



Study of generalized cylindrical polytropes with complexity factor

Shiraz Khan^a, S. A. Mardan^b, M. A. Rehman^c

Department of Mathematics, University of Management and Technology, C-II. Johar Town, Lahore 54590, Pakistan

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Abstract In this paper, complexity factor is used with generalized polytropic equation of state to develop two consistent systems of three differential equations and a general framework is established for modify form of Lane-Emden equations. For this purpose anisotropic fluid distribution is considered in cylindrical static symmetry with two cases of generalized polytropic equation of state (i) mass density μ_o and (ii) energy density μ . A graphical analysis will be carried out for the numerical solution of these systems of three differential equations.

1 Introduction

Polytropes, in the context of general relativity have great importance in study of different changes in characteristics and physical models of a astronomical objects. These characteristics and physical models can easily be described by the polytropic equation of state (PEoS). Polytropes are the solution of Lane-Emden equation (LEe), which is a pair of non linear differential equations so these are always attracted by many researchers in astrophysics and mathematics. Lane [1] used polytropes to give some basic results associated to the modeling of cosmological structures. Chandrasekhar [2] used the idea of polytropes to estimate the maximum mass limit for a stable white dwarf star in Newtonian physics. Tooper [3] made use of PEoS to analysis the solution of basic field equations in the context of general theory of relativity for spherical compressible fluid under gravitational equilibrium. He [4] also found the numerical solutions of hydrostatic equilibrium equation in general relativity using spherical compressible fluid which governed by the relation of pressure-energy density. Kaplan and Lupanov [5] studied the relativistic effects in the theory of the structure of poly-

tropic sphere and they obtained an analytical relation between central density and spherical mass in weak relativity. Managhan and Roxburgh [6] investigated the structure of rotating polytropes for different values of polytropic index. For this purpose they used method of approximation for the matching of two solutions at an interface. Kaufmann [7] used PEoS under static spherical symmetry to obtain a single integro-differential equation and its solution depended on different values of polytropic index n . Occhionero [8] evaluated the impact of rotation on the structure of polytropes for $n \geq 2$, which were in equilibrium to the second order with suitable parameter. Kovetz [9] reviewed the theory of slowly rotating polytropes of Chandrasekhar [2] and removed some inconsistencies in it.

Horedt [10] discussed the instability of weakly distorted polytropic sphere for polytropic index $n > 3$. He also observed that slowly rotating cylinder and polytropic rings did not show any instability under the external pressure. Sharma [11] tabulated the values of radius of static polytropic sphere for polytropic index $n = 0, 1, 3$ and values of other physical parameters for $n = 0, 1$ by using the pade (2, 2) approximation. Singh and Singh [12] carried out a study of rotationally distorted and tidally polytropes by using the method of [6]. Horedt [13] analyzed the properties like mass acceleration gravitational potential and mean density of N -dimensional radially symmetric polytropes with the help of gamma function. He also discussed the mass-radius relation in case cylindrical and spherical symmetry. Pandey [14] calculated various parameters to study the spherical static structure by using PEoS. For a certain region of sun interior, Hendry [15] developed three polytropic models.

Herrera and Barreto [17] evaluated the relativistic polytropes by using the two different definitions of relativistic polytropes giving same Newtonian limit for self gravitating sphere. They [20] also gave a general structure for the modeling of polytropes in the context of general relativity and derived the LEe. Herrera et al. [23] used the PEoS to analyze the spherically symmetric fluid which is distributed confor-

^a e-mail: f2016265004@umt.edu.pk (corresponding author)

^b e-mail: syedalimardanazmi@yahoo.com

^c e-mail: aziz.rehman@umt.edu.pk

mally flat and constructed models of highly compact star for anisotropic polytropes in mass density case. Herrera et al. [24] examined the effects of different variations in energy density and pressure anisotropy under cracking technique for spherical non static compact objects satisfying the PEoS.

The generalized polytropic equation of state (GPEoS) is used to discuss the generalized polytropes (GPs) for the study of gravitating objects. This equation consists of two parts: (i) polytropic part $P_r = K\mu_o^\gamma = K\mu_o^{1+\frac{1}{n}}$ and (ii) linear part $P_r = \alpha_1\mu_o$. Combination of these two parts define the GPEoS [25] as

$$P_r = K\mu_o^\gamma + \alpha_1\mu_o = K\mu_o^{1+\frac{1}{n}} + \alpha_1\mu_o, \tag{1}$$

where P_r , K , γ , α_1 and n are called principal stress, polytropic constant, polytropic exponent, linear coefficient constant and polytropic index respectively. Change of μ_o by μ gives

$$P_r = K\mu^{1+\frac{1}{n}} + \alpha_1\mu. \tag{2}$$

Azam et al. [25,26] carried out spherical and cylindrical symmetric fluid distribution to study the charged polytropes with relativistic GPEoS. Mardan et al. [27,28] used spherical symmetric GPs to investigate some gravitating objects. They found exact solutions of field equations by taking different values of polytropic index n and analyzed some mathematical models which were found physically viable and well behaved. Mardan et al. [29,30] established new classes of polytropic models using spherical symmetry and analyzed the mass and radius of different astronomical objects.

Different equations of state are playing an important role to describe the two very fundamental aspect of universe, dark energy and dark matter. Babichev et al. [35] used a form of linear equation of state, called generalized linear equation of state with perfect fluid distribution to describe the different scenarios for dark energy. Mukhopadhyay et al. [18] discussed the real nature of dark energy through the parameter of PEoS specially in non-dust situation. Chavanis [19,21,22] used GPEoS in his series of papers to study dark energy and dark matter. He considered dark fluid with GPEoS in which its linear and polytropic part describes the dark energy and dark matter respectively and used different values of parameters of GPEoS to explain the early and late universe.

