Local Extensions of Completely Rational Conformal Quantum Field Theories

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1 Introduction

To understand the interplay of quantum theory and special relativity turned out to be a much harder task than expected, which has remained unsolved for already three quarters of a century. At the same time it is of great physical importance, for example because essentially all the crucial information about the micro–structure comes from experiments involving particles at very high energies, i.e. micro–objects with relativistic speeds.

The problem to establish a relativistic quantum theory, more often called a quantum field theory because a prominent role in it is played by quantum fields, was attacked in various constructive ways and many of them had partial success, but none reached the desired goal of a complete and consistent mathematical description. Then it was realized that starting from the first physical principles which must underlie every quantum field theory one already may determine to a great extent its intrinsic characteristics, supplying the constructors with guiding lights. This was a reason a serious deal of the scientific efforts in quantum field theory to be redirected to its axiomatic treatment.

The first system of axioms used in quantum field theory was invented by Wightman and sets up as main objects the fields, which are operator–valued distributions defined on a common dense domain within a Hilbert space [Streater & Wightman, 1964; Jost, 1965]. It is natural to formulate the general dynamical principles in terms of fields, but working with these objects led to considerable technical difficulties and this resulted into introducing a second axiomatic system [Haag, 1996; Araki, 1999]. The local quantum physics framework, sometimes referred to as algebraic quantum field theory, has as primary objects nets of algebras of local observables, relying on the fact that the relevant physical information in a certain theory is carried by its observable content. Although exhibiting a very beautiful structure and providing us with a rich new insight, within this second framework the explicit computation of some physically interesting quantities is sometimes too complicated and at the same time easier in the field–theoretical approach. Also, discussion of concrete models is mostly done in terms of pointlike localized fields. The conclusion is that we must view the two axiomatic approaches as complementary rather than as rivaling and that their joint exploitation may provide us with a broader view in our research.

Obviously then, theoretically interesting is the question how to establish a correspondence between the two axiomatic descriptions, namely to understand how to assign to a Wightman field theory a net of algebras, how to reconstruct the fields from the net of algebras and under which conditions this is possible. In [Fredenhagen & Hertel, 1981; Driessler *et al.*, 1986] receipts for these are given, however they are applicable only to cases in which strong regularity requirements are obeyed and they are in general very difficult to be verified.

In this Ph.D project we make use of the advantages of both approaches to study different features of chiral conformal field theories. A convenient sample of such models consists of those ones with $c < 1$ — on one hand for them exists a complete classification, achieved in the algebraic approach [Kawahigashi & Longo, 2004], on the other hand this sample provides some of the first examples of exactly solvable non-trivial relativistic quantum theories, in the sense that all correlation functions of fields can be computed [Belavin *et al.*, 1984*a*]. Actually, for these models the passage between the two axiomatic approaches is more easy to be understood than in the general case — the switching from the algebraic to the field–theoretical framework is easily accomplishable using the vacuum character for the net which indicates the field content and one obtains it with algebraic methods; the inverse passage is possible using the stress–energy tensor, an important ingredient of each conformal field theory, to define a net of local algebras.

A tensor product of two chiral conformal theories lies in the core of every two dimensional conformal field theory, thus to understand the one dimensional chiral theory is the first step towards understanding of the two dimensional theory.

From the point of view of the "rigorous research", conformal field theories are valuable mainly because of the perspective to exploit them as "toy models" in the quest of constructing a mathematically consistent theory describing relativistic quantum phenomena. While the goal to construct in the axiomatic approaches a "realistic" quantum field theory apart from the free fields ones is unreachable in the present moment, for "easier" models with the simplifying assumption of low space–time dimensions (one or two) and higher symmetry (conformal) a huge sample of exactly solvable models is available. There is a hope that using structural insight from the conformal models, which are better understood, one can understand more deeply the features of the "proper" theories.

Nevertheless, the role of conformal field theories does not limit to a "virtual assisting agent" and one can also describe real physics with them, even though not relativistic quantum physics. Scale invariant systems can be found in two dimensional statistical mechanics. The point is that in these models the absolute scale is set completely by the correlation lengths, which diverge at critical points and hence the absolute scale is lost at these points. For this reason, it becomes possible in separate cases to use the axiomatics of conformal field theory in order to calculate the critical exponents.

The structures which we want to study in our sample of models are the following. Of obvious theoretical interest is to find the possible spaces of states in a theory — superselection sectors — and the algebraic approach is the appropriate framework to treat this problem. On the other hand, for a specific dynamical interpretation the field–theoretical approach might be very useful and for this purpose the commutation relations among the fields will provide the best insight. It is also quite interesting to study the deformation theory of the commutators, because this allows us to uncover whole families of models described by one parameter. *To classify the superselection sectors, to explore the commutation relations among fields and then to study the possibility to deform them will be the three final goals of this Ph.D project.*

In the considered sample of models there is one particularly intriguing — the (A_{28}, E_8) extension of the "minimal" stress–energy tensor model with central charge $c = \frac{144}{145}$. This model is the only one from the whole series, for which there is not found a direct field–theoretical construction as a coset model in terms of well–studied algebras and by simple current extensions, even though it arises as a miror extension of a coset [Xu, 2007]. (Chiral fields apart from the stress energy tensor are interesting, because they are components of conserved two dimensional tensor fields of higher rank.) While the coset construction at least in principle can determine the superselection structure and commutators of the remaining models, this particular one may be studied only by some alternative methods.

Within the algebraic framework to every chiral conformal field theory is assigned a diffeomorphism covariant local isotonous net of von Neumann algebras on S^1. Algebraic techniques allow one to obtain a complete classification of such nets when their associated central charge c is smaller than one. Every irreducible diffeomorphism covariant net is either an irreducible Virasoro net or its local extension of finite index. The number of such extensions is finite and is completely classified in [Kawahigashi & Longo, 2004]. Furthermore, to every extension of a Virasoro net is associated the modular invariant matrix $Z_{\mu\nu}$ [Cappelli *et al.*, 1987], which carries a crucial information about the superselection sectors of the extension.

As we mentioned above, the superselection structure is most efficiently studied with algebraic methods. All the representations of the Virasoro nets are known. They are in bijective correspondence with those of a Virasoro algebra with the same central charge, which are studied with Lie algebraic techniques and their complete classification is obtained in [Friedan *et al.*, 1984]. The sectors are labeled by the pairs (c, h) where the central charge c and the spin h take discrete values. Moreover, their fusion rules, i.e. the decomposition of the tensor product into a direct sum of irreducible sectors, are also known. These data determine also the statistical dimensions (see below).

Then our task reduces to finding the superselection sectors of the local extensions. A first message will be that the conformal nets on S^1 with $c < 1$ are rational [Kawahigashi & Longo, 2004], i.e they possess finitely many inequivalent unitary irreducible sectors with finite statistical dimensions. In further studying of the sector structure one has as a guiding example the representation theory of compact groups because it is very well-understood and especially because it allows defining useful algebraic operations. In four dimensions such an analogy is accomplished in the celebrated papers of Doplicher–Haag–Roberts [Doplicher *et al.*, 1969*a*,*b*, 1971, 1974] who showed that the category of a large class of representations of a QFT net, selected by a special criterion (DHR[1] criterion) is equivalent to the one of the representations of a compact group. This group can be reconstructed by abstract duality theory and hence the aimed superselection structure can be completely determined. However, this is not exactly the case in one and two space–time dimensions due to the specific topological situation there. Indeed, one is still able to define a product of representations as well as a direct sum, a contragradient representation and even a statistical dimension of the representation, that is in general non-integer, but additive under direct sums and multiplicable under products, like the dimensions of representations of a finite group. Especially nice is that with respect to these definitions every reducible representation is decomposable into a direct sum of irreducible ones, which allows us to concentrate our study only on them. The analysis is most efficiently done in terms of "DHR endomorphisms". (Let us remark at this point that although the DHR criterion appears to be very restrictive in 4D QFT, in chiral conformal field theory all positive energy representations satisfy this criterion.) Yet, the analogy with a compact group breaks

[1]DHR are the first letters of Doplicher, Haag and Roberts

at the commutation law for the tensor product, which for a compact group amounts to a permutation. In one and two dimensions $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ still belong to the same sector due to locality, but the unitary intertwiners among them do not square to identity. Instead, they define a representation of the Artin Braid Group. Thus, in one and two dimensions we have braid group statistics, not the Bose–Fermi one. Then the braiding itself produces a unitary representation of the modular group $\mathrm{SL}(2,\mathbb{Z})$ and hence the representations of 1D and 2D conformal nets give rise to modular (completely rational) braided tensor categories.

For the case of 1D conformal nets on S^1 two conjugate to each other braiding operators were constructed explicitly in [Böckenhauer & Evans, 1998]. Subfactor theory, making use of the braiding, provides us with the machinery of α- induction — a technique to produce endomorphisms of the bigger net from the DHR endomorphisms ρ of the subnet [Longo & Rehren, 1995]. Although in general the obtained endomorphisms are not DHR, they contain all the DHR endomorphisms of the extension as submorphisms. The two maps α^+_ρ and α^-_ρ, corresponding to the two different braidings, have nice homomorphic properties, namely they preserve the unitary equivalence and the dimensions of sectors and respect the algebraic operations in the tensor category. The braiding operators of the DHR endomorphisms intertwine their α-induced ones, as well. In parallel, to refine the study of the interrelations of the sectors of the two theories, the restriction of sectors of the larger theory is used. In contrast to α-induction, it preserves the DHR property but it is not a homomorphism. Subfactor theory tells us also that each DHR endomorphism of the extension appears as a submorphism simultaneously of α^+_ρ and α^-_ρ for at least one ρ and that each such "simultaneous submorphism" is DHR [Kawahigashi, 2003]. A decisive information about how to distinguish such submorphisms comes from the modular invariant matrix $Z_{\mu\nu}$ which we associate with the extension and from the dual canonical endomorphism of the net of subfactors θ. With this machinery in hand we can recover all superselection sectors of the extension from the already known sectors of the Virasoro subnet.

At this point we must honestly confess, that after we finished with calculating the superselection sectors for all extensions we observed, that the exact number of superselection sectors for the four higher index extensions is published in [Kawahigashi, 2009]. However, in this work there are no further considerations available apart from the exact number of superselection sectors, which is in principle directly recognizable at first sight of the $Z_{\mu\nu}$ matrix without a deep analysis and exact computations. Moreover, we also computed the fusion rules, which are not available in this article.

Using the vacuum character of a $\mathrm{Diff}(S^1)$-covariant net from the algebraic approach we can determine in the corresponding Wightman theory all the fields which transform covariantly under the whole projective $\mathrm{Diff}(S^1)$ representation. Such fields we call primary and for models with $c < 1$ they are a finite number. In addition to the primaries, there is a larger class of fields which transform covariantly only under the Möbius subgroup and not necessarily under the whole diffeomorphism group. These fields are called quasiprimary and together with their derivatives they produce a basis of the space of fields. All the quasiprimaries and their derivatives, jointly called secondary or descendant, are contained in the OPE of (n copies of) the stress–energy tensor $T(x)$ with some of the primary fields. Hence, they can be obtained as properly defined normal products of $T^n\phi$ which must be constructed in such a way that they are conformally covariant.

To understand the general structure of the commutators among these fields we follow closely the example of the Lüscher–Mack theorem [Mack, 1988], which determined the commutation relations of the stress–energy tensor just on the basis of the most general properties of a relativistic quantum theory and conformal invariance. Using the same argument one can fix the commutators of the stress–energy tensor with an arbitrary primary field and one can almost fix the commutators of the stress–energy tensor with a quasiprimary field. In our work we found it more convenient to work with smeared field operators, for which we can construct a basis entirely from quasiprimary fields. We show that a similar strategy to the Lüscher–Mack theorem allows us to determine the commutation relations between the basis fields up to some structure constants. These structure constants carry the model dependent information of the specific system considered and they are further restricted by Lie algebra structure relations. The anti–symmetry of commutators immediately produces a symmetry rule for the structure constants. However, the Jacobi identity cannot be directly exploited, because the different terms there appear with different test functions, so we must do first some preparatory work.

On the test function level the commutators give rise to (the unique) local intertwiners of the $sl(2,\mathbb{R})$ action on the test function spaces. The spaces of intertwiners from tensor products of representations are finite-dimensional, and we define transformation matrices between their various possible bases (corresponding to subsequent (multiple) action of commutators in different order). These transformation matrices allow us to change between different composite test functions (in particular obtained after actions of commutators) and consequently to be able to strip off the test functions in the field algebra. In this sense we obtain a reduced form of the field space, which is equipped with a new bilinear multi-component bracket obeying a new generalized symmetry rule. The new multi-index Jacobi identity involves certain coefficient matrices multiplying its three terms. These matrices are universal in the sense that they reflect only the underlying representation theory of $sl(2,\mathbb{R})$, but not the specific model.

This reduced version of the Jacobi identity produces an infinite number of constraints for the structure constants of our commutators not involving the test functions anymore. The solutions of these constraints promote potential candidates for chiral conformal field theories. The idea to consider constraints in such form was cherished from [Bowcock, 1991], where a Jacobi identity among structure constants from commutators of Fourier modes of quasiprimary fields was considered. Our approach emphasizes locality of commutators more clearly.

The inspiration to explore the deformation theory of the commutators of the reduced field algebra came from [Hollands, 2008], where deformations in the setting of OPE (operator product expansion) approach to quantum field theory on curved space–time were studied. We consider formal deformations, which are defined as perturbative power series and we work in a setting analogous to that in [Gerstenhaber, 1964], which is the prototype of deformation theory for algebraic structures. Such deformations are naturally related to cohomology complexes, whose cohomology groups may give decisive information about rigidity and about classification of deformations. Thus in all theories of formal deformations of algebraic structures the first step is to relate the deformation problem to a certain cochain complex. In the first examples of such theories [Gerstenhaber, 1964], [Nijenhuis & Richardson, 1967] the second step was to show that the first cohomology groups are directly related to the possibility to deform the algebraic structure considered. The more modern point of view is that the deformation theory in consideration is mastered by a differential graded Lie algebra (or in some cases a homotopy Lie algebra or L_∞-algebra) which can be obtained from the cochain complex by constructing a bracket on this complex, which is skew symmetric with respect to the "grading" by dimension of the cochain spaces and satisfying a graded Jacobi identity [Nijenhuis & Richardson, 1964], [Manetti, 1999], [Borisov, 2005].

The cochain complex, which we constructed, is built out of functions with a complicated symmetry property — Z^ε-symmetry (Section 7.1). The origin of this symmetry can be traced back to the complicated symmetry rules in the reduced algebra. We showed that the first perturbations (also infinitesimal perturbations) of the reduced brackets are classes from the second cohomology group of our complex and we computed the obstruction operators to their integration. We hope that an explicit computation of the cohomology groups in the future will allow us to relate the first of these groups to the problem of rigidity of the bracket and the integrability of the first perturbations.

The thesis is organized as follows. Chapters 1,2 and 3 contain some preliminary general knowledge, comprising the input and the tools for our Ph.D project. Chapter 1 discusses the field content of chiral conformal field theories, conformal generators and their representations. Chapter 2 gives a definition of a conformal field theory from algebraic point of view and the classification of all such theories when $c < 1$. The important message of this chapter is that the study of superselection sectors can be translated into study of DHR endomorphisms and for the later applications very important are the endomorphism calculus and subfactor theory, allowing us to obtain (indirectly) DHR sectors of the extensions from DHR sectors of the subnets. Chapter 3 introduces the Gerstenhaber "muster" theory of formal deformations and explains how it is related to a cohomology complex and its cohomology groups. Here also is discussed the possibility to describe deformation theories of algebraic structures in terms of the deformation theory of a graded Lie algebra. These first chapters are just a review of the literature and the original personal contribution is contained entirely in the last three chapters, every of which is dedicated to one of our three final goals, marked earlier in this section. In Chapter 4 the superselection sectors are found for all local extensions of Virasoro nets with $c < 1$. For the (A_{28}, E_8) extension for $c = \frac{144}{145}$ also the fusion rules and statistical dimensions are computed. In Chapter 5 the general structure of local Möbius covariant commutators is explored and a new axiomatization for a chiral conformal theory is offered. In Chapter 6 we construct a cohomology complex associated to the reduced field algebra for the purpose of describing the deformations of the reduced commutator and we check that this complex is indeed a perspective candidate for this aim. The results of Chapters 5 and 6 are also available in [Kukhtina & Rehren, 2011].

2 Chiral conformal algebras

In this chapter we will review very basic knowledge about chiral conformal field theories on the circle. They arise as subtheories of 2D conformal field theories, either Euclidean or on Minkowski space, and evoke interest because their investigation is the first step towards understanding the two dimensional theories.

In difference with the situation in $D > 2$, the conformal group in $D = 1$ and $D = 2$ is infinite dimensional. Even though the maximal group of unbroken symmetry is its finite dimensional Möbius subgroup, the whole conformal group assists greatly to provide an infinite set of exactly solvable models, in the sense of finding all their correlation functions. Such models were called *minimal models* and their discovery [Belavin *et al.*, 1984*a*], [Belavin *et al.*, 1984*b*] was quite a spectacular event, because it provided some of the first examples of nontrivial relativistic quantum theories. These minimal models are also of special interest for us in this Ph.D thesis and we will provide some introductory knowledge about them later in this chapter.

We will also discuss briefly the conformal group and its representations on the space of fields, as well as the field content of chiral algebras. There are several extensive reviews on the topic [Furlan *et al.*, 1989], [Francesco *et al.*, 1997], [Rehren, Vorlesung Göttingen, WiSe 1997/98], which were useful for us. Throughout this chapter we speak about fields in the sense of Wightman fields [Streater & Wightman, 1964], [Jost, 1965].

2.1 The conformal group in D=1 and D=2 on Minkowski space. 2D conformal field theories and chiral theories

The conformal group consists of all transformations that preserve the angles and the orientation, which means that it leaves also the infinitesimal interval invariant up to a scaling with a positive factor depending on the position.

In $D = 1$ dimensions the conformal group is $\mathrm{Diff}(\mathbb{R})$.

In $D = 1 + 1$ dimensions the conformal group is $\mathrm{Diff}(\mathbb{R})\times\mathrm{Diff}(\mathbb{R})$ — a tensor product of two diffeomorphism groups, each of which acts on one of the light–cones $(t \pm x)$. This is a consequence of the following equality:

$$\mathrm{d}x_\mu \mathrm{d}x^\mu = \mathrm{d}(t + x)\mathrm{d}(t - x) \tag{2.1}$$

where x^μ is a component of the vector (t, x). Clearly, the conformal group in one and two dimensions is infinite dimensional.

In the next section we will see that also the stress–energy tensor, which is responsible for infinitesimal conformal transformations of the fields, splits into two commuting parts, every of them depending on one light–cone variable. This indicates that every D=1+1 conformal field theory has in its core a tensor product of two 1D chiral conformal theories on every of the light cones. Then, the first step towards understanding the two dimensional theory will be to study its chiral subtheories. That is why, in our thesis we will concentrate on the chiral theories.

In all that follows we will identify $\mathbb{R}$ with $S^1 \setminus \{-1\}$ by Cayley transformation and we will regard the fields as distributions on $\mathbb{R}$.

Remark. One can show that the diffeomorphism group cannot have a regular unitary representation with an invariant vector. This is possible only for the Möbius subgroup $SL(2,\mathbb{R})/\mathbb{Z}_2 = SU(1,1)/\mathbb{Z}_2 \subset \mathrm{Diff}(S^1)$, which is therefore the maximal subgroup of unbroken symmetry.

In quantum field theory we are therefore interested in representations not of the group of coordinate transformations but of its covering group.

Definition 2.1 (Conformal covariance of fields). *The covariance law for a conformal chiral field $\Phi(x)$ under a transformation from (a subgroup of) Diff(S^1) is the following:*

$$U(\gamma)\Phi(x)U^{-1}(\gamma) = \left(\frac{d\gamma}{dx}\right)^{d_\Phi} \Phi\left(\gamma(x)\right)$$

where U is a unitary projective representation.

Notation. d_Φ is called the *scaling dimension* of the field.

Note that local chiral fields have integer scaling dimensions.

2.2 The stress–energy tensor

Conformal symmetries in the space of fields are generated by charges, which are integrals of a conserved current. This current has a meaning of an energy and momentum density and is called the stress–energy tensor. In a two dimensional theory this tensor, which we will denote by $T^{\mu\nu}$, has the following properties:

- energy conservation implies that $\partial_\mu T^{\mu\nu} = 0$
- in $D = 1 + 1$ the stress–energy tensor is symmetric
- it is traceless in massless theories
- in $D = 1 + 1$ the stress–energy tensor has scaling dimension 2

In fact, the stress–energy tensor in a dilation invariant theory in any number $s + 1$ of space–time dimensions must be both conserved and traceless.

Observation. The stress–energy tensor in D=1+1 splits into two chiral components on every of the light cones $(t \pm x)$:

1. symmetry and tracelessness imply that the stress–energy tensor has only two independent components:
$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} \\ T^{01} & T^{00} \end{pmatrix} \tag{2.2}$$

2. energy conservation implies that:
$$\begin{array}{rcl} \partial_0 T^{00} + \partial_1 T^{01} & = & 0 \\ \partial_0 T^{01} + \partial_1 T^{00} & = & 0 \end{array} \quad \longrightarrow \quad \begin{array}{rcl} (\partial_0 + \partial_1)(T^{00} + T^{01}) & = & 0 \\ (\partial_0 - \partial_1)(T^{00} - T^{01}) & = & 0 \end{array} \tag{2.3}$$
which means that we have the chiral fields:
$$\begin{aligned} \frac{1}{2}(T^{00} + T^{01}) &= T_R(t - x) \\ \frac{1}{2}(T^{00} - T^{01}) &= T_L(t + x) \end{aligned} \tag{2.4}$$

One can compute the commutators of the stress–energy tensor just on the basis of most general properties of a conformal quantum field theory [Mack, 1988]:

Theorem 2.2 (Lüscher–Mack theorem). *The chiral components of the stress–energy tensor have the following commutation relations:*
$$\begin{aligned} i[T_{R/L}(x), T_{R/L}(y)] &= T'_{R/L}(y)\delta(x - y) - 2T_{R/L}(y)\delta'(x - y) + \frac{c}{24}\delta'''(x - y) \\ i[T_{R/L}(x), T_{L/R}(y)] &= 0 \end{aligned} \tag{2.5}$$
where $c \geq 0$ is a constant, called the central charge.

Proof. The main steps of the proof of this theorem are the following:

1. Locality implies: $[T_{R/L}(x), T_{R/L}(y)] = \sum_{l=0}^{n} \delta^{(l)}(x - y) O_l(y)$

2. Scaling invariance implies $n = 3$, $O_l(y)$ is a local field of scaling dimension $3 - l$. (Note that the scaling dimension of δ is 1, the scaling dimension of T is 2 and every derivative contributes with a scaling dimension 1)

3. Anti-symmetry of commutators and translation covariance allow to determine O_0, O_1, O_2 and O_3

The terms in the commutation relations from the Lüscher–Mack theorem correspond to the singular terms in the operator product expansion of two chiral stress–energy tensors:
$$\begin{aligned} 2\pi T(x_1)T(x_2) &= \frac{c}{4\pi(x_{12} - i\epsilon)^4} - 2\frac{T(x_2)}{(x_{12} - i\epsilon)^2} - \frac{T'(x_2)}{x_{12} - i\epsilon} + O(1) \\ &= \frac{c}{4\pi(x_{12} - i\epsilon)^4} - \frac{T(x_1) + T(x_2)}{(x_{12} - i\epsilon)^2} + O(1) \end{aligned} \tag{2.6}$$
having in mind the relation:
$$(-1)^n n! \left(\frac{1}{(x - i\varepsilon)^{n+1}} - \frac{1}{(x + i\varepsilon)^{n+1}} \right) = 2\pi i \delta^{(n)}(x) \tag{2.7}$$
This expansion determines the possible singularities of the correlation functions.

2.3 Virasoro algebra

The Fourier modes of the stress–energy tensor defined as:

$$L_n = \frac{1}{2}T(f_n^{(2)}) = \frac{1}{2}\int \mathrm{d}x(1-ix)^{1-n}(1+ix)^{1+n}T(x) \tag{2.8}$$

with $f_n^{(2)} := (1-ix)^{1-n}(1+ix)^{1+n}$ a test function, have the following commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{2.9}$$

which follow directly from the Lüscher–Mack theorem from the previous section. This algebra is called Virasoro algebra. It is a central extension of the Witt algebra $[l_m, l_n] = (m-n)l_{m+n}$ — the algebra of $\mathrm{Diff}(S^1)$, which confirms the rule that the algebra of field transformations is a central extension of the algebra of coordinate transformations. c is the central charge, which is a c-number and commutes with all the other generators:

$$[L_n, c] = 0 \tag{2.10}$$

Observation. Certain linear combinations of the modes L_m for $m = 0, \pm 1$:

$$\begin{aligned} P &= \frac{1}{2}(L_{+1}+L_{-1}) + L_0 = \int T(x)dx \\ D &= \frac{1}{2i}(L_{+1}-L_{-1}) = \int xT(x)dx \\ K &= -\frac{1}{2}(L_{+1}+L_{-1}) + L_0 = \int x^2T(x)dx \end{aligned} \tag{2.11}$$

give rise to generators of the Möbius group with the following commutation relations:

$$[P, D] = iP, \quad [P, K] = 2iD, \quad [D, K] = iK \tag{2.12}$$

Here P is the generator of translations, D is the generator of dilations and K is the generator of special conformal transformations.

The transformation laws of the field $\phi(x)$ with scaling dimension d_ϕ under these generators are:

$$\begin{aligned} i\,[P, \phi(x)] &= \partial\phi(x) \\ i\,[D, \phi(x)] &= (x\partial + d_\phi)\phi(x) \\ i\,[K, \phi(x)] &= (x^2\partial + 2d_\phi x)\phi(x) \end{aligned} \tag{2.13}$$

Remark. A positive energy unitary representation of the Virasoro algebra with an invariant vector, i.e. such that $L_m\Omega = 0$, is possible only for $c = 0$, which would lead to $T(x) = 0$, $L_m = 0$, so we have a broken symmetry. Such a representation is possible only for the Möbius group, because for $m = 0, \pm 1$ the central term vanishes.

Representations of the stress–energy tensor are defined through representations of the Virasoro algebra. As we are interested only in the representations with positive energy, we require that P is a positive operator. Then $L_0 = \frac{1}{2}(P + K)$ must be positive as well, because K is related to P by conjugation with an unitary operator, hence it is also positive.

Remark. 2D conformal field theories have two commuting Virasoro algebras which share the same central charge but may be represented with two different lowest weights h and $\overline{h}$. $h + \overline{h}$ gives the scaling dimension of the field, $h - \overline{h}$ gives the spin.

2.4 Representations of the Virasoro algebra

The irreducible positive energy representation spaces of the Virasoro algebra have the Verma module structure — a structure very familiar in physics, which has a "ground state" out of which one can recover the whole space by "rising operators". The "ground state" realizes the lowest (or the highest) eigenvalues of the energy and certain charges and the whole space can be decomposed as a direct sum of simultaneous eigenspaces of these operators. The Verma module is turned into a Hilbert space by factoring out its null vectors. One can assign to such structure a character function which describes the spectrum and determines the representation uniquely.

The role of the energy operator in the Virasoro case will be played by L_0, the creation operators will be L_{-n} and the annihilation operators will be L_n, $\forall n > 0$, which have the following commutation relations with L_0:

$$[L_0, L_{-n}] = nL_{-n}, \quad [L_0, L_n] = -nL_n \tag{2.14}$$

A concrete realization of the ground state for a Virasoro representation may be achieved the following way. Suppose that in the theory exists apart from the stress–energy tensor also a quantum field $\phi(x)$, which obeys the following commutation relations:

$$i[T(x), \phi(y)] = \phi'(y)\delta(x - y) - h\,\phi(y)\delta'(x - y) \tag{2.15}$$

Let us construct the vector $|h\rangle := e^{iPa}\phi(x)\Omega|_{a=i,x=0} = \phi(i)\Omega$. One can show that this vector is an eigenvector of L_0 with eigenvalue h. Moreover, this vector is annihilated by L_n, $\forall n > 0$. Then $|h\rangle$ is a lowest weight vector with lowest weight h of the Verma module, generated by the action of polynomials of L_{-n} on $|h\rangle$. There is one-to-one correspondence between the lowest weight vectors in the different representations and the conformal fields of the theory with commutation relations as above.

One can recover the whole representation space by successive action of the rising operators $L_{-n}(n > 0)$ on $|h\rangle$:

$$V_h := \mathrm{Span}\{L_{-n_1}...L_{-n_r}|h\rangle : \quad n_1 \geq ... \geq n_r > 0\} \tag{2.16}$$

The space V_h can be decomposed as a direct sum of eigenspaces $V_h^{(k)}$ of L_0 with eigenvalue $k + h$ such that:

$$V_h^{(k)} := \mathrm{Span}\{L_{-n_1}...L_{-n_r}|h\rangle : n_1 + ... + n_r = k\} \tag{2.17}$$

To turn the Verma module V_h into a Hilbert space we need to define a scalar product $(\cdot,\cdot)_{h,c}$ such that we have the hermiticity property $L_n^\dagger = L_{-n}$. This property fixes the scalar product up to normalization $\||h\rangle\|^2$ because it yields:

$$(L_{-n_1}...L_{-n_r}|h\rangle, L_{-m_1}...L_{-m_s}|h\rangle)_{h,c} = (|h\rangle, L_{n_1}...L_{n_r}L_{-m_1}...L_{-m_s}|h\rangle)_{h,c} \tag{2.18}$$

Further, we can perform the standard procedure and move by successive commutations all annihilation operators to the right and all creation operators to the left. In the cases when $\sum n_i \gtrless \sum m_j$ there will be L operators "left uncompensated", which will annihilate either $\langle h|$ or $|h\rangle$. This implies that the spaces $V_h^{(k)}$ are pairwise orthogonal. When $\sum n_i = \sum m_j$ one gets in general:

$$(|h\rangle, L_{n_1}...L_{n_r}L_{-m_1}...L_{-m_s}|h\rangle)_{h,c} = (|h\rangle, P_{\underline{n},\underline{m}}(h,c)|h\rangle)_{h,c} \tag{2.19}$$

with $P_{\underline{n},\underline{m}}(h,c)$ polynomials in h and c depending on $(\underline{n},\underline{m})$. If we choose the normalization $(|h\rangle,|h\rangle)_{h,c} = 1$ (for example), then the scalar product will be completely fixed.

In order to ensure that V_h is really a Hilbert space we have to show that the scalar product on this space is positive (semi-)definite. In case that it is positive semi-definite, in order to turn V_h into a Hilbert space we have to factor out the space of null vectors. The null vectors correspond to non-trivial linear combinations of products of creation operators which annihilate the ground state.

It can also happen that the scalar product is indefinite, which means that there is one or more states $|\psi\rangle$ in V_h, such that $\langle\psi|\psi\rangle$ is negative, which we call ghosts. In such case we do not have a Hilbert space representation.

The positivity constraint for the scalar product is equivalent to the requirement that the matrix of scalar products among the basis vectors has no negative eigenvalues. We can consider separately the subspaces $V_h^{(k)}$, since they are orthogonal to each other. We can approach the positivity problem for each $V_h^{(k)}$ by studying the zeros of the Kac determinant $\det(M_k)$, where M_k is the $P(k)\times P(k)$ matrix of inner products of vectors of the form $L_{-n_1}...L_{-n_s}|h\rangle$ with $n_1+...+n_s = k$, such that $P(k)$ is the number of ways the positive integer k can be presented as a sum of positive integers n_i with $i<j \to n_i<n_j$. The formula for this determinant, up to an overall positive normalization constant, is the following:

$$\det(M_k) \sim \prod_{i=1}^{k}\left[\prod_{pq=i}(h-h_{p,q}(c))\right]^{P(k-i)} \tag{2.20}$$

with p, q positive integers and the explicit expression for the functions $h_{p,q}(c)$ will be displayed later.

If for a given representation labelled by the pair (c,h) $\det(M_k)$ is positive for every integer k, we have a positive scalar product on V_h. If the Kac determinant is negative for some k, we have an indefinite scalar product and the corresponding representation cannot occur in any unitary theory. A zero Kac determinant indicates the presence of null vectors, but does not give enough information about the existence of ghost states and we have to perform our analysis in some alternative way.

In the general case, the (semi-)positivity on the subspace $V_h^{(1)}$, meaning that the scalar product of the only basis vector $|1\rangle = L_{-1}|h\rangle$ with itself $\langle 1|1\rangle = 2h$ is non-negative, yields $h \geq 0$. Also, the state $|n\rangle = L_{-n}|h\rangle$ has a scalar product $\langle n|n\rangle = 2nh + cn(n^2-1)/12$ and its (semi-)positivity when n is large requires $c \geq 0$.

Further studying of Kac determinant shows that for $c \geq 1$ and $h \geq 0$ we always have a positive definite scalar product.

More interesting is the situation when $h \geq 0$, $c < 1$. In this region the Kac determinant is almost everywhere negative, with the exception when (c, h) lies on one or more of the curves $h_{p,q}(c)$ — $\det(M_k)$ will be zero on every of these curves for which $k > pq$. After a more involved analysis it was shown in [Friedan *et al.*, 1984] that even on these curves almost all the points correspond to representations containing ghosts and that the only possible candidates for ghost free representations occur for the discrete infinite series of numbers c and h given by:

$$\begin{aligned} h_{p,q}(c) &= \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \quad p \in [1, m-1],\, q \in [1, m] \\ c &= 1 - \frac{6}{m(m+1)}, \quad m = 2, 3, 4, ... \end{aligned} \tag{2.21}$$

Such representations contain necessarily a null state and as we will discuss in the next section such states give rise to infinite sets of linear differential equations on the correlation functions. To obtain a (possibly) unitary representation, we have to factor out the space with null vectors.

For $c = 1$, $h \geq 0$ the scalar product is almost everywhere positive definite, with the exception of the points $(c = 1, h = \frac{k^2}{4})$, $k \in \mathbb{N}$, where it is positive semi-definite.

If $c = 0$ we have $h = 0$ and the only representation (the trivial one) exists with L_n vanishing.

In [Goddard & Olive, 1985], [Goddard *et al.*, 1985] it was shown that every representation from the list (2.21) can be obtained from a coset construction of known algebras. The representations of those algebras are unitary and this guarantees the unitarity of all the representations from the discrete series (for more details see Appendix A).

2.5 Chiral algebras and their field content. Minimal models

One shows that fields with the commutation relations (2.15) transform covariantly (see Definition 2.1) under a projective representation of the whole diffeomorphism group $\mathrm{Diff}(S^1)$ and such fields will be called *primary.* As we saw in the previous section, they are in one-to-one correspondence with the representation spaces of the Virasoro algebra. Primary fields appear as intertwining maps from the vacuum representation to other lowest weight modules.

It was recognized that primary fields cannot exhaust the field content in the theory, for example because in every conformal theory there must be the stress–energy tensor $T(x)$, which is responsible for conformal transformations of the fields. $T(x)$ does not transform covariantly under the whole diffeomorphism group, but only under its Möbius subgroup and such fields are called *quasiprimary.* They obey the following commutation relations with $T(x)$:

$$i[T(x), \phi(y)] = \phi'(y)\delta(x-y) - h\,\phi(y)\delta'(x-y) + \sum_{3 \leq k \leq h+1} \delta^{(k)}(x-y)\phi_k(y) \tag{2.22}$$

where ϕ_k are either quasiprimary fields or derivatives of quasiprimary fields of lower dimensions and h is the scaling dimension of the field.

Further, together with the quasiprimary fields there are additional fields (derivatives of quasiprimary fields) which appear in the operator product expansion of primary fields with $T(x)$. All the fields present in this operator product expansion are called *secondary* or *descendant.*

A primary field together with all of its secondary fields forms a *conformal family.* The conformal family includes naturally all the derivatives of each field involved. A transformation law mixes only among members of the same conformal family — therefore, each conformal family corresponds to some irreducible representation of the conformal algebra on the space of fields. Note that all fields appearing in the right–hand side of (2.22) are from the same conformal family as $\phi(y)$.

If a conformal family contains a null field, then it will be called a *degenerate conformal family.* Also the corresponding primary field is called degenerate.

One can show that every vector in a Verma module can be created from the vacuum by a linear combination of quasiprimary fields and their derivatives for $x = i$. This is called state–field correspondence. Then existence of null vectors corresponds to a linear relation among these fields.

The correlation functions of all fields in the chiral algebra are related via differential operators to the correlation functions of the primary fields. Hence, all the information about the conformal quantum field theory is contained in these correlators. One can show that these correlators are built up out of some basic bricks, called conformal blocks (see [Belavin *et al.*, 1984*a*] for more details). The problem of calculating the conformal blocks is extremely difficult and is worked out completely only in separate cases. The presence of null vectors, however, assists greatly for the solving of concrete models (in the sense of finding of all the correlation functions), because they give rise to additional differential equations for the conformal blocks.

Especially favourable is the situation for the models with a central charge from the discrete series (2.21) where the number of conformal families is finite and each of them is degenerate. Such models contain much less fields than usual and that is why the proposed name for these theories is *minimal models.* Actually, the only observable field there is the stress–energy tensor. All correlation functions in these models can be obtained as solutions of infinitely many differential equations, called Ward identities, which means that these models are completely solvable.

2.6 Correlation functions

A conformally–invariant two point correlation function is restricted to be of the form:

$$\begin{aligned} W^{(2)}(x_1,x_2) &= (\Omega, \varphi_1(x_1)\,\varphi_2(x_2)\,\Omega) \\ &= C_{\varphi_1\varphi_2}\delta_{h_1h_2}\Delta(x_1-x_2)^{2h} := C_{\varphi_1\varphi_2}\delta_{h_1h_2}\left(\frac{-i}{x_1-x_2-i\varepsilon}\right)^{2h} \end{aligned} \tag{2.23}$$

Translation invariance tells us that $W^{(2)}$ is a function only of the difference $x_{12} := x_1 - x_2$, dilation invariance requires that $W^{(2)} \sim \Delta(x_{12})^{(h_1+h_2)}$ and because of the special conformal transformations invariance h_1 must be equal to h_2. The sign in front of $i\epsilon$ is foxed from the spectral condition.

By a similar argument we can find also the conformal three point function $W^{(3)}(x_1, x_2, x_3)$:

$$\begin{aligned} W^{(3)}(x_1, x_2, x_3) &= (\Omega, \varphi_1\,(x_1)\,\varphi_2\,(x_2)\,\varphi_3\,(x_3)\,\Omega) \\ &= C_{\varphi_1\varphi_2\varphi_3}\Delta(x_1 - x_2)^{h_1+h_2-h_3}\Delta(x_1 - x_3)^{h_1+h_3-h_2}\Delta(x_2 - x_3)^{h_2+h_3-h_1} \end{aligned} \tag{2.24}$$

In the general case, conformal invariance fixes $W^{(N)}(x_1, ..., x_N)$ only up to arbitrary functions of the so called conformal ratios $x^{ij}_{kl} := \frac{(x_i - x_j)(x_k - x_l)}{(x_i - x_k)(x_j - x_l)}$, which are Möbius invariant:

$$W^{(N)}(x_1, ..., x_N) = \prod_{s<t} \Delta(x_s - x_t)^{p_{st}} F(x^{ij}_{kl}), \quad \sum_s p_{st} = 2h_t \tag{2.25}$$

Obviously the choice of p_{st} is not unique, but the various choices differ up to factors, which can be absorbed in $F(x^{ij}_{kl})$.

3 Algebraic approach to chiral conformal field theories

The function of this chapter is to prepare theoretically the ground for attacking the first final goal of this thesis — the classification of superselection sectors of conformal field theories for $c < 1$. The algebraic approach has proven to be the right framework for such a study. Moreover, within the algebraic framework we are able to obtain a complete classification of chiral conformal theories on the circle when the associated central charge is smaller than one. Therefore, in this chapter we will review general definitions and results from algebraic quantum field theory, especially in relation to conformal field theories, which we will need in our analysis. We will be minimalistic in our exposition and we will rather focus on the concepts involved in our study and on interrelations among them and their properties. We will omit the proofs and we will rather refer the reader to the original literature.

The main messages of this chapter are the following. All conformal field theories with $c < 1$ correspond either to Virasoro nets or to their local extensions. All these nets are completely rational — i.e. they possess a finite number of inequivalent irreducible sectors with finite statistical dimensions and non-degenerate braiding. The sectors of Virasoro nets are well-known and the sectors of the extensions can be gained with the α-induction and σ-restriction mechanisms of subfactor theory. In analogy to DHR theory (a sector theory for 4D relativistic quantum theories) the analysis was done in terms of localized and transportable von Neumann algebra endomorphisms, which represent the sectors, and it can be shown that they form a tensor category which resembles a lot the category of representations of a compact group.

3.1 Conformal nets on the circle

In this section we will give precise mathematical definitions of conformal field theories and related concepts from algebraic point of view. The basic literature which we used for the main line of section was [Gabbiani & Fröhlich, 1993] and [Kawahigashi & Longo, 2004].

Definition 3.1 (Möbius covariant net). *Let $\{\mathcal{A}(I)\}_{I\subset S^1}$ be a set of von Neumann algebras acting on the Hilbert space $\mathcal{H}$, s.t. I are proper (non-empty, non-dense, open and connected) intervals on the circle. Suppose that the set $\{\mathcal{A}(I)\}_{I\subset S^1}$ satisfy the following properties:*

1. **isotony:** $I_1 \subseteq I_2 \Longrightarrow \mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$
2. **locality:** $I_1 \subseteq I_2' \Longrightarrow \mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)'$, $\quad I' = S^1\backslash\bar{I}$, $\quad \mathcal{A}(I)'$ *denotes the commutant*[1] *of*

[1]Let us remind that the commutant of $\mathcal{A}(I)$ in $\mathcal{B}(\mathcal{H})$ is defined as $\mathcal{A}(I)' := \{x \in \mathcal{B}(\mathcal{H})|\ xy = yx, \forall y \in \mathcal{A}(I)\}$

$\mathcal{A}(I)$ in $\mathcal{B}(\mathcal{H})$

3. **covariance:** *the local algebras $\mathcal{A}(I)$ transform covariantly under a strongly continuous and unitary projective representation U of the Möbius group on $\mathcal{H}$:*

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad \forall g \in PSL(2,\mathbb{R}), \quad I \subset S^1 \tag{3.1}$$

4. **positive energy representation:** *the spectrum of the generator of rotations of $PSL(2,\mathbb{R})$ is positive*

5. **existence of a vacuum:** *there exists a unique vector $\Omega \in \mathcal{H}$ which is invariant under $PSL(2,\mathbb{R})$*

6. **cyclicity of a vacuum:** *Ω is cyclic for the von Neumann algebra $\mathcal{A} := \{\bigcup_{I\subset S^1} \mathcal{A}(I)\}''$*

Then the set $\{\mathcal{A}(I)\}_{I\subset S^1}$ is called a Möbius covariant net.

Definition 3.2 (Conformal net). *A Möbius covariant net is called a conformal (diffeomorphism–covariant) net if there exists a projective unitary representation U of Diff(S^1) on $\mathcal{H}$ extending the unitary representation of PSL(2,$\mathbb{R}$) such that for all $I \subset S^1$ we have:*

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathit{Diff}(S^1) \\ U(g)AU(g)^* = A \quad A &\in \mathcal{A}(I), \quad g \in \mathit{Diff}(I') \end{aligned} \tag{3.2}$$

where Diff(I) denotes the group of smooth endomorphisms g of S^1, such that $g(t) = t$ for all $t \in I'$.

All the physical properties of a theory are encoded in the assignment of local algebras $\mathcal{A}(I)$ to every interval $I \subset S^1$ in such a way that the conditions above hold.

In general $U(g)\Omega = \Omega$ is not true for all $g \in \mathrm{Diff}(S^1)$. Otherwise the Reeh–Schlieder theorem would be violated.

