FORMALISM AND PHENOMENOLOGY OF COMPLEX ANGULAR MOMENTUM

Lectures by: Kerson Huang
Massachusetts Institute of Technology

Notes taken by: Frank E. Paige

Table of Contents

Introduction

In these lectures we discuss two-body hadronic scattering in the high-energy limit, under the hypothesis that it is dominated by Regge singularities, i.e., singularities in the finite parts of the complex angular momentum plane of the partial-wave amplitudes in the crossed channel. In particular we discuss the motivation of the hypothesis, the procedure for putting it into practical use, some of its experimental consequences, and possible glimpses into the dynamics of strong interactions. For general references the following books are recommended.

R.J. Eden, High Energy Collisions of Elementary Particles, (Cambridge, The University Press, (1967)).

E.J. Squires, Complex Angular Momentum and Particle Physics (W.A. Benjamin, New York (1964)).

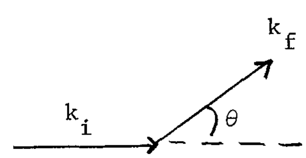
P.D.B. Collins and E.J. Squires, Regge Poles in Particle Physics (Springer-Verlag, Berlin (1968)).

I. Regge Poles in Potential Scattering

A. Regge Poles and Resonances:

As an introduction to the idea of Regge poles, we give a brief review of potential scattering, where they were first introduced as a new way to describe bound states and resonances.

Suppose a spinless non-relativistic particle is scattered by a central potential central V(r), with kinematics as shown in the accompanying sketch.



In units such that $\hbar = 2m = 1$ let E be the energy of the particle and z be the cosine of the scattering angle:

$$E = k^2$$
$$k = |\vec{k}_i| = |\vec{k}_f|$$
$$z = \cos\theta = \vec{k}_i \cdot \vec{k}_f / k^2 \quad . \tag{1.1}$$

The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(E,z)|^2 \tag{1.2}$$

where the scattering amplitude f(E,z) has the familiar partial-wave expansion

$$f(E,z) = \sum_{\ell=0}^{\infty} (2\ell+1) F_\ell(E) P_\ell(z) \quad , \tag{1.3}$$

where the partial-wave amplitude $F_\ell(E)$ is determinable from the solution to the radial equation

$$\frac{d^2u_\ell}{dr^2} + Eu_\ell(r) = \left[V(r) + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r)$$

$$u_\ell(r) \underset{r \to 0}{\to} 0 \quad . \tag{1.4}$$

Asymptotically the solution is of the form

$$u_\ell(r) \to C\sin\left(kr - \frac{\pi\ell}{2} + \delta_\ell(E)\right) \quad . \tag{1.5}$$

The partial-wave ampliitude is then given in terms of the phase shift $\delta_\ell(E)$ by

$$F_\ell(E) = \frac{1}{2ik}\,[e^{2i\delta_\ell(E)}-1] \quad . \tag{1.6}$$

It is a real analytic function of E, and it has a branch cut along the positive real E axis. There are no poles on the physical Riemann sheet except along the negative real axis, where they correspond to bound states of spin ℓ. Complex poles can occur only in conjugate pairs on the second Riemann sheet. If they are close to the branch cut, the one just below the cut is near the physical region, and correspond to a resonance of spin ℓ. Its conjugate partner is far from the physical region, and thus not directly "visible". (Except when the pair of poles are near E=0, but there threshold effects become important.)

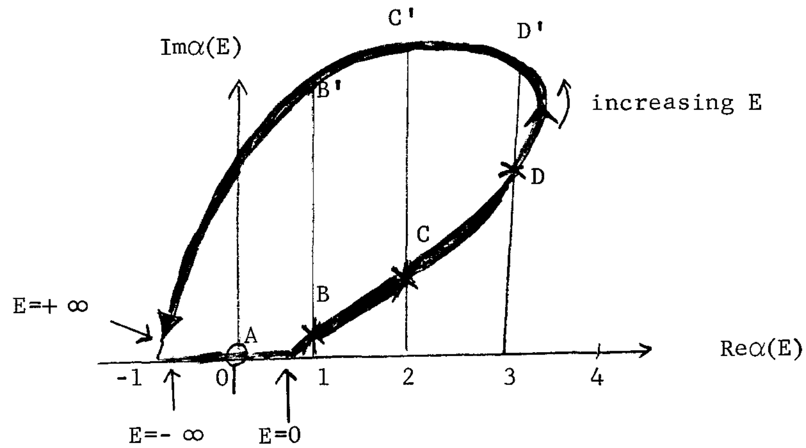
Regge shows that the same bound states and resonances show up as poles of $F_\ell(E)$ in the complex ℓ-plane, in the following way. First, from the radial equation for a superposition of Yukawa potentials, one can show that $F_\ell(E)$ can be uniquely continued to complex ℓ, thereby giving a function $F(E,\ell)$. It has the following properties:

1. $F(E,\ell)$ is meromorphic for $\mathrm{Re}\,\ell > -\frac{1}{2}$,
2. $F(E,\ell) \to 0$ as $|\ell| \to \infty$,
3. The positions of the poles in ℓ move with the energy E.

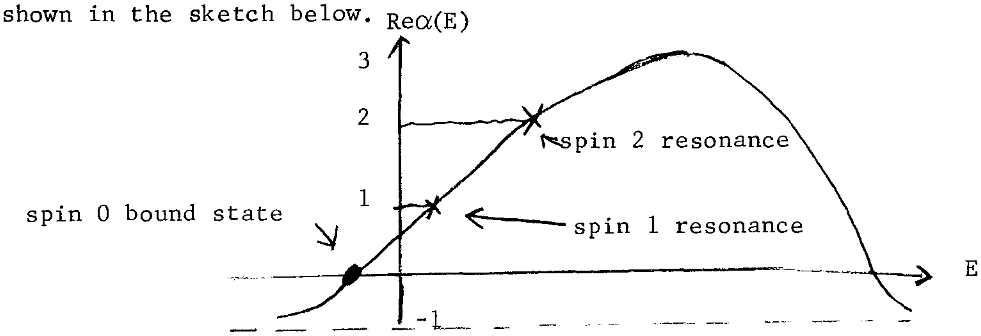
Such a moving pole is called a Regge pole, its locus $\alpha(E)$ a Regge trajectory.

In the usual description, a resonance is identified with a pole of $F(\ell,E)$ in E, at a positive integer value of $\alpha(E)$, which generally occurs at complex E. We now propose to keep E real and associate resonances with the behavior of $\alpha(E)$ in the complex α plane.

A typical locus of $\alpha(E)$ in the complex angular momentum plane is shown in the sketch below.



The imaginary part Imα(E) vanishes for E < 0. Whenever Reα(E) passes through positive integer ℓ with d[Reα(E)]/dE > 0, a bound state or resonance of spin ℓ occurs, provided Imα(E) is small. In the sketch, for example, A is a bound state, and B,C,D are resonances. This family of bound states and resonances appear as recurrences of the same state. Other families can occur as well, and will be characterized by other trajectories. Each family is characterized by the principal quantum number (i.e. the number of codes in the radial wave function). The points B', C', D' do not correspond to identifiable resonances, because the poles corresponding to them are far from the physical region. Actually, each trajectory α(E) has a complex conjugate partner represented by its mirror image with respect to the Re α axis. The mirror images of B,C,D and B',C',D' all lie too far from the physical region to be identifiable as resonances. Another way to exhibit the resonances is to plot Reα(E) against E, as shown in the sketch below.



To see the correspondence between the new way and the old way of describing a resonance, let us examine F(E,ℓ) near $\ell = \alpha(E)$:

$$F(E,\ell) \approx \frac{\beta(E)}{\ell-\alpha(E)} \quad . \tag{1.7}$$

Suppose that for some real value $E=E_n$ we have Re$\alpha(E_n) = n$. Then in the neighborhood of $E = E_n$ we can write

$$\mathrm{Re}\,\alpha(E) \cong n + \alpha'(E-E_n) \ , \qquad \alpha' \equiv \frac{d\alpha}{dE} \ (E=E_n) \quad .$$

Hence

$$F(E,n) \approx \frac{\beta(E_n)}{n-\alpha(E)}$$

$$\approx \frac{\beta(E_n)}{n-[n+\alpha'(E-E_n)]-i\,\mathrm{Im}\alpha(E_n)} \qquad (1.8)$$

$$= -\frac{1}{\alpha}\;\frac{\beta(E_n)}{E-E_n+i\,\frac{\mathrm{Im}\alpha}{\alpha'}}$$

If $\alpha' > 0$, this is the familiar Breit-Wigner formula for a resonance with mass E_n and total width $2\,\frac{\mathrm{Im}\alpha(E_n)}{\alpha'}$. If $\alpha' < 0$, there is still a pole on the second sheet, but it is not close to the physical region.

B. Sommerfeld-Watson Transform: To isolate the contribution of a Regge trajectory to the scattering amplitude, we write the partial-wave series in the form of a contour integral. Noting that $1/\sin\pi\ell$ has poles at integer values of ℓ, and

$$\mathrm{Res}\left[\frac{\pi(-1)^{\ell}}{\sin\pi\ell}\right] = 1, \quad \ell=0,\ \pm 1,\ \pm 2,\ldots$$

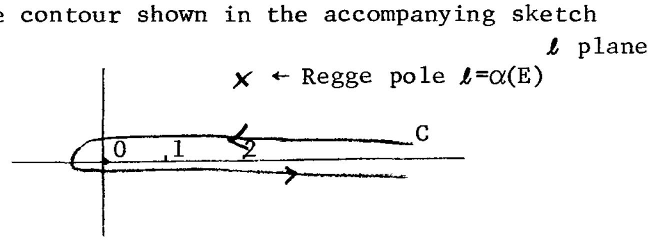
we have

$$f(E,z) = \sum_{\ell=0}^{\infty}(2\ell+1)P_{\ell}(z)F(E,\ell) = \frac{1}{2\pi i}\int_C d\ell\,\frac{\pi}{\sin\pi\ell}\,P_{\ell}(-z)F(E,\ell) \qquad (1.9)$$

where we have used

$$P_{\ell}(-z) = (-1)^{\ell}\,P_{\ell}(z) \qquad (1.10)$$

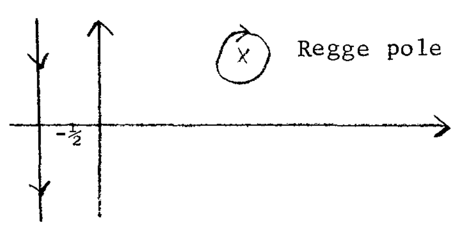
and where C is the contour shown in the accompanying sketch



For fixed $|z| \le 1$,

$$P_{\ell}(-z) \xrightarrow[|\ell|\to\infty]{} \frac{C}{(\ell)^{\frac{1}{2}}}\,e^{-\theta|\mathrm{Im}\ell|}, \quad (z=\cos\theta) \qquad (1.11)$$

Since $F(E,\ell) \to 0$ as $|\ell| \to \infty$, we can expand the contour, drop the piece at infinity, and pick up the Regge poles:



$$f(E,z) = \sum_{\substack{\alpha \\ Re\alpha > -\frac{1}{2}}} \left[- \frac{\pi(2\alpha+1)\beta(E)P_\alpha(-z)}{\sin\pi\alpha} \right] + \frac{1}{2i} \int_{-\frac{1}{2}+i\infty}^{-\frac{1}{2}-i\infty} d\ell \, \frac{P_\ell(-z)}{\sin\pi\ell} \, F(E,\ell) \quad . \tag{1.12}$$

The term in brackets represents the contribution from a Regge trajectory $\alpha(E)$, which contains the effects of a whole family of bound states and resonances.

The original partial-wave expansion converges only for z lying in an ellipse with foci $\pm$ 1 (the Lehmann ellipse), but with (12) we can continue it outside of the ellipse. In particular (12) has a simple asymptotic form for $|z| \to \infty$. The region is of course unphysical for potential scattering; but for relativistic scattering it corresponds to high energy in the crossed channel. To obtain the asymptotic behavior we note

$$P_\alpha(z) \xrightarrow[|z|\to\infty]{} \frac{1}{(\pi)^{\frac{1}{2}}} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 1)} (2z)^\alpha \left[1 + 0(z^{-2}) \right], \ (\mathrm{Re}\ \alpha > -\tfrac{1}{2}) \ . \tag{1.13}$$

Hence as $|z| \to \infty$,

$$f(E,z) \sim \sum_{Re\alpha > -\frac{1}{2}} \left[-(\pi)^{\frac{1}{2}} \frac{(2\alpha + 1)}{\sin\pi\alpha} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 1)} (-2z)^\alpha \right] + 0(z^{-\frac{1}{2}}), \tag{1.14}$$

where the term $0\ (z^{-\frac{1}{2}})$ comes from the "background" integral in (12). If there are Regge poles with $Re\alpha(E) > -\frac{1}{2}$, then the highest one dominates the asymptotic behavior.

If there are no Regge poles with $Re\alpha(E) > -\frac{1}{2}$, then we learn nothing from (14). It won't help to push the background integral further to the left, even if that is possible. The reason is that $P_\alpha(z) = P_{-\alpha-1}(z)$, so that for $Re\alpha < -\frac{1}{2}$ the asymptotic behavior of $P_\alpha(-z)$ is

$$P_\alpha(z) \xrightarrow[|z| \to \infty]{} \frac{1}{(\pi)^{\frac{1}{2}}} \frac{\Gamma(-\alpha-\frac{1}{2})}{\Gamma(-\alpha)} (2z)^{-\alpha-1} [1+0(z^{-2})], (Re\alpha < -\tfrac{1}{2}) \tag{1.15}$$

instead of (13), hence the background integral would still dominate over the pole contributions. Thus, we need to know something about the background integral in (12), and the Mandelstam symmetry comes to our aid.

C. <u>Mandelstam Symmetry</u>: The Mandelstam symmetry states

$$F(E,\ell) = F(E,-\ell-1) \text{ for } \ell = \text{half-integer.} \tag{1.16}$$

Note that the radial equation (4) is invariant under $\ell \rightleftarrows -\ell-1$. If $V(r) \to \infty$ faster than r^{-2} as $r \to 0$, so that it dominates over the centrifugal potential $\ell(\ell+1)/r^2$, then $u_\ell(r)$ vanishes at r=0 in a manner independent of ℓ. In this case it is clear that the Mandelstam symmetry holds not only for half integers, but for all ℓ. If, however, the centrifugal potential dominates over V(r) near r=0, then the two solutions to the radial equation have the respective behaviors

$$u_\ell \xrightarrow[r \to 0]{} \begin{cases} r^\ell \\ r^{-\ell-1} \end{cases} \tag{1.17}$$

For $\ell > 0$ we must choose the first solution, while for $\ell < 0$ we must choose the second solution. It turns out that the Mandelstam symmetry holds only for half-integer ℓ.

We make use of the Mandelstam symmetry to do the Sommerfeld-Watson transform in a different way. First let us define

$$\mathcal{P}_\ell(z) = \begin{cases} P_\ell(z) & \ell = 0,1,2,\ldots \\ 0 & \ell = -1,\ -2,\ldots \end{cases} \tag{1.18}$$

This function can be continued to complex ℓ:

$$\mathcal{P}_\ell(z) = -\frac{\tan\pi\ell}{\pi^{\frac{1}{2}}}\, Q_{-\ell-1}(z) = \frac{1}{(\pi)^{\frac{1}{2}}}\,\frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+1)}\,(2z)^\ell\, F\left(-\frac{\ell}{2},\ \frac{1-\ell}{2};\ \frac{1}{2}-\ell;\ \frac{1}{z^2}\right) \tag{1.19}$$

with asymptotic behavior

$$\mathcal{P}_\ell(z) \xrightarrow[|z|\to\infty]{} \frac{1}{(\pi)^{\frac{1}{2}}}\,\frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+1)}\,(2z)^\ell\left[1 + 0(z^{-2})\right], \tag{1.20}$$

which holds for all ℓ. We note that $\mathcal{P}_\ell(z)$ has simple poles at half-integer ℓ, with residues given by

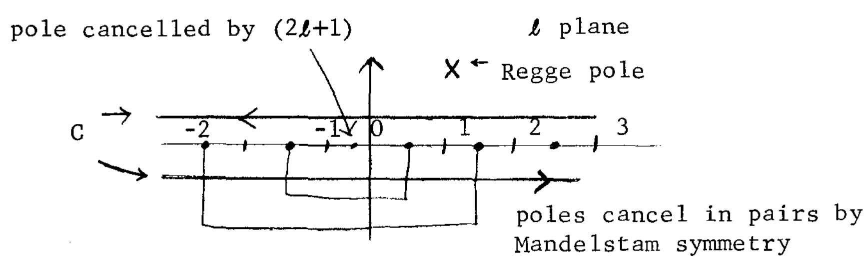
$$\mathrm{Res}\,\mathcal{P}_\ell(z) = \frac{(-)^{\ell-\frac{1}{2}}}{\pi}\, Q_{-\ell-1}(z),\ (\ell = \text{half-integer}) \tag{1.21}$$

$$\mathrm{Res}\,\mathcal{P}_\ell(z) = -\mathrm{Res}\, P_{-\ell-1}(z),\ (\ell = \text{half-integer}) \ . \tag{1.22}$$

The last equality comes from the well-known equality $Q_\ell(z) = Q_{-\ell-1}(z)$ at half-integer ℓ. We now write

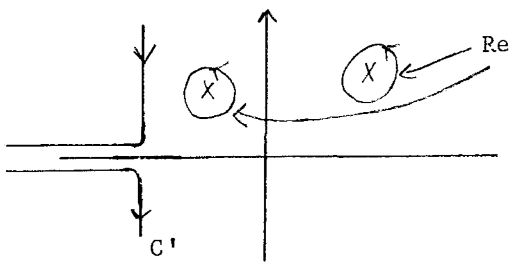
$$f(E,z) = \sum_{\ell=-\infty}^{\infty}(2\ell+1)\,F_\ell(E)\,\mathcal{P}_\ell(z) = \frac{1}{z_1}\int_C \frac{d\ell}{\sin\pi\ell}\,(2\ell+1)\,F(E,\ell)\,\mathcal{P}_\ell(z)\ , \tag{1.23}$$

where the contour C is shown in the accompanying sketch.



The poles of $\mathcal{P}_\ell(z)$ at half-integer ℓ do not give spurious contributions to the integral because the one at $\ell = \frac{1}{2}$ is cancelled by the factor $(2\ell + 1)$, and the other cancels in pairs by (16) and (22).

We now expand the contour and discard the contribution from infinity,



and obtain

$$f(E,z) = \sum_{\text{Re}\alpha > -L} \left[\frac{-\pi\beta(2\alpha+1)}{\sin\pi\alpha} P_\alpha(-z) \right] + \text{background integral.} \tag{1.24}$$

Hence

$$f(E,z) \xrightarrow[|z| \to \infty]{} \sum_{\text{Re}\alpha > -L} \left[-(\pi)^{\frac{1}{2}} \frac{\beta(2\alpha+1)}{\sin\pi\alpha} \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)} (-2z)^\alpha \right] + O(z^{-L}) \quad . \tag{1.25}$$

In this representation, the Regge poles always dominate the background integral.

D. Exchange Potential and Signature: Suppose we have an exchange potential

$$V(r) = V_1(r) + V_2(r)P \tag{1.26}$$

where

$$Pf(\vec{r}) = f(-\vec{r}) \quad . \tag{1.27}$$

Then the effect potential is different for even and odd partial waves, for the radial equation reads

$$\frac{d^2u_\ell}{dr^2} + Eu_\ell = \left[\frac{\ell(\ell+1)}{r^2} + V_1 + (-1)^\ell V_2 \right] u_\ell \quad .$$

Since $(-1)^\ell$ does not have a unique analytic continuation in ℓ , we separately continue the two equations

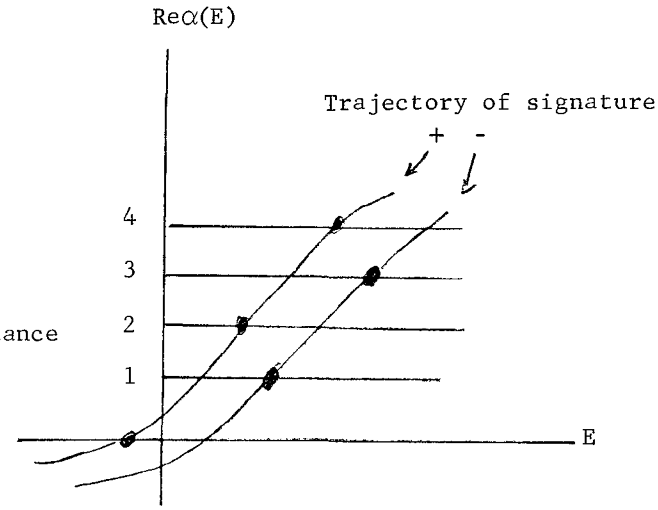
$$\frac{d^2u_\ell^\pm}{dr^2} + Eu_\ell^\pm = \left[\frac{\ell(\ell+1)}{r^2} + V_1 \pm V_2 \right] u_\ell^\pm$$

and obtain from them the two partial wave amplitudes $F^\pm(E,\ell)$. Clearly

$$\begin{aligned} F^+(E,\ell) &= F_\ell(E), (\ell \text{ even}) \\ F^-(E,\ell) &= F_\ell(E). (\ell \text{ odd.}) \end{aligned} \tag{1.28}$$

We refer to $F^\pm(E,\ell)$ as partial-wave amplitudes of even (odd) signature. Regge poles occuring in $F^\pm(E,\ell)$ will be characterized by signature. An even (odd) signatured Regge pole produces a resonance only where it passes through an even (odd) integer value. We illustrate this in the accompanying sketch. The two signatured trajectories become degenerate if either $V_1=0$, or $V_2=0$. Such a degeneracy is called exchange degeneracy.

● denotes a bound state or resonance



To carry out the Watson-Sommerfeld transform write

$$f(E,z) = \sum_{\ell \text{ even}} (2\ell + 1)\ F^{+}(E,\ell)\mathcal{P}_{\ell}(z) + \sum_{\ell \text{ odd}} (2\ell + 1)\ F^{-}(E,\ell)\mathcal{P}_{\ell}(z)$$

$$= \frac{1}{2}\sum_{\ell=-\infty}^{\infty} (2\ell + 1)F^{+}(E,\ell)\ \left[\mathcal{P}_{\ell}(z) + \mathcal{P}_{\ell}(-z)\right]$$

$$+ \frac{1}{2}\sum_{\ell=-\infty}^{\infty} (2\ell + 1)F^{-}(E,\ell)\ \left[\mathcal{P}_{\ell}(z) - \mathcal{P}_{\ell}(-z)\right] \qquad (1.29)$$

Then, in a manner analogous to the earlier development, we obtain

$$+(E,z) \sum_{\alpha \text{ of + signature}} \left\{ \frac{-\pi\beta(2d + 1)}{2\ \sin\pi\alpha} \left[\mathcal{P}_{\alpha}(-z) + \mathcal{P}_{\alpha}(z)\right] \right\}$$

$$+ \sum_{\alpha \text{ of - signature}} \left\{ - \frac{\pi\beta(2\alpha + 1)}{2\ \sin\pi\alpha} \left[\mathcal{P}_{\alpha}(-z) - \mathcal{P}_{\alpha}(-z)\right] \right\} \qquad (1.30)$$

+ (Background integrals).

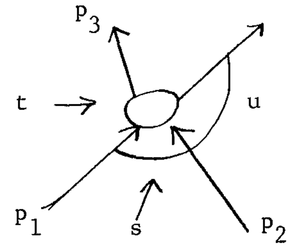
II. Relevance of Regge Poles to Relativistic Scattering

We discuss some motivations for taking over the ideas of Regge poles from the realm of potential scattering, where it is proved, to the realm of relativistic scattering, where it is unproved. There is a practical and a theoretical motivation. The former rests on the hope that Regge poles will lead to a simple description of high energy scattering. The latter is based on the fact that the bootstrap hypothesis seems to find a concrete expression in terms of Regge poles.

A. High-Energy Scattering

To illustrate the role of Regge poles in high-energy scattering, consider the elastic scattering of spinless particles of equal mass, represented schematically by the sketch shown, with

$$s = (p_1+p_2)^2 = 4(k^2+m^2)$$
$$t = (p_1-p_3)^2 = -2k^2(1-\cos\theta)$$
$$u = (p_1-p_4)^2 = -2k^2(1+\cos\theta) \qquad (2.1)$$



where k and θ are the center of mass three-momentum and scattering angle, respectively. Let the scattering amplitude f(s,t) describe the s-channel reaction $p_1+p_2 \rightarrow p_3+p_4$ for $s > 4m^2$, $t < 0$. Then by crossing symmetry, the same function f(s,t) describes the t-channel reaction $p_1+\bar{p}_3 \rightarrow \bar{p}_2+p_4$ when analytically continued to the region $t > 4m^2$, $s < 0$. Similarly, if f(s,t) is analytically continued to the region $u = 4m^2-s-t > 4m^2$, $s < 0$, $t < 0$, it describes the u-channel reaction $p_1+\bar{p}_4 \rightarrow \bar{p}_2+p_3$. Of course no such crossing symmetry exists in the case of potential scattering.

We now make a partial wave expansion in the s-channel:

$$f(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)F_\ell(s)P_\ell(z) \qquad (2.2)$$

where

$$z = \cos\theta = 1 + \frac{t}{2k^2} \quad . \qquad (2.3)$$

Suppose that we can continue $F_\ell(s)$ into the complex ℓ plane and carry out the Sommerfeld-Watson transform. Then, if the only singularities are simple poles, we will obtain as in potential scattering

$$f(s,t) \xrightarrow[(z)\rightarrow\infty]{} -\frac{\pi\beta(2\alpha+1)}{\sin\pi\alpha} P_\alpha(-z) \ , \qquad (2.4)$$

where $\alpha(s)$ is the leading Regge pole in the s-channel. Using (2.3) and the asymptotic form of P_α, we have

$$f(s,t) \xrightarrow[\substack{t \to \infty \\ s \text{ fixed}}]{} C(s)t^{\alpha(s)} \quad , \tag{2.5}$$

which says that the energy dependence of high-energy t-channel scattering at fixed s is governed by the leading Regge pole in the s-channel. Similarly, for the s-channel reaction, forward scattering ($\theta \to 0$) is governed by the leading t-channel Regge pole, and backward scattering ($\theta \to \pi$) is governed by the leading u-channel Regge pole.

$$f(s,t) \xrightarrow[\substack{s \to \infty \\ t \text{ fixed}}]{} C(t)s^{\alpha(t)} \tag{2.6}$$

$$f(s,t) \xrightarrow[\substack{s \to \infty \\ u \text{ fixed}}]{} C(u)s^{\alpha(u)} \quad . \tag{2.7}$$

We have not bothered to distinguish the trajectory α in (2.5), (2.6), (2.7), but of course they need not be the same trajectory. The t-channel trajectory, for example, generates bound states resonances having the quantum numbers of the t-channel, and will be characterized by these quantum numbers. We assume that the trajectory function α is independent of the external particles in the scattering process, and speak of "Regge pole exchange" in analogy with single-particle exchange. As we can see from (2.6) the salient feature of Regge pole exchange is that asymptotically the scattering amplitude is proportional to the α^{th} power of the squared c.m. energy, where α is the variable spin of the object exchanged in the crossed channel. As we change the momentum transfer t, the spin varies along the Regge trajectory. This furnishes a simple and physically attractive picture of high energy scattering.

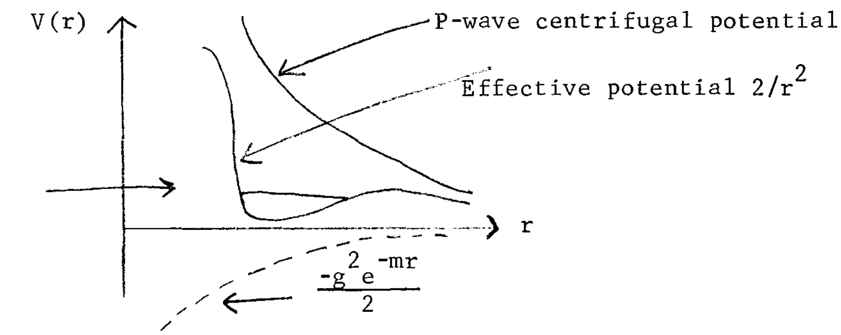
B. The Bootstrap Idea

The bootstrap idea, first proposed by Chew and Mandelstam, is that among the hadrons there are no "elementary" particles, but that they are composite states of one another. It has been difficult to state this idea in a form that is both sufficiently practical and sufficiently precise, so that one may use it in an actual calculation. To appreciate the difficulty, let us look at some attempts at formulation.