Herrera [31] used orthogonal splitting of Riemann tensor into structure scalars for a self gravitating system to present a new concept of complexity factor (CF). Abbas and Nazar [32–35] carried out this concept of vanishing CF for self gravitating object in the context of modify gravity $f(R)$ and expressed the physical behavior of $f(R)$ model for some compact objects in spherically static, dynamical and axially symmetry. Sharif and Iqra [36,37] implemented the CF on cylindrical system and discussed the electromagnetic effect

on this system. Khan et al. [38,39] used the idea of CF to study the GPs and charged GPs for spherical self gravitating fluid.

The layout of this paper will be follow Sect. 2 will contain the detail of basic field equations using cylindrical static symmetry and Tolman–Oppenheimer–Volkoff (TOV) equation. In this section, we will also use the the Weyl tensor to discuss the mass function for gravitating object. Section 3 will be devoted for the study of structure scalars, which are obtained through orthogonal splitting of Riemann tensor and then CF will be defined. In Sect. 4 a discussion will be carried out about GPs and physical conditions. We will give a graphical solution of cylindrical GPs with CF in Sect. 5. In the last Sect. 6 we will summarize our study.

2 Basic equations and mass function

Let us consider a static cylindrical symmetric line element, as

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + C^2 d\theta^2 + a^2 C^2 dz^2, \tag{3}$$

where $A = A(r)$, $B = B(r)$, $C = C(r)$, a is an arbitrary constant and coordinates are: $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = z$. The stress-energy tensor is defined as

$$T_{\mu\nu} = (P_r - P_\perp)s_\mu s_\nu + (P_\perp + \mu)u_\mu u_\nu - P_\perp g_{\mu\nu}, \tag{4}$$

here P_\perp is an other principal stress, four vector and four velocity respectively are defined as $s_\mu = (0, \frac{1}{B}, 0, 0)$ and $u_\mu = (\frac{1}{A}, 0, 0, 0)$, with properties:

$$s^\mu u_\mu = 0, \quad s^\mu s_\mu = 1, \quad u^\mu u_\mu = -1.$$

It will be more convenient if we take stress–energy tensor as

$$T_\gamma^\lambda = \mu u^\lambda u_\gamma + P h_\gamma^\lambda + \Delta_\gamma^\lambda, \tag{5}$$

with

$$\begin{aligned} \Delta_\gamma^\lambda &= \Delta(s^\lambda s_\gamma + \frac{1}{3}h_\gamma^\lambda); \quad 3P = 2P_\perp + P_r \\ \Delta &= -(P_\perp - P_r); \quad h_\gamma^\lambda = \delta_\gamma^\lambda - u^\lambda u_\gamma. \end{aligned} \tag{6}$$

Basic equations are

$$8\pi\mu = \frac{2C^{**}}{B^2C} - \frac{2B^*C^*}{B^3C} + \frac{C^{*2}}{B^2C^2}, \tag{7}$$

$$-8\pi P_r = \frac{C^{*2}}{B^2C^2} + \frac{2A^*C^*}{AB^2C}, \tag{8}$$

$$-8\pi P_\perp = \frac{A^*C^*}{AB^2C} + \frac{C^{**}}{B^2C} - \frac{B^*C^*}{B^3C} + \frac{A^{**}}{AB^2} - \frac{A^*B^*}{AB^3}, \tag{9}$$

where ‘*’ indicates the derivative w. r. t. ‘r’. The corresponding exterior geometry is considered as [40].

$$ds^2 = \frac{2M}{R} dv^2 - 2dRdv + R^2(d\theta^2 + a^2)dz^2, \tag{10}$$

where M represents the total mass in the exterior. On the hyper surface Σ , the necessary and sufficient conditions for smooth matching of two metrics (3) and (10) are given in [40] as $C(r) = r \Rightarrow C^*(r) = 1$ and $C^{**}(r) = 0$, then Eqs. (7–9) become,

$$\mu = \frac{1}{8\pi} \left[-\frac{2B^*}{B^3r} + \frac{1}{B^2r^2} \right], \tag{11}$$

$$P_r = -\frac{1}{8\pi} \left[\frac{1}{B^2r^2} + \frac{2A^*}{AB^2r} \right], \tag{12}$$

$$P_\perp = -\frac{1}{8\pi} \left[\frac{A^{**}}{AB^2} - \frac{A^*B^*}{AB^3} + \frac{A^*}{AB^2r} - \frac{B^*}{B^3r} \right]. \tag{13}$$

Solving Eqs. (11–13) simultaneously we obtain generalized TOV equation as

$$P_r^* = -\frac{A^*}{A}(\mu + P_r) + \frac{2(P_\perp - P_r)}{r}. \tag{14}$$

Thorne [41] defined C-energy in the form of mass function as

$$\mathbf{E} = \frac{1}{8} \left(1 - \frac{1}{l} \nabla_\rho r \nabla^\rho r \right), \tag{15}$$

it yields

$$m(r) \cong \mathbf{E} = l\mathbf{E} = -\frac{ra}{2B^2} \left(1 - \frac{B^2}{4} \right), \tag{16}$$

using Eqs. (11) and (16)

$$m(r) = \frac{ar}{8} - 4\pi \int_0^r \bar{r}^3 \mu d\bar{r}. \tag{17}$$

From Eqs. (11–13) and (16), we have

$$m = \frac{ra}{8} + 8\pi(\mu - P_r + P_\perp) - \frac{1}{B^2} \left(\frac{ra}{2} + \frac{1}{r^2} \right) + \frac{1}{AB^2} \left[A^{**} - \frac{A^*B^*}{B} - \frac{A^*}{r} \frac{A}{r^2} + \frac{AB^*}{rB} \right]. \tag{18}$$

