

GENERAL NUMERICAL  
METHODS

## Estimation of the Remainder Term of the Appell Hypergeometric Series $F_2$

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**Abstract**—Integral representations and estimates for the remainder terms of the Appell double hypergeometric series  $F_2$  are constructed. The found formulas can be used to develop algorithms for computing the Appell functions  $F_1$  and  $F_3$  in  $\mathbb{C}^2$  by applying analytic continuation formulas. The results have applications to problems in mathematical physics and computational function theory, including the construction of conformal mappings of complex polygons based on the Schwarz–Christoffel integral.

**Keywords:** Appell functions, analytic continuation, computation of hypergeometric functions of two variables, estimation of remainder terms of hypergeometric series

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### 1. INTRODUCTION

#### 1.1. Appell Series and Some Formulas for Their Analytic Continuation

In this paper, we consider the function  $F_2(a_1, a_2; b, c_1, c_2; z_1, z_2)$ , which is one of four functions introduced by Appell in [1] (see also [2, 3]) as the most natural generalizations of the Gauss hypergeometric function to the case of two complex variables  $(z_1, z_2) \in \mathbb{C}^2$ . In the domain

$$\mathbb{M} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\} \quad (1.1)$$

the function  $F_2$  is defined as the double hypergeometric series

$$F_2(a_1, a_2; b, c_1, c_2; z_1, z_2) := \sum_{k_1, k_2=0}^{\infty} \frac{(b)_{k_1+k_2} (a_1)_{k_1} (a_2)_{k_2}}{(c_1)_{k_1} (c_2)_{k_2} k_1! k_2!} z_1^{k_1} z_2^{k_2}, \quad (1.2)$$

where  $a_1, a_2, b, c_1, c_2$  are complex parameters of  $F_2$ . The Pochhammer symbol is expressed in terms of the gamma function  $\Gamma(s)$  by the formula

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)} \quad (1.3)$$

and, for nonnegative integer  $k$ , represents a product of the form

$$(a)_0 = 1, \quad (a)_k := (a)_k = a(a+1)\dots(a+k-1), \quad k = 1, 2, \dots$$

The parameters  $c_1, c_2$  in formula (1.2) do not take nonpositive integer values, i.e.,  $c_1, c_2 \notin \mathbb{Z}^-$ . If  $\operatorname{Re} a_1 > 0$ ,  $\operatorname{Re} a_2 > 0$ ,  $\operatorname{Re}(c_1 - a_1) > 0$ , and  $\operatorname{Re}(c_2 - a_2) > 0$ , then in the domain

$$\{(z_1, z_2) \in \mathbb{C}^2 : |\arg(1 - z_j)| < \pi, j = 1, 2\}$$

the function  $F_2$  can be represented as an integral:

$$F_2(a_1, a_2; b, c_1, c_2; z_1, z_2) := \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_1 - a_1)\Gamma(c_2 - a_2)} \times \int_0^1 \int_0^1 \frac{u^{a_1-1} v^{a_2-1} (1-u)^{c_1-a_1-1} (1-v)^{c_2-a_2-1}}{(1-uz_1 - vz_2)^b} dudv. \tag{1.4}$$

It was shown in [4, 5] that the analytic continuations of the Appell functions  $F_1$  and  $F_3$  into some subdomain of  $\mathbb{C}^2$  can be expressed in terms of  $F_2$ . The Appell hypergeometric series  $F_1$  and  $F_3$  have the form (see [1–3])

$$F_1(a_1, a_2; b, c; z_1, z_2) := \sum_{k_1, k_2=0}^{\infty} \frac{(b)_{k_1+k_2} (a_1)_{k_1} (a_2)_{k_2}}{(c)_{k_1+k_2} k_1! k_2!} z_1^{k_1} z_2^{k_2}, \quad (z_1, z_2) \in \mathbb{U}^2, \tag{1.5}$$

$$F_3(a_1, a_2, b_1, b_2, c; z_1, z_2) := \sum_{k_1, k_2=0}^{\infty} \frac{(a_1)_{k_1} (a_2)_{k_2} (b_1)_{k_1} (b_2)_{k_2}}{(c)_{k_1+k_2} k_1! k_2!} z_1^{k_1} z_2^{k_2}, \quad (z_1, z_2) \in \mathbb{U}^2; \tag{1.6}$$

here,  $a_1, a_2, b_1, b_2, b, c$  are complex parameters such that  $b_1, b_2, b, c \notin \mathbb{Z}^-$ . Series (1.5) and (1.6) converge in the unit bidisk

$$\mathbb{U}^2 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}.$$

Outside  $\mathbb{U}^2$ , the Appell functions  $F_1$  and  $F_3$  are understood as analytic continuations of series (1.5) and (1.6), respectively. Before discussing these continuations, we note that the series  $F_1$  regarded as a function of  $z_1$  and  $z_2$  satisfies the following system of partial differential equations [2–6]:

$$\begin{cases} z_1(1-z_1)\partial_{11}u + z_2(1-z_1)\partial_{12}u + [c - (a_1 + b + 1)z_1]\partial_1u - a_1\partial_2u - a_1bu = 0, \\ z_2(1-z_2)\partial_{22}u + z_1(1-z_2)\partial_{12}u + [c - (a_2 + b + 1)z_2]\partial_2u - a_2\partial_1u - a_2bu = 0, \end{cases} \tag{1.7}$$

where  $u = u(z_1, z_2)$  is the unknown function and  $\partial_{jk}u(z, \zeta)$  denotes partial derivatives; for example,  $\partial_{12}u$  and  $\partial_{11}u$  denote  $\partial^2u/\partial z_1\partial z_2$  and  $\partial^2u/\partial z_1^2$ , respectively. The solution space of system (1.7) is three-dimensional (see [6, 7]). At the same time, the Appell series  $F_3$  is the solution to the system of equations

$$\begin{cases} z_1(1-z_1)\partial_{11}u + z_2\partial_{12}u + [c - (a_1 + b_1 + 1)z_1]\partial_1u - a_1b_1u = 0, \\ z_2(1-z_2)\partial_{22}u + z_1\partial_{12}u + [c - (a_2 + b_2 + 1)z_2]\partial_2u - a_2b_2u = 0, \end{cases} \tag{1.8}$$

where  $u = u(z_1, z_2)$  is the unknown function. The solution space of system (1.8) is four-dimensional (see [6, 7]).

It is natural to search for an analytic continuation of series (1.5) and (1.6) in the form of expansions in terms of some bases in the solution space of systems (1.7) and (1.8) in suitable subdomains of  $\mathbb{C}^2$ . For example, if  $b - a_1 - a_2 \notin \mathbb{Z}$ , then the analytic continuation formula for series (1.5) into the domain

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| > 1 + |z_1 - z_2|, |\arg(-z_1)| < \pi\} \tag{1.9}$$

has the following form (see [4], equality (1.8)):

$$F_1(a_1, a_2; b, c; z_1, z_2) = \sum_{s=1}^2 B_s(a_1, a_2; b, c) \mathcal{U}_s^{(\infty)}(a_1, a_2; b, c; z_1, z_2), \tag{1.10}$$

where the functions  $\mathcal{U}_s^{(\infty)}$ ,  $s = 1, 2$ , are expressed in terms of the Appell series  $F_2$  by the formulas

$$\mathcal{U}_1^{(\infty)}(a_1, a_2; b, c; z_1, z_2) := (-z_1)^{-a_1-a_2} F_2\left(1 - c + a_1 + a_2, a_2; a_1 + a_2, 1 - b + a_1 + a_2, a_1 + a_2; \frac{1}{z_1}, \frac{z_1 - z_2}{z_1}\right), \tag{1.11}$$

$$\mathcal{U}_2^{(\infty)}(a_1, a_2; b, c; z_1, z_2) := (-z_1)^{-b} F_2\left(1 - c + b, a_2; b, 1 - a_1 - a_2 + b, a_1 + a_2; \frac{1}{z_1}, \frac{z_1 - z_2}{z_1}\right), \tag{1.12}$$

while the coefficients  $B_1$  and  $B_2$  are given by

$$B_1 := \frac{\Gamma(c)\Gamma(b-a_1-a_2)}{\Gamma(b)\Gamma(c-a_1-a_2)}, \quad B_2 := \frac{\Gamma(c)\Gamma(a_1+a_2-b)}{\Gamma(a_1+a_2)\Gamma(c-b)}.$$

In domain (1.9) the functions  $\mathcal{U}_s^{(\infty)}$ ,  $s = 1, 2$ , defined by (1.11) and (1.12) are linearly independent solutions of system (1.8). Note that, like (1.10)–(1.12), the continuation formulas into the domain

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_2| > 1 + |z_1 - z_2|, |\arg(-z_2)| < \pi\}$$

contain series (1.2).

Provided that neither of the numbers  $(b_j - a_j)$ ,  $j = 1, 2$ , is an integer, the analytic continuation of the Appell series  $F_3$  to the domain

$$\{(z_1, z_2) : |z_1|^{-1} + |z_2|^{-1} < 1, |\arg(-z_j)| < \pi, j = 1, 2\} \quad (1.13)$$

has the following form (see [5], Theorem 4.3 for  $N = 2$ ):

$$F_3(a_1, a_2, b_1, b_2; c; z_1, z_2) = \sum_{s=1}^4 C_s(a_1, a_2, b_1, b_2; c) \mathcal{V}_s^{(\infty)}(a_1, a_2, b_1, b_2; c; z_1, z_2), \quad (1.14)$$

where all four functions  $\mathcal{V}_s^{(\infty)}$ ,  $s = \overline{1, 4}$ , are expressed in terms of  $F_2$  as follows:

$$\mathcal{V}_1^{(\infty)}(a_1, a_2, b_1, b_2; c; z_1, z_2) := (-z_1)^{-a_1} (-z_2)^{-a_2} F_2\left(a_1, a_2, 1 - c + a_1 + a_2; 1 - b_1 + a_1, 1 - b_2 + a_2; \frac{1}{z_1}, \frac{1}{z_2}\right), \quad (1.15)$$

$$\mathcal{V}_2^{(\infty)}(a_1, a_2, b_1, b_2; c; z_1, z_2) := (-z_1)^{-b_1} (-z_2)^{-a_2} F_2\left(b_1, a_2, 1 - c + b_1 + a_2; 1 - a_1 + b_1, 1 - b_2 + a_2; \frac{1}{z_1}, \frac{1}{z_2}\right), \quad (1.16)$$

$$\mathcal{V}_3^{(\infty)}(a_1, a_2, b_1, b_2; c; z_1, z_2) := (-z_1)^{-b_1} (-z_2)^{-b_2} F_2\left(b_1, b_2, 1 - c + b_1 + b_2; 1 - a_1 + b_1, 1 - a_2 + b_2; \frac{1}{z_1}, \frac{1}{z_2}\right), \quad (1.17)$$

$$\mathcal{V}_4^{(\infty)}(a_1, a_2, b_1, b_2; c; z_1, z_2) := (-z_1)^{-a_1} (-z_2)^{-b_2} F_2\left(a_1, b_2, 1 - c + a_1 + b_2; 1 - b_1 + a_1, 1 - b_2 + a_2; \frac{1}{z_1}, \frac{1}{z_2}\right), \quad (1.18)$$

while the coefficients  $C_s$ ,  $s = \overline{1, 4}$ , are given by

$$C_1 = \frac{\Gamma(c)\Gamma(b_1 - a_1)\Gamma(b_2 - a_2)}{\Gamma(c - a_1 - a_2)\Gamma(b_1)\Gamma(b_2)}, \quad C_2 = \frac{\Gamma(c)\Gamma(a_1 - b_1)\Gamma(b_2 - a_2)}{\Gamma(c - b_1 - a_2)\Gamma(a_1)\Gamma(b_2)},$$

$$C_3 = \frac{\Gamma(c)\Gamma(a_1 - b_1)\Gamma(a_2 - b_2)}{\Gamma(c - b_1 - b_2)\Gamma(a_1)\Gamma(a_2)}, \quad C_4 = \frac{\Gamma(c)\Gamma(b_1 - a_1)\Gamma(a_2 - b_2)}{\Gamma(c - a_1 - b_2)\Gamma(b_1)\Gamma(a_2)}.$$

In domain (1.13), the functions  $\mathcal{V}_s^{(\infty)}$ ,  $s = \overline{1, 4}$ , defined in (1.15)–(1.18) form a basis in the solution space of system (1.8).

To construct the analytic continuations of the Appell functions  $F_1$  and  $F_3$ , we can also use Euler-type integral representations. For example, if  $\operatorname{Re} c > \operatorname{Re} b > 0$ , then, in the domain

$$\{(z_1, z_2) \in \mathbb{C}^2 : |\arg(1 - z_j)| < \pi, j = 1, 2\}, \quad (1.19)$$

the Appell function  $F_1$  can be represented by the following integral [3, 6]:

$$F_1(a_1, a_2; b, c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1} (1-t)^{c-b-1}}{(1-z_1 t)^{a_1} (1-z_2 t)^{a_2}} dt. \quad (1.20)$$

If  $\operatorname{Re} b_1 > 0$ ,  $\operatorname{Re} b_2 > 0$ , and  $\operatorname{Re}(c - b_1 - b_2) > 0$ , then the Appell function  $F_3$  in domain (1.19) can be represented by the integral [3]

$$F_3(a_1, a_2, b_1, b_2; c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c-b_1-b_2)} \iint_{\Omega} \frac{t^{b_1-1} \tau^{b_2-1} (1-t-\tau)^{-c-b_1-b_2-1}}{(1-z_1 t)^{a_1} (1-z_2 \tau)^{a_2}} dt d\tau, \quad (1.21)$$

where the domain of integration is defined as  $\Omega := \{(t, \tau) : t > 0, \tau > 0, t + \tau < 1\}$ .

