alpha-1} \log \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}, \tag{4}$$

which was introduced by Rényi as a measure of information that satisfies almost the same axioms as the relative entropy.[28] In particular, the Rényi divergence reduces to the relative entropy in the limit $\alpha \to 1$. On the other hand, one may consider the Rényi divergence (4) as a deformation of the Rényi entropy (2). Indeed, the Rényi entropy can be expressed in terms of the Rényi divergence of P from the uniform distribution $U = (1/n, \ldots, 1/n)$:

$$H_\alpha(P) = H_\alpha(U) - D_\alpha(P\|U) = \log n - D_\alpha(P\|U). \tag{5}$$

Let us study the α-dependence of the Rényi divergence. In order to understand that, one may first look at the α-related Rényi entropy inequalities:

$$\partial_\alpha H_\alpha \leq 0, \tag{6}$$

$$\partial_\alpha \left(\frac{\alpha-1}{\alpha} H_\alpha\right) \geq 0, \tag{7}$$

$$\partial_\alpha ((\alpha-1)H_\alpha) \geq 0, \tag{8}$$

$$\partial_\alpha^2 ((\alpha-1)H_\alpha) \leq 0. \tag{9}$$

For the proof of these inequalities, see Ref. 17. One natural question is: Are there similar inequalities like (6)–(9) for the Rényi divergence D_α? We now prove the following inequalities:

$$\partial_\alpha D_\alpha \geq 0,\ \alpha > 0, \tag{10}$$

$$\partial_\alpha \left(\frac{\alpha-1}{\alpha} D_\alpha\right) \geq 0,\ \alpha > 0, \tag{11}$$

$$\partial_\alpha ((\alpha-1)D_\alpha) \geq 0,\ \alpha \geq 1, \tag{12}$$

$$\partial_\alpha^2 ((\alpha-1)D_\alpha) \geq 0,\ \alpha > 0. \tag{13}$$

Among these four inequalities, (10) and (13) have been proven by van Erven and Harremoës in Ref. 29. We will prove the other two equations (11) and (12) and give an alternative proof of (13). Below we also include the proof of (10) by van Erven and Harremoës for completeness.

2.1. *Monotonicity of D_α*

Now we prove $\partial_\alpha D_\alpha \geq 0$. Let $\alpha < \beta$ be positive real numbers ($\alpha, \beta \neq 1$). Then for $x \geq 0$, the function $f(x) = x^{\frac{\alpha-1}{\beta-1}}$ is strictly convex if $\alpha < 1$ and strictly concave if $\alpha > 1$. Therefore by Jensen's inequality

$$D_\alpha = \frac{1}{\alpha-1} \log \sum_{i=1}^{n} \left(\frac{p_i}{q_i}\right)^{(\beta-1)\frac{\alpha-1}{\beta-1}} p_i \tag{14}$$

$$\leq \frac{1}{\beta-1} \log \sum_{i=1}^{n} \left(\frac{p_i}{q_i}\right)^{\beta-1} p_i \tag{15}$$

$$= D_\beta. \tag{16}$$

Note that the normalization condition $\sum_i q_i = 1$ is not necessary for the proof of $\partial_\alpha D_\alpha \geq 0$. Jensen's inequality states that, if $f(x)$ is a convex function of x, then

$$\mathrm{E}[f(x)] \geq f(\mathrm{E}[x]), \tag{17}$$

where $\mathrm{E}[X]$ means taking the average of variable X under a normalized probability distribution. The inequality is reversed if $f(x)$ is concave.

2.2. *Monotonicity of* $\frac{\alpha-1}{\alpha}D_\alpha$

Now we prove $\partial_\alpha\left(\frac{\alpha-1}{\alpha}D_\alpha\right) \geq 0$. Let $\alpha < \beta$ be positive real numbers. Then for $x \geq 0$, the function $f(x) = x^{\frac{\alpha}{\beta}}$ is strictly concave. Therefore,

$$\frac{\alpha-1}{\alpha}D_\alpha = \frac{1}{\alpha}\log\sum_{i=1}^{n}\left(\frac{p_i}{q_i}\right)^{\beta\frac{\alpha}{\beta}} q_i \tag{18}$$

$$\leq \frac{1}{\beta}\log\sum_{i=1}^{n}\left(\frac{p_i}{q_i}\right)^{\beta} q_i \tag{19}$$

$$= \frac{\beta-1}{\beta}D_\beta, \tag{20}$$

where we have used Jensen's inequality again in the second step. Note that the normalization condition $\sum_i q_i = 1$ is now essential in this proof.