Now with the help of Weyl tensor, we can simplify above expression. For cylindrical symmetric fluid distribution Weyl tensor has electric and magnetic components. For the purpose of simplification, we take electric component as

$$E_{\alpha\beta} = C_{\alpha\gamma\beta\delta} u^\gamma u^\delta, \tag{19}$$

where

$$C_{\mu\nu\kappa\lambda} = (g_{\mu\nu\alpha\beta} g_{\kappa\lambda\gamma\delta} - \eta_{\mu\nu\alpha\beta} \eta_{\kappa\lambda\gamma\delta}) u^\alpha u^\gamma E^{\beta\delta}, \tag{20}$$

with $g_{\mu\nu\alpha\beta} = g_{\mu\alpha} g_{\nu\beta}$ and $\eta_{\mu\nu\alpha\beta}$. Note that

$$E_{\alpha\beta} = E \left(s_\alpha s_\beta + \frac{1}{3} h_{\alpha\beta} \right), \tag{21}$$

and

$$E = A^{**} - \frac{A^*B^*}{B} - \frac{A^*}{r} - \frac{A}{r^2} + \frac{AB^*}{rB}, \tag{22}$$

satisfying the following properties:

$$E_{11} = \frac{1}{3A} E, \quad E_{22} = -\frac{r^2}{6AB^2} E, \quad E_{33} = \frac{a^2 r^2}{6AB^2} E. \tag{23}$$

From Eqs. (18) and (22) we have

$$m = \frac{ra}{8} + 8\pi(\mu - P_r + P_\perp) - \frac{1}{B^2} \left(\frac{ra}{2} + \frac{1}{r^2} \right) + \frac{E}{AB^2}. \tag{24}$$

Using Eqs. (17) in (24)

$$E = \frac{4}{3} \pi a AB^2 \int_0^r \bar{r}^3 \mu^* d\bar{r} - \frac{4}{3} \pi a AB^2 r^3 \mu + A \left(\frac{ra}{2} + \frac{1}{r^2} \right) - 8\pi AB^2 \mu + 8\pi AB^2 (P_r - P_\perp). \tag{25}$$

Then Eq. (24) will be

$$m = \frac{ra}{8} + \frac{4}{3} \pi a \int_0^r \bar{r}^3 \mu^* d\bar{r} - \frac{4}{3} \pi ar^3 \mu. \tag{26}$$

Using Eq. (16) in (12)

$$\frac{A^*}{A} = -\frac{16a\pi r^2 P_r}{ra - 8\pi m} - \frac{1}{2r}. \tag{27}$$

Put Eq. (27) in (12), the TOV equation becomes

$$P_r^* = \left(\frac{16a\pi r^2 P_r}{ra - 8\pi m} + \frac{1}{2r} \right) (\mu + P_r) + \frac{2(P_\perp - P_r)}{r}. \tag{28}$$

3 Structure scalars and vanishing complexity factor

CF is defined [31] through structure scalars, which are obtained from orthogonal splitting of curvature tensor [43]. This splitting of curvature tensor give the following tensors [44,45].

$$Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta} u^\gamma u^\delta, \tag{29}$$

$$X_{\alpha\beta} = \bullet R_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \frac{1}{2} \eta_{\alpha\gamma}^{\epsilon\mu} R_{\epsilon\mu\beta\delta} u^\gamma u^\delta, \tag{30}$$

where \bullet denote the dual tensor i.e. $R_{\alpha\beta\gamma\delta}^\bullet = \frac{1}{2} \eta_{\epsilon\mu\gamma\delta} R_{\alpha\beta}^{\epsilon\mu}$. While the trace free parts (Y_{TF}, X_{TF}) and trace part (Y_T, X_T) of these tensors are related as [44,45], and so the tensors in Eqs. (29) and (30) can be represented as

$$Y_{\alpha\beta} = Y_{TF} \left(\frac{1}{3} h_{\alpha\beta} + s_\alpha s_\beta \right) + \frac{1}{3} Y_T h_{\alpha\beta}, \tag{31}$$

$$X_{\alpha\beta} = X_{TF} \left(\frac{1}{3} h_{\alpha\beta} + s_\alpha s_\beta \right) + \frac{1}{3} X_T h_{\alpha\beta}. \tag{32}$$

After using the field equations, we have

$$X_T = 8\pi\mu, \tag{33}$$

$$X_{TF} = 4\pi \Delta_{\alpha\beta} - E. \tag{34}$$

Using Eq. (25) in (34)

$$X_{TF} = 4\pi\Delta(1 - 2AB^2) - \frac{4}{3}\pi aAB^2 \int_0^r \bar{r}^3 \mu^* d\bar{r} + \frac{4}{3}\pi aAB^2 r^3 \mu - A\left(\frac{ra}{2} + \frac{1}{r^2}\right) + 8\pi AB^2 \mu, \tag{35}$$

$$Y_T = 4\pi(\mu + 3P_r - 2\Delta), \tag{36}$$

$$Y_{TF} = 4\pi\Delta + E, \tag{37}$$

or using Eq. (25) in (37)

$$Y_{TF} = 4\pi\Delta(1 + 2AB^2) + \frac{4}{3}\pi aAB^2 \int_0^r \bar{r}^3 \mu^* d\bar{r} - \frac{4}{3}\pi aAB^2 r^3 \mu + A\left(\frac{ra}{2} + \frac{1}{r^2}\right) - 8\pi AB^2 \mu. \tag{38}$$