However, in the case of arbitrary parameters, integrals of the form (1.20), (1.21), like integral (1.4), as a rule, cannot be expressed in terms of elementary functions. For an adequate representation of the Appell functions  $F_1$  and  $F_3$  in  $\mathbb{C}^2$ , the natural apparatus is provided by analytic continuation formulas, such as (1.10) and (1.14). Moreover, it is the analytic continuation formulas that allow integrals (1.20), (1.21) to be calculated effectively. Note that integral (1.4) in the domain  $\mathbb{M}$  defined by formula (1.1) can be calculated using series (1.2).

Thus, to develop algorithms for computing the hypergeometric series  $F_1$ ,  $F_2$ , and  $F_3$ , it is necessary to construct a reliable estimate for the remainder term of the Appell series  $F_2$  defined by formula (1.2). Integral representations for the remainder term of  $F_2$  are obtained in Sections 2 and 3, and asymptotic expressions are given in Section 4. The main results of this paper are stated in Theorem 3 and Propositions 4 and 5. The auxiliary relations and theorems used to obtain these integral representations and asymptotic expressions are presented in Subsection 1.2.

This paper is a continuation of the studies begun in [8, 9]. The development of algorithms for calculating hypergeometric functions of two or more variables has attracted the attention of specialists (see [10–13]). The appearance of modern works emphasizes the importance of deriving explicit formulas and constructing efficient algorithms for estimating remainder terms of hypergeometric series of several variables. The formulas derived in this paper and in [8, 9] for estimating remainder terms of hypergeometric series of several variables can find applications in numerous problems of mathematical physics where such series arise naturally (see, e.g., [12, 14–23]).

### 1.2. Some Formulas and Asymptotic Estimates to Be Used

To derive integral representations for the remainder of series (1.2), we will use the following well-known definitions and identities for special functions. In what follows, we will need integral representations for the gamma function

$$\Gamma(z) := \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \operatorname{Re} z > 0, \quad (1.22)$$

and for the beta function

$$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \int_0^1 t^{z-1} (1-t)^{w-1} dt =: B(z, w), \quad \operatorname{Re} z > 0, \quad \operatorname{Re} w > 0. \quad (1.23)$$

The Humbert hypergeometric series  ${}_1F_1(a, c; z)$  can be represented as the Laplace integral

$${}_1F_1(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} e^{zt} dt, \quad \operatorname{Re} c > \operatorname{Re} a > 0. \quad (1.24)$$

The confluent hypergeometric series  ${}_2F_2(a_1, a_2; c_1, c_2; t)$  can be represented as an integral with kernel (1.24):

$${}_2F_2(a_1, a_2; c_1, c_2; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n z^n}{(c_1)_n (c_2)_n n!} = \frac{\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)} \int_0^1 t^{a_2-1} (1-t)^{c_2-a_2-1} {}_1F_1(a_1, c_1; tz) dt, \quad (1.25)$$

$$\operatorname{Re} c_2 > \operatorname{Re} a_2 > 0.$$

To derive the asymptotics of the integrals with respect to the parameter, we use the following two statements, which were given in [24].

**Theorem 1** (see [24], p. 58, Theorem 1). *Assume that the functions  $h(x)$  and  $\phi(x)$  have the following asymptotic expansions as  $x \rightarrow a$ :*

$$h(x) \sim h(a) + \sum_{s=0}^{\infty} a_s (x-a)^{s+\mu}, \quad \phi(x) \sim \sum_{s=0}^{\infty} b_s (x-a)^{s+\alpha-1}, \quad (1.26)$$

where  $\mu > 0$  and  $\operatorname{Re} \alpha > 0$ ; moreover,

$$\forall x \in (a, b): \quad h(x) > h(a), \quad \forall \delta > 0: \quad \inf_{x \in [a+\delta, b)} (h(x) - h(a)) > 0.$$

Assume that the integral

$$I(m) = \int_a^b \phi(x) e^{-mh(x)} dx$$

converges absolutely for all sufficiently large  $m$ . Then the following asymptotic expansion holds:

$$I(\lambda) \sim e^{-mh(a)} \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\alpha}{\mu}\right) \frac{c_s}{m^{(s+\alpha)/\mu}}, \quad \lambda \rightarrow \infty, \quad (1.27)$$

where  $c_s$  are expressed in terms of the coefficients of series (1.26); in particular,

$$c_0 = \frac{b_0}{\mu a_0^{\alpha/\mu}}, \quad c_1 = \left( \frac{b_1}{\mu} - \frac{(\alpha+1)a_1 b_0}{\mu^2 a_0} \right) \frac{1}{a_0^{(\alpha+1)/\mu}}.$$

The required asymptotics for the double integral is established by the following assertion.

**Theorem 2** (see [24], formula (11.4)). *Let*

$$I(m) = \iint_D g(x, y) e^{-mf(x, y)} dx dy,$$

where  $g(x, y)$  and  $f(x, y)$  are infinitely differentiable real-valued functions. Assume that  $f$  attains a minimum at some point  $(x_0, y_0)$  of the boundary of  $D$  at which the curvature is 0, while the gradient at  $(x_0, y_0)$  is not 0. Then the following asymptotic formula holds:

$$I(m) = \left( \frac{2\pi}{\kappa_0} \right)^{1/2} g(x_0, y_0) e^{-mf(x_0, y_0)} \frac{1}{m^{3/2}} \left( 1 + O\left(\frac{1}{m}\right) \right),$$

where

$$\kappa_0 = \left( \partial_{xx} f \partial_y f^2 - 2 \partial_{xy} f \partial_x f \partial_y f + \partial_{yy} f \partial_x f^2 \right) \Big|_{(x, y) = (x_0, y_0)}.$$

In the following sections, we first construct integral representations for the remainder term of the hypergeometric series (1.2). Then Theorems 1 and 2 are used to obtain asymptotics of these integral representations. As a large parameter  $m$ , we choose the maximum value of the summation indices, which is used to calculate the partial sums in (1.2).

## 2. SUMMATION OF THE SERIES $F_2$ AND INTEGRAL REPRESENTATION FOR THE REMAINDER TERM

We represent the Appell series (1.5) in the form

$$F_2(a_1, a_2; b, c_1, c_2; z_1, z_2) = \mathcal{F}_2^{(m)}(a_1, a_2; b, c_1, c_2; z_1, z_2) + \mathcal{R}^{(m)}(a_1, a_2; b, c_1, c_2; z_1, z_2),$$

where the first term  $\mathcal{F}_2^{(m)}$  is the sum of a finite number of terms:

$$\mathcal{F}_2^{(m)}(a_1, a_2; b, c_1, c_2; z_1, z_2) := \sum_{k_1, k_2=0}^{m-1} \frac{(b)_{k_1+k_2} (a_1)_{k_1} (a_2)_{k_2}}{(c_1)_{k_1} (c_2)_{k_2} k_1! k_2!} z_1^{k_1} z_2^{k_2},$$

and the second term  $\mathcal{R}^{(m)}$  is the remainder term. In what follows, these quantities are briefly denoted by  $\mathcal{F}_2^{(m)}(z_1, z_2)$  and  $\mathcal{R}^{(m)}(z_1, z_2)$ .

In our approach the function  $F_2$  at a point  $(z_1, z_2) \in \mathbb{M}$ , where  $\mathbb{M}$  is defined in (1.1), can be computed with prescribed accuracy  $\delta$  using the approximate equality

$$F_2(a_1, a_2; b, c_1, c_2; z_1, z_2) \simeq \mathcal{F}_2^{(m)}(a_1, a_2; b, c_1, c_2; z_1, z_2),$$

where  $m$  is a number such that

$$|\mathcal{R}^{(m)}(z_1, z_2)| < \delta.$$

To calculate the remainder term  $\mathcal{R}^{(m)}(z_1, z_2)$ , it is possible to use the integral representation given by Theorem 3 below. The integrals obtained in this theorem can be estimated numerically or using the asymptotic formulas derived in Section 4.

**Theorem 3.** Assume that the parameters of the Appell series  $F_2$  defined by formula (1.2) satisfy the conditions

$$\operatorname{Re} b > 0, \quad \operatorname{Re} c_1 > \operatorname{Re} a_1 > 0, \quad \operatorname{Re} c_2 > \operatorname{Re} a_2 > 0. \tag{2.1}$$

Then the summation formula

$$F_2(a_1, a_2; b, c_1, c_2; z_1, z_2) = \sum_{k_1, k_2=0}^{m-1} \frac{(b)_{k_1+k_2} (a_1)_{k_1} (a_2)_{k_2}}{(c_1)_{k_1} (c_2)_{k_2} k_1! k_2!} z_1^{k_1} z_2^{k_2} + \mathcal{R}^{(m)}(z_1, z_2) \tag{2.2}$$

holds, where the remainder  $\mathcal{R}^{(m)}(z_1, z_2)$  has the following integral representation:

$$\mathcal{R}^{(m)}(z_1, z_2) = z_2^m \frac{(b)_m (a_2)_m}{m! (c_2)_m} I^{(0,m)} + z_1^m \frac{(b)_m (a_1)_m}{m! (c_1)_m} I^{(m,0)} - z_1^m z_2^m \frac{(b)_{2m} (a_1)_m (a_2)_m}{(m!)^2 (c_1)_m (c_2)_m} I^{(m,m)}, \tag{2.3}$$

here,  $I^{(0,m)}$  and  $I^{(m,0)}$  denote the integrals

$$I^{(0,m)} = I^{(0,m)}(a_1, a_2; b, c_1, c_2; z_1, z_2) = \frac{m\Gamma(c_1)\Gamma(c_2 + m)}{\Gamma(a_1)\Gamma(a_2 + m)\Gamma(c_1 - a_1)\Gamma(c_2 - a_2)} \times \int_M d\mu \left( \frac{1}{1 - z_1 u - r z_2 v} \right)^{m+b} (1 - r)^{m-1} u^{a_1-1} (1 - u)^{c_1-a_1-1} v^{a_2+m-1} (1 - v)^{c_2-a_2-1}, \tag{2.4}$$

$$I^{(m,0)} = I^{(m,0)}(a_1, a_2; b, c_1, c_2; z_1, z_2) = \frac{m\Gamma(c_1 + m)\Gamma(c_2)}{\Gamma(a_1 + m)\Gamma(a_2)\Gamma(c_1 - a_1)\Gamma(c_2 - a_2)} \times \int_M d\mu \left( \frac{1}{1 - r z_1 u - z_2 v} \right)^{m+b} (1 - r)^{m-1} u^{a_1+m-1} (1 - u)^{c_1-a_1-1} v^{a_2-1} (1 - v)^{c_2-a_2-1}, \tag{2.5}$$

where  $M = [0, 1] \times [0, 1] \times [0, 1]$  and  $d\mu = drdudv$ , and  $I^{(m,m)}$  denotes the integral

$$I^{(m,m)} = I^{(m,m)}(a_1, a_2; b, c_1, c_2; z_1, z_2) = \frac{m^2\Gamma(c_1 + m)\Gamma(c_2 + m)}{\Gamma(a_1 + m)\Gamma(a_2 + m)\Gamma(c_1 - a_1)\Gamma(c_2 - a_2)} \times \int_\Omega d\omega \left( \frac{1}{1 - \xi z_1 u - r z_2 v} \right)^{2m+b} (1 - \xi)^{m-1} (1 - r)^{m-1} u^{a_1+m-1} (1 - u)^{c_1-a_1-1} v^{a_2+m-1} (1 - v)^{c_2-a_2-1}, \tag{2.6}$$

where  $\Omega = [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$  and  $d\omega = d\xi drdudv$ .

The following result establishes an alternative expression for the remainder.

**Remark 1.** Under conditions (2.1), the integral representation (2.3)–(2.6) for the remainder  $\mathcal{R}^{(m)}(z_1, z_2)$  of series (1.2) can be rewritten as

$$\begin{aligned} \mathcal{R}^{(m)}(z_1, z_2) = & \frac{1}{\Gamma(b)} \left( \frac{(a_2)_m z_2^m}{(c_2)_m m!} \int_0^\infty t^{b+m-1} e^{-t} I_{22}(a_2, c_2, z_2 t, m) I_{11}(a_1, c_1, z_1 t, 0) dt \right. \\ & + \frac{(a_1)_m z_1^m}{(c_1)_m m!} \int_0^\infty t^{b+m-1} e^{-t} I_{22}(a_1, c_1, z_1 t, m) I_{11}(a_2, c_2, z_2 t, 0) dt \\ & \left. - \frac{(a_1)_m (a_2)_m z_1^m z_2^m}{(c_1)_m (c_2)_m (m!)^2} \int_0^\infty t^{b+2m-1} e^{-t} I_{22}(a_1, c_1, z_1 t, m) I_{22}(a_2, c_2, z_2 t, m) dt \right); \end{aligned} \tag{2.7}$$

here,  $I_{22}$  and  $I_{11}$  denote the integrals

$$I_{22}(a, c, z, m) = m \int_0^1 (1 - \xi)^{m-1} I_{11}(a, c, z\xi, m) d\xi, \tag{2.8}$$

$$I_{11}(a, c, z, m) = \frac{\Gamma(c+m)}{\Gamma(a+m)\Gamma(c-a)} \int_0^1 e^{zu} u^{a+m-1} (1-u)^{c-a-1} du. \quad (2.9)$$

If the parameters of the Appell series do not satisfy the constraints imposed in Theorem 3, then for the remainder term, we can obtain a similar integral representation, where the integrals along intervals are replaced by integrals over special Pochhammer loop-shaped contours.