2.3. *Monotonicity of* $(\alpha-1)D_\alpha$

Now we prove $\partial_\alpha((\alpha-1)D_\alpha) \geq 0$ for $\alpha \in [1,\infty)$. Given that $\partial_\alpha^2((\alpha-1)D_\alpha) \geq 0$, which will be proven in the following subsection, we only need to prove

$$\partial_\alpha((\alpha-1)D_\alpha)|_{\alpha\to 1} \geq 0. \tag{21}$$

This can be shown as follows:

$$\partial_\alpha((\alpha-1)D_\alpha) = \partial_\alpha\left(\log\sum_{i=1}^{n}p_i^\alpha q_i^{1-\alpha}\right) \tag{22}$$

$$= \frac{\sum_{i=1}^{n}\left(\frac{p_i}{q_i}\right)^\alpha q_i \log\frac{p_i}{q_i}}{\sum_{i=1}^{n}\left(\frac{p_i}{q_i}\right)^\alpha q_i}. \tag{23}$$

The $\alpha \to 1$ limit of the above formula can be evaluated as follows:

$$\partial_\alpha((\alpha-1)D_\alpha)|_{\alpha\to 1} = D(P\|Q) \geq 0. \tag{24}$$

In the last step we have used the non-negativity of the relative entropy, whose proof requires the normalization conditions for both P and Q.

2.4. *Convexity of* $(\alpha-1)D_\alpha$

Now we prove $\partial_\alpha^2((\alpha-1)D_\alpha) \geq 0$. We take one more derivative of (23) with respect to α

$$\partial_\alpha^2((\alpha-1)D_\alpha) = \frac{\sum_{i=1}^{n}\left(\frac{p_i}{q_i}\right)^\alpha q_i\left(\log\frac{p_i}{q_i}\right)^2}{\sum_{i=1}^{n}\left(\frac{p_i}{q_i}\right)^\alpha q_i} - \frac{\left[\sum_{i=1}^{n}\left(\frac{p_i}{q_i}\right)^\alpha q_i\log\frac{p_i}{q_i}\right]^2}{\left[\sum_{i=1}^{n}\left(\frac{p_i}{q_i}\right)^\alpha q_i\right]^2}. \tag{25}$$

Define a new distribution

$$\bar{\rho}_i := \frac{\left(\frac{p_i}{q_i}\right)^\alpha q_i}{\sum_{i=1}^n \left(\frac{p_i}{q_i}\right)^\alpha q_i}, \tag{26}$$

which automatically satisfies the normalization $\sum_i \bar{\rho}_i = 1$, one can rewrite (25) as follows:

$$\partial_\alpha^2((\alpha-1)D_\alpha) = \sum_{i=1}^n \bar{\rho}_i \left(\log \frac{p_i}{q_i}\right)^2 - \left(\sum_{i=1}^n \bar{\rho}_i \log \frac{p_i}{q_i}\right)^2 \geq 0, \tag{27}$$

where the last step follows from the convexity of the function $f(x) = x^2$ and Jensen's inequality.

3. Inequalities of Supersymmetric Rényi Entropy

In QFTs in flat space, the Rényi entropy can be used to measure the entanglement spectrum between two regions A and $\bar{A}$ separated by the entangling surface Σ. For a state characterized by a density matrix ρ_0 on a spatial slice consisting of A and $\bar{A}$, one can define the Rényi entropy for A using the reduced density matrix $\rho_A = \mathrm{Tr}_{\bar{A}}\, \rho_0$,

$$S_\alpha = \frac{1}{1-\alpha} \log\, \mathrm{Tr}\, \rho_A^\alpha, \tag{28}$$

where $\alpha > 0$. S_α in (28) is the field theory analogy of H_α in (2) in information theory. Note that the previous definition now has been generalized to infinite-dimensional spaces by replacing the probabilities by the reduced density matrix and the sum by a trace. In Euclidean QFT, a state is characterized by a path integral with certain boundary conditions. Therefore, (28) can be expressed in terms of path integrals on a Euclidean spacetime with a conical singularity.