Example. The theories, whose chiral algebras are generated by the stress–energy tensor, give rise to Virasoro nets on the circle. The local algebras of such theories may be defined, for example, in terms of the left-moving part of the stress–energy tensor:

$$\mathcal{A}(I) := \{\exp\left(iT_L\left(f\right)\right) \mid f \text{ a real } C^\infty \text{ function with supp} f \subset I\}'' \tag{3.3}$$

We consider the operator algebra $\mathcal{A}(I)$ generated by exponentiated smeared fields localized in the given interval I of S^1 and take its closure in the weak operator topology.

Definition 3.3 (Isomorphic nets). *Two conformal nets $\{\mathcal{A}_1(I)\}_{I\subset S^1}$ and $\{\mathcal{A}_2(I)\}_{I\subset S^1}$ are called isomorphic if exists a unitary operator V from $\mathcal{H}_1$ to $\mathcal{H}_2$, such that V maps Ω_1 to Ω_2 and $V\mathcal{A}_1(I)V^* = \mathcal{A}_2(I)$ for all $I \subset S^1$. Then V intertwines also the Möbius covariant representations of $\{\mathcal{A}_1(I)\}_{I\subset S^1}$ and $\{\mathcal{A}_2(I)\}_{I\subset S^1}$. In this thesis we will work with a weaker notion of isomorphism, where V is not required to preserve the vacuum vector.*

Let us note that the isomorphism class of a given net corresponds to the Borchers class for the generating field, i.e. (by Haag duality, see below) two fields generate isomorphic nets iff they are relatively local [Haag, 1996].

Definition 3.4 (Irreducible net). *We will call a conformal net $\{\mathcal{A}(I)\}_{I\subset S^1}$ irreducible if:*

$$\vee_{I\in S^1}\mathcal{A}(I) = \mathcal{B}(\mathcal{H}) \tag{3.4}$$

We may always consider only irreducible nets, because we can obtain all reducible nets as direct integrals of irreducible nets.

Definition 3.5 (Vacuum representation of a net). $\{\mathcal{H}, U, \mathcal{A}, \Omega\}$ *determine the vacuum sector of the conformal field theory, or the vacuum representation of the conformal net $\{\mathcal{A}(I)\}_{I\subset S^1}$.*

Next, we review a property of the conformal nets, which is crucial for the applicability of the algebraic framework to the analysis of the superselection structure of quantum field theories:

Definition 3.6 (Haag duality). *In the vacuum sector of a conformal field theory, the net $\{\mathcal{A}(I)\}_{I\subset S^1}$ is said to satisfy Haag duality, if for any proper interval $I \subset S^1$ holds:*

$$\mathcal{A}(I)' = \mathcal{A}(I') \tag{3.5}$$

where $I' := (S^1 \setminus I)^0$ is the interior of the complement of I in S^1.

Haag duality was proven to hold for conformal field theories in [Buchholz & Schulz-Mirbach, 1990]. The authors used arguments of Bisognano and Wichmann [Bisognano & Wichmann, 1975] and proved that some regularity conditions, which guarantee Haag duality, are satisfied. Another proof of Haag duality, which is independent on the underlying Wightman theory, is presented in [Gabbiani & Fröhlich, 1993].

Observation. If $\{\mathcal{A}(I)\}_{I\subset S^1}$ is a local conformal net on S^1, then by Haag duality:

$$U(\mathrm{Diff}(I)) \subset \mathcal{A}(I) \tag{3.6}$$

With arguments of Driessler [Driessler, 1975] one can prove the following lemma:

Lemma 3.7. *In the vacuum sector of a conformal field theory the local algebras $\mathcal{A}(I)$, $I \subset S^1$ are factors of type III_1. Moreover —they are hyperfinite type III_1 factors (which is related to the split property, see Section 3.2). Hence, conformal nets are nets of factors.*

3.2 Classification of conformal nets with $c < 1$ on the circle

A naturally interesting problem is the classification of all conformal nets with $c < 1$ on the circle. We start from the following key fact:

Observation. Because of diffeomorphism covariance, all conformal nets on the circle contain a Virasoro subnet. (A subnet would be the smaller net in a net of subfactors, see Section 3.4.) This subnet is generated by the unitary projective representation of the diffeomorphism group of S^1, and its central charge will be considered also a characteristic of the bigger net. Let us remind that for a given conformal net $\{\mathcal{A}(I)\}_{I\subset S^1}$ we have $U(\mathrm{Diff}(I)) \subset \mathcal{A}(I)$. Then the local algebras of the Virasoro subnet of $\{\mathcal{A}(I)\}_{I\subset S^1}$ are defined as:

$$\mathrm{Vir}_c(I) = U(\mathrm{Diff}(I))'' \quad \big(= (3.3)\big) \tag{3.7}$$

One proves that for $c < 1$ this subnet is of finite index and if the conformal net is irreducible then the subnet is also irreducible.

The classification problem for $c < 1$ then becomes to classify the irreducible local finite-index extensions of Virasoro nets for $c < 1$. Moreover, these nets have another very important property — complete rationality:

Definition 3.8 (Complete rationality). *The following set of conditions is referred to as complete rationality:*

1. **split property:** *given a net $\{\mathcal{A}(I)\}_{I\subset S^1}$ and two intervals $I_1 \subset I_2$, $\overline{I}_1 \subset I_2^o$, we say that $\{\mathcal{A}(I)\}_{I\subset S^1}$ has the split property if exists a type I_∞ factor M such that $\mathcal{A}(I_1) \subseteq M \subseteq \mathcal{A}(I_2)$*
2. **strong additivity:** *let I be an interval and p — a point on it, let I_1, I_2 be two connected components of $I \setminus \{p\}$, then we have $\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$*
3. **finiteness of the Jones index for the 2-interval inclusion:** *(measures the size of the tensor category) let us split the circle to four intervals I_1, I_2, I_3, I_4 in a counterclockwise order, then the μ-index of the net $\mathcal{A}$ is defined to be the Jones–Kosaki index (see Section 3.4) of the subfactor $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subset [\mathcal{A}(I_3) \vee \mathcal{A}(I_4)]'$*

Under the assumption of complete rationality one proves that the conformal net has only finitely many inequivalent irreducible representations, such that all of them have finite statistical dimensions (rationality) and that the associated braiding (defined in Section 3.3.3) is non-degenerate. Non-degeneracy of the braiding is often called modularity, or invertibility of the S-matrix (see Section 3.3.3). Which means that the irreducible DHR endomorphisms of the net (which basically correspond to primary fields, see Section 3.3.2) produce a modular tensor category.

Complete rationality is difficult to prove, but it is inherited by a subnet or an extension with finite index. One can show that the Virasoro nets are completely rational for $c < 1$ as they can be obtained as a coset construction of known algebras, possessing this property:

Proposition 3.9. *The Virasoro net on the circle with central charge $c = 1 - \frac{6}{m(m+1)}$ and the coset net arising from the diagonal embedding $SU(2)_{m-1} \in SU(2)_{m-2} \times SU(2)_1$ are isomorphic.*

Corollary. The Virasoro net on the circle with central charge $c < 1$ is completely rational, then so are also its local extensions of finite index.

Note that Virasoro nets for $c<1$ correspond to minimal models and they are indeed in a sense minimal, because they contain no nontrivial subnet.

Further, an important role plays the modular invariant matrix $Z_{\mu\nu}$, associated to every extension of a Virasoro net, which carries a decisive information about the superselection structure of the theory. It is defined as:

$$Z_{\mu\nu} := \langle \alpha^+_\lambda, \alpha^-_\mu \rangle \tag{3.8}$$

The objects $\alpha^\pm_\lambda$ are induced representations of the extension and will be defined in Section 3.4, the bracket among them (see Section 3.3.2) gives information about their equivalence and decomposability. The matrix $Z_{\mu\nu}$ is in the commutant of the unitary representation of $SL(2,\mathbb{Z})$ produced by the braiding and this gives very strong constraints of the possible extensions of the Virasoro net. $Z_{\mu\nu}$ is a modular invariant, in the sense that $ZS = SZ$, $ZT = TZ$ (where S and T are generators of $PSL(2,\mathbb{Z})$). For a given unitary representation of $SL(2,\mathbb{Z})$ the number of modular invariants is always finite and they are classified in [Cappelli *et al.*, 1987]. They are labelled by a pair of Dynkin diagrams. For each modular invariant of this classification the existence and uniqueness of corresponding extensions is checked [Kawahigashi & Longo, 2004] and the following classification result is achieved with the methods of subfactor theory:

Theorem 3.10. *All irreducible Virasoro nets for $c = 1-\frac{6}{m(m+1)}$ with $m = 4n+1$ and $m' = 4n'$ have a local irreducible index 2 extension which is labelled by the pairs of Dynkin diagrams (A_{4n}, D_{2n+2}) and $(D_{2n'+2}, A_{4n'+2})$ and for them the dual canonical endomorphisms (see Definition 3.38) are $\theta = \lambda_{11} + \lambda_{1m}$ and $\theta = \lambda_{11} + \lambda_{m-1,1}$. For the four exceptional and more complicated cases with $m = 11$, 12, 29 and 30 there exists an additional local extension of larger index with the corresponding labelling (A_{10}, E_6), (E_6, A_{12}), (A_{28}, E_8) and (E_8, A_{30}) and corresponding dual canonical endomorphisms $\theta = \lambda_{11} + \lambda_{17}$, $\theta = \lambda_{11} + \lambda_{71}$, $\theta = \lambda_{11} + \lambda_{1,11}\lambda_{1,19} + \lambda_{1,29}$ and $\theta = \lambda_{11} + \lambda_{11,1}\lambda_{19,1} + \lambda_{29,1}$. The index of the first two are $3+\sqrt{3}$ and the last two have an index $\frac{\sqrt{30-6\sqrt{5}}}{2sin(\pi/30)}$. These, together with the Virasoro nets themselves for any $c<1$, give the complete list of local conformal nets on S^1 with $c<1$.*

In the theorem above by local extension of the conformal net $\mathcal{A}$ is meant a conformal net $\mathcal{B}$, such that there is an inclusion of the local algebras $\mathcal{B}(I) \supset \mathcal{A}(I)$, $U(\mathrm{Diff}(I))_{\mathcal{B}} \restriction_{\mathcal{A}} = U(\mathrm{Diff}(I))_{\mathcal{A}}$ and $\mathcal{B}$ is a local net. Note that there can exist also extensions $\mathcal{B}$ for which the requirement of locality is relaxed and they are called non-local extensions.

In fact, for all these nets exists a coset construction in terms of well–studied algebras, which simplifies our investigation a lot. There is one exception, though — the (A_{28}, E_8) extension of a Virasoro net with $m = 29$ and $c = \frac{144}{145}$, which is not easily identifiable with a coset net (it arises as a mirror extension of a coset [Xu, 2007]) and our aim will be to invent an independent approach to study it.

3.3 Superselection sectors

In this section we will discuss the representations of a conformal net and their properties. Our main literature sources are [Gabbiani & Fröhlich, 1993], [Haag, 1996] and [Araki, 1999]. From now on we will call the net simply $\mathcal{A}$ and will speak about representations of the net $\mathcal{A}$.

One can show that there is a bijective correspondence among positive energy representations of the iniversal C^* algebra $C^*(\mathcal{A})$ (constructed as in [Fredenhagen *et al.*, 1989]) and those of the net $\mathcal{A}$.

Let us provide a rigorous definition for a representation of a conformal net:

Definition 3.11 (Representation of a conformal net). *A representation of the conformal net $\mathcal{A}$ on the separable Hilbert space $\mathcal{H}_\pi$ is a family of representations $\pi = \{\pi_I\}_{I \in S^1}$ of the local algebras $\mathcal{A}(I)$ such that the following conditions are satisfied:*

1. **consistence:** *if $I \subseteq J$ then $\pi_J \restriction_{\mathcal{A}(I)} = \pi_I$*
2. **covariance:** *under projective unitary representation of* $\mathrm{Diff}(S^1)$
3. **positive spectrum:** *the spectrum of the infinitesimal generator of rotations on $\mathcal{H}_\pi$ is positive*

Such representations always respect the local structure of $\mathcal{A}$.

Let us remind that by Definition 3.5 the identity representation of the net is called a vacuum representation (or vacuum sector) and will be denoted by π_0.

We will be interested in classification of irreducible representations: the building blocks of all other representations:

Definition 3.12 (Irreducible representation). *A representation π of the conformal net $\mathcal{A}$ is called irreducible if the von Neumann algebra*

$$\pi(\mathcal{A})' := \{\pi_I(\mathcal{A}(I)),\, I \subset S^1\}' \tag{3.9}$$

is equal to $\mathbb{C} \cdot 1_{\mathcal{H}_\pi}$.

Not all of the irreducible representations really differ from each other, i.e. some of them describe the same physics and will be called equivalent:

Definition 3.13 (Equivalent representations). *Two representations π_1 and π_2 are unitarily equivalent if there exists an unitary operator $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that:*

$$\pi_{2I}(\cdot)U = U\pi_{1I}(\cdot), \quad \forall I \subset S^1 \tag{3.10}$$

Definition 3.14 (Sector). *An equivalence class of representations of a conformal net $\mathcal{A}$ is called a sector. The sector associated to the representation π will be denoted by $[\pi]$.*

The representations of a conformal net have a remarkable property, described by the following lemma [Buchholz *et al.*, 1988]:

Lemma 3.15. *Any representation π of a conformal net $\mathcal{A}$ is locally unitarily equivalent to the vacuum representation:*

$$\pi \restriction_{\mathcal{A}(I)} \cong \pi_0 \restriction_{\mathcal{A}(I)}, \quad \forall I \subset S^1 \tag{3.11}$$

As we will see later, this lemma allows us to describe conveniently all representations of a conformal net in the same space — the Hilbert space $\mathcal{H}$ on which the vacuum representation is defined. Since $\mathcal{H}$ and $\mathcal{H}_\pi$ are separable Hilbert spaces for every representation, there exists a unitary V such that $V : \mathcal{H}_\pi \to \mathcal{H}$. Hence, instead of π acting on $\mathcal{H}_\pi$ we may consider the equivalent representation:

$$\pi_{\mathcal{H}}(A) = V\pi(A)V^*, \quad \forall A \in \mathcal{A} \tag{3.12}$$

acting on $\mathcal{H}$.

In our analysis a central role will be played by the localized representations:

Definition 3.16 (Localized representation). *A representation π of the conformal net $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$ is called localized in the interval I if:*

$$\pi \restriction_{\mathcal{A}(I')} = \pi_0 \restriction_{\mathcal{A}(I')} \tag{3.13}$$

The following lemma tells us that localized representations are very wide–spread and present in every sector:

Lemma 3.17. *In every sector $[\pi]$ of representations of $\mathcal{A}$ for each interval $I \subset S^1$ exists at least one representation π localized in I.*

Indeed, since by Lemma 3.15 $\pi \restriction_{\mathcal{A}(I)} \cong \pi_0 \restriction_{\mathcal{A}(I)}$, $\forall I \subset S^1$, in particular for every commutant I', we can choose one interval I and a bijective isometry $U : \mathcal{H}_\pi \to \mathcal{H}$ such that:

$$\rho_I(A) = U\pi(A)U^* = \pi_0(A), \quad \forall A \in \mathcal{A}(I') \tag{3.14}$$

Then $\rho_I(A) = U\pi(A)U^*$, $\forall A \in \mathcal{A}$ is a representation on $\mathcal{H}$ localized in I. In this way we can construct a representation localized in every interval $I \subset S^1$.

The interpretation is that ρ_I corresponds to the operation of creating some charge in I (excitation of the vacuum).

Observation. One can show a number of useful properties of ρ_I:

- ρ_I is a von Neumann algebra endomorphism of $\mathcal{A}(I)$ and a C^* algebra endomorphism of $C^*(\mathcal{A})$
- ρ_I is a localized endomorphism
- ρ_I is a transportable endomorphism
- the set of all localized in an interval I and transportable endomorphisms of the conformal net $\mathcal{A}$ forms a semi–group and its equivalence classes are in one–to–one correspondence with the equivalence classes of representations of the conformal net

This means that, in analogy to DHR theory [Doplicher *et al.*, 1969*a*], [Doplicher *et al.*, 1969*b*], we can "translate" our study of superselection structure in the language of endomorphisms — this allows us to observe a lot of structure and easy manipulations. Here we give the definitions of localized and of transportable endomorphisms:

Definition 3.18 (Localized endomorphism). *An endomorphism ρ_I of the conformal net $\mathcal{A}$ is called localized in the interval $I \in S^1$ if it acts trivially at space–like distances from I on S:*

$$\rho_I(a) = a, \quad \forall a \in \mathcal{A}(I'), \quad \rho_I\left(\mathcal{A}\left(I\right)\right) \subset \mathcal{A}(I) \tag{3.15}$$

Definition 3.19 (Transportable endomorphism). *A localized endomorphism ρ_I of the conformal net $\mathcal{A}$ is called transportable if for all $J \subset S^1$ there are unitary operators $U_{\rho_I;I,J}$ called charge transporters such that $\rho_J = Ad(U_{\rho_I;I,J}) \circ \rho_I$ is localized in J. With $\Delta_{\mathcal{A}}(I)$ we denote the set of all localized in $I \subset S^1$ transportable endomorphisms of $\mathcal{A}$.*

3.3.1 DHR representations

DHR representations (DHR is an abbreviation taking the first letters of Doplicher, Haag and Roberts) are a large class of representations in 4D relativistic quantum theories selected by the following criterion:

Definition 3.20 (DHR criterion). *We are interested in irreducible representations π of the net $\mathcal{A}$ such that:*

1. **local excitations of vacuum:** *the representation differs from the vacuum representation only on some open double cone region $\mathcal{O}$, i.e.:*

   $$\pi \restriction \mathcal{A}(\mathcal{O}') \simeq \pi_0 \restriction \mathcal{A}(\mathcal{O}') \tag{3.16}$$

   interpretation: if the charged states are observed only in $\mathcal{O}$, they cannot be distinguished from a state carrying a zero charge — i.e. the effect of the charge dies out at a large enough distance, in $\mathcal{O}'$ no effect can be propagated from $\mathcal{O}$.

2. **transportable excitations:** *If (3.16) holds for $\mathcal{O}$ it should be valid also for the transported domain $\mathcal{O} + a$.*

3. **Haag duality:**
   interpretation: $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')'$ by locality anyway, then $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}')'$ indicates some maximality of the local algebras.

It was proven in [Doplicher *et al.*, 1969*a*] and [Doplicher *et al.*, 1969*b*] that the representations, which obey this criterion in four or higher dimensions, form a tensor category equivalent to the tensor category of representations of a compact group. Moreover, they were able to recover exactly this compact group by abstract duality theory and hence to determine completely the superselection structure of DHR representations.

Actually, the DHR criterion appears to be very restrictive in 4D relativistic quantum theory, because it excludes long–range interactions and thus electromagnetic charges because of the Gauss theorem. It also excludes topological charges — charges accompanied by correlation effects which are discernable at arbitrary large distances. The second problem is attacked by Buchholz–Fredenhagen analysis for theories without massless particles [Buchholz & Fredenhagen, 1982]. The idea there is to use cone instead of double cone localization and it is believed that this theory describes topological charges.

In conformal field theory, though, every sector is a DHR sector. This is because every representation is locally equivalent to the vacuum representation by Lemma 3.15, because every conformal field theory is Haag dual and because the representations are transportable due to Haag duality and their localization properties. From solidarity, we will call also in our analysis the localized and transportable endomorphisms DHR endomorphisms.

3.3.2 Endomorphism calculus

In analogy to DHR theory, one can define operations among the DHR endomorphisms ρ of a given conformal net $\mathcal{A}$, similar to operations in the category of representations of a compact group. This subject is well-treated for example in [Gabbiani & Fröhlich, 1993] and [Rehren, Vorlesung Göttingen, WiSe 1997/98]. Before we do that, let us define two related important objects:

Definition 3.21 (Intertwiner). *An intertwiner between two endomorphisms ρ_1 and ρ_2 is an element $U \in \mathcal{A}$ such that:*

$$U \cdot \rho_1(a) = \rho_2(a) \cdot U, \quad \forall a \in \mathcal{A} \tag{3.17}$$

Definition 3.22 (Projection). *A projection of an endomorphism is an intertwiner $E : \rho \to \rho$ such that $E \in \mathcal{A} \cap \rho(\mathcal{A})'$. Then exists an endomorphism ρ_1 and an intertwiner $W : \rho_1 \to \rho$, $W \in \mathcal{A}$ such that $E = WW^*$, $W^*W = 1$ and we will call later in this section ρ_1 a submorphism of ρ.*

Definition 3.23 (Tensor product of intertwiners). *Let us have the two intertwiners $T_1 : \rho_1 \to \sigma_1$ and $T_2 : \rho_2 \to \sigma_2$. Then their tensor product is defined as the intertwiner $T_1 \times T_2 : \rho_1 \circ \rho_2 \to \sigma_1 \circ \sigma_2$. One finds that $T_1 \times T_2 = T_1 \cdot \rho_1(T_2) = \sigma_1(T_2) \cdot T_1$.*

Remark. The operator W is an isometry. Its existence is related to the type III_1 property of the net, which says that there are no finite traces, no minimal projections and that every two projections e_1 and e_2 can be connected:

$$e_1 = \widetilde{W}\widetilde{W}^*, \quad e_2 = \widetilde{W}^*\widetilde{W} \tag{3.18}$$

where $\widetilde{W} \in \mathcal{A}$ is an isometry.

Then one can define the following operations among DHR endomorphisms ρ of the conformal net $\mathcal{A}$:

1. ***equivalence of endomorphisms:*** if two representations are equivalent and ρ_1 and ρ_2 are their corresponding endomorphisms defined as in Section 3.3, then it follows by Haag duality that there exists an unitary intertwiner $U \in \mathcal{A}$ such that $U \cdot \rho_1(a) = \rho_2(a) \cdot U$, $\forall a \in \mathcal{A}$. It gives rise to an equivalence relation, the equivalence classes of endomorphisms $[\rho]$ are called also sectors and are in one–to–one correspondence with the equivalence classes of the related representations. If two sectors are equivalent, we will write $[\rho_1] \sim [\rho_2]$

2. ***product:*** we define a product of two representations $\pi_1 \times \pi_2$ through composition of the corresponding endomorphisms $\rho_1 \circ \rho_2$

3. ***direct sum:*** in order to define the direct sum of two endomorphisms, one chooses an arbitrary projection $E \in \mathcal{A}$ and two isometries W_1, $W_2 \in \mathcal{A}$ such that:

$$W_1 W_1^* = E, \quad W_2 W_2^* = 1 - E, \quad W_i^* W_j = \delta_{ij} \tag{3.19}$$

We construct:

$$\rho(a) := W_1 \cdot \rho_1(a) \cdot W_1^* + W_2 \cdot \rho_2(a) \cdot W_2^*, \quad \forall a \in \mathcal{A} \tag{3.20}$$

$\rho(a)$ is also a localized and transportable endomorphism of $\mathcal{A}$. Then we define:

$$\rho =: \rho_1 \oplus \rho_2 \tag{3.21}$$

This operation is well defined also among sectors $[\rho] = [\rho_1] \oplus [\rho_2]$. Moreover it holds for the related representations:

$$[\pi] \simeq [\pi_1] \oplus [\pi_2] \tag{3.22}$$

We can generalize the construction above to adding an arbitrary number of representations. Let us consider the set of endomorphisms ρ_i, $i = 1, 2, ..., n$. Since we have an infinite algebra $\mathcal{A}$, we can take a set of isometries $W_i \in \mathcal{A}$, $i = 1, 2, ..., n$ forming a Cuntz algebra:

$$W_i^* W_j = \delta_{ij} 1, \quad \sum_{i=1}^{n} W_i W_i^* = 1 \tag{3.23}$$

Then we define the direct sum of ρ_i as:

$$\rho(a) = \oplus_{i=1}^{n} W_i \cdot \rho_i(a) \cdot W_i^*, \quad \forall a \in \mathcal{A} \tag{3.24}$$

4. ***submorphism:*** representations may be reducible. Suppose that π_1 is a subrepresentation of π, then on the level of endomorphisms we will write $\rho_1 \prec \rho$ and we will call ρ_1 a submorphism of ρ. In such case there exists a projection:

$$E : \rho \to \rho, \quad E \cdot \rho(a) = \rho(a) \cdot E \quad \forall a \in \mathcal{A} \tag{3.25}$$

We choose again two isometries W_1 and $W_2 \in \mathcal{A}$ satisfying relations (3.19) and such that:

$$\rho_1(a) = W_1^* \cdot \rho(a) \cdot W_1, \quad E = W_1 W_1^* \tag{3.26}$$

Let us compose $\rho_2(a) = W_2^* \cdot \rho(a) \cdot W_2$, s.t $W_2 W_2^* = 1 - E$. One checks that in case of compact groups and von Neumann algebras always holds that $\rho \simeq \rho_1 \oplus \rho_2$, i.e. the representation is fully decomposable.

In analogy, we define a subsector $[\rho_1] \prec [\rho]$. A related concept is:

Definition 3.24 (Irreducible endomorphism). *The endomorphism λ of $\mathcal{A}$ is called irreducible if it holds:*

$$\lambda(\mathcal{A})' \cap \mathcal{A} = \mathbb{C}1 \tag{3.27}$$

Such an endomorphism does not possess subobjects.

5. ***conjugated endomorphism:***

Definition 3.25 (Conjugated endomorphism). *ρ and $\bar{\rho}$ are conjugated to each other if:*

- **irreducible case:** $1 \prec \rho \circ \bar{\rho}$ *and* $1 \prec \bar{\rho} \circ \rho$
- **reducible case:** *there exist two operators R and $\bar{R}$ such that:*

$$\begin{aligned} &R:\, 1 \to \rho \circ \bar{\rho}, \quad R^* \circ R = 1 \\ &\bar{R}:\, 1 \to \bar{\rho} \circ \rho, \quad \bar{R}^* \circ \bar{R} = 1 \end{aligned}$$

and satisfying certain regularity conditions:

$$\begin{aligned} &(R^* \times 1_\rho) \circ (1_\rho \times \bar{R}):\, \rho \to \rho \quad \textit{is invertible} \\ &(\bar{R}^* \times 1_{\bar{\rho}}) \circ (1_{\bar{\rho}} \times R):\, \bar{\rho} \to \bar{\rho} \quad \textit{is invertible} \end{aligned} \tag{3.28}$$

where 1_ρ is an identity intertwiner for ρ.

The existence of conjugated representations is guaranteed in the conformal case.

We can also speak about a conjugated sector.

6. ***statistical dimension of a representation:***

Definition 3.26 (Statistical dimension). *If the interval of localization of a DHR endomorphism ρ is I, then its statistical dimension $d(\rho)$ is defined just as the Jones-Kosaki index $[\mathcal{A}(I) : \rho(\mathcal{A}(I))]$ (see Section 3.4).*

The statistical dimension obeys the following properties:

- $d(\rho) \geq 1$, $d(\mathrm{id}) = 1$
- $d(\rho \oplus \sigma) = d(\rho) + d(\sigma)$
- $d(\rho \circ \sigma) = d(\rho) \cdot d(\sigma)$
- $d(\overline{\rho}) = d(\rho)$
- $d(\rho)$ is an invariant for a sector

Statistical dimensions are not integers in general, but they are integers for compact groups.

Let ρ and σ be endomorphisms of the conformal net $\mathcal{A}$ and let us denote with $\mathrm{Hom}(\rho, \sigma)$ the space of intertwiners among ρ and σ. It is a vector space, whose dimension we will denote as:

$$\langle \rho, \sigma \rangle := \dim \mathrm{Hom}(\rho, \sigma) = \dim \mathrm{Hom}(\sigma, \rho) \tag{3.29}$$

In case that we want to emphasize that the intertwiners must belong to some space X, we will write also $\mathrm{Hom}_X(\rho, \sigma)$ and $\langle \rho, \sigma \rangle_X$. The bracket $\langle \cdot, \cdot \rangle$ is linear in its two arguments, and it carries information about subobjects and multiplicities. For example, it is easy to see that for two irreducible endomorphisms ρ and σ it holds that $\langle \rho, \sigma \rangle = 1$ if $\rho \sim \sigma$ and $\langle \rho, \sigma \rangle = 0$ if ρ and σ are not equivalent. The following property is quite useful for calculations:

Definition 3.27 (Frobenius reciprocity). *If μ and λ have conjugates, then the following relations are satisfied:*

$$\langle \lambda \circ \mu, \nu \rangle = \langle \lambda, \nu \circ \bar{\mu} \rangle = \langle \mu, \bar{\lambda} \circ \nu \rangle \tag{3.30}$$

Definition 3.28 (Fusion rules). *The decomposition of the tensor product of two irreducible DHR endomorphisms in the basis of all irreducible DHR endomorphisms of a conformal net $\mathcal{A}$ is called fusion rules:*

$$\rho_i \circ \rho_j \simeq \oplus N_{ij}^k \rho_k \tag{3.31}$$

The fusion rules can be expressed in terms of the bracket $\langle \cdot, \cdot \rangle$ as:

$$\langle \rho_i \circ \rho_j, \rho_k \rangle = N_{ij}^k \tag{3.32}$$

3.3.3 Braid statistics operators

Let us again consider the conformal net $\mathcal{A}$ and two of its DHR endomorphisms ρ_1 and ρ_2. Due to locality, there is a canonically defined (see below) unitary operator $\varepsilon(\rho_1, \rho_2) : \rho_1 \circ \rho_2 \to \rho_2 \circ \rho_1$, $\varepsilon(\rho_1, \rho_2) \in \mathcal{A}$ such that:

$$\varepsilon(\rho_1, \rho_2) \cdot \rho_1 \circ \rho_2(a) = \rho_2 \circ \rho_1(a) \cdot \varepsilon(\rho_1, \rho_2) \quad \forall a \in \mathcal{A} \tag{3.33}$$

In $d > 2$ $\rho_1 \circ \rho_2$ and $\rho_2 \circ \rho_1$ are intertwined by a unitary operator which gives rise to a representation of the permutation group.

In $d = 1, 2$ $\rho_1 \circ \rho_2$ and $\rho_2 \circ \rho_1$ still belong to the same sector, however the statistics operator does not square to one due to the topology there, but rather defines a representation of the Artin braid group and we speak about braid group statistics.

Definition 3.29 (Braid group). *Let B_n denote the braid group on n strands with generators $\sigma_1, ..., \sigma_{n-1}$ and relations:*

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, ..., n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \ \textit{if}\ |i - j| > 1, \quad i, j = 1, ..., n-1 \end{aligned} \tag{3.34}$$

In the category of DHR endomorphisms the braiding relations arise when we define $\sigma_n := \rho^{n-1}(\varepsilon)$, $\sigma_1 = \varepsilon$, where ρ is a DHR endomorphism and ε is the statistics operator, intertwining $\rho \circ \rho$ and $\rho \circ \rho$, which we will define constructively below.

The braiding on a rational tensor category produces two finite–dimensional scalar–valued matrices S and T, which are generators of $PSL(2,\mathbb{Z})$. The T-matrix is always unitary, but the S-matrix can be non-invertible in general. When we have this invertibility property, in addition to rationality, we say that the tensor category is modular, since we have a unitary representation of the modular group $SL(2,\mathbb{Z})$.

The following lemma tells us that certain couples of endomorphisms do commute:

Lemma 3.30. *Let I_1, $I_2 \subset S^1$ and $I_1 \cap I_2 = 0$. Let $\lambda_{I_1} \in \Delta_{\mathcal{A}}(I_1)$ and $\lambda_{I_2} \in \Delta_{\mathcal{A}}(I_2)$. Then λ_{I_1} and λ_{I_2} commute, i.e $\lambda_{I_1} \circ \lambda_{I_2} = \lambda_{I_2} \circ \lambda_{I_1}$.*

Assume now that we have two endomorphisms, λ_I and μ_I, which are localized in the same interval $I \subset S^1$ and are transportable, i.e $\lambda_I, \mu_I \in \Delta_{\mathcal{A}}(I)$. Then, in general, they will not commute, but $\lambda_I \circ \mu_I$ and $\mu_I \circ \lambda_I$ are intertwined by a unitary operator. To obtain this operator, choose I_1, $I_2 \subset S^1$ such that $I_1 \cap I_2 = \emptyset$ and consider the unitary operators $U_1 := U_{\lambda_I;I,I_1}$ and $U_2 := U_{\mu_I;I,I_2}$ such that $\lambda_{I_1} = \mathrm{Ad}(U_1) \circ \lambda_I \in \Delta_{\mathcal{A}}(I_1)$ and $\mu_{I_2} = \mathrm{Ad}(U_2) \circ \mu_I \in \Delta_{\mathcal{A}}(I_2)$. Then we construct:

$$\varepsilon^{I_1,I_2}_{U_1,U_2}(\lambda_I, \mu_I) = \mu_I(U_1^*)U_2^*U_1\lambda_I(U_2) \tag{3.35}$$

Let us now consider the point at infinity $z \in S^1$ and let us denote $I_2 >_z I_1$ if the intervals I_1 and I_2 are disjoint, if I_1 lies clockwise to I_2 relative to the point z and if the closure of neither of them contains z.

Lemma 3.31. *The operators $\varepsilon^{I_1,I_2}_{U_1,U_2}(\lambda_I, \mu_I)$ do not depend on the special choice of U_1 and U_2. Moreover, varying I_1 and I_2 such that the relation "$>_z$" is preserved, $\varepsilon^{I_1,I_2}_{U_1,U_2}(\lambda_I, \mu_I)$ remains constant.*

Let us then choose $I_1 = I$ and $U_1 = 1$. Let us set $U_{\mu_I,+} := U_2$ whenever $I_2 >_z I_1$ and $U_{\mu_I,-} := U_2$ whenever $I_1 >_z I_2$. Then we show that there are only two operators ε, which are in general different:

$$\varepsilon^{\pm}(\lambda_I, \mu_I) = U^*_{\mu_I,\pm}\lambda_I(U_{\mu_I,\pm}) \tag{3.36}$$

We call $\varepsilon^{\pm}(\lambda_I, \mu_I)$ statistics operators.

Observation. The statistics operators $\varepsilon^{\pm}(\lambda_I, \mu_I)$ have the following properties:

- they yield the following commutation law:

$$\varepsilon^{\pm}(\lambda_I, \mu_I) \cdot \lambda_I \circ \mu_I(a) = \mu_I \circ \lambda_I(a) \cdot \varepsilon^{\pm}(\lambda_I, \mu_I), \quad a \in \mathcal{A}, \quad \lambda_I,\, \mu_I \in \Delta_{\mathcal{A}}(I) \tag{3.37}$$

- they belong to the local algebra $\mathcal{A}(I)$

- the two statistics operators are related via conjugation:

$$\varepsilon^{+}(\lambda_I, \mu_I) = (\varepsilon^{-}(\mu_I, \lambda_I))^{*} \tag{3.38}$$

- we have the following composition laws:

$$\begin{aligned} \varepsilon^{\pm}(\lambda_I \circ \mu_I, \nu_I) &= \varepsilon^{\pm}(\lambda_I, \nu_I)\lambda_I(\varepsilon^{\pm}(\mu_I, \nu_I)) \\ \varepsilon^{\pm}(\lambda_I, \mu_I \circ \nu_I) &= \mu_I(\varepsilon^{\pm}(\lambda_I, \nu_I))\varepsilon^{\pm}(\lambda_I, \mu_I) \end{aligned} \tag{3.39}$$

- if we assume strong additivity (i.e. irrelevance of points, $A(I) = A(I_1) \vee A(I_2)$ whenever the intervals I_1 and I_2 are obtained from I by removing one single point) we have the naturality equation for λ_I, μ_I, $\rho_I \in \Delta_{\mathcal{A}}(I)$ and $T \in \mathrm{Hom}_{\mathcal{A}(I)}(\lambda_I, \mu_I,)$:

$$\begin{aligned} \rho(T)\varepsilon^{\pm}(\lambda_I, \rho_I) &= \varepsilon^{\pm}(\mu_I, \rho_I)T \\ T\varepsilon^{\pm}(\rho_I, \lambda_I) &= \varepsilon^{\pm}(\rho_I, \mu_I)\rho(T) \end{aligned} \tag{3.40}$$

- for λ_I, μ_I, $\nu_I, \rho_I \in \Delta_{\mathcal{A}}(I)$ and $S \in \mathrm{Hom}_{\mathcal{A}(I)}(\lambda_I \circ \mu_I, \nu_I)$ we have the braiding fusion equations:

$$\begin{aligned} \rho(S)\varepsilon^{\pm}(\lambda_I, \rho_I)\lambda(\varepsilon^{\pm}(\mu_I, \rho_I)) &= \varepsilon^{\pm}(\nu_I, \rho_I)S \\ S\lambda_I(\varepsilon^{\pm}(\rho_I, \mu_I))\varepsilon^{\pm}(\rho_I, \lambda_I) &= \varepsilon^{\pm}(\rho_I, \nu_I)\rho(S) \end{aligned} \tag{3.41}$$

- we have also the Yang–Baxter equation for λ_I, $\mu_I, \nu \in \Delta_{\mathcal{A}}(I)$:

$$\nu(\varepsilon^{\pm}(\lambda_I, \mu_I))\varepsilon^{\pm}(\lambda_I, \nu_I)\lambda_I(\varepsilon^{\pm}(\mu_I, \nu_I)) = \varepsilon^{\pm}(\mu_I, \nu_I)\mu(\varepsilon^{\pm}(\lambda_I, \nu_I))\varepsilon^{\pm}(\lambda_I, \nu_I) \tag{3.42}$$

A similar analysis can be done for the case when $I_1 \cap I_2 \neq 0$, $I_1 \neq I_2$, $\overline{I_1 \cup I_2} \neq S^1$.

The braid group statistics issue in conformal field theories is treated in [Böckenhauer & Evans, 1998], [Fredenhagen *et al.*, 1992], [Fredenhagen *et al.*, 1989].

3.3.4 Superselection structure of the conformal nets for $c < 1$ on the circle

From Section 3.3 we know that conformal nets on the circle with associated central charge smaller than one are completely rational, so they possess finitely many inequivalent superselection sectors with finite statistical dimensions.

The representations of the Virasoro subnet with central charge $c < 1$ are in a bijective correspondence with those of a Virasoro algebra with the same central charge. For every admissible value of the central charge c there exists exactly one irreducible (unitary, positive energy) representation U of the Virasoro algebra (projective unitary representation of $\mathrm{Diff}(S^1)$) such that the lowest eigenvalue of the conformal hamiltonian L_0 is 0 (lowest weight) — this is the vacuum representation with central charge c. Any other unitary irreducible positive energy representation with a given central charge c is determined up to unitary invariance by its spin h, which is the lowest eigenvalue of the conformal hamiltonian. The set of all possible spins for a given central charge $c(m) = 1 - \frac{6}{m(m+1)}$, $(m = 2, 3, 4, ...)$ is the following:

$$\begin{aligned} h_{R,S}(m) &= \frac{\big[(m+1)R - mS\big]^2 - 1}{4m(m+1)}, \quad 1 \leq R \leq m-1,\, 1 \leq S \leq m \\ h_{R,S}(m) &= h_{m-R,m+1-S}(m) \end{aligned} \tag{3.43}$$

We will denote the corresponding DHR endomorphisms with λ_{RS}. Their fusion rules are:

$$\lambda_{RS} \circ \lambda_{R'S'} = \oplus_{p=|R-R'|+1,\, p+R+R':\text{odd}}^{\min(R+R'-1,\, 2m-R-R'-1)} \oplus_{q=|S-S'|+1,\, q+S+S':\text{odd}}^{\min(S+S'-1,\, 2(m+1)-S-S'-1)} \lambda_{pq} \tag{3.44}$$

Knowing the sectors of the Virasoro subnet, we can recover the sectors of the extension using the machinery of subfactor theory, which we will discuss in the next section. The sectors will be computed explicitly in the next chapter.

3.4 Elements of subfactor theory

The main tool in our study of superselection structure of conformal nets with $c < 1$ will be subfactor theory and in this section we will review some basic definitions and theorems of this theory. This section is based mainly on [Longo & Rehren, 1995] and [Böckenhauer & Evans, 1998].

Definition 3.32 (Factor). *We recall, that a von Neumann algebra is a weakly closed subalgebra $M \subset \mathcal{B}(\mathcal{H})$ of the algebra of bounded operators on some Hilbert space $\mathcal{H}$. It is called a factor if its center is trivial: $M' \cap M = \mathbb{C}1$.*

Definition 3.33 (Subfactor). *An inclusion $N \subset M$ of factors with common unit is called a subfactor. A subfactor is called irreducible if the relative commutant is trivial $N' \cap M = \mathbb{C}1$.*

We are interested in pairs of theories, in which one extends the other in a local way., i.e for every space–time region one has the inclusion of the corresponding local algebras.

Definition 3.34 (Nets of subfactors on S^1). *A net of subfactors on S^1 consists of two nets of factors $\mathcal{N}$ and $\mathcal{M}$ such that for every $I \in S^1$ $\mathcal{N}(I) \subset \mathcal{M}(I)$ is an inclusion of factors. A net $\mathcal{M}$ is called standard if there is a vector $\Omega \in \mathcal{H}$ which is cyclic and separating for every $\mathcal{M}(I)$. The net of subfactors $\mathcal{N} \subset \mathcal{M}$ is called standard if $\mathcal{M}$ is standard (on $\mathcal{H}$) and $\mathcal{N}$ is standard (on $\mathcal{H}_0 \subset \mathcal{H}$) with the same cycling and separating vector $\Omega \in \mathcal{H}_0$.*

Important concepts in subfactor theory are the index of the subfactor and the canonical endomorphism, which we clarify in the following set of definitions.

Definition 3.35 (Conditional expectation). *Conditional expectation $\varepsilon : M \to N$ is a completely positive normalized map with the property:*

$$\varepsilon(n^* m n) = n^* \varepsilon(m) n \quad (n \in N,\, m \in M) \tag{3.45}$$

A conditional expectation is called normal if it is weakly continuous. The set of faithful normal conditional expectations is denoted by $C(M,N)$. An arbitrary pair $N \subset M$ may not possess any conditional expectation at all, i.e. $C(M,N)$ may be empty. If there is any normal conditional expectation for an irreducible inclusion, then it is unique and faithful.

Definition 3.36 (Jones-Kosaki index). *The Jones-Kosaki index is defined as:*

$$Ind(\varepsilon) := \varepsilon^{-1}(1) \in [1, \infty] \tag{3.46}$$

It is ∞ when the unity is not in the domain of ε^{-1}. For reducible subfactors:

$$Ind(\varepsilon_0) = inf_{\varepsilon} Ind(\varepsilon) =: [M : N] \tag{3.47}$$

The Jones-Kosaki index is constant in a directed standard net of subfactors with a standard conditional expectation.

Definition 3.37 (Modular conjugation). *Let us consider the von Neumann algebra M acting on a Hilbert space $\mathcal{H}$ and let $\Omega \subset \mathcal{H}$ be a cyclic and separating vector. Then exists a densely defined, unbounded and anti–linear operator $S : m\Omega \to m^*\Omega$ with the polar decomposition $S = J\Delta^{\frac{1}{2}}$, $\Delta = S^*S$, $J^2 = 1$. $\Delta^{\frac{1}{2}}$ is a positive operator and J is an antiunitary operator, which we call a modular conjugation operator with the property:*

$$JMJ = M' \tag{3.48}$$

Definition 3.38 (Canonical endomorphism). *Let $N \subset M$ be an infinite subfactor on a separable Hilbert space $\mathcal{H}$. Then there is a vector $\Xi \in \mathcal{H}$ which is cyclic and separating for both M and N. Let J_M and J_N be the modular conjugations of M and N with respect to Ξ. The endomorphism*

$$\gamma = Ad(J_N J_M)|_M \in End(M) \tag{3.49}$$

satisfies $\gamma(M) \subset N$ and is called a canonical endomorphism from M into N and is unique up to conjugation by a unitary in N. (The freedom of choice comes from the freedom of choice of Ξ.) The restriction $\theta = \gamma|_N$ is called a dual canonical endomorphism.