### 3. PROOF OF THEOREM 3

It is easy to see that the remainder term  $\mathcal{R}^{(m)}(z_1, z_2)$  can be written as a sum of three functions:

$$\mathcal{R}^{(m)}(z_1, z_2) = R^{(0,m)}(z_1, z_2) + R^{(m,0)}(z_1, z_2) - R^{(m,m)}(z_1, z_2), \quad (3.1)$$

where

$$R^{(0,m)}(z_1, z_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=m}^{\infty} \frac{(b)_{k_1+k_2} (a_1)_{k_1} (a_2)_{k_2}}{(c_1)_{k_1} (c_2)_{k_2} k_1! k_2!} z_1^{k_1} z_2^{k_2}, \quad (3.2)$$

$$R^{(m,0)}(z_1, z_2) = \sum_{k_1=m}^{\infty} \sum_{k_2=0}^{\infty} \frac{(b)_{k_1+k_2} (a_1)_{k_1} (a_2)_{k_2}}{(c_1)_{k_1} (c_2)_{k_2} k_1! k_2!} z_1^{k_1} z_2^{k_2}, \quad (3.3)$$

$$R^{(m,m)}(z_1, z_2) = \sum_{k_1, k_2=m}^{\infty} \frac{(b)_{k_1+k_2} (a_1)_{k_1} (a_2)_{k_2}}{(c_1)_{k_1} (c_2)_{k_2} k_1! k_2!} z_1^{k_1} z_2^{k_2}. \quad (3.4)$$

Let us obtain integral representations for  $R^{(0,m)}(z_1, z_2)$ ,  $R^{(m,0)}(z_1, z_2)$ , and  $R^{(m,m)}(z_1, z_2)$ .

#### 3.1. Representation for $R^{(m,m)}(z_1, z_2)$

The following proposition provides two equivalent integral representations for the function  $R^{(m,m)}(z_1, z_2)$ .

**Proposition 1.** (1) *If the parameters of the series  $R^{(m,m)}(z_1, z_2)$  defined by formula (3.4) satisfy conditions (2.1), then the following integral representation holds:*

$$R^{(m,m)}(z_1, z_2) = z_1^m z_2^m \frac{(b)_{2m} (a_1)_m (a_2)_m}{(m!)^2 (c_1)_m (c_2)_m} I^{(m,m)}(z_1, z_2), \quad (3.5)$$

where  $I^{(m,m)}(z_1, z_2)$  is defined in (2.6).

(2) *Under the same constraints on the parameters, the following integral representation holds for  $R^{(m,m)}(z_1, z_2)$ :*

$$R^{(m,m)}(z_1, z_2) = z_1^m z_2^m \frac{1}{(m!)^2} \frac{(a_1)_m (a_2)_m}{(c_1)_m (c_2)_m} \frac{1}{\Gamma(b)} \int_0^{\infty} t^{b+2m-1} e^{-t} I_{22}(a_1, c_1, z_1 t, m) I_{22}(a_2, c_2, z_2 t, m) dt, \quad (3.6)$$

where the integral  $I_{22}(a, c, zt, m)$  is defined by formula (2.8).

**Proof.** Replacing the summation indices in (3.4) so that the summation starts at 0 and using the identities

$$(k+m)! = (1+m)_k m!, \quad (a)_{k+m} = (a)_m (a+m)_k, \quad (3.7)$$

we write  $R^{(m,m)}(z_1, z_2)$  as

$$R^{(m,m)}(z_1, z_2) = \frac{(a_1)_m (a_2)_m}{(c_1)_m (c_2)_m} \frac{z_1^m z_2^m}{m! m!} \sum_{k_1, k_2=0}^{\infty} \frac{(b)_{k_1+k_2+2m}}{(c_1+m)_{k_1} (c_2+m)_{k_2} (1+m)_{k_1} (1+m)_{k_2}} \frac{z_1^{k_1}}{(1+m)_{k_1}} \frac{z_2^{k_2}}{(1+m)_{k_2}}. \quad (3.8)$$

Using the definition of the Pochhammer symbol (1.3) and applying the integral representation of the gamma function (1.22), we rewrite  $(b)_{k_1+k_2+2m}$  as

$$(b)_{k_1+k_2+2m} = \frac{1}{\Gamma(b)} \int_0^\infty t^{b+k_1+k_2+2m-1} e^{-t} dt. \tag{3.9}$$

Substituting (3.9) into (3.8) and changing the order of summation and integration, we obtain the following expression for  $R^{(m,m)}(z_1, z_2)$ :

$$R^{(m,m)}(z_1, z_2) = \frac{1}{\Gamma(b)} \frac{(a_1)_m (a_2)_m}{(c_1)_m (c_2)_m} \frac{z_1^m z_2^m}{(m!)^2} \int_0^\infty t^{b+2m-1} e^{-t} \sum_{k_1=0}^\infty \frac{(a_1+m)_{k_1}}{(c_1+m)_{k_1}} \frac{(z_1 t)^{k_1}}{(1+m)_{k_1}} \sum_{k_2=0}^\infty \frac{(a_2+m)_{k_2}}{(c_2+m)_{k_2}} \frac{(z_2 t)^{k_2}}{(1+m)_{k_2}} dt. \tag{3.10}$$

By using formulas (1.25) and (1.24), the sums are rewritten as integrals:

$$\begin{aligned} \sum_{k_1=0}^\infty \frac{(a_1+m)_{k_1}}{(c_1+m)_{k_1}} \frac{(1)_{k_1}}{(1+m)_{k_1}} \frac{(z_1 t)^{k_1}}{k_1!} &= {}_2F_2(a_1+m, 1; c_1+m, 1+m; z_1 t) \\ &= m \int_0^1 (1-\xi)^{m-1} {}_1F_1(a_1+m, c_1+m, \xi z_1 t) d\xi \\ &= m \int_0^1 (1-\xi)^{m-1} \left( \frac{\Gamma(c_1+m)}{\Gamma(a_1+m)\Gamma(c_1-a_1)} \int_0^1 e^{\xi z_1 t u} u^{a_1+m-1} (1-u)^{c_1-a_1-1} du \right) d\xi. \end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10) and taking into account definitions (2.8) and (2.9), we obtain the sought expression (3.6).

Now we show how to pass from expression (3.6) to (3.5). For this purpose, we extract the integral with respect to  $t$  from (3.6) and transform it as follows:

$$I_t = \int_0^\infty t^{b+2m-1} e^{-t} e^{(\xi x_1 u + r x_2 v)t} dt = \int_0^\infty t^{b+2m-1} e^{-(1-\gamma)t} dt,$$

for convenience, in this section, we use the notation  $\gamma := \xi x_1 u + r x_2 v$ .

Making the substitution  $y = (1-\gamma)t$ , we derive the following expression for  $I_t$ :

$$I_t = \frac{1}{(1-\gamma)^{b+2m}} \int_0^\infty y^{b+2m-1} e^{-y} dy = \frac{\Gamma(b+2m)}{(1-\gamma)^{b+2m}}. \tag{3.12}$$

Substituting (3.12) into (3.6) and recalling definition (1.3), we obtain (3.5).

Proposition 1 is proved.

### 3.2. Representations for $R^{(0,m)}(z_1, z_2)$ and $R^{(m,0)}(z_1, z_2)$

The following proposition provides two equivalent integral representations for the function  $R^{(0,m)}(z_1, z_2)$ .

**Proposition 2.** (1) *If the parameters of the series  $R^{(0,m)}(z_1, z_2)$  defined by formula (3.2) satisfy conditions (2.1), then the following integral representation holds:*

$$R^{(0,m)}(z_1, z_2) = z_2^m \frac{(b)_m (a_2)_m}{m! (c_2)_m} I^{(0,m)}(z_1, z_2), \tag{3.13}$$

where  $I^{(0,m)}(z_1, z_2)$  is defined in (2.4).

(2) *Under the same constraints on the parameters, the following integral representation holds for  $R^{(0,m)}(z_1, z_2)$ :*

$$R^{(0,m)}(z_1, z_2) = z_2^m \frac{(a_2)_m}{m! (c_2)_m \Gamma(b)} \int_0^\infty t^{b+m-1} e^{-t} I_{22}(a_2, c_2, z_2 t, m) I_{11}(a_1, c_1, z_1 t, 0) dt, \tag{3.14}$$

where the integrals  $I_{22}(a, c, z, m)$  and  $I_{11}(a, c, z, m)$  are defined by formulas (2.8) and (2.9), respectively.

**Proof.** Following the same method as in the proof of Proposition 1, we replace the summation indices in expression (3.2) for  $R^{(0,m)}$  so that the sum starts at 0 and use identities (3.7):

$$R^{(0,m)}(z_1, z_2) = \frac{(a_2)_m z_2^m}{(c_2)_m m!} \sum_{k_1, k_2=0}^{\infty} (b)_{k_1+k_2+m} \frac{(a_1)_{k_1}}{(c_1)_{k_1} (k_1)!} z_1^{k_1} \frac{(a_2+m)_{k_2}}{(c_2+m)_{k_2} (1+m)_{k_2}} z_2^{k_2}. \quad (3.15)$$

Rewriting  $(b)_{k_1+k_2+m}$  in formula (3.15) by analogy with (3.9) and changing the order of summation and integration, we obtain

$$R^{(0,m)}(z_1, z_2) = \frac{1}{\Gamma(b)} \frac{(a_2)_m z_2^m}{(c_2)_m m!} \int_0^{\infty} t^{b+m-1} e^{-t} \sum_{k_1=0}^{\infty} \frac{(a_1)_{k_1}}{(c_1)_{k_1} (k_1)!} z_1^{k_1} \sum_{k_2=0}^{\infty} \frac{(a_2+m)_{k_2}}{(c_2+m)_{k_2} (1+m)_{k_2}} (z_2 t)^{k_2} dt. \quad (3.16)$$

Next, the sums with respect to  $k_1$  and  $k_2$  are rewritten as integrals using formulas (1.25) and (1.24). Specifically, the sum with respect to  $k_1$  has the form

$$\sum_{k_1=0}^{\infty} \frac{(a_1)_{k_1}}{(c_1)_{k_1}} \frac{(z_1 t)^{k_1}}{k_1!} = {}_1F_1(a_1, c_1, z_1 t) = \frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)} \int_0^1 u^{a_1-1} (1-u)^{c_1-a_1-1} e^{z_1 t u} du, \quad (3.17)$$

and the sum with respect to  $k_2$  is

$$\begin{aligned} \sum_{k_2=0}^{\infty} \frac{(a_2+m)_{k_2}}{(c_2+m)_{k_2}} \frac{(z_2 t)^{k_2}}{(1+m)_{k_2}} &= {}_2F_2(a_2+m, 1; c_2+m, 1+m; z_2 t) \\ &= m \int_0^1 (1-r)^{m-1} \left( \frac{\Gamma(c_2+m)}{\Gamma(a_2+m)\Gamma(c_2-a_2)} \int_0^1 e^{\xi z_2 t v} v^{a_2+m-1} (1-v)^{c_2-a_2-1} dv \right) dr. \end{aligned} \quad (3.18)$$

Substituting (3.17) and (3.18) into (3.16) yields the desired expression (3.14) for  $R^{(0,m)}(z_1, z_2)$ .

To pass to expression (3.13), we extract the integral with respect to  $t$  from (3.14):

$$I_t = \int_0^{\infty} t^{b+m-1} e^{-t} e^{(x_1 u + r x_2 v) t} dt = \int_0^{\infty} t^{b+m-1} e^{-(1-\gamma)t} dt,$$

where  $\gamma$  denotes, for brevity, the expression  $\gamma := x_1 u + r x_2 v$ . Making the substitution  $y = (1-\gamma)t$  yields

$$I_t = \frac{1}{(1-\gamma)^{b+m}} \int_0^{\infty} y^{b+m-1} e^{-y} dy = \frac{\Gamma(b+m)}{(1-\gamma)^{b+m}} = \frac{\Gamma(b+m)}{(1-x_1 u - r x_2 v)^{b+m}}. \quad (3.19)$$

Substituting (3.19) into (3.14) and recalling definition (1.3), we obtain (3.13). Proposition 2 is proved.

Since  $R^{(m,0)}(a_1, a_2; b, c_1, c_2; z_1, z_2) = R^{(0,m)}(a_2, a_1; b, c_2, c_1; z_2, z_1)$ , Proposition 2 implies the following result.

**Proposition 3.** (1) *If the parameters of the series  $R^{(m,0)}(z_1, z_2)$  defined by formula (3.3) satisfy conditions (2.1), then the following integral representation holds:*

$$R^{(m,0)} = z_1^m \frac{(b)_m (a_1)_m}{m! (c_1)_m} I^{(m,0)}(z_1, z_2), \quad (3.20)$$

where  $I^{(m,0)}(z_1, z_2)$  is defined in (2.5).

(2) *Under the same constraints on the parameters,  $R^{(m,0)}(z_1, z_2)$  has the integral representation*

$$R^{(m,0)}(z_1, z_2) = z_1^m \frac{1}{m! (c_1)_m} \frac{1}{\Gamma(b)} \int_0^{\infty} t^{b+m-1} e^{-t} I_{22}(a_1, c_1, z_1 t, m) I_{11}(a_2, c_2, z_2 t, 0) dt, \quad (3.21)$$

where  $I_{22}(a, c, z, m)$  and  $I_{11}(a, c, z, m)$  are defined in formulas (2.8) and (2.9), respectively.

Substituting expressions (3.5), (3.13), and (3.20) for  $R^{(m,m)}(z_1, z_2)$ ,  $R^{(0,m)}(z_1, z_2)$ , and  $R^{(m,0)}(z_1, z_2)$  (see Propositions 1–3) into formula (3.1), we obtain (2.3). Substituting expressions (3.6), (3.14), and (3.21) for  $R^{(m,m)}(z_1, z_2)$ ,  $R^{(0,m)}(z_1, z_2)$ , and  $R^{(m,0)}(z_1, z_2)$  (see Propositions 1–3) into formula (3.1), we derive (2.7).

Theorem 3 is proved.