We focus on CFTs in $\mathbb{R}^{1,d-1}$, the Rényi entropy (28) associated with a spherical entangling surface ($\Sigma = \mathbb{S}^{d-2}$) can be computed by conformally mapping the Euclidean conic space to a hyperbolic space $\mathbb{S}_\alpha^1 \times \mathbb{H}^{d-1}$, where the previous density matrix ρ_A now becomes $\rho \propto e^{-2\pi H}$ by a unitary transformation and H is the Hamiltonian quantized on $\mathbb{H}^{d-1}$.[12] In this case, S_α can be written as follows:

$$S_\alpha = \frac{1}{1-\alpha} \log\, \mathrm{Tr}\, \rho^\alpha = \frac{1}{1-\alpha} \log \frac{\mathrm{Tr}\, e^{-2\pi\alpha H}}{(\mathrm{Tr}\, e^{-2\pi H})^\alpha}, \tag{29}$$

where we have considered the normalization $\mathrm{Tr}\, \rho = 1$.

We are particularly interested in supersymmetric theories with a conserved $U(1)$ R-symmetry because the computation of S_α is very challenging for interacting theories. We also restrict ourselves to the spherical entangling surface without considering the shape dependence. In the viewpoint of rigid supersymmetry, the spacetime with a conical singularity breaks all the supersymmetries. Equivalently, the space $\mathbb{S}_\alpha^1 \times \mathbb{H}^{d-1}$ for $\alpha \neq 1$ does not preserve any supersymmetry. To proceed further, we twist the Rényi entropy (29) into a supersymmetric one by turning on a background R-symmetry gauge field along $\mathbb{S}^1_\alpha$. This twisting has been first studied in three dimensions[26,27] and then extended to other dimensions.[30–39] The supersymmetric twist can be written[b] as follows:

$$\bar{S}_\alpha = \frac{1}{1-\alpha}\log\frac{\operatorname{Tr} e^{-2\pi\alpha(H-\mu\hat{Q})}}{(\operatorname{Tr} e^{-2\pi H})^\alpha} \tag{30}$$

$$= \frac{1}{1-\alpha}\log\frac{\operatorname{Tr} e^{-2\pi(\alpha H-\hat{Q}(\alpha-1))}}{(\operatorname{Tr} e^{-2\pi H})^\alpha}, \tag{31}$$

where the chemical potential corresponding to the conserved $U(1)$ R-symmetry takes the value

$$\mu = \frac{\alpha-1}{\alpha} \tag{32}$$

as required by the Killing spinor equations. Note that we choose the convention such that the preserved Killing spinors' R-charge $r = 1/2$ in general d-dimensions. For details on how to determine μ by solving Killing spinor equations on conic space in various dimensions, $d = 2, 3, 4, 5, 6$, see Refs. 38, 27, 30, 34 and 36, respectively. Obviously $\bar{S}_\alpha$ returns to S_{EE} at $\alpha \to 1$. By unitarily transforming the effective density matrix in (31), one can rewrite $\bar{S}_\alpha$ in flat space by replacing H with the modular Hamiltonian K and replacing $\hat{Q}$ with the conserved R-charge $\hat{Q}'$ defined in the subregion A. Note that the trace now is taken over the Hilbert space of the subregion A as (28). One can observe a connection between the supersymmetric Rényi entropy and the Rényi divergence D_α in (4). That is, by identifying

$$\rho_A = \frac{e^{-2\pi K}}{\operatorname{Tr} e^{-2\pi K}}, \quad \sigma_A = e^{-2\pi\hat{Q}'}, \tag{33}$$