From Eqs. (35) and (38)

$$8\pi\Delta = X_{TF} + Y_{TF}. \tag{39}$$

In order to discuss stellar structure the basic field Eqs. (11–13) form a system of three ordinary differential equations (DEs) for static cylindrical symmetry in five variables (A, B, μ, P_\perp, P_r). So we implement condition $Y_{TF} = 0$ which gives the vanishing CF Eq. (38), as [36]

$$\Delta = \frac{-1}{4\pi(1 + 2AB^2)} \left[\frac{4}{3}\pi aAB^2 \int_0^r \bar{r}^3 \mu^* d\bar{r} - \frac{4}{3}\pi aAB^2 r^3 \mu + A\left(\frac{ra}{2} + \frac{1}{r^2}\right) - 8\pi AB^2 \mu \right]. \tag{40}$$

but we may still required one more condition to explain the stellar structure for this purpose we will use GPEoS.

4 Generalized cylindrical polytropes

Since GPEoS are widely used [20 – 26, 31, 32] to discuss the different characteristics of inner structure of self gravitating objects. So we use it for cylindrical static symmetry anisotropic fluid as

4.1 Case 1

$$P_r = K\mu_o^{1+\frac{1}{n}} + \alpha_1\mu_o, \tag{41}$$

energy density μ connected with the mass density μ_o [23] as

$$\mu = \mu_o + nP_r. \tag{42}$$

following assumptions are to be considered

$$r = \frac{\xi}{N}, \quad \alpha = \frac{P_{rc}}{\mu_c}, \quad N^2 = \frac{4\pi\mu_c}{(1+n)\alpha}. \tag{43}$$

$$\psi_o = \frac{\mu_o}{\mu_{oc}}, \quad v(\xi) = \frac{m(r)N^3}{4\pi\mu_c}. \tag{44}$$

Then TOV Eq. (15) becomes

$$\xi N^{-4}\alpha^{-2} \left[-16\pi\psi_o^{2n}\xi^3 aP_{rc}^2(\alpha_1 + (\alpha\alpha_1n - \alpha_1 + \alpha)\psi_o - \alpha\alpha_1n) \times ((1+n)\psi_o(\alpha - \alpha_1 + \alpha\alpha_1n) + (1 - \alpha n)(\alpha_1 + \alpha_1n + 1)) - 2(\alpha N^2\xi - 32\pi P_{rc}v)(2\alpha\Delta - \xi P_{rc}\psi_o^{n-1}\psi_o') \times ((1+n)(\alpha - \alpha_1 + \alpha\alpha_1n)\psi_o - \alpha_1n(\alpha n - 1)) \right] = 0. \tag{45}$$

where prime indicates the derivative w. r. t. ξ . Using Eqs. (43, 44) in (17) we have

$$\frac{dv}{d\xi} = \frac{a}{8(n+1)\alpha} - a\xi^2\psi_o^n \left[(1 - n\alpha)(1 + n\alpha_1) + n(\alpha - \alpha_1 + n\alpha\alpha_1) \right]. \tag{46}$$

At boundary surface $\xi = \xi_n$ such that $\psi_o(\xi_n) = 0$ and we have boundary conditions as

$$\psi_o(\xi = 0) = 1 \quad \text{and} \quad v(\xi = 0) = 0. \tag{47}$$

Equations (45, 46) constitute the LEE for this case

$$\frac{1}{\alpha^3 N^4} \left[2 \left(-\frac{2}{n+1} \left[(2\alpha\Delta - \xi P_{rc}\psi_o^{n-1}\psi_o'((n+1)\psi_o\beta + \alpha_1n(1 - \alpha n))) \times (\xi(\alpha^2 N^2(n+1) - 2\pi aP_{rc}) + 16\pi\alpha(n+1)P_{rc} \times (a\xi^3\psi_o^n(n\psi_o\beta + (1 - \alpha n) \times (\alpha_1n + 1)) - v)) \right] - 8\pi a\alpha(n+1)\xi^4 P_{rc}^2\beta\psi_o^{2n}\psi_o' \times (\alpha_1 + \psi_o\beta - \alpha\alpha_1n) + 8\pi a\alpha n\xi^4 P_{rc}^2\psi_o^{2n-1}\psi_o'(\alpha_1 + \psi_o\beta + \alpha\alpha_1(-n))((\alpha n - 1)\beta_1 - (n+1) \times \psi_o\beta) - 8\pi a\alpha\xi^3 P_{rc}^2\psi_o^{2n-1} \times ((n+1)\psi_o\beta - (\alpha n - 1)\beta_1)(\xi\psi_o'((n+1)\psi_o\beta + \alpha_1n(1 - \alpha n)) + 4\psi_o(\alpha_1 + \psi_o\beta + \alpha\alpha_1(-n))) + \alpha\xi P_{rc}\psi_o^{n-2}(\alpha N^2\xi - 32\pi P_{rc}v)(-\alpha_1(n-1)n\xi(\alpha n - 1)\psi_o'^2 + (n+1)\psi_o^2\beta(\xi\psi_o'' + \psi_o')) + n\psi_o(\alpha_1\xi(1 - \alpha n)\psi_o'' + \psi_o' \times (\alpha_1 + (n+1)\xi\beta\psi_o' + \alpha\alpha_1(-n))) \right] \right], \tag{48}$$

where $\beta = (\alpha - \alpha_1 + \alpha\alpha_1n)$ and $\beta_1 = (\alpha_1 + \alpha_1n + 1)$