#### 4. ASYMPTOTICS OF THE REMAINDER TERM OF THE SERIES $F_2$

The modulus  $|\mathcal{R}^m(z_1, z_2)|$  is estimated as follows:

$$|\mathcal{R}^m(z_1, z_2)| \leq |R^{(0,m)}(z_1, z_2)| + |R^{(m,0)}(z_1, z_2)| + |R^{(m,m)}(z_1, z_2)|,$$

where, according to formulas (3.13), (3.20), and (3.5) obtained in Propositions 1–3, respectively, the terms  $|R^{(0,m)}(z_1, z_2)|$ ,  $|R^{(m,0)}(z_1, z_2)|$ , and  $|R^{(m,m)}(z_1, z_2)|$  are given by

$$|R^{(0,m)}(z_1, z_2)| = \left| z_2^m \frac{(b)_m (a_2)_m}{m! (c_2)_m} I^{(0,m)}(z_1, z_2) \right|, \tag{4.1}$$

$$|R^{(m,0)}(z_1, z_2)| = \left| z_1^m \frac{(b)_m (a_1)_m}{m! (c_1)_m} I^{(m,0)}(z_1, z_2) \right|, \tag{4.2}$$

$$|R^{(m,m)}(z_1, z_2)| = \left| z_1^m z_2^m \frac{(b)_{2m} (a_1)_m (a_2)_m}{(m!)^2 (c_1)_m (c_2)_m} I^{(m,m)}(z_1, z_2) \right|, \tag{4.3}$$

where  $I^{(0,m)}$ ,  $I^{(m,0)}$ , and  $I^{(m,m)}$  are defined by formulas (2.4)–(2.6), respectively. Let us obtain asymptotics of the integrals  $|I^{(0,m)}(z_1, z_2)|$ ,  $|I^{(m,0)}(z_1, z_2)|$ , and  $|I^{(m,m)}(z_1, z_2)|$  involved in (4.1)–(4.3).

In what follows, we use the notation  $x_1 = \operatorname{Re} z_1$  and  $x_2 = \operatorname{Re} z_2$ . Note that  $|z_1| + |z_2| < 1$  in the considered domain, so  $|x_1| + |x_2| < 1$ .

##### 4.1. Asymptotics of $I^{(0,m)}$ as $m \rightarrow \infty$

The following proposition provides an estimate for the integral  $I^{(0,m)}(z_1, z_2)$  involved in expression (3.13) for  $R^{(0,m)}(z_1, z_2)$  as  $m \rightarrow \infty$ .

**Proposition 4.** *For  $|z_1| + |z_2| < 1$ , the integral  $I^{(0,m)}(z_1, z_2)$  defined in (2.4) satisfies the estimate  $|I^{(0,m)}(z_1, z_2)| \leq \mathcal{I}^{(0,m)}(x_1, x_2)$ , where  $\mathcal{I}^{(0,m)}(x_1, x_2)$  has the following asymptotics as  $m \rightarrow \infty$ .*

(1) *If  $x_1 = 0$ , then*

$$\mathcal{I}^{(0,m)}(0, x_2) = \frac{K_0}{1 - x_2} \left( 1 + \frac{C_0(0, x_2)}{m} + O\left(\frac{1}{m^2}\right) \right), \tag{4.4}$$

where the coefficient  $K_0$  is given by

$$K_0 = \frac{\Gamma(\operatorname{Re}(c_1 - a_1)) \Gamma(\operatorname{Re}(c_2 - a_2))}{|\Gamma(c_1 - a_1)| |\Gamma(c_2 - a_2)|} \frac{|\Gamma(c_1)|}{\Gamma(\operatorname{Re} c_1)} \frac{\Gamma(\operatorname{Re} a_1)}{|\Gamma(a_1)|} e^{1mb/2}, \tag{4.5}$$

and the function  $C_0(x_1, x_2)$  at  $x_1 = 0$  has the form

$$C_0(0, x_2) = \frac{1}{1 - x_2} \left( (b - 1)x_2 - \frac{c_2 - a_2}{2} ((1 - a_2 - c_2) - x_2(3 - a_2 - c_2)) \right). \tag{4.6}$$

(2) If  $x_1 > 0$ , then

$$\mathcal{I}^{(0,m)}(x_1, x_2) = \frac{K_+}{m^{\operatorname{Re}(c_1-a_1)}(1-x_1)^m} \frac{(1-x_1)^{\operatorname{Re}(c_1-a_1-b)+1}}{x_1^{\operatorname{Re}(c_1-a_1)}(1-x_1-x_2)} \left( 1 + \frac{C_+(x_1, x_2)}{m} + O\left(\frac{1}{m^2}\right) \right), \quad (4.7)$$

where the coefficient  $K_+$  is given by

$$K_+ = \frac{|\Gamma(c_1)|}{|\Gamma(a_1)|} \frac{|\Gamma(\operatorname{Re}(c_1 - a_1))|}{|\Gamma(c_1 - a_1)|} \frac{|\Gamma(\operatorname{Re}(c_2 - a_2))|}{|\Gamma(c_2 - a_2)|} e^{\operatorname{Im}b/2}, \quad (4.8)$$

and the function  $C_+(x_1, x_2)$  has the form

$$\begin{aligned} C_+(x_1, x_2) &= \frac{c_1 - a_1}{2} \left( 1 + a_1 - 2b + c_1 + \frac{2(1-a_1)}{x_1} - \frac{2(1-x_1)}{1-x_1-x_2} \right) \\ &+ \frac{(b-1)x_2}{1-x_1-x_2} + \frac{(c_2-a_2)}{2(1-x_1-x_2)} \left( x_2(3-a_2-c_2) - \frac{(1-x_1)}{(1-a_2-c_2)} \right). \end{aligned} \quad (4.9)$$

(3) If  $x_1 < 0$ , then

$$\mathcal{I}^{(0,m)}(x_1, x_2) = \frac{K_-}{m^{\operatorname{Re}a_1}(-x_1)^{\operatorname{Re}a_1}(1-x_2)} \left( 1 + \frac{C_-(x_1, x_2)}{m} + O\left(\frac{1}{m^2}\right) \right), \quad (4.10)$$

where the coefficient  $K_-$  is given by

$$K_- = \frac{|\Gamma(c_1)|}{|\Gamma(c_1 - a_1)|} \frac{|\Gamma(\operatorname{Re}(c_2 - a_2))|}{|\Gamma(c_2 - a_2)|} \frac{|\Gamma(\operatorname{Re}a_1)|}{|\Gamma(a_1)|} e^{\operatorname{Im}b/2}, \quad (4.11)$$

and the function  $C_-(x_1, x_2)$  has the form

$$\begin{aligned} C_-(x_1, x_2) &= -\frac{a_1}{2}(3+a_1-2b) + \frac{a_1}{x_1}(1+a_1-c_1) + \frac{1}{1-x_2} \\ &\times \left( a_1 + (b-1)x_2 - \frac{c_2-a_2}{2}(-1+a_2+c_2+(-3+a_2+c_2)x_2) \right). \end{aligned} \quad (4.12)$$

If all the parameters  $b, a_1, a_2, c_1, c_2$  are real, then the expressions for  $K_0, K_+$ , and  $K_-$  become

$$K_0 = 1, \quad K_+ = \frac{|\Gamma(c_1)|}{|\Gamma(a_1)|}, \quad K_- = \frac{|\Gamma(c_1)|}{|\Gamma(c_1 - a_1)|}.$$

**Proof.** According to formula (2.4),  $|I^{(0,m)}|$  satisfies the relation

$$\begin{aligned} |I^{(0,m)}| &\leq \int |\dots| = m \left| \frac{\Gamma(c_2+m)}{\Gamma(a_2+m)\Gamma(c_2-a_2)} \frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)} \right| e^{\operatorname{Im}b/2} \\ &\times \int_0^1 dr \int_0^1 du \int_0^1 dv \left( \frac{(1-r)^{m-1}}{(1-x_1u-rx_2v)^{m+\operatorname{Re}b}} u^{\operatorname{Re}a_1-1} (1-u)^{\operatorname{Re}(c_1-a_1)-1} v^{\operatorname{Re}a_2+m-1} (1-v)^{\operatorname{Re}(c_2-a_2)-1} \right). \end{aligned} \quad (4.13)$$

Recall that, by  $x_1$  and  $x_2$ , we understood  $\operatorname{Re} z_1$  and  $\operatorname{Re} z_2$ , respectively. Throughout the rest of this section, we use the following notation:  $\beta = \operatorname{Re} b, \alpha_1 = \operatorname{Re} a_1, \alpha_2 = \operatorname{Re} a_2, \sigma_1 = \operatorname{Re} c_1$ , and  $\sigma_2 = \operatorname{Re} c_2$ . Let the right-hand side of (4.13) be denoted by  $\mathcal{I}^{(0,m)}(x_1, x_2)$ .

We sequentially consider the integrals with respect to  $r, v$ , and  $u$ .

(a) The order of integration on the right-hand side of (4.13) is changed so that the integral with respect to  $r$  comes first. Consider the integral

$$I_r = \int_0^1 \frac{(1-r)^{m-1}}{(1-x_1u-rx_2v)^{m+\beta}} dr.$$

We transform  $I_r$  into a form convenient for applying Theorem 1:

$$I_r = \int_0^1 dr e^{m \ln \frac{1-r}{1-x_1u-rx_2v}} \frac{1}{(1-r)(1-x_1u-rx_2v)^\beta}.$$

Since

$$\frac{d}{dr} \left( \frac{1-r}{1-x_1u-rx_2v} \right) = \frac{-1+ux_1+vx_2}{(1-x_1u-rx_2v)^2} < 0$$

for  $|x_1| + |x_2| < 1$ , the argument of the logarithm in the integration domain  $r \in [0, 1]$  is maximized at the point  $r = 0$ . Applying Theorem 1 to the integral  $I_r$  yields

$$I_r = \frac{1}{m(1-x_1u)^{\beta+m-1}} \frac{1}{(1-x_1u-x_2v)} \left( 1 + \frac{(\beta-1)x_2v}{m(1-x_1u-x_2v)} + O\left(\frac{1}{m^2}\right) \right). \quad (4.14)$$

Substituting this expression for  $I_r$  into (4.13), we find

$$\begin{aligned} \mathcal{I}^{(0,m)}(x_1, x_2) &= \left| \frac{\Gamma(c_2+m)}{\Gamma(a_2+m)\Gamma(c_2-a_2)} \frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)} \right| e^{\text{Im}b/2} \int_0^1 du \int_0^1 dv v^{\alpha_2+m-1} (1-v)^{\sigma_2-\alpha_2-1} \\ &\times u^{\alpha_1-1} (1-u)^{\sigma_1-\alpha_1-1} \frac{1}{(1-x_1u)^{\beta+m-1} (1-x_1u-x_2v)} \left( 1 + \frac{(\beta-1)x_2v}{m(1-x_1u-x_2v)} + O\left(\frac{1}{m^2}\right) \right). \end{aligned} \quad (4.15)$$

(b) Consider the integral with respect to  $v$ . From (4.15) we extract all factors involving  $v$ ; the resulting integral is denoted by  $I_v$ :

$$\begin{aligned} I_v &= \int_0^1 dv v^{\alpha_2+m-1} (1-v)^{\sigma_2-\alpha_2-1} \frac{1}{(1-x_1u-x_2v)} \left( 1 + \frac{(\beta-1)x_2v}{m(1-x_1u-x_2v)} + O\left(\frac{1}{m^2}\right) \right) \\ &= \int_0^1 dv e^{m \ln v} v^{\alpha_2-1} (1-v)^{\sigma_2-\alpha_2-1} \left( 1 + \frac{(\beta-1)x_2v}{m(1-x_1u-x_2v)} + O\left(\frac{1}{m^2}\right) \right). \end{aligned}$$

To apply Theorem 1, it is necessary that the expression in the exponent be maximum in the lower limit of the integral, so we make the substitution  $v = 1-x$ :

$$I_v = \int_0^1 dx e^{m \ln(1-x)} \frac{(1-x)^{\alpha_2-1} x^{\sigma_2-\alpha_2-1}}{(1-x_1u-x_2(1-x))} \left( 1 + \frac{(\beta-1)x_2(1-x)}{m(1-x_1u-x_2(1-x))} + O\left(\frac{1}{m^2}\right) \right).$$

The function  $\ln(1-x)$  for  $x \in [0, 1]$  is maximized at the point  $x = 0$ . Applying Theorem 1 to  $I_v$  yields

$$\begin{aligned} I_v &= \frac{\Gamma(\sigma_2-\alpha_2)}{m^{\sigma_2-\alpha_2}} \frac{1}{(1-x_1u-x_2)} \left( 1 + \frac{1}{m(1-x_1u-x_2)} \left( (b-1)x_2 \right. \right. \\ &\left. \left. - \frac{(\sigma_2-\alpha_2)}{2} ((1-x_1u)(1-\alpha_2-\sigma_2) - x_2(3-\alpha_2-\sigma_2)) \right) + O\left(\frac{1}{m^2}\right) \right). \end{aligned} \quad (4.16)$$

Substituting this  $I_v$  into formula (4.15) for  $\mathcal{I}^{(0,m)}(x_1, x_2)$ , we obtain

$$\begin{aligned} \mathcal{I}^{(0,m)}(x_1, x_2) &= \left| \frac{\Gamma(c_2+m)}{\Gamma(a_2+m)\Gamma(c_2-a_2)} \frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)} \right| \frac{\Gamma(\sigma_2-\alpha_2)}{m^{\sigma_2-\alpha_2}} e^{\text{Im}b/2} \\ &\times \int_0^1 u^{\alpha_1-1} (1-u)^{\sigma_1-\alpha_1-1} \frac{1}{(1-x_1u)^{\beta+m-1} (1-x_1u-x_2)} \left( 1 + \frac{1}{m(1-x_1u-x_2)} \left( (\beta-1)x_2 \right. \right. \\ &\left. \left. - \frac{(\sigma_2-\alpha_2)}{2} ((1-x_1u)(1-\alpha_2-\sigma_2) - x_2(3-\alpha_2-\sigma_2)) \right) + O\left(\frac{1}{m^2}\right) \right) du. \end{aligned} \quad (4.17)$$

(c) In (4.17) it remains to consider the integral with respect to  $u$ :

$$\begin{aligned} I_u &= \int_0^1 \frac{u^{\alpha_1-1} (1-u)^{\sigma_1-\alpha_1-1}}{(1-x_1u)^{\beta+m-1}} \frac{1}{(1-x_1u-x_2)} \left( 1 + \frac{1}{m(1-x_1u-x_2)} \left( (\beta-1)x_2 \right. \right. \\ &\quad \left. \left. - \frac{(\sigma_2-\alpha_2)}{2} ((1-x_1u)(1-\alpha_2-\sigma_2) - x_2(3-\alpha_2-\sigma_2)) \right) + O\left(\frac{1}{m^2}\right) \right) du \\ &= \int_0^1 e^{-m \ln(1-x_1u)} \frac{u^{\alpha_1-1} (1-u)^{\sigma_1-\alpha_1-1}}{(1-x_1u)^{\beta-1} (1-x_1u-x_2)} \left( 1 + \frac{1}{m(1-x_1u-x_2)} \left( (\beta-1)x_2 \right. \right. \\ &\quad \left. \left. - \frac{(\sigma_2-\alpha_2)}{2} ((1-x_1u)(1-\alpha_2-\sigma_2) - x_2(3-\alpha_2-\sigma_2)) \right) + O\left(\frac{1}{m^2}\right) \right) du. \end{aligned} \quad (4.18)$$

It is easy to see that the minimum of the expression under the logarithm sign depends on  $x_1$ . Accordingly, we consider the following three cases:  $x_1 = 0$ ,  $x_1 > 0$ , and  $x_1 < 0$ .