Proposition 3.39. *Let $\mathcal{N} \subset \mathcal{M}$ be a standard net of subfactors with a standard conditional expectation. For every $I \subset S^1$ there is an endomorphism γ of the C^* algebras $\mathcal{M}$ into $\mathcal{N}$ such that $\gamma|_{\mathcal{M}(J)}$ is a canonical endomorphism of $\mathcal{M}(J)$ into $\mathcal{N}(J)$ whenever $I \subset J$.*

In the usual representation theory we have the machinery of induction and restriction of representations for a group G and its subgroup H. For the representations of conformal nets of subfactors on the circle a similar theory was established and we will review some basics of this theory in the rest of this subsection.

Definition 3.40 (α-induction). *Let $\mathcal{N} \subset \mathcal{M}$ be a net of subfactors. For $\lambda \in \Delta_{\mathcal{N}}(I)$ we define the α–induced endomorphism $\alpha_\lambda \in End(\mathcal{M})$ by:*

$$\alpha_\lambda^\pm = \gamma^{-1} \circ Ad(\varepsilon^\pm(\lambda, \theta)) \circ \lambda \circ \gamma \tag{3.50}$$

(In order such a definition to make sense, one has to establish that $Ad(\varepsilon^\pm(\lambda, \theta)) \circ \lambda \circ \gamma$ is in the range of γ.)

Observation. $\alpha^\pm$-induction has the following properties:

- $\alpha_\lambda^\pm$ is an extension of λ, i.e.:

$$\alpha_\lambda(n) = \lambda(n), \quad n \in \mathcal{N} \tag{3.51}$$

- α-induction respects sector structure, i.e. for some $\lambda, \mu \in \Delta_{\mathcal{N}}(I)$:

$$[\lambda] = [\mu] \to \left[\alpha_\lambda^\pm\right] = \left[\alpha_\mu^\pm\right] \tag{3.52}$$

  This means that:

$$t \cdot \lambda(n) = \mu(n) \cdot t \quad \longrightarrow \quad t \in \mathrm{Hom}_{\mathcal{M}}(\alpha_\lambda^\pm, \alpha_\mu^\pm) \tag{3.53}$$

- $\alpha^\pm$ is a homomorphic map, i.e. for any μ, $\nu \in \Delta_{\mathcal{N}}(I)$ we have:

$$\alpha_{\mu\circ\nu}^\pm = \alpha_\mu^\pm \circ \alpha_\nu^\pm \tag{3.54}$$

- α-induction preserves also sums of sectors, i.e for any $\lambda, \lambda_1, \lambda_2 \in \Delta_{\mathcal{N}}(I)$:

$$[\lambda] = [\lambda_1] \oplus [\lambda_2] \to \left[\alpha_\lambda^\pm\right] = \left[\alpha_{\lambda_1}^\pm\right] \oplus \left[\alpha_{\lambda_2}^\pm\right] \tag{3.55}$$

- α-induction preserves sector conjugation, i.e for $\lambda, \bar{\lambda} \in \Delta_{\mathcal{N}}(I)$:

$$[\alpha_{\bar{\lambda}}] = [\overline{\alpha}_\lambda] \tag{3.56}$$

- α-induction preserves the statistical dimension of the sector:

$$d_{[\alpha_\lambda^\pm]} = d_{[\lambda]} \tag{3.57}$$

- α-induction respects the braiding, i.e. for any μ, $\nu \in \Delta_{\mathcal{N}}(I)$ we have:

$$\alpha_\mu^\pm \circ \alpha_\nu^\pm = \mathrm{Ad}(\varepsilon^\pm(\nu, \mu))\alpha_\nu^\pm \circ \alpha_\mu^\pm \tag{3.58}$$

- for λ, $\mu, \nu \in \Delta_{\mathcal{N}}(I)$ we have the Yang–Baxter equation:

$$\alpha_\nu^\pm(\varepsilon^\pm(\lambda, \mu))\varepsilon^\pm(\lambda, \nu)\alpha_\lambda^\pm(\varepsilon^\pm(\mu, \nu)) = \varepsilon^\pm(\mu, \nu)\alpha_\mu^\pm(\varepsilon^\pm(\lambda, \nu))\varepsilon^\pm(\lambda, \mu) \tag{3.59}$$

- $\alpha_\mu^\pm$ is in general not localized (it is localized if and only if the monodromy $\varepsilon(\mu,\theta)\varepsilon(\theta,\mu)$ is trivial)

Theorem 3.41. *For λ, $\mu \in \Delta_{\mathcal{N}}(I)$ we have:*

$$\langle \alpha_\lambda^\pm, \alpha_\mu^\pm \rangle_{\mathcal{M}(I)} = \langle \theta \circ \lambda, \mu \rangle_{\mathcal{N}(I)} \tag{3.60}$$

We will use later the following proposition:

Proposition 3.42. *For $\lambda \in \Delta_{\mathcal{N}}(I)$ the following are equivalent:*

1. $[\alpha_\lambda^+] = [\alpha_\lambda^-]$
2. $\alpha_\lambda^+ = \alpha_\lambda^-$
3. *the monodromy is trivial:* $\varepsilon(\lambda,\theta)\varepsilon(\theta,\lambda) = 1$

Definition 3.43 (σ-restriction). *For $\beta \in End(\mathcal{M})$ the σ-restricted endomorphism $\sigma_\beta \in End(\mathcal{N})$ is defined by:*

$$\sigma_\beta = \gamma \circ \beta|_{\mathcal{N}} \tag{3.61}$$

where γ is the canonical endomorphism of $\mathcal{M}$ into $\mathcal{N}$.

The next lemma tells us, that σ-restriction, in difference with α-induction, respects the DHR property of endomorphisms:

Lemma 3.44. *If $\beta \in \Delta_{\mathcal{M}}(I)$ then $\sigma_\beta \in \Delta_{\mathcal{N}}(I)$.*

In the following we will discuss some properties of σ-restriction.

Observation. For $\lambda \in \Delta_{\mathcal{N}}(I)$ we have:

$$\sigma_{\alpha_\lambda} = \theta \circ \lambda \rightarrow [\lambda] \prec [\sigma_{\alpha_\lambda}] \tag{3.62}$$

It is natural to ask whether also $\beta \in \Delta_{\mathcal{M}}(I)$ is a subsector of α_{σ_β}. The following theorem assures that the answer is positive:

Theorem 3.45 ($\alpha - \sigma$ reciprocity). *For $\lambda \in \Delta_{\mathcal{N}}(I)$ and $\beta \in \Delta_{\mathcal{M}}(I)$ we have the following property, called $\alpha - \sigma$ reciprocity:*

$$\langle \alpha_\lambda, \beta \rangle_{M(I)} = \langle \lambda, \sigma_\beta \rangle_{N(I)} \tag{3.63}$$

Observation. σ-restriction obeys the following properties:

- If $[\beta_1] = [\beta_2]$ then $[\sigma_{\beta_1}] = [\sigma_{\beta_2}]$
- Let β, β_1, $\beta_2 \in \mathrm{End}(\mathcal{M}(I))$. If $[\beta] = [\beta_1] \oplus [\beta_2]$, then $[\sigma_\beta] = [\sigma_{\beta_1}] \oplus [\sigma_{\beta_2}]$

σ-restriction in general does not preserve the sector product, i.e $[\sigma_{\beta_1}] \circ [\sigma_{\beta_2}]$ is in general different from $[\sigma_{\beta_1 \circ \beta_2}]$, for example for $\beta_1 = \beta_2 = \mathrm{id}$ because $\sigma_{\mathrm{id}} = \theta$.

4 Cohomology and deformations of algebraic structures

One of the main goals of this thesis is to study the deformation theory of the commutators (6.1) with “disentangled” test functions. Of interest for us is whether these “reduced” brackets give rise to new inequivalent theories or they are stable under deformations. The deformation question is directly related to the question whether extensions are possible and whether there are whole families of models depending on a continuous parameter.

In our study we will consider formal deformations, i.e. deformations such that the new bracket is obtained from the old as a perturbative formal power series:

$$[\cdot,\cdot]^{\lambda} = [\cdot,\cdot] + \sum_{i=1}^{\infty} [\cdot,\cdot]_i \lambda^i \tag{4.1}$$

We will follow the example of the existing deformation theories of algebras, which we will review shortly in this chapter.

Due to an extensive research from the middle of the last century, the deformation theory of a large class of algebraic structures was well–understood and systematically described in an algebraic–cohomological setting. The pioneering work was done by Gerstenhaber [Gerstenhaber, 1964], who established the deformation theory of associative algebras involving Hochschild cohomology and using a lot of insight and concepts from the deformation theory of complex–analytic structures on compact manifolds. The latter was extensively developed shortly before that in a series of papers mostly by Kodaira, Spencer and Kuranishi [Kodaira *et al.*, 1958], [Kuranishi, 1962] and has been the most prominent deformation theory of mathematical structures. It was observed by Nijenhuis and Richardson [Nijenhuis & Richardson, 1964], [Nijenhuis & Richardson, 1966] that the two theories have in their common core graded Lie algebras. In their further work they showed that the deformation theory of graded Lie algebras can be applied to describe as well the deformations of ordinary Lie algebras [Nijenhuis & Richardson, 1967], commutative and associative algebras (see also [Knudson, 1969]), extensions of all of the types above and representations of all these algebras.

After these first foundational works formal deformation theories of various other algebraic structures were studied in an analogous manner. Further generalization showed that the Hochschild complex controls deformations of a whole family of algebras called A_∞-algebras [Lazarev, 2003], of which the associative algebras are a particular case. There is also an example in theoretical physics, where perturbations of a given quantum field theory are characterized in terms of a certain cohomology ring of a Hochschild type [Hollands, 2008]. Also a number of separate cases were considered, such as Landweber–Novikov algebras (a subset of Hopf algebras) [Yau, 2006], Poisson algebras, Hom-associative and Hom-Lie algebras [Makhlouf &

Silvestrov, 2010]. In all cases a cohomology theory adapted to the deformation theory was constructed and in some cases a mastering differential graded Lie algebra was recognized.

In fact, nowadays there is the basic philosophy that over a field of characteristic 0 every deformation problem is governed by a differential graded Lie algebra via solutions of a deformation equation moduli some gauge action [Manetti, 1999]. However, it has been realized recently that a more sophisticated analysis often involves homotopy Lie algebras and L_∞-algebras instead [Manetti, 1999], [Borisov, 2005].

In our thesis we will be interested in deformation theory of the Lie algebra of field operators in conformal chiral theories (from the sample of minimal models and their extensions) which is reduced in the sense that the test function part is "disentangled" from the operator part. Again, we will rely on the algebraic–cohomological method.

The function of this chapter will be to provide some preliminary knowledge about cohomology and algebraic deformations, which we will use in our analysis. We will not provide a consistent survey on the topic, we will rather be as minimalistic and sketchy as possible in our exposition and we will exhibit technical details only when it is directly needed for our calculations.

In the first section we will give a rough definition of a cohomology theory and examples of three cohomology complexes will be considered — a Hochschild complex, because it is related to associative deformation theory, which is our guiding example; a Chevalley–Eilenberg complex, because the complex in our analysis will be constructed in a close analogy; and finally a complex of a differential Lie algebra, whose deformation theory is believed to dominate every deformation theory. We will also discuss how one can obtain the cohomology groups of a Lie algebra and of an associative algebra from those of a differential graded Lie algebra.

In the second section we will review briefly most general concepts and results of Gerstenhaber's deformation theory and we will generalize them to Lie algebras and differential graded Lie algebras. The most important message will be that the first three cohomology groups of an algebra are tightly related to its deformation theory.

In the last section we will describe very shortly the relation between the second cohomology group of an algebra and its extensions.

4.1 Cohomology of algebras

Cohomology has been in the last fifty years a powerful tool in mathematics with numerous applications reaching even beyond topology and abstract algebra. The cohomology theory of an algebra is in tight relation with its inner structure, deformations and extensions. In this section we will discuss briefly cohomology theories of three algebras which provide important ideas for our computations.

4.1.1 General definitions and concepts of cohomology theories

In this subsection we would like to give a definition in a most general sense for a cohomology associated to a space X. But before that we need to introduce the notion of a cochain complex associated to this space and also of several other concepts related to it:

Definition 4.1 (Cochain complex associated to a space X). *Let us introduce a sequence of abelian groups $C_n(X)$, containing in some sense information about X, and also the sequence of dual groups $C^n(X, A) := Hom(C_n, A)$ with coefficients in A, which consists of the spaces of homomorphisms of C_n into some group A. We define as well the sequence of homomorphisms $\delta^n : C^n \to C^{n+1}$, such that $\delta^n \circ \delta^{n-1} = 0$. Then C^n together with δ^n compose a cochain complex: $C(X, A) := \{C^n(X, A), \delta^n\}$.*

Definition 4.2 (Some concepts from cohomology theory). *We define some objects related to cochain complexes and important for cohomology theories:*

- *the elements of $C^n(X, A)$ are called cochains*
- *the δ^n's are called coboundary operators and also differentials*
- *define $Z^n(X, A) := Ker(\delta^n) = \{\phi^n \in C^n(X, A) |\, \delta^n \phi^n = 0\}$, the elements of $Z^n(X, A)$ are called n-cocycles*
- *define $B^n(X, A) := Im(\delta^{n-1}) = \{\phi^n \in C^n(X, A) |\, \phi^n = \delta^{n-1}\phi^{n-1},\ \phi^{n-1} \in C^{n-1}(X, A)\}$, the elements of $B^n(X, A)$ are called n-coboundaries*

Now we are ready to define a cohomology:

Definition 4.3 (Cohomology associated to a space X). *The n^{th} cohomology group of a cochain complex $C(X, A) = \{C_k(X, A), \delta_k\}$, $k \in \mathbb{Z}$ is defined as $H^n(X, A) = Z^n(X, A)/B^n(X, A)$. The elements of the cohomology group $H^n(X, A)$ are called cohomology classes.*

Remark. A cochain complex is called exact at the position n if $\mathrm{Im}(\delta^n)$ coincides with $\mathrm{Ker}(\delta^n)$. Hence, a cochain complex is exact at the position n iff $H^n(X, A) = 0$. It means that cohomology measures how strong the deviation of a cochain complex is from exactness.

4.1.2 Examples of cohomology complexes

In this subsection we will define cohomology complexes for three algebras so that to prepare the ground for the description of their deformation theories later.

Associative algebra cohomology

Let us consider an associative algebra A, which is not necessary finite–dimensional. A cohomology complex associated to such algebras was constructed by Hochschild [Hochschild, 1945]:

Definition 4.4 (Hochschild complex). *A Hochschild complex consists of:*

1. **Cochain spaces $C^m(A, \mathcal{B})$ of dimension m with coefficients in $\mathcal{B}$:** *the additive groups of m-linear maps $f^m : \underbrace{A \otimes ... \otimes A}_{m} \to \mathcal{B}$, where $\mathcal{B}$ is a two sided A-module.*

2. **Coboundary operators δ_H^m of dimension** m**:** *linear maps from m-dimensional cochains f^m to $m+1$-dimensional cochains f^{m+1} such that:*

$$\begin{aligned}(\delta_H^m f^m)(a_1, ..., a_{m+1}) &= a_1 \cdot f^m(a_2, ..., a_{m+1}) + \sum_{i=1}^{m} (-1)^i f^m(a_1, ..., a_i a_{i+1}, ..., a_{m+1}) \\ &\quad + (-1)^{m+1} f^m(a_1, .., a_m) \cdot a_{m+1} \end{aligned} \tag{4.2}$$

"$\cdot$" is the two sided action of A on $\mathcal{B}$. One proves by induction on the dimension m that $\delta_H^m \circ \delta_H^{m-1} = 0$.

Then the Hochschild cohomology groups of A with coefficients in $\mathcal{B}$ will be defined as:

$$HH^n(A, \mathcal{B}) = \mathrm{Ker}(\delta_H^n)/\mathrm{Im}(\delta_H^{n-1}) \tag{4.3}$$

Notation. We will denote the Hochschild cohomology groups of an algebra A with coefficients into itself simply by $HH^n(A)$.

One can also define the cohomology algebra as a direct sum of the cohomology groups of all dimensions:

$$HH(A) = \oplus_{i \in \mathbb{N}} HH^i(A) \tag{4.4}$$

Observation. The first cohomology group $HH^1(A, \mathcal{B})$ determines all the higher dimension groups $HH^n(A, \mathcal{B})$, which means that the cohomology theory arising from a Hochschild complex is degenerate in a sense. A root of this degeneracy can be found in the relation:

$$HH^m(A, \mathcal{B}) \cong HH^{m-1}\left(A, C^1(A, \mathcal{B})\right), \quad \text{for } m \geq 2 \tag{4.5}$$

where $C^1(A, \mathcal{B})$ is a two sided A-module with a two-sided action defined in a special way [Hochschild, 1945].

The first cohomology group $HH^1(A, \mathcal{B})$ is interpreted as the space of derivations of A into itself modulo the inner derivations.

There are deep relations between the structure of the algebra A and its cohomology groups $HH^n(A, \mathcal{B})$. For example, it is shown that A is separable if and only if its cohomology groups vanish, which is equivalent to the vanishing of $HH^1(A, \mathcal{B})$ for every two-sided A-module $\mathcal{B}$.

Lie algebra cohomology

The Lie algebra cohomology was defined by Chevalley and Eilenberg, originally for the purpose of reducing topological questions concerning Lie groups to algebraic questions concerning Lie algebras [Chevalley & Eilenberg, 1948]. We will consider a Lie algebra L over a field K of characteristic 0. The cochain complex in this case is strongly correlated to the Koszul complex:

Definition 4.5 (Lie algebra cochain complex). *Lie algebra cochain complex consists of:*

1. **Cochain spaces $C^m(L,V)$ of dimension m with coefficients in V:** *the additive groups of m-linear alternating maps $\omega^m : \underbrace{L \otimes ... \otimes L}_{m} \to V$, where V is a vector space over K, such that there is a representation π of L by linear transformations.*

2. **Coboundary operators δ^m_{LA} of dimension m:** *maps of m-dimensional cochains ω^m into $m+1$-dimensional cochains ω^{m+1} such that:*

$$\begin{aligned}(\delta^m_{LA}\omega^m)(x_1,...,x_{m+1}) &= \frac{1}{m+1}\sum_{i=1}^{m+1}(-1)^{i+1}\pi(x_i)\omega^m(x_1,...,\hat{x}_i,...,x_{m+1}) + \\ &+\frac{1}{m+1}\sum_{i<j}(-1)^{i+j+1}\omega^m\left([x_i,x_j],x_1,...,\hat{x}_i,...,\hat{x}_j,...,x_{m+1}\right)\end{aligned} \tag{4.6}$$

Here $[\cdot,\cdot]$ is the Lie bracket, the hat means omitting the corresponding argument. One proves also that $\delta^m_{LA} \circ \delta^{m-1}_{LA} = 0$. A proof for the case of the adjoint representation is displayed in the Appendix B for the purpose of further applications.

Then the Lie algebra cohomology groups of L with coefficients in V will be defined as:

$$LH^n(L,V) = \mathrm{Ker}(\delta^n_{LA})/\mathrm{Im}(\delta^{n-1}_{LA}) \tag{4.7}$$

One can prove that the cohomology groups $LH^n(L,V)$ of a semi-simple Lie algebra vanish for all dimensions n and for all non-trivial irreducible representations π. Moreover, to show that a Lie algebra L over a field of characteristic 0 is semi-simple, it is enough to check only that $LH^1(L,V) = \{0\}$ for every representation π of L. This means essentially that in the case of semi-simple Lie algebras nothing is gained by studying cohomology groups over representations. However, if one constructs the cohomology $LH^n(L,K)$ of a Lie algebra L with coefficients in K and with δ^m_{LA} as in (4.6) restricted to the second term, one proves that $LH^3(L,K) \neq \{0\}$ in the case of semi-simple algebras, even though $LH^n(L,K) = \{0\}$ for $n = 1,2,4$. Moreover, one shows that the cohomology ring $LH(L,K) = \oplus_q LH^q(L,K)$ of a semi-simple Lie algebra L (over K) is isomorphic to the direct sum of the cohomology rings (over K) of a finite number of odd-dimensional spheres.

Differential graded Lie algebra (DGLA) cohomology

Let us first introduce the notion of a graded Lie algebra [Nijenhuis & Richardson, 1964]:

Definition 4.6 (Graded Lie algebra). *A graded Lie algebra consists of:*

1. **Vector space:** *a graded vector space $E = \sum_{n\geq 0} E^n$ over a field K of characteristic 0.*

2. **Multiplication:** *a bilinear bracket $[\cdot,\cdot] : E \times E \to E$ such that:*

   - $[\cdot,\cdot]$ *is skew symmetric for homogeneous elements, which means that for $x \in E^m$ and $y \in E^n$ follows that $[x,y] \in E^{m+n}$ and $[x,y] = -(-1)^{mn}[y,x]$*

- *the nested bracket of every three homogeneous elements $x \in E^m$, $y \in E^n$ and $z \in E^p$ satisfies a graded Jacobi identity in the form:*

$$(-1)^{mp}\,[[x,y],z] + (-1)^{nm}\,[[y,z],x] + (-1)^{pn}\,[[z,x],y] = 0 \tag{4.8}$$

An important part in the theory of graded Lie algebras will be played by the derivations of degree 1. Let us give the following definitions:

Definition 4.7 (Derivation of degree n). *A linear map $f : E \to E$ is called a derivation of degree n if $f(E^i) \subset E^{i+n}$ and if it satisfies the graded Leibniz rule $f([a,b]) = [f(a),b] + (-1)^{na}[a, f(b)]$.*

Definition 4.8 (Differential). *A differential d of a graded Lie algebra $\{E, [\cdot,\cdot]\}$ is a derivation of degree 1 such that $d \circ d = 0$.*

Observation. There are natural candidates for derivations and differentials:

1. If $a \in E^i$ then $\mathrm{ad}_a : E \to E$, $\mathrm{ad}_a(b) = [a,b]$, is a derivation of degree i.
2. Let $M = \{x \in E^1 | [x,x] = 0\}$. Then $\forall x \in M$ holds that $\delta_x := \mathrm{ad}_x$ is a differential of E.

With the notion of a differential we can give a definition for a differential graded Lie algebra [Manetti, 1999]:

Definition 4.9 (Differential graded Lie algebra (DGLA)). *A differential graded Lie algebra $\{E, [\cdot,\cdot], d\}$ is a graded Lie algebra $\{E, [\cdot,\cdot]\}$ equipped with a differential d.*

Now we can already define a cohomology theory of a differential graded Lie algebra:

Definition 4.10 (DGLA Cohomology). *In differential graded Lie algebras arises naturally a cochain complex with a differential d and cochain spaces of dimension m — the homogeneous spaces E^m of degree m. In a standard way we define $Z^n(E,d) = E^n \cap Ker(d)$ and $B^n(E,d) = d(E^{n-1})$, then the cohomology groups will be $DH^n(E,d) = Z^n(E,d)/B^n(E,d)$.*

Remark. $DH(E,d) := \oplus_n DH^n(E,d)$ inherits naturally the structure of a graded Lie algebra.

Lie algebra cohomology and associative algebra cohomology from DGLA cohomology

In this subsection we will demonstrate how one can obtain the cohomology groups of a finite Lie algebra from those of a DGLA [Nijenhuis & Richardson, 1967], [Lecomte, 1987]. We will also explain how one can recognize a graded Lie algebra in a Hochschild complex [Stasheff, 1993].

Let V be a vector space over a field K of characteristic 0. Let $Alt^n(V)$ be the vector spaces of alternating linear maps of V^{n+1} into V and $Alt(V) := \oplus_{n\geq -1} Alt^n(V)$. We define a "$\bar{\wedge}$-product" on $Alt(V)$ of two homogeneous elements $f \in Alt^p(V)$ and $h \in Alt^q(V)$:

$$f \,\bar{\wedge}\, h(u_0, ..., u_{p+q}) = \sum_\eta \mathrm{sign}(\eta) f\left(h(u_{\eta(0)}, ..., u_{\eta(q)}), u_{\eta(q+1)}, ..., u_{\eta(q+p)}\right) \tag{4.9}$$

$f \bar{\wedge} h \in Alt^{p+q}(V)$, η are the possible divisions of $0, ..., p+q$ into two ordered sets $\{\eta(0), ..., \eta(q)\}$ and $\{\eta(q+1), ..., \eta(q+p)\}$. Then a bracket, turning $Alt(V)$ into a graded Lie algebra, will be:

$$[f, h] = f \bar{\wedge} h - (-1)^{pq} h \bar{\wedge} f \tag{4.10}$$

A differential, turning $\{Alt(V), [\cdot, \cdot]\}$ into a DGLA, will be δ_f with $f \in M$ defined as in the previous subsection.

One can prove that the points of M are precisely the Lie algebra multiplications on V. Let $f \in M$ and let L be the Lie algebra with an underlying vector space V equipped with the bracket f. Then a computation shows that:

$$DH^n\left(Alt(V), \delta_f\right) \cong LH^{n+1}(L, L) \tag{4.11}$$

Let us consider the associative case, which is a little bit more tricky due to the infinite dimensionality. The Hochschild cochain spaces of an associative algebra A over K may be identified with the spaces of graded coderivations $Coder(T^c\overline{A})$, where $T^c\overline{A} := \oplus_{n\geq 0}\overline{A}^{\otimes n}$ is a tensor coalgebra and $\overline{A} = A$ as a K-module. We have:

$$\begin{aligned} \Delta(\bar{a}_1 \otimes ... \otimes \bar{a}_n) &= 1 \otimes (\bar{a}_1 \otimes ... \otimes \bar{a}_n) + (\bar{a}_1 \otimes ... \otimes \bar{a}_n) \otimes 1 + \\ &\quad + \sum_{p=1}^{n-1} (\bar{a}_1 \otimes ... \otimes \bar{a}_p) \otimes (\bar{a}_{p+1} \otimes ... \otimes \bar{a}_n) \end{aligned} \tag{4.12}$$

The coderivation h of degree $|h|$ is a k-linear map $A^{\otimes k} \to A^{\otimes k-|h|}$ such that:

$$\begin{aligned} \Delta h[\bar{a}_1|...|\bar{a}_n] &= \sum h[\bar{a}_1|...|\bar{a}_p] \otimes [\bar{a}_{p+1}|...|\bar{a}_n] + \\ &\quad + (-1)^{p|h|}[\bar{a}_1|...|\bar{a}_p] \otimes h[\bar{a}_{p+1}|...|\bar{a}_n] \end{aligned} \tag{4.13}$$

The graded bracket of two coderivations is defined as:

$$[\theta, \phi] = \theta \circ \phi - (-1)^{|\theta||\phi|} \phi \circ \theta \tag{4.14}$$

It remains to provide a differential D:

$$\begin{aligned} D\theta &:= \theta d \pm d\theta \\ d[\overline{a}_1|...|\overline{a}_n] &:= \sum_{i=1}^{n-1} (-1)^i [...|\overline{a}_i \overline{a}_{i+1}|...] \end{aligned} \tag{4.15}$$

Then we have the DGLA $\{Coder(T^c\overline{A}), [\cdot, \cdot], D\}$. As before one can show:

$$DH^{n-1}(Coder(T^c\overline{A}), D) \cong HH^n(A) \tag{4.16}$$

4.2 Formal deformations of algebras

In this section we will review shortly most general concepts and results of the theory of formal deformations of algebraic structures. We will present partially the deformation theory of associative algebras by Gerstenhaber, which was the foundational work on formal deformations of algebras [Gerstenhaber, 1964] and serves a prototype for many theories of algebraic deformations. Most of the definitions and theorems hold or can be straightforward generalized for wider classes of algebras.

Let us give a more explicit definition for a formal deformation of an associative algebra:

Definition 4.11 (Deformation of an associative algebra). *Let A be an associative algebra over a field k, which may be finite or infinite dimensional, let V be its underlying vector space and let $\mu : V \times V \to V$ be its multiplication. Let K contain the power series in t over k an let $V_K = V \otimes_k K$, i.e it is obtained from V by extending the coefficient domain from k to K. Then a deformation of A will be an algebra A_λ with an underlying vector space V_K and with an associative product $\mu_\lambda : V_K \times V_K \to V_K$:*

$$\mu_\lambda(a, b) = \mu(a, b) + \lambda\mu_1(a, b) + \lambda^2\mu_2(a, b) + ... \tag{4.17}$$

such that $\lambda \in \mathbb{R}$, $\mu_i : V_K \times V_K \to V_K$, $i \in \mathbb{N}$ are extensions of bilinear functions $f : V \times V \to V$ and μ on V_K is an extension of μ on V. We consider A_λ as "the generic element of a one-parameter family of deformations of A".

The associativity condition for the deformed product yields:

$$\mu_\lambda\left(\mu_\lambda(a, b), c\right) = \mu_\lambda\left(a, \mu_\lambda(b, c)\right) \quad \to \sum_{\substack{i+j=const \\ i,j\in\mathbb{N}_0}} \mu_i\left(\mu_j(a, b), c\right) - \mu_i\left(a, \mu_j(b, c)\right) = 0 \tag{4.18}$$

This condition must hold $\forall\nu \in \mathbb{N},\ \nu = i + j$.

Among the families of deformations there are some, that will not be considered as "proper deformations":

Definition 4.12 (Trivial deformations). *Deformations A_λ, such that:*

$$\Phi_\lambda\mu_\lambda(a, b) = \mu(\Phi_\lambda a, \Phi_\lambda b) \quad \Longrightarrow \quad \mu_\lambda(a, b) = \Phi_\lambda^{-1}\mu(\Phi_\lambda a, \Phi_\lambda b) \tag{4.19}$$

with a non-singular linear map Φ_λ of the form:

$$\Phi_\lambda(a) = a + \lambda\varphi_1(a) + \lambda^2\varphi_2(a) + ... \tag{4.20}$$

and $\varphi_i : V_K \to V_K$ linear maps, will be called trivial.

Obviously, a trivial deformation amounts to a mere λ-dependent basis redefinition and hence to an isomorphism between A_λ and the algebra A_K with the extended vector space V_K and the extension of the initial product $\mu(a, b)$. The isomorphism is in fact the linear map Φ_λ. Therefore, such deformations do not produce new algebras and we would like to exclude them from our discussion.

There are algebras for which the only admissible deformations are trivial:

Definition 4.13 (Rigid algebra). *An associative algebra is said to be rigid if there exist no deformations obeying the associativity law (4.18) apart from the trivial.*

Naturally comes the definition for equivalent deformations:

Definition 4.14 (Equivalent deformations). *Two one-parameter families of deformations* A^f_λ *and* A^g_λ *will be called equivalent if* $\mu^f_\lambda(a,b) = \Phi^{-1}_\lambda \mu^g_\lambda(\Phi_\lambda a, \Phi_\lambda b)$ *for some* Φ_λ *as in (4.20).*

The "factorization" of trivial deformations gives the possibility to formulate the theory of deformations in an algebraic–cohomological setting. In the rest of this section we will explain how the first three Hochschild cohomology groups control the deformation theory of an associative algebra A. We will use the definitions from Subsection 4.1.2.

In our discussion a special role will be played by μ_1:

Definition 4.15 (Infinitesimal deformation). μ_1 *is viewed as an "infinitesimal deformation" or a "differential" of the family* A_λ.

Observation *(1).* $\delta^2_H \mu_1 = 0$, which is a direct corollary from the associativity law (4.18) written for $i + j = 1$:

$$\mu(id \otimes \mu_1) - \mu(\mu_1 \otimes id) + \mu_1(id \otimes \mu) - \mu_1(\mu \otimes id) = 0 \tag{4.21}$$

Thus $\mu_1 \in Z^2(A, A)$. Moreover, one can argue that, if $\mu_1 = 0$, then the first non-zero μ_i is again in $Z^2(A, A)$.

Observation *(2).* For a trivial deformation as in (4.19) one proves that $\mu_1(a, b) = \delta^1_H \varphi_1(a, b)$, so $\mu_1 \in B^2(A, A)$.

Observation *(3).* One shows that the infinitesimal deformations of two equivalent deformations differ with $\delta^1_H \varphi_1$.

Remark. The equivalence classes of associative deformations of an algebra A correspond to certain cohomology classes from $HH^2(A) = Z^2(A)/B^2(A)$. These cohomology classes can be interpreted as their infinitesimal deformations.

An interesting question is whether every element $[\widetilde{\mu}_1]$ of $HH^2(A)$ is an infinitesimal deformation for some equivalence class of non-trivial deformations. If it is such, then we will say that $[\widetilde{\mu}_1]$ is integrable. Suppose now that we have chosen an element μ_1 of $Z^2(A)$ which is integrable and that we want to lift the perturbation to second order μ_2. In such case the associativity law for $i + j = 2$ must be obeyed:

$$\mu_1(\mu_1(a, b), c) - \mu_1(a, \mu_1(b, c)) = [\delta^2_H \mu_2](a, b, c) \tag{4.22}$$

For a general element $\widetilde{\mu}_1 \in Z^2(A)$ we define:

$$G^2[\widetilde{\mu}_1](a, b, c) := \widetilde{\mu}_1(\widetilde{\mu}_1(a, b), c) - \widetilde{\mu}_1(a, \widetilde{\mu}_1(b, c)), \quad \widetilde{\mu}_1 \in HH^2(A) \tag{4.23}$$

It is easy to verify that $\delta^3_H G^2[\widetilde{\mu}_1] = 0 \quad \forall \widetilde{\mu}_1 \in Z^2(A)$, thus $G^2(a,b,c) \in Z^3(A)$. If $\widetilde{\mu}_1$ is also integrable, then G^2 must be of the form $G^2 = \delta^2_H \widetilde{\mu}_2$ and thus $G^2(a,b,c) \in B^3(A)$. Therefore, for an integrable $\widetilde{\mu}_1$ the cohomology class of $G^2[\widetilde{\mu}_1]$ in $HH^3(A)$ must vanish. This cohomology class is viewed as the first obstruction to the integration of $\widetilde{\mu}_1$.

In analogy, provided that we have lifted the perturbation to order $m-1$, we examine under which circumstances we can lift the perturbation to order m. Assume that we have a set of perturbations $\widetilde{\mu}_1, ..., \widetilde{\mu}_{m-1} \in C^2(A)$ which obey the associativity conditions:

$$\mu(id \otimes \widetilde{\mu}_i) - \mu(\widetilde{\mu}_i \otimes id) + \widetilde{\mu}_i(id \otimes \mu) - \widetilde{\mu}_i(\mu \otimes id) = -\sum_{j=1}^{i-1} \widetilde{\mu}_{i-j}(id \otimes \widetilde{\mu}_j) - \widetilde{\mu}_{i-j}(\widetilde{\mu}_j \otimes id) \quad (4.24)$$

for $i \in [1, m-1]$. Then the obstruction to lift the perturbation to order m is:

$$G^m[\widetilde{\mu}_1, ..., \widetilde{\mu}_{m-1}](a,b,c) = -\sum_{j=1}^{m-1} \widetilde{\mu}_{m-j}(a, \widetilde{\mu}_j(b,c)) - \widetilde{\mu}_{m-j}(\widetilde{\mu}_j(a,b), c) \quad (4.25)$$

A theorem by Gerstenhaber states that $\delta^3_H G^m[\widetilde{\mu}_1, ..., \widetilde{\mu}_{m-1}] = 0$, then $G^m(a,b,c) \in Z^3(A)$. Again, in case that we are allowed to proceed the integration to the m^{th} order, associativity requires that G^m is of the form $G^m = \delta^2_H \mu_m$ and consequently that its cohomology class vanishes in $HH^3(A)$.

In summary, the second cohomology group $HH^2(A)$ of an associative algebra A may be interpreted as the group of infinitesimal deformations of A and the obstructions to their integration lie in $HH^3(A)$. It follows that the vanishing of $HH^2(A)$ is a sufficient condition for the rigidity of A and the vanishing of $HH^3(A)$ implies that every $[\mu_1] \in HH^2(A)$ is integrable.

In a close parallel to the argument above it was shown that the first cohomology group $HH^1(A)$ is interpreted as the group of infinitesimal automorphisms of A and the obstructions to their integration are elements of $HH^2(A)$. Hence, the vanishing of $HH^2(A)$ implies not only the rigidity of A but also the integrability of any $\widetilde{\varphi}_1 \in HH^1(A)$ to a one-parameter family of automorphisms of A.

In the Lie algebra case one obtains analogous results to the ones of Gerstenhaber [Nijenhuis & Richardson, 1967]. Namely, a Lie algebra L is rigid if $LH^2(L) = \{0\}$ and the obstructions to integrating an infinitesimal deformation μ_1 from $Z^2(L)$ to a one-parameter family of deformations μ_λ lie in $LH^3(L)$.

The deformation theories both of associative and Lie algebras can be derived from the deformation theory of graded Lie algebras associated with them as in Subsection 4.1.2. In this way the computations are greatly simplified and put onto a more conceptual basis, also one may apply the already existing general theorems for deformations of graded Lie algebras [Nijenhuis & Richardson, 1966]. In this theory rigidity is a property of the points $a \in M_0 \subset E_1$ (E is the graded vector space of the graded Lie algebra in consideration) such that M_0 is the algebraic set of all solutions of a deformation equation $Da + \frac{1}{2}[a,a] = 0$, where D is a differential. In specific applications M_0 will be the set of Lie algebra brackets, associative products, etc. We say that an element a is rigid provided that $DH^1(E, D_a) = \{0\}$, with $D_a b = Db + [a,b]$ for

$b \in E$. Let G be the group of automorphisms of E, whose Lie algebra is E_0, acting on E by the adjoint representation. M_0 is stable under an appropriate action of G and the points of M_0 on the same orbit under G are called equivalent. The vanishing of $DH^1(E, D_a)$ is in direct relation to the fact that the G-orbit of a is a neighbourhood of a in M_0 and this result is a bridge to the deformation theory of complex analytic structures. The deformation theorems from the complex case have their precise algebraic analogues.

From the discussion above follows that every associative separable algebra and every semi-simple Lie algebra over $\mathbb{C}$ are rigid. Also, the Lie algebra of endomorphisms of a finite–dimensional vector space is rigid.

4.3 Cohomology and extensions

The 2-dimensional cohomology groups of an algebra are directly related to the extensions of this algebra. Let us consider the following definitions [Hochschild, 1945], [Chevalley & Eilenberg, 1948]:

Definition 4.16 (Extension of an algebra). *Let A be an algebra over K. An extension of A is a pair $(\mathcal{B}, \sigma)$ such that $\mathcal{B}$ is an algebra over K and σ is a homomorphism of $\mathcal{B}$ onto A.*

Definition 4.17 (Inessential extension). *An extension is called inessential if there exists a subalgebra A' of $\mathcal{B}$ such that $\sigma(A') \simeq A$. Then $\mathcal{B} \simeq A + Ker(\sigma)$.*

In the case of associative algebras one has:

Definition 4.18 (Singular extension). *An extension $(\mathcal{B}, \sigma)$ of an associative algebra A is called singular if $\mathcal{K} := Ker(\sigma)$ satisfies $\mathcal{K}^2 = \{0\}$.*

It is possible to prove [Hochschild, 1945]:

Theorem 4.19. *There is one-to-one correspondence between the classes of isomorphic singular extensions of the associative algebra A and the 2-dimensional cohomology classes of A. In order every extension of A to be inessential it is necessary and sufficient that $HH^2(A, \mathcal{B}) = \{0\}$ for every two-sided A-module $\mathcal{B}$.*

Very similar results are obtained for Lie algebras [Chevalley & Eilenberg, 1948].

Definition 4.20 (Extension of L by π). *Suppose that the space V and the representation π of the Lie algebra L in V are given. Any extension (L^*, ϕ), such that $Ker(\phi) = V$ and $[V, V] = 0$ will be called an extension of L by π.*

The classification theorem is:

Theorem 4.21. *The elements of $LH^2(L, \pi)$ are in a one-to-one correspondence with the isomorphism classes of extensions of L by π. In order every extension of a Lie algebra L to be inessential, it is necessary and sufficient to prove that $LH^2(L, \pi) = 0$ for every representation π.*

Corollary. Having in mind the Theorems 4.19 and 4.21 and remembering from Section 4.1.2 that the corresponding cohomology groups vanish, it follows immediately that:

1. Every extension of an associative separable algebra is inessential.
2. Every extension of a semi-simple Lie algebra over a field characteristic 0 is inessential.

Remark. In the context of this section an extension is related to a quotient of an algebra over a kernel of a homomorphism, while in the chapters that follow the extension of an algebra $\mathcal{A}$ is simply an algebra $\mathcal{B}$ containing $\mathcal{A}$ as a subalgebra.

5 Superselection sectors of conformal nets for $c < 1$

In this chapter our task will be to find all the irreducible DHR-sectors $[\beta]$ of all local extensions $\mathcal{B} \supset \mathcal{A}$ (see Section 3.2) such that $\mathcal{A}$ is a Virasoro net with central charge $c(m) = 1 - \frac{6}{m(m+1)}, m = 2, 3, 4....$ To remind, the sectors of the Virasoro net for a given central charge are labelled by $h_{R,S}(m) = \frac{\left[(m+1)R - mS\right]^2 - 1}{4m(m+1)}$ such that $1 \leq R \leq m-1$, $1 \leq S \leq m$, $h_{R,S}(m) = h_{m-R,m+1-S}(m)$. We denote the corresponding DHR endomorphism sectors by $\lambda_{R,S}$ (we will omit the sector brackets of λ for simplicity throughout this section).

We will consider first the series $m_1(n) = 4n+1$, $m_2(n) = 4n+2$, which have one local extension of index 2 and after that — the four special cases with higher indices for $m = 11, 12, 29$ and 30. A special case is (A_{28}, E_8) extension for $m = 29$ — the only extension which cannot be presented as a coset model of familiar models and thus the only extensions for which there is no available other method to compute the superselection sectors.

We will use for this purpose the machinery of α-induction and σ-restriction introduced in Section 3.4, and in our case $\mathcal{A}$ and $\mathcal{B}$ will play the roles of $\mathcal{N}$ and $\mathcal{M}$. In all cases we use a similar strategy, which we will explain in the following.

Observation. The key observation is that the DHR-sectors $[\beta]$ of the extension $\mathcal{B}$ are subsectors of $[\alpha^+_{\sigma_\beta}]$ and of $[\alpha^-_{\sigma_\beta}]$ simultaneously. Indeed, σ_β is a localized and transportable endomorphism of the subnet (see Lemma 3.18 [Böckenhauer & Evans, 1998]), even though it is in general reducible. Then, by α–σ reciprocity:

$$\langle \alpha^{\pm}_{\sigma_\beta}, \beta \rangle_{\mathcal{B}} = \langle \sigma_\beta, \sigma_\beta \rangle_{\mathcal{A}} \geq 1 \tag{5.1}$$

which means that $[\beta] \prec [\alpha^{\pm}_{\sigma_\beta}]$.

We can reduce the statement above to:

Lemma 5.1. *If $\beta \in \Delta_{\mathcal{B}}(I)$ then $[\beta] \prec [\alpha^+_\mu]$ and $[\beta] \prec [\alpha^-_\mu]$ for at least one $\mu \in \Delta_{\mathcal{A}}(I)$*

Proof. The proof of this lemma is based on the following observations:

1. $[\sigma_\beta]$ is in general reducible. Then we can write:

   $$[\sigma_\beta] \cong \oplus_i [\sigma^i_\beta] \tag{5.2}$$

   such that $\sigma^i_\beta \in \Delta_{\mathcal{A}}(I)$ are irreducible. Then by linearity:

   $$[\alpha^+_{\sigma_\beta}] \cong \oplus_i [\alpha^+_{\sigma^i_\beta}], \quad [\alpha^-_{\sigma_\beta}] \cong \oplus_i [\alpha^-_{\sigma^i_\beta}] \tag{5.3}$$

   where $[\alpha^+_{\sigma^i_\beta}]$ and $[\alpha^-_{\sigma^i_\beta}]$ are not necessarily irreducible.

2. We know that $[\beta] \prec [\alpha^+_{\sigma_\beta}]$, then $[\beta] \prec [\alpha^+_{\sigma^i_\beta}]$ for at least one i. Analogously, $[\beta] \prec [\alpha^-_{\sigma_\beta}]$, then $[\beta] \prec [\alpha^-_{\sigma^j_\beta}]$ for at least one j.