4.2 Case 2

Now we consider [25]

$$P_r = K\mu^{1+\frac{1}{n}} + \alpha_1\mu, \tag{49}$$

in this case energy density μ and mass density μ_o are expressed as [46]

$$\mu(K\mu_o^{1/n} - 1)^n = (-1)^n \mu_o. \tag{50}$$

By taking $\psi^n = \frac{\mu}{\mu_o}$ TOV equation becomes

$$\begin{aligned} & \frac{\xi}{\alpha^2 N^4} \left[-16\pi a \xi^3 P_{rc}^2 \psi^{2n} ((\alpha - \alpha_1)\psi + \alpha_1) \right. \\ & \quad \times ((\alpha - \alpha_1)\psi + \alpha_1 + 1) \\ & \quad - 2(\alpha N^2 \xi - 32\pi P_{rc} v)(2\alpha \Delta - \xi P_{rc} \psi^{n-1} \psi'(\xi)) \\ & \quad \left. \times ((n+1)(\alpha - \alpha_1) + \alpha_1 n) \right] = 0. \end{aligned} \tag{51}$$

and from Eq. (17) we have

$$\frac{dv}{d\xi} = \frac{a}{8(n+1)\alpha} - a\xi^2 \psi^n. \tag{52}$$

Equations (51, 52) together give the generalized LEE

$$\begin{aligned} & \frac{2}{\alpha^3 N^4} \left[\frac{-1}{n+1} \left[2(2\pi a \xi P_{rc} (8\alpha(n+1)\xi^2 \psi^n - 1) \right. \right. \\ & \quad + \alpha^2 N^2 (n+1)\xi \\ & \quad - 16\pi \alpha (n+1) P_{rc} v)(2\alpha \Delta + \xi P_{rc} \psi^{n-1} \psi' \\ & \quad \times ((1+n)(\alpha_1 - \alpha)\psi \\ & \quad - \alpha_1 n)) \left. \right] - 8\pi a \alpha \xi^4 P_{rc}^2 (\alpha - \alpha_1) \psi^{2n} \psi' \\ & \quad \times ((\alpha - \alpha_1)\psi + \alpha_1) \\ & \quad - 8\pi a \alpha n \xi^4 P_{rc}^2 \psi^{2n-1} \psi' ((\alpha - \alpha_1)\psi \\ & \quad + \alpha_1) ((\alpha - \alpha_1)\psi + \alpha_1 + 1) \\ & \quad - 8\pi a \alpha \xi^3 P_{rc}^2 \psi^{2n-1} ((\alpha - \alpha_1)\psi \\ & \quad + \alpha_1 + 1)(4\psi ((\alpha - \alpha_1)\psi + \alpha_1) \\ & \quad + \xi \psi' ((n+1)(\alpha - \alpha_1)\psi + \alpha_1 n)) \\ & \quad + \alpha \xi P_{rc} \psi^{n-2} (\alpha N^2 \xi \\ & \quad - 32\pi P_{rc} v)((n+1)(\alpha - \alpha_1)\psi^2 (\xi \psi'' + \psi') \\ & \quad + n\psi (\alpha_1 \xi \psi'' + \psi'(\alpha_1 \\ & \quad + (1+n)\xi \psi'(\alpha - \alpha_1))) - \alpha_1 \\ & \quad \left. \left. \times (1-n)n\xi \psi'^2 \right] \right] = 0. \end{aligned} \tag{53}$$

Both cases have to satisfy the physical conditions

$$\mu > 0, \quad \frac{P_r}{\mu} \leq 1, \quad \frac{P_\perp}{\mu} \leq 1. \tag{54}$$

Conditions (54) takes the form for case 1 as

$$\begin{aligned} & (1 - n\alpha) + (1 + n\alpha_1) + n(\alpha - \alpha_1 + n\alpha\alpha_1)\psi_o > 0, \\ & \frac{\alpha_1}{\psi_o} + (\alpha_1 + \alpha\psi_o) \leq (1 - n\alpha) + (1 + n\alpha_1) \end{aligned}$$

$$\begin{aligned} & + (\alpha n \alpha_1 - \alpha_1 + \alpha) n \psi_o, \\ & \frac{v}{\xi^3 \psi_o^n} - \frac{a}{8\alpha(n+1)\xi^2 \alpha_o^n} \leq (1 - a^2) \\ & \quad \times [n(1 - n\psi_o)(\alpha - \alpha_1 n \alpha \alpha_1) - 1] \\ & \quad + a[\alpha_1(1 - n\alpha) + n\alpha^2 \psi_o]. \end{aligned} \tag{55}$$

and in Case 2 these conditions (54) becomes as

$$\begin{aligned} & \mu > 0, \quad \alpha_1(1 - \psi) + \alpha\psi \leq 1, \\ & \frac{v}{a\xi^3 \psi^{-n}} - \frac{1}{8\alpha(n+1)\xi^2 \psi^{-n}} \leq \alpha_1(1 - n\alpha)(\psi - 1) \\ & \quad - (1 + \alpha\psi - a). \end{aligned} \tag{56}$$

5 Vanishing complexity factor with generalized cylindrical polytropes

As we already have discussed CF as a single scalar and now it is merged with cylindrical GPEoS so that we are able to develop a consistent system of DEs for both cases.