(1) Let  $x_1 = 0$ . In this case, there is no dependence on  $m$  and, with the help of (1.23), the integral can be represented as a product of gamma functions:

$$\begin{aligned} I_u &= \int_0^1 u^{\alpha_1-1} (1-u)^{\sigma_1-\alpha_1-1} du \frac{1}{1-x_2} \left( 1 + \frac{1}{m(1-x_2)} \left( (\beta-1)x_2 \right. \right. \\ &\quad \left. \left. - \frac{\sigma_2-\alpha_2}{2} ((1-\alpha_2-\sigma_2) - x_2(3-\alpha_2-\sigma_2)) \right) + O\left(\frac{1}{m^2}\right) \right) \\ &= \frac{\Gamma(\alpha_1)\Gamma(\sigma_1-\alpha_1)}{\Gamma(\sigma_1)} \frac{1}{1-x_2} \left( 1 + \frac{1}{m(1-x_2)} \left( (\beta-1)x_2 \right. \right. \\ &\quad \left. \left. - \frac{\sigma_2-\alpha_2}{2} ((1-\alpha_2-\sigma_2) - x_2(3-\alpha_2-\sigma_2)) \right) + O\left(\frac{1}{m^2}\right) \right). \end{aligned} \quad (4.19)$$

Substituting this result into (4.17) and taking into account

$$\left| \frac{\Gamma(c_2+m)}{\Gamma(a_2+m)} \right| = m^{\sigma_2-\alpha_2} \left( 1 + O\left(\frac{1}{m}\right) \right), \quad m \rightarrow \infty, \quad (4.20)$$

which follows from the Stirling formula, for  $x_1 = 0$ , we obtain

$$\begin{aligned} \mathcal{I}^{(0,m)}(x_1, x_2) &= \frac{K_0}{1-x_2} \left( 1 + \frac{1}{m(1-x_2)} \left( (\beta-1)x_2 \right. \right. \\ &\quad \left. \left. - \frac{\sigma_2-\alpha_2}{2} ((1-\alpha_2-\sigma_2) - x_2(3-\alpha_2-\sigma_2)) \right) + O\left(\frac{1}{m^2}\right) \right), \end{aligned} \quad (4.21)$$

where  $K_0$  is defined in (4.5). Note that, if  $b, a_1, a_2, c_1, c_2$  are real, then  $K_0 = 1$ .

In view of notation (4.6), expression (4.21) coincides with the desired expression (4.4).

(2) Now consider the case  $x_1 > 0$ . The expression  $\ln(1-x_1u)$  in the exponent in formula (4.18) has a minimum at the point  $u = 1$  for  $u \in [0, 1]$ . Making the substitution  $u = 1-x$ , we obtain

$$\begin{aligned} I_u &= \int_0^1 e^{-m \ln(1-x_1(1-x))} \frac{(1-x)^{\alpha_1-1} x^{\sigma_1-\alpha_1-1}}{(1-x_1(1-x))^{\beta-1} (1-x_1(1-x)-x_2)} \left( 1 + \frac{1}{m(1-x_1(1-x)-x_2)} \right. \\ &\quad \left. - \left( (\beta-1)x_2 - \frac{\sigma_2-\alpha_2}{2} ((1-x_1(1-x))(1-\alpha_2-\sigma_2) - x_2(3-\alpha_2-\sigma_2)) \right) + O\left(\frac{1}{m^2}\right) \right) dx. \end{aligned}$$

Applying Theorem 1 yields the following asymptotics for  $I_u$  as  $m \rightarrow \infty$ :

$$I_u = \frac{\Gamma(\sigma_1 - \alpha_1)}{m^{\sigma_1 - \alpha_1}} \frac{(1 - x_1)^{\sigma_1 - \alpha_1 - b - m + 1}}{x_1^{\sigma_1 - \alpha_1} (1 - x_1 - x_2)} \left( 1 + \frac{1}{m} \left( \frac{\sigma_1 - \alpha_1}{2} \left( 1 + \alpha_1 - 2\beta + \sigma_1 + \frac{2(1 - \alpha_1)}{x_1} - \frac{2(1 - x_1)}{1 - x_1 - x_2} \right) + \frac{(\beta - 1)x_2}{1 - x_1 - x_2} + \frac{\sigma_2 - \alpha_2}{2(1 - x_1 - x_2)} \left( x_2(3 - \alpha_2 - \sigma_2) - \frac{1 - x_1}{1 - \alpha_2 - \sigma_2} \right) \right) + O\left(\frac{1}{m^2}\right) \right) \tag{4.22}$$

Substituting (4.22) into (4.17) and taking into account (4.20), for  $x_1 > 0$ , we obtain

$$\mathcal{I}^{(0,m)} = \frac{K_+}{m^{\sigma_1 - \alpha_1} (1 - x_1)^m} \frac{(1 - x_1)^{\sigma_1 - \alpha_1 - b + 1}}{x_1^{\sigma_1 - \alpha_1} (1 - x_1 - x_2)} \left( 1 + \frac{1}{m} \left( \frac{\sigma_1 - \alpha_1}{2} \left( 1 + \alpha_1 - 2\beta + \sigma_1 + \frac{2(1 - \alpha_1)}{x_1} - \frac{2(1 - x_1)}{1 - x_1 - x_2} \right) + \frac{(\beta - 1)x_2}{1 - x_1 - x_2} + \frac{\sigma_2 - \alpha_2}{2(1 - x_1 - x_2)} \left( x_2(3 - \alpha_2 - \sigma_2) - \frac{1 - x_1}{1 - \alpha_2 - \sigma_2} \right) \right) + O\left(\frac{1}{m^2}\right) \right), \tag{4.23}$$

where  $K_+$  is defined in (4.8). Note that, if  $b, a_1, a_2, c_1, c_2$  are real, then

$$K_+ = \left| \frac{\Gamma(c_1)}{\Gamma(a_1)} \right|.$$

In view of notation (4.9), expression (4.23) coincides with the desired expression (4.7).

(3) Consider the case  $x_1 < 0$ . Then  $\ln(1 - x_1 u)$  in the exponent in formula (4.18) has a minimum at the point  $u = 0$  for  $u \in [0, 1]$ . Applying Theorem 1 from [24], p. 58, to the integral for  $I_u$ , we obtain the following asymptotics as  $m \rightarrow \infty$ :

$$I_u = \frac{\Gamma(\alpha_1)}{m^{\alpha_1} (-x_1)^{\alpha_1} (1 - x_2)} \left( 1 - \frac{1}{m} \left( -\frac{\alpha_1}{2} (3 + \alpha_1 - 2\beta) + \frac{\alpha_1}{x_1} (1 + \alpha_1 - \sigma_1) + \frac{1}{1 - x_2} \left( \alpha_1 + (\beta - 1)x_2 - \frac{\sigma_2 - \alpha_2}{2} (-1 + \alpha_2 + \sigma_2 + (-3 + \alpha_2 + \sigma_2)x_2) \right) \right) \right) + O\left(\frac{1}{m^2}\right). \tag{4.24}$$

Substituting the resulting expression into formula (4.17) for  $\mathcal{I}^{0,m}$ , in the case  $x_1 < 0$ , we obtain

$$\mathcal{I}^{0,m} = \frac{K_-}{m^{\alpha_1} (-x_1)^{\alpha_1} (1 - x_2)} \left( 1 - \frac{1}{m} \left( -\frac{\alpha_1}{2} (3 + \alpha_1 - 2\beta) + \frac{\alpha_1}{x_1} (1 + \alpha_1 - \sigma_1) + \frac{1}{1 - x_2} \left( \alpha_1 + (\beta - 1)x_2 - \frac{\sigma_2 - \alpha_2}{2} (-1 + \alpha_2 + \sigma_2 + (-3 + \alpha_2 + \sigma_2)x_2) \right) \right) \right) + O\left(\frac{1}{m^2}\right), \tag{4.25}$$

where  $K_-$  is defined in formula (4.11). If  $b, a_1, a_2, c_1, c_2$  are real, then

$$K_- = \left| \frac{\Gamma(c_1)}{\Gamma(c_1 - a_1)} \right|.$$

In view of notation (4.12), expression (4.25) coincides with the desired expression (4.10).

Proposition 4 is proved.

#### 4.2. Asymptotics $I^{(m,m)}$ As $m \rightarrow \infty$

The following proposition provides an estimate of the integral  $I^{(m,m)}(z_1, z_2)$  involved in expression (3.13) for  $R^{(m,m)}(z_1, z_2)$  as  $m \rightarrow \infty$ . Recall that  $x_1 = \operatorname{Re} z_1$  and  $x_2 = \operatorname{Re} z_2$ .

**Proposition 5.** For  $|z_1| + |z_2| < 1$ , the integral  $I^{(m,m)}(z_1, z_2)$  defined in (2.6) satisfies the estimate  $|I^{(m,m)}(z_1, z_2)| \leq \mathcal{I}^{(m,m)}(x_1, x_2)$ , where  $\mathcal{I}^{(m,m)}(x_1, x_2)$  has the following asymptotics as  $m \rightarrow \infty$ .

(1) If  $x_1 < 1/2, x_2 < 1/2$ , then

$$\mathcal{I}^{(m,m)}(x_1, x_2) = \frac{K}{(1 - 2x_1)(1 - 2x_2)} \left( 1 + O\left(\frac{1}{m}\right) \right). \tag{4.26}$$

(2) If  $x_1 > 1/2$ , then

$$\mathcal{I}^{(m,m)}(x_1, x_2) = 2^{1-b} \sqrt{\pi} K \frac{\sqrt{m}}{4^m x_1^m (1-x_1)^m} \frac{(1-x_1)^{\operatorname{Re}[c_1-a_1]+1}}{x_1^{\operatorname{Re}[c_1-a_1]} (1-x_1-x_2)} \left(1 + O\left(\frac{1}{m}\right)\right). \quad (4.27)$$

(3) If  $x_2 > 1/2$ , then

$$\mathcal{I}^{(m,m)}(x_1, x_2) = 2^{1-b} \sqrt{\pi} K \frac{\sqrt{m}}{4^m x_2^m (1-x_2)^m} \frac{(1-x_2)^{\operatorname{Re}[c_2-a_2]+1}}{x_2^{\operatorname{Re}[c_2-a_2]} (1-x_1-x_2)} \left(1 + O\left(\frac{1}{m}\right)\right). \quad (4.28)$$

(4) If  $x_1 = 1/2$ , then

$$\mathcal{I}^{(m,m)}(x_1, x_2) = \sqrt{\pi} K \frac{\sqrt{m}}{1-2x_2} \left(1 + O\left(\frac{1}{m}\right)\right). \quad (4.29)$$

(5) If  $x_2 = 1/2$ , then

$$\mathcal{I}^{(m,m)}(x_1, x_2) = \sqrt{\pi} K \frac{\sqrt{m}}{1-2x_1} \left(1 + O\left(\frac{1}{m}\right)\right). \quad (4.30)$$

In formulas (4.26)–(4.30), the coefficient  $K$  is given by

$$K = \frac{\Gamma(\operatorname{Re}(c_1 - a_1)) \Gamma(\operatorname{Re}(c_2 - a_2))}{|\Gamma(c_1 - a_1)| |\Gamma(c_2 - a_2)|} e^{\operatorname{Im} b/2}. \quad (4.31)$$

**Proof.** The following estimate holds:

$$\begin{aligned} |I^{(m,m)}| &\leq \int |\dots| = m^2 \left| \frac{\Gamma(c_1 + m)}{\Gamma(a_1 + m)\Gamma(c_1 - a_1)} \frac{\Gamma(c_2 + m)}{\Gamma(a_2 + m)\Gamma(c_2 - a_2)} \right| e^{\operatorname{Im} b/2} \\ &\times \int_0^1 d\xi \int_0^1 dr \int_0^1 du \int_0^1 dv \left( \left( \frac{1}{1 - \xi \operatorname{Re} z_1 u - r \operatorname{Re} z_2 v} \right)^{2m+b} (1-\xi)^{m-1} (1-r)^{m-1} \right. \\ &\quad \left. \times u^{\operatorname{Re} a_1 + m - 1} (1-u)^{\operatorname{Re}(c_1 - a_1) - 1} v^{\operatorname{Re} a_2 + m - 1} (1-v)^{\operatorname{Re}(c_2 - a_2) - 1} \right). \end{aligned} \quad (4.32)$$

Throughout the rest of this section, we use the following notation:  $\beta = \operatorname{Re} b$ ,  $\alpha_1 = \operatorname{Re} a_1$ ,  $\alpha_2 = \operatorname{Re} a_2$ ,  $\sigma_1 = \operatorname{Re} c_1$ , and  $\sigma_2 = \operatorname{Re} c_2$ .