one can express $\bar{S}_\alpha$ in terms of the Rényi divergence[c] of ρ_A from σ_A

$$\bar{S}_\alpha = \frac{1}{1-\alpha}\log \operatorname{Tr}\rho_A^\alpha\sigma_A^{1-\alpha} \tag{34}$$

$$= -D_\alpha(\rho_A\|\sigma_A). \tag{35}$$

[b]This trace formula for supersymmetric Rényi entropy has passed nontrivial tests. For instance, one can check a relation derived from it[37]

$$S'_{\alpha=1} = -V_{d-1}\left(\frac{\pi^{\frac{d}{2}+1}\Gamma(\frac{d}{2})(d-1)}{(d+1)!}C_T - \frac{\pi^{\frac{d+3}{2}}}{2^{d-3}(d-1)\Gamma(\frac{d-1}{2})}C_J\right),$$

where C_T and C_J are defined from the stress tensor 2-point correlator and the R-current 2-point correlator, respectively. Note that $\hat{Q}$ in this paper is equal to $\alpha\hat{Q}$ in Ref. 37. The charged Rényi entropy defined in Ref. 40 is not supersymmetric.

[c]Both K and $\hat{Q}'$ are Hermitian in real-time quantization. The chemical potential is kept to be real in our convention and it is unchanged under Weyl transformations. A path integral approach to Rényi divergence of different distributions is given in Refs. 41 and 42.

Note that to make the identification (35) we temporarily abandon the normalization condition for σ_A. Also note that in our case $[\rho_A, \sigma_A] = 0$ because of $[K, \hat{Q}'] = 0$, therefore we do not distinguish Rényi divergence and quantum Rényi divergence. As one can see from the previous section, the normalization condition for the second distribution is not necessary for the proof of (10) and (13). Therefore, the following two inequalities follow directly by replacing D_α by $-\bar{S}_\alpha$ in (10) and (13),

$$\partial_\alpha \bar{S}_\alpha \leq 0, \tag{36}$$

$$\partial_\alpha^2((\alpha - 1)\bar{S}_\alpha) \leq 0. \tag{37}$$

This proves the monotonicity of $\bar{S}_\alpha$ and the concavity of $(\alpha - 1)\bar{S}_\alpha$. One can think that they are the analogies of the properties (6) and (9) of the Rényi entropy.

Now we study the other two analogies of (7) and (8) for $\bar{S}_\alpha$:

$$\partial_\alpha\left(\frac{\alpha - 1}{\alpha}\bar{S}_\alpha\right) \geq 0, \tag{38}$$

$$\partial_\alpha((\alpha - 1)\bar{S}_\alpha) \geq 0. \tag{39}$$

As one can see, they cannot be deduced from (11) or (12), because the normalization condition of σ now is crucial.

We instead give a physical proof of (38) and (39) by following the way in Ref. 18. We begin with the supersymmetric partition function Z on $\mathbb{S}^1_\alpha \times \mathbb{H}^{d-1}$ with a $U(1)$ R-symmetry chemical potential. We work in grand canonical ensemble

$$Z[\beta, \mu] = \mathrm{Tr}\big[e^{-\beta(H - \mu\hat{Q})}\big], \tag{40}$$

where the inverse temperature β and the chemical potential μ are background parameters, $\beta = 2\pi\alpha$, $\mu = \frac{\alpha-1}{\alpha}$. Define $I := -\log Z$, the state variables can be worked out from (40) as follows:

$$E = \left(\frac{\partial I}{\partial \beta}\right)_\mu - \frac{\mu}{\beta}\left(\frac{\partial I}{\partial \mu}\right)_\beta, \tag{41}$$

$$S = \beta\left(\frac{\partial I}{\partial \beta}\right)_\mu - I, \tag{42}$$

$$Q = -\frac{1}{\beta}\left(\frac{\partial I}{\partial \mu}\right)_\beta. \tag{43}$$