5.1 Case No. 1

In this case Eqs. (43, 44) are used with $Y_{TF} = 0$, as

$$\begin{aligned} & 4\pi(\Delta' + 2B^2(\Delta A' + A\Delta')) + 4AB\Delta B' \\ & \quad - \frac{1}{24\alpha^2 N^3 \xi^3} \left[\alpha \xi A' (\xi^2 B^2 (32\pi P_{rc} \right. \\ & \quad \times ((6N^3 - a\xi^3)\psi_o^n (n(\alpha - \alpha_1)\psi_o \\ & \quad + \alpha(-n) + \alpha_1 n + 1) - 3v) + a\alpha \xi (3N^2 \\ & \quad + 32\pi \xi^2)) - 12\alpha N^2 (a\xi^3 + 2N^3)) + A \left(12\alpha^2 N^2 (4N^3 - a\xi^3) \right. \\ & \quad + \xi^3 B \left(2\alpha B' (32\pi P_{rc} ((6N^3 - a\xi^3)\psi_o^n \right. \\ & \quad \times (n(\alpha - \alpha_1)\psi_o + \alpha(-n) + \alpha_1 n + 1) - 3v) + a\alpha \xi \\ & \quad \times (3N^2 + 32\pi \xi^2)) + B \left(32\pi \alpha n P_{rc} \psi_o^{n-1} ((6N^3 - a\xi^3) P_{rc} \psi_o' \right. \\ & \quad \times ((n+1)(\alpha - \alpha_1)\psi_o \\ & \quad + \alpha(-n) + \alpha_1 n + 1) + 3a\alpha \alpha_1 n \xi^2 (\psi_o - 1)\psi_o) \\ & \quad \left. \left. \left. + 3a \left(32\pi \alpha^2 \xi^2 + \alpha^2 N^2 - \frac{4\pi P_{rc}}{n+1} \right) \right) \right) \right] = 0. \end{aligned} \tag{57}$$

Now Eqs. (45, 46, 57) form a system of ordinary DEs having three variables v , ψ_o and Δ . This system of ordinary DEs is solved numerically and its solution is described graphically. Figures 1, 2 and 3 depicted the behavior of v , ψ_o and Δ for $\alpha_1 = .5$, $\alpha = .5$, $A = .5$, $B = .5$ and $a = 5$.