Let the right-hand side of (4.32) be denoted by  $\mathcal{I}^{(m,m)}(x_1, x_2)$ . To construct asymptotics of the integral  $\mathcal{I}^{(m,m)}(x_1, x_2)$ , we sequentially consider the integrals with respect to  $u$ ,  $v$ ,  $\xi$ , and  $r$ .

(I) We begin with the integrals with respect to  $du$  and  $dv$ . From (4.32) we extract the factors containing  $u$  and  $v$ , i.e., consider the repeated integral

$$\int_0^1 dv \int_0^1 du \left( \frac{1}{1 - \xi x_1 u - r x_2 v} \right)^{2m+\beta} u^{\alpha_1 + m - 1} (1-u)^{\sigma_1 - \alpha_1 - 1} v^{\alpha_2 + m - 1} (1-v)^{\sigma_2 - \alpha_2 - 1}. \quad (4.33)$$

(a) Consider the integral with respect to  $u$  first:

$$\begin{aligned} I_u &= \int_0^1 du \left( \frac{1}{1 - \xi x_1 u - r x_2 v} \right)^{2m+\beta} u^{\alpha_1 + m - 1} (1-u)^{\sigma_1 - \alpha_1 - 1} \\ &= \int_0^1 du \exp\left( m \ln \frac{u}{(1 - \xi x_1 u - r x_2 v)^2} \right) \frac{u^{\alpha_1 - 1} (1-u)^{\sigma_1 - \alpha_1 - 1}}{(1 - \xi x_1 u - r x_2 v)^\beta}. \end{aligned}$$

Since

$$\frac{d}{du} \left( \frac{u}{(1 - \xi x_1 u - r x_2 v)^2} \right) = \frac{1 + \xi x_1 u - r v x_2}{(1 - \xi x_1 u - r x_2 v)^3} > 0$$

on the interval  $u = [0, 1]$ , the function in the exponent attains its maximum at the point  $u = 1$ . Making the substitution  $u = 1 - x$  to transfer the maximum to the lower limit of the integral and applying Theorem 1, we obtain

$$I_u = \frac{\Gamma(\sigma_1 - \alpha_1)}{m^{\sigma_1 - \alpha_1}} \frac{1}{(1 - \xi x_1 - r v x_2)^{2m + \beta}} \left( \frac{1 - \xi x_1 - r v x_2}{1 + \xi x_1 - r v x_2} \right)^{\sigma_1 - \alpha_1} \left( 1 + O\left(\frac{1}{m}\right) \right), \quad m \rightarrow \infty. \tag{4.34}$$

(b) Substituting the expression obtained for  $I_u$  into (4.33) yields the following integral with respect to  $v$ :

$$I_v = \frac{\Gamma(\sigma_1 - \alpha_1)}{m^{\sigma_1 - \alpha_1}} \int_0^1 dv \frac{v^m}{(1 - \xi x_1 - r v x_2)^{2m}} \frac{v^{\alpha_2 - 1} (1 - v)^{\sigma_2 - \alpha_2 - 1}}{(1 - \xi x_1 - r v x_2)^b} \left( \frac{1 - \xi x_1 - r v x_2}{1 + \xi x_1 - r v x_2} \right)^{\sigma_1 - \alpha_1} \left( 1 + O\left(\frac{1}{m}\right) \right), \quad m \rightarrow \infty.$$

Making the substitution  $v = 1 - x$  and applying Theorem 1, we obtain the following asymptotics as  $m \rightarrow \infty$ :

$$I_v = \frac{\Gamma(\sigma_1 - \alpha_1) \Gamma(\sigma_2 - \alpha_2)}{m^{\sigma_1 - \alpha_1} m^{\sigma_2 - \alpha_2}} \frac{1}{(1 - \xi x_1 - r x_2)^{2m + \beta}} \left( \frac{1 - \xi x_1 - r x_2}{1 + \xi x_1 - r x_2} \right)^{\sigma_1 - \alpha_1} \left( \frac{1 - \xi x_1 - r x_2}{1 - \xi x_1 + r x_2} \right)^{\sigma_2 - \alpha_2} \left( 1 + O\left(\frac{1}{m}\right) \right). \tag{4.35}$$

Substituting expression (4.35) for the integral  $I_v$  into (4.32) and taking into account (4.20), we obtain the following asymptotics for  $\mathcal{I}^{(m,m)}(x_1, x_2)$ :

$$\begin{aligned} \mathcal{I}^{(m,m)}(x_1, x_2) &= m^2 K \int_0^1 d\xi \int_0^1 dr \left( (1 - \xi)^{m-1} (1 - r)^{m-1} \frac{1}{(1 - \xi x_1 - r x_2)^{2m}} \right. \\ &\times \left. \frac{(1 - \xi x_1 - r x_2)^{\sigma_1 - \alpha_1 + \sigma_2 - \alpha_2 - b}}{(1 + \xi x_1 - r x_2)^{\sigma_1 - \alpha_1} (1 - \xi x_1 + r x_2)^{\sigma_2 - \alpha_2}} \right) \left( 1 + O\left(\frac{1}{m}\right) \right), \quad m \rightarrow \infty, \end{aligned} \tag{4.36}$$

where  $K$  is defined by formula (4.31). Note that if  $c_1, a_1, c_2, a_2, b$  have zero imaginary parts, then  $K = 1$ .

(II) Consider the integrals with respect to  $\xi$  and  $r$  in (4.36):

$$I_{\xi r} = \int_0^1 d\xi \int_0^1 dr \left( \frac{(1 - \xi)(1 - r)}{(1 - \xi x_1 - r x_2)^2} \right)^m \frac{(1 - \xi x_1 - r x_2)^{\sigma_1 - \alpha_1 + \sigma_2 - \alpha_2 - b}}{(1 - \xi)(1 - r)(1 + \xi x_1 - r x_2)^{\sigma_1 - \alpha_1} (1 - \xi x_1 + r x_2)^{\sigma_2 - \alpha_2}}. \tag{4.37}$$

We introduce the following notation:

$$Q(\xi, r) := \ln \left( \frac{(1 - \xi)(1 - r)}{(1 - \xi x_1 - r x_2)^2} \right), \tag{4.38}$$

$$\phi(\xi, r) := \frac{(1 - \xi x_1 - r x_2)^{\sigma_1 - \alpha_1 + \sigma_2 - \alpha_2 - b}}{(1 - \xi)(1 - r)(1 + \xi x_1 - r x_2)^{\sigma_1 - \alpha_1} (1 - \xi x_1 + r x_2)^{\sigma_2 - \alpha_2}}. \tag{4.39}$$

Then  $I_{\xi r}$  becomes

$$I_{\xi r} = \int_0^1 d\xi \int_0^1 dr e^{m Q(\xi, r)} \phi(\xi, r). \tag{4.40}$$

To derive the asymptotics of integral (4.40) as  $m \rightarrow \infty$ , we examine the behavior of  $Q(\xi, r)$  in the integration domain depending on  $x_1$  and  $x_2$ . Recall that  $|x_1| + |x_2| < 1$ .

**Proposition 6.** (1) *If  $-1 < x_1 \leq 1/2$  and  $-1 < x_2 \leq 1/2$ , then the maximum of  $Q(\xi, r)$  is attained at the point  $(\xi, r) = (0, 0)$ .*

(2) *If  $1/2 < x_1 < 1$ , then the maximum of  $Q(\xi, r)$  is attained at the point  $(\xi, r) = (\xi_A, 0)$ , where  $\xi_A := (-1 + 2x_1)/x_1$ .*

(3) *If  $1/2 < x_2 < 1$ , then the maximum of  $Q(\xi, r)$  is attained at the point  $(\xi, r) = (0, r_A)$ , where  $r_A := (-1 + 2x_2)/x_2$ .*

**Proof.** According to (4.38), the partial derivatives of  $Q(\xi, r)$  have the form

$$\frac{\partial Q(\xi, r)}{\partial \xi} = \frac{-1 + rx_2 + 2x_1 - \xi x_1}{(1 - \xi)(1 - \xi x_1 - rx_2)}, \quad (4.41)$$

$$\frac{\partial Q(\xi, r)}{\partial r} = \frac{-1 + \xi x_1 + 2x_2 - rx_2}{(1 - r)(1 - \xi x_1 - rx_2)}. \quad (4.42)$$

It follows from (4.41) and (4.42) that  $\text{grad}Q$  does not vanish in the domain  $(0, 1) \times (0, 1)$ ; hence,  $Q(\xi, r)$  has no maximum in this domain. Now we check whether it has local maxima on the boundary of the integration domain.

(a) Consider the boundary of the integration domain for  $r = 0$ . Setting the first derivative (4.41) with  $r = 0$  to zero, we obtain an extremum point:

$$\left. \frac{\partial Q(\xi, r)}{\partial \xi} \right|_{r=0} = \frac{-1 + 2x_1 - \xi x_1}{(1 - \xi)(1 - \xi x_1)} = 0 \quad \text{for} \quad \xi_A = \frac{-1 + 2x_1}{x_1}. \quad (4.43)$$

It is easy to see that  $\xi_A \in (0, 1)$  only if  $x_1 > 1/2$ . Since

$$\left. \frac{\partial^2 Q(\xi, r)}{\partial \xi^2} \right|_{(\xi_A, 0)} = -\frac{x_1^2}{2(-1 + x_1)^2} < 0, \quad (4.44)$$

we conclude that the point  $(\xi_A, 0)$  is a maximizer of  $Q(\xi, r)$  on the boundary of the integration domain for  $r = 0$ .

Let us show that, for  $x_1 \leq 1/2$ , the maximum of  $Q(\xi, r)|_{r=0}$  is reached at the point  $(0, 0)$ . Indeed, for  $x_1 < 0$ , it is true that  $\xi_A > 1$ . Calculating  $\partial Q / \partial \xi|_{\xi=0} = -1 + 2x_1$ , we conclude that the derivative is negative for  $x_1 < 0$ . In this case, the point  $(0, 0)$  is a maximizer. For  $x_1 = 0$ , derivative (4.41) is negative on the entire interval, so the point  $(0, 0)$  is a maximizer. For  $0 < x_1 < 1/2$ , it is true that  $\xi_A < 0$ . Calculating  $\partial Q / \partial \xi|_{\xi=0}$ , we conclude that the derivative is negative for  $x_1 < 1/2$ . In this case, the point  $(0, 0)$  is also a maximizer. For  $x_1 = 1/2$ , we have  $\xi_A = 0$ . The first derivative of  $Q$  at this point is zero, while the second derivative is negative. In this case, the point  $(0, 0)$  is a maximizer. The point  $\xi_A$  cannot be equal to 1, since  $|x_1| < 1$ .

(b) Considering the boundary of the integration domain for  $\xi = 0$ , by analogy with the case  $r = 0$ , we conclude that there is a maximizer on the boundary at the point  $(0, r_A)$ , where  $r_A = (-1 + 2x_2)/x_2$ . For  $x_2 \leq 1/2$ , there is a maximizer at  $(0, 0)$ , while there cannot be a maximizer for  $r = 1$ , since  $|x_2| < 1$ .

(c) On the parts of the boundary where  $\xi = 1$  or  $r = 1$ , there cannot be a maximum, because in this case the function  $Q(\xi, r)$  tends to  $-\infty$ .

Thus, Proposition 6 is proved.

To obtain the asymptotics of the integral  $I_{\xi, r}$  (see (4.40)), we sequentially consider the cases identified in Proposition 6.

(1) In the case of  $-1 < x_1 < 1/2$  and  $-1 < x_2 < 1/2$ , when, according to Proposition 6 the maximum of  $Q(\xi, r)$  is reached at the point  $(\xi, r) = (0, 0)$ , we pass to the polar coordinates  $\xi = \rho \cos t$  and  $r = \rho \sin t$ :

$$I_{\xi, r} = \left( \int_0^{\pi/4} dt \int_0^{1/\cos t} d\rho + \int_{\pi/4}^{\pi/2} dt \int_0^{1/\sin t} d\rho \right) e^{m \ln g(\rho, t)} \phi(\rho, t) := I_1 + I_2, \quad (4.45)$$

where

$$g(\rho, t) := \frac{(1 - \rho \cos t)(1 - \rho \sin t)}{(1 - \rho(x_1 \cos t + x_2 \sin t))^2},$$

$$\phi(\rho, t) = \frac{\rho(1 - \rho(x_1 \cos t + x_2 \sin t))^{\sigma_1 - \alpha_1 + \sigma_2 - \alpha_2 - b}}{(1 - \rho \cos t)(1 - \rho \sin t)(1 + \rho(x_1 \cos t - x_2 \sin t))^{\sigma_1 - \alpha_1}(1 - \rho(x_1 \cos t - x_2 \sin t))^{\sigma_2 - \alpha_2}}.$$

Fixing  $t$ , we consider the integrals with respect to  $\rho$ . To derive their asymptotics, we examine the behavior of  $g(\rho, t)$  on the intervals  $\rho = [0, 1/\cos t]$  and  $\rho = [0, 1/\sin t]$ . The denominator  $g(\rho, t) \neq 0$ . Let us find an extremum point, i.e., the solution of the equation  $\partial g(\rho, t)/\partial \rho = 0$ :

$$\rho = \rho_0 := \frac{\cos t(2x_1 - 1) + \sin t(2x_2 - 1)}{\cos t \sin t(-2 + x_1 + x_2) + x_1 \cos^2 t + x_2 \sin^2 t}. \tag{4.46}$$

The following proposition determines the position of  $\rho_0$  relative to the intervals  $\rho = [0, 1/\cos t]$  and  $\rho = [0, 1/\sin t]$  depending on the values of  $x_1, x_2$ , and  $t$ .