3. Suppose that exist $i \neq j$ such that $[\beta] \prec [\alpha^+_{\sigma^i_\beta}]$ and $[\beta] \prec [\alpha^-_{\sigma^j_\beta}]$. Then Lemma 3.1 from [Böckenhauer & Evans, 1999] implies that also $[\beta] \prec [\alpha^-_{\sigma^i_\beta}]$ and $[\beta] \prec [\alpha^+_{\sigma^j_\beta}]$.

Observation. $[\alpha^\pm_\mu]$ localized is equivalent to $\varepsilon(\mu,\theta)\varepsilon(\theta,\mu) = 1$ ([Longo & Rehren, 1995], Prop 3.9) and $[\alpha^+_\mu] = [\alpha^-_\mu]$ is equivalent to $\varepsilon(\mu,\theta)\varepsilon(\theta,\mu) = 1$ ([Böckenhauer & Evans, 1998], Prop), where θ is the dual canonical endomorphism from Definition 3.38. It means that $[\alpha^\pm_\mu]$ localized is equivalent to $[\alpha^+_\mu] = [\alpha^-_\mu]$.

There is the following, even stronger statement by [Kawahigashi, 2003]:

Proposition 5.2. *If $\{\lambda_{pq}\}$ is the set of inequivalent irreducible DHR sectors of a Virasoro net with $c < 1$, then the intersection of the irreducible endomorphisms appearing in the decompositions of $\alpha^+_{\lambda_{pq}}$ and $\alpha^-_{\lambda_{pq}}$ is exactly the system of irreducible inequivalent DHR endomorphisms of the local extension of the Virasoro net.*

So, our **general strategy** will contain the following steps, which will be worked out in detail in the following sections (in the following we will omit the sector brackets of $\alpha^\pm_{\lambda_{pq}}$ for simplicity):

1. check reducibility of $\alpha^\pm_{\lambda_{pq}}$ as well as common content and equivalence among $\alpha^+_{\lambda_{pq}}$-sectors and among $\alpha^-_{\lambda_{pq}}$-sectors. For this purpose calculate the dimensions of intertwiner spaces between them. Use that:

$$\langle \alpha^\pm_{\lambda_{pq}}, \alpha^\pm_{\lambda_{p'q'}} \rangle = \langle \lambda_{pq}, \theta \circ \lambda_{p'q'} \rangle$$

2. check common content and equivalence among α^+- and α^--sectors. Use that:

$$\langle \alpha^+_{\lambda_{pq}}, \alpha^-_{\lambda_{p'q'}} \rangle = Z_{\lambda_{pq}\lambda_{p'q'}}$$

   Use the explicit expression for $Z(\tau)$ from [Cappelli *et al.*, 1987] such that:

$$Z(\tau) = \sum_{\substack{p,q \\ p',q'}} Z_{\lambda_{pq}\lambda_{p'q'}} \chi_{\lambda_{pq}} \chi^*_{\lambda_{p'q'}} \tag{5.4}$$

With this information we will be able to classify all irreducible DHR representations of the extensions of Virasoro subnets.

5.1 DHR sectors for extensions of nets with $m = 4n + 1$ and $m = 4n + 2$

All nets with $m = 4n + 1$ have (A_{4n}, D_{2n+2}) extensions with index 2 and $\theta = \lambda_{11} + \lambda_{1m}$. Respectively, all nets with $m = 4n + 2$ have (D_{2n+2}, A_{4n+2}) extensions with index 2 and $\theta = \lambda_{11} + \lambda_{1m}$.

The following observation will assist us at choosing the convenient set of irreducible sectors of the Virasoro subnet and at studying the decompositions of the induced sectors:

Observation. If we have a model with central charge $c(m)$, then $\lambda_{1m} \equiv \lambda_{m-1,1}$ has a specific action on the lattice of $m \times (m-1)$ irreducible Virasoro sectors:

- **m-odd:** according to the fusion rules (3.44):

$$\lambda_{1m}\lambda_{pq} = \lambda_{p,m+1-q}, \quad \lambda_{1m}\lambda_{p,\frac{m+1}{2}} = \lambda_{p,\frac{m+1}{2}} \tag{5.5}$$

  i.e λ_{1m}, acting on the lattice of irreducible Virasoro sectors (with central charge $c(m)$) via composition, behaves like a reflection operator with respect to the axis $\lambda_{f,\frac{m+1}{2}}$, $f = 1, ..., m-1$, preserving the elements on this axis (the red axis on the figure below).

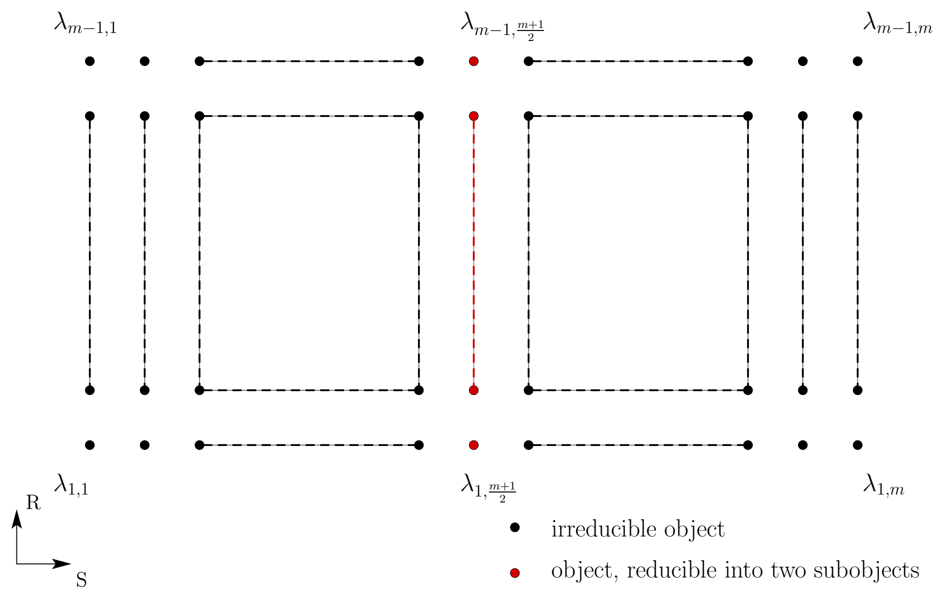

Figure 5.1: *Lattice of irreducible sectors for minimal models with $m = 2n + 1$*

- **m-even:** according to the fusion rules (3.44):

$$\lambda_{m-1,1}\lambda_{pq} = \lambda_{m-p,q}, \quad \lambda_{m-1,1}\lambda_{\frac{m}{2},q} = \lambda_{\frac{m}{2},q} \tag{5.6}$$

i.e $\lambda_{m-1,1}$, acting on the lattice of irreducible Virasoro sectors (with central charge $c(m)$) via composition, behaves like a reflection operator with respect to the axis $\lambda_{\frac{m}{2},g}$, $g = 1,...,m$, preserving the elements on this axis (the red axis on the figure below).

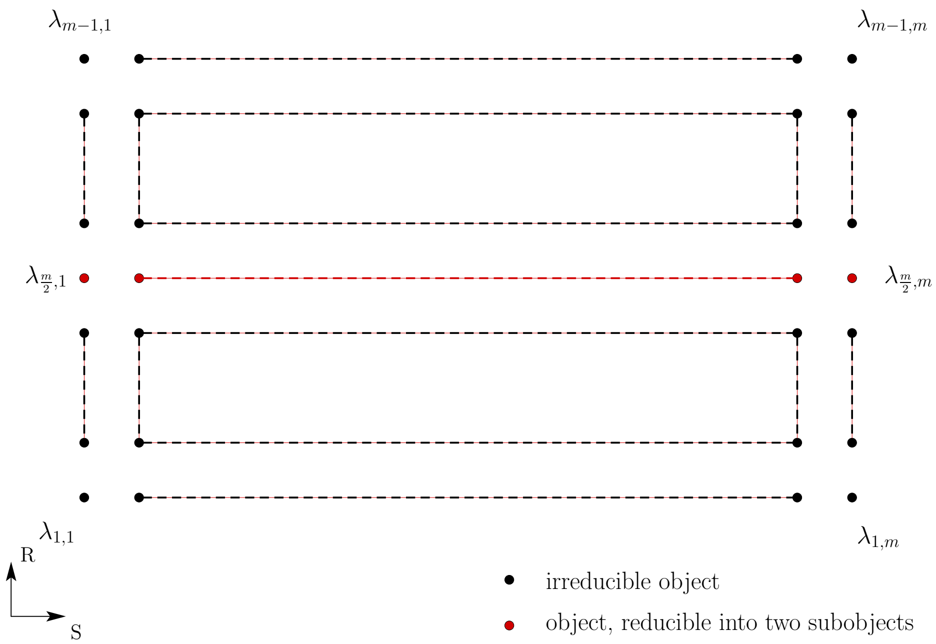

Figure 5.2: *Lattice of irreducible sectors for minimal models with $m = 2n$*

Let us now remind, that the sectors in the $m \times (m-1)$ lattice are pairwise equivalent — every sector from the lattice is equivalent to the inverse sector with respect to the "center" of the lattice (this center is not a knot of the lattice itself). Having in mind the observation above, we will choose the following complete sets of inequivalent irreducible sectors of the Virasoro subnet:

- **m-odd:** we will work with $\{\lambda_{pq}\}_{\substack{p=1...\frac{m-1}{2} \\ q=1...m}}$

- **m-even:** we will work with $\{\lambda_{pq}\}_{\substack{p=1...m-1 \\ q=1...\frac{m}{2}}}$

5.1.1 Reducibility and common content among α^+ and among α^- sectors

In this subsection we will perform Step 1 of the strategy discussed in the introductory part of this chapter. We will calculate $\langle \alpha^{\pm}_{\lambda_{pq}}, \alpha^{\pm}_{\lambda_{p'q'}} \rangle$ which is equal to $\langle \lambda_{pq}, \theta \circ \lambda_{p'q'} \rangle$ (Theorem 3.41) and for this reason we will use essentially the observation from Section 5.1 and the explicit expression $\theta = \lambda_{11} + \lambda_{1m}$. Let us also remind that $\lambda_{11} \circ \lambda_{pq} = \lambda_{pq}$.

Let us first calculate $\langle \lambda_{pq}, \theta \circ \lambda_{p'q'} \rangle$ for **odd** m:

$$\langle \lambda_{pq}, \theta \circ \lambda_{p'q'} \rangle = \underbrace{\langle \lambda_{pq}, \lambda_{p'q'} \rangle}_{\mathrm{I}} + \underbrace{\langle \lambda_{pq}, \lambda_{1m} \circ \lambda_{p'q'} \rangle}_{\mathrm{II}} \tag{5.7}$$

Since we work with a set of inequivalent sectors, it holds that:

$$\mathrm{I} = \langle \lambda_{pq}, \lambda_{p'q'} \rangle = \begin{cases} 1 & \text{if } p = p',\ q = q' \\ 0 & \text{otherwise} \end{cases} \tag{5.8}$$

For the second term in the r.h.s of (5.7) we obtain:

$$\mathrm{II} = \langle \lambda_{pq}, \lambda_{1m} \circ \lambda_{p'q'} \rangle = \langle \lambda_{pq}, \lambda_{p',m+1-q'} \rangle = \begin{cases} 1 & \text{if } p = p',\ q = m + 1 - q' \\ 0 & \text{otherwise} \end{cases} \tag{5.9}$$

Then the sum of these terms gives:

$$\mathrm{I{+}II} = \langle \lambda_{pq}, \theta \circ \lambda_{p'q'} \rangle = \langle \alpha^+_{\lambda_{pq}}, \alpha^+_{\lambda_{p'q'}} \rangle = \langle \alpha^-_{\lambda_{pq}}, \alpha^-_{\lambda_{p'q'}} \rangle = \begin{cases} 1 & \text{if } p = p',\ q = q' \neq \frac{m+1}{2} \\ 1 & \text{if } p = p',\ q = m + 1 - q' \neq \frac{m+1}{2} \\ 2 & \text{if } p = p',\ q = q' = \frac{M+1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{5.10}$$

Then we arrive at the following:

Results for odd m:

i. $\alpha^+_{\lambda_{pq}}$ and $\alpha^-_{\lambda_{pq}}$ are:

- irreducible iff $q \neq \frac{m+1}{2}$
- reducible into two inequivalent sectors for $q = \frac{m+1}{2}$

ii. $\alpha^+_{\lambda_{pq}} \sim \alpha^+_{\lambda_{p'q'}}$ and $\alpha^-_{\lambda_{pq}} \sim \alpha^-_{\lambda_{p'q'}}$ iff λ_{pq} is the inverse of $\lambda_{p'q'}$ w.r.t reflection by the axis $\lambda_{f,\frac{m+1}{2}}$

iii. $\alpha^+_{\lambda_{p,\frac{m+1}{2}}}$ does not contain any sector $\alpha^+_{\lambda_{p'q'}}$ and $\alpha^-_{\lambda_{p,\frac{m+1}{2}}}$ does not contain any sector $\alpha^-_{\lambda_{p'q'}}$ if $q' \neq \frac{m+1}{2}$

Let us now calculate $\langle \lambda_{pq}, \theta \circ \lambda_{p'q'} \rangle$ for **even** m with completely analogous strategy:

$$\langle \lambda_{pq}, \theta \circ \lambda_{p'q'} \rangle = \underbrace{\langle \lambda_{pq}, \lambda_{p'q'} \rangle}_{\mathrm{I}'} + \underbrace{\langle \lambda_{pq}, \lambda_{m-1,1} \circ \lambda_{p'q'} \rangle}_{\mathrm{II}'} \tag{5.11}$$

Again, we show that:

$$\mathrm{I}' = \langle \lambda_{pq}, \lambda_{p'q'} \rangle = \begin{cases} 1 & \text{if } p = p',\ q = q' \\ 0 & \text{otherwise} \end{cases} \tag{5.12}$$

The second term in the r.h.s of (5.11) becomes:

$$\mathrm{II}\ ' = \langle \lambda_{pq}, \lambda_{m-1,1} \circ \lambda_{p'q'} \rangle = \langle \lambda_{pq}, \lambda_{m-p',q'} \rangle = \begin{cases} 1 & \text{if } p = m - p',\ q = q' \\ 0 & \text{otherwise} \end{cases} \tag{5.13}$$

Then we sum these two terms and obtain:

$$\mathrm{I}'+\mathrm{II}' = \langle \lambda_{pq}, \theta \circ \lambda_{p'q'} \rangle = \langle \alpha^+_{\lambda_{pq}}, \alpha^+_{\lambda_{p'q'}} \rangle = \langle \alpha^-_{\lambda_{pq}}, \alpha^-_{\lambda_{p'q'}} \rangle = \begin{cases} 1 & \text{if } p = p' \neq \frac{m}{2},\ q = q' \\ 1 & \text{if } p = m - p' \neq \frac{m}{2},\ q = q' \\ 2 & \text{if } p = p' = \frac{m}{2},\ q = q' \\ 0 & \text{otherwise} \end{cases} \tag{5.14}$$

Then we obtain the following:

Results for even m:

i. $\alpha^+_{\lambda_{pq}}$ and $\alpha^-_{\lambda_{pq}}$ are:

- irreducible iff $p \neq \frac{m}{2}$
- reducible into two inequivalent sectors for $p = \frac{m}{2}$

ii. $\alpha^+_{\lambda_{pq}} \sim \alpha^+_{\lambda_{p'q'}}$ and $\alpha^-_{\lambda_{pq}} \sim \alpha^-_{\lambda_{p'q'}}$ iff λ_{pq} is the inverse of $\lambda_{p'q'}$ w.r.t reflection by the axis $\lambda_{\frac{m}{2},g}$

iii. $\alpha^+_{\lambda_{\frac{m}{2},q}}$ does not contain any sector $\alpha^+_{\lambda_{p'q'}}$ and $\alpha^-_{\lambda_{\frac{m}{2},q}}$ does not contain any sector $\alpha^-_{\lambda_{p'q'}}$ if $p' \neq \frac{m}{2}$

5.1.2 Common content and equivalence among α^+ and α^- sectors

To be able to understand which of the irreducible sectors in the decomposition of the $\alpha^\pm_{\lambda_{pq}}$ sectors are DHR-sectors, we have to know the quantities $\langle \alpha^+_{\lambda_{pq}}, \alpha^-_{\lambda_{p'q'}} \rangle$, which are equal to the matrix elements $Z_{\lambda_{pq},\lambda_{p'q'}}$. These matrix elements can be obtained from the sesquilinear form $Z(\tau) = \sum_{p,q,p',q'} Z_{\lambda_{pq}\lambda_{p'q'}} \chi_{\lambda_{pq}} \chi^*_{\lambda_{p'q'}}$ associated to the concrete extension we work with. Classification of all sesquilinear forms is given in [Cappelli *et al.*, 1987].

Before we make use of this classification, let us first settle down a small notation issue. In the notations of the article, the central charge and the spin are parameterized by two successive integers p and p' instead of m:

$$\begin{aligned} c(p,p') &= 1 - \frac{6(p-p')^2}{pp'}, \qquad p' = \widetilde{m}-1,\ p = \widetilde{m} \quad \text{or } p' = \widetilde{m},\ p = \widetilde{m}-1 \\ h_{r,s}(p,p') &= \frac{(rp-sp')^2-(p-p')^2}{4pp'} \equiv h_{p'-r,p-s}(p,p'), \qquad 0 < r < p',\ 0 < s < p \end{aligned} \tag{5.15}$$

Let us remind, that the notation which we have chosen for this thesis is:

$$\begin{aligned} c(m) &= 1 - \frac{6}{m(m+1)} \\ h_{R,S}(m) &= \frac{[(m+1)R - mS]^2 - 1}{4m(m+1)} \equiv h_{m-R,m+1-S}(m), \qquad 1 \le R \le m-1,\ 1 \le S \le m \end{aligned}$$

It is clear that:

$$\frac{(p-p')^2}{pp'} = \frac{1}{m(m+1)} \quad \Rightarrow \quad \frac{1}{(\widetilde{m}-1)\widetilde{m}} = \frac{1}{m(m+1)} \quad \to \quad \widetilde{m} = m+1 \tag{5.16}$$

In [Cappelli *et al.*, 1987] the cases $p' = 4\rho$ and $p' = 4\rho + 2$ are classified. Then we obtain for the interesting for us cases:

- $m = 4n+1$ is realized for

$$\begin{aligned} p' &= 4\rho + 2 = \widetilde{m} = m+1 = 4n+2, \quad \rho = n \ge 1 \\ p &= \widetilde{m} - 1 = m = 4n+1 \end{aligned}$$

- $m = 4n+2$ is realized for

$$\begin{aligned} p' &= 4\rho + 2 = \widetilde{m} - 1 = m = 4n+2, \quad \rho = n \ge 1 \\ p &= \widetilde{m} = m+1 = 4n+3 \end{aligned}$$

Let us first consider the case $m = 4n+1$. With $r = S$, $s = R$ and $\chi_{RS} := \chi_{\lambda_{RS}}$ the relevant sesquilinear form is:

$$Z(\tau) = \frac{1}{2}\sum_{R=1}^{m-1}\left[\sum_{\substack{S=1,\text{odd} \\ S \neq \frac{m+1}{2}}} |\chi_{RS}|^2 + 2|\chi_{R,\frac{m+1}{2}}|^2 + \sum_{S=1,\text{odd}}^{\frac{m-1}{2}-1}\left(\chi_{RS}\chi^*_{R,m+1-S} + \chi_{R,m+1-S}\chi^*_{RS}\right)\right] \tag{5.17}$$

Let us remind that the sectors in the $m \times (m-1)$ lattice are pairwise equivalent, then we want to exclude half of them and we rewrite $Z(\tau)$ as:

$$\begin{aligned} Z(\tau) &= \frac{1}{2}\sum_{R=1}^{\frac{m-1}{2}}\Bigg[\sum_{\substack{S=1,\mathrm{odd}\\ S\neq\frac{m+1}{2}}}|\chi_{RS}|^2+2|\chi_{R,\frac{m+1}{2}}|^2+\sum_{S=1,\mathrm{odd}}^{\frac{m-1}{2}-1}\left(\chi_{RS}\chi^*_{R,m+1-S}+\chi_{R,m+1-S}\chi^*_{RS}\right)\Bigg]+ \\ &+\frac{1}{2}\sum_{R=\frac{m+1}{2}}^{m-1}\Bigg[\sum_{\substack{S=1,\mathrm{odd}\\ S\neq m+1-\frac{m+1}{2}}}|\chi_{m-R,m+1-S}|^2+2|\chi_{m-R,m+1-\frac{m+1}{2}}|^2+ \\ &\qquad+\sum_{S=1,\mathrm{odd}}^{\frac{m+1}{2}-1}\left(\chi_{m-R,m+1-S}\chi^*_{m-R,S}+\chi_{m-R,S}\chi^*_{m-R,m+1-S}\right)\Bigg] \end{aligned} \tag{5.18}$$

It is easy to check that:

$$\begin{aligned} \sum_{R'=\frac{m+1}{2}}^{m-1}F(\chi_{m-R',f(S)}) &= \sum_{R=1}^{\frac{m-1}{2}}F(\chi_{R,f(S)}), \quad \text{for a fixed } S \text{ and } R:=m-R' \\ \sum_{\substack{S'=1,\mathrm{odd}\\ S'\neq m+1-\frac{m+1}{2}}}^{m}G(\chi_{g(R),m+1-S'}) &= \sum_{\substack{S=1,\mathrm{odd}\\ S=\frac{m+1}{2}}}G(\chi_{g(R),S}), \quad \text{for a fixed } R \text{ and } S:=m+1-S' \end{aligned} \tag{5.19}$$

and then we can prove that the two big terms in the r.h.s of (5.18) are equal. Further, we show that:

$$\sum_{S'=1,\mathrm{odd}}^{\frac{M-1}{2}-1}\chi_{R,m+1-S'}\chi^*_{R,S'} = \sum_{S=\frac{M+1}{2},\mathrm{odd}}^{m}\chi_{R,S}\chi^*_{R,m+1-S} \tag{5.20}$$

and all of this allow us to simplify $Z(\tau)$ significantly:

$$Z(\tau)=\sum_{R=1}^{\frac{m-1}{2}}\Bigg[\sum_{\substack{S=1,\mathrm{odd}\\ S\neq\frac{m+1}{2}}}^{m}|\chi_{RS}|^2+2|\chi_{R,\frac{m+1}{2}}|^2+\sum_{\substack{S=1,\mathrm{odd}\\ S\neq\frac{m+1}{2}}}^{m}\chi_{RS}\chi^*_{R,m+1-S}\Bigg] \tag{5.21}$$

Results for $m = 4n + 1$:

The non-zero entries of the Z-matrix are:

- $Z_{\lambda_{RS}\lambda_{RS}} = 1$ for $R = 1 \ldots \frac{M-1}{2}$; $\quad S = 1 \ldots M, \quad S = \text{odd}, \; S \neq \frac{M+1}{2}$

- $Z_{\lambda_{R,\frac{M+1}{2}}\lambda_{R,\frac{M+1}{2}}} = 2$ for $R = 1 ... \frac{M-1}{2}$
- $Z_{\lambda_{R,M+1-S}\lambda_{RS}} = 1$ for $R = 1 ... \frac{M-1}{2}$; $\quad S = 1 ... M, \quad S = \text{odd}, \; S \neq \frac{M+1}{2}$

It means that:

i. $\alpha^+_{\lambda_{pq}} = \alpha^-_{\lambda_{pq}}$ for odd q

ii. $\alpha^+_{\lambda_{pq}} = \alpha^-_{\lambda_{p'q'}}$ if λ_{pq} is inverse to $\lambda_{p'q'}$ with respect to reflection by $\lambda_{f,\frac{M+1}{2}}$ axis

Conclusion for $m = 4n+1$:

The complete set of irreducible DHR sectors of the index 2 extension for a fixed $m = 4n+1$ is
$\{\alpha^+_{\lambda_{pq}}\}_{p=1...\frac{m-1}{2},q=1...\frac{m-1}{2},q:\,odd} \equiv \{\alpha^-_{\lambda_{pq}}\}_{p=1...\frac{m-1}{2},q=1...\frac{m-1}{2},q:\,odd}$ *plus the two irreducible sectors of every* $\alpha^+_{\lambda_{p\frac{m+1}{2}}}$, $p = 1, ..., \frac{m-1}{2}$.

Let us now consider the case $m = 4n+2$. With $r = R$ and $s = S$ the corresponding sesquilinear form is:

$$Z(\tau) = \frac{1}{2}\sum_{S=1}^{m}\left[\sum_{\substack{R=1,\,\text{odd}\\ R\neq\frac{m}{2}}}|\chi_{RS}|^2 + 2|\chi_{\frac{m}{2},S}|^2 + \sum_{R=1,\,\text{odd}}^{\frac{m}{2}-2}\left(\chi_{RS}\chi^*_{m-R,S} + \chi_{m-R,S}\chi^*_{RS}\right)\right] \quad (5.22)$$

Following the same way as in the previous case we are able to simplify $Z(\tau)$:

$$Z(\tau) = \sum_{S=1}^{\frac{m}{2}}\left[\sum_{\substack{R=1,\,\text{odd}\\ R\neq\frac{m}{2}}}^{m}|\chi_{RS}|^2 + 2|\chi_{\frac{m}{2},S}|^2 + \sum_{\substack{R=1,\,\text{odd}\\ R\neq\frac{m}{2}}}^{m}\chi_{RS}\chi^*_{m-R,S}\right] \quad (5.23)$$

Results for $m = 4n+2$:

The non-zero entries of the Z-matrix are:

- $Z_{\lambda_{RS},\lambda_{RS}} = 1$ for $R = 1 ... M-1$, $R = \text{odd}$, $R \neq \frac{M}{2} \quad S = 1 ... \frac{M}{2}$
- $Z_{\lambda_{\frac{M}{2},S},\lambda_{\frac{M}{2},S}} = 2$ for $S = 1 ... \frac{M}{2}$
- $Z_{\lambda_{RS},\lambda_{M-R,S}} = 1$ for $R = 1 ... M-1$, $R = \text{odd}$, $R \neq \frac{M}{2} \quad S = 1 ... \frac{M}{2}$

It means that:

i. $\alpha^+_{\lambda_{pq}} = \alpha^-_{\lambda_{pq}}$ for odd p

ii. $\alpha^+_{\lambda_{pq}} = \alpha^-_{\lambda_{p'q'}}$ if λ_{pq} is inverse to $\lambda_{p'q'}$ with respect to reflection by $\lambda_{\frac{M}{2},g}$ axis

Conclusion for $m = 4n + 2$:

The complete set of irreducible DHR sectors of the index 2 extension for a fixed $m = 4n + 2$ is
$\{\alpha^+_{\lambda_{pq}}\}_{p=1\ldots\frac{m}{2}-1,\, q=1\ldots\frac{m}{2},p:\, odd} \equiv \{\alpha^-_{\lambda_{pq}}\}_{p=1\ldots\frac{m}{2}-1,\, q=1\ldots\frac{m}{2},p:\, odd}$ *plus the two irreducible sectors of every* $\alpha^+_{\lambda_{\frac{m+1}{2},q}}$, $p = 1, ..., \frac{m}{2}$.

5.2 DHR sectors for (A_{28}, E_8) extension with $m = 29$

In this section we will calculate the DHR sectors of the only model from the minimal series, which does not correspond to Virasoro and which is not proven to be expressible as a coset model.

The (A_{28}, E_8) extension with $m = 29$, $\theta = \lambda_{1,1} + \lambda_{1,29} + \lambda_{1,11} + \lambda_{1,19}$ and index $\frac{\sqrt{30-6\sqrt{5}}}{2\sin(\pi/30)}$, is realized in the [Cappelli *et al.*, 1987] notation as $p' = 30 = \widetilde{m}$, $p = 29 = \widetilde{m} - 1$, for $s = R,\ r = S$ and the corresponding sesquilinear form is:

$$\begin{aligned} Z(\tau) &= \frac{1}{2}\sum_{R=1}^{28}\left[|\chi_{R,1} + \chi_{R,11} + \chi_{R,19} + \chi_{R,29}|^2 + |\chi_{R,7} + \chi_{R,13} + \chi_{R,17} + \chi_{R,23}|^2\right] \\ &= \sum_{R=1}^{14}\left[|\chi_{R,1} + \chi_{R,11} + \chi_{R,19} + \chi_{R,29}|^2 + |\chi_{R,7} + \chi_{R,13} + \chi_{R,17} + \chi_{R,23}|^2\right] \end{aligned} \tag{5.24}$$

Having in mind Proposition 5.2, one can see straightforward from this formula that the DHR sectors of the extension are 28, half of them is contained in or coincides with $\alpha^\pm_{\lambda_{R,1}}$, $\alpha^\pm_{\lambda_{R,11}}$, $\alpha^\pm_{\lambda_{R,19}}$, $\alpha^\pm_{\lambda_{R,29}}$, the other half is contained in or coincides with $\alpha^\pm_{\lambda_{R,7}}$, $\alpha^\pm_{\lambda_{R,13}}$, $\alpha^\pm_{\lambda_{R,17}}$, $\alpha^\pm_{\lambda_{R,23}}$. So from now on we will concentrate our investigation for this sectors.

Let us now compute $\langle\alpha^\pm_{\lambda_{pq}}, \alpha^\pm_{\lambda_{p'q'}}\rangle$, which is equal to $\langle\lambda_{pq}, \theta \circ \lambda_{p'q'}\rangle$ (Theorem 3.41) and in this case $\theta = \lambda_{1,1} + \lambda_{1,29} + \lambda_{1,11} + \lambda_{1,19}$. To calculate these brackets we take into consideration the following fusion rules:

$$\begin{aligned} \lambda_{1,11} \circ \lambda_{pq} &= \oplus^{S}_{\substack{s=|11-q|+1 \\ s\mathrm{mod}2=q\mathrm{mod}2}} \lambda_{ps} \quad S = \min\begin{cases} 11 + q - 1 = 10 + q \\ 60 - q + 12 = 48 - q \end{cases} \\ \lambda_{1,19} \circ \lambda_{pq} &= \oplus^{S}_{\substack{s=|19-q|+1 \\ s\mathrm{mod}2=q\mathrm{mod}2}} \lambda_{ps} \quad S = \min\begin{cases} 19 + q - 1 = 18 + q \\ 60 - q + 20 = 40 - q \end{cases} \end{aligned} \tag{5.25}$$

and the observation from Section 5.1. As we discussed before, the explicit computation of the set of the brackets $\langle\alpha^\pm_{\lambda_{pq}}, \alpha^\pm_{\lambda_{p'q'}}\rangle$ gives us the complete information about the reducibility and common content of the induced sectors $\alpha^\pm_{\lambda_{RS}}$. This information, combined with the analysis of the Z-matrix above, allows us to find the complete set of DHR sectors of the extension and we present them on the following picture:

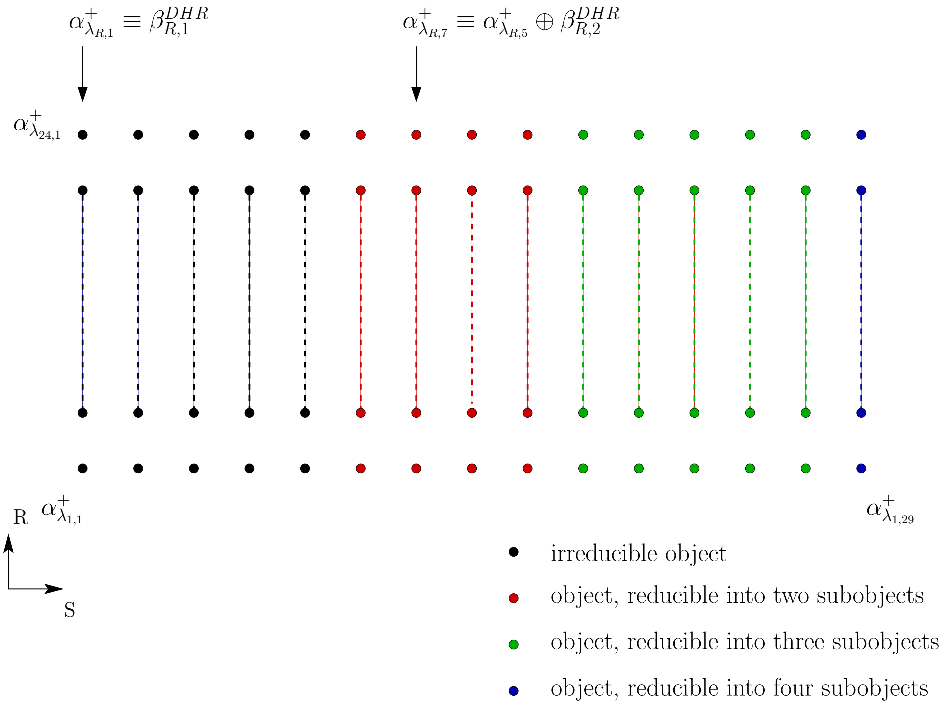

Figure 5.3: *Irreducible sectors $\beta^{DHR}_{R,1}$ and $\beta^{DHR}_{R,2}$ of the (A_{28}, E_8) extension for $m = 29$*

On the picture above each knot on the lattice accounts for the induced endomorphism $\alpha^+_{\lambda_{RS}}$. (We obtain exactly the same scheme for the $\alpha^-_{\lambda_{RS}}$ endomorphisms, although $\alpha^+_{\lambda_{RS}}$ in general do not coincide with $\alpha^-_{\lambda_{RS}}$.) We see the set of inequivalent and irreducible DHR endomorphisms of the extension is $\beta^{DHR}_{R,1} := \alpha^+_{\lambda_{R,1}}$, $\beta^{DHR}_{R,2}$ is hidden in the decomposition $\alpha^+_{\lambda_{R,7}} = \alpha^+_{\lambda_{R,5}} \oplus \beta^{DHR}_{R,2}$ and in both cases $R = 1...14$. (Similarly, $\beta^{DHR}_{R,1} := \alpha^-_{\lambda_{R,1}}$ and $\alpha^-_{\lambda_{R,7}} = \alpha^-_{\lambda_{R,5}} \oplus \beta^{DHR}_{R,2}$).

We can also calculate the fusion rules of $\{\beta^{DHR}_{R,i}\}_{\substack{R=1...14\\ i=1,2}}$, using the homomorphism property $\alpha_\lambda \circ \alpha_\mu = \alpha_{\lambda\circ\mu}$ and linearity $\alpha_{\lambda\oplus\mu} = \alpha_\lambda \oplus \alpha_\mu$:

$$\beta^{DHR}_{R,i} \circ \beta^{DHR}_{R',j} = \oplus^{\min(R+R'-1,\, 2m-R-R'-1)}_{\substack{p=|R-R'|+1\\ R+R'+1:\text{odd}}} \oplus^{\max(i,j)}_{q=|i-j|+1} \beta^{DHR}_{pq} \tag{5.26}$$

We also compute the statistical dimensions in terms of the statistical dimensions of the Virasoro subnet, using that α-induction preserves the dimensions and that the dimension of a sum of two sectors is equal to the sum of the dimensions of these two sectors:

$$d(\beta^{DHR}_{R,1}) = d(\lambda_{R,1}), \quad d(\beta^{DHR}_{R,2}) = d(\lambda_{R,7}) - d(\lambda_{R,5}) \tag{5.27}$$

5.3 DHR sectors for the remaining higher index extensions

The strategy to find the inequivalent irreducible DHR sectors of the remaining three higher local extensions is the same as for the (A_{29}, E_8)-extension for $m = 29$. Therefore, in this section we will just present the graphics with final results with a short comment on them. Again, these graphics will represent only the induced endomorphisms $\alpha^+_{\lambda_{RS}}$ and they are equivalent to the graphics representing the induced endomorphisms $\alpha^-_{\lambda_{RS}}$, even though $\alpha^+_{\lambda_{RS}}$ and $\alpha^-_{\lambda_{RS}}$ do not coincide in general.

The DHR sectors of the (E_8, A_{30}) extension with $m = 30$, $\theta = \lambda_{1,1} + \lambda_{29,1} + \lambda_{11,1} + \lambda_{19,1}$ and index $\frac{\sqrt{30-6\sqrt{5}}}{2\sin(\pi/30)}$ are presented on fig.5.4. Note that fig.5.3 and fig.5.4 are equivalent up to exchange of the axes S and R and the second figure has one more row of sectors. This equivalence is due to the specific structure of the fusion rules, the very similar canonical endomorphisms of the two models and the fact, that the two extensions share the same sesquilinear from $Z(\tau)$ from the list in [Cappelli *et al.*, 1987]. Note also that the presence of $\lambda_{1,m}$ in the expressions for θ together with the symmetry of $Z(\tau)$ associated to the extensions in consideration allow us to find all the inequivalent DHR sectors just in one quarter of the lattice $\alpha^{\pm}_{\lambda_{RS}}$, $R \in [1..m-1]$, $S \in [1..m]$.

The same equivalence is observed for the second couple of models: (A_{10}, E_6) extension corresponding to $m = 11$ with index $3 + \sqrt{3}$ and $\theta = \lambda_{1,1} + \lambda_{1,7}$ and (E_6, A_12) extension corresponding to $m = 12$ with index $3 + \sqrt{3}$ and $\theta = \lambda_{1,1} + \lambda_{7,1}$, whose DHR sectors are presented respectively on fig.5.5 and fig.5.6. Here we have to consider a larger part of the lattice $\alpha^{\pm}_{\lambda_{RS}}$ because the canonical endomorphism now does not contain $\lambda_{1,m}$ and the corresponding sesquilinear form does not possess the symmetry from the previous case.

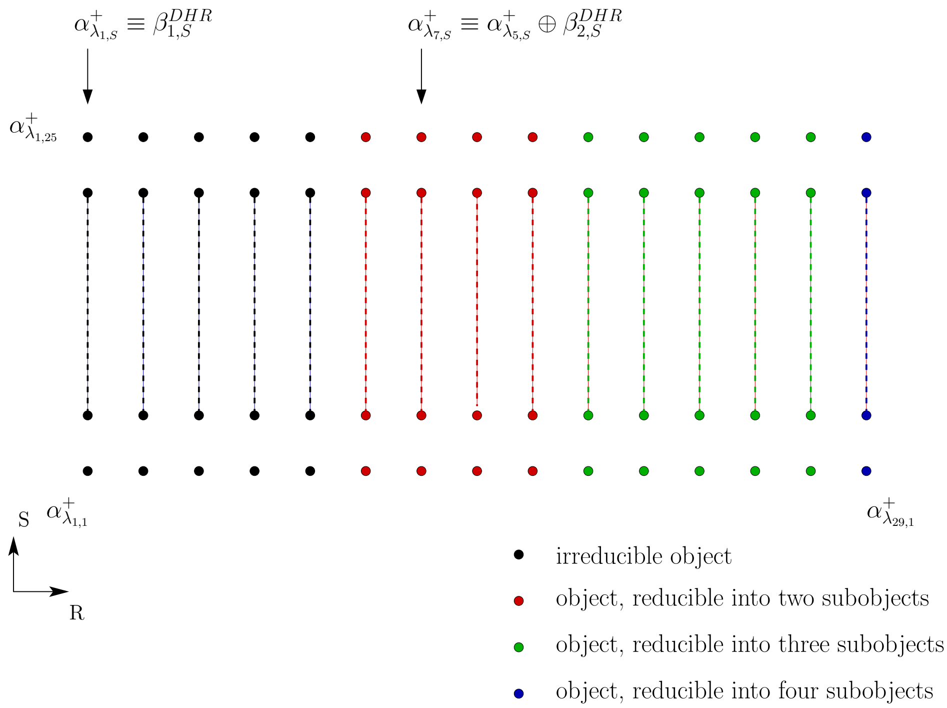

Figure 5.4: *Irreducible sectors $\beta^{DHR}_{1,S}$ and $\beta^{DHR}_{2,S}$ of the (E_8, A_{30}) extension for $m = 30$*

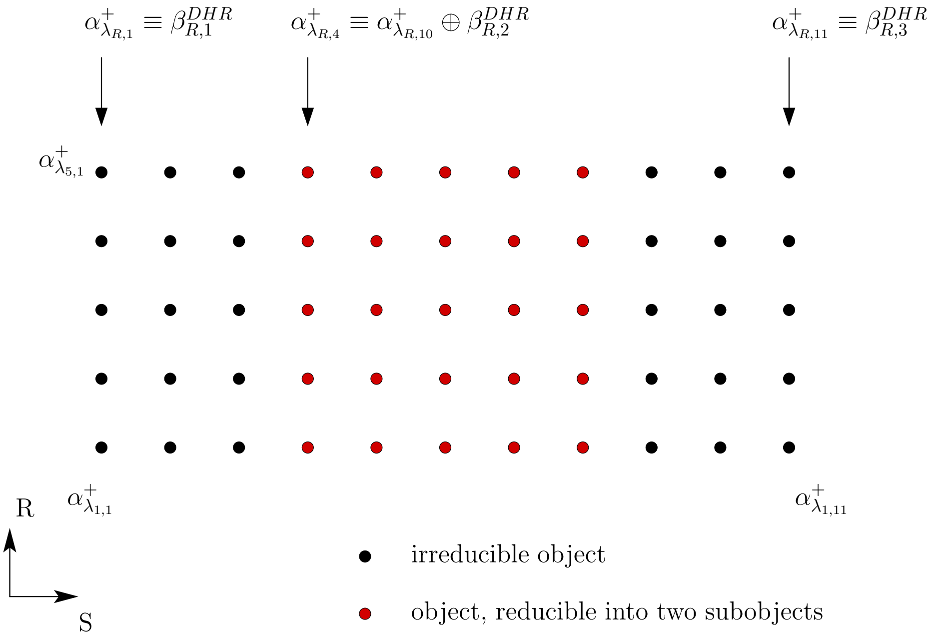

Figure 5.5: *Irreducible sectors $\beta^{DHR}_{R,1}$, $\beta^{DHR}_{R,2}$ and $\beta^{DHR}_{R,3}$ of the (A_{10}, E_6) extension for $m = 11$*

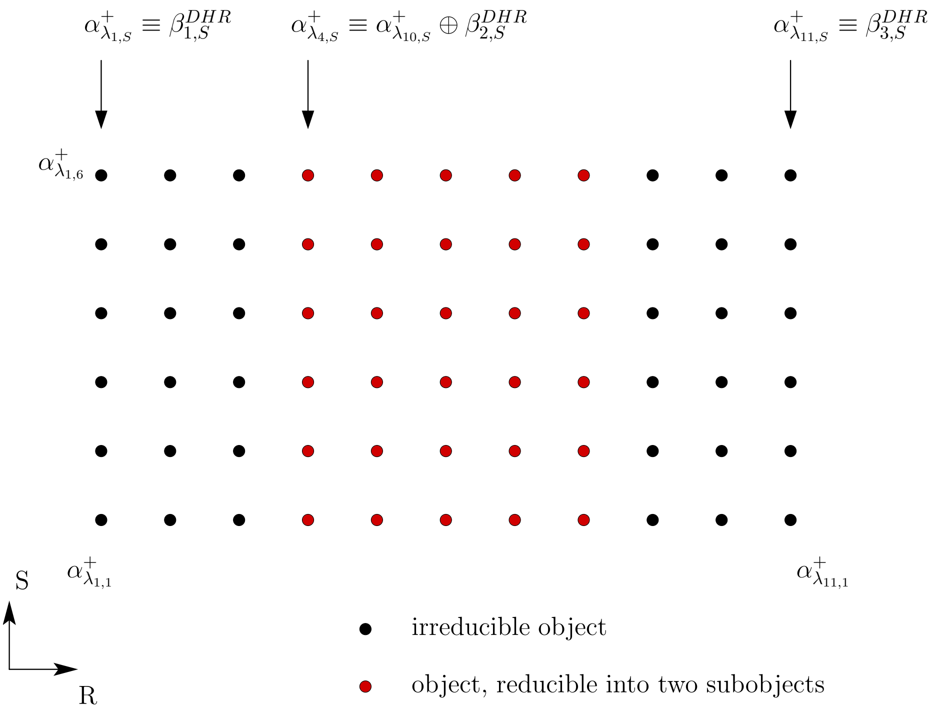

Figure 5.6: *Irreducible sectors $\beta^{DHR}_{1,S}$, $\beta^{DHR}_{2,S}$ and $\beta^{DHR}_{3,S}$ of the (E_6, A_{12}) extension for $m = 12$*

6 The local commutation relations

In this chapter we will explore the commutation relations among quasiprimary fields in chiral conformal invariant theories. We will perform our study following closely the example of the Lüscher–Mack theorem, which determines completely the commutation relations of the stress–energy tensor with itself. We will show that our commutators are intrinsically determined up to model dependent structure constants. These structure constants must further obey an infinite set of constraints coming from the Lie algebra structure relations.