A simple-minded example, which illustrates the idea, but which does not give a consistent scheme for calculation, is the following. Suppose we calculate π-π scattering by solving a non-relativistic Schrödinger equation with an attractive Yukawa potential

$$V(r) = -g^2 \frac{e^{-mr}}{r} \quad , \tag{2.8}$$

which we regard as the adiabatic potential due to the exchange of a ρ meson of mass m and coupling constant g. The P-wave phase shift, $\delta_1(E;m,g)$ will then depend on the energy E of the $\pi\pi$ system, as well as on the parameters m and g. If m and g are appropriately chosen the P-wave effective potential, as shown in the sketch below, can accommodate a resonance, whose position and width depend on m and g. The bootstrap requires that this resonance be the ρ meson that generated the potential in the first place. Thus



δ_1 should pass through $\pi/2$ at E = m, with a slope consistent with the decay width Γ for $\rho \to \pi\pi$:

$$\Gamma = g^2 C(\mu,m) \quad , \tag{2.9}$$

where C depends on the pion mass μ and the ρ mass m in a known way. The relation between Γ and the phase shift may be obtained by noting

$$e^{i\delta_1}\sin\delta_1 = \frac{1}{\cot\delta_1 - i} \approx \frac{1}{\delta_1'(E_0)} \; \frac{1}{E-E_0+[i/\delta_1'(E_0)]} \tag{2.10}$$

where $\delta_1' = \partial\delta_1/\partial E$, and E_0 is such that $\delta_1(E_0) = \pi/2$. Thus we require

$$\begin{aligned} \delta_1(m;m,g) &= \pi/2 \\ \delta_1'(m;m,g) &= \frac{2}{g^2 C(\mu,m)} \end{aligned} \tag{2.11}$$

from which m/μ and g can be determined. This, however, is not a real example because the potential (2.8) is actually incorrect for spin 1 exchange, and there is no simple way to find a "correct" version. Also, pions don't obey the Schrödinger equation. A general way to state the bootstrap idea is that the requirements of analyticity, crossing symmetry, and unitarity, plus "boundary conditions" of some kind, should completely determine all scattering amplitudes, including the existence of particle poles, and their location and residues. To make this precise, one has to be more specific about the "boundary conditions". A suggestion that has underlined many practical calculations (the so-called N/D calculations) is to impose Levinson's theorem, taken over from potential scattering:

$$\delta_\ell(E=0)-\delta_\ell(E=\infty) = \pi N_\ell \tag{2.12}$$

where $\delta_\ell(E)$ is the ℓ^{th} wave phase shift, and N_ℓ is the number of bound states (not resonances, but bound states), of spin ℓ. When inelastic channels are open, one would replace $\delta_\ell(E)$ by eigen-phase shifts. The idea expressed by (2.12) is that, since $N_\ell=0$ when there is no interaction (i.e., when $\delta_\ell\equiv 0$), there would be no "elementary" bound state. In mathematical examples* in which (2.12) can be rigorously imposed, one does find that it determines the number of bound states and resonances that can occur, and places restrictions on their positions and coupling constants. But its general consequences has not been fully explored, owing to the difficulty in using it in a full relativistic scattering problem.

Instead of the Levinson theorem, it seems far simpler, and more satisfactory to take over from potential scattering the idea that all particles lie on Regge trajectories. The statement is precise, and is independent of a detailed formulation of the dynamical equations. It has the immediate experimental consequence that all known p hadrons should be classifiable according to Regge trajectories, which should also control the asymptotic behavior of scattering amplitudes.

C. Chew-Frautschi Plot

We can immediately test the hypothesis that all hadrons lie on Regge trajectories by plotting the spin vs. $(\text{mass})^2$ for known hadrons, resulting in what is known as Chew-Frautschi plots, as shown in the following figures. The trajectories that one might postulate from such a plot can be tested experimentally by analyzing high energy scattering data. A striking feature is that all known trajectories seem to be straight lines. The presence of the f^0 at spin 2 on the ρ trajectory suggests that there is exchange degeneracy of the ρ and f trajectories.

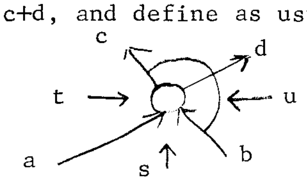
---

* K. Huang and A.H. Mueller, Phys. Rev. 140, B365 (1965).

III. Relativistic Scattering of Spinless Particles

A. Preliminaries

We consider the two body scattering process $a+b \to c+d$, and define as usual



$$s = (p_a+p_b)^2 = ((p_{ab}^2+m_a^2)^{\frac{1}{2}}+(p_{ab}^2+m_b^2)^{\frac{1}{2}})^2$$
$$t = (p_a-p_c)^2$$
$$u = (p_b-p_c)^2 \qquad (3.1)$$

where p_{ab} is the magnitude of the three-momentum in the center of mass of a and b. These variables satisfy

$$s+t+u = \sum_{i=1}^{4} m_i^2 \quad . \qquad (3.2)$$

We write the S matrix for this process as

$$s = 1+iT \qquad (3.3)$$

where

$$< cd|T|ab > = (2\pi)^4\delta^4(p_a+p_b-p_c-p_d)f(s,t) \qquad (3.4)$$

where $f(s,t)$ is Lorentz invariant, provided single-particle states are so normalized that the phase-space volume for one particle is invariant:

$$\sum_{\substack{\text{one-particle}\\ \text{states}}} = \frac{1}{(2\pi)^3}\int \frac{d^3p}{E_p}\sum_{\alpha} \qquad (3.5)$$

where α indicates quantum numbers other than momentum. The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2 s}\frac{p_{cd}}{p_{ab}}|f(s,t)|^2 \ . \qquad (3.6)$$

Crossing symmetry states that $f(s,t)$ describes different reactions in different domains of its arguments. The three reactions, or channels, are as follows:

$$\text{s-channel: } a+b \to c+d, \text{for } s > \max\,[(m_a+m_b)^2,(m_c+m_d)^2] \quad ,$$
$$\text{t-channel: } a+\bar{c} \to b+\bar{d}, \text{for } t > \max\,[(m_a+m_c)^2,(m_b+m_d)^2] \quad ,$$
$$\text{u-channel: } b+\bar{c} \to \bar{a}+d, \text{for } u > \max\,[(m_b+m_c)^2,(m_a+m_d)^2] \quad . \qquad (3.7)$$

We assume that $f(s,t)$ can be analytically continued from one of these domains to another.

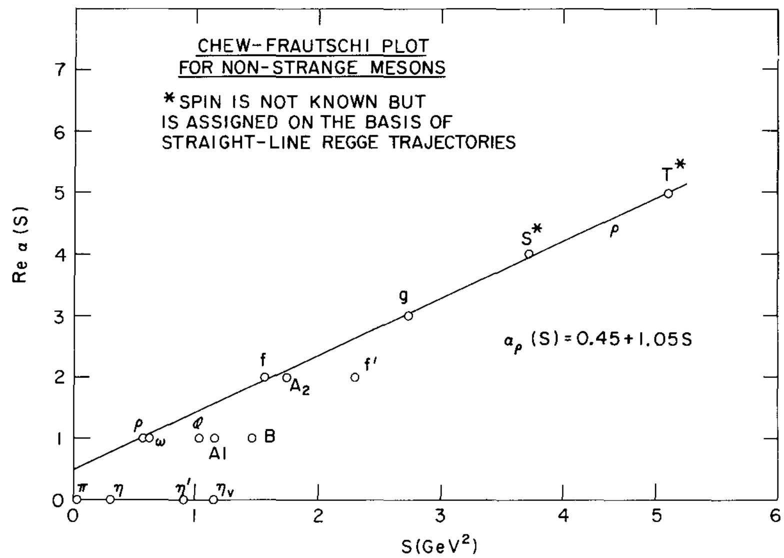
Unitarity states that $S^+S = 1$, or

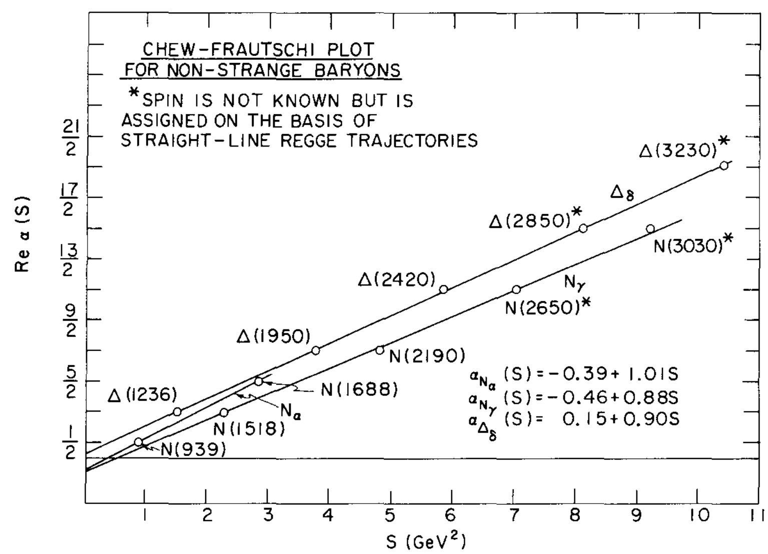
$$\frac{1}{2i}(T-T^+) = \frac{1}{2}T^+T \quad , \qquad (3.8)$$

and time reversal invariance implies

$$< \alpha|T|\beta > = \ < \beta|T|\alpha > \ . \qquad (3.9)$$

Then, taking the matrix element of (3.8) we obtain


CHEW-FRAUTSCHI PLOT
FOR NON-STRANGE MESONS
*SPIN IS NOT KNOWN BUT
IS ASSIGNED ON THE BASIS OF
STRAIGHT-LINE REGGE TRAJECTORIES
Re α (S)
S(GeV²)
α_ρ (S) = 0.45 + 1.05S
T*
S*
ρ
g
f
f′
A2
B
A1
ρ
ω
π
η
η′
ηv


CHEW-FRAUTSCHI PLOT
FOR NON-STRANGE BARYONS
*SPIN IS NOT KNOWN BUT IS
ASSIGNED ON THE BASIS OF
STRAIGHT-LINE REGGE TRAJECTORIES
Re α (S)
S (GeV²)
Δ(3230)*
Δδ
Δ(2850)*
N(3030)*
Δ(2420)
Nγ
N(2650)*
Δ(1950)
N(2190)
Δ(1236)
N(1688)
Nα
N(1518)
N(939)
α_{N_α} (S) = −0.39 + 1.01S
α_{N_γ} (S) = −0.46 + 0.88S
α_{Δ_δ} (S) = 0.15 + 0.90S

$$\mathrm{Im} f_{ab \to cd}(s,t) = \frac{1}{2} \sum_n f^*_{cd \to n} (2\pi)^4 \delta^4 (p_a - p_c - p_d) f_{ab \to n} . \quad (3.10)$$

The optical theorem reads

$$\mathrm{Im} f_{ab \to ab}(s,0) = \frac{1}{2} k(s)^{\frac{1}{2}} \sigma_T(s) \quad (3.11)$$

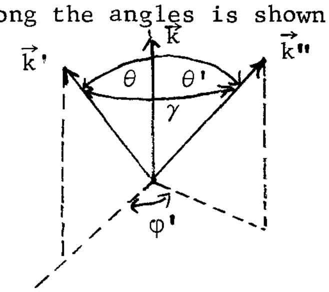
where $\sigma_T(s)$ is the total cross section for $a+b \to$ anything. By virtue of (3.10) $f(s,t)$ has a series of branch cuts in s, with branch points at the various thresholds for $ab \to n$. The right hand side of (3.10) gives the discontinuities across the cuts. The discontinuity across the elastic cut is given by the elastic unitarity relation

$$\mathrm{Im} f(s,t) = \frac{1}{8\pi^2} \frac{k}{(s)^{\frac{1}{2}}} \int d\Omega' f^*(s,t_1) f(s,t_2) \quad (3.12)$$

where, in the equal mass case,

$$\begin{aligned} t &= -2k^2(1-\cos\theta) \\ t_1 &= -2k^2(1-\cos\theta') \\ t_2 &= -2k^2(1-\cos\gamma) \\ \cos\gamma &= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi' \quad . \end{aligned} \quad (3.13)$$

The geometrical relationship among the angles is shown below.



We can expand the two-body scattering amplitude $f(s,t)$ in partial waves:

$$f(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z_s) F_\ell(s) \quad . \quad (3.14)$$

If f is an elastic amplitude, then elastic unitarity takes the simple form

$$\mathrm{Im} F_\ell(s) = \frac{1}{2\pi} \frac{k}{(s)^{\frac{1}{2}}} |F_\ell(s)|^2 \quad , \quad (3.15)$$

with the solution

$$F_\ell(s) = \frac{\pi(s)^{\frac{1}{2}}}{ik} (e^{2i\delta_\ell(s)} - 1), \quad \mathrm{Im}\delta_\ell(s) \geq 0 \quad . \quad (3.16)$$

If the elastic threshold is the lowest threshold then $\mathrm{Im}\delta_\ell(s) = 0$. Using the orthognality of the Legendre polynomials, we can invert the partial wave expansion to obtain

$$F_\ell(s) = \frac{1}{2} \int_{-1}^{+1} dz P_\ell(z) f(s,t(z)), \quad (\ell=0,1,2,\ldots) \quad (3.17)$$

B. Froissart-Gribov Continuation

We wish to continue $F_\ell(s)$ into the complex plane so that we may study its poles and other singularities there. In potential scattering this could be done by solving the Schrödinger equation for complex ℓ. In relativistic scattering we must make use of dynamical assumptions. A guide to the analytic continuation is

Carlson's Theorem: Let f(z) be analytic for Rez $\geq$ 0. Suppose f(z) = 0 for z = 0,1,2,..., and that $|f(z)| < \text{const} \times e^{\pi|z|}$ as $|z| \to \infty$. Then f(z) = 0 for all Rez $\geq$ 0.

Hence if we can find an analytic function $F(E,\ell)$ which reduces to $F_\ell(E)$ for ℓ = 0,1, 2,.... and which grows less fast than $e^{\pi|\ell|}$, then we know that any other analytic continuation must grow at least as fast as $e^{\pi|\ell|}$. Since $P_\ell(z)$ grows essentially like $e^{\pi z|\ell|}$ for $-1 < z < 1$, and therefore does not possess a unique continuation, we have to examine the properties of f(s,t) to see whether $F_\ell(s)$ has a unique continuation. We assume that f(s,t) satisfies an N-times subtracted dispersion relation at fixed s:

$$f(s,t) = \frac{t^N}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'^N}\frac{A_t(s,t')}{t'-t} + \frac{u^N}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'^N}\frac{A_u(s,u')}{u'-u} + \sum_{i=0}^{N-1} a_i(s)t^i . \qquad (3.18)$$

The first term gives rise to the analog of a potential, and the second term an exchange potential. Now both t and u are linear functions of $z \equiv z_s$, of the forms $t = az+b$, $u = -a'z+b'$, where $a > 0$, $a' > 0$ in the s-channel physical region. Hence (3.18) may be rewritten in the s-channel physical region, as

$$f(s,z) = \frac{z^N}{\pi}\int_{z_1}^{\infty}\frac{dz'}{z'^N}\frac{D_1(s,z')}{z'-z} + (-1)^N\frac{z^N}{\pi}\int_{z_2}^{\infty}\frac{dz'}{z'^N}\frac{D_2(s,z')}{z'+z} + \sum_{i=0}^{N-1} c_i(s)z^i ,$$

$$D_1(s,z) = A_t(s,t(z)),\ D_2(s,z) = A_u(s,u(-z)),\ z_1 \geq 1,\ z_2 \geq 1. \qquad (3.19)$$

For ℓ = 0,1,2,..., (3.17) is certainly valid, and we can substitute (3.19) into it and interchange the order of integration to obtain

$$F_\ell(s) = \frac{1}{\pi}\int_{z_1}^{\infty}\frac{dz'}{z'^N} D_1(s,z')\frac{1}{2}\int_{-1}^{+1}dz\,\frac{z^N}{z'-z}P_\ell(z) + (-1)^N\frac{1}{\pi}\int_{z_2}^{\infty}\frac{dz'}{z'^N}D_2(s,z')$$
$$\frac{1}{2}\int_{-1}^{+1}dz\,\frac{z^N}{z'+z}P_\ell(z) + \sum_{i=0}^{N-1} c_i(s)\frac{1}{2}\int_{-1}^{+1}dz\,z^i P_\ell(z) \qquad . \qquad (3.20)$$

Now

$$\frac{z^N}{z'-z} = \frac{[(z-z')+z']^N}{z'-z} = -(z-z')^{N-1} - Nz'(z-z')^{N-2} - \ldots + \frac{z'^N}{z'-z}$$

$$\frac{z^N}{z'+z} = \frac{[(z+z')-z']^N}{z'+z} = +(z+z')^{N-1} - Nz'(z+z')^{N-2} + \ldots + (-1)^N \frac{z'^N}{z'+z} . \qquad (3.21)$$

If $\ell \geq N$, then the polynomials do not contribute, and we obtain

$$F_\ell(s) = \frac{1}{\pi}\int_{z_1}^{\infty} dz' D_1(s,z') \frac{1}{2}\int_{-1}^{+1} dz \frac{P_\ell(z)}{z'-z} + \frac{1}{\pi}\int_{z_2}^{\infty} dz' D_2(s,z') \frac{1}{2}\int_{-1}^{+1} dz \frac{P_\ell(z)}{z'+z} . \qquad (3.22)$$

Noting that

$$\frac{1}{2}\int_{-1}^{+1} dz \frac{P_\ell(z)}{z'-z} = Q_\ell(z) \qquad (3.23)$$

we have

$$F_\ell(s) = \frac{1}{\pi}\int_{z_1}^{\infty} dz' D_1(s,z') Q_\ell(z') + \frac{1}{\pi}\int_{z_2}^{\infty} dz' D_2(s,z') Q_\ell(-z') ,$$

$$(\ell = N, N+1, N+2, \ldots) . \qquad (3.24)$$

Recall that for integer ℓ,

$$Q_\ell(-z) = (-1)^\ell Q_\ell(z) \qquad (3.25)$$

so that

$$F_\ell(s) = \frac{1}{\pi}\int_{z_1}^{\infty} dz' D_1(s,z') Q_\ell(z') + (-1)^\ell \frac{1}{\pi}\int_{z_2}^{\infty} dz' D_2(s,z') Q_\ell(z') . \qquad (3.26)$$

As in potential scattering, we are therefore led to define the signatured partial wave amplitudes

$$F^{\pm}(s,\ell) = \frac{1}{\pi}\int_{z_1}^{\infty} dz D_1(s,z) Q_\ell(z) \pm \frac{1}{\pi}\int_{z_2}^{\infty} dz D_2(s,z) Q_\ell(z) . \qquad (3.27)$$

This is the Froissart-Gribov formula. As in potential scattering we have

$$F_\ell(s) = \begin{cases} F^+(s,\ell) & (\ell = 0,2,4,\ldots) \\ F^-(s,\ell) & (\ell = 1,3,5,\ldots) \end{cases} . \qquad (3.28)$$

We must still show that (3.27) defines a unique continuation of $F_\ell(s)$ to complex ℓ. By hypothesis, the dispersion relation (3.19) requires only N subtractions. Hence the integrals in (3.27) converge at least for $\mathrm{Re}\,\ell \geq N$ and so define an analytic fraction there. Also, as $|\ell| \to \infty$, $Q_\ell(z) \sim C\ell^{-\frac{1}{2}}[z+(z^2-1)^{\frac{1}{2}}]^{-\ell-\frac{1}{2}}$. Hence, since $z_1 > 1$ and $z_2 > 1$, $F^{\pm}(s,\ell) \to 0$ as $|\ell| \to \infty$ and so satisfies the hypotheses of Carlson's Theorem. It therefore gives a unique analytic continuation.

The Mandelstam symmetry

$$F^{\pm}(s,\ell) = F^{\pm}(s,-\ell-1) \quad (\ell = \text{half integer}) \tag{3.29}$$

is assumed to hold. It is formally true of (3.27), for Q_ℓ has this property. However, the integral do not converge as they stand and the assumption is that the analytic continuation still maintains this property.

Since $Q_\ell(z)$ has simple poles at $\ell = -1,-2,\ldots$, $F^{\pm}(s,\ell)$ would, in general, have fixed poles (i.e., s-independent poles) at these values of ℓ. These are inadmissible by the elastic unitarity realtion (3.15). For, by similar arguments given above, (3.15) can be uniquely continued to complex ℓ to read

$$\frac{1}{2i} \lim_{\epsilon \to 0} \left[F^{\pm}(\ell,s+i\epsilon) - F^{\pm}(\ell,s-i\epsilon)\right] = \frac{1}{2\pi} \frac{k}{(s)^{\frac{1}{2}}} F^{\pm}(\ell,s+i\epsilon)F^{\pm}(\ell,s-i\epsilon) \quad . \tag{3.30}$$

This cannot be satisfied if $F^{\pm}(\ell,s)$ has real fixed poles in ℓ.

To get rid of them, we require their residues to vanish, namely

$$\int_{z_1}^{\infty} dz D_1(s,z)P_\ell(z) = 0 \ . \quad (\ell=0,1,2,\ldots,i=1,2) \ . \tag{3.31}$$

C. Regge Poles

We have seen that (3.27) defines an analytic function of ℓ for $\mathrm{Re}\ell \geq N$. For $\mathrm{Re}\ell < N$, singularities may occur, the simplest being Regge poles. They arise from a failure of the integrals in (3.27) to converge at the upper limit. Suppose

$$D_i(s,z) \underset{z\to\infty}{\rightarrow} \tilde{\beta}_i(s) z^{\alpha(s)} \quad (i=1,2), \tag{3.31}$$

We split the integrals into two parts, for example

$$\frac{1}{\pi}\int_{z_1}^{\infty} dz D_1(s,z)Q_\ell(z) = \frac{1}{\pi}\left(\int_{z_1}^{Z} + \int_{Z}^{\infty}\right) dz D_1(s,z)Q_\ell(z) \tag{3.32}$$

where Z is fixed but arbitrarily large. The first part, being a finite integral, defines an analytic function. The second part can be evaluated using (3.31) and the fact

$$Q_\ell(z) \underset{Z\to\infty}{\rightarrow} C\ z^{-\ell-1} \ .$$

We then obtain

$$F^{\pm}(s,\ell) = \frac{\beta_1(s) \pm \beta_2(s)}{\ell - \alpha(s)} + \quad [\text{Terms regular at } \ell = \alpha(s)] \ . \tag{3.33}$$

Thus $F^{\pm}(s,\ell)$ has a Regge pole at $\ell = \alpha(s)$, if $\beta_1(s) \pm \beta_2(s) \neq 0$.

To examine the singularities of $\alpha(s)$ and $\beta(s)$ we keep only the parts of (3.27) that contribute to a Regge pole:

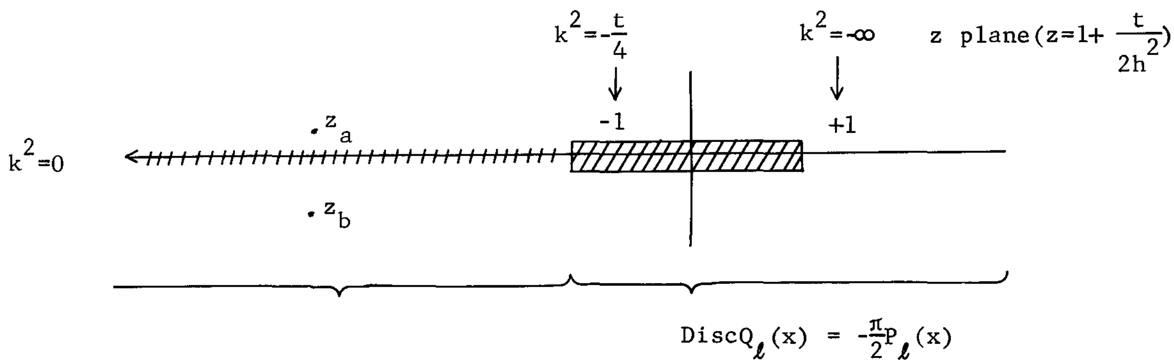
$$F^{\pm}_{pole}(s,\ell) = \frac{1}{\pi}\int_Z^{\infty} dz\ [D_1(s,z) \pm D_2(s,z)]Q_\ell(z) \quad . \tag{3.34}$$

This is valid in the s-channel physical region. To continue in s, we must first restore t and u as integration variables. We carry this out explicitly for the simple case of equal-mass scattering:

$$F^{\pm}_{pole}(s,\ell) = \frac{1}{2\pi k^2}\int_T^{\infty} dt[A_t(s,t) \pm A_u(s,t)]Q_\ell(1-\frac{t}{2k^2}) \tag{3.35}$$

where T is positive and arbitrarily large. This can now be continued in s.

We first note that the function $Q_\ell(z)$ has a cut from z = +1 to z = -1, and one from z = -1 to z = $-\infty$, with discontinuities as indicated in the sketch.



$Q_\ell(z_a) = -e^{-i\pi\ell}Q_\ell(-z)$

$Q_\ell(z_b) = -e^{-\pi\ell}Q_\ell(-z)$

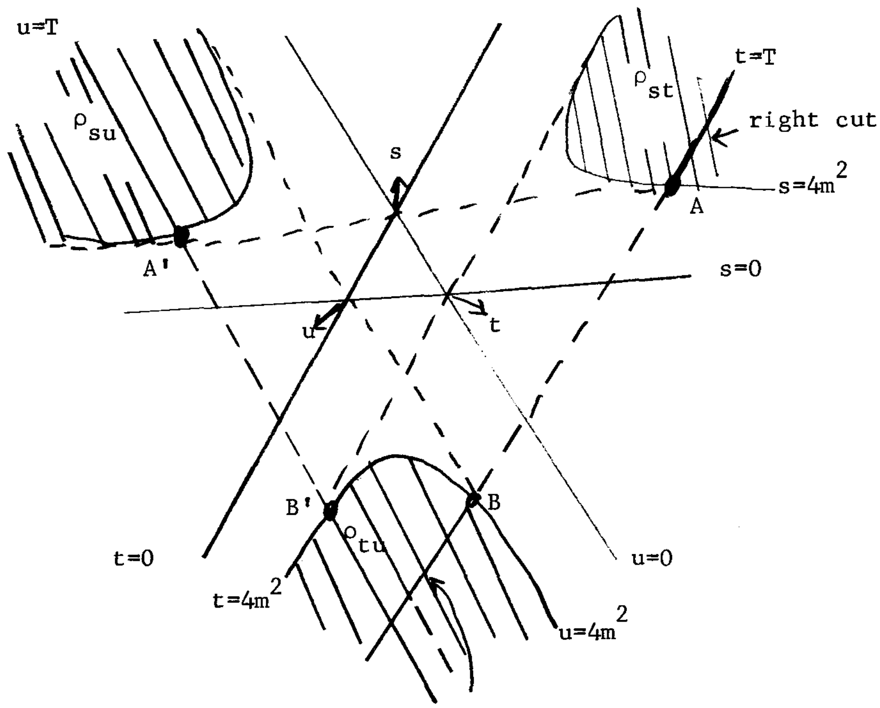
$(-z_a = -z_b = -z)$

The cut from -1 to $-\infty$ gives rise to a s-cut in $F^{\pm}_{pole}(s,\ell)$ from $k^2 = 0$ to $k^2 = -t/4$. It is present only when ℓ is non-integer. The combination $(z-1)^\ell Q_\ell(z)$, however, has no cut from -1 to $-\infty$, even for non-integer ℓ. Since $k^2 = t/2(z-1)$, we see that the combination $F^{\pm}_{pole}(s,\ell)/k^{2\ell}$ has no cut from $k^2 = 0$ to $k^2 = -t/4$. This means that for a Regge pole the reduced residue function

$$\bar{\beta}(s) = \beta(s)/k^{2\alpha(s)} \text{(equal mass case)} \tag{3.36}$$

and the trajectory $\alpha(s)$ can have only the cut from $k^2 = -t/4$ to $k^2 = -\infty$, (coming from the cut of $Q_\ell(z)$ from z = -1 to z = +1), plus other cuts coming from $A_t \pm A_u$. The former cut is, in fact, absent because $t > T$, and $T \to \infty$. The factor $k^{2\ell}$ in (3.36) corresponds to the threshold condition $\delta_\ell(k) \underset{k\to 0}{\to} k^{2\ell+1}$, familiar from potential scattering.

We now examine singularities of $F^{\pm}_{pole}$ due to those of $A_t \pm A_u$, which for fixed t has right and left hand cuts in s, and is real analytic. At fixed t, the s-cuts of A_t run from $s_A(t)$ to ∞, and from $s_B(t)$ to $-\infty$, as shown in the sketch below. Similarly the s-cuts of A_u run from $s_{A'}(u)$ to ∞, and from $s_{B'}(u)$ to $-\infty$. Since t is



integrated from T up, and $T \to \infty$, it is clear from the sketch that only the right cut remains in $F^{\pm}_{pole}(s,\ell)$, and it runs from $4m^2$ to ∞. The left hand branch point recedes to $-\infty$ because both B and B' reced to $s = -\infty$ as $T \to \infty$. Therefore $\alpha(s)$ and $\bar{\beta}(s)$ can have a right cut, but no left cut. Since they are real analytic functions, they are real for $s < 4m^2$.

For the general mass case similar results are obtained. The reduced residue function is given by

$$\bar{\beta}(s) = \beta(s)/(p_{ab}p_{cd})^{\alpha(s)} \tag{3.37}$$

as a generalization of (3.36). Both $\alpha(s)$ and $\bar{\beta}(s)$ are real analytic functions, with possibly a right cut from the lowest s-channel threshold to ∞, but no left cut. Below threshold both $\alpha(s)$ and $\bar{\beta}(s)$ are real.