5.2 Case No. 2

CF for case 2 will be read the as

$$4\pi(\Delta' + 2B^2(\Delta A' + A\Delta')) + 4AB\Delta B'$$

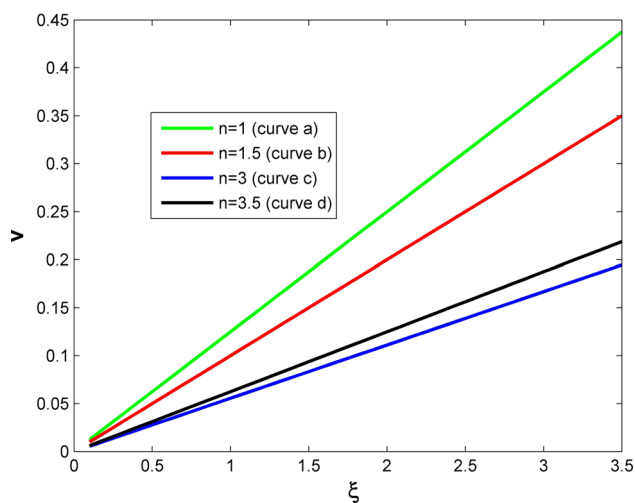


Fig. 1 Graphs between ξ and v

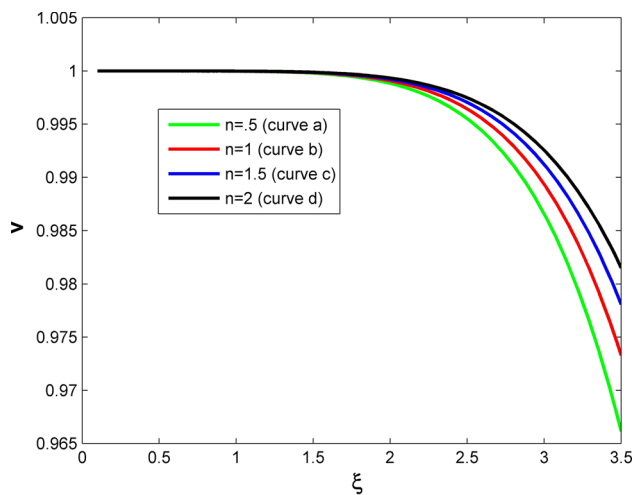


Fig. 4 Graphs between ξ and v

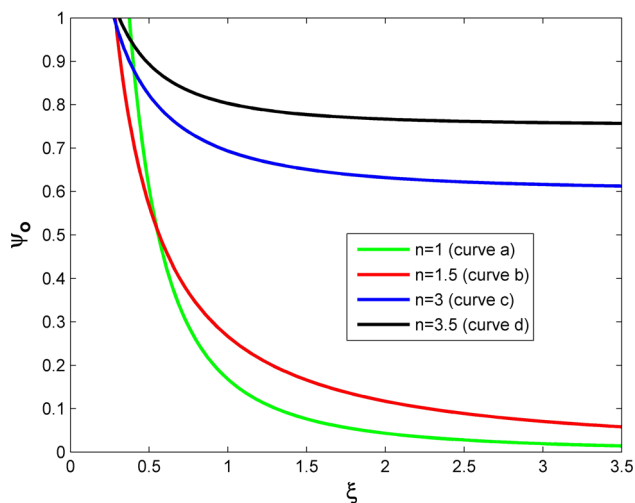


Fig. 2 Graphs between ξ and ψ_0

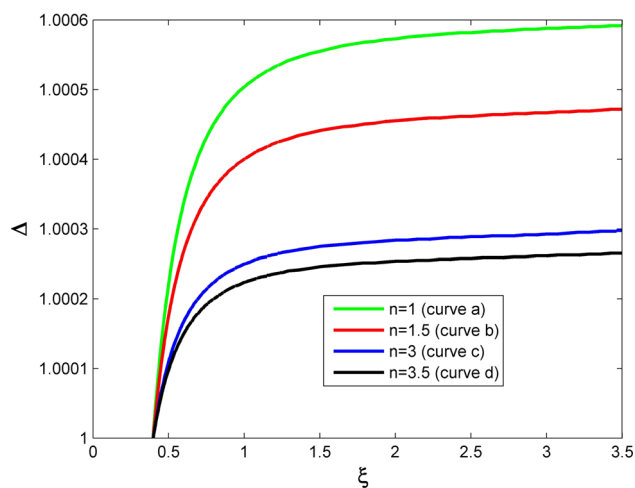


Fig. 3 Graphs between ξ and Δ

$$\begin{aligned}
 & - \frac{1}{24N^3\xi^5} \left[\left(2AB \left(B \left(\frac{1}{\alpha^2(n+1)} \right) \right) \right. \right. \\
 & \times \left[\pi(a+1)\xi^3 P_{rc} \right] + 6N^5 \Big) - 3N^5 \xi B' \Big) \\
 & + N^5(4 - 3\xi B^2 A') \Big) + \frac{1}{\alpha^2} \left[\xi^2 \left(\alpha \xi A' \right. \right. \\
 & \times (\xi^2 B^2 (\alpha \alpha \xi (3N^2 + 32\pi \xi^2)) - 32\pi P_{rc} \\
 & \times ((a\xi^3 - 6N^3)\psi^n + 3v)) - 12\alpha N^2 (a\xi^3 \\
 & + 2N^3) \Big) + A \left(12\alpha^2 N^2 (4N^3 - a\xi^3) \right. \\
 & + \xi^2 B \left(2\alpha \xi B' (\alpha \alpha \xi (3N^2 + 32\pi \xi^2)) \right. \\
 & \left. \left. - 32\pi P_{rc} ((a\xi^3 - 6N^3)\psi^n + 3v) \right) \right. \\
 & \left. + B \left(4\alpha (-8\pi n \xi P_{rc} (a\xi^3 - 6N^3)\psi^{n-1}\psi'(\xi)) \right. \right. \\
 & \left. \left. + 3a\xi \left(32\pi \alpha^2 \xi^2 + \alpha^2 N^2 \right. \right. \right. \\
 & \left. \left. \left. - \frac{4\pi P_{rc}}{n+1} \right) \right) \right) \Big] = 0.
 \end{aligned} \tag{58}$$

Equations (51, 52, 58) constitute a system of ordinary DEs containing three variables Δ , v and ψ . This system of ordinary DEs is numerically solved and its solution is explained through graphs in Figs. 4, 5 and 6 for $\alpha_1 = .5$, $\alpha = .5$, $A = .5$, $B = .5$ and $a = 5$.

6 Summary

We have used the CF to provide a general framework for the development of modify form of LEEs for both cases of GPEoS. For this purpose static anisotropic fluid distribution with cylindrical symmetry is used for stellar structure. Basic

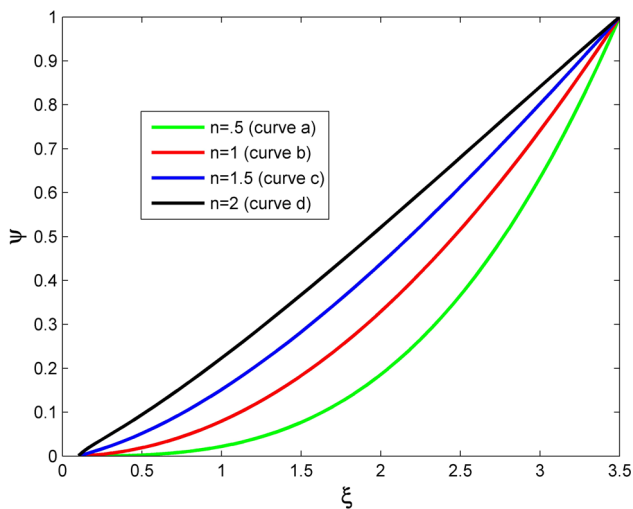


Fig. 5 Graphs between ξ and ψ

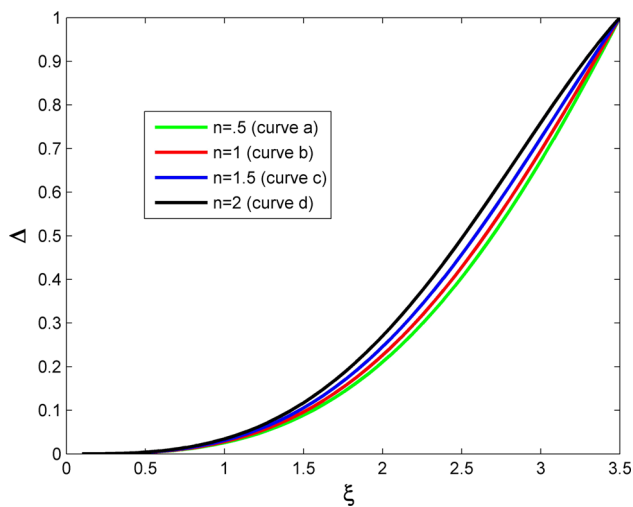


Fig. 6 Graphs between ξ and Δ

field equations and TOV equation is established. The C-energy is applied to develop an expression for mass function. Curvature and Weyl tensors are brought into play to calculate the structure scalars. These structure scalars are used to define vanishing CF. We applied assumptions Eqs. (43, 44) to establish the LEs for static cylindrical fluid distribution for two cases (1) mass density (μ_o) and (2) energy density (μ) to study the physical characteristics of GPs. The physical conditions have also been studied for these two cases under the same assumptions. These two sets of LEs led us to the two systems of three ordinary DEs with CF. These systems of DEs were numerically solved and discussed as