Furthermore, on the level of test functions commutators give rise to (the unique) local intertwiners of the corresponding $sl(2,\mathbb{R})$ action and one verifies that the various intertwiner spaces of tensor products of representations are finite dimensional. With the help of transformation matrices among these intertwiner spaces we are able to achieve "reduction" of the field algebra, which amounts to stripping off the test functions, thus disentangling the "kinematic" representation details. On the new "reduced space" the commutator turns into a multicomponent "reduced bracket". The idea how to achieve a reduced version of the Jacobi identity was cherished from [Bowcock, 1991], where a Jacobi identity among structure constants of commutators among Fourier modes of quasiprimary fields was considered.

Finally, in our theory there must be a symmetric positive quadratic form representing the vacuum state and we show that the reduced bracket imposes an invariance condition on this bracket. Then a new axiomatization of a chiral conformal QFT must consist of a reduced space, a reduced bracket and an invariant quadratic form, of course subject to some additional conditions, which we will discuss in the following.

6.1 The general form of the local commutation relations in 2D chiral conformal field theories

In this section we will show that the local commutation relations in conformal chiral quantum field theories are intrinsically determined up to numerical factors ("structure constants") by locality, conformal invariance and Wightman positivity, and that the Lie algebra structure imposes further constrains on the possible values of the structure constants.

It will be more convenient for us to work with smeared fields. Since $A'(f) = -A(f')$, we do not consider the derivatives of quasiprimary fields as independent fields. Hence, a basis of the field algebra is an infinite set of quasiprimary fields. In a decent theory, e.g. such that $e^{-\beta L_0}$ is a trace-class operator, the number of quasiprimary fields of a given dimension is finite (because each field of dimension a contributes a power series in $t = e^{-\beta}$ with leading term t^a). We shall denote the basis of fields of scaling dimension a by W_a and assume without loss of generality that all $A \in W_a$ are hermitian fields.

It will be enough to find just the commutators among the basis quasiprimary fields. Our strategy to understand the general structure of Möbius covariant commutators in chiral conformal field theories is similar to that of the Lüscher–Mack theorem:

Proposition 6.1. *Locality, scale invariance and Wightman positivity imply the following general form of the commutator of two smeared quasiprimary fields $A(f)$ and $B(g)$:*

$$-i\,[A(f),B(g)] = \sum_{c<a+b}\sum_{C\in W_c} F^C_{AB} C\left(\lambda^c_{ab}(f,g)\right) \tag{6.1}$$

where a, b are the scaling dimensions of A and B, the sum runs over a basis of quasiprimary fields of scaling dimension $c<a+b$, F^C_{AB} are structure constants and

$$\lambda^c_{ab}(f,g) = \sum_{\substack{p,q\geq 0\\ p+q=a+b-c-1}} \lambda^c_{ab}(p,q)\,\partial^p f\cdot\partial^q g \tag{6.2}$$

are bilinear maps on the test functions such that $\mathrm{supp}\,\lambda^c_{ab}\subset \mathrm{supp}\,f\cap\mathrm{supp}\,g$*, i.e. λ^c_{ab} preserves the supports. The maps depend only on the dimensions of the fields involved.*

Proof. We present here the main steps of the proof:

1. Locality implies that the commutator $-i\,[A(x),B(y)]$ has support on the line $x=y$. Then follows that $-i\,[A(x),B(y)] = \sum_{l=0}^n \delta^{(l)}(x-y)O_l(y)$, where O_l are linear combinations of quasiprimary fields and derivatives. This means that in the smeared version $-i\,[A(f),B(g)]$ a quasiprimary field C must appear with a test function of the form $\sum_{p,q\geq 0} d^C_{AB}(p,q)\partial^p f\cdot\partial^q g$. The coefficients $d^C_{AB}(p,q)$ satisfy a recursion in p and q, coming from Möbius invariance, and the solution of this recursion is fixed, up to some numerical constant, only by the scaling dimensions of the fields A, B, C. The numerical constant can be absorbed in the coefficients F^C_{AB}.

2. Scaling invariance implies $p+q=n$, $C(y)$ is a local field of scaling dimension $a+b-n-1$.

3. Wightman positivity implies that the scaling dimension of the fields in the theory must be non-negative (unitarity bound), hence $c\in[0,a+b-1]$.

Observation. The recursion for $\lambda^c_{ab}(p,q)$ coming from the Möbius invariance for fixed $a,b\geq 1$ and positive c is solved by:

$$\lambda^c_{ab}(p,q) = \binom{p+q}{p}(-1)^q\frac{(c+b-a)_p(c+a-b)_q}{(2c)_{p+q}} \tag{6.3}$$

where $(x)_p$ denotes the Pocchammer symbol:

$$(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} \tag{6.4}$$

In particular, the maps $\lambda^c_{ab}(f,g) = \sum_{p+q=a+b-c-1} \lambda^c_{ab}(p,q)\partial^p f \cdot \partial^q g$ possess the graded symmetry property:

$$\lambda^c_{ab}(f,g) = (-1)^{a+b-c-1}\lambda^c_{ba}(g,f) \tag{6.5}$$

Note that this (anti-)symmetry respects the $\mathbb{Z}_2$ grading of the source and range spaces, but the linear maps λ^c_{ab} themselves are not $\mathbb{Z}_2$ graded.

The denominator $(2c)_{p+q}$ in (6.3) is a mere convention and was chosen such that for $c = 0$ the expression $\lambda^c_{ab}(p,q)$ can be defined as a continuous limit from $c > 0$ that vanishes if $a \neq b$, thus expressing the fact that the unit operator can contribute to the commutator of the two fields only if these have equal dimensions.

It is noteworthy to recognize that λ^c_{ab} coincide with the Rankin–Cohen brackets arising in the theory of modular forms. The latter are bilinear differential maps $[f,g]_n : M_{2k} \times M_{2l} \to M_{2k+2l+2n}$ on the spaces of modular forms of weights $2k$, $2l$ ([Rankin, 1956; Cohen, 1975; Cohen *et al.*, 1996]). In this context, of course, the test functions have to be replaced by modular forms, and the emphasis is on the discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$ of $\mathrm{SL}(2,\mathbb{R})$, under which modular forms are invariant. The precise relation is (with notations as in [Cohen *et al.*, 1996]):

$$[f,g]^{(k=1-a,l=1-b)}_{n=a+b-c-1} \equiv \rho^c_{ab}(f,g) = \frac{(2c)_{a+b-c-1}}{(a+b-c-1)!} \cdot \lambda^c_{ab}(f,g) \tag{6.6}$$

We will give some more comments later in Section 6.4.1.

It becomes clear that the overall structure of the commutators in conformal chiral field theories is to a great extent fixed — we know fields of which dimensions contribute to the commutator of any pair of fields and with which test functions these fields are smeared. The only unknown ingredients are the structure constants F^C_{AB}, which are numbers. We shall now investigate further restrictions of the structure constants due to the Lie algebra structure relations of the commutator.

Observation. The anti–symmetry of commutators together with the symmetry property (6.5) of λ^c_{ab} implies the following symmetry rule for the structure constants:

$$F^C_{AB} = (-1)^{a+b-c}\, F^C_{BA} \tag{6.7}$$

Taking adjoints and recalling that the basis consists of hermitian fields, one finds that F^C_{AB} are real numbers.

Further restrictions for the structure constants F^C_{AB} come from the Jacobi identity for commutators of smeared field operators, as we will see in Section 6.6. We cannot derive this restrictions directly, because Jacobi identity in its original form would produce constraints burdened with test functions. A reduction of the field algebra, performed in Section 6.5, will allow us to strip off the test functions and to achieve a reduced Jacobi identity involving only the structure constants F^C_{AB} To prepare the ground for that, in the next subsections we study the effect of the commutator on the test functions level.

The structure constants F^C_{AB} are also related to the amplitudes of 2- and 3-point functions as we will elaborate in Section 6.7.

We also pursued the idea that with the help of the coproduct of a Lie algebra of generating quasiprimary fields we could recognize the compounds of the quasi–primary fields appearing in the sum above for some fixed pair of A, B. However, working out some simple examples showed that this method is not giving us the whole information about the building blocks of these fields and that we have to search for some more powerful tools.

We then concentrated on the minimal Lie algebra containing the stress–energy tensor T — its enveloping Lie algebra $E(T)$. We believed that at least there F_C^{AB} must be intrinsically determined. The original idea was to determine F_C^{AB} in $E(T)$ just on the basis of Jacobi identities involving at least one operator T. The scheme was the following. We wanted first to obtain recursively the coefficients F_C^{TB} as functions of $N_{C''}^{TC'}$, where both C' and C'' have lower dimensions than B. This relation should come out from imposing Jacobi identity on two copies of T and a "vector", consisting of all fields of lower dimensions and B itself. Once having the coefficients F_C^{TB} in hand and applying Jacobi identity to T and two quasi–primary fields, one should get also the overall coefficients F_C^{AB} in $E(T)$. However, after some computations by hand for low dimensions, we were convinced that the Jacobi identity does not lead to sufficient knowledge about the desired normalization factors and additional model information is needed as an input.

6.2 λ_{ab}^c are intertwiners

Quasiprimary fields of scaling dimension a extend to a larger test function space than just the Schwartz functions, namely to the space π_a of smooth functions on $\mathbb{R}$ for which $x^{2-2a}f(x^{-1})$ extends smoothly to $x = 0$. We regard this space as a representation of $sl(2,\mathbb{R})$ with generators p, d, and k such that:

$$(pf)(x) = i\partial f(x), \qquad (df)(x) = i(x\partial+1-a)f(x), \qquad (kf)(x) = i(x^2\partial+2(1-a)x)f(x) \tag{6.8}$$

We must remark that π_a is neither irreducible nor unitary. In particular, the inner product induced by the 2-point function annihilates the $(2a-1)$-dimensional subspace of polynomials of order $2a-2$.

The direct product $\pi_a\times\pi_b$ equals $\pi_a\otimes\pi_b$ as a space and carries the representation $(\pi_a\otimes\pi_b)\circ\Delta$, where the Δ is the Lie algebra coproduct.

Then the maps

$$\lambda_{ab}^c : \pi_a \times \pi_b \to \pi_c, \qquad f\otimes g \to \lambda_{ab}^c(f,g) = \sum_{p+q=a+b-c-1} \lambda_{ab}^c(p,q)\,\partial^p f\cdot\partial^q g \tag{6.9}$$

intertwine the corresponding $sl(2,\mathbb{R})$ actions on the spaces of test functions. Their distinguishing feature among all such intertwiners is that they preserve supports (see above), for which we call them *local intertwiners*. The constructive argument in the proof of Proposition 6.1 means that they are actually the unique local intertwiners of the $sl(2,\mathbb{R})$ action. Therefore, our task will be to understand the category of representations π_a of $sl(2,\mathbb{R})$ equipped with the local intertwiners.

6.3 Bases for the intertwiner spaces

One important observation is that the bound $c < a + b$ for λ^c_{ab} guarantees that the intertwiner spaces $\pi_{a_1} \times \pi_{a_2} \times ... \times \pi_{a_n} \to \pi_e$, where $e \leq \sum_{i=1}^n a_i - n + 1$, are finite–dimensional. In this subsection we will construct bases for intertwiner spaces and will describe the relevant matrices for a switch between bases.

Our "default" choice of basis, adapted to the structures which appear in our calculations (nested commutators), will be the following:

Definition 6.2 (Default basis for intertwiners $\left(T_{\underline{a}_n}\right)^{\underline{m}_{n-1}}$). *Let us define the operator:*

$$\left(T_{\underline{a}_n}\right)^{\underline{m}_{n-1}} = \lambda^e_{a_1\epsilon_1} \circ \left(1_{a_1} \times \lambda^{\epsilon_1}_{a_2\epsilon_2} \circ \left(1_{a_1} \times 1_{a_2} \times \lambda^{\epsilon_2}_{a_3\epsilon_3} \circ \left(... \circ \left(1_{a_1} \times 1_{a_2} \times ... \times 1_{a_{n-2}} \times \lambda^{\epsilon_{n-2}}_{a_{n-1}a_n}\right) ...\right)\right)\right) \tag{6.10}$$

in which $\underline{x}_n$ stands for the n-tuple $(x_1, ..., x_n)$, $\underline{a}_n$ is the n-tuple of scaling dimensions a_i and the indices $m_i \in \mathbb{N}_0$ are defined as:

$$\begin{aligned} &m_{n-1} := a_{n-1} + a_n - \epsilon_{n-2} - 1, \quad m_1 = a_1 + \epsilon_1 - e - 1, \\ &m_i := a_i + \epsilon_i - \epsilon_{i-1} - 1 \quad \text{for} \quad i = 2...n-2 \end{aligned} \tag{6.11}$$

Then the set of operators $\left(T_{\underline{a}_n}\right)^{\underline{m}_{n-1}}$, such that $M(\underline{a}_n, e) := m_1+...+m_{n-1} = \sum_{i=1}^n a_i - e - n + 1$, constitute a basis for the intertwiner space $\pi_{a_1} \times \pi_{a_2} \times ... \times \pi_{a_n} \to \pi_e$.

Observation. The indices ϵ_i are subject to a restriction, originating from the bound $c < a+b$ for λ^c_{ab}:

$$\epsilon_{n-2} \leq a_{n-1} + a_n - 1, \quad \epsilon_1 \geq e - a_1 + 1, \quad \epsilon_i \leq \sum_{k=i+1}^{n} a_k - n + i + 1 \quad \text{for} \quad i = 1...n-3 \tag{6.12}$$

For a fixed n-tuple of the m's we can obtain the ϵ's recursively:

$$\epsilon_i = \sum_{s=i+1}^{n} a_s - \sum_{t=i+1}^{n-1} m_t - n + i + 1, \qquad e = \sum_{s=1}^{n} a_s - \sum_{t=1}^{n-1} m_t - n + 1 \tag{6.13}$$

It should be noted that some of the dimensions ϵ_i may be negative. We shall ignore unitarity bound (admitting only non-negative dimensions) at this point. It will be imposed later (Section 6.8).

Remark. The operators $\left(T_{\underline{a}_n}\right)^{\underline{m}_{n-1}}$ are multilinear maps on functions $(f_1, ..., f_n)$ such that $f_i \in \pi_{a_i}$. The images $\left(T_{\underline{a}_n}\right)^{\underline{m}_{n-1}}(f_1, ..., f_n)$ are test functions belonging to the space π_e with $e = \sum_{s=1}^n a_s - \sum_{t=1}^{n-1} m_t - n + 1$.

Occasionally it will be necessary to consider nested brackets in different order:

Definition 6.3 (Basis for intertwiners $\left(T_{B,\underline{a}_n}\right)^{\underline{m}_{n-1}}$). *With $\left(T_{B,\underline{a}_n}\right)^{\underline{m}_{n-1}}$ we denote the elements of an alternative basis of intertwiners, which are constructed similarly to $\left(T_{\underline{a}_n}\right)^{\underline{m}_{n-1}}$, but where λ's couple arguments and results of previous couplings in a different way and B carries information about the coupling order. We call B a bracket scheme.*

Example. An alternative basis for the intertwiner space $\pi_a \times \pi_b \times \pi_c \to \pi_e$ may be:

$$\left(T_{S,abc}\right)^{m_1 m_2} := \lambda^e_{\epsilon c} \circ (\lambda^\epsilon_{ab} \times 1_c), \quad m_1 + m_2 = a + b + c - e - 2 \tag{6.14}$$

6.4 Transformation matrices

From (6.5) one immediately has:

$$\left(T_{abc}\right)^{m_1 m_2}(f,g,h) = (-1)^{m_1} \left(T_{S,bca}\right)^{m_1 m_2}(g,h,f) = (-1)^{m_2} \left(T_{acb}\right)^{m_1 m_2}(f,h,g) \tag{6.15}$$

For the analysis of the Jacobi identity, however, we shall need relations among $\left(T_{abc}\right)^{m_1 m_2}(f,g,h)$, $\left(T_{bca}\right)^{m_1 m_2}(g,h,f)$ and $\left(T_{cab}\right)^{m_1 m_2}(h,f,g)$, not covered by (6.15). In this subsection we introduce the transformation matrices for general permutations and rebracketings.

Definition 6.4 (The matrix $\left(Z_{B_1 B_2,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\underline{\widetilde{m}}_{n-1}}_{\underline{m}_{n-1}}$). *Let us define the matrix $\left(Z_{B_1 B_2,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\underline{\widetilde{m}}_{n-1}}_{\underline{m}_{n-1}}$ which relates two bases T_{B_1} and T_{B_2} with permuted arguments:*

$$\left(T_{B_1,\sigma_{\underline{i}_n}(\underline{a}_n)}\right)^{\underline{\widetilde{m}}_{n-1}} \circ \tau_{\sigma_{\underline{i}_n}} = \left(Z_{B_1 B_2,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\underline{\widetilde{m}}_{n-1}}_{\underline{m}_{n-1}} \left(T_{B_2,\underline{a}_n}\right)^{\underline{m}_{n-1}} \tag{6.16}$$

where $\sigma_{\underline{i}_n}$ is the permutation of labels $(x_1,...,x_n) \to (x_{i_1},...,x_{i_n})$ and $\tau_{\sigma_{\underline{i}_n}} : (f_1,...,f_n) \to (f_{i_1},...,f_{i_n})$ the corresponding permutation on $\pi_{a_1} \times ... \times \pi_{a_n}$. In other words, permutations act on the intertwiner spaces $\pi_{a_1} \times ... \times \pi_{a_n} \to \pi_e$ by permutation of the factors, $\sigma(T) := T \circ \tau_\sigma$ and $\left(Z_{B_1 B_2,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\underline{\widetilde{m}}_{n-1}}_{\underline{m}_{n-1}}$ are the matrix elements of these linear maps between intertwiner spaces in various bases of the latter.

Of particular interest for us will be the matrix $\left(Y_{bca}\right)^{\widetilde{m}_1 \widetilde{m}_2}_{m_1 m_2}$ which describes the cyclic permutations of $\left(T_{abc}\right)^{m_1 m_2}(f,g,h)$:

$$\left(T_{bca}\right)^{\widetilde{m}_1 \widetilde{m}_2}(g,h,f) = \left(Y_{bca}\right)^{\widetilde{m}_1 \widetilde{m}_2}_{m_1 m_2} \left(T_{abc}\right)^{m_1 m_2}(f,g,h) \tag{6.17}$$

By (6.15) the transposition of the last two entries is described by the diagonal matrix:

$$I^{\widetilde{m}_1 \widetilde{m}_2}_{m_1 m_2} := \delta^{\widetilde{m}_1}_{m_1} \delta^{\widetilde{m}_2}_{m_2} (-1)^{m_2} \tag{6.18}$$

From the definition follows directly that $Y_{abc} \cdot Y_{cab} \cdot Y_{bca} = 1$ and $Y_{abc} \cdot I \cdot Y_{cba} \cdot I = 1$, i.e. that the matrices Y and I generate a representation of S^3. In particular we have:

$$T_{bac}(g, f, h) = IT_{bca}(g, h, f) = IY_{bca}T_{abc}(f, g, h) \tag{6.19}$$

A calculation and explicit expression for the quite non-trivial matrix $\Big(Y_{abc}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1 m_2}$ will be given in the next subsection. This matrix is closely related to the matrix $\Big(X_{abc}\Big)^{m_1 m_2}_{\widetilde{m}_1\widetilde{m}_2}$, which describes the passage from the basis $\Big(T_{abc}\Big)^{m_1 m_2}$ to the basis $\Big(T_{S,abc}\Big)^{m_1 m_2}$ without a permutation ("re-bracketing"):

$$\Big(T_{abc}\Big)^{m_1 m_2}(f, g, h) = \Big(X_{abc}\Big)^{m_1 m_2}_{\widetilde{m}_1\widetilde{m}_2}\Big(T_{S,abc}\Big)^{\widetilde{m}_1\widetilde{m}_2}(f, g, h) \tag{6.20}$$

Namely, by (6.15) one has:

$$T_{S,abc}(f, g, h) = (-1)^M IT_{cab}(h, f, g) = (-1)^M IY^{-1}_{abc}T_{abc}(f, g, h), \quad M = M(a, b, c; e) = a+b+c-e-2(= m_1+m_2) \tag{6.21}$$

hence:

$$X_{abc} = (-1)^{M(a,b,c;e)} \cdot Y_{abc}I \tag{6.22}$$

Moreover, we claim that the matrix elements $\Big(Y_{abc}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1 m_2}$ are the building blocks of every matrix element $(Z_{B,\underline{a}_n,\sigma_{\underline{i}_n}})^{\underline{\widetilde{m}}_{n-1}}_{\underline{m}_{n-1}}$. Namely, one can achieve every bracket scheme from the default bracket scheme (6.10) by a sequence of applications of (6.5) ("flips"), at the price of a permutation of the arguments. The flips will produce signs $(-1)^{m_i}$, where the label m_i refers to the flipped intertwiner. Now, the permutations can be undone by a sequence of transpositions without changing the bracket scheme. One sees from (6.19) that in the default basis (6.5) the transposition $k \leftrightarrow k+1$ is described by the matrix $I_{k+1}Y_{a_{k+1}\epsilon_{k+1}a_k}$, where $\Big(I_{k+1}\Big)^{\widetilde{m}_k\widetilde{m}_{k+1}}_{m_k m_{k+1}}$ is the diagonal matrix with entries $(-1)^{m_{k+1}}$.

6.4.1 The matrix $\Big(Y_{abc}\Big)^{m_1 m_2}_{\widetilde{m}_1\widetilde{m}_2}$

In this subsection we will explain how we determined the matrix $\Big(Y_{abc}\Big)^{m_1 m_2}_{\widetilde{m}_1\widetilde{m}_2}$ which transforms $\Big(T_{cab}\Big)^{\widetilde{m}_1\widetilde{m}_2}(h, f, g)$ into $\Big(T_{abc}\Big)^{m_1 m_2}(f, g, h)$, and which is the essential ingredient of the reduced Jacobi identity (6.65). This problem was reduced just to a linear algebra task to find the entries of a matrix which switches between known vectors with numerical components. The specific structure of the T's allowed us to derive a recursion formula for the entries of $\Big(Y_{abc}\Big)^{m_1 m_2}_{\widetilde{m}_1\widetilde{m}_2}$, which we were able to solve afterwards.

An elegant method would have been to exploit the associativity of a nontrivial, one-parameter family of products on $\oplus_k M_{2k}$ defined in terms of Rankin–Cohen brackets [Cohen *et al.*, 1996], generalizing an unpublished observation by Eholzer. Varying the parameter, one obtains linear relations between $\rho^e_{dc} \circ \rho^d_{ab} \times 1_c$ and $\rho^e_{ad'} \circ 1_a \times \rho^{d'}_{bc}$ for every fixed a, b, c, e from which one would read off the matrix X_{abc} that describes the re-bracketing and then by (6.22) the matrix Y_{abc}. Unfortunately, due to a symmetry with respect to the parameter, varying the parameter gives only one half of the necessary relations. This is another puzzling surprise, since one would have expected that an associative product rather encodes twice as much information than a (generalized) commutator. Instead, we have to adopt much more down-to-earth linear algebra approach.

The explicit formulae obtained below are meromorphic functions which may have poles at real positive values of the dimensions a, b, c. In other words, the intertwiner bases may become degenerate at these points. These singularities can be regularized, e.g by letting the scaling dimensions have small positive imaginary parts, while keeping the summation indices p, q and m integer. While the representation theory of $\mathrm{SL}(2,\mathbb{R})$ is perfectly meaningful for complex a, b, c, the physical dimensions are of course positive integers. For the removal of regularization in QFT see Section 6.6.

Using (6.3) we write the explicit expression for the two vectors:

$$\begin{aligned}\Big(T_{abc}\Big)^{m_1m_2}(f,g,h) &= \lambda^{e_1}_{ae_2} \circ \big(1_a \times \lambda^{e_2}_{bc}\big)(f,g,h)\\ &= \sum_{\substack{p+q=m_1\\ s+t=m_2}} (-1)^{q+t}\binom{m_1}{p}\binom{m_2}{s}\frac{(2b+2c-m_1-2m_2-3)_p(2a-m_1-1)_q}{\big[2(a+b+c-m_1-m_2-2)\big]_{m_1}}\times\\ &\quad\times\frac{(2c-m_2-1)_s(2b-m_2-1)_t}{\big[2(b+c-m_2-1)\big]_{m_2}}\partial^p f\partial^q(\partial^s g\partial^t h)\end{aligned} \tag{6.23}$$

$$\begin{aligned}\Big(T_{cab}\Big)^{\widetilde{m}_1\widetilde{m}_2}(h,f,g) &= \lambda^{\widetilde{e}_1}_{c\widetilde{e}_2} \circ \big(1_c \times \lambda^{\widetilde{e}_2}_{ab}\big)(h,f,g)\\ &= \sum_{\substack{p+q=\widetilde{m}_1\\ s+t=\widetilde{m}_2}} (-1)^{q+t}\binom{\widetilde{m}_1}{p}\binom{\widetilde{m}_2}{s}\frac{(2a+2b-\widetilde{m}_1-2\widetilde{m}_2-3)_p(2c-\widetilde{m}_1-1)_q}{\big[2(a+b+c-\widetilde{m}_1-\widetilde{m}_2-2)\big]_{\widetilde{m}_1}}\times\\ &\quad\times\frac{(2b-\widetilde{m}_2-1)_s(2a-\widetilde{m}_2-1)_t}{\big[2(a+b-\widetilde{m}_2-1)\big]_{\widetilde{m}_2}}\partial^p h\partial^q(\partial^s f\partial^t g)\end{aligned} \tag{6.24}$$

It is not obvious from first sight how to operate with such complicated expressions. However, the following two observations give us a clue how to treat this problem:

1. Let us organize (6.23) and (6.24) as:

$$\begin{aligned}\Big(T_{abc}\Big)^{m_1m_2}(f,g,h) &= \sum_{r_1+r_2+r_3=m_1+m_2}\Big(T_{abc}\Big)^{m_1m_2}_{r_1r_2r_3}\partial^{r_1}f\partial^{r_2}g\partial^{r_3}h\\ \Big(T_{cab}\Big)^{\widetilde{m}_1\widetilde{m}_2}(h,f,g) &= \sum_{\widetilde{r}_1+\widetilde{r}_2+\widetilde{r}_3=\widetilde{m}_1+\widetilde{m}_2}\Big(T_{cab}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{\widetilde{r}_1\widetilde{r}_2\widetilde{r}_3}\partial^{\widetilde{r}_1}f\partial^{\widetilde{r}_2}g\partial^{\widetilde{r}_3}h\end{aligned} \tag{6.25}$$

with $\left(T_{abc}\right)^{m_1 m_2}_{r_1 r_2 r_3}$ and $\left(T_{cab}\right)^{\widetilde{m}_1 \widetilde{m}_2}_{\widetilde{r}_1 \widetilde{r}_2 \widetilde{r}_3}$ numerical coefficients. The entries of $\left(Y_{abc}\right)^{m_1 m_2}_{\widetilde{m}_1 \widetilde{m}_2}$ are also pure numbers and contain no derivatives, then this matrix cannot mix terms with different orders of derivatives and hence it must switch also between $\left(T_{abc}\right)^{m_1 m_2}_{r_1 r_2 r_3}$ and $\left(T_{cab}\right)^{\widetilde{m}_1 \widetilde{m}_2}_{r_1 r_2 r_3}$ for any fixed triple (r_1, r_2, r_3):

$$\left(Y_{abc}\right)^{m_1 m_2}_{\widetilde{m}_1 \widetilde{m}_2} \left(T_{cab}\right)^{\widetilde{m}_1 \widetilde{m}_2}_{r_1 r_2 r_3} = \left(T_{abc}\right)^{m_1 m_2}_{r_1 r_2 r_3} \tag{6.26}$$

Clearly, this is possible only if $r_1 + r_2 + r_3 = \widetilde{r}_1 + \widetilde{r}_2 + \widetilde{r}_3$, which is equivalent to $m_1 + m_2 = \widetilde{m}_1 + \widetilde{m}_2$. In fact, such an equality was expected, because it reflects the intuitive idea, that it is not possible to decompose $\left(T_{abc}\right)^{m_1 m_2}$ in the basis of $\left(T_{cab}\right)^{\widetilde{m}_1 \widetilde{m}_2}$ if they map to representations with different scaling dimensions. Then we can relax two of the indices of $\left(Y_{abc}\right)^{m_1 m_2}_{\widetilde{m}_1 \widetilde{m}_2}$:

$$\left(Y_{abc}\right)^{m_1 m_2}_{\widetilde{m}_1 \widetilde{m}_2} = \delta_{m_1+m_2, \widetilde{m}_1+\widetilde{m}_2} \left(Y_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}, \quad n := m_1 + m_2 \tag{6.27}$$

It means that $\left(Y_{abc}\right)^{m_1 m_2}_{\widetilde{m}_1 \widetilde{m}_2}$ consists of a set of two-index matrices labelled by n. Every two-index matrix $\left(Y_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}$ will relate two vectors $\left(T_{abc}\right)^{n-m_2, m_2}_{r_1 r_2 r_3}$ and $\left(T_{cab}\right)^{n-\widetilde{m}_2, \widetilde{m}_2}_{r_1 r_2 r_3}$ so that:

$$\left(Y_{abc}(n)\right)^{m_2}_{\widetilde{m}_2} \left(T_{cab}\right)^{n-\widetilde{m}_2, \widetilde{m}_2}_{r_1 r_2 r_3} = \left(T_{abc}\right)^{n-m_2, m_2}_{r_1 r_2 r_3} \tag{6.28}$$

We could as well relax the indices m_2 and $\widetilde{m}_2$ instead of m_1 and $\widetilde{m}_1$, it is just a matter of choice.

2. (6.28) taken for $n+1$ triples (r_1, r_2, r_3) and m_2, $\widetilde{m}_2 \in [0, n]$ gives a system of $(n+1) \times (n+1)$ equations for $(n+1) \times (n+1)$ unknown quantities and if the equations are linearly independent it is enough to fix all the entries of $\left(Y_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}$. For us the most convenient choice appeared to be the triples $(k, 0, n-k)$ with $k \in [0, n]$. The vectors $\left(T_{cab}\right)^{n-\widetilde{m}_2, \widetilde{m}_2}_{k, 0, n-k}$ have the nice property that their components are zero if $\widetilde{m}_2 > k$, which allows us to establish a recursion, such that the component $\left(Y_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}$ is recovered recursively from the components $\left(Y_{abc}(n)\right)^{m_2}_{\widehat{m}_2}$ with $\widehat{m}_2 < \widetilde{m}_2$.

In the following we derive the recursion formula for $\left(Y_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}$. We start with the triple $(0, 0, n)$. Since the vector $\left(T_{cab}\right)^{n-\widetilde{m}_2, \widetilde{m}_2}_{0, 0, n}$ has only one non-zero component, the one with $\widetilde{m}_2 = 0$, (6.28) allows us to compute the entries $\Big(Y_{abc}(n)\Big)_0^{m_2}$ for $m_2 \in [0,n]$ immediately:

$$\Big(Y_{abc}(n)\Big)_0^{m_2}\Big(T_{cab}\Big)_{0,0,n}^{n,0} = \Big(T_{abc}\Big)_{0,0,n}^{n-m_2,m_2} \Longrightarrow \Big(Y_{abc}(n)\Big)_0^{m_2} = \frac{\Big(T_{abc}\Big)_{0,0,n}^{n-m_2,m_2}}{\Big(T_{cab}\Big)_{0,0,n}^{n,0}} \tag{6.29}$$

Next, $\Big(T_{cab}\Big)_{1,0,n-1}^{n-\widetilde{m}_2,\widetilde{m}_2}$ has two non-zero entries, the ones with $m_2 = 0$ and $m_2 = 1$, so from (6.28) we obtain an expression for $\Big(Y_{abc}(n)\Big)_1^{m_2}$ involving the already known entries $\Big(Y_{abc}(n)\Big)_0^{m_2}$ from the same row:

$$\begin{aligned}\Big(Y_{abc}(n)\Big)_0^{m_2}\Big(T_{cab}\Big)_{1,0,n-1}^{n,0} + \Big(Y_{abc}(n)\Big)_1^{m_2}\Big(T_{cab}\Big)_{1,0,n-1}^{n-1,1} &= \Big(T_{abc}\Big)_{1,0,n-1}^{n-m_2,m_2}\\ \Longrightarrow \Big(Y_{abc}(n)\Big)_1^{m_2} &= \frac{\Big(T_{abc}\Big)_{1,0,n-1}^{n-m_2,m_2} - \Big(Y_{abc}(n)\Big)_0^{m_2}\Big(T_{cab}\Big)_{1,0,n-1}^{n,0}}{\Big(T_{cab}\Big)_{1,0,n-1}^{n-1,1}}\end{aligned} \tag{6.30}$$

In analogy, we proceed to $\Big(T_{cab}\Big)_{k,0,n-k}^{n-\widetilde{m}_2,\widetilde{m}_2}$, which has $k+1$ non-zero entries for $m_2 \in [0,k]$, and we express $\Big(Y_{abc}(n)\Big)_k^{m_2}$ in terms of the preceding entries in the matrix row, which have been previously determined. We arrive at the following result:

Proposition 6.5. *The entries of* $\Big(Y_{abc}(n)\Big)_{\widetilde{m}_2}^{m_2}$ *satisfy the recursion formula:*

$$\Big(Y_{abc}(n)\Big)_{\widetilde{m}_2}^{m_2} = \frac{\Big(T_{abc}\Big)_{\widetilde{m}_2,0,n-\widetilde{m}_2}^{n-m_2,m_2} - \sum_{j=0}^{\widetilde{m}_2-1}\Big(Y_{abc}(n)\Big)_j^{m_2}\Big(T_{cab}\Big)_{\widetilde{m}_2,0,n-\widetilde{m}_2}^{n-j,j}}{\Big(T_{cab}\Big)_{\widetilde{m}_2,0,n-\widetilde{m}_2}^{n-\widetilde{m}_2,\widetilde{m}_2}} \tag{6.31}$$

$\{j_l\}_s^m$ are the possible sets $\{j_1 = s,\, j_k < j_{k+1},\, j_l = m\}$, including $\{s,\, m\}$.

Our next task is to solve this recursion and to obtain an explicit formula for $\Big(Y_{abc}(n)\Big)_{\widetilde{m}_2}^{m_2}$.

For this purpose we "insert repeatedly the expression for $\left(Y_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}$ into itself" and obtain:

$$
\begin{aligned}
\left(Y_{abc}(n)\right)^{m_2}_{\widetilde{m}_2} &= \frac{1}{\left(T_{cab}\right)^{n-\widetilde{m}_2,\widetilde{m}_2}_{\widetilde{m}_2,0,n-\widetilde{m}_2}}\left[\left(T_{abc}\right)^{n-m_2,m_2}_{\widetilde{m}_2,0,n-\widetilde{m}_2} - \sum_{j=0}^{\widetilde{m}_2-1}\left(T_{abc}\right)^{n-m_2,m_2}_{j,0,n-j}\frac{\left(T_{cab}\right)^{n-j,j}_{\widetilde{m}_2,0,n-\widetilde{m}_2}}{\left(T_{cab}\right)^{n-j,j}_{j,0,n-j}} + \right. \\
&\qquad \left. + \sum_{j=0}^{\widetilde{m}_2-1}\sum_{s=0}^{j-1}\left(Y_{abc}(n)\right)^{m_2}_{s}\left(T_{cab}\right)^{n-s,s}_{j,0,n-j}\frac{\left(T_{cab}\right)^{n-j,j}_{\widetilde{m}_2,0,n-\widetilde{m}_2}}{\left(T_{cab}\right)^{n-j,j}_{j,0,n-j}}\right] = \\
&= \frac{1}{\left(T_{cab}\right)^{n-\widetilde{m}_2,\widetilde{m}_2}_{\widetilde{m}_2,0,n-\widetilde{m}_2}}\left[\left(T_{abc}\right)^{n-m_2,m_2}_{\widetilde{m}_2,0,n-\widetilde{m}_2} - \sum_{j=0}^{\widetilde{m}_2-1}\left(T_{abc}\right)^{n-m_2,m_2}_{j,0,n-j}\frac{\left(T_{cab}\right)^{n-j,j}_{\widetilde{m}_2,0,n-\widetilde{m}_2}}{\left(T_{cab}\right)^{n-j,j}_{j,0,n-j}} + \right. \\
&\qquad \left. + \sum_{j=0}^{\widetilde{m}_2-1}\sum_{s=0}^{j-1}\left(T_{abc}\right)^{n-m_2,m_2}_{s,0,n-s}\frac{\left(T_{cab}\right)^{n-s,s}_{j,0,n-j}}{\left(T_{cab}\right)^{n-s,s}_{s,0,n-s}}\frac{\left(T_{cab}\right)^{n-j,j}_{\widetilde{m}_2,0,n-\widetilde{m}_2}}{\left(T_{cab}\right)^{n-j,j}_{j,0,n-j}} + \ldots\right] = \\
&= \frac{\sum_{s=0}^{\widetilde{m}_2}\left(T_{abc}\right)^{n-m_2,m_2}_{s,0,n-s}\left[\sum_{\{j_l\}_s^{\widetilde{m}_2}}(-1)^{l-1}\frac{\left(T_{cab}\right)^{n-j_1,j_1}_{j_2,0,n-j_2}}{\left(T_{cab}\right)^{n-j_1,j_1}_{j_1,0,n-j_1}}\frac{\left(T_{cab}\right)^{n-j_2,j_2}_{j_3,0,n-j_3}}{\left(T_{cab}\right)^{n-j_2,j_2}_{j_2,0,n-j_2}}\cdots\frac{\left(T_{cab}\right)^{n-j_{l-1},j_{l-1}}_{j_l,0,n-j_l}}{\left(T_{cab}\right)^{n-j_{l-1},j_{l-1}}_{j_{l-1},0,n-j_{l-1}}}\right]}{\left(T_{cab}\right)^{n-\widetilde{m}_2,\widetilde{m}_2}_{\widetilde{m}_2,0,n-\widetilde{m}_2}}
\end{aligned}
\tag{6.32}
$$

We can also define the operators $\left(T^{\rho}_{abc}\right)^{m_1m_2}$ in terms of the Rankin-Cohen brackets ρ instead of λ:

$$
\begin{aligned}
\left(T^{\rho}_{abc}\right)^{m_1m_2}(f,g,h) &= \rho^{e_1}_{ae_2}\circ\left(1_a\times\rho^{e_2}_{bc}\right)(f,g,h) \\
&= \frac{[2(a+b+c-m_1-m_2-2)]_{m_1}}{m_1!}\frac{[2(b+c-m_2-1)]_{m_2}}{m_2!}\left(T_{abc}\right)^{m_1m_2}(f,g,h)
\end{aligned}
\tag{6.33}
$$

$$
\begin{aligned}
\left(T^{\rho}_{cab}\right)^{m_1m_2}(h,f,g) &= \rho^{e_1}_{ce_2}\circ\left(1_c\times\rho^{e_2}_{ab}\right)(h,f,g) \\
&= \frac{[2(a+b+c-m_1-m_2-2)]_{m_1}}{m_1!}\frac{[2(a+b-m_2-1)]_{m_2}}{m_2!}\left(T_{cab}\right)^{m_1m_2}(h,f,g)
\end{aligned}
\tag{6.34}
$$

The expression for the matrix $\left(Y^{\rho}_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}$ switching between $\left(T^{\rho}_{cab}\right)^{m_1m_2}$ and $\left(T^{\rho}_{abc}\right)^{m_1m_2}$ will be analogous to (6.32) with the substitution $T\to T^{\rho}$. The relation between the two matrices will be:

$$
\begin{aligned}
\left(Y_{abc}(n)\right)^{m_2}_{\widetilde{m}_2} &= \left(U_{abc}(n)\right)^{m_2}_{\bar{m}_2}\left(Y^{\rho}_{abc}(n)\right)^{\bar{m}_2}_{\hat{m}_2}\left(U^{-1}_{cab}(n)\right)^{\hat{m}_2}_{\widetilde{m}_2} \qquad (6.35) \\
\left(U_{abc}(n)\right)^{m_2}_{\widetilde{m}_2} &= \delta_{m_2\widetilde{m}_2}\frac{(n-m_2)!}{[2(a+b+c-n-2)]_{n-m_2}}\frac{m_2!}{[2(b+c-m_2-1)]_{m_2}}
\end{aligned}
$$

$\Big(U_{abc}\Big)^{m_2}_{\widetilde{m}_2}$ is the matrix which transforms $\Big(T^{\rho}_{abc}\Big)^{n-\widetilde{m}_2,\widetilde{m}_2}$ into $\Big(T_{abc}\Big)^{n-m_2,m_2}$.

It will be technically more convenient first to derive a formula for $\Big(Y^{\rho}_{abc}(n)\Big)^{m_2}_{\widetilde{m}_2}$ and then to use (6.35) to compute $\Big(Y_{abc}(n)\Big)^{m_2}_{\widetilde{m}_2}$.

To calculate expressions of the sort $\dfrac{\Big(T^{\rho}_{cab}\Big)^{n-j_t,j_t}_{j_k,0,n-j_k}}{\Big(T^{\rho}_{cab}\Big)^{n-j_t,j_t}_{j_t,0,n-j_t}}$ we use formulae (6.24) and (6.34) and get:

$$\begin{aligned}
\frac{\Big(T^{\rho}_{cab}\Big)^{n-j_r,j_r}_{j_{r+1},0,n-j_{r+1}}}{\Big(T^{\rho}_{cab}\Big)^{n-j_r,j_r}_{j_r,0,n-j_r}} &= (-1)^{j_{r+1}-j_r}\binom{n-j_r}{j_{r+1}-j_r}\frac{(\gamma-(n-j_r)+1)_{j_{r+1}-j_r}}{(\alpha+\beta-(j_{r+1}+j_r)+1)_{j_{r+1}-j_r}}\\
\frac{\Big(T^{\rho}_{cab}\Big)^{n-s,s}_{j_2,0,n-j_2}}{\Big(T^{\rho}_{cab}\Big)^{n-s,s}_{s,0,n-s}} &= (-1)^{j_2-s}\binom{n-s}{j_2-s}\frac{(\gamma-(n-s)+1)_{j_2-s}}{(\alpha+\beta-(j_2+s)+1)_{j_2-s}}\\
\frac{\Big(T^{\rho}_{cab}\Big)^{n-j_{l-1},j_{l-1}}_{j_{\widetilde{m}_2},0,n-j_{\widetilde{m}_2}}}{\Big(T^{\rho}_{cab}\Big)^{n-j_{l-1},j_{l-1}}_{j_{l-1},0,n-j_{l-1}}} &= (-1)^{j_{\widetilde{m}_2}-j_{l-1}}\binom{n-j_{l-1}}{j_{\widetilde{m}_2}-j_{l-1}}\frac{(\gamma-(n-j_{l-1})+1)_{j_{\widetilde{m}_2}-j_{l-1}}}{(\alpha+\beta-(j_{\widetilde{m}_2}+j_{l-1})+1)_{j_{\widetilde{m}_2}-j_{l-1}}}
\end{aligned}\tag{6.36}$$

Here $\alpha := 2a-2$, $\beta := 2b-2$, $\gamma := 2c-2$.