Therefore, we obtained the energy expectation value $E = \mathrm{Tr}(e^{-\beta(H-\mu\hat{Q})}H)/\mathrm{Tr}(e^{-\beta(H-\mu\hat{Q})})$ by (41) and the charge expectation value $Q = \mathrm{Tr}(e^{-\beta(H-\mu\hat{Q})}\hat{Q})/\mathrm{Tr}(e^{-\beta(H-\mu\hat{Q})})$ by (43). The thermal entropy S is given by

$$S = \beta(E - \mu Q) - I. \tag{44}$$

In the presence of supersymmetry, both the inverse temperature β and the chemical potential μ are functions of a single variable α and therefore I is considered as follows:

$$I_\alpha := I[\beta(\alpha), \mu(\alpha)]. \tag{45}$$

The supersymmetric Rényi entropy is defined as follows:

$$\bar{S}_\alpha = \frac{\alpha}{1-\alpha}\left(I_1 - \frac{I_\alpha}{\alpha}\right) = \frac{\alpha}{1-\alpha}\int_\alpha^1 \partial_{\alpha'}\left(\frac{I_{\alpha'}}{\alpha'}\right). \tag{46}$$

From this expression, one can write

$$\partial_\alpha\left(\frac{\alpha-1}{\alpha}\bar{S}_\alpha\right) = \partial_\alpha\left(\frac{I_\alpha}{\alpha}\right) \tag{47}$$

$$= \frac{\beta(E-Q) - I_\alpha}{\alpha^2}. \tag{48}$$

When $\hat{Q}$ vanishes, the numerator is exactly the thermal entropy (44), which was assumed to be positive in Ref. 18 to prove the inequality (7) of the Rényi entropy. In the presence of supersymmetry, one may extend the positive thermal entropy condition to be

$$S \geq 2\pi Q. \tag{49}$$

That is, the thermal entropy is bounded from below by the charge. While a general field theory argument for this bound is still lacking, we give a holographic derivation for CFTs having gravity duals in Appendix A. The holographic derivation shows that this bound comes from the causality in gravitational physics. Then

$$\beta(E-Q) - I_\alpha = S - 2\pi Q \geq 0, \tag{50}$$

which ensures that

$$\partial_\alpha\left(\frac{\alpha-1}{\alpha}\bar{S}_\alpha\right) \geq 0. \tag{51}$$

In fact, one can rewrite the proved inequality (37) as $\partial_\alpha(E-Q) \leq 0$. Note that the first α-derivative of (50), $\partial_\alpha(\beta(E-Q) - I_\alpha) = \beta\partial_\alpha(E-Q) \leq 0$, one can prove (50) and therefore (38) with the minimal assumption $(S - 2\pi Q)_{\alpha\to\infty} \geq 0$.

We are left to prove (39). Given the non-positivity of the second derivative of $(\alpha-1)\bar{S}_\alpha$, (37), the only thing we have to show is that

$$\partial_\alpha[(\alpha-1)\bar{S}_\alpha]\big|_\infty \geq 0. \tag{52}$$

By using the last expression in (46), we have

$$\partial_\alpha[(\alpha-1)\bar{S}_\alpha]\big|_\infty = \left[\int_1^\alpha \partial_{\alpha'}\left(\frac{I_{\alpha\prime}}{\alpha'}\right) + \alpha\partial_\alpha\left(\frac{I_\alpha}{\alpha}\right)\right]_\infty \tag{53}$$

$$\geq 0, \tag{54}$$

where the last step follows from the positivity of $\partial_\alpha\left(\frac{I_\alpha}{\alpha}\right)$ (47).

In summary, we have shown that the inequalities (36)–(39) hold for supersymmetric Rényi entropy under the assumption that the thermal entropy is bounded from below by the charge at $\alpha \to \infty$. One can also express (36) in terms of the thermal entropy and the energy:

$$\frac{S - 2\pi E + I_1}{(\alpha - 1)^2} \leq 0, \tag{55}$$

which is equivalent to ($\alpha \neq 1$)

$$\Delta S \leq 2\pi \Delta E, \tag{56}$$

where $\Delta S := S - S_{\alpha=1}$ and $\Delta E := E - E_{\alpha=1}$. (56) is the Bekenstein bound under the deformation parametrized by $\delta = \alpha - 1$ in the spirit of Ref. 7. This bound is independent of the charge therefore it can also be derived from the ordinary Rényi entropy property. One may also write (39) equivalently as $2\pi(E - Q) - I_1 \geq 0$.