**Proposition 7.** *For  $\rho_0$  defined by formula (4.46), the following inequalities hold:*

(1) *If  $x_1 \leq 0$ , then  $\rho_0 > 1/\cos t$ , while if  $0 < x_1 < 1/2$ , then*

$$\begin{cases} \rho_0 < 0 & \text{if } 0 < \tan t < \tau_1, \\ 1/\cos t < \rho_0 & \text{if } \tau_1 < \tan t < 1, \end{cases}$$

where

$$\tau_1 = \frac{2 - x_1 - x_2 - \sqrt{(2 - x_1 - x_2)^2 - 4x_1x_2}}{2x_2}. \tag{4.47}$$

(2) *If  $x_2 \leq 0$ , then  $\rho_0 > 1/\sin t$ , while if  $0 < x_2 < 1/2$ , then*

$$\begin{cases} \rho_0 < 0 & \text{if } 0 < \cot t < \zeta_1, \\ 1/\sin t < \rho_0 & \text{if } \zeta_1 < \cot t < 1, \end{cases}$$

where  $\zeta_1$  is given by

$$\zeta_1 = \frac{2 - x_1 - x_2 - \sqrt{(2 - x_1 - x_2)^2 - 4x_1x_2}}{2x_1}.$$

**Proof.** (1) First, we compare  $\rho_0$  with 0 and  $1/\cos t$ . By noting that  $\cos t > 0$  for  $t \in [0, \pi/2]$ , the expression for  $\rho_0$  can be rewritten as

$$\rho_0 = \frac{1}{\cos t} \frac{2x_1 - 1 + \tan t(2x_2 - 1)}{\tan t(-2 + x_1 + x_2) + x_1 + x_2 \tan^2 t}. \tag{4.48}$$

We introduce the following notation:

$$\begin{aligned} N(\tan t, x_1, x_2) &= 2x_1 - 1 + \tan t(2x_2 - 1), \\ D(\tan t, x_1, x_2) &= \tan t(-2 + x_1 + x_2) + x_1 + x_2 \tan^2 t. \end{aligned} \tag{4.49}$$

(a) Since  $\cos t > 0$ , to compare  $\rho_0$  with 0, it suffices to compare the ratio  $N/D$  with 0. For this purpose, we separately compare the numerator  $N$  and the denominator  $D$  with 0.

(a.1) It is easy to show that, for  $-1 < x_1 < 1/2$  and  $-1 < x_2 < 1/2$ , the numerator  $N$  is always negative for  $\tan t \in [0, 1]$ .

(a.2) Setting the denominator  $D(\tan t, x_1, x_2)$  to zero and denoting its zeros with respect to  $\tan t$ , i.e., the solutions of the equation  $D(\tan t) = 0$ , by  $\tau_1$  (see (4.47)) and  $\tau_2$ , we obtain the following results: if  $x_1 > 0$  and  $x_2 \neq 0$ , then  $D > 0$  for  $0 \leq \tan t < \tau_1$ , but  $D < 0$  for  $\tau_1 < \tan t \leq 1$ . If  $x_1 > 0$  and  $x_2 = 0$ , then  $D > 0$  for  $0 \leq \tan t < \tau_3$ , but  $D < 0$  for  $\tau_3 < \tan t \leq 1$ ; here,  $\tau_3 = x_1/(2 - x_1)$ . If  $x_1 < 0$ , then  $D < 0$  for  $0 \leq \tan t \leq 1$ . If  $x_1 = 0$ , then  $D < 0$  for  $0 < \tan t \leq 1$  and  $D = 0$  for  $\tan t = 0$ .

Examining the sign of  $N/D$ , we conclude that  $\rho_0 > 0$  for  $x_1 \leq 0$ , but  $\rho_0 < 0$  for  $0 < x_1 < 1/2$  in the case  $0 < \tan t < \tau_1$ , while  $\rho_0 > 0$  in the case  $\tau_1 < \tan t < 1$ .

(b) Now we compare  $\rho_0$  with  $1/\cos t$ . It follows from (4.48) that, instead of comparing  $\rho_0$  with  $1/\cos t$ , it suffices to compare  $N/D$  with 1. Since  $D - N > 0$  on the interval  $\tan t \in [0, 1]$  for  $x_2 > 0$  and  $x_2 < 0$ , we conclude that  $\rho_0 < 1/\cos t$  for  $D > 0$  and  $\rho_0 > 1/\cos t$  for  $D < 0$ . Comparing this result with that obtained in (a), we derive the assertion of item (1).

(2) To prove the assertion in item (2), we represent  $\rho_0$  in the form

$$\rho_0 = \frac{1}{\sin t} \frac{2x_2 - 1 + \cot t(2x_1 - 1)}{\cot t(-2 + x_1 + x_2) + x_2 + x_1 \cot^2 t}.$$

Then, using the results of item (1) and taking into account that  $N = N(\cot t, x_2, x_1)$  and  $D = D(\cot t, x_2, x_1)$ , we obtain assertion (2).

Proposition 7 is proved.

It is easy to show that the point  $\rho_0$  is a maximizer of  $g(\rho, t)$  if  $D > 0$  and a minimizer of  $g(\rho, t)$  if  $D < 0$ .

Now, given the behavior of  $g(\rho, t)$ , we apply Theorem 1 to the integrals  $I_1$  and  $I_2$  defined in (4.45). Consider different domains with respect to  $x_1, x_2$ .

(a) Let  $x_1 \leq 0, x_2 \leq 0$ . For these  $x_1, x_2$ , we have  $D < 0 \forall \tan t \in (0, 1)$  and  $\rho_0 > 1/\cos t$ , so the function  $g(\rho, t)$  on the interval  $\rho \in [0, 1/\cos t]$  has a maximum at the point  $\rho = 0$ . Passing from the variable  $t$  to  $\tan t$  in the first integral  $I_1$  and applying Theorem 1 to the integral with respect to  $\rho$ , we obtain

$$I_1 = \frac{1}{m^2} \int_0^1 \frac{d \tan t}{((1 - 2x_1) + (1 - 2x_2) \tan t)^2} \left(1 + O\left(\frac{1}{m}\right)\right). \quad (4.50)$$

Calculating the integral with respect to  $\tan t$  yields the following estimate for the first integral:

$$I_1 = \frac{1}{m^2} \frac{1}{2(1 - 2x_1)(1 - x_1 - x_2)} \left(1 + O\left(\frac{1}{m}\right)\right). \quad (4.51)$$

Similarly, for the second integral  $I_2$ , the function  $g(\rho, t)$  on the interval  $\rho \in [0, 1/\sin t]$  has a maximum at the point  $\rho = 0$ . Passing from the variable  $t$  to  $\cot t$  and applying Theorem 1 to the integral with respect to  $\rho$ , we find that

$$I_2 = \frac{1}{m^2} \int_0^1 \frac{d \cot t}{((1 - 2x_1) \cot t + (1 - 2x_2))^2} \left(1 + O\left(\frac{1}{m}\right)\right) = \frac{1}{m^2} \frac{1}{2(1 - 2x_2)(1 - x_1 - x_2)} \left(1 + O\left(\frac{1}{m}\right)\right). \quad (4.52)$$

Adding (4.51) to (4.52) yields

$$I_{\xi_r} = \frac{1}{m^2} \frac{1}{(1 - 2x_1)(1 - 2x_2)} \left(1 + O\left(\frac{1}{m}\right)\right). \quad (4.53)$$

(b) Consider the case of  $0 < x_1 < 1/2, 0 < x_2 < 1/2$ . Here, in the integrals with respect to  $t$  in (4.45), we need to pass to  $\tan t$  and  $\cot t$  and then to divide each of the integrals into two. For example, the integral  $I_1$  is divided into integrals from 0 to  $\tau_1$  and from  $\tau_1$  to 1:

$$\begin{aligned} I_1 &= \int_0^1 \cos^2 t d \tan t \int_0^{1/\cos t} d\rho e^{m \ln g(\rho, t)} \phi(\rho, t) \\ &= \left( \int_0^{\tau_1} \cos^2 t d \tan t \int_0^{1/\cos t} d\rho + \int_{\tau_1}^1 \cos^2 t d \tan t \int_0^{1/\cos t} d\rho \right) e^{m \ln g(\rho, t)} \phi(\rho, t). \end{aligned} \quad (4.54)$$

In the first of these integrals,  $\rho_0$  is a maximizer of  $g(\rho, t)$  such that  $\rho_0 < 0$ , so the maximum of  $g(\rho, t)$  on the interval  $(0, 1/\cos t)$  is reached at 0. In the second integral in (4.54), the point  $\rho_0$  is a minimizer of  $g(\rho, t)$ , but  $\rho_0 > 1/\cos t$ , so the maximum of  $g(\rho, t)$  on the interval  $(0, 1/\cos t)$  is also reached at 0. Applying Theorem 1 to both integrals and then integrating the result with respect to  $\tan t$ , we obtain

$$I_1 = \frac{1}{2m^2} \frac{1}{(1 - 2x_2)(1 - x_1 - x_2)}. \quad (4.55)$$

Similarly, for  $I_2$ , we find

$$I_2 = \frac{1}{2m^2} \frac{1}{(1 - 2x_1)(1 - x_1 - x_2)}. \quad (4.56)$$

Adding (4.55) to (4.56) yields  $I_{\xi_r}$  in the domain  $0 < x_1 \leq 1/2, 0 < x_2 < 1/2$ :

$$I_{\xi_r} = \frac{1}{m^2} \frac{1}{(1 - 2x_1)(1 - 2x_2)}. \tag{4.57}$$

Substituting (4.53) and (4.57) into (4.36), we obtain the desired asymptotics (4.26) for the integral  $I^{(m,m)}(x_1, x_2)$  as  $m \rightarrow \infty$ .

(c) In the case of  $0 < x_1 < 1/2, x_2 < 0$  for  $I_1$  and  $I_2$ , we add (4.55) to (4.52) to obtain the same result as (4.57) for  $I_{\xi_r}$ . For  $0 < x_2 < 1/2, x_1 < 0$ , we also obtain (4.57) for  $I_{\xi_r}$ .

(2) Consider the case  $x_1 > 1/2$ . Then the maximizer of  $Q(\xi, r)$  (see (4.38)) lies at the boundary  $r = 0$  at the point  $(\xi_A, 0)$ . To obtain the asymptotics of the integral  $I_{\xi_r}$  in (4.40) we apply Theorem 2:

$$I_{\xi_r} = \left(\frac{2\pi}{\kappa_0}\right)^{1/2} \phi(\xi_A, 0) e^{mQ(\xi_A, 0)} \frac{1}{m^{3/2}} \left(1 + O\left(\frac{1}{m}\right)\right), \tag{4.58}$$

here,

$$\kappa_0 = - \left( \frac{\partial^2 Q}{\partial \xi^2} \left(\frac{\partial Q}{\partial r}\right)^2 - 2 \frac{\partial^2 Q}{\partial \xi \partial r} \frac{\partial Q}{\partial \xi} \frac{\partial Q}{\partial r} + \frac{\partial^2 Q}{\partial r^2} \left(\frac{\partial Q}{\partial \xi}\right)^2 \right) \Big|_{(\xi_A, 0)}.$$

The functions  $Q(\xi, r)$  and  $\phi(\xi, r)$  are defined by formulas (4.38) and (4.39), respectively, so

$$\phi(\xi_A, 0) = \frac{(1 - x_1)^{\sigma_1 - \alpha_1 - \beta - 1}}{2^\beta x_1^{\sigma_1 - \alpha_1 - 1}}, \quad e^{mQ(\xi_A, 0)} = \frac{1}{(4x_1(1 - x_1))^m}. \tag{4.59}$$

Since  $\partial Q / \partial r(\xi_A, 0) = -(1 - x_1 - x_2) / (1 + x_1)$ ,  $\partial Q / \partial \xi(\xi_A, 0) = 0$ , and the second derivative  $\partial^2 Q / \partial \xi^2(\xi_A, 0)$  is given by formula (4.44), we obtain

$$\kappa_0 = \frac{x_1^2(1 - x_1 - x_2)^2}{2(1 - x_1)^4}. \tag{4.60}$$

Substituting (4.59) and (4.60) into (4.58), we obtain the following asymptotics of  $I_{\xi_r}$  as  $m \rightarrow \infty$  in the case  $x_1 > 1/2$ :

$$I_{\xi_r} = \frac{1}{m^{3/2}} \frac{\sqrt{\pi}}{2^{\beta-1} 4^m x_1^m (1 - x_1)^m} \frac{(1 - x_1)^{\sigma_1 - \alpha_1 - \beta + 1}}{(1 - x_1 - x_2) x_1^{\sigma_1 - \alpha_1}} \left(1 + O\left(\frac{1}{m}\right)\right). \tag{4.61}$$

Substituting this result into (4.36) yields the sought asymptotics (4.27).

Similarly, considering  $x_2 > 1/2$ , when the maximum of  $Q(\xi, r)$  (4.38) lies at the boundary  $\xi = 0$  at the point  $(0, r_A)$ , we obtain the asymptotics of  $I_{\xi_r}$ , and substituting this result into (4.36) yields the desired asymptotics (4.28) as  $m \rightarrow \infty$ .