Multiplication of two terms of this kind gives:

$$\begin{aligned}
\frac{\Big(T^{\rho}_{cab}\Big)^{n-j_r,j_r}_{j_{r+1},0,n-j_{r+1}}}{\Big(T^{\rho}_{cab}\Big)^{n-j_r,j_r}_{j_r,0,n-j_r}}\frac{\Big(T^{\rho}_{cab}\Big)^{n-j_{r+1},j_{r+1}}_{j_{r+2},0,n-j_{r+2}}}{\Big(T^{\rho}_{cab}\Big)^{n-j_{r+1},j_{r+1}}_{j_{r+1},0,n-j_{r+1}}} &= (-1)^{j_{r+2}-j_r}\binom{n-j_r}{n-j_{r+1}}\binom{n-j_{r+1}}{n-j_{r+2}}\times\\
&\times\frac{(\gamma-(n-j_r)+1)_{j_{r+2}-j_r}\,(\alpha+\beta-2j_{r+1}+1)_{j_{r+1}-j_r}}{(\alpha+\beta-(j_{r+2}+j_{r+1})+1)_{j_{r+2}+j_{r+1}-2j_r}}
\end{aligned}\tag{6.37}$$

Here we have used the property of the Pochhammer symbol that $(a)_m(a+m)_n = (a)_{m+n}$. When we multiply all the terms we get:

$$\begin{aligned}
\frac{\Big(T^{\rho}_{cab}\Big)^{n-s,s}_{j_2,0,n-j_2}}{\Big(T^{\rho}_{cab}\Big)^{n-s,s}_{s,0,n-s}}\cdots\frac{\Big(T^{\rho}_{cab}\Big)^{n-j_{l-1},j_{l-1}}_{\widetilde{m}_2,0,n-\widetilde{m}_2}}{\Big(T^{\rho}_{cab}\Big)^{n-j_{l-1},j_{l-1}}_{j_{l-1},0,n-j_{l-1}}} &= (-1)^{\widetilde{m}_2-s}\frac{(n-s)!}{(n-\widetilde{m}_2)!}\frac{(\gamma-(n-s)+1)_{\widetilde{m}_2-s}}{(\alpha+\beta-2\widetilde{m}_2+1)_{2\widetilde{m}_2-2s}}\times\\
&\times\prod_{j_r\in\{j_l\}^{\widetilde{m}_2}_s}\frac{(\alpha+\beta-2j_{r+1}+1)_{j_{r+1}-j_r}}{(j_{r+1}-j_r)!}
\end{aligned}\tag{6.38}$$

Then the expression for $\left(Y^{\rho}_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}$ becomes:

$$\left(Y^{\rho}_{abc}(n)\right)^{m_2}_{\widetilde{m}_2} = \sum_{s=0}^{\widetilde{m}_2} \frac{\left(T^{\rho}_{abc}\right)^{n-m_2,m_2}_{s,0,n-s}}{\left(T^{\rho}_{cab}\right)^{n-\widetilde{m}_2,\widetilde{m}_2}_{\widetilde{m}_2,0,n-\widetilde{m}_2}} \left[(-1)^{\widetilde{m}_2-s}\frac{(n-s)!}{(n-\widetilde{m}_2)!}\frac{(\gamma-(n-s)+1)_{\widetilde{m}_2-s}}{(\alpha+\beta-2\widetilde{m}_2+1)_{2\widetilde{m}_2-2s}}\times\right.$$
$$\left.\times\sum_{\{j_l\}^{\widetilde{m}_2}_s}(-1)^{l-1}\prod_{j_r\in\{j_l\}^{\widetilde{m}_2}_s}\frac{(\alpha+\beta-2j_{r+1}+1)_{j_{r+1}-j_r}}{(j_{r+1}-j_r)!}\right] \tag{6.39}$$

Claim 6.6. *On the basis of calculations for several small l's we claim that:*

$$\sum_{\{j_l\}^{\widetilde{m}_2}_s}(-1)^{l-1}\prod_{j_r\in\{j_l\}^{\widetilde{m}_2}_s}\frac{(\alpha+\beta-2j_{r+1}+1)_{j_{r+1}-j_r}}{(j_{r+1}-j_r)!} = (-1)^{\widetilde{m}_2-s}\frac{(\alpha+\beta-2\widetilde{m}_2+1)(\alpha+\beta-\widetilde{m}_2-s+2)_{\widetilde{m}_2-s-1}}{(\widetilde{m}_2-s)!} \tag{6.40}$$

In this case we reach the following formula:

Proposition 6.7. *We obtain the following expression for the matrix $\left(Y^{\rho}_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}$:*

$$\left(Y^{\rho}_{abc}(n)\right)^{m_2}_{\widetilde{m}_2} = (-1)^{n-\widetilde{m}_2}\frac{\binom{n}{m_2}}{\binom{n}{\widetilde{m}_2}}\frac{(-\beta)_{m_2}}{(-\beta)_{\widetilde{m}_2}}\frac{1}{(2\widetilde{m}_2-\alpha-\beta)_{n-\widetilde{m}_2}}\times$$
$$\times\sum_{s=0}^{\widetilde{m}_2}\frac{\binom{n-m_2}{s}(n+m_2-s-\beta-\gamma)_s(s-\alpha)_{n-m_2-s}\binom{n-s}{\widetilde{m}_2-s}(n-\widetilde{m}_2-\gamma)_{\widetilde{m}_2-s}}{(\alpha+\beta-2\widetilde{m}_2+2)_{\widetilde{m}_2-s}} \tag{6.41}$$

Relying only on the recursion formula (6.31), without referring to Claim 6.6, we observed the following interesting property of the matrix $\left(Y^{\rho}_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}$:

Proposition 6.8. *The entries from an arbitrary column of the matrix $\left(Y^{\rho}_{abc}(n)\right)^{m_2}_{\widetilde{m}_2}$ sum to $(-1)^{n+\widetilde{m}_2}$, where $\widetilde{m}_2$ is the number of the column.*

Proof. We will prove this statement by induction in the number of the column.
Let us first consider the column $\widetilde{m}_2 = 0$. The entries from this column are expressed as:

$$\left(Y^{\rho}_{abc}(n)\right)^{m_2}_{0} = (-1)^n\frac{(-\alpha)_{n-m_2}}{(n-m_2)!}\frac{(-\beta)_{m_2}}{m_2!}\frac{n!}{[-(\alpha+\beta)]_n} \tag{6.42}$$

Then, using the property $\frac{(a+b)_n}{n!} = \sum_{i=1}^{n}\frac{(a)_i}{i!}\frac{(b)_{n-i}}{(n-i)!}$ we compute $\sum_{m_2=0}^{n}\left(Y^{\rho}_{abc}(n)\right)^{m_2}_{0} = (-1)^n$, i.e. the statement of the proposition holds for $\widetilde{m}_2 = 0$.

Now let us assume that $\sum_{m_2=0}^{n}\left(Y^{\rho}_{abc}(n)\right)_j^{m_2} = (-1)^{n+j}$ is true for every $j \leq k-1$. We will prove that then $\sum_{m_2=0}^{n}\left(Y^{\rho}_{abc}(n)\right)_k^{m_2} = (-1)^{n+k}$. We use formula (6.31) and write:

$$\begin{aligned}
\sum_{m_2=0}^{n}\left(Y^{\rho}_{abc}(n)\right)_k^{m_2} &= \sum_{m_2=0}^{n}\frac{\left(T^{\rho}_{abc}\right)_{k,0,n-k}^{n-m_2,m_2} - \sum_{j=0}^{k-1}\left(Y^{\rho}_{abc}(n)\right)_j^{m_2}\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-j,j}}{\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-k,k}} \\
&= \frac{1}{\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-k,k}}\left\{\sum_{m_2=0}^{n-k}\left(T^{\rho}_{abc}\right)_{k,0,n-k}^{n-m_2,m_2} - \sum_{j=0}^{k-1}\left[\sum_{m_2=0}^{n}\left(Y^{\rho}_{abc}(n)\right)_j^{m_2}\right]\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-j,j}\right\} \\
&= \frac{1}{\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-k,k}}\left\{\sum_{m_2=0}^{n-k}\left(T^{\rho}_{abc}\right)_{k,0,n-k}^{n-m_2,m_2} - \sum_{j=0}^{k-1}(-1)^{n+j}\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-j,j}\right\}
\end{aligned} \tag{6.43}$$

Let us write the second sum as:

$$\sum_{j=0}^{k-1}(-1)^{n+j}\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-j,j} = \sum_{j=0}^{k}(-1)^{n+j}\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-j,j} - (-1)^{n+k}\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-k,k} \tag{6.44}$$

Then we obtain:

$$\sum_{m_2=0}^{n}\left(Y^{\rho}_{abc}(n)\right)_k^{m_2} = \frac{1}{\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-k,k}}\left\{\sum_{m_2=0}^{n-k}\left(T^{\rho}_{abc}\right)_{k,0,n-k}^{n-m_2,m_2} - \sum_{j=0}^{k}(-1)^{n+j}\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-j,j}\right\} + (-1)^{n+k} \tag{6.45}$$

Hence, we have to prove that the expression in the brackets vanishes. Let us write this expression explicitly:

$$\begin{aligned}
&\sum_{m=0}^{n-k}\left(T^{\rho}_{abc}\right)_{k,0,n-k}^{n-m,m} - \sum_{j=0}^{k}(-1)^{n+j}\left(T^{\rho}_{cab}\right)_{k,0,n-k}^{n-j,j} = \\
&= (-1)^{n+k}\Bigg\{\sum_{m=0}^{n-k}\frac{(\beta+\gamma-(n+m)+1)_k}{k!}\frac{(\alpha-(n-m)+1)_{n-m-k}}{(n-m-k)!}\frac{(\beta-m+1)_m}{m!} - \\
&\qquad - \sum_{m=0}^{k}\frac{(\alpha+\beta-(n+m)+1)_{n-k}}{(n-k)!}\frac{(\gamma-(n-m)+1)_{k-m}}{(k-m)!}\frac{(\beta-m+1)_m}{m!}\Bigg\}
\end{aligned} \tag{6.46}$$

Let us now derive one useful formula:

$$\left.\begin{aligned}
\frac{(a+b)_n}{n!} &= \sum_{i=0}^{n}\frac{(a)_i}{i!}\frac{(b)_{n-i}}{(n-i)!} \\
(a-n+1)_n &= (-1)^n(-a)_n
\end{aligned}\right\} \longrightarrow \quad \frac{(A+B+1)_n}{n!} = \sum_{j=0}^{n}\frac{(A+j+1)_{n-j}}{(n-j)!}\frac{(B-j+1)_j}{j!} \tag{6.47}$$

Using this formula we can develop the first sum in (6.46) as:

$$\begin{aligned}
&\sum_{m=0}^{n-k} \frac{(\beta+\gamma-(n+m)+1)_k}{k!} \frac{(\alpha-(n-m)+1)_{n-m-k}}{(n-m-k)!} \frac{(\beta-m+1)_m}{m!} = \\
&= \sum_{m=0}^{n-k} \sum_{i=0}^{k} \frac{(\gamma-n+i+1)_{k-i}}{(k-i)!} \frac{(\beta-m-i+1)_i}{i!} \frac{(\alpha-(n-m)+1)_{n-m-k}}{(n-m-k)!} \frac{(\beta-m+1)_m}{m!}
\end{aligned} \tag{6.48}$$

The product of the two terms which contain the parameter β may be rewritten using the property $(a)_{m+n} = (a)_m(a+m)_n = (a)_n(a+n)_m$:

$$\frac{(\beta-m-i+1)_i}{i!} \frac{(\beta-m+1)_m}{m!} = \frac{(\beta-m-i+1)_m}{m!} \frac{(\beta-i+1)_i}{i!} \tag{6.49}$$

Then formula (6.48) becomes:

$$\begin{aligned}
&\sum_{m=0}^{n-k} \frac{(\beta+\gamma-(n+m)+1)_k}{k!} \frac{(\alpha-(n-m)+1)_{n-m-k}}{(n-m-k)!} \frac{(\beta-m+1)_m}{m!} = \\
&= \sum_{i=0}^{k} \frac{(\gamma-n+i+1)_{k-i}}{(k-i)!} \frac{(\beta-i+1)_i}{i!} \sum_{m=0}^{n-k} \frac{(\alpha-(n-m)+1)_{n-m-k}}{(n-m-k)!} \frac{(\beta-m-i+1)_m}{m!} = \\
&= \sum_{i=0}^{k} \frac{(\gamma-n+i+1)_{k-i}}{(k-i)!} \frac{(\beta-i+1)_i}{i!} \frac{(\alpha+\beta-(n+i)+1)_{n-k}}{(n-k)!}
\end{aligned} \tag{6.50}$$

This means that the first sum is equal to the second sum in (6.46), hence their difference is zero and formula (6.45) gives:

$$\sum_{m_2=0}^{n} \left(Y^{\rho}_{abc}(n)\right)^{m_2}_{k} = (-1)^{n+k} \tag{6.51}$$

This proves the induction hypothesis and the proposition.

6.5 Reduction of the field algebra

The field algebra, which we will denote with $\mathcal{V}$, decomposes as a linear space into a direct sum of unitary representations via commutators of $sl(2,\mathbb{R})$, which is a subalgebra of $\mathcal{V}$:

$$\mathcal{V} = \bigoplus_{a\in\mathbb{N}} \mathcal{V}_a \tag{6.52}$$

Every subspace $\mathcal{V}_a$ is a span of finitely many quasiprimary fields with the same integer scaling dimension $a > 0$ and is isomorphic to $V_a \otimes \pi_a$. As in Section 6.3, π_a is a test functions space, which is a representation space for $sl(2,\mathbb{R})$. V_a is a finite–dimensional multiplicity space with basis W_a, which accounts for the number of fields with scaling dimension a. The isomorphism above is realized by the map ϕ_a which acts as:

$$\phi_a : A \otimes f \to A(f), \quad A \in V_a, \quad f \in \pi_a \tag{6.53}$$

We leave out the identity operator I (of dimension $a = 0$) from the reduced space for several reasons. First, (6.53) fails to be an isomorphism in this case because $I(f) = (\int f(x)dx) \cdot 1$ depends only on the integral of f. Second, the unit operator is central in the field algebra, so its commutator with other fields contains no information. Third, the contribution of the unit operator to the commutator of two fields is completely determined by the 2-point function, which we shall treat as an independent structure element in Section 6.7.

Definition 6.9 (The reduced space V). *The direct sum of all multiplicity spaces $V = \oplus_{a\in\mathbb{N}} V_a$ will be called the reduced space V.*

In the following we will show that the Lie algebra structure of $\mathcal{V}$ is enciphered into a multi–component structure on the reduced space V.

Definition 6.10 (The reduced Lie bracket $\Gamma^*(\cdot,\cdot)_m$). *On the reduced space $V = \oplus_a V_a$ the commutator $[\cdot,\cdot]$ in $\mathcal{V}$ is represented by the multi–component $*$-bracket $[\cdot,\cdot]^*_m$ or $\Gamma^*(\cdot,\cdot)_m : V_a \times V_b \to V_{a+b-1-m}$, $m \geq 0$:*

$$\Gamma^*(A,B)_m := \sum_{C\in W_{a+b-1-m}} F^C_{AB} C, \quad m := a+b-c-1 \tag{6.54}$$

Indeed, if we rewrite the Lie commutators (6.1) using (6.53) we find (suppressing the detailed form of the contribution from the unit operator):

$$\begin{aligned} -i[\phi_a(A\otimes f), \phi_b(B\otimes g)] &= \sum_{c<a+b} \phi_c\left(\sum_{C\in W_c} F^C_{AB} C \otimes \lambda^c_{ab}(f,g)\right) + (\text{unit operator}) \\ &= \sum_{c<a+b} \phi_c\left(\sum_{c} \Gamma^*(A,B)_m \otimes \lambda^c_{ab}(f,g)\right) + (\text{unit operator}) \end{aligned} \tag{6.55}$$

Observation. The anti–symmetry property of the commutator is encoded in the graded symmetry property of the $*$-bracket:

$$\Gamma^*(X_1,X_2)_m = (-1)^{m+1}\Gamma^*(X_1,X_2)_m \tag{6.56}$$

(6.56) actually reproduces the graded symmetry of the structure constants F^C_{AB} (6.7).

Remark. The reduction of the algebra may be interpreted as disentangling the $sl(2,\mathbb{R})$ "kinematic" representation details from the structure constants F^C_{AB}. The former are completely dictated by the conformal symmetry, whereas the latter specify the model (together with the dimension dim V_a).

In order to perform a complete reduction of the field algebra $\mathcal{V}$ we must also "reduce" the Jacobi identity and this will be done in the next section.

6.6 The reduced Jacobi identity and further constraints on the structure constants F_{AB}^{C}

In this section we will examine what becomes of the Jacobi identity of the commutators under the "space reduction". In this way we will complete the reduction of the field algebra and we will find further restrictions on the coefficients F_{AB}^{C}.

The Jacobi identity in its full form among three quasiprimary fields $A(f) \in \mathcal{V}_a$, $B(g) \in \mathcal{V}_b$ and $C(h) \in \mathcal{V}_c$ is:

$$\Big[A(f), \big[B(g), C(h)\big]\Big] + \Big[B(g), \big[C(h), A(f)\big]\Big] + \Big[C(h), \big[A(f), B(g)\big]\Big] = 0 \tag{6.57}$$

Now let us concentrate on the first term. As in (6.54), we want to detach the test function contribution from the operator part. Using the construction of intertwiners for multiple products of representations (6.10) and the relation (6.53) we write:

$$\Big[A(f), \big[B(g), C(h)\big]\Big] \cong \sum_{m_1 m_2} \Gamma^*\Big(A, \Gamma^*(B,C)\Big)_{m_1 m_2} \otimes \Big(T_{abc}\Big)^{m_1 m_2}(f,g,h) \tag{6.58}$$

$$\Gamma^*\Big(A, \Gamma^*(B,C)\Big)_{m_1 m_2} = \sum_{\substack{E_1 \in W_{e_1} = W_{a+b+c-m_1-m_2-2} \\ E_2 \in W_{e_2} = W_{b+c-m_2-1}}} F_{AE_2}^{E_1} F_{BC}^{E_2} E_1 \tag{6.59}$$

Here and everywhere in the rest of this section the relation between the e's and the m's are as in Section 6.3.

The same considerations + (6.17) and bilinearity of the tensor product give for the second term:

$$\begin{aligned}
\Big[B(g), \big[C(h), A(f)\big]\Big] &\cong \sum_{\widetilde{m}_1, \widetilde{m}_2} \Gamma^*\Big(B, \Gamma^*(C,A)\Big)_{\widetilde{m}_1 \widetilde{m}_2} \otimes \Big(T_{bca}\Big)^{\widetilde{m}_1 \widetilde{m}_2}(g,h,f) \\
&= \sum_{\widetilde{m}_1, \widetilde{m}_2} \Gamma^*\Big(B, \Gamma^*(C,A)\Big)_{\widetilde{m}_1 \widetilde{m}_2} \otimes \sum_{m_1, m_2} \Big(Y_{bca}\Big)_{m_1 m_2}^{\widetilde{m}_1 \widetilde{m}_2} \Big(T_{abc}\Big)^{m_1 m_2}(f,g,h) \\
&= \sum_{\widetilde{m}_1, \widetilde{m}_2, m_1, m_2} \Gamma^*\Big(B, \Gamma^*(C,A)\Big)_{\widetilde{m}_1 \widetilde{m}_2} \Big(Y_{bca}\Big)_{m_1 m_2}^{\widetilde{m}_1 \widetilde{m}_2} \otimes \Big(T_{abc}\Big)^{m_1 m_2}(f,g,h)
\end{aligned} \tag{6.60}$$

$$\Gamma^*\Big(B, \Gamma^*(C,A))\Big)_{\widetilde{m}_1 \widetilde{m}_2} = \sum_{\substack{E_1 \in W_{\tilde{e}_1} = W_{a+b+c-\widetilde{m}_1-\widetilde{m}_2-2} \\ E_2 \in W_{\tilde{e}_2} = W_{c+a-\widetilde{m}_2-1}}} F_{BE_2}^{E_1} F_{CA}^{E_2} E_1 \tag{6.61}$$

In analogy, we obtain for the third term:

$$\Big[C(h), \big[A(f), B(g)\big]\Big] \cong \sum_{\widehat{m}_1, \widehat{m}_2, \widetilde{m}_1, \widetilde{m}_2, m_1, m_2} \Gamma^*\Big(C, \Gamma^*(A,B)\Big)_{\widehat{m}_1 \widehat{m}_2} \Big(Y_{cab}\Big)_{\widetilde{m}_1 \widetilde{m}_2}^{\widehat{m}_1 \widehat{m}_2} \Big(Y_{bca}\Big)_{m_1 m_2}^{\widetilde{m}_1 \widetilde{m}_2} \otimes \Big(T_{abc}\Big)^{m_1 m_2}(f,g,h) \tag{6.62}$$

$$\Gamma^*\Big(C, \Gamma^*(A,B)\Big)_{\widehat{m}_1 \widehat{m}_2} = \sum_{\substack{E_1 \in W_{\hat{e}_1} = W_{a+b+c-\widehat{m}_1-\widehat{m}_2-2} \\ E_2 \in W_{\hat{e}_2} = W_{a+b-\widehat{m}_2-1}}} F_{CE_2}^{E_1} F_{AB}^{E_2} E_1 \tag{6.63}$$

Now we sum all these terms and we get:

$$\sum_{m_1 m_2} \Big\{ \Gamma^*\Big(A, \Gamma^*(B,C)\Big)_{m_1 m_2} + \Gamma^*\Big(B, \Gamma^*(C,A)\Big)_{\widetilde{m}_1 \widetilde{m}_2} \Big(Y_{bca}\Big)_{m_1 m_2}^{\widetilde{m}_1 \widetilde{m}_2} + \\ + \Gamma^*\Big(C, \Gamma^*(A,B)\Big)_{\widehat{m}_1 \widehat{m}_2} \Big(Y_{cab}\Big)_{\widetilde{m}_1 \widetilde{m}_2}^{\widehat{m}_1 \widehat{m}_2} \Big(Y_{bca}\Big)_{m_1 m_2}^{\widetilde{m}_1 \widetilde{m}_2} \Big\} \otimes \Big(T_{abc}\Big)^{m_1 m_2}(f,g,h) = 0 \tag{6.64}$$

Having in mind that the basis components $\Big(T_{abc}\Big)^{m_1 m_2}(f,g,h)$ for different values of m_1 and m_2 are linearly independent functionals of the test functions, and the test functions are arbitrary, we conclude for any fixed pair (m_1, m_2):

$$\Gamma^*\Big(A, \Gamma^*(B,C)\Big)_{m_1 m_2} + \Gamma^*\Big(B, \Gamma^*(C,A)\Big)_{\widetilde{m}_1 \widetilde{m}_2} \Big(Y_{bca}\Big)_{m_1 m_2}^{\widetilde{m}_1 \widetilde{m}_2} + \Gamma^*\Big(C, \Gamma^*(A,B)\Big)_{\widehat{m}_1 \widehat{m}_2} \Big(Y_{cab}\Big)_{\widetilde{m}_1 \widetilde{m}_2}^{\widehat{m}_1 \widehat{m}_2} \Big(Y_{bca}\Big)_{m_1 m_2}^{\widetilde{m}_1 \widetilde{m}_2} = 0 \tag{6.65}$$

Let us denote the left-hand side of the reduced Jacobi identity (6.65) with $\mathrm{RJI}(A,B,C)_{m_1 m_2}$. Clearly, because $Y_{abc} \cdot Y_{cab} \cdot Y_{bca} = 1$, one has the following symmetry rule:

$$\mathrm{RJI}(A,B,C)_{m_1 m_2} = \mathrm{RJI}(B,C,A)_{\widetilde{m}_1 \widetilde{m}_2} \Big(Y_{bca}\Big)_{m_1 m_2}^{\widetilde{m}_1 \widetilde{m}_2} \tag{6.66}$$

i. e. the vanishing of $\mathrm{RJI}(A,B,C)_{m_1 m_2}$ is invariant under cyclic permutations, as it is expected. If we use the explicit expressions for the nested (Γ^*)'s from above, the reduced Jacobi identity becomes for every quadruple of quasiprimary fields A, B, C and E and for every pair m_1, m_2 such that $m_1 + m_2 = a + b + c - e - 2$:

$$\left[\sum_{E_2 \in W_{e_2}} F_{AE_2}^{E} F_{BC}^{E_2} \right]_{m_1 m_2} + \left[\sum_{E_2 \in W_{\tilde{e}_2}} F_{BE_2}^{E} F_{CA}^{E_2} \right]_{\widetilde{m}_1 \widetilde{m}_2} \Big(Y_{bca}\Big)_{m_1 m_2}^{\widetilde{m}_1 \widetilde{m}_2} + \\ + \left[\sum_{E_2 \in W_{\hat{e}_2}} F_{CE_2}^{E} F_{AB}^{E_2} \right]_{\widehat{m}_1 \widehat{m}_2} \Big(Y_{cab}\Big)_{\widetilde{m}_1 \widetilde{m}_2}^{\widehat{m}_1 \widehat{m}_2} \Big(Y_{bca}\Big)_{m_1 m_2}^{\widetilde{m}_1 \widetilde{m}_2} = 0 \tag{6.67}$$

Observation. The reduced form of the Jacobi identity gives further restrictions on the structure constants F_{AB}^{C}. Every solution of this infinite set of constraints provides a candidate for the commutator algebra of a local chiral conformal field theory.

As noted in Section 6.4.1, the matrix elements of Y_{abc} can have vanishing denominators, that have to be regularized (e.g by giving small imaginary parts to the dimensions). As it turns out, in (6.67) these singularities will not be suppressed in general by the vanishing of the structure constants involving negative scaling dimensions. To make sense of the singular Jacobi identities, one has to multiply by the singular denominators and then remove the regulators. The effect will be that only one or two of the three terms of the Jacobi identity may survive, so that the general appearance of the Jacobi identity may be quite different from the usual "three-term" form. Notice that anyway, due to the multi–component structure of the bracket, each of the three terms is in general a sum over different "intermediate" representations.

6.7 Relation between F_{AB}^{C} and 2- and 3-point amplitudes

As we know from previous sections, the 2-point function of the fields $A(x)$ and $B(x)$ has the form:

$$\langle A(x_1)B(x_2)\rangle = \langle\langle AB\rangle\rangle \left(\frac{-i}{x_{12}-i\varepsilon}\right)^{2a} \equiv \frac{\langle\langle AB\rangle\rangle}{(ix_{12})_{\varepsilon}^{2a}} \tag{6.68}$$

The map $A, B \to \langle\langle AB\rangle\rangle$ is a real bilinear map on the reduced space which is:

- symmetric: $\langle\langle AB\rangle\rangle = \langle\langle BA\rangle\rangle$
- respects the grading: $\langle\langle AB\rangle\rangle = 0$ if the scaling dimensions $a \neq b$
- positive definite: $\langle\langle AB\rangle\rangle > 0$ unless $A = 0$

The first property reflects locality of the QFT, the second is a consequence of Möbius invariance, and the last one is Wightman positivity, i.e., the positive-definiteness of the Hilbert space inner product.

Similarly, 3-point function has the following form:

$$\langle A(x)B(y)C(z)\rangle = \langle\langle ABC\rangle\rangle \frac{(-i)^{a+b+c}}{(x-y-i\varepsilon)^{a+b-c}(y-z-i\varepsilon)^{b+c-a}(x-z-i\varepsilon)^{a+c-b}} \tag{6.69}$$

and by locality its amplitude must satisfy:

$$\langle\langle BAC\rangle\rangle = (-1)^{a+b-c}\langle\langle ABC\rangle\rangle \tag{6.70}$$

We will show that the amplitudes of the 2- and the 3-point functions are not independent on each other. For this purpose, let us consider the 3-point function $\langle [A,B]C\rangle$. We can find it as $\langle [A,B]C\rangle = \langle ABC\rangle - \langle BAC\rangle$:

$$\langle [A,B]C\rangle = \frac{(-i)^{a+b+c}\langle\langle ABC\rangle\rangle}{(x-z-i\varepsilon)^{a+c-b}(y-z-i\varepsilon)^{b+c-a}}\left[\frac{1}{(x-y-i\varepsilon)^{a+b-c}} - \frac{1}{(x-y+i\varepsilon)^{a+b-c}}\right] \tag{6.71}$$

Then we remember:

$$\begin{aligned} \left(\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}\right) &= 2\pi i\delta(x) \\ (-1)^n n!\left(\frac{1}{(x-i\varepsilon)^{n+1}} - \frac{1}{(x+i\varepsilon)^{n+1}}\right) &= 2\pi i\delta^{(n)}(x) \end{aligned} \tag{6.72}$$

and obtain:

$$\begin{aligned} \langle [A(x),B(y)]C(z)\rangle &= \frac{(-i)^{a+b+c}\langle\langle ABC\rangle\rangle}{(x-z-i\varepsilon)^{a+c-b}(y-z-i\varepsilon)^{b+c-a}}\frac{2\pi i}{(-1)^{a+b-c-1}(a+b-c-1)!}\delta^{(a+b-c-1)}(x-y) + \\ &\quad +\text{lower derivatives of } \delta \end{aligned} \tag{6.73}$$

Using the following expansion:

$$\delta^{(m)}(x-y)f(x,y) = \delta^{(m)}(x-y)f(x,x) + \text{lower derivatives of } \delta \tag{6.74}$$

we end up with:

$$\langle [A,B]C\rangle = \frac{(-i)^{a+b+c}\langle\langle ABC\rangle\rangle}{(x-z-i\varepsilon)^{2c}} \frac{2\pi i}{(-1)^{a+b-c-1}(a+b-c-1)!}\delta^{(a+b-c-1)}(x-y) \tag{6.75}$$

On the other hand, we remember that $-i[A,B] = \sum F^C_{AB} C$. We use the following translation formula:

$$C(\partial^p \cdot \partial^q g) \rightarrow (-1)^{p+q}\partial^q_y(\partial^p_x \delta(x-y)\cdot C(y)) \tag{6.76}$$

and get:

$$\begin{aligned} -i[A(x),B(y)] &= \sum_{c<a+b}\sum_{C\in W_c} F^C_{AB} \sum_{\substack{p,q\geq 0\\ p+q=a+b-c-1}} \lambda^c_{ab}(p,q)(-1)^{p+q}\partial^q_y(\partial^p_x\delta(x-y)\cdot C(y)) \\ &= \sum_{c<a+b}\sum_{C\in W_c} F^C_{AB} \sum_{\substack{p,q\geq 0\\ p+q=a+b-c-1}} \lambda^c_{ab}(p,q)(-1)^{p}\partial^{p+q}_x\delta(x-y)C(y) + \\ &\qquad + \text{lower derivatives of } \delta \end{aligned} \tag{6.77}$$

We can calculate:

$$\begin{aligned} \sum_{\substack{p+q=a+b-c-1\\ p,q\geq 0}} (-1)^{p+q}\lambda^c_{ab}(p,q) &= (-1)^{a+b-c-1}\frac{(a+b-c-1)!}{(2c)_{a+b-c-1}} \sum_{\substack{p,q\geq 0\\ p+q=a+b-c-1}} \frac{(c+b-a)_p}{p!}\frac{(c+a-b)_q}{q!} \\ &= (-1)^{a+b-c-1}\frac{(a+b-c-1)!}{(2c)_{a+b-c-1}}\frac{(2c)_{a+b-c-1}}{(a+b-c-1)!} \\ &= (-1)^{a+b-c-1} \end{aligned} \tag{6.78}$$

Then we have:

$$-i[A(x),B(y)] = \sum_{c<a+b}\sum_{C\in W_c} F^C_{AB}(-1)^{a+b-c-1}\delta^{(a+b-c-1)}(x-y)C(y) \tag{6.79}$$

and can write $\langle [A,B]C\rangle$ as:

$$\begin{aligned} \langle [A(x),B(y)]C(z)\rangle &= \sum_{C'\in W_c} F^{C'}_{AB}(-1)^{a+b-c-1}\delta^{(a+b-c-1)}(x-y)\langle\langle C'C\rangle\rangle\left(\frac{-i}{x-z-i\varepsilon}\right)^{2c} + \\ &\quad + \text{lower derivatives of } \delta \end{aligned} \tag{6.80}$$

Comparing (6.73) and (6.80) we obtain:

$$\frac{(-i)^{a+b+c}}{(a+b-c-1)!}2\pi i\langle\langle ABC\rangle\rangle = \sum_{C'\in W_c} F^{C'}_{AB}\langle\langle C'C\rangle\rangle(-i)^{2c} \tag{6.81}$$

With the same considerations for $\langle A[B,C]\rangle$ we obtain:

$$\frac{(-i)^{a+b+c}}{(b+c-a-1)!}2\pi i\langle\langle ABC\rangle\rangle = \sum_{A'\in W_a} F^{A'}_{BC}\langle\langle A'A\rangle\rangle(-i)^{2a} \tag{6.82}$$

The last two formulae allow us to find a new condition on the structure constants F^C_{AB} involving only 2-point amplitudes:

$$(a+b-c-1)!(-1)^c \sum_{C'\in W_c} F^{C'}_{AB}\langle\langle C'C\rangle\rangle = (b+c-a-1)!(-1)^a \sum_{A'\in W_a} F^{A'}_{BC}\langle\langle A'A\rangle\rangle \tag{6.83}$$

or

$$(a+b-c-1)!(-1)^c\langle\langle\Gamma^*(A,B)_{a+b-c-1},C\rangle\rangle = (b+c-a-1)!(-1)^a\langle\langle A,\Gamma^*(B,C)_{b+c-a-1}\rangle\rangle \tag{6.84}$$

There are two ways how to look at this condition: either one assumes a given quadratic form $\langle\langle\cdot,\cdot\rangle\rangle$, which amounts to fixing bases of the finite-dimensional reduced field spaces V_a: then (6.83) is indeed an additional constraint on the structure constants F^C_{AB}. Or one regards the reduced bracket (6.10) subject to the structure relations (6.7) and (6.67) as the primary structure: then (6.84) is an invariance condition on the quadratic form, in the same way as the invariance condition $g([X,Y],Z) = g(X,[Y,Z])$ on a quadratic form on a Lie algebra. This invariant quadratic form on the reduced Lie algebra corresponds to the vacuum expectation functional on the original commutator algebra.

6.8 Axiomatization of chiral conformal QFT

The upshot of the previous analysis is a new axiomatization of chiral conformal quantum field theory. It consists of the three data:

- a graded reduced space of fields $V = \bigoplus_{a\in\mathbb{N}} V_a$,
- a generalized Lie bracket $\Gamma^* = \sum_{m\geq 0}\Gamma^*_m : V\times V\to V$
- and a quadratic form $\langle\langle\cdot\,\cdot\rangle\rangle : V\times V\to\mathbb{R}$

These data should enjoy the features outlined before: V_a are real linear spaces; the bracket is filtered: $\Gamma^*(V_a\times V_b)\subset\bigoplus_{m\geq 0} V_{a+b-1-m}$, and satisfies the graded symmetry (6.56) and generalized Jacobi identity (6.65); the quadratic form is symmetric, positive definite, respects the grading, and is invariant (6.84) with respect to the bracket.

Notice that the unitarity bound (absence of negative scaling dimensions) has been imposed through the specification of the reduced space V. Although the local intertwiner bases, and therefore also the coefficient matrices Y in the Jacobi identity do involve "intermediate" representations of negative dimensions ($a + b - 1 - m$ may be < 0), these do not contribute to the present axiomatization because they multiply non-existent structure constants. Recall also that the possibly singular instances of the Jacobi identity have to be understood as explained in the end of Sect. 6.6.

One may impose further physically motivated constraints, e.g., the existence of a stress-energy tensor as a distinguished field $T \in V_2$ whose structure constants F^A_{TA} take canonical values; or the generation of the entire reduced space by iterated brackets of a finite set of fields, formulated as a surjectivity property of the bracket.

As a simple example, one may consider the constraints on the structure constants for the commutator of two fields A, B of dimension one. The only possibility in this case is $\dim C = 1$. The generalized Jacobi identity just reduces to the classical Jacobi identity for the structure constants of some Lie algebra g. Likewise, the invariance property of the quadratic form becomes the classical g-invariance of the quadratic form $h(A, B) = \langle\langle AB \rangle\rangle$ on g. The positivity condition on the quadratic form implies that g must be compact, and that h is a multiple of the Cartan-Killing metric. In other words: one obtains precisely the Kac-Moody algebras as solutions to this part of the constraints. The quantization of the level is expected to arise by the interplay between the positivity condition with the higher generalized Jacobi identities.

Other approaches [Zamolodchikov, 1986; Bouwknegt, 1988; Blumenhagen *et al.*, 1991] to the classification of "W-algebras" have, of course, exploited essentially the same consistency relations for a set of generating fields. Our focus here is, however, on the entire structure including all "composite" fields, and the possibility to formulate a deformation theory, to which we turn in the next chapter.

7 Cohomology and deformations of the reduced Lie algebra

In this chapter we will study the deformation theory of the reduced Lie bracket. The motivating example for us was [Hollands, 2008], where deformations in the setting of OPE(operator product expansion) approach to quantum field theory on curved space–time were studied. We consider formal deformations, defined as a perturbative series, such that the reduced Jacobi identity is respected. Following the standard strategy from other deformation theories of algebraic structures, we first construct a cohomology complex related to our deformation problem. To construct this complex, which we call reduced Lie algebra cohomology complex, we will adapt the scheme used to define a Lie algebra cohomology to our case. The functions, which build the cochain spaces of our cochain complex, will possess a complicated symmetry property, a generalization of (anti-)symmetry, which we will define in the first section of this chapter and we will call it Z_B^{ε}-symmetry.

We still have not calculated the cohomology groups of this complex, but we have shown that the non-trivial first order perturbations in the deformation theory belong to the second cohomology group and we have computed the obstruction operators to their integration. As our cohomology complex is obtained merely by stripping off the test functions from the field Lie algebra cohomology complex, we expect that it inherits all the nice properties of the Lie case, in particular that the obstructions to lift a perturbation to higher order lie in the third cohomology group. In such case we would be immediately able to relate the rigidity of the reduced Lie bracket to the content of the second cohomology group and the integrability of the elements of this group to the content of the third cohomology group.

7.1 Z_B^{ε}-symmetry

The reduced bracket (6.54) obeys the symmetry rule (6.7). The reduced Jacobi identity (6.65) obeys the symmetry rule (6.66). A symmetry rule, generalizing the last two rules for structures with more arguments, will be the following:

Definition 7.1 (Z_B^{ε}-symmetry). *Let V be the reduced space as in Section 6.5 and let us consider the maps $\omega_B^{*n}(\cdot,...,\cdot)_{m_1...m_{n-1}} : \underbrace{V \times ... \times V}_{n} \to V$. Let $\underline{a}_n$ be the n-tuple of scaling dimensions a_i, let $X_i \in V_{a_i}$ and let $\underline{m}_{n-1}$ will be the n-tuple $(m_1, ..., m_{n-1})$. Let $\omega_B^{*n}(X_1, ..., X_n)_{\underline{m}_{n-1}}$ be non-zero only for $m_i \leq \sum_{s=i}^{n} a_s - \sum_{t=i+1}^{n-1} m_t - n + i$. We will say that $\omega_B^{*n}(X_1, ..., X_n)_{\underline{m}_{n-1}}$*

are Z^{ε}_B-symmetric if:

$$\omega^{*n}_B(X_1, ..., X_n)_{\underline{m}_{n-1}} = \frac{1}{n!}\sum_{\underline{i}_n} \omega^{*n}_B(X_{i_1}, ..., X_{i_n})_{\widetilde{\underline{m}}_{n-1}} \left(Z^{\varepsilon}_{BB,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \tag{7.1}$$

or equivalently:

$$\omega^{*n}_B(X_1, ..., X_n)_{\underline{m}_{n-1}} = \omega^{*n}_B(X_{i_1}, ..., X_{i_n})_{\widetilde{\underline{m}}_{n-1}} \left(Z^{\varepsilon}_{BB,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \tag{7.2}$$

where $\left(Z^{\varepsilon}_{BB,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} := \varepsilon_{i_1...i_n}\left(Z_{BB,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}}$, *with* $\left(Z_{BB,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}}$ *as in Definition 6.4 and $\sigma_{\underline{i}_n}$ is the permutation $\{i_1, ..., i_n\}$ of the indices $\{1, ..., n\}$.*

From now on we will use the notation $\sum_{\underline{i}_n} := \sum_{\substack{i_1\neq...\neq i_n \\ i_1...i_n\in[1,...n]}}$

Notation. We will be interested in those Z^{ε}_B-symmetric maps for which B is the default bracket scheme as in Section 6.3. We will call such maps Z^{ε}-symmetric. From now on, whenever the label B stands for the default bracket scheme, we will just omit it.

Observation. It follows from the definition that:

$$\omega^{*n}_B(X_1, ..., X_n)_{\widehat{\underline{m}}_{n-1}} \left(Z^{\varepsilon}_{B,\underline{a}_n,1}\right)^{\widehat{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} = \omega^{*n}_B(X_{i_1}, ..., X_{i_n})_{\widetilde{\underline{m}}_{n-1}} \left(Z^{\varepsilon}_{B,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \tag{7.3}$$

To show this, one uses:

$$\left(Z^{\varepsilon}_{B,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} = \left(Z^{\varepsilon}_{BB,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\widehat{\underline{m}}_{n-1}} \left(Z^{\varepsilon}_{B,\underline{a}_n,1}\right)^{\widehat{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \tag{7.4}$$

Proposition 7.2. *The Z^{ε}_B-symmetry of $\omega^{*n}_B(X_1, ..., X_n)_{\underline{m}_{n-1}}$ ensures that the function*

$$\omega^n(X_1(f_1), ..., X_n(f_n)) := \sum_{\sum m_i<\sum a_k-n+1} \omega^{*n}_B(X_1, ..., X_n)_{\underline{m}_{n-1}} \otimes \left(T_{B,\underline{a}_n}\right)^{\underline{m}_{n-1}}(f_1, ..., f_n) \tag{7.5}$$

is completely anti–symmetric in the arguments $X_i(f_i)$.

Proof. It follows directly from the definitions that:

$$\begin{aligned}
\omega^n(X_1(f_1), ..., X_n(f_n)) &= \sum \omega^{*n}_B(X_1, ..., X_n)_{\widehat{\underline{m}}_{n-1}} \otimes \left(T_{B,\underline{a}_n}\right)^{\widehat{\underline{m}}_{n-1}}(f_1, ..., f_n) \\
&= \sum \omega^{*n}_B(X_1, ..., X_n)_{\widehat{\underline{m}}_{n-1}} \otimes \left(Z^{\varepsilon}_{B,\underline{a}_n,1}\right)^{\widehat{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \left(T_{\underline{a}_n}\right)^{\underline{m}_{n-1}}(f_1, ..., f_n) \\
&= \sum \omega^{*n}_B(X_{i_1}, ..., X_{i_n})_{\widetilde{\underline{m}}_{n-1}} \otimes \left(Z^{\varepsilon}_{B,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \left(T_{\underline{a}_n}\right)^{\underline{m}_{n-1}}(f_1, ..., f_n) \\
&= \sum \omega^{*n}_B(X_{i_1}, ..., X_{i_n})_{\widetilde{\underline{m}}_{n-1}} \varepsilon_{i_1...i_n} \otimes \left(T_{B,\sigma_{\underline{i}_n}(\underline{a}_n)}\right)^{\widetilde{\underline{m}}_{n-1}}(f_{i_1}, ..., f_{i_n}) \\
&= \varepsilon_{i_1...i_n}\omega^{*n}(X_{i_1}(f_{i_1}), ..., X_{i_n}(f_{i_n}))
\end{aligned} \tag{7.6}$$

which proves the proposition.

Example. The two natural examples for Z^ε-symmetric functions are the $*$-bracket and the reduced Jacobi identity.