D. Reggeization

By Reggeization we mean the isolation of Regge pole contributions to the scattering amplitude. The way to do this is to perform the Sommerfeld-Watson transform. We write the partial wave expansion in the t-channel:

$$f(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z_t) G_\ell(t) \tag{3.38}$$

where

$$z_t = \frac{t(s-u)+(m_d{}^2-m_b{}^2)(m_c{}^2-m_a{}^2)}{[(t-(m_c-m_a)^2)(t-(m_c+m_a)^2)(t-(m_b-m_d)^2)(t-(m_b+m_d)^2)]^{\frac{1}{2}}}$$

$$\underset{s\to\infty}{\rightarrow} \frac{s-u}{4p_{ac}p_{bd}} \quad . \tag{3.39}$$

We assume that the signatured amplitudes have only simple poles of the form

$$G^{\pm}(s,\ell) \sim \frac{\beta(s)}{\ell-\alpha(s)} \quad .$$

Then, repeating the steps of Sec. I-C, we find that as $s \to \infty$

$$f(s,t) = \sum_{\alpha} R_\alpha(s,t) + \text{(background integral)}$$

$$R_\alpha(s,t) \sim -\frac{\pi[2\alpha(t)+1]}{\sin\pi\alpha(t)} \beta(t)\tfrac{1}{2}[\mathcal{P}_\alpha(-z_t) \pm \mathcal{P}_\alpha(z_t)] \tag{3.40}$$

where the $\pm$ sign corresponds to signature $= \pm 1$. As a function of z_t, (3.38) converges in the Lehmann ellipse of the t-channel, which includes the t-channel physical region but not the s-channel physical region. We can now continue it to the s-channel physical region using (3.40). Before we do this, we must determine the phase of $\mathcal{P}_\alpha(-z_t)$, with the help of the relation

$$\mathcal{P}_\alpha(-z) = e^{\mp i\pi\alpha}\mathcal{P}_\alpha(z), \quad (\text{Im}z \gtrless 0) \quad . \tag{3.41}$$

In the physical region of the t channel, Imt > 0 and s < 0. Hence, $\text{Im}z_t > 0$, so that

$$R_\alpha(s,t) \sim -\frac{\pi\beta(t)(2\alpha(t)+1)}{\sin\pi\alpha(t)} \tfrac{1}{2}(e^{-i\pi\alpha(t)} \pm 1)\mathcal{P}_\alpha(z_t) \tag{3.42}$$

and this can be continued to the s-channel physicsl region. The reason we must use (3.41) before the continuation is that the path of continuation passes through a branch point of $\mathcal{P}_\alpha(z)$ in z, and the phase, if not determined beforehand, becomes ambiguous thereafter.

We now examine the singularities of $R_\alpha(s,t)$ in t:

1) $R_\alpha(s,t)$ has poles at the integers from the factor $[\sin\pi\alpha(t)]^{-1}$. We discuss

separately two types of poles:

(a) Those poles at $\alpha=0,1,2,\ldots.$ correspond to physical particles of spin α. Because of the signature factor, only the even or odd ones are actually present in R_α. However, if $\alpha(0) > 0$, and the signature is positive, then $\alpha=0$ corresponds to a particle of negative mass, a "ghost", which may be removed by assuming that $\beta(t) \propto \alpha(t)$. (There are other mechanisms to deal with this problem when the extrenal particles have spin. See discussion later, in Sec. VII).

(b) Those poles at $\alpha=-1,-2,-3,\ldots,$ correspond to unphysical, or "nonsense", values of singular momentum. They are automatically removed from R_α because $\mathcal{P}_\alpha(z)$ vanishes at these points. The signature factor then produces zeroes in $R_\alpha(s,t)$ at nonsense wrong-signature values of α, unless $\beta(t)$ has poles at these values of α. (See discussion later, in Sec. IVC).

2) $R_\alpha(s,t)$ has poles at $\alpha = \pm\frac{1}{2}, +3/2, \ldots,$ arising from the poles of $\mathcal{P}_\alpha(z)$. The pole at $\alpha=-\frac{1}{2}$ is cancelled by the factor $(2\alpha+1)$. For the others there are two possibilities.

(a) The residue of the pole may vanish.

(b) If the residue does not vanish, then the Mandelstam symmetry (3.29) requires that there be another trajectory at $-\alpha-1$ with the same residue. The pole from this trajectory exactly cancels the original pole. This is known as a "compensating trajectory".

For negative values of α, the compensating trajectory would lie above the original one. For this reason it is customary to assume that (a) is the correct choice and to take

$$\beta(t) \propto \frac{1}{\Gamma(\alpha(t)+\frac{3}{2})} \quad .$$

To obtain the required analyticity properties, the correct threshold behavior, and the absence of a ghost at $\alpha(t) = 0$, we write

$$\beta(t) = \frac{\alpha(t)}{\Gamma(\alpha(t)+\frac{3}{2})} \left(\frac{4p_{ac}p_{bd}}{s_0}\right)^{\alpha(t)} \gamma(t) \tag{3.43}$$

where s_0 is an arbitrary scale factor. Then $\gamma(t)$ is real analytic with no left hand cut. Recalling (1.20) and using the properties of the gamma function, we find that as $s \to \infty$,

$$R_\alpha(s,t) \sim -\frac{\gamma(t)}{(\pi)^{\frac{1}{2}}} \Gamma(1-\alpha(t))(e^{-i\pi\alpha(t)} \pm 1)\left(\frac{s}{s_0}\right)^{\alpha(t)} \quad . \tag{3.44}$$

This is the formula which is used in practical applications. Note that the threshold factor in (3.) is cancelled by a similar factor in z_t [see (3.38)]. This is of course no accident, for $R_\alpha(s,t)$ is expected on general grounds to be a real analytic function of s. If the ghost-killing zero is not needed, and we do not put it in, then $\Gamma(1-\alpha)$ in (3.44) is replaced by $-\Gamma(-\alpha)$.

E. Khuri Poles

We have kept only the leading term in the asymptotic expression (3.44) for the contribution of a single Regge pole. Keeping the full asymptotic expansion of the hypergeometric function in the definition (1.19) of $\mathcal{P}_\alpha(z_t)$, we obtain

$$R_\alpha(s,t) = -\frac{\gamma(t)}{(\pi)^{\frac{1}{2}}}\Gamma(1-\alpha(t))(e^{-i\pi\alpha(t)}\pm 1)(\frac{s}{s_0})^{\alpha(t)}\left[1+\sum_{n=1}^{\infty} d_n(t)(\frac{s}{s_0})^{-n}\right] \quad (3.45)$$

The $d_n(t)$ are just such that

$$R_\alpha(s,t) \to \text{const} \times \frac{P_\ell(z_t)}{\ell-\alpha(t)} \quad , \text{ as } \alpha(t) \to \ell \quad , \quad (3.46)$$

which is required for a resonance to have a definite spin. Thus (3.45) is significant if resonance positions are non-degenerate. If, however at the same energy there exist resonances of various spins, then the residue function in(3.46) could be an arbitrary polynomial in z_t, and the combination (3.45) is not particularly significant. Since we do not have full knowledge of all the resonances present, and since asymptotically only the leading term in (3.45) is significant, it would be advantageous to have an alternative expansion to the partial-wave expansion, such that the result of a Sommerfeld-Watson transformation would lead naturally to just one term in the infinite n sum in (3.45). Such an expansion is supplied by Khuri*.

One can expand f(s,t) in a power series of s instead of in a series of Legendre polynomials in z_t in the form

$$f(s,t) = \sum_{n=0}^{\infty} c_n(t)s^n \quad , \quad (3.47)$$

which converges in some circle in s. One then analytically continues $c_n(t)$ in n to obtain $c^{\pm}(t,n)$, defined in the complex n plane (with signature introduced in the usual way). Assuming that $c^{\pm}(t,n)$ has poles in n whose positions depend on t, (which might be called Khuri poles), one can pick up their contribution to (3.47) by doing the Sommerfeld-Watson transformation and obtain

$$f(s,t) = \sum_{\alpha} K_\alpha(s,t) + (\text{background integral})$$

$$K_\alpha(s,t) = -\frac{\gamma(t)}{(\pi)^{\frac{1}{2}}}\Gamma(1-\alpha(t))(e^{-i\pi\alpha(t)}\pm 1)(\frac{s}{s_0})^{\alpha(t)}$$

where $\alpha(t)$ is the trajectory of a Khuri pole. Clearly, one Regge pole corresponds to an infinite family of Khuri poles, spaced successively by one unit. The leading

---

* N.N. Khuri, Phys. Rev. 132, 914 (1963).

member of this family of Khuri poles coincides with the Regge pole. Conversely, one Khuri pole corresponds to an infinite family of Regge poles. As long as we do not have a dynamical theory, there is little to choose between the point of view of Regge poles and that of Khuri poles. In either case the requirements of analyticity and unitarity in all channels probably can only be satisfied with an infinite number of poles, Regge or Khuri. For formal considerations, however, Khuri poles are often convenient.

Instead of (3.47), we can, in fact, consider a power series in some other variable, for example in $\nu = (s-u)/2s_0$. Then we could arrive at (3.48) with $K_\alpha(s,t)$ replaced by

$$K_\alpha(\nu,t) = -\frac{\gamma(t)}{(\pi)^{\frac{1}{2}}}\Gamma(1-\alpha(t))(e^{-i\pi\alpha(t)}\pm 1)\nu^{\alpha(t)} \tag{3.48a}$$

which is convenient when it is important to take into account the symmetry of the scattering amplitude under s-u interchange.

F. Factorizability of Regge Residues

The residue function $\beta(s)$ of a Regge pole can be written as a product of two factors in a manner similar to coupling constants in field theory. This is a consequence of elastic unitarity, and we shall prove it for the case of the following set of s-channel reactions:

1. $\pi+\pi \to \pi+\pi$ with partial wave amplitude $F_1(s,\ell)$
2. $\pi+\pi \to N+\bar{N}$ with partial wave amplitude $F_2(s,\ell)$
3. $N+\bar{N} \to N+\bar{N}$ with partial wave amplitude $F_3(s,\ell)$.

The spin of N is ignored for simplicity, and signature is understood. For $4m_\pi^2 < s < 16m_\pi^2$ the 2π state is the only intermediate state in the unitarity relation (3.10), for all three reactions. Therefore in that interval of s, the unitarity relations for the partial waves, continued in ℓ are simple generalizations of (3.30):

$$\begin{aligned} \text{Im}F_1(s,\ell) &= \rho(s)F_1^*(s,\ell)F_1(s,\ell) \\ \text{Im}F_2(s,\ell) &= \rho(s)F_2^*(s,\ell)F_1(s,\ell) \\ \text{Im}F_3(s,\ell) &= \rho(s)F_2^*(s,\ell)F_2(s,\ell) \end{aligned} \tag{3.49}$$

where

$$\rho(s) \equiv \frac{1}{4\pi}\left(\frac{s-4m_\pi^2}{s}\right)^{\frac{1}{2}}$$

$$\text{Im}F_n(s,\ell) \equiv \frac{1}{2i}[F_n(s+i\epsilon,\ell)-F_n(s-i\epsilon,\ell)] \quad (n=1,2,3) \quad . \tag{3.50}$$

Since all three reactions have the same quantum numbers, the same Regge pole $\alpha(s)$ occurs in $F_n(s,\ell)$, (n=1,2,3). Thus near $\ell = \alpha(s+i\epsilon)$,

$$F_n(s+i\epsilon,\ell) \sim \frac{\beta_n(s+i\epsilon)}{\ell-\alpha(s+i\epsilon)}$$

$$F_n(s-i\epsilon,\ell) \sim \frac{\beta_n(s-i\epsilon)}{\alpha(s+i\epsilon)-\alpha(s-i\epsilon)} \quad . \tag{3.51}$$

Substituting these into (3.49), multiplying through by $\ell-\alpha(s+i\epsilon)$, and taking the limit $\ell-\alpha(s+i\epsilon) \to 0$, we obtain

$$\beta_1(s+i\epsilon) = \frac{\rho(s)}{\mathrm{Im}\alpha(s)} \beta_1(s-i\epsilon)\beta_1(s+i\epsilon)$$

$$\beta_2(s+i\epsilon) = \frac{\rho(s)}{\mathrm{Im}\alpha(s)} \beta_2(s-i\epsilon)\beta_1(s+i\epsilon)$$

$$\beta_3(s+i\epsilon) = \frac{\rho(s)}{\mathrm{Im}\alpha(s)} \beta_2(s-i\epsilon)\beta_2(s+i\epsilon) \quad . \tag{3.52}$$

Taking the quotient of the last two equations, we obtain

$$\beta_2(s+i\epsilon)^2 = \beta_1(s+i\epsilon)\beta_3(s+i\epsilon), (4m_\pi^{\ 2} < s < 16m_\pi^{\ 2}) \quad . \tag{3.53}$$

It is to be noted that our proof depends on the fact that there is no other state degenerate with the 2π state. Similarly a generalization of the proof to take the spin of the nucleon into account works only because the pion has spin zero, and would not go through if there is spin degeneracy. If the 2π state were degeneracy, the proof would have to be modified by considering new linear combinations of the degenerate states. Since (3.53) is analytic in s, we can continue it into the complex s plane. It therefore holds for all s. The reduced residue $\gamma(s)$ defined in (3.43) also satisfies (3.53), because the factors in its definition trivially factorize. We can therefore write, as a solution to (3.53),

$$\gamma_1(s) = g_{\pi\pi}(s)g_{\pi\pi}(s)$$

$$\gamma_2(s) = g_{\pi\pi}(s)g_{N\overline{N}}(s)$$

$$\gamma_3(s) = g_{N\overline{N}}(s)g_{N\overline{N}}(s) \quad . \tag{3.54}$$

The same proof can be used to show that the discontinuity function of a Regge cut has similar factorizability, for a Regge cut may be thought of as a continuous distribution of Regge poles.

G. Complication Due to Spin and Intrinsic Quantum Numbers

In order to apply the formulas we have derived to actual experiments, we have to understand, at least qualitatively, how our results are affected by the spin and intrinsic quantum numbers of the external particles. We now give a brief discussion of this. A detailed consideration of spin will be postponed till later.

If the particles have spin, we must specify their helicities λ_a, λ_b, λ_c, and λ_d

as well as their momenta. We do this by using the helicity amplitudes $f_{cd;ab}(s,t)$ of Jacob and Wick*, where cd;ab is an abbreviation of $\lambda_c,\lambda_d;\lambda_a,\lambda_b$. The s- and t-channel amplitudes are no longer identical but are related by a crossing matrix. That is,

$$f_H^s(s,t) = \sum_{H'} \mathcal{M}_{HH'}(s,t) f_{H'}^t(s,t) \quad , \tag{3.55}$$

where H or H' denotes the relevant set of helicity indices. The crossing matrix $\mathcal{M}_{HH'}$ has been calculated by Trueman and Wick**. For our present purposes we only need to know that it is a real orthogonal matrix:

$$\mathcal{M}^T\mathcal{M} = 1 \ . \tag{3.56}$$

The unpolarized differential cross section in the s channel is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2 s} \frac{k_f}{k_i} \frac{1}{(2J_a+1)(2J_b+1)} \sum_H |f_H^s(s,t)|^2 , \tag{3.57}$$

where J_a and J_b are the incident spins. Because $\mathcal{M}^T\mathcal{M} = 1$, this is equivalent to

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2 s} \frac{k_f}{k_i} \frac{1}{(2J_a+1)(2J_b+1)} \sum_H |f_H^t(s,t)|^2 \quad . \tag{3.58}$$

If $f_H^s(s,t)$ describes elastic scattering, $a+b \rightarrow a+b$, then the optical theorem states

$$\text{Im} < f^s(s,0) > = \tfrac{1}{2}k(s)^{\frac{1}{2}}\sigma_T(s) \tag{3.59}$$

where $\sigma_T(s)$ is the total unpolarized cross section for $a+b \rightarrow$ anything and $< >$ denotes the following helicity average:

$$< f^s(s,0) > = \frac{1}{(2J_a+1)(2J_b+1)} \sum_{a,b} f_{ab;ab}^s(s,0) . \tag{3.60}$$

It can be shown that

$$< f^s(s,0) > = \frac{1}{(2J_a+1)(2J_b+1)} \sum_{a,b} f^t_{b,-b;a,-a}(s,0) \quad , \tag{3.61}$$

so that we can compute the s-channel total cross section directly in terms of t-channel Regge poles.

The helicity amplitudes are particularly appropriate for Regge pole analysis because they have simple partial wave expansions:

---

* M. Jacob and G.C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959).

** T.L. Trueman and G.C. Wick, Ann. Phys. (N.Y.) 26, 322 (1964).

$$f_{cd,ab}{}^{t}(s,t) = \sum_{J} (2J+1) F_{cd,ab}{}^{J}(t) d_{\lambda\mu}{}^{J}(z_t)$$

$$\lambda = a-b, \quad \mu = c-d \quad z_t = \cos\theta_t \tag{3.62}$$

where $d_{\lambda\mu}{}^{J}(z)$ is a rotation coefficient. If a=b=c=d=0, then (3.62) reduces to the partial wave expansion of the spinless case. Regge poles occur as J-poles of $F_{cd,ab}{}^{J}$, suitably continued into the J plane. The contribution of a single Khuri pole $\alpha(t)$ to (3.62) has the same form as in the spinless case, except that the reduced residue now acquires helicity indices:

$$f_H{}^{t}(\text{pole}) = -\frac{\gamma_H(t)}{(\pi)^{\frac{1}{2}}} \Gamma(1-\alpha(t))(e^{-i\pi\alpha(t)} \pm 1)(\frac{s}{s_0})^{\alpha(t)} . \tag{3.63}$$

Actually the helicity amplitudes contain kinematic singularities and satisfy constraint equations that did not exist in the spinless case. This means that $\gamma_H(t)$ have kinematic singularities, and that the $\gamma_H(t)$ of different Regge poles may be related to one another at some value of t. The simplest of these constraints come from the requirement that in s-channel forward or backward scattering the total helicity be conserved. Through crossing this forces certain linear combinations of t-channel helicity amplitudes to vanish at these kinematic points.

We now turn to intrinsic quantum numbers, and use isospin as an example. If we do not work with scattering amplitudes of definite isospin, the no further complication arises. For example, consider the s-channel reaction $\pi^- p \to \pi^- p$, with the corresponding t-channel reaction $\pi^-\pi^+ \to p\bar{p}$. Crossing between the two channels is simply given by (3.55), in which f^s refers to $\pi^- p \to \pi^- p$ and f^t refers to $\pi^-\pi^+ \to p\bar{p}$. If we decompose all scattering amplitudes into amplitudes with definite total isospin, however, then an isospin crossing matrix enters into the crossing relation. For example, let $f_{H,I}{}^{s}$ denote the helicity amplitude for $\pi^- p \to \pi^- p$ in the total isospin state I, and let $f_{H',I'}{}^{t}$ denote the corresponding amplitude for $\pi^+\pi^- \to p\bar{p}$. Then the crossing relation reads

$$f_{H,I}{}^{s}(s,t) = \sum_{H'I'} \mathcal{M}_{HH'}(s,t) C_{II'} f_{H'I'}{}^{t}(s,t) \quad , \tag{3.64}$$

where $C_{II'}$ is the isospin crossing matrix. It is a constant matrix independent of s and t. We merely outline the procedure to derive it.

Suppressing helicity indices, and denoting a two-particle state by $|p_1, I_1, m_1; p_2, I_2, m_2\rangle$, where I is the particle isospin and m its z-component, crossing symmetry states

$$\langle p_3, I_3, m_3; p_4, I_4, m_4 | T | p_1, I_1, m_1; p_2, I_2, m_2 \rangle$$
$$= \langle -p_2, I_2, -m_2; p_4, I_4, m_4 | T | p_1, I_1, m_1; -p_3, I_3, -m_3 \rangle \quad . \tag{3.65}$$

Now, both sides can be decomposed into linear combinations of amplitudes of definite total isospin and give a relation of the form

$$\sum_{I} a^{I}_{m_3 m_4, m_1, m_2} f_I^{\ s} = \sum_{I'} b^{I'}_{-m_2, m_4; -m_1, -m_3} f_{I'}^{\ t} \qquad (3.66)$$

where a,b, are certain Clebsch-Gordan coefficients. We may now use the orthogonality relations of Clebsch-Gordan coefficients to solve (3.66), resulting in the crossing relation (3.64). The only delicate problem in the derivation is the choice of phases for the coefficients a and b. A clear and elementary discussion of this is given by Carruthers and Krisch.* They have worked out isospin crossing matrices for many useful cases. For reference we cited some of these in Table I.

* P. Carruthers and J. Krisch, Ann. Phys. (N.Y.) 33, 1 (1965).

Table I - I-spin Crossing Matrices

$$f_I^{\,s}(s,t) = \sum_{I'} C_{II'}^{\,st} f_{I'}^{\,t}(s,t)$$

---

s: $\pi\pi \to \pi\pi$
t: $\pi\pi \to \pi\pi$
u: $\pi\pi \to \pi\pi$

$$C^{st} = \begin{array}{c|ccc} I_s \backslash I_t & 0 & 1 & 2 \\ \hline 0 & \frac{1}{3} & 1 & \frac{5}{3} \\ 1 & \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ 2 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{array}$$

---

s: $\pi N \to \pi N$
t: $\pi\pi \to N\bar{N}$
u: $\pi N \to \pi N$

$$C^{st} = \begin{array}{c|cc} I_s \backslash I_t & 0 & 1 \\ \hline \frac{1}{2} & \frac{1}{(6)^{\frac{1}{2}}} & 1 \\ \frac{3}{2} & \frac{1}{(6)^{\frac{1}{2}}} & -\frac{1}{2} \end{array}$$

same for
s: $\pi K \to \pi K$ etc.

$$C^{sn} = \begin{array}{c|cc} I_s \backslash I_t & \frac{1}{2} & \frac{3}{2} \\ \hline \frac{1}{2} & -\frac{1}{3} & \frac{4}{3} \\ \frac{3}{2} & \frac{2}{3} & \frac{1}{3} \end{array}$$

---

s: $NN \to NN$
t: $N\bar{N} \to N\bar{N}$
u: $NN \to NN$

$$C^{st} = \begin{array}{c|cc} I_s \backslash I_t & \frac{1}{2} & \frac{3}{2} \\ \hline \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

same for
s: $KK \to KK$ etc.

$$C_{sn} = \begin{array}{c|cc} I_s \backslash I_t & \frac{1}{2} & \frac{3}{2} \\ \hline \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{array}$$

IV. Some Simple Physical Consequences

A. Single Pole Dominance

Suppose $\alpha(t)$ is the leading Regge trajectory which can be exchanged in the t-channel. If there are no Regge cuts, it alone will dominate the s-channel scattering when s is sufficiently large. From (3.58) and (3.63) we obtain the asymptotic differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2 s} \frac{k_f}{k_i} b(t)\Gamma^2(1-\alpha(t)) \left| \frac{e^{-i\pi\alpha(t)} \pm 1}{2} \right|^2 \left(\frac{s}{s_0}\right)^{2\alpha(t)} ,$$

$$= \frac{1}{4\pi^2 s} \frac{k_f}{k_i} b(t)\Gamma^2(1-\alpha(t)) \begin{Bmatrix} \cos^2 \frac{\pi\alpha(t)}{2} \\ \sin^2 \frac{\pi\alpha(t)}{2} \end{Bmatrix} \left(\frac{s}{s_0}\right)^{2\alpha(t)} \quad ,(\text{signature} = \pm 1), \qquad (4.1)$$

where

$$b(t) = \frac{1}{\pi} \frac{1}{(2J_a+1)(2J_b+1)} \sum_{a,b,c,d} \gamma^2_{cd;ab}(t) \quad . \qquad (4.2)$$

Hence $(k_i/k_f)(d\sigma/d\Omega)$ has a very simple asymptotic s-dependence:

$$\frac{k_i}{k_f} \frac{d\sigma}{d\Omega} = C(t) \left(\frac{s}{s_0}\right)^{2\alpha(t)-1} \quad . \qquad (4.3)$$

If we plot $\ln\left(\frac{k_i}{k_f}\frac{d\sigma}{d\Omega}\right)$ vs. $\ln\left(\frac{s}{s_0}\right)$ at fixed t, we should obtain a straight line whose slope is $2\alpha(t)-1$. This would enable us to determine the trajectory $\alpha(t)$ for negative values of t by comparison with experiments. Most of the trajectories determined so far conform remarkably well to a straight line:

$$\alpha(t) = \alpha_0 + \alpha' t \quad . \qquad (4.4)$$

B. Total Cross Sections

By using the optical theorem (3.59) and the formulas (3.61) and (3.63), we can calculate the asymptotic total cross section in the s-channel in terms of the leading Regge pole $\alpha(t)$ in the t-channel:

$$\sigma_T(s) \underset{s\to\infty}{\to} \frac{c_a c_b}{\Gamma(\alpha_0)} \left(\frac{s}{s_0}\right)^{\alpha_0 - 1} \quad , \qquad (4.5)$$

where σ_T is the total cross section for $a+b \to$ anything, and $\alpha_0 = \alpha(0)$, and c_n $(n=a,b)$ is defined by

$$c_n = \frac{(4\pi^{\frac{1}{2}} s_0^{-1})^{\frac{1}{2}}}{(2J_n+1)} \sum_{\lambda} g^{n\bar{n}}_{\lambda,-\lambda}(0) \quad , \qquad (4.6)$$

where $g^{n\bar{n}}_{\lambda\mu}(t)$ is the coupling [in the sense of Eq. (3.54)] of the t-channel Regge pole to the $n\bar{n}$ system with helicities λ,μ. Since the Froissart bound* requires

---

* See Khuri's lectures in this Summer School.

$\sigma_T < c(\ell ns)^2$, we see that no Regge trajectory can have an intercept α_0 greater than unity. If α_0=1 then according to (4.5) σ_T approaches a finite constant, otherwise it approaches zero by a power law.

It seems attractive to assume that there is a trajectory with α_0=1, having the quantum numbers of the vacuum. It should have positive signature so that it does not create a zero-mass spin-one hadron. If such a trajectory exists, it would be exchanged in all elastic scatterings, and by (4.5) all total cross sections will approach constants as $s \to \infty$. Furthermore, the total cross sections for a+b and a+$\bar{b}$ will be equal in that limit, because the trajectory will in both cases be coupled to $a\bar{a}$ and $b\bar{b}$ pairs, thus giving the same $c_a c_b$. These are just the conclusions of the Pomeranchuk theorem, and this trajectory is named the Pomeranchuk trajectory or the Pomeron and is denoted by $\alpha_P(t)$. However, experimental data so far have neither clearly confirmed nor ruled out the Pomeron. If it exists, then the factorized form of the coefficient in (4.5) predicts relations among asymptotic cross section, for example

$$\sigma_{\pi N}^{2}(\infty) = \sigma_{\pi\pi}(\infty)\,\sigma_{NN}(\infty) \quad . \tag{4.7}$$

Assuming that at p_{lab} = 30 GeV/c, the total cross sections have essentially attained their asymptotic limit, as is consistent with the trends in the experimental data, one finds

$$\sigma_{\pi\pi}(\infty) = 16 \text{ mb}. \tag{4.8}$$

This number, of course, has not been measured experimentally.

While the Pomeron (assuming that it exists) gives the asymptotic constant cross section. The way this limit is approached depends on lower-lying Regge trajectories. Their effect on the total cross section is simple to calculate via the optical theorem, because the latter involves the amplitude linearly, so the contributions from different trajectories are simply additive. Consider, for example, pion-nucleon scattering. The s channel is $\pi+N \to \pi+N$, and the t channel is $\pi+\pi \to N+\bar{N}$. The quantum numbers of the t channel are $P = +(-1)^J$, $G = +1$, and $I = 0,1$. The known trajectories with these quantum numbers are

$$\begin{aligned} I = 0&: P,\ f^0 \quad (\text{signature} = +1) \\ I = I&: \rho \qquad\ \ (\text{signature} = -1) \quad . \end{aligned} \tag{4.8}$$

Hence for large s

$$\begin{aligned} f_0^{\pm}(s,t) &= K_P + K_f \\ f_1^{t}(s,t) &= K_\rho \end{aligned} \tag{4.9}$$

where $K_P \equiv K_{\alpha_P}(s,t)$, with $K_\alpha(s,t)$ given by (3.48). Using the isospin crossing matrices of Table I, in Sec. III, we find

$$f_{1/2}{}^s(s,t) = \frac{1}{(6)^{\frac{1}{2}}}(K_P+K_f)+K_\rho$$

$$f_{3/2}{}^s(s,t) = \frac{1}{(6)^{\frac{1}{2}}}(K_P+K_f)-\tfrac{1}{2}K_\rho \tag{4.10}$$

which leads to

$$\sigma_{\pi^+p} = \frac{1}{(6)^{\frac{1}{2}}}\sigma_P + \frac{1}{(6)^{\frac{1}{2}}} - \tfrac{1}{2}\sigma_\rho$$

$$\sigma_{\pi^-p} = \frac{1}{(6)^{\frac{1}{2}}}\sigma_P + \frac{1}{(6)^{\frac{1}{2}}} - \tfrac{1}{2}\sigma_\rho \quad . \tag{4.11}$$

Using the approximate value $\alpha_\rho(0) \approx \alpha_{fo}(0) \approx \frac{1}{2}$, we have

$$\sigma_{\pi^+p}(s) = \sigma_\infty+(c_f-c_\rho)s^{-\frac{1}{2}}$$

$$\sigma_{\pi^-p}(s) = \sigma_\infty+(c_f+c_\rho)s^{-\frac{1}{2}} \quad , \tag{4.12}$$

where σ_∞, c_f, c_ρ are constants. The constants c_f and c_ρ are proportional to residue functions evaluated at t=0. These residue functions must be positive when t as at the squared mass of a particle, but may change sign by the time we extrapolate to t=0. Assuming, however, that c_f and c_ρ are positive, we have

$$\sigma_{\pi^-p}(s) > \sigma_{\pi^+p}(s) \tag{4.13}$$

which happen to be experimentally correct so far.