(3) We consider the cases  $x_1 = 1/2$  and  $x_2 = 1/2$  separately.

(a) If  $x_2 = 1/2$ , then Theorem 1 is applied to the integral with respect to  $d\xi$ , the result is substituted into (4.37), and Theorem 1 is applied to the integral with respect to  $dr$ . As a result, we obtain

$$I_{\xi_r} = \frac{\sqrt{\pi}}{m^{3/2}(1 - 2x_1)} \left(1 + O\left(\frac{1}{m}\right)\right). \tag{4.62}$$

Substituting (4.62) into (4.36) yields (4.30).

(b) If  $x_1 = 1/2$ , then Theorem 1 is applied to the integral with respect to  $dr$  and then to the integral with respect to  $d\xi$ . As a result, we obtain

$$I_{\xi_r} = \frac{\sqrt{\pi}}{m^{3/2}(1 - 2x_2)} \left(1 + O\left(\frac{1}{m}\right)\right). \tag{4.63}$$

Substituting (4.63) into (4.36) yields (4.29).

Proposition 5 is proved.

### 4.3. Relations between the Absolute Values of $R^{(0,m)}$ , $R^{(m,0)}$ , and $R^{(m,m)}$

In this section, we set

$$z_1 = x_1, \quad z_2 = x_2; \quad b = \beta; \quad a_1 = \alpha_1, \quad a_2 = \alpha_2, \quad c_1 = \sigma_1, \quad c_2 = \sigma_2, \quad (4.64)$$

i.e., the variables and parameters are real. The following propositions show which of the terms  $R^{(0,m)}$ ,  $R^{(m,0)}$ , and  $R^{(m,m)}$  in sum (3.1) representing the remainder  $\mathcal{R}^{(m)}$  dominates depending on  $x_1, x_2$ .

**Proposition 8.** *If conditions (4.64) hold, then the absolute values of  $R^{(0,m)}(x_1, x_2)$  and  $R^{(m,0)}(x_1, x_2)$  defined by formulas (3.2) and (3.3), respectively, satisfy the following relations for  $|x_1| + |x_2| < 1$ .*

(1) *If*

$$\begin{cases} x_2 < x_1, \\ x_2 < x_1(x_1 - 1) \end{cases} \quad \text{or} \quad \begin{cases} x_1 < x_2, \\ x_2(x_2 - 1) < x_1, \end{cases} \quad (4.65)$$

then  $|R^{(m,0)}(x_1, x_2)| = |R^{(0,m)}(x_1, x_2)|O(m^{-2})$ ,  $m \rightarrow \infty$ .

(2) *If*

$$\begin{cases} x_1 < x_2, \\ x_1 < x_2(x_2 - 1) \end{cases} \quad \text{or} \quad \begin{cases} x_2 < x_1, \\ x_1(x_1 - 1) < x_2, \end{cases} \quad (4.66)$$

then  $|R^{(0,m)}(x_1, x_2)| = |R^{(m,0)}(x_1, x_2)|O(m^{-2})$ ,  $m \rightarrow \infty$ .

**Proof.** If equalities (4.64) hold, then

$$|I^{(0,m)}| = |\mathcal{I}^{(0,m)}|, \quad |I^{(m,0)}| = |\mathcal{I}^{(m,0)}|,$$

so, using formulas (4.1)–(4.10), (1.3), and (4.20), the relation  $|R^{(m,0)}(x_1, x_2)/R^{(0,m)}(x_1, x_2)|$  can be written as follows.

(1) If  $x_1 > 0$  and  $x_2 > 0$ , then

$$\frac{|R^{(m,0)}|}{|R^{(0,m)}|} = \left( \frac{|x_1|}{|x_2|} \frac{(1-x_1)^m}{(1-x_2)^m} \right) \widetilde{K}_1 \left( 1 + O\left(\frac{1}{m}\right) \right), \quad (4.67)$$

where  $\widetilde{K}_1 = \widetilde{K}_1(x_1, x_2; \alpha_1, \alpha_2; \beta; \sigma_1, \sigma_2)$  is a coefficient independent of  $m$ . Taking into account that  $|z|^m$  for  $|z| < 1$  decreases faster than any power function, we expand the moduli to obtain the following results for  $x_1 > 0, x_2 > 0$ :

$$\begin{aligned} |R^{(m,0)}(x_1, x_2)| &= |R^{(0,m)}(x_1, x_2)|O(m^{-2}) & \text{if } 0 < x_1 < x_2, \\ |R^{(0,m)}(x_1, x_2)| &= |R^{(m,0)}(x_1, x_2)|O(m^{-2}) & \text{if } 0 < x_2 < x_1. \end{aligned}$$

(2) If  $x_1 < 0$  and  $x_2 < 0$ , then

$$\frac{|R^{(m,0)}|}{|R^{(0,m)}|} = \frac{|x_1|^m}{|x_2|^m} \frac{m^{\sigma_2 - 2\alpha_2}}{m^{\sigma_1 - 2\alpha_1}} \widetilde{K}_2 \left( 1 + O\left(\frac{1}{m}\right) \right), \quad m \rightarrow \infty, \quad (4.68)$$

where  $\widetilde{K}_2 = \widetilde{K}_2(x_1, x_2; \alpha_1, \alpha_2; \beta; \sigma_1, \sigma_2)$  is a coefficient independent of  $m$ . Taking into account that  $|z|^m$  decreases faster than any power  $m^{-k}$ , we expand the moduli to obtain the following results for  $x_1 < 0, x_2 < 0$ :

$$\begin{aligned} |R^{(m,0)}(x_1, x_2)| &= |R^{(0,m)}(x_1, x_2)|O(m^{-2}) & \text{if } x_2 < x_1 < 0, \\ |R^{(0,m)}(x_1, x_2)| &= |R^{(m,0)}(x_1, x_2)|O(m^{-2}) & \text{if } x_1 < x_2 < 0. \end{aligned}$$

(3) If  $x_1 > 0, x_2 < 0$ , then

$$\left| \frac{R^{(m,0)}}{R^{(0,m)}} \right| = \left( \frac{x_1}{x_2} (1-x_1) \right)^m m^{\sigma_2 - 2\alpha_2} \widetilde{K}_3 \left( 1 + O\left(\frac{1}{m}\right) \right), \quad m \rightarrow \infty, \tag{4.69}$$

where  $\widetilde{K}_3 = \widetilde{K}_3(x_1, x_2; \alpha_1, \alpha_2; \beta; \sigma_1, \sigma_2)$  is a coefficient independent of  $m$ . Taking into account that  $|z|^{-m}$  decreases faster than any power  $m^{-k}$ , we expand the moduli to obtain the following results for  $x_1 > 0, x_2 < 0$ :

$$\begin{aligned} |R^{(m,0)}(x_1, x_2)| &= |R^{(0,m)}(x_1, x_2)| O(m^{-2}) && \text{if } x_2 < x_1(x_1 - 1), \\ |R^{(0,m)}(x_1, x_2)| &= |R^{(m,0)}(x_1, x_2)| O(m^{-2}) && \text{if } x_1(x_1 - 1) < x_2. \end{aligned}$$

(4) Similarly, for  $x_1 < 0, x_2 > 0$ , we obtain

$$\begin{aligned} |R^{(m,0)}(x_1, x_2)| &= |R^{(0,m)}(x_1, x_2)| O(m^{-2}) && \text{if } x_2(x_2 - 1) < x_1, \\ |R^{(0,m)}(x_1, x_2)| &= |R^{(m,0)}(x_1, x_2)| O(m^{-2}) && \text{if } x_1 < x_2(x_2 - 1). \end{aligned}$$

(5) If  $x_1 = 0$ , then  $|R^{(m,0)}(x_1, x_2)| = 0$ ; if  $x_2 = 0$ , then  $|R^{(0,m)}(x_1, x_2)| = 0$ .

Summarizing the above results, we obtain Proposition 8.

**Proposition 9.** *If conditions (4.64) hold, then the absolute values of  $R^{(0,m)}(z_1, z_2), R^{(m,0)}(z_1, z_2)$ , and  $R^{(m,m)}(z_1, z_2)$  defined by formulas (3.2)–(3.4), respectively, satisfy the following relations for  $|x_1| + |x_2| < 1$ .*

(1) *If*

$$\begin{cases} 1/2 < x_2, \\ x_2(x_2 - 1) < x_1 \end{cases} \quad \text{or} \quad \begin{cases} x_2 < 1/2, \\ -1/4 < x_1 < 1/2, \end{cases} \tag{4.70}$$

then  $|R^{(m,m)}(x_1, x_2)| = |R^{(0,m)}(x_1, x_2)| O(m^{-2}), m \rightarrow \infty$ , and if

$$\begin{cases} 1/2 < x_1, \\ x_1(x_1 - 1) < x_2 \end{cases} \quad \text{or} \quad \begin{cases} x_1 < 1/2, \\ -1/4 < x_2 < 1/2, \end{cases} \tag{4.71}$$

then  $|R^{(m,m)}(x_1, x_2)| = |R^{(m,0)}(x_1, x_2)| O(m^{-2}), m \rightarrow \infty$ .

(2) *If*

$$\begin{cases} 1/2 < x_2, \\ x_1 < x_2(x_2 - 1) \end{cases} \quad \text{or} \quad \begin{cases} x_2 < 1/2, \\ x_1 < -1/4, \end{cases} \tag{4.72}$$

then  $|R^{(0,m)}(x_1, x_2)| = |R^{(m,m)}(x_1, x_2)| O(m^{-2}), m \rightarrow \infty$ , and if

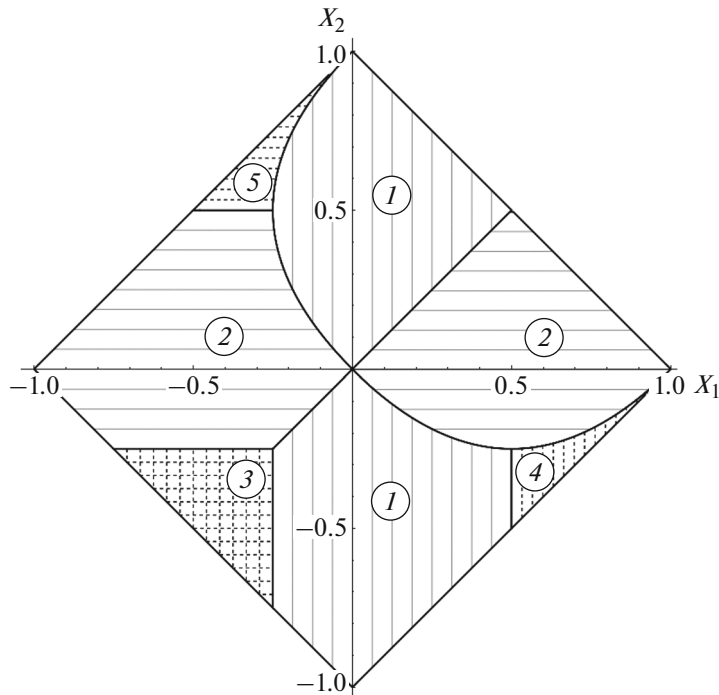
$$\begin{cases} 1/2 < x_1, \\ x_2 < x_1(x_1 - 1) \end{cases} \quad \text{or} \quad \begin{cases} x_1 < 1/2, \\ x_2 < -1/4, \end{cases} \tag{4.73}$$

then  $|R^{(m,0)}(x_1, x_2)| = |R^{(m,m)}(x_1, x_2)| O(m^{-2}), m \rightarrow \infty$ .

(3)  $|R^{(0,m)}(x_1, x_2)|$  and  $|R^{(m,m)}(x_1, x_2)|$  are comparable for  $x_1 > 1/2$ , while  $|R^{(m,0)}(x_1, x_2)|$  and  $|R^{(m,m)}(x_1, x_2)|$  are comparable for  $x_2 > 1/2$ .

**Proof.** It is similar to the proof of Proposition 8.

The results of Propositions 8 and 9 are illustrated in Fig. 1. Specifically,  $|R^{(0,m)}(x_1, x_2)|$  dominates in domain 1;  $|R^{(m,0)}(x_1, x_2)|$  dominates in domain 2; and  $|R^{(m,m)}(x_1, x_2)|$ , in domain 3.  $|R^{(m,m)}(x_1, x_2)|$  and  $|R^{(0,m)}(x_1, x_2)|$  are comparable in domain 4, while  $|R^{(m,m)}(x_1, x_2)|$  and  $|R^{(m,0)}(x_1, x_2)|$  are comparable in domain 5.



**Fig. 1.** Comparison of  $|R^{(m,m)}(x_1, x_2)|$ ,  $|R^{(m,0)}(x_1, x_2)|$ , and  $|R^{(0,m)}(x_1, x_2)|$ .  $|R^{(0,m)}(x_1, x_2)|$  dominates in domain 1;  $|R^{(m,0)}(x_1, x_2)|$  dominates in domain 2; and  $|R^{(m,m)}(x_1, x_2)|$ , in domain 3.  $|R^{(m,m)}(x_1, x_2)|$  and  $|R^{(0,m)}(x_1, x_2)|$  are comparable in domain 4, while  $|R^{(m,m)}(x_1, x_2)|$  and  $|R^{(m,0)}(x_1, x_2)|$  are comparable in domain 5.

Thus, based on the results obtained in this paper, the function  $F_2(z_1, z_2)$  in the domain  $\mathbb{M}$  defined in (1.1) can be computed with prescribed accuracy  $\delta$ . Specifically, by applying Theorem 3 or Propositions 4 and 5, it is possible to find  $m$  such that the absolute value of the remainder term is less than  $\delta$ . Then the partial sum of the series calculated up to terms of order  $m - 1$  differs from  $F_2(z_1, z_2)$  by less than  $\delta$ .

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#### CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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