For case 1 the solutions of the system of three DEs (45, 46, 57) are shown in Figs. 1, 2 and 3. These solutions reveal the response of variables v , ψ_o and Δ corresponding to different values of parameters. The curves of Fig. 1 indicate that v has zero value at the center and uniformly increases in a straight

line. It can also be observed from Fig. 1 that for a small value of polytropic index n , v has its maximum value (curve a) but when the value of n increases, the value of v decreases at the boundary (curves b, c, d). In Fig. 2, the curves show the pattern of ψ_o , which has its maximum value at the center and gradually decreases towards the boundary and becomes zero in the case of curve (a) and (b). While curves (c) and (d) show that the value of ψ_o is considerably high. In Fig. 3, the anisotropic factor Δ has zero value near the center of the compact star and gradually increases but with the increase of the polytropic index n , the value of Δ decreases at the boundary surface.

For case 2, Figs. 4, 5 and 6 display the pattern of v , ψ and Δ . It can be seen from the curves of Fig. 4 that the value of v is minimum at the boundary and gradually increases towards the center with the increase of the polytropic index n . The values of ψ and Δ in Figs. 5, 6 behave in the same manner; both variables have zero values at the center and maximum values at the boundary of the surface.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: No data is required for this study.]

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References

1. J.H. Lane, *Am. J. Sci. Arts.* **50**, 148 (1870)
2. S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (University of Chicago, Chicago, 1939)
3. R.F. Tooper, *Astrophys. J.* **140**, 434 (1964)
4. R.F. Tooper, *Astrophys. J.* **142**, 1541 (1965)
5. S.A. Kaplan, G.A. Lupanov, *Sov. Astron.* **9**, 233 (1965)
6. J.J. Managhan, W. Roxburgh, *Mon. Not. R. Astron. Soc.* **131**, 13 (1965)
7. W.J. Kaufmann III, *Astrophys. J.* **72**, 754 (1967)
8. F. Occhionero, *Mem. Soc. Astron. Ital.* **38**, 331 (1967)
9. A. Kovetz, *Astrophys. J.* **154**, 999 (1968)
10. G.P. Horedt, *Astron. Astrophys.* **23**, 303 (1973)
11. J.P. Sharma, *Gen. Relativ. Gravity* **13**, 663 (1981)
12. M. Singh, G. Singh, *Astrophys. Space Sci.* **96**, 313 (1983)
13. G.P. Horedt, *Astrophys. Space Sci.* **133**, 81 (1987)
14. S.C. Pandey et al., *Astrophys. Space Sci.* **180**, 75 (1991)
15. A.W. Hendry, *Am. J. Phys.* **61**, 906 (1993)
16. E. Babichev et al., *Class. Quantum Gravity* **22**, 143 (2004)
17. L. Herrera, W. Barreto, *Gen. Relativ. Gravity* **36**, 127 (2004)

18. U. Mukhopadhyay et al., *Mod. Phys. Lett. A* **23**, 37 (2008)
19. P.H. Chavanis, *Astron. Astrophys.* **127**, 537 (2012)
20. L. Herrera, W. Barreto, *Phys. Rev. D* **88**, 084022 (2013)
21. P.H. Chavanis, *Eur. Phys. J. Plus* **129**, 38 (2014)
22. P.H. Chavanis, *Eur. Phys. J. Plus* **129**, 222 (2014)
23. L. Herrera et al., *Gen. Relativ. Gravity* **46**, 1827 (2014)
24. L. Herrera et al., *Phys. Rev. D* **93**, 024047 (2016)
25. M. Azam et al., *Eur. Phys. J. C* **76**, 315 (2016)
26. M. Azam, S.A. Mardan, *Eur. Phys. J. C* **77**, 113 (2017)
27. S.A. Mardan et al., *Eur. Phys. J. C* **78**, 516 (2018)
28. S.A. Mardan et al., *Eur. Phys. J. Plus* **134**, 242 (2019)
29. S.A. Mardan et al., *Eur. Phys. J. Plus* **135**, 3 (2020)
30. S.A. Mardan et al., *Eur. Phys. J. C* **80**, 119 (2020)
31. L. Herrera, *Phys. Rev. D* **97**, 044010 (2018)
32. G. Abbas, H. Nazar, *Eur. Phys. J. C* **78**, 510 (2018)
33. G. Abbas, H. Nazar, *Eur. Phys. J. C* **78**, 957 (2018)
34. G. Abbas, H. Nazar, *Int. J. Geom. Methods Mod. Phys.* **16**(11), 1950170 (2019)
35. G. Abbas, H. Nazar, *Int. J. Geom. Methods Mod. Phys.* **17**(03), 2050043 (2020)
36. M. Sharif, Iqra *Eur. Phys. J. C* **78**, 850 (2018)
37. M. Sharif, Iqra *Chin. J. Phys.* **61**, 238 (2019)
38. S. Khan et al., *Eur. Phys. J. C* **1037**, 79 (2019)
39. S. Khan et al., *Eur. Phys. J. Plus* **136**, 404 (2021)
40. M. Sharif, G. Abbas, *AstoPhys. Space Sci* **335**, 515 (2011)
41. K.S. Thorne, *Phy. Rev.* **138**, B251 (1965)
42. K. S. Thorne, *Ibid* **139**, B244 (1965)
43. L. Bel, *Ann. Inst. Henri Poincaré* **17**, 37 (1961)
44. L. Herrera et al., *Phys. Rev. D* **79**, 064025 (2009)
45. L. Herrera et al., *Phys. Rev. D* **84**, 107501 (2011)
46. L. Herrera, W. Barreto, *Phys. Rev. D* **87**, 087303 (2013)