C. Diffraction Scattering

In any elastic scattering, we expect the amplitude to be dominated by Pomeron exchange for small t and large s (i.e., high energy scattering near the forward direction):

$$f(s,t) = -\frac{1}{(\pi)^{\frac{1}{2}}}\gamma_P(t)\Gamma(1-\alpha_P(t))\,\frac{e^{-i\pi\alpha_P(t)}\pm 1}{2}\left(\frac{s}{s_0}\right)^{\alpha_P(t)} \tag{4.14}$$

as $t \to 0$, $\alpha_P(t) \to 1$ by hypothesis. Then

$$\Gamma(1-\alpha_P(t))\,\frac{e^{-i\pi\alpha_P(t)}\pm 1}{2} = \frac{\pi}{\sin\pi\alpha_P(t)\Gamma(\alpha_P(t))}\,e^{-i\frac{\pi}{2}\alpha_P(t)}\cos\tfrac{\pi}{2}\alpha_P(t)$$

$$= \frac{\pi}{2\sin\frac{\pi\alpha_P(t)}{2}\cos\frac{\pi\alpha_P(t)}{2}\Gamma(\alpha_P(t))}\,e^{-i\frac{\pi}{2}\alpha_P(t)}\cos\tfrac{\pi}{2}\alpha_P(t)$$

$$\xrightarrow[t\to 0]{} -\frac{i\pi}{2} \quad .$$

Hence the amplitude is pure imaginary at t=0:

$$f(s,0) = i\frac{(\pi)^{\frac{1}{2}}}{2}\gamma_P(0)\left(\frac{s}{s_0}\right)^{\alpha_P(0)} \quad . \qquad (4.15)$$

This means that the ratio of the forward elastic cross section to the total cross section is as small as possible consistent with unitarity. That is to say, one may physically attribute the elastic scattering to the effect of all inelastic reactions. One calls this diffraction scattering because the same picture holds in the diffraction of light by a completely absorptive sphere. In that classical example, the incident light casts a shadow behind the sphere. The shadow is of course "caused" by the absorption (inelastic effects), but its existence requires that there be a definite amount of elastic scattering to cancel the incident wave behind the sphere.

Since the Pomeron has positive signature, the elastic cross section is

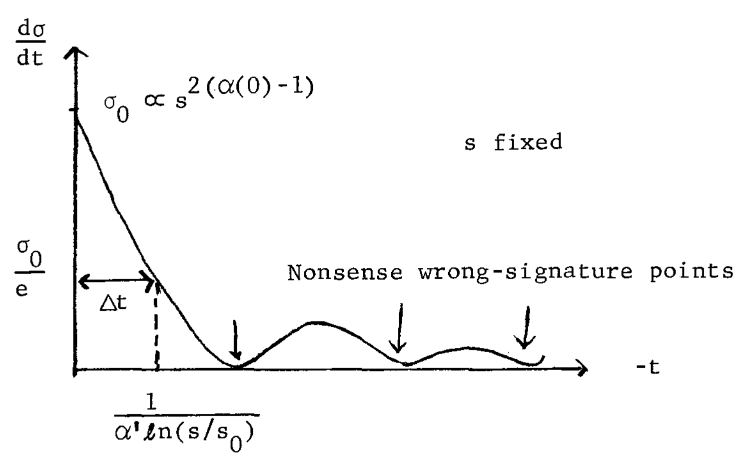
$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2 s}\gamma_P{}^2(t)\Gamma^2(1-\alpha_P(t))\cos^2[\tfrac{\pi}{2}\alpha_P(t)]\left(\frac{s}{s_0}\right)^{2\alpha_P(t)} . \qquad (4.16)$$

It is convenient to define

$$\frac{d\sigma}{dt} = \frac{\pi}{k^2}\frac{d\sigma}{d\Omega} \approx c(t)\left(\frac{s}{s_0}\right)^{2(\alpha_P(t)-1)}$$

$$c(t) = \pi\gamma_P{}^2(t)\Gamma^2(1-\alpha_P(t))\cos^2[\tfrac{\pi}{2}\alpha_P(t)] \ . \qquad (4.17)$$

A qualitative sketch of $d\sigma/dt$ is given below.



This cross section exhibits certain characteristic features.

1. The value of $\sigma_0 = (d\sigma/dt)_{t=0}$ varies with s like $\left(\frac{s}{s_0}\right)^{2(\alpha_p(0)-1)}$, so that it is independent of s if $\alpha_p(0)=1$. The constantcy of σ_0 is indeed experimentally

observed in all elastic scatterings.

2. Suppose that the trajectory is linear in t,

$$\alpha_p(t) = \alpha_0 + \alpha' t \quad , \tag{4.18}$$

then (4.16) becomes

$$\frac{d\sigma}{dt} = c(t)\left(\frac{s}{s_0}\right)^{2(\alpha_0-1)} e^{\alpha' t \ell n \frac{s}{s_0}} \quad . \tag{4.19}$$

If c(t) varies slowly for small t, then the dominent t dependence comes from the exponential factor. Hence the cross section will show a forward peak with a characteristic width

$$\Delta t = \frac{1}{\alpha' \ell n \frac{s}{s_0}} \tag{4.20}$$

which shrinks logarithmically with s. This shrinkage is observed in some but not all elastic scattering, possible because, in existing experiments, the energy is not sufficiently high, so that lower trajectories are still important.

3. $d\sigma/dt$ vanishes at the nonsense wrong-signature points, $\alpha_p(t) = -1,-3,-5,..,$ where the signature factor is zero, provided that $\beta(t)$ has no poles there. This would produce dips in the cross section, similar to the diffraction minima outside of the central maximum in Fraunhofer diffraction. It was, however, pointed out by Jones and Teplitz* and Mandelstam and Wang,** that $\beta(t)$ may have poles at precisely the nonsense-wrong signature points. The residues of these poles are proportional to certain integrals over the "third double spectral function" ρ_{tu}. Whether or not these poles actually exists is a dynamical question. We can only say that there is no general reason to expect a dip to occur except at nonsense wrong-signature points. If a dip does occur at such a point, then the type of pole mentioned above is either absent for some reason, or that its residue is small.

It is interesting to compare the characteristic features discussed above with that of the optical model of scattering, which includes the Fraunhofer diffraction of light. We start with the partial-wave expansion

$$f(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z) F_\ell(s)$$

$$= \frac{\pi(s)^{\frac{1}{2}}}{ik} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z) \left(e^{2i\delta_\ell(s)} - 1\right) \quad . \tag{4.21}$$

At high energies assume that many partial waves contribute, so that for small angle

---

* C.E. Jones and V.L. Teplitz, Phys. Rev. 159, 1271 (1967).

** S. Mandelstam and L.L. Wang, Phys. Rev. 160, 1490 (1967).

scattering we can use the approximation

$$P_\ell(\cos\theta) \underset{\substack{\ell\to\infty\\ \theta\to 0}}{\rightarrow} J_0(\ell\theta) = J_0(b(-t)^{\frac{1}{2}}),$$

$$b = \ell/k \quad , \tag{4.22}$$

where b is the classical impact parameter. We further assume that absorption effects are important, so that $\delta_\ell(s)$ is pure imaginary, and that it is only a function of b. Thus

$$f(s,t) \approx \frac{4\pi k^2}{i}\int_0^\infty dbbJ_0(b(-t)^{\frac{1}{2}})\chi(b) \tag{4.23}$$

where

$$\chi(b) = e^{2i\delta_\ell(s)} - 1 \tag{4.24}$$

is real by assumption. The model is then specified by the choice of $\chi(b)$.

Suppose that the target is a black sphere with a sharp edge. Then all partial waves are completely absorbed if the impact parameter is less than the radius of the sphere, and completely unmodified otherwise. This corresponds to choosing

$$\chi(b) = \begin{cases} -\chi_0 \; , & b < R \\ 0 \; , & b > R \; . \end{cases} \tag{4.25}$$

Then

$$f(s,t) = i4\pi k^2\chi_0\int_0^R dbbJ_0(\ell(-t)^{\frac{1}{2}})$$

$$= i4\pi k^2\chi_0 \frac{R}{(-t)^{\frac{1}{2}}} J_1(R(-t)^{\frac{1}{2}}) \quad . \tag{4.26}$$

This gives a diffraction peak of half width $\Delta t \sim 1/R^2$, with diffraction minima occurring at the zeroes of $J_1(R(-t)^{\frac{1}{2}})$. The first zero is at $R(-t)^{\frac{1}{2}} = 3.83$ which corresponds to a scattering single

$$\theta = 1.22 \left(\frac{\lambda}{2R}\right) \quad , \tag{4.27}$$

a formula well-known to amateur telescope makers.

As a second example, let us consider an absorptive sphere with a fuzzy edge, represented by

$$\chi(b) = -\chi_0 e^{-b^2/R^2} \quad . \tag{4.28}$$

This leads to

$$f(s,t) = i4\pi k^2\chi_0 \frac{R^2}{2} e^{\frac{1}{4}R^2 t} \tag{4.29}$$

and the cross section exhibits a diffraction peak of width $\Delta t \sim 1/R^2$ but no diffraction minima.

From these examples we gather that the width of the diffraction peak is related to the size of the target, while the depth of the diffraction minima is related to the sharpness of the edge of the target. If we compare this with Pomeron exchange, we see that the effective radius of a hadron as seen by another is

$$R \approx \left(4\alpha' \ln \frac{s}{s_0}\right)^{\frac{1}{2}} \qquad (4.30)$$

which increases slowly with energy. We cannot say, however, that the presence of nonsense wrong-signature dips implies that hadrons have sharp edges, because this mechanism for dips is entirely different from that in the optical model. The scattering amplitude in the optical model is pure imaginary for all t - a consequence of the assumption that $\chi(b)$ is real. In Pomeron exchange, however, the scattering amplitude is pure imaginary only at t=0. Away from t=0 a real part comes in through the signature factor. It is precisely the interference between the real and imaginary parts that give rise to nonsense wrong-signature dips. If we must make a classical picture of a hadron according to the Regge picture, we would have to say that a hadron is a fuzzy black sphere surrounded by a real potential which exerts a direct and an exchange force.

D. The ρ Trajectory

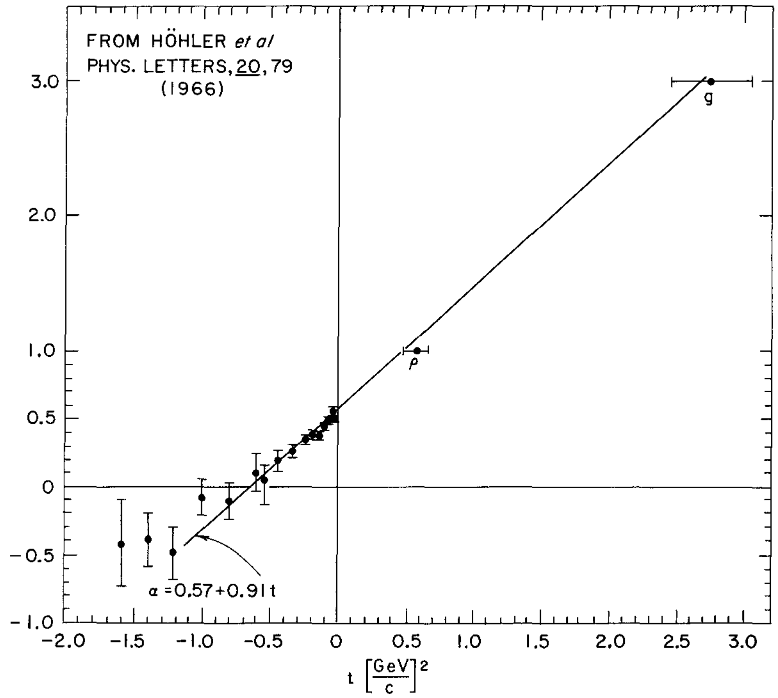
In the charge-exchange scattering $\pi^- p \to \pi^0 n$, the t-channel is $\pi^0 \pi^+ \to n\bar{p}$, with quantum numbers I=1, $P=+(-)^J$, G=+1. The only known Regge trajectory with these quantum number is the ρ. Hence one may hope to extract its properties unambiguously from experiments, using the procedure described earlier. The result of such an analysis is shown in the accompanying figure, and we note that $\alpha(t)$ is consistent with a straight line which extrapolates through the ρ and g mesons.

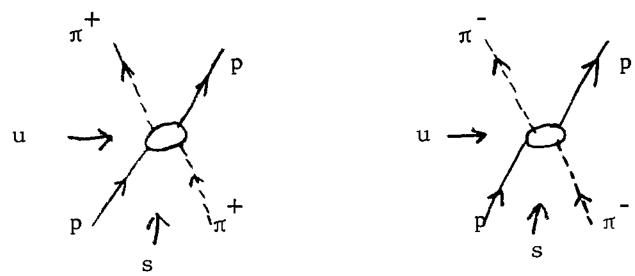
In the experimentally cross section, a marked dip is observed at $t = -0.58\text{GeV}^2$, which is consistent with the first nonsense wrong-signature point, where $\alpha_\rho = 0$.

The single pole model predicts that the spin-flip and the spin-nonflip amplitudes have the same phase, which comes entirely from the signature factor. Hence it predicts that the polarization is zero. Experimentally, however, the polarization is not zero. This indicates that perhaps a second Regge pole with the same quantum numbers is the ρ, or a cut is present.

E. The N and Δ Trajectories

The N and Δ trajectories may be studied in the backward scatterings $\pi^+ p \to p\pi^+$ and $\pi^- p \to p\pi^-$ (i.e., in the region of small u and large s), as illustrated in the sketch below.





At large s and fixed u, both reactions are controlled by trajectories with baryon number B=+1. The u channel for $\pi^- p \to p\pi^-$ is a pure I = 3/2 state and so contains only the Δ trajectory. The u channel for $\pi^+ p \to p\pi^+$ is a mixture of I = 1/2 and I = 3/2 and so contains both the N and the Δ trajectories. However, this cross section is much larger than that for $\pi^- p \to p\pi^-$, so we assume that the contribution of the I = 3/2 state can be neglected, with the result that only N is exchanged. Thus the relevant amplitudes may be written

$$f_{\pi^+p}{}^s(s,t) = -\pi\beta_N(u)(2\alpha_N(s)+1)\left(\frac{s}{s_0}\right)^{\alpha_N(u)} \frac{[e^{-i\pi(\alpha_N(u)-\frac{1}{2})}+1]}{\sin\pi(\alpha_N(u)-\frac{1}{2})}$$

$$f_{\pi^-p}{}^s(s,t) = -\pi\beta_\Delta(u)(2\alpha_\Delta(u)+1)\left(\frac{s}{s_0}\right)^{\alpha_\Delta(u)} \frac{[e^{-i\pi(\alpha_\Delta(u)-\frac{1}{2})}-1]}{\sin\pi(\alpha_\Delta(u)-\frac{1}{2})} \quad . \qquad (4.31)$$

The salient feature of these formula is that, owing to the difference in signature of N and Δ, $f_{\pi^+p}{}^s$ has a dip at the nonsense wrong-signature point $\alpha_N(u)=-\frac{1}{2}$, whereas no dip is expected for $f_{\pi^-p}{}^s$ at $\alpha_\Delta(u)=-\frac{1}{2}$ because that is not a nonsense wrong-signature point. This expectation is dramatically verified by experiments. Thus we see in both the cases of the ρ and the N trajectories that the poles of the residue function, which theoretically may occur at nonsesne wrong-signature values, do not seem to be present.

If we adopt the point of view of Regge poles (rather than Khuri poles), then (4.31) merely represents the first term in the expansion of $\mathcal{P}_\alpha(z_u)$ in powers of z_μ. For equal mass scattering this is sufficient, for $z_u \to \infty$ as $s \to \infty$. In the present case, however, the last property does not hold, for

$$z_u = \frac{u(s,t)-(m^2-\mu^2)^2}{[u-(m+\mu)^2][u-(m-\mu)^2]} \approx \frac{u(s-t)-m^4}{(u-m^2)^2} \tag{4.32}$$

where m and μ are respectively the nucleon and pion mass. In the exact backward direction $\theta_s = \pi$ we have $u \approx 2m^4/s$, hence

$$z_u \underset{s\to\infty}{\to} 1+0(\frac{1}{s}). \quad (\text{at } \theta_s=\pi) \quad . \tag{4.33}$$

Therefore we must keep all terms in the expansion of $\mathcal{P}_\alpha(z_u)$. This leads to a difficulty,namely when we re-expand the series in powers of s, the coefficients of all but the leading term diverge at $\theta_s=\pi$. Since this would violate analyticity, the non-leading powers must, in fact, be absent. This would call for the existence of an infinite family of Regge poles, spaced successively one unit beneath the leading one, with residue functions so arranged to effect the cancellation of all terms except the leading one. These new trajectories are called daughter trajectories. In this case, the leading pole plus the infinite family of daughters just precisely make up one Khuri pole. The interest of this theoretical problem lies in the fact that it illustrates a constraint placed on the existence of Regge poles by analyticity: You must take the whole family or none.

If we take the point of view of Khuri poles from the beginning, then this particular problem does not arise. However, when the trajectory of the nucleon Khuri pole passes through $\frac{1}{2}$, it calls for an infinite family of daughter Khuri poles to make up precisely one Regge pole, in order to make a nucleon of spin $\frac{1}{2}$. Thus it seems that a Regge pole or Khuri pole is generally accompanied by an infinite family. A more detailed study of daughter trajectories is given by Freedman and Wang.*

---

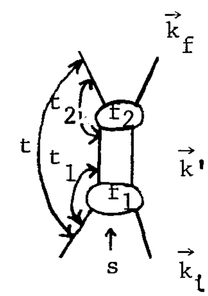
* D. Freedman and J.M. Wang. Phys. Rev. 153, 1596 (1967).

V. REGGE CUTS

A. Regge Cut from Two-Particle Unitarity

Although we have assumed until now that there are only poles in the ℓ plane, elastic unitarity strongly suggests that there exist Regge cuts as well. To see this let us consider equal mass spin zero scattering, and consider a term in the unitarity relation corresponding to intermediate states containing two particles of the same mass as the external particles:

$$\mathrm{Im} f(s,t) = \frac{1}{8\pi^2} \frac{k}{(s)^{\frac{1}{2}}} \int d\Omega' f_2^*(s,t_2) f_1(s,t_1) \qquad (5.1)$$



The kinematics is illustrated in the sketch, with

$$t = -|\vec{k}_f - \vec{k}_i|^2 = -2k^2(1-\cos\theta)$$

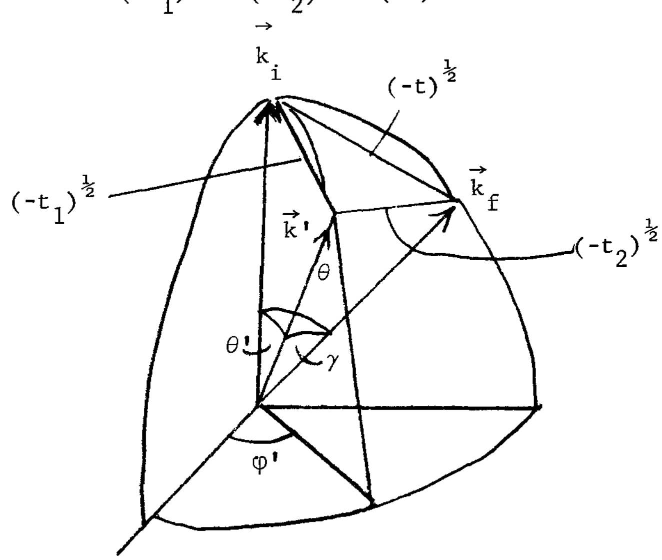
$$t_1 = -|\vec{k}' - \vec{k}_i|^2 = -2k^2(1-\cos\theta')$$

$$t_2 = -|\vec{k}_f - \vec{k}'|^2 = -2k^2(1-\cos\gamma)$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi' \qquad (5.2)$$

A geometrical construction of t, t_1 and t_2 is given in the sketch below, from which we see that

$$(-t_1)^{\frac{1}{2}} + (-t_2)^{\frac{1}{2}} \geq (-t)^{\frac{1}{2}} \qquad (5.3)$$



The equality is actually never attainable, but as $s \to \infty$ at fixed t, we have $\theta \to 0$, and the equality is almost fulfilled when $\vec{k}_i$, $\vec{k}_f$, $\vec{k}'$ are coplanar:

$$(-t_1)^{\frac{1}{2}} + (-t_2)^{\frac{1}{2}} = (-t)^{\frac{1}{2}} + 0(\frac{1}{s})$$

$$[\text{for } \theta=\theta'+\gamma,\ s \to \infty,\ t \text{ fixed (i.e., } \theta \to 0] \quad . \quad (5.4)$$

We now let s become large and assume that $f_1(s,t_1)$ and $f_2(s,t_2)$ are each dominated by a single Regge pole:

$$f_1(s,t_1) = A_1(t_1)s^{\alpha_1(t_1)}$$
$$f_2(s,t_2) = A_2(t_2)s^{\alpha_2(t_2)} \quad . \qquad (5.5)$$

Hence

$$\mathrm{Im}f(s,t) = \frac{1}{8\pi^2}\frac{k}{(s)^{\frac{1}{2}}}\int_{-4k^2}^{0}\frac{dt_1}{2k^2}\int_0^{2\pi}d\varphi' A_2^*(t_2)A_1(t_1)s^{\alpha_1(t_1)+\alpha_2(t_2)}$$

$$\approx \frac{1}{8\pi^2}\int_{-\infty}^{0}dt_1\int_0^{2\pi}d\varphi' A_2^*(t_2)A_1(t_1)s^{\alpha_1(t_1)+\alpha_2(t_2)-1} \quad . \qquad (5.6)$$

Let the maximum of the exponent of s be denoted by

$$\alpha_c(t) = \max[\alpha_1(t_1)+\alpha_2(t_2)-1],\ ((-t_1)^{\frac{1}{2}}+(-t_2)^{\frac{1}{2}} \geq (-t)^{\frac{1}{2}}). \qquad (5.7)$$

We can then transform the integral to the form

$$\mathrm{Im}f(s,t) = \int_{-\infty}^{\alpha_c(t)} d\ell D(\ell,t)s^{\ell} \qquad (5.8)$$

where

$$D(\ell,t) = \frac{1}{8\pi^2}\int_{-\infty}^{0}dt_1\int_0^{2\pi}d\varphi' A_2^*(t_2)A_1(t_1)\delta(\ell-\alpha_1(t_1)-\alpha_2(t_2)+1) \quad . \qquad (5.9)$$

The right hand side of (5.8) looks like the contribution of a continuous line of Regge poles in the ℓ plane starting at $\alpha_c(t)$. Hence there is a Regge cut from $\ell = \alpha_c(t)$ to $\ell = -\infty$.

Assume that $\alpha_1(t)$ and $\alpha_2(t)$ are increasing functions of t, so that the maximum in (5.8) occurs at $(-t_1)^{\frac{1}{2}} + (-t_2)^{\frac{1}{2}} = (-t)^{\frac{1}{2}}$. Putting $x = (-t_1)^{\frac{1}{2}}$, $y = (-t_2)^{\frac{1}{2}}$, the maximazation condition reads

$$\delta\{\alpha_1(-x^2)+\alpha_2(-y^2)-1-\lambda(x+y)\} = 0 \qquad (5.10)$$

where λ is a Lagrange multiplier. The solution is

$$\alpha_c(t) = \alpha_1(-x^2)+\alpha_2(-y^2)-1 \qquad (5.11)$$

where x,y are such that

$$\frac{d\alpha_1(-x^2)}{dx} = \frac{d\alpha_2(-y^2)}{dy} \quad . \qquad (5.12)$$

For linear trajectories $\alpha_i = \alpha_{0i}+\alpha_i{}'t$ (i=1,2), the explicit solution is

$$\alpha_c(t) = \alpha_{01}+\alpha_{02}-1) + \frac{\alpha_1{}'\alpha_2{}'}{\alpha_1{}'+\alpha_2{}'}\, t \quad . \tag{5.13}$$

In particular, for $\alpha_1(t) = \alpha_2(t) = \alpha_0+\alpha' t$, we obtain

$$\alpha_c(t) = (2\alpha_0-1) + \frac{1}{2}\alpha' t \quad . \tag{5.14}$$

This result was first derived in a slightly different way by Amati, Fubini and Stanghellini.* The type of Regge cuts obtained here is usually referred to as an AFS cut.

Little is known about the discontinuity $D(\ell,t)$ in (5.) except that it must vanish at $\ell = \alpha_c(t)$.** If we assume

$$D(\ell,t) \underset{\ell \to \alpha_c}{\to} c(t)(\alpha_c(t)-\ell)^a \quad , \tag{5.15}$$

where $a > 0$, then for large ℓns we have

$$\mathrm{Im} f(s,t) \underset{\ell ns \to \infty}{\to} c(t) s^{\alpha_c(t)} \int_0^\infty dx\, x^a e^{-x\ell ns} = \Gamma(a+1) c(t) \frac{s^{\alpha_c(t)}}{(\ell ns)^{a+1}} \quad . \tag{5.16}$$

Thus a Regge cut contribution differs from that of a Regge pole by a logarithmic factor. How high the energy should be in order that (5.16) be a good approximation depends on a more detailed knowledge of $D(\ell,t)$. Since ℓns is a slowly varying function, (5.16) can hardly be distinguished from a Regge pole contribution over a limited range of s.

The argument we have given for the AFS cut is of course not rigorous, for the inelastic contributions to unitarity, which have been neglected, may alter our conclusion. These contributions consist of additive terms to the right side of (5.6), and they are positive at t=0. They may cancel the AFS cut, and replace it by a higher-lying Regge singularity. All we can say is that this seems implausible.

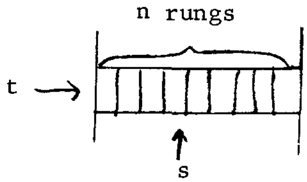
B. Some Model Calculations

The argument given earlier for the AFS cut is based only on elastic unitarity, and no appeal has been made to any detailed dynamical theory. We would like to give a brief qualitative description of some calculations based on Feynman diagrams. Any single Feynman diagram behaves asymptotically like $s^p(\ell ns)^q$, where p and q are fixed integers, and so does not exhibit Regge behavior. However, if we compute the leading asymptotic behavior of the n-rung ladder shown in the sketch and sum over n,

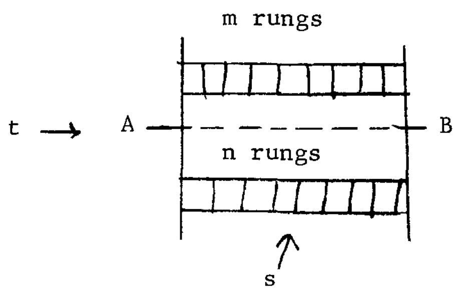
---

* D. Amati, S. Fubini and A. Stanghellini, Physics Letters 1, 29 (1962).

** J. Bronzan and C.E. Jones, Phys. Rev. 160, 1494 (1967).

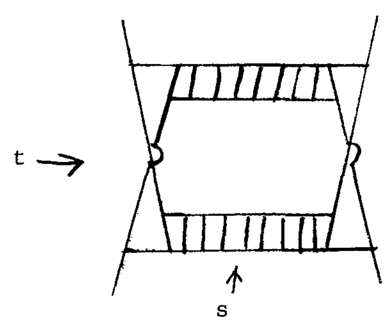


we do obtain asymptotic behavior of the form $s^{\alpha(t)}$. Therefore this sum of Feynman diagrams contains a Regge pole. The original Amati, Fubini and Stanghellini work was, in fact, based on a sum of Feynman diagrams of the type shown in the sketch below; but they only made an



approximate calculation. The exact sum of graphs can be written in the form of a dispersion integral, in which the absorptive parts are to be obtained by "cutting" the graph (i.e., replacing propagators by δ-functions) in all possible ways and adding the contributions. The original AFS calculation retains only the two-particle absorptive part by cutting the graphs along AB. This is not the same as two-particle unitarity, but the mathematics is similar and they obtained the cut whose branch point is given by (5.).