4. Applications

Now we discuss the applications of the inequalities (36)–(39). Our main concern is a spherical entangling surface in CFTs in flat space, the universal part of Rényi entropy (or supersymmetric) is invariant on $\mathbb{H}^{d-p} \times \mathbb{S}^p_\alpha$ for different integer p, where α denotes a conical singularity and $1 \leq p \leq d$, since these geometries are related by Weyl transformations.[12,37] We mainly focus on $\mathbb{S}^1_\alpha \times \mathbb{H}^{d-1}$ but it is equivalent to working on other geometries such as conic sphere $\mathbb{S}^d_\alpha$. In order to avoid a sign ambiguity coming from the regularization of the volume V_{d-1} of the hyperbolic space $\mathbb{H}^{d-1}$, we instead consider $s_\alpha := \bar{S}_\alpha / V_{d-1}$ as the true quantity in applying the inequalities (36)–(39).

4.1. *Even-dimensional SCFTs*

- $d = 2$, $\mathcal{N} = (2, 2)$ SCFT

For these theories, $\bar{S}_\alpha$ has been computed from the partition function on branched two sphere[38] or the correlation function of twisted fields.[39] $\bar{S}_\alpha$ is independent of α and coincides with the entanglement entropy, whose log term is $\frac{c}{3} \log \frac{R}{\epsilon}$ where c is the $2d$ central charge and R is the length of a single interval. Therefore, the inequalities (36)–(39) trivially hold

$$0 = 0, \quad 0 = 0, \quad \frac{c}{\alpha^2} \geq 0, \quad c \geq 0. \tag{57}$$

- $d = 4$, $\mathcal{N} = 1$ SCFT

For these theories, there is a conserved $U(1)$ R-symmetry. We consider Lagrangian theories in flat space with the entangling surface being a round 2-sphere with radius R, $\bar{S}_\alpha$ enjoys a universal behavior at $\alpha \ll 1$[35]

$$\bar{S}_{\alpha \ll 1} = \frac{4}{27\alpha^2}(3c - 2a)\frac{V_3}{2\pi}, \quad V_3 = -2\pi \log \frac{R}{\epsilon}, \tag{58}$$

where V_3 is the regularized volume of $\mathbb{H}^3$ and a, c are the Weyl anomaly coefficients defined from the anomalous trace of the stress tensor in $4d$ curved background

$$\langle T_\mu^\mu \rangle = \frac{1}{(4\pi)^2}(aE - cW^2). \tag{59}$$

Equation (58) was derived from the free field computation with a nontrivial R-symmetry background and shown[35] to be universal for SCFTs by matching to the $4d$ supersymmetric Casimir energy[43] on an extremely squashed sphere. Plugging (58) into the four different inequalities (36)–(39), one obtains a single constraint $3c - 2a \geq 0$, which is the Hofman–Maldacena upper bound for general $\mathcal{N} = 1$ SCFTs.[44] Together with the unitarity bound $c > 0$ and the positivity of the universal spherical entanglement entropy $S_{\mathrm{EE}} \propto a$,[12,d] we have

$$\frac{3}{2} \geq \frac{a}{c} \geq 0. \tag{60}$$

Note that this is not as tight as the $\mathcal{N} = 1$ Hofman–Maldacena bounds, $\frac{1}{2} \leq \frac{a}{c} \leq \frac{3}{2}$. For recent approaches to a proof of Hofman–Maldacena bounds, see Refs. 46 and 47.