Proposition 7.3. *Every Z^ε_B-symmetric map gives rise to a Z^ε-symmetric map. The function*

$$\omega^{*n}(X_1,...,X_n)_{\underline{m}_{n-1}} = \omega^{*n}_B(X_1,...,X_n)_{\widetilde{\underline{m}}_{n-1}}\left(Z^\varepsilon_{B,\underline{a}_n,1}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \tag{7.7}$$

is Z^ε-symmetric.

Proof. One observes that:

$$\left(Z^\varepsilon_{B,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} = \left(Z^\varepsilon_{B,\sigma_{\underline{i}_n}(\underline{a}_n),1}\right)^{\widetilde{\underline{m}}_{n-1}}_{\widehat{\underline{m}}_{n-1}}\left(Z^\varepsilon_{\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widehat{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \tag{7.8}$$

Then one writes:

$$\begin{aligned}
\omega^{*n}(X_1,...,X_n)_{\underline{m}_{n-1}} &= \omega^{*n}_B(X_1,...,X_n)_{\widehat{\underline{m}}_{n-1}}\left(Z^\varepsilon_{B,\underline{a}_n,1}\right)^{\widehat{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \\
&= \omega^{*n}_B(X_{i_1},...,X_{i_n})_{\widetilde{\underline{m}}_{n-1}}\left(Z^\varepsilon_{B,\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \\
&= \omega^{*n}_B(X_{i_1},...,X_{i_n})_{\widetilde{\underline{m}}_{n-1}}\left(Z^\varepsilon_{B,\sigma_{\underline{i}_n}(\underline{a}_n),1}\right)^{\widetilde{\underline{m}}_{n-1}}_{\widehat{\underline{m}}_{n-1}}\left(Z^\varepsilon_{\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widehat{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \\
&= \omega^{*n}(X_{i_1},...,X_{i_n})_{\widehat{\underline{m}}_{n-1}}\left(Z^\varepsilon_{\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\widehat{\underline{m}}_{n-1}}_{\underline{m}_{n-1}}
\end{aligned} \tag{7.9}$$

and this proves the proposition.

Corollary. One can construct a Z^ε-symmetric map out of any set of Z^ε_B-symmetric maps $\omega^{*n}_{B_i}(X_1,...,X_n)_{\underline{m}_{n-1}}$ the following way:

$$\omega^{*n}(X_1,...,X_n)_{\underline{m}_{n-1}} = \sum_i \omega^{*n}_{B_i}(X_1,...,X_n)_{\widetilde{\underline{m}}_{n-1}}\left(Z^\varepsilon_{B_i,\underline{a}_n,1}\right)^{\widetilde{\underline{m}}_{n-1}}_{\underline{m}_{n-1}} \tag{7.10}$$

Remark. In an analogous way one shows that a $Z^\varepsilon_{B_1}$-symmetric map gives rise to a $Z^\varepsilon_{B_2}$-symmetric map for every choice of bracket schemes B_1 and B_2.

All this shows, that if we consider the set of Z^ε symmetric maps, we automatically take into account also the Z^ε_B-symmetric maps for any bracket scheme B.

7.2 Reduced Lie algebra cohomology

In this section we will introduce the reduced Lie algebra cohomology complex:

Definition 7.4 (reduced Lie algebra cohomology). *We define the reduced Lie algebra cohomology as:*

- **Cochain complex:**

1. **Cochain spaces $C^n(V)$ of dimension n:**

   The n-cochains in the cochain complex will be the tensor–valued Z^ε-symmetric maps $\omega^{*n}(\cdot,...,\cdot)_{\underline{m}_{n-1}}$. The spaces $C^n(V)$ of all Z^ε-symmetric ω^{*n}'s for a fixed n will compose the cochain sequence $C := (C^n(V))_{n\in\mathbb{N}}$.

2. **Coboundary operators b^n:**

   We define the coboundary operator $b^n : C^n(V) \to C^{n+1}(V)$ through the following component–wise action, provided that $m_i \leq \sum_{s=i}^n a_s - \sum_{t=i+1}^{n-1} m_t - n + i$:

$$
\begin{aligned}
[b^n\omega^{*n}](\underline{X}_{n+1})_{\underline{m}_n} \quad := \quad & \frac{(-1)^n}{n!}\sum_{\underline{i}_{n+1}}\left[\Gamma^*\Big(X_{i_1},\omega^{*n}(X_{i_2},...,X_{i_{n+1}})\Big)\right]_{\widetilde{\underline{m}}_n}\Big(Z^\varepsilon_{\underline{a}_{n+1},\sigma_{\underline{i}_{n+1}}}\Big)^{\widetilde{\underline{m}}_n}_{\underline{m}_n} + \\
& + \frac{1}{2(n-1)!}\sum_{\underline{j}_{n+1}}\left[\omega^{*n}\Big(X_{j_1},...,X_{j_{n-1}},\Gamma^*(X_{j_n},X_{j_{n+1}})\Big)\right]_{\widetilde{\underline{m}}_n}\Big(Z^\varepsilon_{\underline{a}_{n+1},\sigma_{\underline{j}_{n+1}}}\Big)^{\widetilde{\underline{m}}_n}_{\underline{m}_n}
\end{aligned}
\tag{7.11}
$$

   or equivalently:

$$
\begin{aligned}
[b^n\omega^{*n}](\underline{X}_{n+1})_{\underline{m}_n} \quad := \quad & (-1)^n\sum_{i=1}^{n+1}\left[\Gamma^*\Big(X_i,\omega^{*n}(X_1,...,\hat{X}_i,...,X_{n+1})\Big)\right]_{\widetilde{\underline{m}}_n}\Big(Z^\varepsilon_{\underline{a}_{n+1},\sigma_{\widehat{i}}}\Big)^{\widetilde{\underline{m}}_n}_{\underline{m}_n} + \\
& + \sum_{k>j=1}^{n}\left[\omega^{*n}\Big(X_1,...,\hat{X}_j,...,\hat{X}_k,...,X_{n+1},\Gamma^*(X_j,X_k)\Big)\right]_{\widetilde{\underline{m}}_n}\Big(Z^\varepsilon_{\underline{a}_{n+1},\sigma_{\widehat{j}\widehat{k}}}\Big)^{\widetilde{\underline{m}}_n}_{\underline{m}_n}
\end{aligned}
\tag{7.12}
$$

   where $\sigma_{\widehat{i}}$ is the permutation $\{i,1,...,\widehat{i},...,n{+}1\}$ and $\sigma_{\widehat{j}\widehat{k}}$ is the permutation $\{1,...,\widehat{j},...,\widehat{k},...,n+1,j,k\}$.

   Again $\sum_{\underline{i}_{n+1}} := \sum_{\substack{i_1\neq...\neq i_{n+1}\\ i_k\in[1,...n+1]}}$.

   For those n-tuples $(m_1,...,m_n)$, for which the condition $m_i \leq \sum_{s=i}^n a_s - \sum_{t=i+1}^{n-1} m_t - n + i$ does not hold, $[b^n\omega^{*n}](X_1,...,X_{n+1})_{\underline{m}_n}$ will be set to 0.

   We will show later that $b^{n+1}\circ b^n = 0$.

- **Cohomology group:**

  We define:

$$
\begin{aligned}
Z^n(V) \quad &:= \quad Ker(b^n) = \Big\{\omega^{*n}\in C^n(V) \mid \quad [b^n\omega^{*n}](X_1,...,X_{n+1})_{\underline{m}_n} = 0,\ \forall \underline{m}_n\in\mathbb{N}_0^{\otimes n}\Big\} \\
B^n(V) \quad &:= \quad Im(b^n) = \Big\{\omega^{*n}\in C^n(V) \mid \quad \omega^{*n} = b^{n-1}\omega^{*n-1},\ \omega^{*n-1}\in C^{n-1}(V)\Big\}
\end{aligned}
\tag{7.13}
$$

Clearly, $b^{n+1} \circ b^n = 0$ implies $B^n(V) \subseteqq Z^n(V)$. Then we define the n^{th} reduced Lie algebra cohomology group as the quotient:

$$RLH^n(V) = Z^n(V)/B^n(V) \tag{7.14}$$

In order to guarantee that the structure above really gives rise to a cohomology theory, one has to verify that the operators b are really differentials and this will be our occupation until the end of this section.

Proposition 7.5. *$b^{n+1} \circ b^n = 0$ applied to any map ω^{*n} from the cochain complex.*

Proof. Before we proved this proposition, we proved as an exercise a similar statement for Lie algebras (see Appendix B) and throughout the current proof we use the guiding example of the Lie algebra case. This proof will be at places sketchy, because a complete proof would be too much overloaded with subtle technical details and the main ideas would become hardly observable. Before the concrete calculations, let us make the following observations:

Observation *(1).* Suppose that we know the map ω^{*n} in the form:

$$\omega^{*n}(X_1, ..., X_n)_{\underline{m}_{n-1}} = \frac{1}{n!} \sum_{\underline{i}_n} \Omega^{*n}(X_{i_1}, ..., X_{i_n})_{\underline{\widetilde{m}}_{n-1}} \Big(Z^{\varepsilon}_{\underline{a}_n, \sigma_{\underline{i}_n}}\Big)^{\underline{\widetilde{m}}_{n-1}}_{\underline{m}_{n-1}} \tag{7.15}$$

and Ω^{*n} is a composite function in terms of other Z^{ε}-symmetric maps (for example, the two terms of (7.11)). Then we want to write (7.11) and (7.12) in a form, which applied to such maps to produce immediately a more concrete result, that may be more conveniently simplified.

Let us focus on the second term of (7.11). Assumed that we know ω^{*n} in the form (7.15), the map $\Big[\omega^{*n}\Big(X_1, ..., X_{n-1}, \Gamma^*(X_n, X_{n+1})\Big)\Big]_{\underline{\widetilde{m}}}$ can be decomposed into a sum of blocks:

$$\Big[\omega^{*n}\Big(X_1, ..., X_{n-1}, \Gamma^*(X_n, X_{n+1})\Big)\Big]_{B_\kappa, \underline{m}_n} =$$

$$= \frac{1}{n!} \sum_{\underline{i}_{n-1}} (-1)^{n-k} \Omega^{*n}\Big(X_{i_1}, ..., \Gamma^*(X_n, X_{n+1})_{m_n}, X_{i_\kappa} ..., X_{i_{n-1}}\Big)_{\underline{\widetilde{m}}_{n-1}} \Big(Z^{\varepsilon}_{B_\kappa B_\kappa, \underline{a}_{n+1}, \sigma_{\underline{i}^{\kappa}_{n-1}}}\Big)^{\underline{\widetilde{m}}_n}_{\underline{m}_n} \tag{7.16}$$

where $\sigma_{\underline{i}^{\kappa}_{n-1}}$ denotes the permutation $\{i_1, ..., i_{\kappa-1}, n, n+1, i_\kappa, ..., i_{n-1}\}$ and T_{B_κ} will be a basis of the type:

$$\Big(T_{\sigma_{\underline{i}^{\widetilde{a};\kappa}_{n-1}}(\underline{a}_{n-1})}\Big)^{\underline{m}_{n-1}} \circ (1_{a_{i_1}} \times 1_{a_{i_2}} \times ... \times \lambda^{\widetilde{a}}_{a_n, a_{n+1}} \times ... \times 1_{a_{i_{n-1}}})^{m_n} \tag{7.17}$$

where $\sigma_{\underline{i}^{\widetilde{a};\kappa}_{n-1}}(\underline{a}_{n-1})$ denotes the permutation $\{a_{i_1}, .., a_{i_{\kappa-1}}, \widetilde{a}, a_{i_\kappa}, ..., a_{i_{n-1}}\}$. We obtain the following Z^{ε}-symmetrized version of $\Big[\omega^{*n}\Big(X_1, ..., X_{n-1}, \Gamma^*(X_n, X_{n+1})\Big)\Big]_{B_\kappa, \underline{\widetilde{m}}}$:

$$\frac{1}{(n+1)!} \sum_{\underline{j}_{n+1}} \Big[\omega^{*n}\Big(X_{j_1}, ..., X_{j_{n-1}}, \Gamma^*(X_{j_n}, X_{j_{n+1}})\Big)\Big]_{B_\kappa, \underline{\widetilde{m}}_n} \Big(Z^{\varepsilon}_{B_\kappa, \underline{a}_{n+1}, \sigma_{\underline{j}_{n+1}}}\Big)^{\underline{\widetilde{m}}_n}_{\underline{m}_n} \tag{7.18}$$

Note that a decomposition of ω^{*n} in the form (7.15) always exists, and we know for every ω^{*n} at least one such decomposition (see (7.1)). Of course, such decomposition is not unique and there could be more than one Ω^{*n} producing the same ω^{*n}. Suppose that we know two such maps, for example Ω^{*n}_1 and Ω^{*n}_2. Then, using the expression (7.15), we obtain immediately:

$$\sum_{\underline{i}_n}(\Omega^{*n}_1-\Omega^{*n}_2)(X_{i_1},...,X_{i_n})_{\underline{\widetilde{m}}_{n-1}}\left(Z^{\varepsilon}_{\underline{a}_n,\sigma_{\underline{i}_n}}\right)^{\underline{\widetilde{m}}_{n-1}}_{\underline{m}_{n-1}}=0 \tag{7.19}$$

Then the map $\Omega^{*n}_1-\Omega^{*n}_2$ is "Z^{ε}-nilpotent". This means that although the blocks $[\ \ ]_{B,\underline{m}_n}$ are different for different maps Ω^{*n}, their Z^{ε}-symmetrization produces the same result. As we use only symmetrized versions of these blocks, we will avoid additional labelling indicating the concrete choice of Ω^{*n}.

Then we rewrite (7.11) and (7.12) in the form:

$$\begin{aligned}[b^n\omega^{*n}](\underline{X}_{n+1})_{\underline{m}_n} &:= \frac{(-1)^n}{n!}\sum_{\underline{i}_{n+1}}\left[\Gamma^*\Big(X_{i_1},\omega^{*n}(X_{i_2},...,X_{i_{n+1}})\Big)\right]_{\underline{\widetilde{m}}_n}\left(Z^{\varepsilon}_{\underline{a}_{n+1},\sigma_{\underline{i}_{n+1}}}\right)^{\underline{\widetilde{m}}_n}_{\underline{m}_n}+\\ &+\frac{1}{2(n-1)!}\sum_{\kappa=1}^{n}\sum_{\underline{j}_{n+1}}\left[\omega^{*n}\Big(X_{j_1},...,X_{j_{n-1}},\Gamma^*(X_{j_n},X_{j_{n+1}})\Big)\right]_{B_\kappa,\underline{\widetilde{m}}_n}\left(Z^{\varepsilon}_{B_\kappa,\underline{a}_{n+1},\sigma_{\underline{j}_{n+1}}}\right)^{\underline{\widetilde{m}}_n}_{\underline{m}_n}\end{aligned} \tag{7.20}$$

or equivalently:

$$\begin{aligned}[b^n\omega^{*n}](\underline{X}_{n+1})_{\underline{m}_n} &:= (-1)^n\sum_{i=1}^{n+1}\left[\Gamma^*\Big(X_i,\omega^{*n}(X_1,...,\hat{X}_i,...,X_{n+1})\Big)\right]_{\underline{\widetilde{m}}_n}\left(Z^{\varepsilon}_{\underline{a}_{n+1},\sigma_{\widehat{i}}}\right)^{\underline{\widetilde{m}}_n}_{\underline{m}_n}+\\ &+\sum_{\kappa=1}^{n}\sum_{k>j=1}^{n}\left[\omega^{*n}\Big(X_1,...,\hat{X}_j,...,\hat{X}_k,...,X_{n+1},\Gamma^*(X_j,X_k)\Big)\right]_{B_\kappa,\underline{\widetilde{m}}_n}\left(Z^{\varepsilon}_{B_\kappa,\underline{a}_{n+1},\sigma_{\widehat{jk}}}\right)^{\underline{\widetilde{m}}_n}_{\underline{m}_n}\end{aligned} \tag{7.21}$$

where $\sigma_{\widehat{i}}$ is the permutation $\{i,1,...,\widehat{i},...,n+1\}$ and $\sigma_{\widehat{jk}}$ is the permutation $\{1,...,\widehat{j},...,\widehat{k},..., n+1,j,k\}$.

Observation *(2).* The map, appearing in the first term of (7.20):

$$W^{*n+1}(X_1,...,X_{n+1})_{\underline{m}_n}=\frac{(-1)^n}{n!}\sum_{\underline{i}_{n+1}}\left[\Gamma^*\Big(X_{i_1},\omega^{*n}(X_{i_2},...,X_{i_{n+1}})\Big)\right]_{\underline{\widetilde{m}}_n}\left(Z^{\varepsilon}_{\underline{a}_{n+1},\sigma_{\underline{i}_{n+1}}}\right)^{\underline{\widetilde{m}}_n}_{\underline{m}_n} \tag{7.22}$$

is Z^{ε}-symmetric. Supposed that we know the decomposition (7.15) of ω^{*n}, we can write:

$$\begin{aligned}W^{*n+1}(X_1,...,X_{n+1})_{\underline{m}_n} &= \frac{1}{(n+1)!}\sum_{\underline{i}_{n+1}}\Omega^{*n+1}(X_{i_1},...,X_{i_{n+1}})_{\underline{\hat{m}}_n}\left(Z^{\varepsilon}_{\underline{a}_{n+1},\sigma_{\underline{i}_{n+1}}}\right)^{\underline{\hat{m}}_n}_{\underline{m}_n}\\ \Omega^{*n+1}(X_{i_1},...,X_{i_{n+1}})_{\underline{m}_n} &:= (-1)^n(n+1)\left[\Gamma^*\Big(X_{i_1},\omega^{*n}(X_{i_2},...,X_{i_{n+1}})\Big)\right]_{\underline{m}_n}\\ &= (-1)^n(n+1)\left[\Gamma^*\Big(X_{i_1},\Omega^{*n}(X_{i_2},...,X_{i_{n+1}})\Big)\right]_{\underline{m}_n}\end{aligned} \tag{7.23}$$

The map, appearing in the second term of (7.20):

$$W^{*n+1}_{\kappa}(X_1,...,X_{n+1})_{\underline{m}_n}=\frac{1}{2(n-1)!}\sum_{\kappa=1}^{n}\sum_{\underline{j}_{n+1}}\Big[\omega^{*n}\Big(X_{j_1},...,X_{j_{n-1}},\Gamma^*(X_{j_n},X_{j_{n+1}})\Big)\Big]_{B_\kappa,\widetilde{\underline{m}}_n}\Big(Z^{\varepsilon}_{B_\kappa,\underline{a}_{n+1},\sigma_{\underline{j}_{n+1}}}\Big)^{\widetilde{\underline{m}}_n}_{\underline{m}_n} \tag{7.24}$$

is Z^ε-symmetric. We observe that:

$$\Big(Z^{\varepsilon}_{B_\kappa,\underline{a}_{n+1},\sigma_{\underline{j}_{n+1}}}\Big)^{\widetilde{\underline{m}}_n}_{\underline{m}_n}=\Big(Z^{\varepsilon}_{B_\kappa,\sigma_{\underline{j}_{n+1}}(\underline{a}_{n+1}),1}\Big)^{\widetilde{\underline{m}}_n}_{\hat{\underline{m}}_n}\Big(Z^{\varepsilon}_{\underline{a}_{n+1},\sigma_{\underline{j}_{n+1}}}\Big)^{\hat{\underline{m}}_n}_{\underline{m}_n} \tag{7.25}$$

Thus $W^{*n+1}_{\kappa}(X_1,...,X_{n+1})$ can be written as:

$$\begin{aligned}W^{*n+1}_{\kappa}(X_1,...,X_{n+1})_{\underline{m}_n} &= \frac{1}{(n+1)!}\sum_{\underline{i}_{n+1}}\Omega^{*n+1}_{\kappa}(X_{i_1},...,X_{i_{n+1}})_{\hat{\underline{m}}_n}\Big(Z^{\varepsilon}_{\underline{a}_{n+1},\sigma_{\underline{i}_{n+1}}}\Big)^{\hat{\underline{m}}_n}_{\underline{m}_n}\\ \Omega^{*n+1}_{\kappa}(X_{i_1},...,X_{i_{n+1}})_{\underline{m}_n} &:= \frac{n(n+1)}{2}\Big[\omega^{*n}\Big(X_{j_1},...,X_{j_{n-1}},\Gamma^*(X_{j_n},X_{j_{n+1}})\Big)\Big]_{B_\kappa,\widetilde{\underline{m}}_n}\Big(Z^{\varepsilon}_{B_\kappa,\sigma_{\underline{j}_{n+1}}(\underline{a}_{n+1}),1}\Big)^{\widetilde{\underline{m}}_n}_{\underline{m}_n}\end{aligned} \tag{7.26}$$

Then we find:

$$\begin{aligned}&\Big[W^{*n+1}_{\kappa_1}\Big(X_1,...,X_n,\Gamma^*(X_{n+1},X_{n+2})\Big)\Big]_{B_{\kappa_2},\widetilde{\underline{m}}_{n+1}}=\\ &=\frac{1}{(n+1)!}\sum_{\underline{i}_n}(-1)^{n+1-\kappa_2}\Omega^{*n+1}_{\kappa_1}\Big(X_{i_1},...,\Gamma^*(X_{n+1},X_{n+2})_{m_{n+1}},X_{i_{\kappa_2}}...,X_{i_n}\Big)_{\widetilde{\underline{m}}_n}\Big(Z^{\varepsilon}_{B_{\kappa_2}B_{\kappa_2},\underline{a}_{n+2},\sigma_{\underline{i}_n^{\kappa_2}}}\Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}}\end{aligned} \tag{7.27}$$

Taking into account (7.15) and (7.16) after some quite cumbersome computations we arrive at:

$$\begin{aligned}&\sum_{\kappa_1=1}^{n+1}\sum_{\kappa_2=1}^{n+1}\sum_{k>j=1}^{n+1}\Big[W^{*n+1}_{\kappa_1}\Big(X_1,...,\hat{X}_j,...,\hat{X}_k,...,X_{n+2},\Gamma^*(X_j,X_k)\Big)\Big]_{B_{\kappa_2},\widetilde{\underline{m}}_{n+1}}\Big(Z^{\varepsilon}_{B_{\kappa_2},\underline{a}_{n+2},\sigma_{\widehat{jk}}}\Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}}=\\ &=\frac{n(n-1)}{4(n!)}\sum_{\kappa_1\neq\kappa_2=1}^{n}\sum_{\underline{i}_{n+2}}\Big[\omega^{*n}\Big(X_{i_1},...,\Gamma^*(X_{i_{n-1}},X_{i_n}),\Gamma^*(X_{i_{n+1}},X_{i_{n+2}})\Big)\Big]_{B_{\kappa_1\kappa_2},\widetilde{\underline{m}}_{n+1}}\Big(Z^{\varepsilon}_{B_{\kappa_1\kappa_2},\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}}\Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}}+\\ &+\frac{n}{2(n!)}\sum_{\kappa=1}^{n}\sum_{\underline{i}_{n+2}}\Big[\omega^{*n}\Big(X_{i_1},...,X_{i_{n-1}},\Gamma^*\Big(X_{i_n},\Gamma^*(X_{i_{n+1}},X_{i_{n+2}})\Big)\Big)\Big]_{\widetilde{B}_\kappa,\widetilde{\underline{m}}_{n+1}}\Big(Z^{\varepsilon}_{\widetilde{B}_\kappa,\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}}\Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}}\end{aligned} \tag{7.28}$$

In the following we will explain the two terms in the right hand side of this equality. Let us denote:

$$\begin{aligned}&\sigma_{\underline{i}_{n-2}^{\kappa_1\kappa_2}}:\ \{1,...,n\}\to\{i_1,...,i_{\kappa_1-1},n-1,n,i_{\kappa_1},...,i_{\kappa_2-2},n+1,n+2,i_{\kappa_2-1},...,i_{n-2}\}\\ &\sigma_{\underline{i}_{n-2}^{\widetilde{a}_1\widetilde{a}_2;\kappa_1\kappa_2}}(\underline{a}_{n-2})\equiv\{a_{i_1},...,a_{i_{\kappa_1-1}},\widetilde{a}_1,a_{i_{\kappa_1}},...,a_{i_{\kappa_2-2}},\widetilde{a}_2,a_{i_{\kappa_2-1}},...,a_{i_{n-2}}\}\end{aligned} \tag{7.29}$$

and let $T_{B_{\kappa_1\kappa_2}}$ is a basis of the type:

$$\Big(T_{\sigma_{\underline{i}_{n-2}^{\widetilde{a}_1\widetilde{a}_2;\kappa_1\kappa_2}}(\underline{a}_{n-2})}\Big)^{\underline{m}_{n-1}}\circ(1_{a_{i_1}}\times...\times1_{a_{i_{\kappa_1-1}}}\times\lambda^{\widetilde{a}_1}_{a_{n-1},a_n}\times1_{a_{i_{\kappa_1}}}\times...\times1_{a_{i_{\kappa_2-2}}}\times\lambda^{\widetilde{a}_2}_{a_{n+1},a_{n+2}}\times1_{a_{i_{\kappa_2-1}}}\times...\times1_{a_{i_{n-2}}})^{m_nm_{n+1}}$$

(7.30)

Then, supposed that we know a decomposition of ω^{*n} in the form (7.15), we construct the blocks:

$$\left[\omega^{*n}\Big(X_1,...,X_{n-2},\Gamma^*(X_{n-1},X_n),\Gamma^*(X_{n+1},X_{n+2})\Big)\right]_{B_{\kappa_1\kappa_2},\underline{m}_{n+1}} :=$$
$$\frac{1}{n!}\sum_{\underline{i}_{n-2}}(-1)^{\kappa_1+\kappa_2-1}\Omega^n(X_{i_1},...,\underbrace{\Gamma^*(X_{n-1},X_n)_{m_n}}_{\kappa_1-\text{position}},...,\underbrace{\Gamma^*(X_{n+1},X_{n+2})_{m_{n+1}}}_{\kappa_2-\text{position}},...,X_{i_{n-2}})_{\underline{\widetilde{m}}_{n-1}}\cdot$$
$$\cdot\left(Z^{\varepsilon}_{B_{\kappa_1\kappa_2}B_{\kappa_1\kappa_2},\underline{a}_{n+2},\sigma_{\underline{i}^{\kappa_1\kappa_2}_{n-2}}}\right)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}} \tag{7.31}$$

which are Z^{ε}-symmetrizable as:

$$\frac{1}{(n+2)!}\sum_{\underline{i}_{n+2}}\left[\omega^{*n}\Big(X_{i_1},...,X_{i_{n-2}},\Gamma^*(X_{i_{n-1}},X_{i_n}),\Gamma^*(X_{i_{n+1}},X_{i_{n+2}})\Big)\right]_{B_{\kappa_1\kappa_2},\underline{\widetilde{m}}_{n+1}}\left(Z^{\varepsilon}_{B_{\kappa_1\kappa_2},\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}}\right)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}} \tag{7.32}$$

Let us now denote:

$$\begin{aligned}&\widetilde{\sigma}_{\underline{i}^{\kappa}_{n-1}}:\ \{1,...,n\}\to\{i_1,...,i_{\kappa-1},n,n+1,n+2,i_\kappa,...i_{n-1}\}\\&\widetilde{\sigma}_{\underline{i}^{\widetilde{a};\kappa}_{n-1}}(\underline{a}_{n-1})\equiv\{a_{i_1},...,a_{i_{\kappa-1}},\widetilde{a},a_{i_\kappa},...,a_{i_{n-1}}\}\end{aligned} \tag{7.33}$$

and let $T_{\widetilde{B}_{\kappa_1\kappa_2}}$ is a basis of the type:

$$\left(T_{\widetilde{\sigma}_{\underline{i}^{\widetilde{a};\kappa}_{n-1}}(\underline{a}_{n-1})}\right)^{\underline{m}_{n-1}}\circ\left[1_{a_{i_1}}\times...\times 1_{a_{i_{\kappa-1}}}\times\Big(T_{a_na_{n+1}a_{n+2}}\Big)^{\widetilde{a}}\times 1_{a_{i_\kappa}}\times...\times 1_{a_{i_{n-1}}}\right]^{m_nm_{n+1}} \tag{7.34}$$

Again, supposed that we know a decomposition of ω^{*n} in the form (7.15), we construct the blocks:

$$\left[\omega^{*n}\Big(X_1,...,X_{n-1},\Gamma^*(X_n,\Gamma^*(X_{n+1},X_{n+2}))\Big)\right]_{\widetilde{B}_\kappa,\underline{m}_{n+1}} :=$$
$$\frac{1}{n!}\sum_{\underline{i}_{n-2}}\Omega^n(X_{i_1},...,\underbrace{\Gamma^*(X_n,\Gamma^*(X_{n+1},X_{n+2}))_{m_nm_{n+1}}}_{\kappa-\text{position}},...,X_{i_{n-1}})_{\underline{\widetilde{m}}_{n-1}}\left(Z^{\varepsilon}_{\widetilde{B}_\kappa\widetilde{B}_\kappa,\underline{a}_{n+2},\widetilde{\sigma}_{\underline{i}^{\kappa}_{n-1}}}\right)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}} \tag{7.35}$$

and they are Z^{ε}-symmetrizable as:

$$\frac{1}{(n+2)!}\sum_{\underline{i}_{n+2}}\left[\omega^{*n}\Big(X_{i_1},...,X_{i_{n-1}},\Gamma^*(X_n,\Gamma^*(X_{n+1},X_{n+2}))\Big)\right]_{\widetilde{B}_\kappa,\underline{\widetilde{m}}_{n+1}}\left(Z^{\varepsilon}_{\widetilde{B}_\kappa,\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}}\right)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}} \tag{7.36}$$

Now we are ready for concrete manipulations. When we apply b^{n+1} on the r.h.s. of (7.21) we obtain:

$$
\begin{array}{ll}
 & (b^{n+1}\circ b^n\omega^{*n})(X_1,...,X_{n+2})_{\underline{m}_{n+1}} = \\
\textbf{(I)} & -\dfrac{(n+1)}{(n+1)!}\displaystyle\sum_{\underline{i}_{n+2}}\Big[\Gamma^*\Big(X_{i_1},\Gamma^*\Big(X_{i_2},\omega^{*n}(X_{i_3},...,X_{i_{n+2}})\Big)\Big)\Big]_{\underline{\widetilde{m}}_{n+1}}\Big(Z^{\varepsilon}_{\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}}\Big)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}}+ \\
\textbf{(II)} & +(-1)^{n+1}\dfrac{(n+1)n}{2(n+1)!}\displaystyle\sum_{\kappa=1}^{n}\sum_{\underline{i}_{n+2}}\Gamma^*\Big(X_{i_1},\Big[\omega^{*n}\Big(X_{i_2},...,X_{i_n},\Gamma^*(X_{i_{n+1}},X_{i_{n+2}})\Big)\Big]_{B_\kappa}\Big)_{\underline{\widetilde{m}}_{n+1}}\Big(Z^{\varepsilon}_{B_{\kappa+1},\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}}\Big)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}}+ \\
\textbf{(III)} & +(-1)^{n}\dfrac{n}{2(n!)}\displaystyle\sum_{\kappa=1}^{n}\sum_{\underline{i}_{n+2}}\Gamma^*\Big(X_{i_1},\Big[\omega^{*n}\Big(X_{i_2},...,X_{i_n},\Gamma^*(X_{i_{n+1}},X_{i_{n+2}})\Big)\Big]_{B_\kappa}\Big)_{\underline{\widetilde{m}}_{n+1}}\Big(Z^{\varepsilon}_{B_{\kappa+1},\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}}\Big)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}}+ \\
\textbf{(IV)} & +\dfrac{1}{2(n!)}\displaystyle\sum_{\underline{i}_{n+2}}\Gamma^*\Big(\Gamma^*(X_{i_1},X_{i_2}),\omega^{*n}(X_{i_3},...,X_{i_{n+2}})\Big)_{B_1,\underline{\widetilde{m}}_{n+1}}\Big(Z^{\varepsilon}_{B_1,\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}}\Big)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}}+ \\
\textbf{(V)} & +\dfrac{n(n-1)}{4(n!)}\displaystyle\sum_{\kappa_1\neq\kappa_2=1}^{n}\sum_{\underline{i}_{n+2}}\Big[\omega^{*n}\Big(X_{i_1},...,\Gamma^*(X_{i_{n-1}},X_{i_n}),\Gamma^*(X_{i_{n+1}},X_{i_{n+2}})\Big)\Big]_{B_{\kappa_1\kappa_2},\underline{\widetilde{m}}_{n+1}}\Big(Z^{\varepsilon}_{B_{\kappa_1\kappa_2},\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}}\Big)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}}+ \\
\textbf{(VI)} & +\dfrac{n}{2(n!)}\displaystyle\sum_{\kappa=1}^{n}\sum_{\underline{i}_{n+2}}\Big[\omega^{*n}\Big(X_{i_1},...,X_{i_{n-1}},\Gamma^*\Big(X_{i_n},\Gamma^*(X_{i_{n+1}},X_{i_{n+2}})\Big)\Big)\Big]_{\widetilde{B}_\kappa,\underline{\widetilde{m}}_{n+1}}\Big(Z^{\varepsilon}_{\widetilde{B}_\kappa,\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}}\Big)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}}
\end{array}
\tag{7.37}
$$

B_1 is a bracket scheme B_κ with $\kappa = 1$.

We will show that these terms cancel each other in pairs or alone. For this purpose we need the following lemmas:

Lemma 7.6. *The following structural relation can be proven:*

$$
\begin{aligned}
&\omega^{*2}\Big(\omega^{*s}(X_{i_1},...,X_{i_s}),\omega^{*t}(X_{i_{s+1}},...,X_{i_{\tilde{n}}})\Big)_{\underline{m}_{\tilde{n}-1}}\Big(Z^{\varepsilon}_{B_a,\underline{a}_{\tilde{n}},\sigma_{\underline{i}_{\tilde{n}}}}\Big)^{\underline{m}_{\tilde{n}-1}}_{\underline{\widetilde{m}}_{\tilde{n}-1}} = \\
&= (-1)^{st-1}\omega^{*2}\Big(\omega^{*t}(X_{i_{s+1}},...,X_{i_{\tilde{n}}}),\omega^{*s}(X_{i_1},...,X_{i_s})\Big)_{\underline{\widehat{m}}_{\tilde{n}-1}}\Big(Z^{\varepsilon}_{B_b,\underline{a}_{\tilde{n}},\sigma^{\sharp}_{\underline{i}^s_{\tilde{n}}}}\Big)^{\underline{\widehat{m}}_{\tilde{n}-1}}_{\underline{\widetilde{m}}_{\tilde{n}-1}}
\end{aligned}
\tag{7.38}
$$

where $\tilde{n} = s + t$, with $\sigma^{\sharp}_{\underline{i}^s_{\tilde{n}}}$ we denote the permutation $\{i_{s+1},...,i_{\tilde{n}},i_1,...,i_s\}$ and the bases T_{B_a} and T_{B_b} are:

$$
\begin{aligned}
\Big(T_{B_a,\underline{a}_{s+t}}\Big)^{\underline{m}_{s+t-1}} &= \lambda^{e}_{\epsilon_1\epsilon_2}\circ\Big[\Big(T_{a_1...a_s}\Big)^{m_2,...,m_s}\otimes\Big(T_{a_{s+1}...a_{s+t}}\Big)^{m_{s+1},...,m_{s+t-1}}\Big] \\
\Big(T_{B_b,\underline{a}_{s+t}}\Big)^{\underline{m}_{s+t-1}} &= \lambda^{e}_{\epsilon'_1\epsilon'_2}\circ\Big[\Big(T_{a_1...a_t}\Big)^{m_2,...,m_t}\otimes\Big(T_{a_{t+1}...a_{s+t}}\Big)^{m_{t+1},...,m_{s+t-1}}\Big]
\end{aligned}
\tag{7.39}
$$

Proof. Let for some fixed $\underline{m}_{\tilde{n}-1}$ and $\sigma_{\underline{i}_{\tilde{n}}}(\underline{a}_{\tilde{n}})$ $\Big(T_{a_{i_1}...a_{i_s}}\Big)^{m_2,...,m_s}$ maps to a representation with scaling dimension u, $\Big(T_{a_{i_{s+1}}...a_{i_{s+t}}}\Big)^{m_{s+1},...,m_{s+t-1}}$ maps to a representation with scaling dimension v and let $\omega^{*2}\Big(\omega^{*s}(X_{i_1},...,X_{i_s}),\omega^{*t}(X_{i_{s+1}},...,X_{i_{\tilde{n}}})\Big)_{\underline{m}_{\tilde{n}-1}}$, $\omega^{*s}(X_{i_1},...,X_{i_s})_{m_2...m_s}$ and

$\omega^{*t}(X_{i_{s+1}},...,X_{i_{\tilde{n}}})_{m_{s+1}...m_{s+t-1}}$ have scaling dimensions e, u and v. Let us also denote $\underline{\check{m}}_{\tilde{n}-1} := m_1, m_{s+1}...m_{s+t-1}, m_2...m_s$ The graded symmetry of the reduced space yields:

$$\omega^{*2}\Big(\omega^{*s}(X_{i_1},...,X_{i_s}),\omega^{*t}(X_{i_{s+1}},...,X_{i_{\tilde{n}}})\Big)_{\underline{m}_{\tilde{n}-1}} = (-1)^{u+v-e}\omega^{*2}\Big(\omega^{*t}(X_{i_{s+1}},...,X_{i_{\tilde{n}}}),\omega^{*s}(X_{i_1},...,X_{i_s})\Big)_{\underline{\check{m}}_{\tilde{n}-1}} \tag{7.40}$$

Because of the graded symmetry of λ^e_{uv} the elements of intertwiner bases for B_a and B_b (from the l.h.s. of (7.38)) are related as:

$$\Big(T_{B_a,\sigma_{\underline{i}_{\tilde{n}}}(\underline{a}_{\tilde{n}})}\Big)^{\underline{m}_{\tilde{n}-1}} = (-1)^{u+v-e-1}\Big(T_{B_b,\sigma^{\sharp}_{\underline{i}^s_{\tilde{n}}}(\underline{a}_{\tilde{n}})}\Big)^{\underline{\check{m}}_{\tilde{n}-1}} \tag{7.41}$$

Then for the two Z-matrices defined as in (7.6) holds:

$$\Big(Z_{B_a,\underline{a}_{\tilde{n}},\sigma_{\underline{i}_{\tilde{n}}}}\Big)^{\underline{m}_{\tilde{n}-1}}_{\underline{\widetilde{m}}_{\tilde{n}-1}} = (-1)^{u+v-e-1}\Big(Z_{B_b,\underline{a}_{\tilde{n}},\sigma^{\sharp}_{\underline{i}^s_{\tilde{n}}}}\Big)^{\underline{\check{m}}_{\tilde{n}-1}}_{\underline{\widetilde{m}}_{\tilde{n}-1}} \tag{7.42}$$

Taking into consideration that $\varepsilon_{i_1,...,i_{\tilde{n}}} = (-1)^{st}\varepsilon_{i_{s+1},...,i_{\tilde{n}},i_1,...,i_s}$ we obtain:

$$\Big(Z^{\varepsilon}_{B_a,\underline{a}_{\tilde{n}},\sigma_{\underline{i}_{\tilde{n}}}}\Big)^{\underline{m}_{\tilde{n}-1}}_{\underline{\widetilde{m}}_{\tilde{n}-1}} = (-1)^{u+v-e-1}(-1)^{st}\Big(Z^{\varepsilon}_{B_b,\underline{a}_{\tilde{n}},\sigma^{\sharp}_{\underline{i}^s_{\tilde{n}}}}\Big)^{\underline{\check{m}}_{\tilde{n}-1}}_{\underline{\widetilde{m}}_{\tilde{n}-1}} \tag{7.43}$$

Then follows immediately:

$$\begin{aligned}&\Big[\omega^{*2}\Big(\omega^{*s}(X_{i_1},...,X_{i_s}),\omega^{*t}(X_{i_{s+1}},...,X_{i_{\tilde{n}}})\Big)\Big]_{B_a,\underline{m}_{\tilde{n}-1}}\Big(Z^{\varepsilon}_{B_a,\underline{a}_{\tilde{n}},\sigma_{\underline{i}_{\tilde{n}}}}\Big)^{\underline{m}_{\tilde{n}-1}}_{\underline{\widetilde{m}}_{\tilde{n}-1}} = \\ &= (-1)^{st-1}\Big[\omega^{*2}\Big(\omega^{*t}(X_{i_{s+1}},...,X_{i_{\tilde{n}}}),\omega^{*s}(X_{i_1},...,X_{i_s})\Big)\Big]_{B_b,\underline{\check{m}}_{\tilde{n}-1}}\Big(Z^{\varepsilon}_{B_b,\underline{a}_{\tilde{n}},\sigma^{\sharp}_{\underline{i}^s_{\tilde{n}}}}\Big)^{\underline{\check{m}}_{\tilde{n}-1}}_{\underline{\widetilde{m}}_{\tilde{n}-1}}\end{aligned} \tag{7.44}$$

and the sum over $\underline{m}_{\tilde{n}-1}$ on both sides of the equality above produces exactly (7.38), which proves the lemma.

Lemma 7.7. *The following relation holds:*

$$\begin{aligned}&\Big[\omega^{*n}\Big(X_{i_1},...,X_{i_{n-2}},\omega^{*t}(X_{i_{n-1}},...,X_{i_{t+n-2}}),\omega^{*t}(X_{i_{t+n-1}},...,X_{i_{\tilde{n}}})\Big)\Big]_{B_{\kappa_1\kappa_2},\underline{\widehat{m}}_{\tilde{n}-1}}\Big(Z^{\varepsilon}_{B_{\kappa_1\kappa_2},\underline{a}_{\tilde{n}},\sigma_{\underline{i}_{\tilde{n}}}}\Big)^{\underline{\widehat{m}}_{\tilde{n}-1}}_{\underline{m}_{\tilde{n}-1}} = \\ &= -\Big[\omega^{*n}\Big(X_{i_1},...,X_{i_{n-2}},\omega^{*t}(X_{i_{t+n-1}},...,X_{i_{\tilde{n}}}),\omega^{*t}(X_{i_{n-1}},...,X_{i_{n+t-2}})\Big)\Big]_{B_{\kappa_1\kappa_2},\underline{\widetilde{m}}_{\tilde{n}-1}}\Big(Z^{\varepsilon}_{B_{\kappa_1\kappa_2},\underline{a}_{\tilde{n}},\sigma^{\flat}_{\underline{i}^t_{\tilde{n}}}}\Big)^{\underline{\widetilde{m}}_{\tilde{n}-1}}_{\underline{m}_{\tilde{n}-1}}\end{aligned} \tag{7.45}$$

where t is an even number and $\sigma^{\flat}_{\underline{i}^t_{\tilde{n}}}$ denotes the permutation $\{i_1,...,i_{n-2},i_{t+n-1},...,i_{\tilde{n}},i_{n-1},...,i_{n+t-2}\}$.

Proof. The proof of this statement follows the same philosophy as the proof of Lemma 7.6. We will not display it here because it is too technical.