Mandelstam* has shown, however, that if one takes into account all of the multiparticle absorptive parts in the AFS calculation, the discontinuity of the AFS cut $D(\ell,s)$ is identically zero for s on the physical sheet. He considers another class of Feynman diagrams, of the type shown in the sketch below, and shows that this does give rise to a Regge cut with the same branch point as the AFS cut. The essential difference between the new class of diagram and the old one is that the new class consists of non-planar graphs, representing an amplitude having a non-vanishing third double spectral function ρ_{tu}, whereas $\rho_{tu} = 0$ for the AFS graphs. The lesson learned from these calculations seems to be that Regge



cuts owe their existence to the third double spectral function. In this respect, they has a common root with the poles of Regge residues at nonsense wrong-signature points.

C. Effect of Regge Cuts in Scattering

If we accept the existence of the Pomeron and that of AFS cuts, then the Pomeron would generate an infinite family of cuts, which would have an appreciable effect on elastic scattering as $s \to \infty$.

Let us first see how the Pomeron generates cuts, and for this purpose assume

* S. Mandelstam, Nuovo Cimento 30, 1127 (1963).

that the Pomeron trajectory is linear:

$$\alpha_p(t) = 1+\alpha' t \quad . \tag{5.17}$$

The AFS cut generated by the exchange of two Pomeron in the t-channel has branch points at

$$\alpha_2(t) = 1+\frac{1}{2}\alpha' t \quad . \tag{5.18}$$

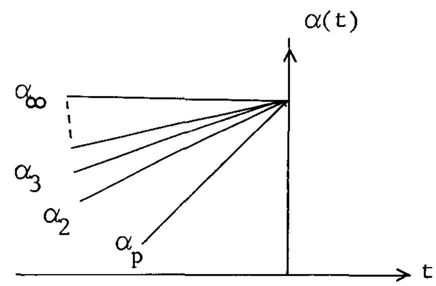
We can now take $f_1(s,t_1)$ in (5.1) to be dominated by the PP cut and $f_2(s,t_2)$ to be dominated by the Pomeron. Then we find a new AFS cut which may be looked upon as the effect of triple Pomeron exchange:

$$\alpha_3(t) = 1 + \frac{\alpha'(\frac{1}{2}\alpha')t}{\alpha'+\frac{1}{2}\alpha'} = 1+\frac{1}{3}\alpha' t \ . \tag{5.19}$$

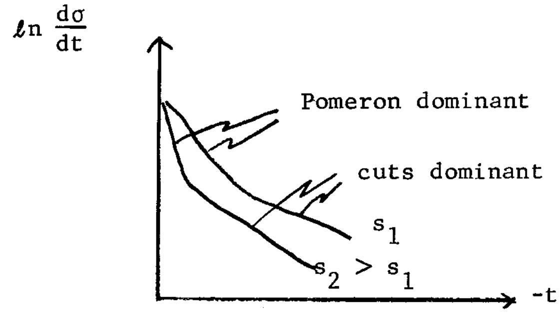
By repeating this argument, we find that the exchange of n Pomerons gives rise to an AFS cut with branch point at

$$\alpha_n(t) = 1+\frac{1}{n}\alpha' t \quad . \tag{5.20}$$

The trajectories of the family of cuts are shown in the sketch below.



How these cuts may affect high-energy scattering, of course, cannot be predicted before we have some dynamical information. Let us, however, make a reasonable guess. Let us assume that the coupling of the PP cut is much weaker than that of the Pomeron, and that the couplings of the higher cuts are pregressively weaker still. Then at small t, the separation of P-P and P becomes greater, and the P-P will take over. But by then the higher cuts also become well-separated, so that their total effect may be more important than that from any single one. Thus for a given large s, there is a small neighborhood of t=0 in which the Pomeron dominates, and the cross section will have a diffraction peak which shrinks logarithmically with increasing s. Outside of this neighborhood, the PP cut and possible other higher cuts too, become important. The cross section then falls off less rapidly with -t in this region, since the slope of the cut trajectores are smaller. Furthermore, as s is increased, the separation between $\alpha_p(t)$ and $\alpha_{pp}(t)$ becomes greater, and so the neighborhood in which the Pomeron dominates shrinks with increasing s. Thus the cross section may have a qualitative behavior as illustrated in the sketch below.



We can make a crude calculation by assuming that the contribution of the n^{th} cut to the scattering amplitude has the form $g^n s^{\alpha_n(t)}$, where g may have a weak dependence on s. Then the scattering amplitude can be written as

$$f(s,t) = \sum_{n=0}^{\infty} g^n s^{1+\frac{1}{n}\alpha' t} = s \sum_{n=0}^{\infty} \exp[n\ell ng+ \frac{\alpha' t \ell ns}{n}] \ . \tag{5.21}$$

As $s \to \infty$, we convert the sum into an integral which we evaluate by the method of steepest descent:

$$f(s,t) \approx s \int_0^{\infty} dn \ \exp[n\ell ng+ \frac{\alpha' t\ell ns}{n}] \approx s \ \exp[\bar{n}\ell ng+ \frac{\alpha' t\ell ns}{\bar{n}}] \tag{5.22}$$

where $\bar{n}$ is the value of n which maximizes the exponent:

$$\bar{n} = [\alpha' t\ell ns/\ell ng]^{\frac{1}{2}} \ . \tag{5.23}$$

Hence

$$f(s,t) \approx se^{-c(-t)^{\frac{1}{2}}} \ ,$$

$$\frac{d\sigma}{dt} \approx \frac{1}{\pi} e^{-2c(-t)^{\frac{1}{2}}} \ ,$$

$$c = 2[\alpha'\ell ns(-\ell ng)]^{\frac{1}{2}} \ . \tag{5.24}$$

It is interesting that the t dependence is the fastest decrease allowed by the Cerrulus-Martin bound.* If we assume that $-(\ell ns)(\ell ng)$ is a constant, then at a fixed t, the cross section $d\sigma/dt$ would fall with increasing s towards a limiting envelop. Experiments on pp scattering indicated that this might be so, but a more definite conclusion must await future experiments at higher energies.

---

* F. Cerrulus and A. Martin, Physics Letters 8, 80 (1964). Also see Khuri's lectures in this Summer School.

VI. TOWARDS DYNAMICS?

One of the motivations that we have mentioned for studying Regge poles is the hope that it helps to formulate the bootstrap hypothesis. We now discuss some important advances in this respect.

A. Finite-Energy Sum Rules.

By combining analyticity and Regge asymptotic behavior, one can deduce an interesting sum rule that relates s-channel resonances to t-channel Regge poles. For this purpose note that Regge asymptotic behavior holds along any direction in the s-plane, if it holds at all. This is because $\mathcal{P}_\alpha(z) \underset{z\to\infty}{\to} z^\alpha$ along any direction in the z-plane, hence the ratio of Regge to background terms is of the same order in any direction.

It is convenient to introduce

$$\nu = \frac{s-u}{2s_o} \tag{6.1}$$

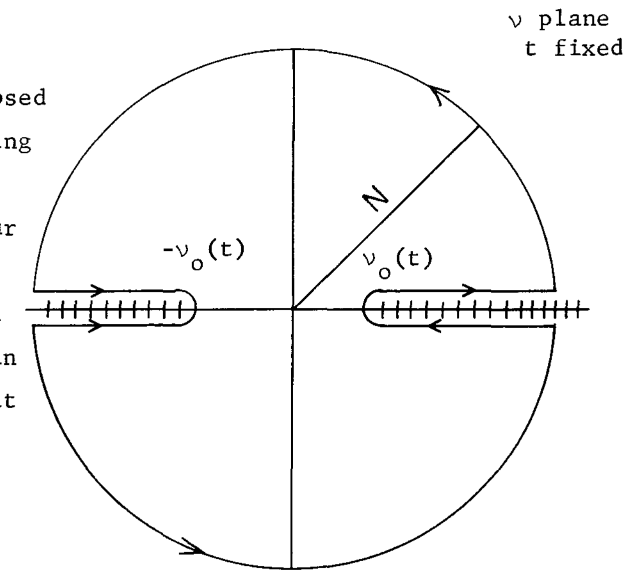
where s_o is an arbitrary scale, and use ν, t as independent variables. We decompose the scattering amplitude into terms symmetric and antisymmetric in ν:

$$\begin{aligned} f(\nu,t) &= f^+(\nu,t) + f^-(\nu,t), \\ f^\pm(\nu,t) &= \pm f^\pm(-\nu,t). \end{aligned} \tag{6.2}$$

Clearly $f^\pm(\nu,t)$ admits only t-channel Regge poles of signature ± 1. In the complex ν plane, $f^\pm(\nu,t)$ has cuts along the real axis and no other singularity. If there are bound state poles, we include them as part of the cuts. The branch points of the cuts are functions of t, and for some t the right and left cuts may overlap. In that event we carry out our development for a value t for which they do not overlap and continue the results to the desired t. By Cauchy's theorem, then,

$$\frac{1}{2\pi i} \oint \nu^n f^\pm(\nu,t) = 0, \tag{6.3}$$

where n is an integer, and the closed contour is shown in the accompanying sketch. Note the circle is of finite radius N. Since the contour has reflection symmetry with respect to $\nu = 0$, (6.3) is a trivial identity unless the integrand is an odd function of ν. This means that (6.3) has content only for



$$n = \begin{cases} \text{even integer for } f^{+}(\nu,t) \\ \text{odd integer for } f^{-}(\nu,t), \end{cases} \qquad (6.4)$$

and we shall only consider these values of n. Now separate the integral into an integral around the cuts plus that along the circle. Using the antisymmetry of $\nu^n f^{\pm}(\nu,t)$, we obtain

$$\frac{2}{\pi}\int_{\nu_o}^{N} d\nu\, \nu^n \, \mathrm{Im} f^{\pm}(\nu,t) + \frac{1}{2\pi i}\int_C d\nu\, \nu^n f^{\pm}(\nu,t) = 0 \ , \qquad (6.5)$$

where C denotes a circle of radius N, excluding the two points on the real axis, and

$$\mathrm{Im} f^{\pm}(\nu,t) = \frac{1}{2i}\lim_{\epsilon\to 0^+}\left[f^{\pm}(\nu+i\epsilon,t) - f^{\pm}(\nu-i\epsilon,t)\right] \ . \qquad (6.6)$$

Regge asymptotic behavior states that for large ν

$$f^{\pm}(\nu,t) = \sum_{\substack{\alpha > L \\ \mathrm{sgn}=\pm}} K_\alpha(\nu,t) + 0(\nu^{-L}) \ , \qquad (6.7)$$

where $K_\alpha(\nu,t)$ is given earlier in (3.48a). The integral of $\nu^n K_\alpha$ over C is elementary:

$$\int_C d\nu\, \nu^n K_\alpha(\nu,t) = A\int_C d\nu\, \nu^n[(-\nu)^\alpha \pm \nu^\alpha]$$

$$= \pm\, 2A\int_C d\nu\, \nu^{n+\alpha} = \pm\, 2iA\, N^{\alpha+A+1}\int_{-\pi}^{\pi} d\theta\, e^{i(\alpha+n+1)\theta}$$

$$= \pm 4iA \sin\pi(\alpha+n+1)\, N^{\alpha+n+1}/(\alpha+n+1) \qquad (6.8)$$

where $A = -\gamma\Gamma(1-\alpha)/(\pi)^{\frac{1}{2}}$.

Therefore (6.5) becomes

$$\int_{\nu_o}^{N} d\nu\, \nu^n\, \mathrm{Im}f^{\pm}(\nu,t) \pm \sum_{\substack{\alpha>-L \\ sgn=\pm 1}} \left[+ \frac{\gamma}{(\pi)^{\frac{1}{2}}} \frac{(-)^n}{\Gamma(\alpha)} \frac{N^{\alpha+n+1}}{\alpha+n+1}\right] + O(N^{-L}) = 0 \quad . \qquad (6.9)$$

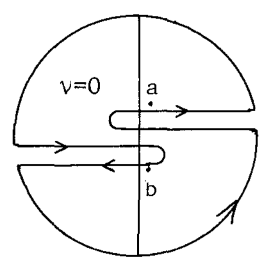
Neglecting $O(N^{-L})$, and writing the above explicitly for f^+ and f^-, we have for sufficiently large N:

$$\int_{\nu_o}^{N} d\nu\, \nu^n\, \mathrm{Im}f^{+}(\nu,t) \cong \sum_{\alpha, sgn=+1} \frac{\gamma\, N^{\alpha+n+1}}{(\pi)^{\frac{1}{2}}\, (\alpha+n+1)\, \Gamma(\alpha)}, (n \text{ odd})$$

$$\int_{\nu_o}^{N} d\nu\, \nu^m\, \mathrm{Im}f^{-}(\nu,t) \cong \sum_{\alpha, sgn=-1} \frac{\gamma\, N^{\alpha+m+1}}{(\pi)^{\frac{1}{2}}(\alpha+m+1)\, \Gamma(\alpha)} \cdot (m \text{ even}) \qquad (6.10)$$

These are the finite-energy sum rules (FESR) first derived by Dolen, Horn and Schmid* by a slightly different method. They have given some actual numerical examples, which we shall not go into.

In the s-channel physical region, $\nu_o(t)$ often becomes negative. The analytic continuation of (6.10) means that the original contour of integration actually looks like that shown in the sketch below. We can, in fact, replace ν_o by 0, if we understand $\mathrm{Im}f^{\pm}$ to be the discontinuity taken between points a and b shown in the sketch.



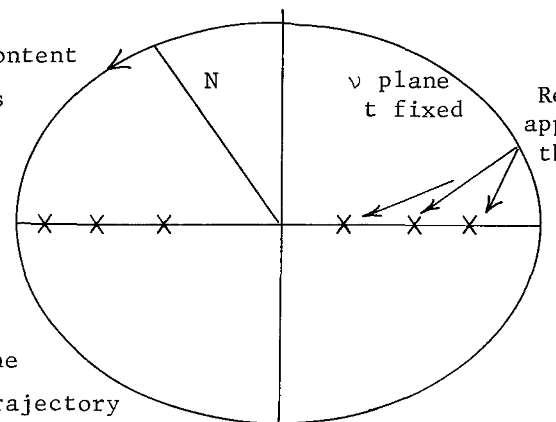
In these FESR, an integral of the amplitude extending over the s-channel low-energy region, which contains s-channel resonances, is approximately equated with the sum of t-channel Regge poles, which dominate the s-channel high-energy scattering. It therefore connects low-energy and high-energy phenomena, and connects exchanged particles (which produces a "potential") with resonances (which are "due" to the potential). Thus by combining analyticity with the Regge hypothesis, we begin to see some manifestations of the bootstrap.

----

*R. Dolen, D. Horn, and C. Schmid, Phys. Rev. 166, 1768 (1968).

B. Dualitу.

To explore the dynamical implications of the FESR, we have to make simplications in order to form an approximate picture of their real content. Suppose that in (6.10) the integrals on the left side can be separated in some manner into contributions from narrow resonances and a background. Harari* conjectures that the background approximates the Pomeron contribution on the right side, while the narrow resonances add up approximately to the rest of the Regge poles. This division is of course ambiguous and cannot be made more precise until a dynamical theory emerges. We accept this conjecture, however, as a first approximation. That is, we approximate $f^{\pm}$ on the left side by a sum of narrow resonances, and leave out the Pomeron on the right side, if it is there. Then in the ν plane for $f^{\pm}(\nu,t)$, the right and left cuts are replaced by a series of poles that were originally on unphysical Riemann sheets, as indicated in the sketch below.



In this approximation the content of the FESR may be stated as follows: At a given t the sum of residues (generally t-dependent) of all the poles within a large circle of radius N is proportional to $N^{\alpha+1}$ where $\alpha = \alpha(t)$ is the leading non-Pomeron Regge trajectory in the t-channel. The criterion for large N is that the leading trajectory dominates over the next one. Thus, although any one of the poles produces for large N a contribution $\propto N^{-1}$, the sum total of them gives $N^{\alpha+1}$. We say that the direct-channel resonances add up to a Regge pole in the crossed channel (which generates crossed-channel resonances). Conversely, a crossed-channel Regge pole already contains the contributions from all direct-channel resonances below a large energy N. This phenomenon is referred to as duality.

As defined above, duality is an immediate consequence of the FESR plus the narrow-resonance approximation. Of these, the FESR are on relative firm ground, both theoretically and experimentally. Thus a test of duality in this form is mainly a test of the narrow-resonance approximation. An interesting experimental test has been made by Schmid.† He calculated numerically the

----

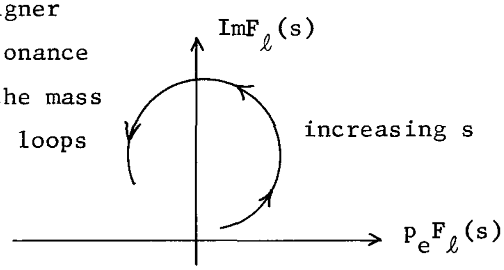
*H. Harari, Phys. Rev. Lett. 20, 1395 (1968). See also Harari's lectures in this summer school.

†C. Schmid, Phys. Rev. Lett. 20, 684 (1968).

partial wave projection of the amplitude for πN charge exchange scattering from ρ-trajectory exchange, which we write in a simplified way ignoring spin:

$$F_\ell(s) = \tfrac{1}{2}\int_{-1}^{+1} dz\ P_\ell(z)\ [-\gamma(t)\Gamma(1-\alpha(t)(e^{-i\pi\alpha(t)}-1)\nu^{\alpha(t)}], \qquad (6.11)$$

where the parameters of the ρ trajectory are taken from fits to actual scattering data. He found that $F_\ell(s)$ when plotted in the Argand diagram moves in a loop as a function of s, as shown in the sketch below. Such loops are also made by a Breit-Wigner resonance of spin ℓ. For a narrow resonance the top of the loop corresponding to the mass of the resonance. By interpreting the loops as resonances, Schmid found a semi-quantitative correspondence between his loops and the known direct channel resonances. Thus, although the Regge-exchange amplitude has no poles in s, its partial-mass projections mimics resonances. This is just what one would expect if one believes in duality. The mathematical reason why (6.12) gives rise to the loops is essentially the linearity of the ρ trajectory; namely, since $\alpha(t) = \alpha_o - 2\alpha'(1-z)k^2$, the phase of the signature factor, which is solely responsible for the phase of $F_\ell(s)$, increases with s.



One might wonder whether the concept of duality is fundamental and can be stated as a general principle independent of the narrow-resonance approximation. We do not yet know the answer to this question. More likely, duality occupies a place similar to that of complementarity. Before quantum mechanics, complementarity cannot be precisely formulated, after quantum mechanics its precise formulation becomes uninteresting; but it served as a useful working principle that guided the way to quantum mechanics.

C. <u>Exchange Degeneracy</u>

The FESR (6.10) treats the even and odd parts of $f(\nu,t)$ separately. To obtain a sum rule for f itself, multiply the first equation in (6.10) by N^m, the second by N^n, add the two equations and re-express $f^{\pm}$ in terms of f by (6.2). We find in this manner

$$\tfrac{1}{2}\int_{\nu_o}^{N} d\nu\ \mathrm{Im}[(\nu^n N^m + \nu^m N^n)f(\nu,t) + (\nu^n N^m - \nu^m N^n)f(-\nu,t)]$$

$$\cong \sum_\alpha \frac{\gamma\ N^{\alpha+n+m+1}}{(\pi)^{\frac{1}{2}}(\alpha+n+m+1)\ \Gamma(\alpha)}\ , \quad \begin{pmatrix} n = \text{odd integer} \\ m = \text{even integer} \end{pmatrix}, \qquad (6.12)$$

where on the right side we sum over all trajectories $\alpha > -L$, of both signatures.

In the narrow-resonance approximation, we replace $f(\nu,t)$ by a sum of zero-width resonance poles, and leave out the Pomeron contribution on the right side. Furthermore, we neglect the second term, which contains resonances in the u-channel, arguing that the factors $\nu^n N^m - \nu^m N^n$ averages to something small, (i.e. of the same order as terms already neglected in the narrow-resonance approximation). Consider now a two-body system that has no s-channel resonances. Examples are pp, $\pi^+\pi^+$, pK^+. In our approximation the left side of (6.12) is zero. Therefore, the sum of Regge poles on the right side vanishes for all N. This means that if there is a Regge pole of a given signature, there must exist one of opposite signature, with the same trajectory function $\alpha(t)$, and equal and opposite residue function $-\gamma(t)$. It cannot have the same signature, for that would cancel the original Regge pole identically. This degeneracy between two Regge poles of opposite signature is called exchange degeneracy. It has the same physical meaning as in potential scattering.

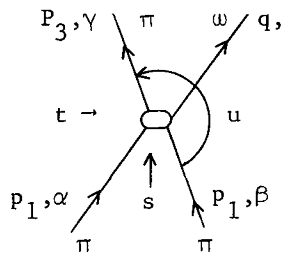
The requirement that exchange-degenerate trajectories have equal and opposite residue functions depends on one sign convention (3.48a), which has the signature factor in the form $e^{-i\pi\alpha} \pm 1$. If one redefines $\gamma(t)$ to make the signature factor $1 \pm e^{-i\pi\alpha}$, then we would require equal residue functions. The exchange-degenerate trajectories must be such that when their contribution is odded together, the term $e^{-i\pi\alpha}$ is cancelled.

For $\pi^+\pi^+$ scattering we know that the ρ trajectory is exchanged. Therefore a degenerate trajectory of opposite signature (i.e. positive) is called for. In the $\pi^+\pi^-$ system in the t-channel, even signature means that the amplitude is symmetric under $\pi^+\pi^-$ interchange, hence $I = 0$ or $I = 2$. It cannot have $I = 2$, for in that case it would also couple to $\pi^+\pi^+$, contradicting the fact that there are no resonances in $\pi^+\pi^+$. Hence the exchange-degenerate partner of ρ has $I = 0$, and the only known trajectory with $I = 0$, $G = +1$, $P = (-)^J$ is the f trajectory. Experimentally, the f meson lies remarkably close to the ρ trajectory, taken as the straight line passing through the ρ and g mesons.

In general, however, we should not be surprised if exchange degeneracy is only approximately realized in nature for its theoretical basis depends not only on the narrow-resonance approximation, but also on the neglect of the effects of u-channel resonances.

D. Bootstrap of the ρ Trajectory.

In a very interesting calculation, Ademollo et al.* try to bootstrap the ρ trajectory using the FESR in the narrow-resonance approximation. They were able to do this only by introducing further assumptions. Let us see what they do in some detail, for the true significance of such schemes is not yet clear at the present time. They consider the reaction



α, β, γ are isospin tensor indices

λ is the ω helicity

$\pi\,\pi \to \pi\,\omega$, for which the s,t,u channels are identical and have $I = 1$, $G = +1$, $P = +(-1)^J$. Hence in each channel only the ρ trajectory can contribute. This is a particularly happy choice because the Pomeron is not present, and we are spared the task of ejecting it forcefully.

Let the helicity amplitude be $f^s_{\lambda,\alpha\beta\gamma}(s,t)$. It must be antisymmetric in α and β because I = 1 in the s channel. Similarly, it is antisymmetric in α and γ and in β and γ. Hence

$$f^s_{\lambda,\alpha\beta\gamma}(s,t) = \epsilon_{\alpha\beta\gamma} f^s_\lambda(s,t). \tag{6.13}$$

By Bose statistics, $f^s_\lambda(s,t)$ is then antisymmetric in p_1 and p_2, p_1 and p_3. It is linear in the polarization vector $\epsilon^{(\lambda)}$ of the ω. Hence

$$f^s_{\lambda,\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\sigma\tau} p_1^\mu p_2^\nu p_3^\sigma \epsilon^{(\lambda)\tau}\, A(s,t,u), \tag{6.14}$$

where the invariant amplitude is totally symmetric in s,t, and u, with

$$s + t + u = \Sigma = 3m_\pi^2 + m_\omega^2. \tag{6.15}$$

One can evaluate the coefficient of A(s,t,u) explicitly and show

$$f^s_{o,\alpha\beta\gamma} = 0$$

$$f^s_{+1,\alpha\beta\gamma} = f^s_{-1,\alpha\beta\gamma}$$

$$= \pm\tfrac{1}{4}\,\epsilon_{\alpha\beta\gamma}(stu - m_\pi^2(m_\omega^2 - m_\pi^2)^2)^{\frac{1}{2}}\, A(s,t,u). \tag{6.16}$$

Since $f^s_{1,\alpha\beta\gamma}(s,t) \sim s^{\alpha(t)}$, the asymptotic behavior for A is

---

*M. Ademollo, H. R. Rubenstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. 176, 1904 (1968).

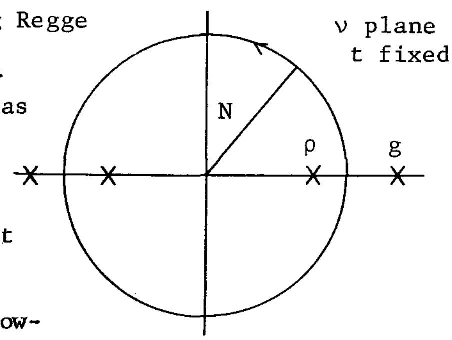
$$A(s,t,u) \sim s^{\alpha(t)-1}. \qquad (6.17)$$

We again introduce the variable $\nu = (s-u)/2s_o$ and write $A(\nu,t)$ for the invariant amplitude. We consider the FESR from (6.10) with n=1:

$$\int_0^N d\nu\ \nu\ \mathrm{Im}\ A(\nu,t) = \frac{\gamma\ N^{\alpha+1}}{(\pi)^{\frac{1}{2}}(\alpha+1)\Gamma(\alpha)} = \frac{\bar{\gamma}\ \alpha\ N^{\alpha+1}}{\Gamma(\alpha+2)} \qquad (6.18)$$

where $\bar{\gamma} = \gamma/(\pi)^{\frac{1}{2}}$, and $\alpha = \alpha(t)$ is the ρ trajectory.

Since we have omitted from the right hand side of (6.18) any lower-lying trajectories that may be present, it is valid only for sufficiently large N. Now Ademollo et al. make the additional assumption that for at least a limited range of t, (6.18) is valid even for N so small that in the interval $0 < \nu < N, A(\nu,t)$ has only one resonance, the ρ resonance. In terms of properties in the complex ν plane, the assumption is that for at least a limit range of t, the contour integral of $\nu A(\nu,t)$ over the circle shown in the sketch is well approximated by that of the leading Regge pole contribution. There is no a priori justification for this assumption. It was introduced partly as an inspired guess, partly as a calculational convenience. But it turns out to be the condition that bootstraps the ρ with brilliant success. Since this requires the FESR in the narrow-resonance approximation to be satisfied in a non-asymptotic region of N, it may be called a condition of strong duality. We adopt this word as a shorthand for the assumption described and refrain from philosophizing. The input assumptions are that the ρ trajectory is linear and passes through 1 at $t=m_\rho^2 \approx 0.5(\mathrm{GeV}/c)^2$:



$$\alpha(t) = \alpha_o + \alpha' t\ ,\quad \alpha(m_\rho^2) = 1 \qquad (6.19)$$

This leaves only one unknown constant among α_o and α'. Now $\alpha(t)$ makes t-channel resonances at $\alpha = 1,3,\ldots$, corresponding to $t = m_\rho^2, m_g^2, \ldots$ By crossing symmetry, there are s-channel resonances at $s = m_\rho^2, m_g^2, \ldots$, with corresponding ν values at

$$\nu_\rho(t) = [(s-u)/2s_o]_{s=m_\rho^2} = (t-t_o)/2s_o$$

$$\nu_g(t) = \nu_\rho(t) + 2/\alpha' s_o \qquad (6.20)$$

where

$$t_o = m_\omega^2 + 3m_\pi^2 - 2m_\rho^2 = -0.53(\text{GeV}/c)^2. \qquad (6.21)$$

Assuming strong duality, we cut off the integral in (6.18) at same point between the ρ and the g meson, i.e.,

$$\nu_\rho(t) < N < \nu_g(t). \qquad (6.22)$$

To do the integral, we have to know the residue of the ρ pole at $\nu = \nu_\rho(t)$. This can be obtained from the input ρ trajectory through crossing symmetry, as follows. The ρ trajectory exchanged in the t-channel contributes to $A(\nu,t)$ the Khuri term

$$K_\alpha(\nu,t) = \bar{\gamma}\,\Gamma(1-\alpha)(1-e^{-i\pi\alpha})\nu^{\alpha-1}, \qquad (6.23)$$

which has a pole at $\alpha = 1$:

$$K_\alpha(\nu,t) \xrightarrow[\alpha\to 1]{} \frac{2\bar{\gamma}}{1-\alpha} = -\frac{2\bar{\gamma}}{\alpha'}\frac{1}{t-m_\rho^2} \quad . \qquad (6.24)$$

Hence the residue is $-2\bar{\gamma}/\alpha'$. By crossing symmetry, the ρ pole in the s-channel must have the same residue. Hence

$$A(\nu,t) \xrightarrow[\nu\to\nu_\rho]{} -\frac{2\gamma}{\alpha'}\frac{1}{s-m_\rho^2} = -\frac{2\bar{\gamma}}{\alpha' s_o}\frac{1}{\nu-\nu_\rho}\ , \qquad (6.25)$$

which gives

$$\text{Im}A(\nu,t) = \frac{2\pi\bar{\gamma}}{\alpha' s_o}\,\delta(\nu-\nu_\rho) \qquad (6.26)$$

in the range (6.22), in the narrow-resonance approximation. Substituting (6.26) into the FESR (6.18), we obtain

$$\nu_\rho(t) = \frac{\alpha' s_o\ \alpha\ N^{\alpha+1}}{2\Gamma(\alpha+2)}. \qquad (6.27)$$

Note that the residue function $\bar{\gamma}(t)$ drops out.