- $d = 4$, $\mathcal{N} = 2$ SCFT

For these theories, the R-symmetry is $SU(2)_R \times U(1)_R$. The $U(1)_R$ may be broken for the purpose of defining sphere partition functions.[48] We turn on the background field corresponding to $U(1)_J \subset SU(2)_R$ to twist the Rényi entropy. Note that we focus on the universal logarithmic term of the supersymmetric Rényi entropy. For Lagrangian theories, $\bar{S}_\alpha$ has been determined completely in terms of $4d$ Weyl anomaly coefficients a, c[35]

$$\bar{S}_\alpha = \left(\frac{c}{\alpha} + 4a - c\right)\frac{V_3}{2\pi}. \tag{61}$$

This result was first derived from the free field computation and shown to be universal[35] by matching to the localization results in Refs. 30 and 49. By plugging (61) into the inequalities (36)–(39), one obtains

$$c \geq 0, \quad c \geq 0, \quad c + (2a - c)\alpha \geq 0, \quad 4a - c + \frac{c}{\alpha^2} \geq 0. \tag{62}$$

The large α limit of the third inequality gives $2a - c \geq 0$, which is the Hofman–Maldacena lower bound for general $\mathcal{N} = 2$ SCFTs. Together with $a/c \leq 3/2$ one obtains

$$\frac{1}{2} \leq \frac{a}{c} \leq \frac{3}{2}. \tag{63}$$

The upper bound comes from describing $\mathcal{N} = 2$ theories as $\mathcal{N} = 1$ ones. Note that (63) is not as tight as the $\mathcal{N} = 2$ Hofman–Maldacena bounds, $\frac{1}{2} \leq \frac{a}{c} \leq \frac{5}{4}$. Four-dimensional $\mathcal{N} = 4$ Super-Yang Mills (SYM) always has positive $a = c$ and its

[d] For the entangling surface with a nontrivial topology, this positivity is not guaranteed.[45]

universal supersymmetric Rényi entropy can be derived either from the free field computation or from the holographic computation on $5d$ BPS charged topological AdS black holes.[30]

- $d = 6$, $\mathcal{N} = (2, 0)$ SCFT

For these theories, the R-symmetry group is $SO(5)$ and the two Cartans are on the equal footing. $\bar{S}_\alpha$ has been determined completely[37] in terms of the $6d$ Weyl anomaly coefficients a, c,[50,51] which are defined from the anomalous trace of the $6d$ stress tensor in curved background (with the normalization such that a free tensor multiplet has units a and c)

$$\bar{S}_\alpha \frac{\pi^2}{V_5} = \frac{r_1^2 r_2^2}{12} \frac{7a - 3c}{4} (\gamma - 1)^3 + \frac{r_1 r_2}{12} c(\gamma - 1)^2 + \frac{1 + 2r_1 r_2}{12} c(\gamma - 1) + \frac{7}{12} a, \quad (64)$$

where $\gamma := 1/\alpha$ and $r_{1,2} \geq 0$ are the weights of the two chemical potentials with a constraint $r_1 + r_2 = 1$. This result was obtained by making use of $\bar{S}_\alpha$ of a free tensor multiplet,[36] the 2- and 3-point functions of the stress tensor multiplet[50] and the $6d$ supersymmetric Casimir energy[52] on an extremely squashed sphere. The large N limit of (64) agrees with the holographic result from $7d$ BPS charged topological AdS black holes.[37] Plugging (64) into (36)–(39) and demanding that the inequalities hold for any positive α, one can get

$$\frac{a}{c} \geq \frac{3}{7}, \quad c \geq 0. \quad (65)$$

The lower bound of a/c together with the unitarity bound $c > 0$ also proves the positivity of a.

4.2. *Other examples*

In odd-dimensional CFTs, the finite parts of the entanglement entropy and the Rényi entropy (or supersymmetric) associated with a spherical entangling surface in flat space are considered to be universal and physical. One can compute them by mapping to a branched sphere $\mathbb{S}^d_\alpha$ because there is no conformal anomaly. For $d = 3, \mathcal{N} = 2$ superconformal Chern–Simons gauge theories with M-theory duals, $\bar{S}_\alpha$ in the large N limit has the scaling $\bar{S}_\alpha/\bar{S}_1 = (3\alpha + 1)/4\alpha$, which satisfies all the four inequalities (36)–(39) as observed in Ref. 26. For $d = 5, \mathcal{N} = 1$ superconformal theory with AdS_6 dual,[53] $\bar{S}_\alpha$ in the large N limit has the scaling[32–34] $\bar{S}_\alpha/\bar{S}_1 = (19\alpha^2 + 7\alpha + 1)/27\alpha^2$, which also satisfies the four inequalities as observed in Ref. 34. One can also numerically check the inequalities for other $5d$ or $3d$ superconformal examples including ABJM with finite N.