With these lemmas we are ready to rewrite the terms of (7.37) in such form so we are able to cancel them:

- obviously **(II)** + **(III)** $= 0$

- with Lemma 7.6 we show that **(I)** + **(IV)** gives rise to a reduced Jacobi identity between $\omega^{*n}(X_{i_3},...,X_{i_{n+2}})_{\underline{m}_{n-1}}$, X_{i_1} and X_{i_2} for fixed $i_1,...,i_{n+2}$, we omit the indices of the matrices Y for simplicity:

$$
\begin{aligned}
\mathbf{(I)}+\mathbf{(IV)} \quad \sim \quad & \sum_{\underline{i}_{n+2}} \Bigg\{ \Bigg[\Gamma^*\Big(X_{i_1}, \Gamma^*\Big(X_{i_2}, \omega^{*n}(X_{i_3},...,X_{i_{n+2}}) \Big)\Big) \Bigg]_{\widetilde{\underline{m}}_{n+1}} \Big(Z^{\varepsilon}_{\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}} \Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}} + \\
& +(-1)^{n+1} \Bigg(\Gamma^*\Big(X_{i_2}, \Gamma^*\Big(\omega^{*n}(X_{i_3},...,X_{i_{n+2}}), X_{i_1} \Big)\Big) \Bigg)_{B_I,\widetilde{\underline{m}}_{n+1}} \Big(Z^{\varepsilon}_{B_I,\underline{a}_{n+2},\sigma'_{\underline{i}_{n+2}}} \Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}} + \\
& + \Bigg(\Gamma^*\Big(\omega^{*n}(X_{i_3},...,X_{i_{n+2}}), \Gamma^*(X_{i_1},X_{i_2}) \Big) \Bigg)_{B_{II},\widetilde{\underline{m}}_{n+1}} \Big(Z^{\varepsilon}_{B_{II}\underline{a}_{n+2},\sigma''_{\underline{i}_{n+2}}} \Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}} \Bigg\} \\
\sim \quad & \sum_{\underline{i}_{n+2}} \Bigg\{ \Bigg[\Gamma^*\Big(X_{i_1}, \Gamma^*\Big(X_{i_2}, \omega^{*n}(X_{i_3},...,X_{i_{n+2}}) \Big)\Big) \Bigg]_{\widetilde{\underline{m}}_{n+1}} + \\
& + \Gamma^*\Big(X_{i_2}, \Gamma^*\Big(\omega^{*n}(X_{i_3},...,X_{i_{n+2}}), X_{i_1} \Big)\Big)_{\widehat{\underline{m}}_{n+1}} Y + \\
& + \Gamma^*\Big(\omega^{*n}(X_{i_3},...,X_{i_{n+2}}), \Gamma^*(X_{i_1},X_{i_2}) \Big)_{\overline{\underline{m}}_{n+1}} YY \Bigg\} \cdot \Big(Z^{\varepsilon}_{\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}} \Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}} = 0
\end{aligned}
\tag{7.46}
$$

B_I and B_{II} are the relevant bases and $\sigma'_{\underline{i}_{n+2}}$ and $\sigma''_{\underline{i}_{n+2}}$ are the relevant permutations.

- the terms in **(V)** for a fixed $B_{\kappa_1\kappa_2} = B$ can be rewritten with the help of Lemma 7.7 as a sum of pairs of terms which cancel each other:

$$
\begin{aligned}
\mathbf{(V)} \quad \sim \quad & \sum_{\underline{i}_{n-2}} \Bigg\{ \Bigg[\omega^{*n}\Big(X_{i_1},...,X_{i_{n-2}}, \Gamma^*(X_{i_r},X_{i_q}), \Gamma^*(X_{i_s},X_{i_t}) \Big) \Bigg]_{B,\widetilde{\underline{m}}_{n+1}} \Big(Z^{\varepsilon}_{B,\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}} \Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}} + \\
& + \Bigg[\omega^{*n}\Big(X_{i_1},...,X_{i_{n-2}}, \Gamma^*(X_{i_s},X_{i_t}), \Gamma^*(X_{i_r},X_{i_q}) \Big) \Bigg]_{B,\widetilde{\underline{m}}_{n+1}} \Big(Z^{\varepsilon}_{B,\underline{a}_{n+2},\sigma^{\flat}_{\underline{i}^2_{n+2}}} \Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}} \Bigg\} \\
\sim \quad & \sum_{\underline{i}_{n-2}} \Bigg\{ \Bigg[\omega^{*n}\Big(X_{i_1},...,X_{i_{n-2}}, \Gamma^*(X_{i_r},X_{i_q}), \Gamma^*(X_{i_s},X_{i_t}) \Big) \Bigg]_{B,\widetilde{\underline{m}}_{n+1}} \Big(Z^{\varepsilon}_{B,\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}} \Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}} - \\
& - \Bigg[\omega^{*n}\Big(X_{i_1},...,X_{i_{n-2}}, \Gamma^*(X_{i_r},X_{i_q}), \Gamma^*(X_{i_s},X_{i_t}) \Big) \Bigg]_{B,\widetilde{\underline{m}}_{n+1}} \Big(Z^{\varepsilon}_{B,\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}} \Big)^{\widetilde{\underline{m}}_{n+1}}_{\underline{m}_{n+1}} \Bigg\} = 0
\end{aligned}
\tag{7.47}
$$

- for a fixed $\widetilde{B} = B$, **(VI)** can be grouped in triples that contain a reduced Jacobi identity which cancels them:

$$
\begin{aligned}
\textbf{(VI)} \sim & \sum_{\underline{i}_{n-1}} \Bigg\{ \bigg[\omega^{*n}\Big(X_{r_1}, ..., X_{r_{n-1}}, \Gamma^*\Big(X_{r_n}, \Gamma^*(X_{r_{n+1}}, X_{r_{n+2}}) \Big) \Big) \bigg]_{B,\underline{\widetilde{m}}_{n+1}} \Big(Z^{\varepsilon}_{B,\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}} \Big)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}} + \\
& + \bigg[\omega^{*n}\Big(X_{r_1}, ..., X_{r_{n-1}}, \Gamma^*\Big(X_{r_{n+1}}, \Gamma^*(X_{r_{n+2}}, X_{r_n}) \Big) \Big) \bigg]_{B,\underline{\widetilde{m}}_{n+1}} \Big(Z^{\varepsilon}_{B,\underline{a}_{n+2},\sigma'_{\underline{i}_{n+2}}} \Big)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}} + \\
& + \bigg[\omega^{*n}\Big(X_{r_1}, ..., X_{r_{n-1}}, \Gamma^*\Big(X_{r_{n+2}}, \Gamma^*(X_{r_n}, X_{r_{n+1}}) \Big) \Big) \bigg]_{B,\underline{\widetilde{m}}_{n+1}} \Big(Z^{\varepsilon}_{B,\underline{a}_{n+2},\sigma''_{\underline{i}_{n+2}}} \Big)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}} \Bigg\} \\
\sim & \sum_{\underline{i}_{n-1}} \bigg[\omega^{*n}\bigg(X_{r_1}, ..., X_{r_{n-1}}, \Big\{ \Gamma^*\Big(X_{r_n}, \Gamma^*(X_{r_{n+1}}, X_{r_{n+2}}) \Big) + \Gamma^*\Big(X_{r_{n+1}}, \Gamma^*(X_{r_{n+2}}, X_{r_n}) \Big) Y + \\
& + \Gamma^*\Big(X_{r_n}, \Gamma^*(X_{r_{n+1}}, X_{r_{n+2}}) \Big) YY \Big\} \bigg) \bigg]_{B,\underline{\widetilde{m}}_{n+1}} \Big(Z^{\varepsilon}_{B,\underline{a}_{n+2},\sigma_{\underline{i}_{n+2}}} \Big)^{\underline{\widetilde{m}}_{n+1}}_{\underline{m}_{n+1}} = 0
\end{aligned}
\tag{7.48}
$$

$\sigma'_{\underline{i}_{n+2}}$ and $\sigma''_{\underline{i}_{n+2}}$ are the relevant permutations.

Then the sum of all terms in (7.37) is 0 and this completes the proof of Proposition 7.5.

7.3 Deformations of the reduced Lie algebra

In the previous section we constructed a cohomology complex designed in a special way to serve in the description of formal deformations of the reduced Lie bracket, which we introduced in Definition 6.10. We will describe those formal deformations in quite a similar way to the description of deformations of an associative algebra.

Definition 7.8 (Formal deformations of the reduced bracket). *A formal deformation of the bracket $\Gamma^* : V \otimes V \to V$ will be defined as a one-parameter family of brackets $\Gamma^*(X, Y, \lambda)_m$ with $\lambda \in \mathbb{R}$ and $\Gamma^*(X, Y, 0)_m \cong \Gamma^*(X, Y)_m$. The deformed bracket will be defined as a formal power series:*

$$\Gamma^*(X, Y, \lambda)_m := \sum_{i=0}^{\infty} \Gamma^*_i(X, Y)_m \, \lambda^i \tag{7.49}$$

and the i^{th} order perturbations of the bracket will be:

$$\Gamma^*_i(X, Y)_m := \frac{1}{i!} \frac{d^i}{d\lambda^i} \Gamma^*(X, Y, \lambda)_m \tag{7.50}$$

Here $\Gamma^*_0(X, Y)_m \equiv \Gamma^*(X, Y)_m$.

We will be interested only in those deformations which are consistent with the generalized Jacobi identity (6.65). This will lead to a number of constraints which will single out the admissible perturbations. The first order perturbations $\Gamma_1^*(X,Y)_m$ must obey:

$$\begin{gathered}
\Gamma_0^*\Big(A,\Gamma_1^*(B,C)\Big)_{m_1m_2}+\Gamma_0^*\Big(B,\Gamma_1^*(C,A)\Big)_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}+\Gamma_0^*\Big(C,\Gamma_1^*(A,B)\Big)_{\widehat{m}_1\widehat{m}_2}\Big(Y_{cab}\Big)^{\widehat{m}_1\widehat{m}_2}_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}+\\
+\Gamma_1^*\Big(A,\Gamma_0^*(B,C)\Big)_{m_1m_2}+\Gamma_1^*\Big(B,\Gamma_0^*(C,A)\Big)_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}+\Gamma_1^*\Big(C,\Gamma_0^*(A,B)\Big)_{\widehat{m}_1\widehat{m}_2}\Big(Y_{cab}\Big)^{\widehat{m}_1\widehat{m}_2}_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}=0
\end{gathered}\tag{7.51}$$

The higher order perturbations must satisfy the following condition:

$$\begin{aligned}
&\Gamma_0^*\Big(A,\Gamma_j^*(B,C)\Big)_{m_1m_2}+\Gamma_0^*\Big(B,\Gamma_j^*(C,A)\Big)_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}+\Gamma_0^*\Big(C,\Gamma_j^*(A,B)\Big)_{\widehat{m}_1\widehat{m}_2}\Big(Y_{cab}\Big)^{\widehat{m}_1\widehat{m}_2}_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}+\\
&+\Gamma_j^*\Big(A,\Gamma_0^*(B,C)\Big)_{m_1m_2}+\Gamma_j^*\Big(B,\Gamma_0^*(C,A)\Big)_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}+\Gamma_j^*\Big(C,\Gamma_0^*(A,B)\Big)_{\widehat{m}_1\widehat{m}_2}\Big(Y_{cab}\Big)^{\widehat{m}_1\widehat{m}_2}_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}=\\
&=-\sum_{k=1}^{j-1}\Bigg\{\Gamma_k^*\Big(A,\Gamma_{j-k}^*(B,C)\Big)_{m_1m_2}+\Gamma_k^*\Big(B,\Gamma_{j-k}^*(C,A)\Big)_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}+\\
&\qquad+\Gamma_k^*\Big(C,\Gamma_{j-k}^*(A,B)\Big)_{\widehat{m}_1\widehat{m}_2}\Big(Y_{cab}\Big)^{\widehat{m}_1\widehat{m}_2}_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}\Bigg\}
\end{aligned}\tag{7.52}$$

We want to exclude from our considerations the "trivial" deformations, i.e. the simple λ-dependent changes of the basis $Q^*:V\to V$, such that:

$$\Gamma^*(X,Y,\lambda)_m=Q^{*-1}(\Gamma^*(Q^*X,Q^*Y)_m),\quad Q^*=1+\lambda q_1^*+\lambda^2q_2^*+... \tag{7.53}$$

Written in a series over λ up to first order, the deformed bracket becomes:

$$\begin{aligned}
Q^{*-1}(\Gamma^*(Q^*X,Q^*Y)_m) &= (1-\lambda q_1^*)\Gamma_0^*\Big((1+\lambda q_1^*)X,(1+\lambda q_1^*)Y\Big)_m+0(\lambda^2)\\
&= \Gamma_0^*(X,Y)_m+\lambda\Gamma_1^*(X,Y)_m+0(\lambda^2) \qquad (7.54)\\
\Gamma_1^*(X,Y)_m &= \Gamma_0^*(X,q_1^*Y)_m+\Gamma_0^*(q_1^*X,Y)_m-q_1^*\Gamma_0^*(X,Y)_m \qquad (7.55)
\end{aligned}$$

So, we have to "factorize" the set of admissible deformations over the set of trivial deformations. In the case of associative algebra such a factorization gave the opportunity to relate the deformations and the conditions for the i^{th}-order perturbation to a Hochschild cohomology complex. We will show that also in our case the deformations are described in terms of a cohomology complex, namely the reduced Lie algebra complex from the previous section.

In the following we formulate in cohomological language some of the formulas above:

Observation *(1).* The Jacobi identity can be rewritten in the compact form:

$$(b^2\Gamma_0^*)(A,B,C)_{m_1m_2}=0 \tag{7.56}$$

Observation *(2).* Jacobi identity gives the following restriction for the first order perturbation:

$$(b^2\Gamma_1^*)(A,B,C)_{m_1m_2}=0 \tag{7.57}$$

i.e. $\Gamma_1^* \in Z^2(V)$. The first trivial perturbation is:

$$\Gamma_1^*(X,Y)_m = (b^1 q_1^*)(X,Y)_m \tag{7.58}$$

which means $\Gamma_1^* \in B^2(V)$. Then it follows, that the non-trivial first order perturbations correspond to non-trivial classes $[\Gamma_1^*] \in RLH^2(V)$.

Observation *(3)*. Jacobi identity yields for higher order perturbations:

$$\begin{aligned} b^2\Gamma_j^*(A,B,C)_{m_1m_2} &= -\sum_{k=1}^{j-1}\Big\{\Gamma_k^*\Big(A,\Gamma_{j-k}^*(B,C)\Big)_{m_1m_2} + \Gamma_k^*\Big(B,\Gamma_{j-k}^*(C,A)\Big)_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2} + \\ &\quad + \Gamma_k^*\Big(C,\Gamma_{j-k}^*(A,B)\Big)_{\widehat{m}_1\widehat{m}_2}\Big(Y_{cab}\Big)^{\widehat{m}_1\widehat{m}_2}_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}\Big\} \end{aligned} \tag{7.59}$$

An interesting question is whether every element $\widetilde{\Gamma}_1^* \in RLH^2(V)$ is integrable, i.e. whether every $\widetilde{\Gamma}_1^* \in Z^2(V)$ serves as the first perturbation for some one-parameter family of deformations of V. In analogy to deformation theory of associative algebras we introduce the first obstruction operator G^2:

$$\begin{aligned} G^2[\widetilde{\Gamma}_1^*](A,B,C)_{m_1m_2} &:= -\Big\{\widetilde{\Gamma}_1^*\Big(A,\widetilde{\Gamma}_1^*(B,C)\Big)_{m_1m_2} + \widetilde{\Gamma}_1^*\Big(B,\widetilde{\Gamma}_1^*(C,A)\Big)_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2} + \\ &\quad + \widetilde{\Gamma}_1^*\Big(C,\widetilde{\Gamma}_1^*(A,B)\Big)_{\widehat{m}_1\widehat{m}_2}\Big(Y_{cab}\Big)^{\widehat{m}_1\widehat{m}_2}_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{cbca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}\Big\} \end{aligned} \tag{7.60}$$

Continuing the analogy, a next task would be to show that $b^3G^2[\widetilde{\Gamma}_1^*] = 0$, $\forall\widetilde{\Gamma}_1^* \in Z^2(V) \;\rightarrow\; G^2(A,B,C) \in Z^3(V)$, which is at the moment still an open question. If it is possible to lift the perturbation $\widetilde{\Gamma}_1^*$ to second order it must hold also that $G^2(A,B,C) = b^2\Gamma_2^*(A,B,C) \in B^3(V)$ because of (7.59). Then, the cohomology class of $G^2[\widetilde{\Gamma}_1^*]$ in $RLH^3(V)$ must vanish for integrable $\widetilde{\Gamma}_1^*$ and this class is viewed as the first obstruction to the integration of $\widetilde{\Gamma}_1^*$.

Let us assume now that we have lifted the perturbation up to order $n-1$ so that we have the set of perturbations $\widetilde{\Gamma}_1^*, ..., \widetilde{\Gamma}_{n-1}^* \in C^2(V)$, which obey the Jacobi identity conditions (7.59). We want to check whether it is possible to lift the perturbation to order n and for this purpose we write the corresponding obstruction operator:

$$\begin{aligned} G^n[\widetilde{\Gamma}_1^*,...,\widetilde{\Gamma}_{n-1}^*](A,B,C)_{m_1m_2} &= -\sum_{k=1}^{n-1}\Big\{\widetilde{\Gamma}_k^*\Big(A,\widetilde{\Gamma}_{n-k}^*(B,C)\Big)_{m_1m_2} + \widetilde{\Gamma}_k^*\Big(B,\widetilde{\Gamma}_{n-k}^*(C,A)\Big)_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2} + \\ &\quad + \widetilde{\Gamma}_k^*\Big(C,\widetilde{\Gamma}_{n-k}^*(A,B)\Big)_{\widehat{m}_1\widehat{m}_2}\Big(Y_{cab}\Big)^{\widehat{m}_1\widehat{m}_2}_{\widetilde{m}_1\widetilde{m}_2}\Big(Y_{bca}\Big)^{\widetilde{m}_1\widetilde{m}_2}_{m_1m_2}\Big\} \end{aligned} \tag{7.61}$$

Again, we hope that it is possible to prove:

$$b^3G^n[\widetilde{\Gamma}_1^*,...,\widetilde{\Gamma}_{n-1}^*] = 0, \quad \widetilde{\Gamma}_1^*,...,\widetilde{\Gamma}_{n-1}^* \text{ as above} \quad \rightarrow \quad G^n(A,B,C) \in Z^3(V) \tag{7.62}$$

Again, for integrable $\widetilde{\Gamma}^*_1, ..., \widetilde{\Gamma}^*_{n-1} \in C^2(V)$ it must hold that $G^n(A, B, C) \in B^3(V)$ because of (7.59) and the cohomology class of $G^n[\widetilde{\Gamma}^*_1, ..., \widetilde{\Gamma}^*_{n-1}]$ in $RLH^3(V)$ must vanish. This class is viewed as the $(n-1)^{\text{st}}$ obstruction to the integration of $\widetilde{\Gamma}^*_1 \in Z^2(V)$.

From all this follows that, very probably, the possibility of formal continuous deformations of a given reduced Lie algebra is decided by second and the third cohomology groups of the associated reduced Lie algebra cohomology complex. A "non-rigid" deformation theory would require a non-zero second cohomology group. A zero third cohomology group would indicate that every class from the second cohomology group may be integrated to an equivalence class of non-trivial deformations. The calculation of these groups is outside the scope of this thesis.

8 Conclusions and outlook

In the present work we addressed several important tasks regarding local extensions of chiral conformal quantum field theories.

First, we showed that algebraic techniques determine completely the superselection structure of all local extensions of Virasoro nets for $c < 1$ (classified in Theorem 3.10). The analysis was done in terms of "DHR-endomorphisms" and the clues to the solution were the following. 1D (and 2D) conformal nets with $c < 1$ are rational, i.e. they have a finite number of inequivalent sectors with finite statistical dimensions. They form a tensor category with similar features to the representation category of a compact group, but instead with braid group statistics. The statistic operators give rise to the methods of α^+- and α^-- induction which together with the method of σ-restriction allow to obtain all sectors of the extension from the well-know sectors λ_{pq} of the Virasoro subnet. It is known that the sectors of the extension are precisely the simultaneous subsectors of $\alpha^+_{\lambda_{pq}}$ and $\alpha^-_{\lambda_{pq}}$ where λ_{pq} are the Virasoro sectors. Then, the common content and equivalence among α^+_ρ- and α^-_ρ-sectors can be checked using the α–σ-reciprocity and the relation between the dimensions of their intertwining spaces and the Cappelli–Itzykson–Zuber modular invariant partition functions.

The classification of all irreducible DHR sectors of the index 2 extensions for $m = 4n+1$ and $m = 4n + 2$ is presented in Section 5.1.2. The classifications for the higher index extensions are presented on figures 5.3–5.6. As expected, the number of the sectors of the extensions is always less than that of the sectors of their Virasoro subnets. In the case of (A_{28}, E_8) extension for $m = 29$ also the fusion rules and the statistical dimensions are computed (formulae (5.26) and (5.27)) using just the homomorphism properties and linearity of the maps $\alpha^\pm$ + fusion rules of the Virasoro sectors. In this way we can also easily find fusion rules and statistical dimensions in almost all other cases. Exception would occur in the cases of index 2 extensions. Every red point in the figures 5.1 and 5.2 gives rise to $\alpha^\pm$-induced sectors which are equivalent and are a direct sum of two inequivalent DHR sectors. The method described above does not allow us to find the statistical dimensions and the fusion rules involving these DHR sectors.

Second, we explored local Möbius invariant commutators in chiral theories following closely the example of the Lüscher–Mack theorem. We showed that these commutators are fixed intrinsically up to structure constants, carrying the model-dependent information and related to the 2-point amplitudes (Proposition 6.1 and formula (6.83)). Furthermore, these structure constants are subject to an infinite number of constraints, originating from the anti–symmetry of commutators (formula (6.7)) and Jacobi identity (formula (6.67)), as well as from Hilbert space positivity (formula (6.83)). What still remains is to analyze these constraints more carefully. In the easiest case, the solution of the constraints for fields of dimension 1 reproduce the well-known Kac–Moody algebras, including the necessary compactness of the underlying Lie algebra.

To be able to derive these constraints, we had to rather consider a reduced version of the field algebra (Section 6.5), in the sense that the test functions are stripped off. In this reduced version the new bracket is multi-component and obeys a new generalized symmetry rule and the three terms in the Jacobi identity appear multiplied with universal (model independent) coefficient matrices. The data, defining a new axiomatization of a chiral conformal field theory, is the reduced space of fields, a reduced bracket, a quadratic form (Section 6.7), solving the constraints discussed above and subject to some other reasonable restrictions, discussed in the previous chapters.

Finally, we proceeded to explore the rigidity of the reduced commutator, in other words to check whether there exist models in the neighborhood of a certain model. For this purpose we have to solve the problem whether formal deformations of the reduced bracket exist. Following the general strategy, we constructed a cohomology complex related to the deformation problem (Definition 7.4). We have not, however, been able to actually compute the cohomology groups associated to this complex and this has to be done before developing a more complete deformation theory. Nevertheless, since our cohomology complex has been derived from a Lie algebra cohomology complex (considering the adjoint representation), we expect that the deformation theory also inherits the features of the Lie algebra deformation theories. In particular, we expect that the cohomology groups $RLH^2(V)$ and $RLH^3(V)$ determine the existence and integrability of non-trivial deformations. To prove this completely it has remained to show that our obstruction operators (7.61) are cocyles.

Another way would be to try to construct a differential graded Lie algebra out of the cohomology complex, whose deformation theory would be tightly related to the deformation theory of the reduced bracket. For this purpose, one has to construct a bracket in this complex, such that it is skew symmetric with respect to the "grading" by dimension of the cochain spaces and satisfies a graded Jacobi identity.

A Kac-Moody algebras and coset models

The theory of Kac–Moody algebras provide powerful mathematical tools for studying symmetries in theoretical physics. An amazing feature of these algebras is that they are compatible with locality, thus they provide a natural framework for unified consideration of symmetry and locality. This section is based mainly on [Goddard & Olive, 1985; Goddard *et al.*, 1985; Goddard & Olive, 1986; Goddard *et al.*, 1986].

Let us consider the affine untwisted Kac–Moody algebra $\widehat{g}$ with generators T^i_m, $m, n \in \mathbb{N}$ and commutation relations:

$$[T^i_m, T^j_n] = i f^{ijl} T^l_{m+n} + k\delta^{ij}\delta_{m+n,0} \tag{A.1}$$

where the central term k commutes with all T^i_m. f^{ijl} are the totally antisymmetric structure constants of the compact Lie algebra g with generators T^i_0, $i = 1, ..., \dim(g)$.

Associated with $\widehat{g}$ is a Virasoro algebra with generators L_m which satisfy:

$$\begin{aligned} \left[L_m, T^j_n\right] &= -nT^j_{m+n} \\ [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \end{aligned} \tag{A.2}$$

with c — another central term.

An interesting question is which values for c and k are allowed and what their interrelation is. A theorem states that a necessary unitarity condition is that k is quantized in terms of the structure constant $\frac{\psi^2}{2}$, where ψ is the highest (lowest) root of g in its adjoint representation, i.e. $\kappa := \frac{2k}{\psi^2} = 0, 1, 2, ...$ and is called the level of the Kac–Moody algebra. T^j_m vanish for $\kappa = 0$.

The apriori possible values of c were discussed in Section 2.4.

Equations (A.1) and (A.2) constitute the semi-direct product of the Kac–Moody algebra $\widehat{g}$ with a Virasoro algebra and are written for the general case, in which their generators may be totally independent. One can also construct the Virasoro generators in terms of bilinears in the Kac–Moody generators. The idea was born in the theory of current algebras, where the aim was to formulate the full dynamics of the theory in terms of currents, the energy–momentum tensor inclusive. The original construction was proposed in [Sugawara, 1968] and for simple groups g the stress–energy tensor $L(z)$ and its Fourier modes L_n are expressed as:

$$\begin{aligned} L(z) &= \sum_{n\in\mathbb{Z}} z^{-n}L_n = \frac{1}{2k+Q_\psi} : \sum_{i=1}^{\dim(g)} T^i(z)T^i(z) : \\ L_n &= \frac{1}{2k+Q_\psi} \sum_{m\in\mathbb{Z}} : \sum_{i=1}^{\dim(g)} T^i_{m+n}T^i_{-m} : \end{aligned} \tag{A.3}$$

Here we have the Kac–Moody field which is constructed as $T^j(z) = \sum_{m\in\mathbb{Z}} z^{-m}T^j_m$, where z is a complex variable lying on the unit circle $\approx \mathbb{R}\cup\infty$. It is worth mentioning at this point that in theories with chiral inner symmetries the symmetry generators are chiral currents, whose Fourier modes satisfy an affine untwisted Kac–Moody algebra and form a semi-direct product with the Virasoro algebra, exactly like $T^j(z)$. The normal ordering for the Kac-Moody current is defined as:

$$T^i(z)T^i(\zeta) =: T^i(z)T^i(\zeta) : + \frac{kz\zeta}{(z-\zeta)^2} \tag{A.4}$$

The normal ordering in the second formula from (A.3) means that T^i_m with positive suffices are moved to the right of those with negative suffices.

Q_ψ is the quadratic Casimir in the adjoint representation of g, ψ denotes its lowest weight. It can be found as:

$$\sum_{k,l=1}^{\dim(g)} f^{ikl}f^{jkl} = \delta^{ij}Q_\psi \tag{A.5}$$

Defined in such way, $L(z)$ and L_n satisfy the commutation relations of Virasoro algebra with central charge:

$$c_g = \frac{2k\dim(g)}{2k+Q_\psi} = \frac{\kappa\dim(g)}{\kappa+\widetilde{h}(g)} \tag{A.6}$$

where $\widetilde{h}(g) = \frac{Q_\psi}{\psi^2}$ is called the dual Coxeter number. One can prove that $\widetilde{h}(g)$ is an integer.

The stress–energy tensor obtained by Sugawara construction is automatically unitary in a certain representation if the Kac–Moody generators are. In this case also L_0 is positive.

If the Lie algebra g is not simple but semi–simple with $g = \oplus g_i$, then the Sugawara construction is achieved as:

$$L^g = L^{g_1} + L^{g_2} + ... \tag{A.7}$$

where L^{g_i} is the Sugawara construction for g_i. Also:

$$c_g = c_{g_1} + c_{g_2} + ... \tag{A.8}$$

These results hold even for not semi–simple groups g.

For abelian g we have simply $Q_\psi = 0$ and:

$$c_g = \dim(g) = \mathrm{rank}(g) \tag{A.9}$$

In the general case for simple g, c_g is a rational number lying between $\dim(g)$ and $\mathrm{rank}(g)$. The lowest bound $\mathrm{rank}(g)$ can be achieved only if g is simply laced and a level 1 representation of $\widehat{g}$ is considered. This is also true for a semi–simple Lie algebra with each g_i simply laced.

Hence, by Sugawara construction we cannot achieve a central charge less than unity, as it must exceed rank(g).

Let us now discuss the case when g has a subalgebra $h \subset g$ and both of them are simple. Then the algebras $\widehat{g}$ and $\widehat{h}$ have the same central term k, but not necessarily the same level, because the lowest roots of the two algebras may have different lengths. In general, the $\widehat{h}$ level must be greater or equal to the $\widehat{g}$ level. The Sugawara construction can be applied to both $\widehat{g}$ and $\widehat{h}$ to obtain Virasoro generators L^g_n and L^h_n, which in general have different central charges c_g and c_h. Since:

$$\begin{aligned} \left[L^g_m, T^j_n\right] &= -nT^j_{m+n}, \quad j = 1, ..., \dim(g) \\ \left[L^h_m, T^j_n\right] &= -nT^j_{m+n}, \quad j = 1, ..., \dim(h) \end{aligned} \tag{A.10}$$

one can construct $K_m := L^g_m - L^h_m$ which commutes with the Kac–Moody algebra $\widehat{h}$:

$$[L^g_m - L^h_m, T^j_n] = 0, \quad j = 1, ..., \dim(g) \tag{A.11}$$

and thus:

$$[L^g_m - L^h_m, L^h_n] = 0 \tag{A.12}$$

which means that L^g_m splits into two mutually commuting components:

$$L^g_m = L^h_m + K_m \tag{A.13}$$

We can think of K_m as related to the coset G/H (such that g and h are the Lie algebras of G and H) Moreover:

$$[L^g_m, L^g_n] = [L^h_m, L^h_n] + [K_m, K_n] \tag{A.14}$$

so K_m also satisfies the Virasoro algebra commutation relations, whose central charge is:

$$c_K = c_g - c_h = \frac{2k\dim(g)}{2k + Q_\psi} - \frac{2k\dim(h)}{2k + Q_\phi} \tag{A.15}$$

ϕ denotes the highest (lowest) weight of $\widehat{h}$ and Q_ϕ is the adjoint representation Casimir operator of $\widehat{h}$.

c_K must be non-negative and it is possible to obtain c_K smaller than 1. It was shown that all central charges from the discrete series can be obtained this way. Moreover, the corresponding Virasoro algebras inherit unitarity of their representations from the two algebras in the coset. Further, as the eigenvalues of L^g_0 are bounded from below, so are those of K_0.

In order to find a suitable coset construction, such that all central charges from the discrete series are recovered, let us first consider the algebra $su(2)$ with generators T^i such that:

$$[T^i, T^l] = \varepsilon_{ilk} T^k \tag{A.16}$$

The associated Kac–Moody algebra will have commutation relations:

$$[T^i_m, T^l_n] = \varepsilon_{ilk} T^k_{m+n} + \frac{K}{2} m \delta^{il} \delta_{m,-n} \tag{A.17}$$

Let us now consider $\widehat{g} = \widehat{su(2) \oplus su}(2)$ such that the two factors have commutation relations (A.17) with $K = N$ and $K = 1$ respectively. We also choose $\widehat{h} = \mathrm{diag}((\widehat{su(1)}) \oplus \widehat{su(1)}) = \widehat{su(2)}$ with commutation relations (A.17) with $K = N + 1$. We denote the corresponding coset with:

$$SU(2)_N \times SU(2)_1 / SU(2)_{N+1} \tag{A.18}$$

With this choice of $\widehat{g}$ and $\widehat{h}$ we can obtain all the central charges from the discrete series $c(m) = 1 - \frac{6}{m(m+1)}$, $m = 2, 3, 4...$ for the stress–energy tensor related to the coset. We use character arguments to show that also all spins from the list $h_{p,q}(c) = \frac{[(m+1)p-mq]^2-1}{4m(m+1)}, \quad p \in [1, m-1]$, $q \in [1, m]$ can be won from the representations of $\widehat{g}$ and $\widehat{h}$.

B Lie algebra cochain complex

Cochain spaces:
$\Omega^n \ni \omega^n : V \otimes ... \otimes V \to V$ with $\omega^n(X_1, ..., X_n)$ completely anti–symmetric functions.

Coboundary map:

$$\begin{aligned}
[b\omega^n](X_1, ..., X_{n+1}) \;&:=\; (n+1)\Big\{\Big[\omega^n(X_1, ..., X_n), X_{n+1}\Big]\Big\}_A - \frac{n(n+1)}{2}\Big\{\omega^n\Big(X_1, ..., X_{n-1}, [X_n, X_{n+1}]\Big)\Big\}_A \\
&=\; (n+1)\frac{1}{(n+1)!}\Big[\sum_{i=1}^{n+1}(-1)^{i+n+1} n!\,\omega^n(X_1, ..., \hat{X}_i, ..., X_{n+1}), X_i\Big] - \\
&\quad -\frac{n(n+1)}{2}\frac{1}{(n+1)!}\sum_{\substack{k,i=1\\k>i}}^{n+1}(-1)^{i+k+1} 2(n-1)!\,\omega^n\Big(X_1, ..., \hat{X}_i, ..., \hat{X}_k, ..., X_{n+1}, [X_i, X_k]\Big) \\
&=\; \sum_{i=1}^{n+1}(-1)^{i+n+1}\Big[\omega^n(X_1, ..., \hat{X}_i, ..., X_{n+1}), X_i\Big] - \\
&\quad - \sum_{\substack{k,i=1\\k>i}}^{n+1}(-1)^{i+k+1}\,\omega^n\Big(X_1, ..., \hat{X}_i, ..., \hat{X}_k, ..., X_{n+1}, [X_i, X_k]\Big)
\end{aligned} \tag{B.1}$$

Proposition: $[b \circ b\,\omega^n](X_1, ..., X_{n+2}) = 0$

Proof:

$$\begin{aligned}
[b \circ b\,\omega^n](X_1, ..., X_{n+2}) = \;& (n+2)\Big\{\Big[\textstyle\sum_{i=1}^{n+1}(-1)^{i+n+1}\Big[\omega^n(X_1, ..., \hat{X}_i, ..., X_{n+1}), X_i\Big], X_{n+2}\Big]\Big\}_A - && (\mathbf{I}) \\
& -(n+2)\Big\{\Big[\textstyle\sum_{k>i=1}^{n+1}(-1)^{i+k+1}\omega^n\Big(X_1, ..., \hat{X}_i, ..., \hat{X}_k, ..., X_{n+1}, [X_i, X_k]\Big), X_{n+2}\Big]\Big\}_A - && (\mathbf{II}) \\
& -\tfrac{(n+1)(n+2)}{2}\Big\{\textstyle\sum_{i=1}^{n}(-1)^{i+n+1}\Big[\omega^n\Big(X_1, ..., \hat{X}_i, ..., X_n, [X_{n+1}, X_{n+2}]\Big), X_i\Big]\Big\}_A - && (\mathbf{III}) \\
& -\tfrac{(n+1)(n+2)}{2}\Big\{\Big[\omega^n(X_1, ..., X_n), [X_{n+1}, X_{n+2}]\Big]\Big\}_A + && (\mathbf{IV}) \\
& +\tfrac{(n+1)(n+2)}{2}\Big\{\textstyle\sum_{k>i=1}^{n}(-1)^{i+k+1}\omega^n\Big(X_1, ..., \hat{X}_i, ..., \hat{X}_k, ..., [X_{n+1}, X_{n+2}], [X_i, X_k]\Big)\Big\}_A + && (\mathbf{V}) \\
& +\tfrac{(n+1)(n+2)}{2}\Big\{\textstyle\sum_{i=1}^{n}(-1)^{i+n}\omega^n\Big(X_1, ..., \hat{X}_i, ..., X_n, \big[X_i, [X_{n+1}, X_{n+2}]\big]\Big)\Big\}_A && (\mathbf{VI})
\end{aligned} \tag{B.2}$$

$$
\begin{aligned}
(\mathbf{I})+(\mathbf{IV}) \quad = \quad & (n+2)(n+1)\frac{1}{(n+2)!}\sum_{\substack{k,i=1\\k>i}}^{n+2}(-1)^{i+k+1}n!\Big[\Big[\omega^n(X_1,...,\hat{X}_i,...,\hat{X}_k,...,X_{n+2}),X_i\Big],X_k\Big]- \\
& -(n+2)(n+1)\frac{1}{(n+2)!}\sum_{\substack{k,i=1\\k>i}}^{n+2}(-1)^{i+k+1}n!\Big[\Big[\omega^n(X_1,...,\hat{X}_i,...,\hat{X}_k,...,X_{n+2}),X_k\Big],X_i\Big]+ \\
& +\frac{(n+2)(n+1)}{2}\frac{1}{(n+2)!}\sum_{\substack{k,i=1\\k>i}}^{n+2}(-1)^{i+k+1}2n!\Big[[X_i,X_k],\omega^n(X_1,...,\hat{X}_i,...,\hat{X}_k,...,X_{n+2})\Big] \\
= \quad & \sum_{\substack{k,i=1\\k>i}}^{n+2}\Bigg\{\Big[\Big[\omega^n(X_1,...,\hat{X}_i,...,\hat{X}_k,...,X_{n+2}),X_i\Big],X_k\Big]+ \\
& +\Big[\Big[X_k,\omega^n(X_1,...,\hat{X}_i,...,\hat{X}_k,...,X_{n+2})\Big],X_i\Big]+\Big[[X_i,X_k],\omega^n(X_1,...,\hat{X}_i,...,\hat{X}_k,...,X_{n+2})\Big]\Bigg\}=0
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
(\mathbf{II})+(\mathbf{III}) \quad = \quad & -(n+2)\frac{n(n+1)}{2}\Bigg\{\Big[\omega^n\Big(X_1,...,X_{n-1},[X_n,X_{n+1}]\Big),X_{n+2}\Big]\Bigg\}_A+ \\
& +\frac{(n+1)(n+2)}{2}n\Bigg\{\Big[\omega^n\Big(X_1,...,X_{n-1},[X_n,X_{n+1}]\Big),X_{n+2}\Big]\Bigg\}_A=0
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
(\mathbf{V}) \quad \sim \sum & \Bigg\{\omega^n\Big(X_{i_1},...,X_{i_{n-2}},[X_{i_r},X_{i_q}],[X_{i_s},X_{i_t}]\Big)+\omega^n\Big(X_{i_1},...,X_{i_{n-2}},[X_{i_s},X_{i_t}],[X_{i_r},X_{i_q}]\Big)\Bigg\}= \\
& \text{(because of anti-symmetrization)} \\
= & \Bigg\{\omega^n\Big(X_{i_1},...,X_{i_{n-2}},[X_{i_r},X_{i_q}],[X_{i_s},X_{i_t}]\Big)-\omega^n\Big(X_{i_1},...,X_{i_{n-2}},[X_{i_r},X_{i_q}],[X_{i_s},X_{i_t}]\Big)\Bigg\}=0 \\
& \text{(because of anti–symmetry of } \omega^n\text{)}
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
(\mathbf{VI}) \quad \sim \sum & \Bigg\{\omega^n\Big(...,\big[X_{i_1},[X_{i_2},X_{i_3}]\big]\Big)+\omega^n\Big(...,\big[X_{i_3},[X_{i_1},X_{i_2}]\big]\Big)+\omega^n\Big(...,\big[X_{i_2},[X_{i_3},X_{i_1}]\big]\Big)\Bigg\}= \\
& \text{(because of anti-symmetrization)} \\
= \sum & \Bigg\{\omega^n\Big(...,\big[X_{i_1},[X_{i_2},X_{i_3}]\big]+\big[X_{i_3},[X_{i_1},X_{i_2}]\big]+\big[X_{i_2},[X_{i_3},X_{i_1}]\big]\Big)\Bigg\}=0
\end{aligned}
\tag{B.6}
$$

$$
\text{(because of linearity of } \omega^n \text{ and Jacobi identity)} \tag{B.7}
$$

C Pochhammer symbol $(x)_n$ and useful properties

The concept Pochhammer symbol denotes the rising factorial:

$$
\begin{aligned}
(x)_n &:= x(x+1)(x+2)(x+3)...(x+n-1) = \\
&= \frac{\Gamma(x+n)}{\Gamma(x)}, \qquad n \in \mathbb{N}_0 \\
&= \frac{(x+n-1)!}{(x-1)!} \qquad \text{for } x, n \in \mathbb{N}_0
\end{aligned}
\tag{C.1}
$$

We will display here some of the properties of the Pochhammer symbol, especially those which we used to calculate the matrix Y in Section 6.4.1:

1.

$$
\sum_{p+q=n} \frac{(x)_p}{p!} \frac{(y)_q}{q!} = \frac{(x+y)_n}{n!}
\tag{C.2}
$$

(a generalization for the expansion of binomial coefficients)

2.

$$
\begin{aligned}
(x-p)_p &= (x-p)(x-p+1)(x-p+2)...(x-p+p-1) = \\
&= (x-1)(x-2)...(x-p) = \\
&= (-1)^p(1-x)(1-x+1)(1-x+2)...(1-x+p-1) = \\
&= (-1)^p(1-x)_p
\end{aligned}
\tag{C.3}
$$

alternatively:

$$
\begin{aligned}
(x)_n &= \frac{\Gamma(x+n)}{\Gamma(x)} = (-1)^n \frac{\Gamma(-x+1)}{\Gamma(-x-n+1)} = \\
&= (-1)^n(1-x-n)_n
\end{aligned}
\tag{C.4}
$$

3.

$$
\begin{aligned}
\frac{(x)_n}{(x)_m} &= (x+m)_{n-m} \quad \text{if } n > m \\
(x+p)_q(x)_p &= (x)_{p+q}
\end{aligned}
\tag{C.5}
$$

4.

$$\begin{aligned}\frac{(s-x)_{q-s}}{(s-x)_{p-s}} &= \frac{\Gamma(q-x)}{\Gamma(s-x)}\frac{\Gamma(s-x)}{\Gamma(p-x)} = \\ &= \frac{\Gamma(q-x)}{\Gamma(t-x)}\frac{\Gamma(t-x)}{\Gamma(p-x)} = \frac{(t-x)_{q-t}}{(t-x)_{p-t}} \qquad s,t\in\mathbb{N}_0,\ s,t<p,q \qquad \text{(C.6)}\end{aligned}$$

5.

$$\begin{aligned}\frac{(a)_n}{(a-m)_n} &= \frac{\Gamma(a+n)}{\Gamma(a)}\frac{\Gamma(a-m)}{\Gamma(a-m+n)} = \\ &= \frac{\Gamma(a+n)}{\Gamma(a-m+n)}\frac{\Gamma(a-m)}{\Gamma(a)} = \frac{(a+n-m)_m}{(a-m)_m} \qquad n>m \qquad \text{(C.7)}\end{aligned}$$

6.

$$\begin{aligned}(x)_{a-i} &= \frac{\Gamma(x+a-i)}{\Gamma(x)} = \frac{\Gamma(x+a-i)}{\Gamma(x+a-b)}\frac{\Gamma(x+a-b)}{\Gamma((x)} = \\ &= \frac{\Gamma(x+a-i)}{\Gamma(x+a-b)}\frac{\Gamma(x+a-b)}{\Gamma(x+a)}\frac{\Gamma(x+a)}{\Gamma((x)} = \\ &= \frac{(x)_a}{(x+a-b)_b}(x+a-b)_{b-i} \qquad \text{(C.8)}\end{aligned}$$

7.

$$\begin{aligned}(x)_n &= \frac{\Gamma(x+n)}{\Gamma(x)} = (-1)^n\frac{\Gamma(-x+1)}{\Gamma(-x-n+1)} = \\ &= (-1)^n\frac{\Gamma(-x+1)}{\Gamma(-x-k-1)}\frac{\Gamma(-x-k+1)}{\Gamma(-x-k+1+k-n)} = \\ &= (-1)^{n-k}\frac{(x)_k}{(1-k-x)_{k-n}} \qquad \text{(C.9)}\end{aligned}$$

8.

$$\begin{aligned}\frac{(x)_n}{n!} &= \binom{x+n-1}{n} \qquad \text{for } x,n\in\mathbb{N}_0 \\ (-1)^n\frac{(-x)_n}{n!} &= \binom{x}{n} \qquad \text{(C.10)}\end{aligned}$$

Definitions, Propositions, Theorems, etc...

Definitions

Theorems

Propositions

Lemmas

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