We first note that $\nu_\rho(t_o) = 0$. Hence

$$\alpha(t_o) = 0, \qquad (6.28)$$

and this completely determines the ρ trajectory, leading to

$$\alpha_o = \frac{-t_o}{m_\rho^2 - t_o} = \frac{2m_\rho^2 - 3m_\pi^2 - m_\omega^2}{3m_\rho^2 - 3m_\pi^2 - m_\omega^2} \approx 0.5$$

$$\alpha' = \frac{1}{m_\rho^2 - t_o} = \frac{1}{3m_\rho^2 - 3m_\pi^2 - m_\omega^2} \approx 1(\text{GeV/c})^2 \quad , \qquad (6.29)$$

which are in remarkably good agreement with experiments. The condition (6.28) can be re-expressed in an amusing form by noting that $\alpha_o + \alpha' t_o = \alpha(s) + \alpha(t) + \alpha(u) - 2$. Hence, (6.28) is equivalent to the following condition for the ρ trajectory:

$$\alpha(s) + \alpha(t) + \alpha(u) = 2, \qquad (6.30)$$

which is of course very well satisfied experimentally.

With $\alpha(t)$ determined, it remains to be seen whether (6.27) can be satisfied for a range of t. Using (6.28), we can write $\nu_\rho(t) = \alpha(t)/2\alpha' s_o$, and substituting into (6.27) yields the condition

$$\frac{(\alpha' s_o)^2 \quad N^{\alpha(t)+1}}{\Gamma(\alpha(t)+2)} = 1. \qquad (6.31)$$

It is now noted that the following is a miraculously good approximation:

$$\frac{(1 + \frac{1}{2}\alpha)^{\alpha+1}}{\Gamma(\alpha+2)} \approx 1 \qquad (-1 < \alpha < +1) \qquad (6.32)$$

Therefore a solution of (6.31) for $-1 < \alpha(t) < 1$ is

$$s_o = 1/\alpha' \qquad (6.33)$$
$$N = 1 + \tfrac{1}{2}\alpha(t) \quad .$$

Thus the arbitrary scale s_o is now fixed. The cutoff N happens to fall exactly halfway between the ρ and the g meson, for using the $\alpha(t)$ and s_o now determined, we find that

$$\tfrac{1}{2}[\nu_\rho(t) + \nu_g(t)] = 1 + \tfrac{1}{2}\alpha(t). \qquad (6.34)$$

Ademollo et al. went on to investigate how they might extend the range of t in which the FESR is satisfied. It turns out that this involves pushing the cutoff N higher to include more resonances on the right-hand side, and at the same time including lower-lying trajectories on the right-hand side.

The most interesting aspect of this calculation is the fact that strong duality, which seems to be an ad hoc assumption, leads miraculously to some good results. We shall return to it in the Veneziano model, which is a crystallization of all the ideas we have discussed.

E. The Veneziano Model.

As we have seen, the FESR in the narrow-resonance approximation can be satisfied unexpectedly for a limited range of t by using a low cutoff as the left-hand side and only one Regge pole as the right-hand side. To extend the range of t, the cutoff has to be increased, and more Regge poles have to be included. The Veneziano model is a simple formula that incorporates all these features. In short, it is a simple solution to the FESR in the narrow-resonance approximation.

Recalling what FESR means in the narrow-resonance approximation, we see that for the process $\pi\pi \to \pi\omega$ a solution consists of finding an amplutide completely symmetric in s,t,u, having no cuts but only simple poles in s, and behaving like $s^{\alpha(t)-1}$ as $s \to \infty$. Veneziano suggests the form

$$A(s,t,u) = +\gamma[V(s,t) + V(s,u) + V(t,u)] \tag{6.35}$$

where

$$V(s,t) = \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)} = B(1-\alpha_s,1-\alpha_t), \tag{6.36}$$

where $\alpha_s \equiv \alpha(s)$, $\alpha_t \equiv \alpha(t)$, and where B(z,w) is the Beta function. Since $\Gamma(z)$ is a meromorphic function with simple poles at $z = 0,1,2\ldots$, V(s,t) has no cuts but has poles at $\alpha(s) = 1,2,3,\ldots$. Because of the gamma functions in the denominator, there are no simultaneous poles in s and t.

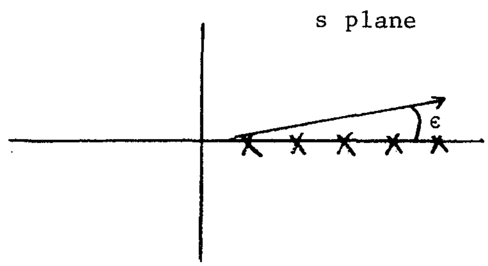
To compute the asymptotic behavior, we need the formula

$$\Gamma(a+bz) \xrightarrow[|z|\to\infty]{} (\pi)^{\frac{1}{2}} e^{-bz} (bz)^{a+bz-\frac{1}{2}}, \quad b > 0, |\arg z| \leq \pi-\epsilon. \tag{6.37}$$

We first rewrite (6.36) in the form

$$v(s,t) = \Gamma(1-\alpha_t) \frac{\Gamma(\alpha_s + \alpha_t - 1)}{\Gamma(\alpha_s)} \frac{\sin \pi \alpha_s}{\sin \pi (\alpha_s + \alpha_t - 1)} \tag{6.38}$$

The limit $s \to \infty$ does not exist along the real axis because V(s,t) has an infinite number of poles there. To avoid this difficulty, which is inherent in the narrow-resonance approximation, we take the limit along a ray in the complex s planes at an arbitrarily small angle ϵ with respect to the real axis.



Then

$$\sin\pi\,\alpha(s) = \frac{1}{2i}\left[e^{i\pi\alpha_s} - e^{-i\pi\alpha_s}\right]$$
$$\to -\frac{1}{2i}\,e^{-i\pi\alpha_s}\left[1 + 0(e^{-2\pi\epsilon s})\right] , \qquad (6.39)$$

and

$$\sin\pi(\alpha_s + \alpha_t - 1) \to -\frac{1}{2i}\,e^{-i\pi(\alpha_s + \alpha_t - 1)}\left[1 + 0(e^{-2\pi\epsilon s})\right], \qquad (6.40)$$

so that

$$V(s,t) \to \Gamma(1-\alpha_t)\;e^{-i\pi(\alpha_t - 1)}(\alpha' s)^{\alpha_t - 1} \quad . \qquad (6.41)$$

Note that to get this result, the linearity of α_s is crucial, at least asymptotically. For V(t,u) we can straightforwardly apply (6.37) to obtain

$$V(t,u) \to \Gamma(1-\alpha(t))\;(\alpha' s)^{\alpha(t)-1}. \qquad (6.42)$$

Finally,

$$V(s,u) = \frac{1}{\Gamma(1-2\alpha_o-\alpha'(\Sigma-t))}\;\frac{\Gamma(1-\alpha(u))}{\Gamma(\alpha(s))}\;\frac{\pi}{\sin\pi\,\alpha(s)}$$
$$= 0(e^{-\pi\epsilon s}) \qquad (6.43)$$

Hence

$$A(s,t,u) \underset{s\to\infty}{\longrightarrow} -\gamma\,\Gamma(1-\alpha(t))\;(e^{-i\pi\alpha(t)}-1)\;(\alpha' s)^{\alpha(t)-1} , \qquad (6.44)$$

which is the proper Regge behavior. If we had used the complete asymptotic expansion for $\Gamma(z)$, we would have obtained in place of (6.44)

$$A(s,t,u) \underset{z\to\infty}{} -\gamma\,\Gamma(1-\alpha_t)\;(e^{-i\pi\alpha_t}-1)\;(\alpha' s)^{\alpha_t - 1}\left[1 + \sum_{n=1}^{\infty} c_n(t) s^{-n}\right].$$

Thus there are an infinite number of parallel "daughter" trajectories $\alpha_n(t) = \alpha(t)-n$ $(n = 1,2,\ldots)$.

From the asymptotic behavior of the amplitude we would expect that at each mass there would be particles of all odd spins up to the leading trajectory. This is in fact the case. From (6.36) and the integral representation of the Beta function, we have

$$V(s,t) = \int_o^1 dx\;x^{-\alpha_s}(1-x)^{-\alpha_t} \quad . \qquad (6.45)$$

Using the binomial theorem, we obtain

$$V(s,t) = \sum_{n=0}^{\infty} (-1)^n \binom{-\alpha_t}{n} \int_0^1 dx\, x^{-\alpha_s + n}$$
$$= \sum_{n=0}^{\infty} (-1)^n \binom{-\alpha_t}{n} \frac{1}{n + 1 - \alpha_s} \qquad (6.47)$$

where

$$(-1)^n \binom{-\alpha_t}{n} = (-1)^n \frac{\Gamma(1-\alpha_t)}{n!\,\Gamma(1-\alpha_t-n)}$$
$$= \frac{1}{n!} \alpha_t (\alpha_t +1)(\alpha_t +2)\ldots(\alpha_t +n-1)$$
$$= \frac{1}{n!} R_n(\alpha_t). \qquad (6.48)$$

$R_n(x)$ is called a Pochammer polynomial of degree n. Hence

$$V(s,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{R_n(\alpha_t)}{n + 1 - \alpha_s} . \qquad (6.49)$$

As $\alpha_s \to n + 1$, therefore,

$$A(s,t,u) \to \gamma \frac{R_n(\alpha_t) + R_n(\alpha_u)}{n + 1 - \alpha_s} . \qquad (6.50)$$

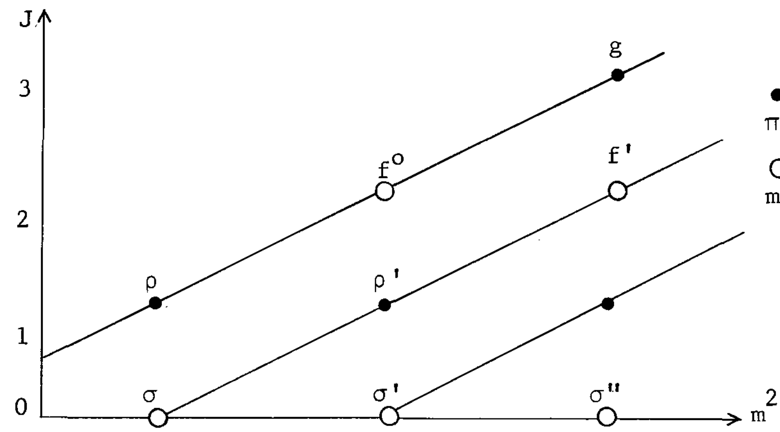
Since α_t and α_u are linear in t, the residue is a polynomial in t symmetric under $t \leftrightarrow u$. To find the spin of the resonances at $\alpha_s = n+1$, we have to express the residue in Legendre polynomials of z_s:

$$z_s = \frac{s(t-u)}{[s(s-4m_\pi^2)(s-(m_\omega-m_\pi)^2)(s-(m_\omega+m_\pi)^2)]^{\frac{1}{2}}} . \qquad (6.51)$$

We note that this is linear in t, and odd under $t \longleftrightarrow u$. Hence the residue is a polynomial in z_s containing only even powers. Since the ω meson has spin one, this implies* that at a mass m satisfying $\alpha(m^2) = n+1$, there are resonances of all odd spins up to n+1. Thus the Veneziano model requires that the mass spectrum forms a regular lattice on the Chew-Frantschi plot, as shown below.

----

*See Chapter 7 for partial-wave expansions.



It is clear that the Veneziano model satisfies the FESR because it has analyticity and Regge asymptotic behavior. With narrow resonance built in, it represents an elegant example of duality. However, while the FESR are satisfied, the trajectories are not completely determined. If one of the meson masses (say that of the ρ meson) is supposed to be given, we still have an arbitrary slope α'. This again demonstrates, as in the previous calculation of Ademollo et al. that the FESR alone is not enough to bootstrap. In the previous case, the bootstrap comes from the ad hoc assumption of strong duality, which turns out to be equivalent to the requirement that not only even-spin mesons like the f^o be decoupled, but also all mesons (of whatever spin) at the same mass. For example, referring to the previous sketch, we would require that ρ' be decoupled also. From (6.50), we see that this would require

$$R_n(\alpha_t) + R_n(\alpha_u) \equiv 0 \quad \text{(for n odd)} . \tag{6.52}$$

By (6.48), this is equivalent to

$$\alpha_t(\alpha_t+1) \cdots (\alpha_t+n-1) \equiv -\alpha_u(\alpha_u+1) \cdots (\alpha_u+n-1), \tag{6.53}$$

(for n odd)

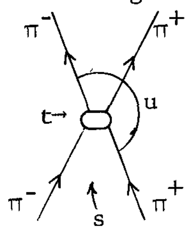
and is solved by setting $\alpha_t = -(\alpha_u+n-1)$. Noting that $n+1 = \alpha_s$, we obtain the condition

$$\alpha_s + \alpha_t + \alpha_u = 2, \tag{6.54}$$

which is the same as the consequence of strong duality in the earlier calculation of Ademollo et al., and which agrees well with experiments. In this model, however, there seems to be no compelling reason to require it.* For the present, therefore, strong duality remains a tantalizing idea not yet fully understood.

F. Veneziano Model for π-π Scattering.

Lovelace† has made an interesting application of the Veneziano model to π-π scattering. To take care of isospin complications, we first show that all 3 isospin amplitudes can be expressed in terms of a single symmetric function of s and t. Consider first $\pi^+\pi^-$ scattering as illustrated in the sketch, and let



$$f^s_{\pi^+\pi^-}(s,t) = \varphi(s,t) \qquad (6.55)$$

Since the t-channel also corresponds to $\pi^+\pi^-$ scattering,

$$\varphi(s,t) = \varphi(t,s) \quad . \qquad (6.56)$$

Since the u-channel corresponds to $\pi^+\pi^+$ scattering,

$$f^s_{\pi^+\pi^+}(s,t) = \varphi(u,t) \quad . \qquad (6.57)$$

Now decompose the $\pi^+\pi^-$ amplitude into amplitudes $f^s_I(s,t)$ of definite isospin I in the s-channel:

$$\varphi(s,t) = \frac{1}{6}\, f^s_2(s,t) + \frac{1}{2}\, f^s_1(s,t) + \frac{1}{3}\, f^s_o(s,t) \quad , \qquad (6.58)$$

where f^s_2 and f^s_o are even, and f^s_1 is odd, under $t \leftrightarrow u$:

$$f^s_I(s,t) = (-1)^I\, f^s_I(s,u). \qquad (6.59)$$

Thus

$$\varphi(s,u) = \frac{1}{6}\, f^s_2(s,t) - \frac{1}{2}\, f^s_1(s,t) + \frac{1}{3}\, f^s_o(s,t). \qquad (6.60)$$

Subtracting (6.60) from (6.58), we obtain $f^s_1(s,t) = \varphi(s,t) - \varphi(t,u)$. We also know that $\pi^+\pi^+$ is pure I = 2, hence by (6.57) $f^s_2(s,t) = \varphi(u,t)$. Substituting these results into (6.58), we find $f^s_o(s,t)$. The final results are:

$$\begin{aligned} f^s_o(s,t) &= \frac{3}{2}[\varphi(s,t) + \varphi(s,u)] - \frac{1}{2}\,\varphi(t,u) \quad , \\ f^s_1(s,t) &= \varphi(s,t) - \varphi(s,u) \ , \qquad (6.61) \\ f^s_2(s,t) &= \varphi(u,t) \quad . \end{aligned}$$

---

*In the original paper of Veneziano, (6.54) was invoked to obtain signatured trajectories; but we have seen that signature emerges automatically without this condition.

†C. Lovelace, Phys. Letters, 28B, 264 (1968).

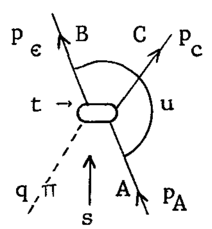
Therefore specifying $\varphi(s,t)$ completely specifies π-π scattering.

Lovelace constructed the Veneziano model for π-π scattering by taking

$$\varphi(s,t) = -\gamma \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1 - \alpha_s - \alpha_t)} \tag{6.62}$$

where $\Gamma(1 - \alpha_s - \alpha_t)$ rather than $\Gamma(2 - \alpha_s - \alpha_t)$ appears in the denominator because this amplitude should behave like $s^{\alpha(t)}$ as $s \to \infty$. With this choice there are no resonances in the I = 2 amplitude since $\varphi(u,t)$ has no poles in s. There are resonances of both even and odd spin on α_s in the I = 0 amplitude, but only resonances of odd spin occur in the I = 1 amplitude. The trajectory α_s is identified as the exchange degenerate ρ-f^o trajectory. This exchange degeneracy corresponds to the absence of I = 2 resonances.

One of the most interesting aspects of this model is the prediction of a zero in the amplitude coinciding with that required by the Adler self-consisting condition. In general, in the reaction $\pi A \to BC$, where A B C are hadrons, the hypothesis of PCAC (partial conservation of axial vector current), plus some assumption about the absence of poles, leads to the conclusion that the scattering amplitude must vanish as the four-momentum q of the pion approaches zero. This result is known as the Adler self-consisting condition. In terms of s,t,u, the zero is located at



$$s = (p_A)^2 = m_A^2$$
$$t = (p_B)^2 = m_B^2 \tag{6.63}$$
$$u = (p_A - p_B)^2 = m_C^2$$

which of course does not satisfy the constraint $s+t+u = \Sigma m^2$, because the pion is taken off the mass shell. For π-π scattering (6.63) becomes

$$s=t=u = m_\pi^2 \tag{6.64}$$

Let us rewrite (6.62) in the form

$$\varphi(s,t) = -\gamma(1-\alpha_s - \alpha_t)\, B(1-\alpha_s,\ 1-\alpha_t) \tag{6.65}$$

At $s=t=u = m_\pi^2$, the Beta function cannot vanish, but the factor $1-\alpha_s-\alpha_t$ vanishes if

$$\alpha(m_\pi^2) = \tfrac{1}{2}. \tag{6.66}$$

Combining this with $\alpha(m_\rho^2) = 1$, we find

$$\alpha_o = 0.483$$
$$\alpha' = 0.83 \qquad (6.67)$$

which is in excellent agreement with experiments.

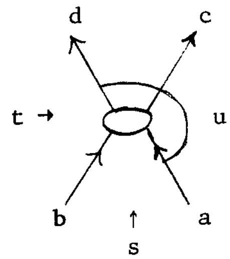
VII. SPIN

We now consider the full complications of spin. In particular we emphasize those features that owe their existence to spin, such as kinematic singularities, constraints, and sense-nonsense.

A. Kinematics

For a general two-body process a+b → c+d with arbitrary masses and spins, we specify single-particle states by their momenta and helicities. As usual let

$$s = (p_a + p_b)^2$$
$$t = (p_b - p_d)^2$$
$$u = (p_a - p_d)^2 \tag{7.1}$$



which satisfy the relation

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \equiv \Sigma \quad . \tag{7.2}$$

The cosines of the center-of-mass scattering angles in the s and t channels are given by

$$z_s = \cos\theta_s = \frac{s(2t+s-\Sigma) + (m_a^2-m_b^2)(m_c^2-m_d^2)}{\mathcal{S}_{ab}\,\mathcal{S}_{cd}} \quad ,$$

$$z_t = \cos\theta_t = \frac{t(2s+t-\Sigma) + (m_d^2-m_b^2)(m_c^2-m_a^2)}{\mathcal{T}_{bd}\,\mathcal{T}_{ca}} \quad , \tag{7.3}$$

where

$$\mathcal{S}_{ab} = \sqrt{[s-(m_a-m_b)^2]\,[s-(m_a+m_b)^2]} = \sqrt{4sp_{ab}^2} \quad ,$$

$$\mathcal{T}_{ca} = \sqrt{[t-(m_c-m_a)^2]\,[t-(m_c+m_a)^2]} = \sqrt{4tp_{ca}^2} \quad , \tag{7.4}$$

where the square roots are positive for positive values of their arguments. The physical region corresponds to

$$\varphi(s,t) \geq 0 \ , \tag{7.5}$$

where $\varphi(s,t)$ is the Kibble function:

$$\varphi(s,t) = stu - s(m_b^2-m_d^2)(m_a^2-m_c^2) - t(m_a^2-m_b^2)(m_c^2-m_d^2)$$
$$- (m_a^2m_d^2-m_c^2m_b^2)(m_a^2+m_d^2-m_c^2-m_b^2) \quad . \tag{7.6}$$

B. Helicity Amplitudes

For the purpose of Regge analysis, it is particularly convenient to use the helicity amplitudes of Jacob and Wick,* because they have simple partial-wave expansions. The helicity amplitude for s-channel scattering will be denoted by $f_{cd,ab}{}^{s}(s,t)$, where the subscripts denote both the particles and their helicities. For its definition and properties we refer to the original paper of Jacob and Wick. Our normalization is such that the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2 s}\frac{p_{cd}}{p_{ab}}\left|f_{cd;ab}{}^{s}(s,t)\right|^2 \quad . \tag{7.7}$$

Our amplitude is related to that of Jacob and Wick by

$$f_{cd;ab}{}^{JW}(s,t) = \sqrt{\frac{1}{4\pi^2 s}\frac{p_{cd}}{p_{ab}}}\, f_{cd;ab}{}^{s}(s,t) \quad . \tag{7.8}$$

The partial-wave expansion reads

$$f_{cd,ab}{}^{s}(s,t) = \sum_{J=\lambda_m}^{\infty} (2J+1)\, F^J_{cd;ab}(s)\, d^J_{\lambda\mu}(z_s) \tag{7.9}$$

$$\lambda = a-b,\ \mu = c-d,\ \lambda_m = \max(\lambda,\mu)$$

where $d^J_{\lambda\mu}(z_s)$ are the usual rotation coefficients. The partial-wave amplitude $F^J_{cd,ab}(s)$ is a matrix element taken between helicity states of definite total angular momentum J and z component M:

$$F^J_{cd;ab}(s) = \langle J,M;c,d|T(s)|J,M;a,b\rangle \quad . \tag{7.10}$$

These helicity states transform under spatial reflection P according to

$$P|J,M;a\ b\rangle = \eta_a\eta_b(-)^{J-J_a-J_b}|J,M;-a,-b\rangle \tag{7.11}$$

where J_a,J_b are the spins of the particles a,b, and η_a,η_b their intrinsic parities.

---

* M. Jacob and G.C. Wick, Annals of Physics 7, 404 (1959).

Thus parity conservation implies

$$F^J_{-c,-d;-a,-b}(s) = \eta_a\eta_b\eta_c\eta_d(-)^{J_c+J_d-J_a-J_b}\, F^J_{cd;ab}(s) \quad . \tag{7.12}$$

Time reversal invariance implies

$$F^J_{cd;ab}(s) = F^J_{ab;cd}(s) \quad . \tag{7.13}$$

Equations (7.12) and (7.13) serve to reduce the number of independent helicity amplitudes.

The crossing relation between the s and t channel helicity amplitudes is*

$$f^s_{cd,ab}(s,t) = \sum_{C'A'D'b'} d^{J_a}_{A'a}(\chi_a)\, d^{J_b}_{b'b}(\chi_b)\, d^{J_c}_{c'c}(\chi_c)\, d^{J_d}_{D'd}(\chi_d)\, f^t_{c'A',D'b'}(s,t), \tag{7.14}$$

where

$$\begin{aligned}
\cos\chi_a &= -\frac{(s+m_a^2-m_b^2)(t+m_a^2-m_c^2) - 2m_a^2(m_c^2-m_a^2+m_b^2-m_d^2)}{\mathcal{S}_{ab}\mathcal{T}_{ac}}, \\
\cos\chi_b &= \frac{(s+m_b^2-m_a^2)(t+m_b^2-m_d^2) - 2m_b^2(m_c^2-m_a^2+m_b^2-m_d^2)}{\mathcal{S}_{ab}\mathcal{T}_{bd}}, \\
\cos\chi_c &= \frac{(s+m_c^2-m_d^2)(t+m_c^2-m_a^2) - 2m_c^2(m_c^2-m_a^2+m_b^2-m_d^2)}{\mathcal{S}_{cd}\mathcal{T}_{ac}}, \\
\cos\chi_d &= -\frac{(s+m_f^2-m_c^2)(t+m_d^2-m_b^2) - 2m_d^2(m_c^2-m_a^2+m_b^2-m_d^2)}{\mathcal{S}_{cd}\mathcal{T}_{bd}},
\end{aligned} \tag{7.15}$$

and

$$\begin{aligned}
\sin\chi_a &= \frac{2m_a\sqrt{\varphi(s,t)}}{\mathcal{S}_{ab}\mathcal{T}_{ac}} , \\
\sin\chi_b &= \frac{2m_b\sqrt{\varphi(s,t)}}{\mathcal{S}_{ab}\mathcal{T}_{bd}} , \\
\sin\chi_c &= \frac{2m_c\sqrt{\varphi(s,t)}}{\mathcal{S}_{cd}\mathcal{T}_{ac}} , \\
\sin\chi_d &= \frac{2m_d\sqrt{\varphi(s,t)}}{\mathcal{S}_{cd}\mathcal{T}_{bd}} .
\end{aligned} \tag{7.16}$$

---

* L. Trueman and G.C. Wick, Annals of Physics 26, 322 (1964).

We may write symbolically

$$f_H^{\,s}(s,t) = \sum_{H'} \mathcal{M}_{HH'}(s,t)\; f^{t}_{H'}(s,t) \quad , \tag{7.17}$$

where $\mathcal{M}$ is a real orthogonal matrix: $\mathcal{M}^T\mathcal{M} = 1$. It should be noted that (7.14) is not valid for an amplitude that differs from ours by a normalization factor that depends on s and t. In particular it is not valid for f^{JW} of (7.8).

The main advantages of helicity amplitudes are the following. (a) The number of independent amplitudes can be easily enumerated and written down for an arbitrary reaction. (b) By (7.11) it is easy to form helicity states of definite parity, and Regge trajectories couple to them independently. (c) It is straightforward to carry out the Sommerfeld-Watson transform on (7.9) to isolate Regge pole contributions.

Helicity amplitudes, however, have kinematic singularities and satisfy constraint equations at certain values of s and t. These are intrinsic in their definition and give rise to complicated structures in Regge residues that were not present in the spinless case.

Instead of helicity amplitudes one can describe the scattering process in terms of invariant amplitudes, which by definition is a set of independent amplitudes completely free of kinematic singularities and constraints. We shall not discuss them in general but merely illustrate them in specific examples. Although it can be proven that invariant amplitudes exist for an arbitrary reaction, there is yet no known method for their explicit construction in the general case. From our point of view the main disadvantage of invariant amplitudes is that the same Regge trajectory generally couples to more than one amplitude, so that Regge residues in different amplitudes cannot be independent.

C. Kinematic Singularities and Constraints

According to Jacob and Wick, a general helicity state is defined as follows. First define the helicity state of a single particle at rest. Then define that for a moving particle by applying the boost operator of a Lorentz transformation. The helicity state for two particles is the product of two of the above, rotated in a standard way by the application of a total rotation operator. The helicity amplitudes are defined as T-matrix elements with respect to two-particle helicity states, and singularities and constraints generally arise from the fact that the boost and rotation operators become singular at certain kinematic points. These have nothing to do with the interactions of particles, and we call them kinematic singularities and constraints. An analysis from this point of view is given by Trueman,* who shows from general principles that kinematic singularities in s can occur only at one of the following places:

(a) At $\mathcal{S}_{ab} = 0$ or $\mathcal{S}_{cd} = 0$, namely $s = (m_a \pm m_b)^2$ or $s = (m_c \pm m_d)^2$. The values corresponding to the + sign are thresholds, the others are called pseudothresholds.

(b) Boundary of the physical region $\varphi(s,t) = 0$.

(c) The point $s = 0$.

Kinematic constraints can occur only at pseudothresholds or at $\varphi(s,t) = 0$. Most important for our purpose, the kinematic singularities at (a) or (b) above can be factored out of the helicity amplitudes. Those at (c) can be factored out except for fermion-boson scattering in the general mass case, where there is a non-factorizable singularity of the type $s^{\frac{1}{2}}$. For this case, however, one can circumvent it by using $W = s^{\frac{1}{2}}$ as independent variable.

Another approach, more elementary but less satisfactory from the point of view of general principles, is due to Wang.** It makes use only of the crossing relation for helicity amplitudes and is a relatively straightforward constructive recipe in specific cases. We shall briefly describe this approach here.

Going back to the partial wave expansion

$$f_H^{\,s}(s,t) = \sum_{J=\lambda_m}^{\infty} (2J+1)\, F_H^{\,J}(s)\, d^J_{\lambda\mu}(z_s) \quad , \tag{7.18}$$

we see that the t dependence is contained in z_s in the rotation coefficient $d^J_{\lambda\mu}(z_s)$. Now

$$d^J_{\lambda\mu}(z) = D_{\lambda\mu}(z)\, e^J_{\lambda\mu}(z) \quad , \tag{7.19}$$

where

$$D_{\lambda\mu}(z) \equiv (1+z)^{\frac{1}{2}|\lambda+\mu|}\,(1-z)^{\frac{1}{2}|\lambda-\mu|} \quad , \tag{7.20}$$

and $e^J_{\lambda\mu}(z)$ is a polynomial in z. (We use the notation of GGLMZ.***) Since the factor $D_{\lambda\mu}(z)$ is independent of J, it can be factored out of the sum in (7.18):

$$f_H^{\,s}(s,t) = D_{\lambda\mu}(z_s)\, \bar{f}_H^{\,s}(s,t) \quad , \tag{7.21}$$

---

* T.L. Trueman, Phys. Rev. 173, 1684 (1968). Errata, Phys. Rev. 181, 2154 (1969).