Appendix A. A Holographic Derivation of $S \geq 2\pi Q$

We consider a $(d + 1)$-dimensional BPS charged topological AdS black hole, which is the gravity dual of the ground state in SCFT_d on supersymmetric $\mathbb{S}^1_\alpha \times \mathbb{H}^{d-1}$

and used to compute the holographic supersymmetric Rényi entropy. Below we will take $5d$ $\mathcal{N}=1$ supersymmetric $USp(2N)$ gauge theory with N_f fundamental hypermultiplets and a single hypermultiplet in the antisymmetric representation as an example, but the argument also goes well in other dimensions. The gravity dual of the ground state of this $5d$ SCFT on $\mathbb{S}^1_\alpha \times \mathbb{H}^4$ is given by a $6d$ BPS charged topological AdS black hole[32–34,54]

$$\begin{gathered} ds^2 = -H^{-3/2} f\, dt^2 + H^{1/2}(f^{-1}\, dr^2 + r^2\, d\Omega^2_{4,-1}), \\ f = -1 + \frac{r^2}{R^2} H^2, \quad H = 1 + \frac{q}{r^3}, \end{gathered} \tag{A.1}$$

together with the scalar and the gauge field

$$X = H^{-1/4}, \quad A = \left(\sqrt{2}(H^{-1}-1) + \mu\right) d\tau, \tag{A.2}$$

where $d\Omega^2_{4,-1}$ denotes the metric on $\mathbb{H}^4$ and $t = -i\tau$. We define a rescaled charge $\kappa = q/r_h^3$, where the event horizon r_h is the largest root of the equation $f(r_h) = 0$. The Hawking temperature, the Bekenstein–Hawking entropy, the total charge and the chemical potential can be worked out as follows:

$$T = \frac{1}{2\pi R}\frac{2-\kappa}{2(1+\kappa)^2}, \tag{A.3}$$

$$S = \frac{V_4 R}{4G_6} r_h^3, \tag{A.4}$$

$$Q = -3\sqrt{2}\kappa \frac{V_4\, r_h^3}{16\pi G_6}, \tag{A.5}$$

$$\mu = \frac{\sqrt{2}}{\kappa^{-1}+1}, \tag{A.6}$$

where G_6 is the 6-dimensional Newton constant and V_4 is the regularized volume of unit $\mathbb{H}^4$. We choose a new normalization for μ and Q such that when $T = \frac{1}{2\pi R\alpha}$, $\tilde{\mu}$ takes the value $\frac{\alpha-1}{\alpha}$ matching to that in (32). In this case, the normalized charge is given by

$$\tilde{Q} = -\kappa \frac{V_4\, r_h^3}{8\pi G_6}. \tag{A.7}$$

The horizon radius r_h should be positive, $r_h > 0$. Then the positivity of

$$S - 2\pi R\tilde{Q} = \frac{V_4 R}{4G_6} r_h^3 H(r_h) \tag{A.8}$$

is guaranteed by the causality, since the sign flip of $H(r)$ in the metric (A.1) is forbidden before reaching to the horizon. $H(r_h) \geq 0$ ensures $S \geq 2\pi R\tilde{Q}$. The same argument goes well in other dimensions, $d = 3, 4, 6$. Note that we restored the length scale R, which has been omitted in the body part. The holographic supersymmetric Rényi entropy can be computed straightforwardly by employing the formula derived in Ref. 27. Recently, there is a holographic study of the Rényi entropy inequalities[56] based on Ref. 55, it would be interesting to consider our bound $S \geq 2\pi RQ$ along that way.

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