** L.L. Wang, Phys. Rev. 142, 1187 (1965).

*** M. Gell-Mann, M. Goldberger, F.E. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964).

where

$$\bar{f}_H^{\ s}(s,t) = \sum_{J=\lambda_m}^{\infty} (2J+1)\ F_H^J(s)\ e^J_{\lambda_\mu}(z_s) \quad . \tag{7.22}$$

The only t-singularities of $\bar{f}_H^{\ s}(s,t)$ come from the possible divergences of the whole series. We presume that these are dynamical and not kinematic singularities. Similarly, if we put

$$f_H^{\ t}(s,t) = D_{\lambda_\mu}(z_s)\ \bar{f}_H^{\ t}(s,t) \quad , \tag{7.23}$$

then $\bar{f}_H^{\ t}(s,t)$ has no s-kinematic singularities.

The new amplitudes $\bar{f}^s$ and $\bar{f}^t$, however, are related through a crossing relation of the form

$$\bar{f}_H^{\ s}(s,t) = \sum_{H'} \mathcal{N}_{HH'}(s,t)\ \bar{f}_{H'}^{\ t}(s,t) \quad , \tag{7.24}$$

where the matrix $\mathcal{N}$ can be deduced from the matrix $\mathcal{M}$ in (7.17). Since $\bar{f}^t(s,t)$ has no s-kinematic singularities by construction, all of the s-kinematic singularities of $\bar{f}^s(s,t)$ must come from the known matrix $\mathcal{N}(s,t)$. Furthermore, $\mathcal{N}(s,t)$ must cancel all of the t-kinematic singularities of $\bar{f}^t(s,t)$, because $\bar{f}^s$ can have no such singularities. Thus by studying the matrix $\mathcal{N}$ all the kinematic singularities in s and t can be recognized. In general this is an extremely tedious procedure, but one arrives at the same conclusion as mentioned before. In particular, we can factor out the s-kinematic singularities from each component of $\bar{f}_H^{\ s}$:

$$\bar{f}_H^{\ s}(s,t) = \mathcal{K}_H(s)\ \hat{f}_H^{\ s}(s,t) \tag{7.25}$$

where $\hat{f}_H^{\ s}(s,t)$ is now free of all kinematic singularities, s or t (except for a $s^{\frac{1}{2}}$ branch point for fermion-boson scattering in the general mass case). Similarly we factor out all t-kinematic singularities from $\bar{f}_H^{\ t}$:

$$\bar{f}_H^{\ t}(s,t) = \mathcal{J}_H(t)\ \hat{f}_H^{\ t}(s,t) \tag{7.26}$$

where $\hat{f}_H^{\ t}(s,t)$ is free of all kinematic singularities, s or t.

Substituting (7.25) and (7.26) into (7.24), we obtain a crossing relation of the form

$$\hat{f}_H^{\ s}(s,t) = \sum_{H'} \mathcal{L}_{HH'}(s,t)\ \hat{f}_H^{\ t}(s,t) \tag{7.27}$$

where the matrix $\mathcal{L}(s,t)$ generally has singularities in s and t, although $\hat{f}_H{}^s$ cannot have these singularities. For values of s,t in the neighborhood of a singularity of $\mathcal{L}(s,t)$, let us write $\mathcal{L}(s,t)$ as a matrix product between a matrix $\mathcal{L}_1(s,t)$ containing the singularity, and a regular matrix $\mathcal{L}_2(s,t)$ (of course $\mathcal{L}_2$ may be simply the unit matrix):

$$\mathcal{L}(s,t) = \mathcal{L}_1(s,t)\ \mathcal{L}_2(s,t) \quad . \tag{7.28}$$

Then at the singularity, say $s = s_0$, $t = 0$, we must have

$$\sum_{H'} [\mathcal{L}_2(s_0,t_0)]_{HH'}\ \hat{f}_{H'}{}^t(s_0,t_0) = 0 \quad , \tag{7.29}$$

which is called a kinematic constraint.

The kinematic singularities and constraints discussed above lead to kinematic singularities and constraints in the t-channel partial-wave amplitudes $G_H{}^J(t)$. Since a Regge pole is a J-pole of the latter, with t-dependent residues, it follows that the Regge residues have known kinematic singularities and satisfy known constraints. In particular the constraints relate the residues of Regge poles of different quantum numbers at certain values of t.

D. Example: $\pi\pi \to \pi\omega$

Earlier we have discussed the reaction $\pi\pi \to \pi\omega$ in terms of an invariant amplitude (See Eq. (6.14)). Let us discuss it in terms of helicity amplitudes as an illustration.

There are three s-channel helicity amplitudes (See Eq. (6.13)) $f_\lambda{}^s(s,t)$, where $\lambda = 1, 0, -1$ is the helicity of ω. The partial-wave expansion reads

$$f_\lambda{}^s(s,t) = \sum_{J=|\lambda|}^{\infty} (2J+1)\ F_\lambda{}^J(s)\ d_{0\lambda}{}^J(z_s) \quad , \tag{7.30}$$

where the partial-wave amplitude $F_\lambda{}^J(s)$ is a matrix element between helicity states:

$$F_\lambda{}^J(s) = \langle\ \pi\omega;J,\lambda|T(s)|\pi\pi;J\ \rangle \quad . \tag{7.31}$$

Under the parity operation, the helicity states concerned transform as follows:

$$P|\pi\pi;J\rangle = (-)^J|\pi\pi;J\rangle$$

$$P|\pi\omega;J,\lambda\rangle = -\ (-)^J|\pi\omega;J,-\lambda\rangle \quad . \tag{7.32}$$

The eigenstates of parity for the $\pi\omega$ system are

$$|\pi\omega;J\pm> = \frac{1}{\sqrt{2}}\,[\,|\pi\omega;J,1> \mp |\pi\omega;J,-1>] \quad , \tag{7.33}$$

with

$$P|\pi\omega;J\pm> = \pm\,(-)^J\,|\pi\omega;J\pm> \quad . \tag{7.34}$$

By parity conservation the only non-vanishing matrix element between $\pi\omega$ and $\pi\pi$ states is

$$<\pi\omega;J+|T(s)|\pi\pi;J> \equiv F^J(s) \quad , \tag{7.35}$$

in terms of which the partial-wave amplitudes are

$$F_1^{\ J}(s) = -\,F_{-1}^{\ J}(s) = \frac{1}{\sqrt{2}}\,F^J(s) \quad ,$$

$$F_0^{\ J}(s) = 0 \quad . \tag{7.36}$$

Therefore there are only two non-vanishing helicity amplitudes $f_1^{\ s}(s,t)$ and $f_{-1}^{\ s}(s,t)$. Furthermore, owing to the fact that $d^J_{\lambda\mu}(z) = (-)^{\lambda-\mu}\,d^J_{-\lambda,-\mu}(z)$, they are equal to each other. Hence there is only one independent helicity amplitude $f_1^{\ s}(s,t)$. Since the reaction is the same for the s, t, and u channels,

$$f_1^{\ s}(s,t) = f_1^{\ t}(s,t) \tag{7.37}$$

up to a constant phase factor.

To factor out the kinematic singularities, we follow (7.21) and put

$$f_1^{\ s}(s,t) = (1-z_s^{\ 2})^{\frac{1}{2}}\,\bar{f}_1^{\ s}(s,t) \quad , \tag{7.38}$$

$$f_1^{\ t}(s,t) = (1-z_t^{\ 2})^{\frac{1}{2}}\,\bar{f}_1^{\ t}(s,t) \quad , \tag{7.39}$$

where $\bar{f}_1^{\ s}(s,t)$ has no t-kinematic singularities, and $\bar{f}_1^{\ t}(s,t)$ has no s-kinematic singularities. By (7.37), we have

$$(1-z_s^{\ 2})^{\frac{1}{2}}\,\bar{f}_1^{\ s}(s,t) = (1-z_t^{\ 2})^{\frac{1}{2}}\,\bar{f}_1^{\ t}(s,t) \quad . \tag{7.40}$$

Now we need to work out some kinematics:

$$z_s = (t-u)/4p_sq_s$$

$$z_t = (s-u)/4p_tq_t \quad , \tag{7.41}$$

where

$$p_s = \frac{1}{2}(s-4m_\pi^{\ 2})^{\frac{1}{2}}$$

$$q_s = \{[s - (m_\omega+m_\pi)^2]\ [s - (m_\omega-m_\pi)^2]/4s\}^{\frac{1}{2}} \quad . \tag{7.42}$$

From these we find that

$$(1-z_s^{\ 2})^{\frac{1}{2}} = \frac{1}{4p_s q_s}\left[\frac{\varphi(s,t)}{s}\right]^{\frac{1}{2}}$$

$$(1-z_t^{\ 2})^{\frac{1}{2}} = \frac{1}{4p_t q_t}\left[\frac{\varphi(s,t)}{t}\right]^{\frac{1}{2}} \quad , \tag{7.43}$$

where

$$\varphi(s,t) = stu - m_\pi^{\ 2}(m_\omega^{\ 2}-m_\pi^{\ 2})^2 \ . \tag{7.44}$$

Substituting (7.43) into (7.40) we have

$$\frac{\bar{f}_1^{\ s}(s,t)}{4p_s q_s\sqrt{s}} = \frac{\bar{f}_1^{\ t}(s,t)}{4p_t q_t\sqrt{t}} \quad . \tag{7.45}$$

Since $\bar{f}_1^{\ s}$ has no t-kinematical singularity, and $\bar{f}_1^{\ t}$ has no s-kinematical singularity, each side must be free of all kinematic singularities. That is

$$\frac{1}{4p_s q_s\sqrt{s}}\ \bar{f}_1^{\ s}(s,t) = A(s,t,u) \tag{7.46}$$

where A(s,t,u) is an invariant amplitude. Thus

$$f_1^{\ s}(s,t) = (1-z_s^{\ 2})^{\frac{1}{2}}\ 4p_s q_s s^{\frac{1}{2}}\ A(s,t,u) = [\varphi(s,t)]^{\frac{1}{2}}\ A(s,t,u) \ , \tag{7.47}$$

which is identical with (6.16).

E. Conspiracy

As mentioned before, kinematic constraints on t-channel amplitudes can occur only at pseudothresholds, or on the boundary of the physical region. When the external masses are equal in pairs, the latter includes the point t = 0. A constraint occurring at this point is physically interesting, because it corresponds to forward scattering in the s-channel. Indeed, in many cases, such a constraint is a direct consequence of angular momentum conservation in the s-channel.

For concreteness, let us consider nucleon-nucleon scattering, for which there are $2^4 = 16$ helicity amplitudes. Parity conservation and time-reversal invariance reduce the independent to 5, which we can choose to be $f^s_{++;++}$, $f^s_{++;--}$, $f^s_{+-;+-}$, $f^s_{+-;-+}$, $f^s_{++;+-}$, where the subscripts $\pm$ correspond to the helicity $\pm\frac{1}{2}$ of a nucleon. Thus in $f^s_{cd;ab}$, (a-b) and (c-d) are the components of the total angular momentum along the relative momentum for the initial and final state, respectively. Conservation of angular momentum tells us that for forward scattering (t = 0) we must have $f_H^s(s,0) = 0$ if $(a-b) \neq (c-d)$. Therefore

$$f^s_{+-,-+}(s,0) = 0$$

$$f^s_{++,+-}(s,0) = 0 \quad . \tag{7.48}$$

These also follow formally from (7.21) due to the fact that the corresponding $D_{\lambda\mu}(z_s)$ vanish at t = 0 ($z_s = 1$). Using the crossing relation (7.14), we can convert these into linear relations imposed on f_H^t. When this is done in detail, we find that the second requirement of (7.48) is in fact satisfied identically, owing to parity conservation and the conservation of total spin, (the latter being a special feature of nucleon-nucleon scattering.) The first of (7.48) leads to a non-trivial constraint:

$$f^t_{++,++} + f^t_{+-,-+} - f^t_{++,--} - f^t_{+-,+-} = 0, \text{ (at } t = 0). \tag{7.49}$$

Everything we have said so far applies equally well to backward scattering u = 0.

By the analyticity considerations outlined in our earlier discussion, we would of course arrive at the same constraint equation. However, we would also obtain other constraints at pseudothresholds, which cannot be deduced by such a simple physical argument.

If we assume that at high energies ($s \to \infty$) the amplitudes occurring in (7.49) are dominated by t-channel Regge poles, then (7.49) relates the residues of various Regge poles at t = 0. The Regge poles are said to "conspire" if their individual residues do not vanish, and are said to "evade" otherwise.

The case of conspiracy is of special interest when one of the conspirators is the pion Regge pole. Because t = 0 is so close to the physical pion pole $t = 4\mu^2$, a conspiring pion would give rise to an extremely sharp forward peak whose width is of order μ^2. Such sharp peaks have been experimentally observed in forward np charge exchange scattering np → pn, and in charged pion photoproduction $\gamma p \to \pi^+ n$. Although in principle this could be explained by pion conspiracy with another Regge pole (which would correspond to a scalar meson), actual calculations using the known π-N coupling constant $g^2/4\pi = 15$ have failed to reproduce the numerical magnitudes of the forward peaks. It is possible that in these processes Regge cuts are important.* So far, therefore, there is no clear evidence for conspiracy involving Regge poles only.

---

* K. Huang and I.J. Muzinich, Phys. Rev. 164, 1726 (1967);
D. Gordon and J. Froyland, Phys. Rev. 177, 2500 (1969).

F. Reggeization of Helicity Amplitudes

We begin with the t-channel partial-wave expansion

$$f^t_{cd;ab}(s,t) = \sum_{J=\lambda}^{\infty} (2J+1)\, G^J_{cd;ab}(t)\, d^J_{\lambda\mu}(z_t) \tag{7.50}$$

$$\lambda \equiv a-b,\ \mu \equiv c-d,\ \lambda \geq \mu \geq 0,\ J = \text{integer}.$$

The object is to calculate Regge pole contributions to this helicity amplitude. We restrict our discussions to integer J (and not half-integer) and assume that $\lambda \geq \mu \geq 0$. The case of half-integer J requires only trivial modifications and is discussed in GGLMZ. The restriction $\lambda \geq \mu \geq 0$ represents no loss in generality, for all other cases can be reduced to this case by using properties of the rotation coefficients:

$$d^J_{\lambda\mu}(z) = d^J_{-\mu,-\lambda}(z) = (-)^{\lambda-\mu}\, d^J_{\mu\lambda}(z)$$

$$d^J_{\lambda\mu}(z) = (-)^{J+\lambda}\, d_{\lambda,-\mu}(-z) \quad . \tag{7.51}$$

The discussion here follows closely that of GGLMZ, especially the Appendices of that paper. All the special functions used here conform to the notation of GGLMZ, which also contains useful tables for them. We put

$$f^t_{cd;ab}(s,t) = (1+z_t)^{\frac{1}{2}|\lambda+\mu|}(1=z_t)^{\frac{1}{2}|\lambda-\mu|}\bar{f}_{cd;ab}{}^t(s,t) \quad , \tag{7.52}$$

and recall from our earlier discussion that

$$\bar{f}_{cd;ab}{}^t(s,t) = \sum_{J=\lambda_m}^{\infty} (2J+1)\, G^J_{cd;ab}(t)\, e^J_{\lambda\mu}(z_t) \tag{7.53}$$

has no s-kinematic singularities. There are still t-kinematic singularities contained in $G^J_{cd;ab}(t)$. The functions $e^J_{\lambda\mu}$ satisfy the properties (7.51).

In general, $G^J_{cd;ab}(t)$ does not have definite parity, so trajectories of both parities will couple to it. To separate their contributions, we now introduce the parity-conserving helicity amplitudes. Using (7.11), we define helicity states of definite parity by

$$|J;a,b>_{\pm} = \frac{1}{\sqrt{2}}\,\{|J;a,b> \pm \eta_a\eta_b(-1)^{J_a+J_b}|J;-a,-b>\} \tag{7.54}$$

with

$$P|J;a,b>_{\pm} = \pm\,(-1)^J|J;a,b>_{\pm} \quad . \tag{7.55}$$

We define partial-wave amplitudes of definite parity by

$$G^{J,\pm}_{cd,ab}(t) = {}_{\pm}\langle J;c,d|G^J(t)|J;a,b\rangle_{\pm} \quad , \tag{7.56}$$

which couples only to Regge poles of parity $\pm(-)^J$. The original partial-wave amplitudes are then given by

$$G^J_{cd;ab}(t) = \langle J;cd|G(t)|J;ab\rangle = \tfrac{1}{2}[G^{J+}_{cd;ab}(t) + G^{J-}_{cd;ab}(t)] \quad . \tag{7.57}$$

Next we define new linear combinations of the amplitudes $\bar{f}$ that are more convenient for reggeization. The motivation is the following. We note that $G^{J\pm}$ at most changes sign when we reverse the sign of all initial helicities, or all final helicities, or both. The coefficient $e^J_{\lambda\mu}$, however, does not have such a simple behavior (See Eq. (7.51)). Hence it is convenient to define new coefficients with simple behavior under helicity reversal and use them to define new helicity amplitudes. We define

$$e^{J\pm}_{\lambda\mu}(z) = \tfrac{1}{2}[e^J_{\lambda\mu}(z) \pm e^J_{\lambda,-\mu}(z)] \quad , \tag{7.58}$$

which at most changes sign when $\mu \to -\mu$. Then by (7.51)

$$e^{J\pm}_{\lambda\mu}(-z) = \pm\, (-)^{J+\lambda}\, e^{J\pm}_{\lambda\mu}(z) \quad . \tag{7.59}$$

The original coefficients are expressible as

$$e^J_{\lambda\mu}(z) = e^{J+}_{\lambda\mu}(z) + e^{J-}_{\lambda\mu}(z) \quad . \tag{7.60}$$

Now define new helicity amplitudes (the "good" amplitudes)

$$g^{\pm}_{cd;ab}(s,t) = \sum_{J=\lambda}^{\infty} (2J+1)\, [G^{J\pm}_{cd;ab}(t)\, e^{J+}_{\lambda\mu}(z_t) + G^{J\mp}_{cd;ab}(t)\, e^{J-}_{\lambda\mu}(z_t)] \quad . \tag{7.61}$$

Then (leaving helicity indices understood)

$$\bar{f}^t(s,t) = \tfrac{1}{2}[g^+(s,t) + g^-(s,t)] \quad . \tag{7.62}$$

Although $g^{\pm}$ contains contributions from Regge poles of both parities, g^+ is dominated by parity $(-)^J$ and g^- by parity $-(-)^J$ as $z_t \to \infty$. The reason is that e^{J+} dominates over e^{J-} asymptotically.

Before we can do the Watson-Sommerfeld transform on (7.61), we have to discuss how $G^{J\pm}$ can be analytically continued into the J-plane. For this we have to invert (7.61) to obtain the analog of the Froissart-Gribov formula. Recall first the orthonormality property

$$\int_{-1}^{+1} dz \; d^{J}_{\lambda\mu}(z) \; d^{J'}_{\lambda\mu}(z) = \frac{2}{2J+1} \, \delta_{JJ'} \quad . \tag{7.63}$$

Defining a new coefficient

$$c^{J}_{\lambda\mu}(z) = (1+z)^{\frac{1}{2}|\lambda+\mu|} \; (1-z)^{\frac{1}{2}|\lambda-\mu|} \; d^{J}_{\lambda\mu}(z) \quad , \tag{7.64}$$

we rewrite (7.63) in the form

$$\int_{-1}^{+1} dz \; e^{J}_{\lambda\mu}(z) \; c^{J'}_{\lambda\mu}(z) = \frac{2}{2J+1} \, \delta_{JJ'} \quad . \tag{7.65}$$

In analogy with (7.58) define

$$c^{J\pm}_{\lambda\mu}(z) = \frac{1}{2}[c^{J}_{\lambda\mu}(z) \pm c^{J}_{\lambda,-\mu}(z)] \tag{7.66}$$

with the property

$$c^{J\pm}_{\lambda\mu}(-z) = \pm \, (-)^{J+\lambda} \; c^{J\pm}_{\lambda\mu}(z) \quad . \tag{7.67}$$

Then we have the orthonormal relations

$$\begin{aligned} &\int_{-1}^{+1} dz \; [e^{J+}_{\lambda\mu}(z) \; c^{J'+}_{\lambda\mu}(z) + e^{J-}_{\lambda\mu}(z) \; c^{J'-}_{\lambda\mu}(z)] = \frac{2}{2J+1} \, \delta_{JJ'} \quad , \\ &\int_{-1}^{+1} dz \; [e^{J+}_{\lambda\mu}(z) \; c^{J'-}_{\lambda\mu}(z) + e^{J-}_{\lambda\mu}(z) \; c^{J'+}_{\lambda\mu}(z)] = 0 \quad . \end{aligned} \tag{7.68}$$

With the help of this, (7.61) can be inverted:

$$G^{J,\pm}(t) = \frac{1}{2} \int_{-1}^{+1} dz_t [c^{J+}_{\lambda\mu}(z_t) \; g^{\pm}(s,t) + c^{J-}_{\lambda\mu}(z_t) \; g^{\mp}(s,t)] \quad , \tag{7.69}$$

where helicity indices on $G^{J\pm}$ and $g^{\pm}$ are understood. Since $g^{\pm}(s,t)$ has no s-kinematic singularities, it satisfies the dispersion relation

$$g^{\pm}(s,t) = \frac{1}{\pi} \int_{z_0}^{\infty} dz' \; \frac{A^{\pm}(t,z')}{z'-z_t} + \frac{1}{\pi} \int_{z_0}^{\infty} dz' \; \frac{B^{\pm}(t,z')}{z'+z_t} \quad , \tag{7.70}$$

where we have ignored possible subtractions, since they will not contribute to the final result, just as in the spinless case. Substituting (7.70) into (7.69) we obtain

$$G^{J\pm}(t) = \frac{1}{\pi}\int_{z_o}^{\infty} dz'\, A^{\pm}(t,z')\, \frac{1}{2}\int_{-1}^{+1} dz_t\, \frac{c^{J+}_{\lambda\mu}(z_t)}{z'-z_t}$$

$$+ \frac{1}{\pi}\int_{z_o}^{\infty} dz'\, B^{\pm}(t,z')\, \frac{1}{2}\int_{-1}^{+1} dz_t\, \frac{c^{J+}(z_t)}{z'+z_t}$$

$$+ \frac{1}{\pi}\int_{z_o}^{\infty} dz'\, A^{\mp}(t,z')\, \frac{1}{2}\int_{-1}^{+1} dz_t\, \frac{c^{J-}(z_t)}{z'-z_t}$$

$$+ \frac{1}{\pi}\int_{z_o}^{\infty} dz'\, B^{\mp}(t,z')\, \frac{1}{2}\int_{-1}^{+1} dz_t\, \frac{c^{J-}(z_t)}{z'+z_t} \quad . \tag{7.71}$$

Let

$$C^{J\pm}_{\lambda\mu}(z) = \frac{1}{2}\int_{-1}^{+1} dz'\, \frac{c^{J\pm}_{\lambda\mu}(z')}{z-z'} \quad , \tag{7.72}$$

with the reflection property

$$C^{J\pm}_{\lambda\mu}(-z) = \pm\, (-)^{J+\lambda}\, C^{J\pm}_{\lambda\mu}(z) \quad , \tag{7.73}$$

which follows from (7.67). Then

$$G^{J\pm}(t) = \frac{1}{\pi}\int_{z_o}^{\infty} dz'[A^{\pm}(t,z')\, C^{J+}_{\lambda\mu}(z') + A^{\mp}(t,z')\, C^{J-}_{\lambda\mu}(z')]$$

$$+ \frac{1}{\pi}\int_{z_o}^{\infty} dz'[B^{\pm}(t,z')\, C^{J+}_{\lambda\mu}(-z') + B^{\mp}(t,z')\, C^{J-}_{\lambda\mu}(-z')] \quad . \tag{7.74}$$

Using (7.73), we rewrite this as

$$G^{J\pm}(t) = \frac{1}{\pi}\int_{z_o}^{\infty} dz'[A^{\pm}(t,z')\, C^{J+}_{\lambda\mu}(z') + A^{\mp}(t,z')\, C^{J-}_{\lambda\mu}(z')]$$

$$+ (-1)^{J+\lambda}\, \frac{1}{\pi}\int_{z_o}^{\infty} dz'[B^{\pm}(t,z')\, C^{J+}_{\lambda\mu}(z') - B^{\mp}(t,z')\, C^{J-}_{\lambda\mu}(z')] \quad . \tag{7.75}$$

To continue this to complex J, we need to know some properties of $C^{J\pm}_{\lambda\mu}$.

The functions $C^{J\pm}_{\lambda\mu}(z)$ are studied in GGLMZ and in greater detail in Andrews and Gunson.* We need to know the following properties:

(i) $C^{J\pm}_{\lambda\mu}(z)$ is a linear combination of Legendre functions of the second kind, $Q_\ell(z)$, with $J-\lambda \le \ell \le J+\lambda$.

---

* Andrews and Gunson, J. Math. Phys. 5, 1391 (1964). They study a function $e^{\lambda\mu}_J(z)$, which is related to ours by

$$C^J_{\lambda\mu}(z) = (-)^{\lambda-\mu}\, (1+z)^{\frac{1}{2}(\lambda+\mu)}\, (1-z)^{\frac{1}{2}(\lambda-\mu)}\, e^{\lambda\mu}_J(z) \quad .$$

(ii) $C^{J\pm}_{\lambda\mu}(z)$ has square root branch points in J for integer values of J satisfying $-\lambda \le J \le -\mu-1$ or $\mu \le J \le \lambda-1$.

(iii) $C^{J\pm}_{\lambda\mu}(z)$ has no other singularities in J except those coming from the Legendre functions $Q_\ell(z)$, $J-\lambda \le \ell \le J+\lambda$. In particular the apparent poles at half-integer values of J in the explicit forms tabulated by GGLMZ are in fact absent: They cancel by virtue of the symmetry property $Q_\ell(z) = Q_{-\ell-1}(z)$ at ℓ = half-integer. The fixed J-poles coming from those of Q_ℓ at $\ell = -1,-2,\ldots$ remain. They occur at $J = \lambda-1, \lambda-2, \ldots$.

The analytic continuation of (7.75) to complex J proceeds in the same manner as the continuation in the spinless case. If the functions $A^\pm$, $B^\pm$ in (7.75) are polynomial bounded, then each integral defines a unique continuation in J which is analytic for sufficiently large Re J. Since $(-1)^{J+\lambda}$ does not have a unique analytic continuation, we introduce the signatured amplitudes

$$\begin{aligned}{}_\eta G^\pm(J,t) = \frac{1}{\pi}\int_{z_o}^\infty dz'[A^\pm(t,z')\, C^{J+}_{\lambda\mu}(z') + A^\mp(t,z')\, C^{J-}_{\lambda\mu}(z')] \\ + \frac{\eta}{\pi}\int_{z_o}^\infty dz'[B^\pm(t,z')\, C^{J+}_{\lambda\mu}(z') - B^\mp(t,z')\, C^{J-}_{\lambda\mu}(z')] \quad , \qquad (7.76)\end{aligned}$$

where $\eta = \pm 1$. This can now be continued to complex J and is the generalization of the Froissart-Gribov formula. It is related to $G^{J\pm}$ for integer J by

$$G^{J\pm}(t) = \begin{cases} {}_+G^\pm(J,t), & \text{for } J+\lambda \text{ even} \\ {}_-G^\pm(J,t), & \text{for } J+\lambda \text{ odd} \end{cases} \quad . \qquad (7.77)$$

We may call η the "apparent signature." It is the same as the signature if λ = even integer and is opposite of the signature if λ = odd integer. The functions $C^{J\pm}_{\lambda\mu}$ have the property

$$C^{J\pm}_{\lambda\mu}(z) = C^{(-J-1)\pm}_{\lambda\mu}(z) \qquad (J = \text{half-integer}) \quad . \qquad (7.78)$$

Hence formally

$${}_\eta G^\pm(J,t) = {}_\eta G^\pm(-J-1,t) \qquad (J = \text{half-integer}) \quad , \qquad (7.79)$$

which is the Mandelstam symmetry. To get rid of fixed J-poles coming from those in $C^{J\pm}_{\lambda\mu}$, we assume

$$\int_{-1}^{+1} dz\, P_J(z) \begin{Bmatrix} A^\pm(t,z) \\ B^\pm(t,z) \end{Bmatrix} = 0. \quad (J = \lambda-1, \lambda-2, \ldots) \qquad (7.80)$$

To carry out the Watson-Sommerfeld transform on (7.61), we first rewrite it as

$$g^{\pm}(s,t) = \frac{1}{2}\sum_{J=\lambda}^{\infty}(2J+1)\ \{[{}_{+}G^{\pm}(J,t)]\ [e^{J+}_{\lambda\mu}(z_t) + e^{J+}_{\lambda\mu}(-z_t)]$$
$$+ [{}_{-}G^{\pm}(J,t)]\ [e^{J+}_{\lambda\mu}(z_t) - e^{J+}_{\lambda\mu}(-z_t)]$$
$$+ [{}_{+}G^{\mp}(J,t)]\ [e^{J-}_{\lambda\mu}(z_t) - e^{J-}_{\lambda\mu}(-z_t)]$$
$$+ [{}_{-}G^{\mp}(J,t)]\ [e^{J-}_{\lambda\mu}(z_t) + e^{J-}_{\lambda\mu}(-z_t)]\} \quad . \qquad (7.81)$$

To take advantage of the Mandelstam symmetry, we proceed as in the spinless case to replace $e^{J\pm}_{\lambda\mu}(z)$ by a special continuation in J. The function $e^{J\pm}_{\lambda\mu}(z)$ is a linear combination of Legendre polynomials and their derivatives. We define $E^{J\pm}_{\lambda\mu}(z)$ as the function obtained from $e^{J\pm}_{\lambda\mu}(z)$ by replacing all $P_\ell(z)$ by $\mathcal{P}_\ell(z)$. This function is discussed in more detail in GGLMZ. It has the following properties. For integer values of J, and $\lambda \geq \mu \geq 0$:

$$E^{(J+x)\pm}_{\lambda\mu}(z) \xrightarrow[x\to 0]{} \begin{cases} e^{J\pm}_{\lambda\mu}(z) & (J \geq \lambda) \qquad (7.82a) \\ \sim 0(x^{\frac{1}{2}}) & (\mu \leq J \leq \lambda-1) \qquad (7.82b) \\ \text{Finite number} & (-\mu \leq J \leq \mu-1) \qquad (7.82c) \\ \sim 0(x^{\frac{1}{2}}) & (-\lambda \leq J \leq -\mu-1) \qquad (7.82d) \\ \sim 0(x) & (J \leq -\lambda-1) \qquad (7.82e) \end{cases}$$

At J = half-integer, it has J-poles with residues satisfying

$$\text{Res}\ E^{J\pm}_{\lambda\mu} = -\ \text{Res}\ E^{(-J-1)\pm}_{\lambda\mu} \quad , \quad (J = \text{half-integer}) \quad . \qquad (7.83)$$

It also has square root branch points in J. In the partial-wave expansion these will always be cancelled by corresponding ones in ${}_{\eta}G^{\pm}(J,t)$ arising from those of $C^{J\pm}_{\lambda\mu}$.

To simplify our discussion, we pretend for the moment that the range (7.82c) does not exist. This will be discussed separately in the next section on the problem of sense and nonsense.

If we ignore the range (7.82c), the discussion proceeds in parallel with that of the spinless case. We replace $e^{J\pm}_{\lambda\mu}$ by $E^{J\pm}_{\lambda\mu}$ in (7.81), extend the J-sum from $-\infty$ to ∞, and replace it by a contour integral:

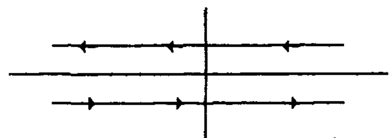
$$g^{\pm}(s,t) = \frac{1}{2}\,\frac{1}{2\pi i}\int_C dJ\,\frac{\pi(2J+1)}{\sin\pi(J+\lambda)}\,\{[{}_{+}G^{\pm}(J,t)]\,[E^{J+}_{\lambda\mu}(z_t) + E^{J+}_{\lambda\mu}(-z_t)]$$

$$- [{}_{-}G^{\pm}(J,t)]\,[E^{J+}_{\lambda\mu}(z_t) - E^{J+}_{\lambda\mu}(-z_t)]$$

$$+ [{}_{+}G^{\mp}(J,t)]\,[E^{J-}_{\lambda\mu}(z_t) - E^{J-}_{\lambda\mu}(-z_t)]$$

$$- [{}_{-}G^{\mp}(J,t)]\,[E^{J-}_{\lambda\mu}(z_t) + E^{J-}_{\lambda\mu}(-z_t)]\} \quad , \qquad (7.84)$$

where C is the contour shown below.



The factor $(-1)^{J+\lambda}$ from the residue of $[\sin\pi(J+\lambda)]^{-1}$ has been absorbed into the $E^{J\pm}_{\lambda\mu}(z)$ functions by using (7.59). In addition to the poles of $[\sin\pi(J+\lambda)]^{-1}$, which reproduce the original sum, the integral also picks up the poles of $E^{J\pm}_{\lambda\mu}(z)$ at the half-integers. The one at $J = -\frac{1}{2}$ is cancelled by $(2J+1)$. By virtue of the Mandelstam symmetry (7.79) and the property (7.83), the rest cancel in pairs as in the spinless case.

A Regge pole of parity $\pm(-)^J$, apparent signature η [signature $= \eta(-)^{\lambda}$] occurs in the form

$$_{\eta}G^{\pm}_{cd;ab} \sim \frac{\beta_{cd;ab}(t)}{J-\alpha(t)} \quad . \qquad (7.85)$$

Its contribution to $g^{\pm}(s,t)$ is obtained by unfolding the contour in (7.84) in the same manner as in the spinless case. This is trivial to do for any particular Regge pole. It seems pointless to give a general formula, for we would merely drown in a sea of superscripts and subscripts.

The asymptotic behavior of $g^{\pm}(s,t)$ for large z_t can be worked out in particular cases from the explicit formulas for $E^{J\pm}_{\lambda\mu}$ tabulated in GGLMZ. In the asymptotic formulas, the true signature (instead of the apparent signature) always appears in the usual factor $(e^{-i\pi\alpha} \pm 1)$.

We give a list of factors that $\beta_{cd;ab}(t)$ should contain:

(1) Threshold factor $[2p_{ab}p_{cd}/s_o]^{\alpha(t)}$, where s_o is an arbitrary scale.

(2) A factor $[\Gamma(\alpha(t) - 3/2)]^{-1}$, for the same reason as in the spinless case.

(3) A factor coming from the J-branch points of $C^{J\pm}_{\lambda\mu}$,

$$\prod_{n=\mu}^{\lambda-1}[\alpha(t)-n]^{\frac{1}{2}} \prod_{n=-\lambda}^{-\mu-1}[\alpha(t)-n]^{\frac{1}{2}} \quad ,$$

which will exactly cancel a corresponding factor in $E_{\lambda\mu}^{\alpha\pm}$.

(4) A factor $K_{cd;ab}(t)$ containing all the t-kinematic singularities of $\overline{f}_{cd;ab}(t)$.

(5) A factor $S(\alpha)$, explained in the next section, having to do with "choosing sense."

In addition, at certain values of t, the residues of various Regge poles may satisfy kinematic constraints. Factorizability requires that $\beta_{cd;ab}(t)$ have the form

$$\beta_{cd;ab}(t) = g_{cd}(t)\; g_{ab}(t) \quad . \tag{7.86}$$

G. Sense and Nonsense

The discussion of the Sommerfeld-Watson transform in the last section is incomplete, because we ignored the fact that $E_{\lambda\mu}^{J\pm} \neq 0$ for integer J-values in the range $-\mu \le J \le \mu-1$. These terms are included in the representation (7.84), although they were not in the original partial-wave expansion (7.81) and should not be included. Actually there is a cancellation among these terms, and (7.84) is still correct; but this cancellation implies constraints on Regge poles that we have to take into account.

The cancellation occurs between the various terms in (7.84), made possible by certain symmetry properties of $E_{\lambda\mu}^{J\pm}$ and $C_{\lambda\mu}^{J\pm}$, namely, for integer J in the range $-\mu \le J \le \mu-1$,

$$E_{\lambda\mu}^{J\pm}(z) = E_{\lambda\mu}^{(-J-1)\mp}(z) \quad , \tag{7.87}$$

$$C_{\lambda\mu}^{J\pm}(z) = C_{\lambda\mu}^{(-J-1)\mp}(z) \quad . \tag{7.88}$$

The first can be proved by using the explicit formula Eq. (A9) of GGLMZ, and the second can be proved by induction by using the recursion formula, Eqs. (A13), (A14) of GGLMZ. The second relation leads via (7.76) and (7.80) to

$$_{\eta}G^{\pm}(J,t) = {}_{-\eta}G^{\mp}(-J-1,t) \tag{7.89}$$

for the same range of J values. Referring to (7.84) we note that the residues of $(2J+1)/\sin\pi(J+\lambda)$ at J_o and $-J_o-1$ are equal to each other. Hence the contributions from $J = J_o$ and $J = -J_o-1$ cancel in pairs: the first term in the curly bracket cancels the fourth, the second against the third. Therefore (7.84) is correct.

The equality (7.89) implies that if $_{\eta}G^{\pm}(J,t)$ has a pole at $J = \alpha(t)$, such that $\alpha(t)$ is an integer with $-\mu-1 < \alpha(t) < \mu$, then $_{-\eta}G^{\mp}(-J-1,t)$ must have a pole at $J = -\alpha(t)-1$. That is, whenever t is such that a Regge trajectory $\alpha(t)$ passes through integer value between $-\mu-1$ and μ, there must be another trajectory passing through $-\alpha(t)-1$, of the same residue but opposite parity and signature, unless the residue of $\alpha(t)$ vanishes. Here, as in the discussion of the phenomenon associated with $\alpha(t)$ passing through half-integer values, we have the alternatives of compensating trajectories vs. vanishing residues. In this case, however, the compensating trajectory has opposite parity and signature.

The integer J-values for $J < \lambda$ are called nonsense values, a definition we have already introduced in the spinless case, where $\lambda = 0$. The range $-\mu-1 \leq J \leq \mu$ therefore contains nonsense values of J, since in our convention $\lambda \geq \mu \geq 0$. When a trajectory passes through these values, it is said to "choose sense" if its residue vanishes, and to "choose nonsense" otherwise. These represent different dynamical possibilities and one cannot decide in favor of either without a theory. The simpler of the two seems to be to choose sense, for that avoids introducing a compensating trajectory.

The factor $S(\alpha)$ listed at the end of the last section is designed to make the residue vanish at the appropriate nonsense values of α, if the trajectory chooses sense. If the trajectory chooses nonsense, then $S(\alpha) = 1$, and we must specifically include compensating trajectories in the analysis. Since $S(\alpha)$ must not introduce singularities in α, it is an entire function of α, usually taken to be a polynomial.

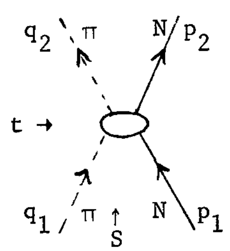
Nonsense values of α also occur in the spinless case, of course. But there we were not faced with choosing sense or nonsense because the function $\mathcal{P}_\alpha$ does not have the peculiarity (7.82c), and consequently nonsense values of α never give rise to a pole contribution to the Watson-Sommerfeld transform. An explicit example of sense and nonsense is given in GGLMZ, Appendix B.

VIII. PION-NUCLEON SCATTERING

As a non-trivial example of reggeization with spin we shall consider pion-nucleon scattering in some detail. This example also gives us a chance to see the detailed relation between invariant and helicity amplitudes and the origin of the kinematic singularities and constraints.

A. Invariant Amplitudes

Let us consider π-N scattering in which the individual particles are in definite charge states. Analysis in terms of total I-spin states may be easily obtained from what we do here and will not be discussed. Suppose we calculate the scattering amplitude by summing all Feynman graphs, then we would obtain a Feynman amplitude of the form



$$f = \bar{u}(p_2,s_2)\ T(p_2,q_2;\ p_1,q_1)\ u(p_1,s_1) \quad , \tag{8.1}$$

where u(p,s) is a Dirac spinor of momentum $\vec{p}$ and z-component of spin s, and T is a 4x4 matrix. We can write T as a linear combination of the 16 Dirac matrices*

$$1,\ \gamma^\mu,\ \gamma^\mu\gamma^\nu,\ \gamma_5\gamma^\mu,\ \gamma_5 \quad , \tag{8.2}$$

with coefficients constructed from the 3 available independent momenta

$$(p_1+p_2)_\mu,\ (p_1-p_2)_\mu,\ (q_1+q_2)_\mu \quad , \tag{8.3}$$

in such a manner to insure that $\bar{u}Tu$ is a Lorentz scalar. Thus terms proportional to $\gamma_5\gamma_\mu$ and γ_5 are immediately ruled out, for they would require pseudovector and pseudoscalar coefficients, and none can be constructed from (8.3). Any invariant constructed from $\gamma^\mu\gamma^\nu$ and (8.3) reduces to one constructed from 1 or γ^μ, when the nucleons are on the mass shell. Under the same condition, the only independent invariant constructed from γ^μ is $\gamma\cdot(q_1+q_2)$. Thus the most general form is

$$T(p_2,q_2;p_1,q_1) = -A(s,t) + \frac{i}{2}\,\gamma\cdot(q_1+q_2)\ B(s,t) \quad , \tag{8.4}$$

---

* We use the convention in S. Gasiorowicz, Elementary Particle Physics, (John Wiley & Sons, New York, 1967), Chap. 2.

where A, B are functions having no singularities except the unitarity cuts and particle poles. By definition they have no kinematic singularities and no kinematic zeros and are called invariant amplitudes.

From the crossing property of Feynman graphs, the amplitude (8.1) also describes $\pi\pi \to N\bar{N}$ when we continue the q_2 and p_2 to the region where their components change sign. For the invariant amplitudes, this simply means that we continue the values of s,t from the s-channel physical region to the t-channel physical region.

B. Helicity Amplitudes

The s-channel helicity amplitudes are

$$f_{\lambda_2,0;\lambda_1,0}{}^{s}(s,t) = \sum_{J=\frac{1}{2}}^{\infty} (2J+1)\ F^J_{\lambda_2,0;\lambda_1,0}(s)\ d^J_{\lambda_1\lambda_2}(z_s) \tag{8.5}$$

where λ_1 and λ_2 assume the values $\pm\frac{1}{2}$. We introduce a shorthand notation in which the amplitudes are labeled only by the signs of λ_2 and λ_1; for example,

$$f_{++}{}^{s}(s,t) = f_{+\frac{1}{2},0;+\frac{1}{2},0}{}^{s}(s,t) \quad . \tag{8.6}$$

Then by (7.11) parity conservation implies

$$F^J_{++}(s) = F^J_{--}(s)$$

$$F^J_{+-}(s) = F^J_{-+}(s) \quad . \tag{8.7}$$

Using this and the properties (7.51) of the rotation coefficients, we find that there are only two independent helicity amplitudes, which we choose to be

$$f_1{}^{s}(s,t) = f_{++}{}^{s}(s,t) = f_{--}{}^{s}(s,t)$$

$$f_2{}^{s}(s,t) = f_{+-}{}^{s}(s,t) = -f_{-+}{}^{s}(s,t) \quad . \tag{8.8}$$

Identical formulas hold for the t-channel amplitudes, if we change the superscripts from s to t.

The helicity amplitudes are in fact the amplitudes (8.1) with specific choices of the Dirac spinors:

$$f_{\lambda_2\lambda_1}{}^{s}(s,t) = \bar{u}(p_2,\lambda_2)\ T(p_2,q_2;p_1,q_1)\ u(p_1,\lambda_1) \quad ,$$

$$f_{\lambda_2\lambda_1}{}^{t}(s,t) = \bar{u}(p_2,\lambda_2)\ T(p_2,-p_1;-q_2,q_1)\ v(-p_1,-\lambda_1) \quad , \tag{8.9}$$

where $u(p,\lambda)$ and $v(p,\lambda)$ are respectively positive and negative energy spinors satisfying

$$\bar{u}(p,\lambda)\ u(p,\lambda) = -\bar{v}(p,\lambda)\ v(p,\lambda) = +1 \tag{8.10}$$

and

$$(\hat{p}\cdot\vec{\sigma})\ u(p,\lambda) = \lambda u(p,\lambda)$$

$$(\hat{p}\cdot\vec{\sigma})\ v(p,\lambda) = \lambda v(p,\lambda) \quad . \tag{8.11}$$

To find the relation between helicity and invariant amplitudes we use (8.4) and find after a lengthy but straightforward calculation

$$f_1{}^s(s,t) = -\sqrt{\frac{1+z_s}{2}}\left[A(s,t) + \frac{s-m^2-\mu^2}{2m}B(s,t)\right] \quad ,$$

$$f_2{}^s(s,t) = -\sqrt{\frac{1-z_s}{2}}\sqrt{\frac{m^2}{4s}}\left[\frac{s+m^2-\mu^2}{m^2}A(s,t) + \frac{s-m^2+\mu^2}{m}B(s,t)\right] \quad ; \tag{8.12}$$

and

$$f_1{}^t(s,t) = -\frac{1}{\sqrt{t-4m^2}}\left[-\frac{(t-4m^2)}{2m}A(s,t) + \frac{s-u}{2}B(s,t)\right] \quad ,$$

$$f_2{}^t(s,t) = -\frac{\sqrt{t}\sqrt{su-(m^2-\mu^2)^2}}{2m\sqrt{t-4m^2}}\left[B(s,t)\right] \quad . \tag{8.13}$$

Since A and B are by definition free of kinematic singularities, the kinematic singularities of the helicity amplitudes are hereby explicitly displayed. The following amplitudes are therefore free of all kinematic singularities:

$$\hat{f}_1{}^s = [\tfrac{1}{2}(1+z_s)]^{-\frac{1}{2}}\ f_1{}^s \quad , \quad \hat{f}_2{}^s = (\frac{s}{m^2})^{\frac{1}{2}}\ [\tfrac{1}{2}(1-z_s)]^{-\frac{1}{2}}\ f_2{}^s \quad ; \tag{8.14}$$

$$\hat{f}_1{}^t = (t-4m^2)^{\frac{1}{2}}\ f_1{}^t \quad , \quad \hat{f}_2{}^t = \frac{2m^2}{t^{\frac{1}{2}}}\left[\frac{t-4m^2}{su-(m^2-\mu^2)^2}\right]^{\frac{1}{2}} f_2{}^t \quad . \tag{8.15}$$

However, they are not completely independent. To see this we solve for A and B in terms of the set $\hat{f}_1{}^s$, $\hat{f}_2{}^s$, and alternatively the set $\hat{f}_1{}^t$, $\hat{f}_2{}^t$:

$$A(s,t) = \frac{2m^2}{\mathcal{S}^2}\left[-(s-m^2+\mu^2)\ \hat{f}_1{}^s(s,t) + (s+m^2-\mu^2)\ \hat{f}_2{}^s(s,t)\right] \quad ,$$

$$B(s,t) = \frac{4m^3}{\mathcal{S}^2}\left[\frac{s+m^2-\mu^2}{2m^2}\ \hat{f}_1{}^s(s,t) - \hat{f}_2{}^s(s,t)\right] \quad , \tag{8.16}$$

with

$$\mathcal{S}^2 = [s-(m+\mu)^2]\ [s-(m-\mu)^2] \quad . \tag{8.17}$$

Since A(s,t) and B(s,t) have no kinematic singularities, the square brackets must vanish at $s = (m \pm \mu)^2$. Similarly, in terms of the t-channel amplitudes

$$A(s,t) = -\frac{2m^2}{t-4m^2}\left[\hat{f}_1{}^t(s,t) + \frac{s-u}{2m^2}\hat{f}_2{}^t(s,t)\right] \quad ,$$

$$B(s,t) = -\frac{1}{m}\hat{f}_2{}^t(s,t) \quad , \tag{8.18}$$

so the square bracket must vanish at $t = 4m^2$. Note that no constraint is needed at $t = 0$ and hence there is no conspiracy condition. What happens is that parity conservation in the t-channel automatically implies conservation of angular momentum in the forward direction in the s-channel.

The crossing relation for the helicity amplitudes is obtained by eliminating A(s,t) and B(s,t) between (8.16) and (8.18). Since the arguments of the square roots change signs during the continuation from the s-channel to the t-channel physical region, the calculation requires a careful consideration of phases. This is the whole point of the paper of Trueman and Wick. We take their result from (7.14):

$$f_{ab}{}^s(s,t) = \sum_{c=-\frac{1}{2}}^{+\frac{1}{2}} \sum_{d=-\frac{1}{2}}^{+\frac{1}{2}} d^{\frac{1}{2}}_{db}(\chi)\, d^{\frac{1}{2}}_{ca}(\pi-\chi)\, f_{cd}{}^t(s,t) \quad , \tag{8.19}$$

where

$$\cos\chi = -\frac{(s+m^2-\mu^2)t}{\mathcal{S}\mathcal{T}}$$

$$\sin\chi = \frac{2m\sqrt{\varphi(s,t)}}{\mathcal{S}\mathcal{T}}$$

$$\mathcal{S} = [s-(m+\mu)^2]\,[s-(m-\mu)^2]$$

$$\mathcal{T} = t(t-4m^2)$$

$$\varphi(s,t) = stu - t(m^2-\mu^2)^2 \quad . \tag{8.20}$$

The rotation coefficients are given in the following matrix

$$d^{\frac{1}{2}}(\chi) = \begin{bmatrix} \cos\frac{\chi}{2} & -\sin\frac{\chi}{2} \\ \sin\frac{\chi}{2} & \cos\frac{\chi}{2} \end{bmatrix} \quad . \tag{8.21}$$

Using this and (8.8), we obtain

$$\begin{bmatrix} f_1^{\,s}(s,t) \\ f_2^{\,s}(s,t) \end{bmatrix} = \begin{bmatrix} \sin\chi & -\cos\chi \\ \cos\chi & \sin\chi \end{bmatrix} \begin{bmatrix} f_1^{\,t}(s,t) \\ f_2^{\,t}(s,t) \end{bmatrix} \quad . \tag{8.22}$$

C. Reggeization of Helicity Amplitudes

Let us now illustrate the procedure discussed in Section VII D by following it step by step for the present case. First we define

$$\bar{f}_1^{\,t}(s,t) = \frac{1}{\sqrt{1+z_t}}\, f_1^{\,t}(s,t)$$

$$\bar{f}_2^{\,t}(s,t) = \frac{1}{\sqrt{1-z_t}}\, f_2^{\,t}(s,t) \quad . \tag{8.23}$$

These amplitudes have no s-kinematic singularities and have the partial wave expansions

$$\bar{f}_2^{\,t}(s,t) = \sum_{J=0}^{\infty} (2J+1)\; G^J_{++}(t)\; e^J_{00}(z_t)$$

$$\bar{f}_2^{\,t}(s,t) = \sum_{J=1}^{\infty} (2J+1)\; G^J_{+-}(t)\; e^J_{01}(z_t) \quad . \tag{8.24}$$

Let $|J,\lambda_1\lambda_2\rangle$ be the $N\bar{N}$ state with angular momentum J and helicities λ_2,λ_1. Then

$$P|J;a,b\rangle = +\,(-1)^J\,|J;-a,-b\rangle \quad . \tag{8.25}$$

The parity eigenstates are therefore

$$|J;a,b\rangle_{\pm} = \frac{1}{\sqrt{2}}\,\left[\,|J;a,b\rangle \pm |J;-a,-b\rangle\right] \tag{8.26}$$

with

$$P|J;a,b\rangle_{\pm} = \pm\,(-1)^J\,|J;a,b\rangle_{\pm} \quad . \tag{8.27}$$

The parity-conserving partial wave amplitudes are

$$G^{J\pm}_{ab}(t) = {}_{\pm}\langle J;a,b|G(t)|J;0,0\rangle \quad , \tag{8.28}$$

$|J,00\rangle$ being the pion state. In terms of these we have

$$G^J_{ab}(t) = \tfrac{1}{2}[G^{J+}_{ab}(t) + G^{J-}_{ab}(t)] \quad . \tag{8.29}$$

The states of $N\bar{N}$ and the trajectories coupled to them are given below.

Parity eigenstate	Abbreviation	Parity	G-parity	Trajectories
$\frac{1}{\sqrt{2}}[\,\|J,++\rangle + \|J,--\rangle]$	$\|J,0+\rangle$	$+(-1)^J$	+1	P, ρ, f^o
			-1	ω, A_2
$\frac{1}{\sqrt{2}}[\,\|J,++\rangle - \|J,--\rangle]$	$\|J,0-\rangle$	$-(-1)^J$	+1	B
			-1	π
$\frac{1}{\sqrt{2}}[\,\|J,+-\rangle + \|J,-+\rangle]$	$\|J,1+\rangle$	$+(-1)^J$	+1	P, ρ, f^o
			-1	ω, A_2
$\frac{1}{\sqrt{2}}[\,\|J,+-\rangle - \|J,-+\rangle]$	$\|J,1-\rangle$	$-(-1)^J$	+1	——
			-1	A_1

Since the $\pi\pi$ states all have $P = +(-1)^J$ and $G = +1$, the only non-vanishing partial wave amplitudes are

$$G_{00}^{J+} \equiv \langle J,0+|G(t)|J,\pi\pi\rangle$$

$$G_{01}^{J+} \equiv \langle J,1+|G(t)|J,\pi\pi\rangle \quad . \tag{8.30}$$

Substituting this, via (8.29), into (8.24), we have the partial wave expansions

$$\bar{f}_1^{\,t}(s,t) = \frac{1}{2}\sum_{J=0}^{\infty} (2J+1)\; G_{00}^{J+}(t)\; e_{00}^{J}(z_t)$$

$$\bar{f}_2^{\,t}(s,t) = \frac{1}{2}\sum_{J=1}^{\infty} (2J+1)\; G_{01}^{J+}(t)\; e_{01}^{J}(z_t) \quad . \tag{8.31}$$

In the general discussion, we had further decomposed the above into the amplitudes $g^{\pm}(s,t)$. But for pion-nucleon scattering, only states with parity $+(-1)^J$ couple and this is unnecessary. If we do it anyway, we find that

$$e_{00}^{J-}(z_t) = e_{01}^{J-}(z_t) = 0$$

$$e_{00}^{J+}(z_t) = e_{00}^{J}(z_t),\; e_{01}^{J+}(z_t) = e_{01}^{J}(z_t) \quad , \tag{8.32}$$

and we are back to (8.31).

From the table in GGLMZ, we find

$$e^J_{00}(z_t) = P_J(z_t) \qquad c^J_{00}(z_t) = P_J(z_t)$$
$$e^J_{01}(z_t) = \frac{P'_J(z_t)}{\sqrt{J(J+1)}} \qquad c^J_{01}(z_t) = \frac{\sqrt{J(J+1)}}{2J+1}[P_{J-1}(z_t) - P_{J+1}(z_t)] \quad . \tag{8.33}$$

Hence

$$E^J_{00}(z_t) = \mathcal{P}_J(z_t) \qquad C^J_{00}(z_t) = Q_J(z_t)$$
$$E^J_{01}(z_t) = \frac{\mathcal{P}'_J(z_t)}{\sqrt{J(J+1)}} \qquad C^J_{01}(z_t) = \frac{\sqrt{J(J+1)}}{2J+1}[Q_{J-1}(z_t) - Q_{J+1}(z_t)] \quad . \tag{8.34}$$

Then from (7.79) we obtain

$$\bar{f}_1{}^t(s,t) = \sum_{i=P,\rho,f^o} - \frac{\pi\, \beta^i_{00}(t)(\alpha_i(t)+\frac{1}{2})}{\sin\pi\, \alpha_i(t)}[E_{00}^{\alpha_i(t)}(-z_t) + \eta_i E_{00}^{\alpha_i(t)}(z_t)]$$
$$\bar{f}_2{}^t(s,t) = \sum_{i=P,\rho,f^o} - \frac{\pi\, \beta^i_{01}(t)(\alpha_i(t)+\frac{1}{2})}{\sin\pi(\alpha_i(t)+1)}[E_{00}^{\alpha_i(t)}(-z_t) - \eta_i E_{00}^{\alpha_i(t)}(z_t)] \quad . \tag{8.35}$$

The signature η_i which appears here is the true signature.

We take the residue functions to be

$$\beta^i_{0\lambda}(t) = \frac{1}{\Gamma(\alpha_i(t)+\frac{3}{2})}\left(\frac{2p_{\pi\pi}p_{N\bar{N}}}{s_i}\right)^{\alpha_i(t)} \gamma^i_{0\lambda}(t) \tag{8.36}$$

where s_i are arbitrary scales, the first factor provides the compensation required by the Mandelstam symmetry, and the second is the threshold factor. In $\gamma^i_{00}(t)$, we factor out the kinematic singularities of $\bar{f}_1{}^t(s,t)$:

$$\gamma^i_{00}(t) = \frac{1}{\sqrt{4m^2-t}}\,\bar{\gamma}^i_{00}(t) \quad . \tag{8.37}$$

Factorization requires

$$\bar{\gamma}^i_{00}(t) \geq 0 \text{ for } t < 4\mu^2 \quad . \tag{8.38}$$

In $\gamma^i_{01}(t)$, we must factor out both the kinematic singularities of $\bar{f}_1{}^t(s,t)$ and the branch points coming from $C^J_{01}(z_t)$:

$$\gamma^i_{01}(t) = \sqrt{t(4\mu^2-t)}\,\sqrt{\alpha_i(t)(\alpha_i(t)+1)}\,\bar{\gamma}_{01}(t) \quad . \tag{8.39}$$

Finally we obtain the s-channel amplitudes by using (8.35) and (8.22).